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Generalized Classical Dynamical Yang-Baxter Equations and Moduli Spaces of Flat Connections on Surfaces

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Abstract: In this paper, we explain how generalized dynamical r-matrices can be obtained by (quasi-)Poisson reduction. New examples of Poisson structures, Poisson G-spaces and Poisson groupoid actions naturally appear in this setting. As an application, we use a generalized dynamical r-matrix, induced by the gauge fixing procedure, to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space of flat connections on a surface. Using this, we find a Poisson groupoid symmetry of the moduli space.

1. Introduction

The classical Yang-Baxter equation (CYBE) plays a key role in the theory of integrable systems. A geometric interpretation of CYBE was given by Drinfeld and gave rise to the theory of Poisson-Lie groups. The classical dynamical Yang-Baxter equation (CDYBE) is a differential equation analogue to CYBE and was introduced by Felder in [8] as the consistency condition for the differential Knizhnik-Zamolodchikov-Bernard equations for correlation functions in conformal field theory on tori. It was shown by Etingof and Varchenko [7] that dynamical r-matrices correspond to Poisson-Lie groupoids (a notion introduced by Weinstein [17]) in much the same way as r-matrices correspond to Poisson-Lie groups. In the meantime, the classical dynamical Yang-Baxter equation is proven to be closely connected with the theory of homogeneous Poisson spaces [5], Dirac structures and Lie bialgebroids [16], see [13,14] and references therein. Inspired by the study of Lie bialgebroids, the notion of generalized classical dynamical Yang-Baxter equations was introduced by Liu and Xu [13] in which the base manifold underlying the CDYBE can be a general Poisson manifold. Despite its importance, this subject suffered from the lack of examples for a long time.

Since Atiyah and Bott introduced canonical symplectic structures on the moduli spaces of flat connections on Riemann surface in [3], a lot of attention has been paid to the moduli spaces by mathematicians and physicists due to their rich mathematical structures and their links with a variety of topics. From the physics perspective, a major motivation for their study is their role in Chern-Simons theory. An independent mathematical motivation for investing moduli spaces of flat connections arises from Poisson geometry. The Atiyah-Bott symplectic structure on the moduli of flat G-connections over oriented surface Σ admits several finite dimensional descriptions. The first such description appears in Goldman's study of symplectic structures on character varieties $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, see [11]. Another possibility is to obtain the moduli space of flat G-connections on a surface $\Sigma_{g,n}$ of genus g with n punctures by (quasi-)Poisson reduction from an enlarged ambient G^{n+2g}. In the Fock-Rosly's approach [10], the Poisson structure on G^{n+2g} is described using a classical r-matrix. In the Alekseev-Malkin-Meinrenken's theory of Lie group valued moment maps [2], the moduli space is obtained by a reduction of a canonical quasi-Poisson tensor on G^{n+2g}.

These two subjects of dynamical Yang-Baxter equations and moduli spaces of flat connections appear to be very different. However, some recent works indicate the possible connection between them. From the viewpoint of Hamiltonian formalism, the moduli spaces of flat connections can be viewed as constrained Hamiltonian systems. Dirac gauge fixing for the moduli space of flat ISO(2, 1)-connections on a Riemann surface has been shown to give rise to generalized classical dynamical r-matrices in [15]. On the other hand, gauge fixing in the Poisson-Lie context has been shown to give rise to classical dynamical r-matrices in some cases [9].

In this paper, we deepen the connection between these two subjects by giving a systematic investigation of the theory of generalized classical dynamical r-matrices. We explain how generalized dynamical r-matrices can be obtained by (quasi-)Poisson reduction. Furthermore, we show that new examples of Poisson structures, Poisson G-spaces and Poisson groupoid actions naturally appear in this setting. As a result, associated to a classical dynamical r-matrix, there is naturally a Poisson manifold carrying simultaneously a Hamiltonian action and a Poisson action, whose Hamiltonian reduction gives a homogeneous Poisson space. After that, we take the canonical quasi-Poisson manifold $G \circledast G$ as an example and concretely analyze the dynamical r-matrices arising from the quasi-Poisson reduction of $G \circledast G$. We also introduce the notion of gauge transformations for generalized dynamical r-matrices. As an application, we use a dynamical r-matrix, induced by the gauge fixing procedure, to give a new finite dimensional description of the symplectic structure on the moduli space. Using this, we find a Poisson groupoid symmetry of the moduli space. We end up with two examples, one of them was previously studied by Meusburger-Schönfeld in the framework of the ISO(2, 1)-Chern-Simons theory of (2 + 1)-dimensional gravity.

Our paper is structured as follows. In Sect. 2, we recall the definition of generalized classical dynamical Yang-Baxter equations and present some examples. After that, we give new examples of generalized dynamical r-matrices from (quasi-)Poisson reduction. We show that new Poisson structures, Poisson G-spaces and Poisson groupoid actions naturally appear in this setting. Moduli space dynamical r-matrices and gauge transformations for generalized dynamical r-matrices are introduced. In Sect. 3, we use a generalized dynamical r-matrix induced by the gauge fixing procedure to give a new finite dimensional description of the Atiyah-Bott symplectic structure on the moduli space of flat connections on surfaces.

2. Generalized Classical Dynamical Yang-Baxter Equations

First we fix some notations. Let M be a manifold, then a Poisson bivector π on M gives rise to a Lie algebroid structure on T^*M, denoted by (T^*M, π), where the anchor map is $\pi^\sharp : T^*M \to TM$ and the Lie bracket is

$$[\alpha, \beta] = L_{\pi^\sharp\alpha} - L_{\pi^\sharp\beta} - d(\pi(\alpha, \beta)),\ \forall\alpha, \beta \in \Omega^1(M). \tag{1}$$

Let G be a Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\{e_i\}_{i=1,\dots,n}$ be a basis of $\mathfrak{g}$. Given a tensor $\theta = \sum X_i \otimes e_i \in \Gamma(TM \otimes \mathfrak{g})$, a smooth map $r : M \to \mathfrak{g} \wedge \mathfrak{g}$ and a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, we define the following operations

$$\delta\theta = \sum X_i \otimes \delta e_i, \quad [r, \theta] = \sum X_i \otimes [r, e_i], \tag{2}$$

$$[\theta, \theta] = \sum [X_i, X_j] \otimes e_i \wedge e_j, \quad \theta \wedge \theta = \sum X_i \wedge X_j \otimes [e_i, e_j]. \tag{3}$$

We denote by $\theta^\sharp : T^*M \to M \times \mathfrak{g}$ the morphism associated with $\theta \in \Gamma(TM \otimes \mathfrak{g})$. With these preparatory notations, we can now give the following definition.

Definition 2.1. [13] For a Poisson manifold (M, π) and a Lie algebra $\mathfrak{g}$, assume that there exists a tensor $\theta \in \Gamma(TM \otimes \mathfrak{g})$ such that $\theta^\sharp : (T^*M, \pi) \to \mathfrak{g}$ is a Lie algebroid morphism. A function $r \in C^\infty(M, \wedge^2\mathfrak{g})$ is called a dynamical r-matrix coupled with the Poisson manifold (M, π) via θ if

$$\frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^\sharp(dr), \tag{4}$$

and the generalized DYBE is satisfied:

$$\mathrm{Alt}(\theta^\sharp dr) + \frac{1}{2}[r, r] = \Omega, \tag{5}$$

where $\mathrm{Alt}(\theta^\sharp dr(x))$ is the skew-symmetrization of $\theta^\sharp dr(x) \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ for all $x \in M$, and $\Omega \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$ is an invariant element and regarded as a constant section of $\wedge^3(M \times \mathfrak{g})$.

We call r a triangular dynamical r-matrix coupled with M via θ if the corresponding $\Omega = 0$ in (5). Throughout this paper, we will also use the triple $((M, \pi), \theta, r)$ to denote a generalized dynamical r-matrix.

Example 2.2. Let η be a Lie subalgebra of $\mathfrak{g}$ and $M = \eta^*$ with the natural linear Poisson structure. Let $\theta^\sharp : T^*\eta^* \to \mathfrak{g}$ be the natural projection: $(\xi, v) \to v$, for all $(\xi, v) \in \eta^* \times \eta$. We can write $\theta = \sum_{i=1}^k \frac{\partial}{\partial x_i} \otimes e_i$ in a basis $\{e_1, \dots, e_k\}$ of η and a dual basis $\{x_1, \dots, x_k\}$ of η^*. One checks that $\theta^\sharp$ is a Lie algebroid morphism, and $[\theta, \theta] = 0$. Therefore, the condition $\frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^*(dr)$ reduces to $[r, \theta] = \pi^*(dr)$, which says that the map $r : \eta^* \to \mathfrak{g} \wedge \mathfrak{g}$ is η-equivariant. In this case, equation $\mathrm{Alt}(\theta^\sharp dr) + \frac{1}{2}[r, r] = \Omega \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$ becomes the classical dynamical Yang-Baxter equation and a solution r is a classical dynamical r-matrix [8].

Similarly, we introduce a notion of the generalized Poisson-Lie dynamical Yang-Baxter equation.

Definition 2.3. For a Poisson manifold (M, π) and a Lie bialgebra $(\mathfrak{g}, \delta)$, assume that there exists a tensor $\theta \in \Gamma(TM \otimes \mathfrak{g})$ such that $\theta^\sharp : (T^*M, \pi) \to \mathfrak{g}$ is a Lie algebroid morphism. A function $r \in C^\infty(M, \wedge^2\mathfrak{g})$ is called a Poisson-Lie dynamical r-matrix coupled with the Poisson manifold (M, π) via θ if

(a) $\delta\theta + \frac{1}{2}[\theta, \theta] = [r, \theta] - \pi^\sharp(dr)$,
(b) the generalized Poisson-Lie DYBE is satisfied:

$$\mathrm{Alt}(\theta^\sharp dr) + \frac{1}{2}[r, r] + \delta r = \Omega \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}. \tag{6}$$

Example 2.4. Let G be a Poisson-Lie group and (G^*, π) the simply connected dual Poisson-Lie group. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g}^* = \mathrm{Lie}(G^*)$. To consider the generalized dynamical r-matrices on (G^*, π), we take a natural section θ of $TG^* \otimes \mathfrak{g}$, which is induced by the isomorphism $T^*G^* \to G^* \times \mathfrak{g}$. Then one checks that $\theta^\sharp : T^*G^* \to \mathfrak{g}$ is a Lie algebroid morphism. A direct calculation shows that equation (a) and (b) in Definition 2.3 reduces to

$$\mathrm{dress}_a^L(r) + [a \otimes 1 + 1 \otimes a, r] = 0,$$
$$[r, r] + \mathrm{Alt}(d^L r) + \mathrm{Alt}((\delta \otimes \mathrm{id})(r)) = \Omega,$$

for $r : G^* \to \mathfrak{g} \wedge \mathfrak{g}$ and any $a \in \mathfrak{g}$, where dress_a^L denotes the left dressing vector field generated by a and $d^L r(g) := e_i \otimes \frac{d}{dt}|_{t=0} r(\mathrm{e}^{-te_i} g)$ for each $g \in G^*$ and an orthonormal basis $\{e_i\}$ of $\mathfrak{g}$. Thus a map $r : G^* \to \mathfrak{g} \wedge \mathfrak{g}$ is a generalized dynamical r-matrix coupled with G^* via θ if and only if r is a Poisson-Lie dynamical r-matrix [6].

2.1. Generalized classical dynamical r-matrices from (quasi-)Poisson reduction. In this subsection, we explain that generalized classical dynamical r-matrices naturally appear in the theory of (quasi-)Poisson reduction. First, let us recall the definition of quasi-Poisson G-manifolds.

We assume that $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a quadratic Lie algebra, ϕ is the Cartan 3-tensor. In terms of an orthogonal basis $\{e_a\}$ of $\mathfrak{g}$, $\phi \in \wedge^3\mathfrak{g}$ is given by

$$\phi = \frac{1}{12} f_{abc} e_a \wedge e_b \wedge e_c, \tag{7}$$

where $f_{abc} = \langle e_a, [e_b, e_c] \rangle$ are the structure constants of $\mathfrak{g}$. Given a G-manifold M, the Lie algebra homomorphism $\rho : \mathfrak{g} \to TM$ can be extended to an equivariant map,

$$\rho : \wedge^\bullet \mathfrak{g} \to \wedge^\bullet TM,$$

preserving wedge products and Schouten brackets.

Definition 2.5. [1] A quasi-Poisson manifold is a G-manifold M, equipped with an invariant bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that

$$[\pi, \pi] = \rho(\phi). \tag{8}$$

Example 2.6. Let G be a Lie group and $\{e_a\}_{a \in I}$ be an orthogonal basis of its Lie algebra $\mathfrak{g}$. Define a bivector field on G by $\pi_G = \frac{1}{2}\sum_{a \in I} R_a \wedge L_a$, where R_a and L_a are right and left invariant vectors generated by e_a. Then (G, π_G) is a quasi-Poisson G-manifold, where G acts on itself by conjugation.

Generally, the G-invariant functions on a quasi-Poisson manifold (M, π) form a Poisson algebra under the binary bracket induced by π. Thus we obtain a Poisson algebra on $C^\infty(M)^G$. Given M a G-manifold with G acting locally freely and $\rho : \mathfrak{g} \to TM$ the corresponding infinitesimal action, we will use the same symbol ρ to denote the following natural extension map:

$$\rho : \wedge^\bullet(TM \oplus \mathfrak{g}) \to \wedge^\bullet TM. \tag{9}$$

Throughout this paper, we denote the skew-symmetrization of any section $A \in \Gamma(\wedge^\bullet(TM) \otimes \wedge^\bullet \mathfrak{g})$ by $\hat{A} \in \Gamma(\wedge^\bullet(TM \oplus \mathfrak{g}))$. Thus, if $\theta = f^{ia}(x)\frac{\partial}{\partial x_i} \otimes e_a \in \Gamma(TM \otimes \mathfrak{g})$ in local coordinates $\{x_i\}$ of U and a basis $\{e_a\}$ of $\mathfrak{g}$, we have $\hat{\theta} = f^{ia}(x)\frac{\partial}{\partial x_i} \wedge e_a$.

Theorem 2.7. *Let $U \subset M$ be a cross-section of the G action and π_M be a bivector field on M. Then there exists a unique triple (π_U, θ, r), where $\pi_U \in \Gamma(\wedge^2 TU)$, $\theta \in \Gamma(TU \otimes \mathfrak{g})$ and $r : U \to \mathfrak{g} \wedge \mathfrak{g}$ such that*

$$\pi_M|_U = \pi_U - \rho(\hat{\theta})|_U + \rho(r)|_U. \tag{10}$$

Moreover,

(a) *if π_M is a G-invariant Poisson tensor on M, then (U, π_U) is a Poisson manifold and r is a triangular dynamical r-matrix coupled with U via θ.*
(b) *if π_M is a quasi-Poisson tensor on M, then (U, π_U) is a Poisson manifold and r is a dynamical r-matrix coupled with U via θ with respect to the Cartan 3-tensor, i.e., $\Omega = -\frac{1}{2}\phi$ in the generalized CDYBE.*

Proof. Because G acts locally freely and U is a cross-section, for any $x \in U$ there is a canonical splitting $T_x M = T_x U \oplus \rho_x(\mathfrak{g})$ of the sequence

$$0 \to \mathfrak{g} \to T_x M \to T_x U \to 0.$$

Thus, there exists unique $\pi_U \in \Gamma(\wedge^2 TU)$, $\theta \in \Gamma(TU \otimes \mathfrak{g})$ and $r : U \to \mathfrak{g} \wedge \mathfrak{g}$ such that

$$\pi_M|_U = \pi_U - \rho(\hat{\theta})|_U + \rho(r)|_U,$$

where π_U is tangent to U.

If π_M is a G-invariant Poisson tensor or quasi-Poisson tensor, it induces a Poisson bracket $\{\cdot, \cdot\}$ on U. From the expression (10) and the fact $\rho(e)f' = 0$ for any $e \in \mathfrak{g}$ and $f' \in C^\infty(M)^G$, we have that $\{f, g\} = \pi_U(df, dg)$ for any $f, g \in C^\infty(U)$. This is to say (U, π_U) is a Poisson manifold. The remaining thing is to check that the triple (π_U, θ, r) satisfies the required compatibility condition and the generalized CDYBE. This can be seen from the proofs of Theorem 2.8 and Theorem 2.12. □

Theorem 2.7 suggests the following construction which generalizes the construction for ordinary classical dynamical r-matrices in [18]. Given a manifold M, $M \times G$ carries natural right and left G-actions defined respectively by $(x, p) \cdot g = (x, pg)$ and $g \cdot (x, p) = (x, gp)$ for all $x \in M$, $p, g \in G$. Let ρ^L denote the infinitesimal left G-actions. Then we have

Theorem 2.8. *Let (M, π_M) be a Poisson manifold and $\theta \in \Gamma(TM \otimes \mathfrak{g})$. Then any smooth function $r : M \to \mathfrak{g} \wedge \mathfrak{g}$ induces a left G-invariant bivector π_r on $M \times G$ which is given by*

$$\pi_r = \pi_M + \rho^L(\hat{\theta}) + \rho^L(r), \tag{11}$$

and

(a) *π_r is a Poisson tensor iff r is a triangular generalized dynamical r-matrix.*
(b) *π_r is a quasi-Poisson tensor iff r is a generalized dynamical r-matrix with respect to the Cartan 3-tensor.*

Proof. Note that the vector field on $M \times G$ has a natural bigrading: elements in TM have degree $(1, 0)$ while elements in TG have degree $(0, 1)$. It is simple to see that $[\pi_M, \pi_M]$ is of degree $(3, 0)$, $[\pi_M, \rho^L(\hat{\theta})]$ is of degree $(2, 1)$, $[\pi_M, \rho^L(r)]$ is of degree $(1, 2)$ and $[\rho^L(r), \rho^L(r)]$ is of degree $(0, 3)$. On the other hand, $[\rho^L(\hat{\theta}), \rho^L(r)]$ consists of elements of degree $(1, 2)$ and of $(0, 3)$ and $[\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]$ consists of elements of degree $(2, 1)$ and of $(1, 2)$. For any $S \in \wedge^3(TM \oplus TG)$, let $S = \sum_{0 \le i, j \le 3} S^{(i,j)}$ be its decomposition with respect to this bigrading. The following equations can be verified by a direct computation:

$$[\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]^{(1,2)} = \rho^L(\widehat{[\theta, \theta]}), \qquad [\rho^L(\hat{\theta}), \rho^L(\hat{\theta})]^{(2,1)} = 2\rho^L(\widehat{\theta \wedge \theta}), \tag{12}$$

$$[\rho^L(\hat{\theta}), \rho^L(r)]^{(0,3)} = \rho^L(\mathrm{Alt}(\theta^* dr)), \quad [\pi_M, \rho^L(r)] = \rho^L(\pi_M^\sharp(dr)), \tag{13}$$

$$[\rho^L(\hat{\theta}), \rho^L(r)]^{(1,2)} = -\rho^L(\widehat{[r, \theta]}), \qquad [\pi_M, \rho^L(\hat{\theta})] = \rho^L(d_{\pi_M}\theta) \tag{14}$$

where the operations $[\theta, \theta]$, $\theta \wedge \theta$ and $[r, \theta]$ for $\theta \in \Gamma(TM \otimes \mathfrak{g})$ and $r \in C^\infty(M, \wedge^2\mathfrak{g})$ are defined as (2) and (3). Eventually, by using the facts $[\rho^L(r), \rho^L(r)] = \rho^L([r, r])$ and $[\pi_M, \pi_M] = 0$ we have

$$\begin{aligned}
[\pi_r, \pi_r] &= [\pi_M + \rho^L(\hat{\theta}) + \rho^L(r), \pi_M + \rho^L(\hat{\theta}) + \rho^L(r)] \\
&= [\rho^L(\hat{\theta}), \rho^L(\hat{\theta})] + 2[\pi_M, \rho^L(\hat{\theta})] + 2[\rho^L(\hat{\theta}), \rho^L(r)] + 2[\pi_M, \rho^L(r)] + [\rho^L(r), \rho^L(r)] \\
&= 2\rho^L(\mathrm{Alt}(\theta^* dr) + \frac{1}{2}[r, r]) + 2\rho^L(\widehat{[\theta, \theta]} - \widehat{[r, \theta]} + \widehat{\pi^*(dr)}) + 2\rho^L(\widehat{\theta \wedge \theta} + d_{\pi_M}\theta).
\end{aligned}$$

Note that the map $\theta^\sharp : T^*M \to \mathfrak{g}$ is a Lie algebroid morphism if and only if

$$\widehat{\theta \wedge \theta} + d_{\pi_M}\theta = 0.$$

Therefore we have that $[\pi_r, \pi_r] = 0$ iff r is a triangular generalized dynamical r-matrix and $[\pi_r, \pi_r] = \rho^R(\phi)$ iff r is a generalized dynamical r-matrix w.r.t $\Omega = -\frac{1}{2}\phi$. $\square$

Similarly, we have the following theorem.

Theorem 2.9. *Let (N, π_N) be a quasi-Poisson G-space and $\rho : \mathfrak{g} \to TN$ be the infinitesimal action. Then for any generalized dynamical r-matrix coupled with (M, π_M) via θ with respect to $\Omega = -\frac{1}{2}\phi$,*

$$\pi := \pi_M + \pi_N + \rho(\hat{\theta}) + \rho(r) \tag{15}$$

is a Poisson tensor on $M \times N$.

Proof. We need to prove $[\pi,\pi]=0$. Note that $[\rho(r),\pi_N]=[\rho(\hat{\theta}),\pi_N]=0$ because of the invariance of π_N. Thus we have

$$\begin{aligned}[\pi,\pi] &= [\pi_M+\pi_N+\rho(\hat{\theta})+\rho(r),\pi_M+\pi_N+\rho(\hat{\theta})+\rho(r)]\\ &= [\pi_M,\pi_M]+[\pi_N,\pi_N]+\rho([r,r])+2[\rho(\hat{\theta}),\rho(r)]+2[\pi_M,\rho(\hat{\theta})]+2[\pi_M,r]\\ &= [\pi_N,\pi_N]+2\rho(\mathrm{Alt}(\theta^* dr)+\frac{1}{2}[r,r])\\ &= \rho(\phi)-\rho(\phi)=0.\end{aligned}$$

This finishes the proof. □

Now we discuss the relation between generalized dynamical r-matrices and the reduction of the fusion product of two quasi-Poisson manifolds. Let M, N be two G-manifolds and ρ_M, ρ_N be the corresponding infinitesimal G action. We define a bivector field on $M\times N$ by

$$\Phi=\sum_{a\in I}\rho_M(e_a)\wedge\rho_N(e_a),$$

where $\{e_a\}_{a\in I}$ is an orthogonal basis of $\mathfrak{g}$.

Proposition 2.10. [1] *If (M,π_M) and (N,π_N) are two quasi-Poisson G-manifolds, then $\pi_M+\pi_N-\Phi$ gives a quasi-Poisson structure on $M\times N$ for the diagonal G-action. This quasi-Poisson manifold, denoted by $M\circledast N$, is called the fusion product of M and N.*

Example 2.11. Let (G,π_G) be the quasi-Poisson G-manifold given in Example 2.6. By doing the fusion product with itself, we get a quasi-Poisson manifold $D(G):=G\circledast G$. Let R_a^i and L_a^i denote the right and left invariant vector fields on the i-th copy of $G\times G$ generated by e_a, then the quasi-Poisson tensor takes the form:

$$\pi_{G^2}=\frac{1}{2}\sum_a(R_a^1\wedge L_a^1+R_a^2\wedge L_a^2+(L_a^1-R_a^1)\wedge(L_a^2-R_a^2)). \tag{16}$$

Let (M,π_M) and (N,π_N) be two quasi-Poisson G-manifolds. We assume G acts locally freely on M. Let $U\subset M$ be any cross-section. By Theorem 2.7, associated to U, there is a triple (π_U,θ,r) such that r is a generalized dynamical r-matrix coupled with (U,π_U) via θ. On the other hand, $U\times N$ is a cross-section of the diagonal G action on $M\times N$. So it inherits a Poisson structure π_{red} by the reduction from the quasi-Poisson structure on $M\circledast N$.

Theorem 2.12. *The Poisson tensor π_{red} on $U\times N$ takes the form of*

$$\pi_{\mathrm{red}}=\pi_U+\rho_N(\hat{\theta})+\rho_N(r)+\pi_N. \tag{17}$$

Before giving a proof, we show the following lemma which says that on the reduction level, fusion product and direct product give rise to the same Poisson structure.

Lemma 2.13. *Let M and N be two quasi-Poisson G-manifolds. Then for any diagonal G-invariant functions $f,g\in C^\infty(M\times N)^G$, $\Phi(df,dg)=0$. Moreover, $\pi_M+\pi_N$ induces a Poisson algebra structure on $C^\infty(M\times N)^G$ which is the same as the Poisson algebra on $C^\infty(M\circledast N)^G$.*

Proof. Note that $\Phi = \sum \rho_M(e_a) \wedge \rho_N(e_a)$, and $(\rho_M(e_a) + \rho_N(e_a))f = 0$ for all $f \in C^\infty(M \times N)^G$. So $\Phi = -\sum \rho_N(e_a) \wedge \rho_N(e_a) = 0$ when restricts to G-invariant functions. □

Proof of Theorem 2.12. For any $f, g \in C^\infty(U \times N)$, let $f', g' \in C^\infty(M \times N)^G$ be the diagonal G-invariant extension of f, g respectively. Then by Lemma 2.13, $\pi_{\mathrm{red}}(df, dg) = (\pi_M + \pi_N)(df', dg')|_{U\times N}$. Following Theorem 2.7, we have

$$\pi_M|_U = \pi_U - \rho_M(\hat{\theta})|_U + \rho_M(r)|_U.$$

Together with the fact that $(\rho_M(e_a) + \rho_N(e_a))F = 0$ for any $F \in C^\infty(M \times N)^G$, we get

$$(\pi_M + \pi_N)(df', dg')|_{U\times N} = (\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df', dg')|_{U\times N}.$$

Note that $\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N$ is tangent to $U \times N$ and $f'|_U = f$, $g'|_U = g$. Therefore,

$$(\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df', dg')|_{U\times N} = (\pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)(df, dg).$$

This is to say $\pi_{\mathrm{red}} = \pi_U + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N$. □

2.2. Classical dynamical r-matrices and Poisson G-spaces. Let G be a complex semi-simple Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$, $\kappa \in (S^2\mathfrak{g})^{\mathfrak{g}}$ the element corresponding to the Killing form on $\mathfrak{g}$ and $\Lambda = \frac{1}{2}\kappa + r_0$ any classical quasi-triangular r-matrix with skew-symmetric part r_0. Let ρ^R and ρ^L denote respectively the infinitesimal actions of the right and left translations. We have

$$\pi_G := \rho^L(r_0) - \rho^R(r_0) \tag{18}$$

defines a Lie Poisson structure on G.

Let H be a Lie subgroup of G with $\mathfrak{h} = \mathrm{Lie}(H)$ and $r = \frac{1}{2}\kappa + A_r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ a classical dynamical r-matrix with the skew-symmetric part A_r, i.e., r is an H-equivariant map and satisfies the classical dynamical Yang-Baxter equation

$$\mathrm{Alt}(\theta^\sharp(dA_r)) + \frac{1}{2}[A_r, A_r] = \frac{1}{4}[\kappa^{12}, \kappa^{23}] = -\frac{1}{2}\phi, \tag{19}$$

where ϕ is the Cartan 3-tensor, $\theta = \frac{\partial}{\partial x^i} \wedge e_i$ for a basis $\{e_i\}$ of $\mathfrak{h}$ and the corresponding coordinates $\{x^i\}$ on $\mathfrak{h}^*$.

On the other hand, we can think of r_0 as a dynamical r-matrix over a point. Therefore, similar to Theorem 2.9, we have

Proposition 2.14. *Associated to the dynamical r-matrix A_r, there is a Poisson structure π_r on $\mathfrak{h}^* \times G$ defined by*

$$\pi_r = \pi_{\mathrm{KKS}} + \rho^L(\theta) + \rho^L(A_r) - \rho^R(r_0), \tag{20}$$

where π_{KKS} is the Kirillov-Kostant-Souriau (KKS) Poisson structure on $\mathfrak{h}^*$.

Proof. Note that the Schouten brackets $[\pi_{KKS}, \rho^R(r_0)]$, $[\rho^L(\theta), \rho^R(r_0)]$ and $[\rho^L(A_r), \rho^R(r_0)]$ are zero, and $[\rho^R(r_0), \rho^R(r_0)] = -\rho^R[r_0, r_0] = \rho^R(\phi) = \rho^L(\phi)$. A straightforward calculation shows that $[\pi_r, \pi_r] = 0$. □

Now let us consider the H action on $\mathfrak{h}^* \times G$ given by

$$h \cdot (x, g) = (\mathrm{Ad}_h^* x, gh), \ \forall h \in H, x \in \mathfrak{h}^*, g \in G. \tag{21}$$

Proposition 2.15. *The Poisson tensor π_r on $\mathfrak{h}^* \times G$ is H-invariant.*

Proof. In the expression of the Poisson tensor π_r on $\mathfrak{h}^* \times G$, the components π_{KKS}, $\rho^L(\theta)$ and $\rho^R(r_0)$ are obviously H-invariant. Therefore, π_r is H-invariant as long as $\rho^L(A_r)$ is H-invariant which can be seen from the fact that the map $A_r : \mathfrak{h}^* \to \mathfrak{g} \wedge \mathfrak{g}$ is H-equivariant. □

Recall that an action of the Poisson-Lie group (G, π_G) on a Poisson manifold M is said to be Poisson if the action map $G \times M \to M$ is a Poisson map, where $G \times M$ is equipped with the product Poisson structure. In our case, $\mathfrak{h}^* \times G$ carries a natural left G-action, i.e.,

$$g_1 \cdot (x, g_2) = (x, g_1 g_2), \ \forall x \in \mathfrak{h}^*, g_1, g_2 \in G. \tag{22}$$

Proposition 2.16. *$(\mathfrak{h}^* \times G, \pi_r)$ is a Poisson (G, π_G)-space with respect to the left G-action.*

Proof. By definition, $G \times (\mathfrak{h}^* \times G) \to \mathfrak{h}^* \times G$ is a Poisson map if and only if for all $g_1 \in G$ and $a = (x, g_2) \in \mathfrak{h}^* \times G$,

$$\pi_r(g_1 \cdot a) = g_{1*}(\pi_r(a)) + a_* \pi_G(g_1), \tag{23}$$

where a in the last term denotes the map $a : G \to \mathfrak{h}^* \times G$, $a(g_1) := (x, g_1 g_2)$, for any $g_1 \in G$. Equation (23) can be obtained by the following calculation

$$\begin{aligned}
\pi_r(g_1 \cdot a) &= \pi_r(x, g_1 g_2) \\
&= \pi_{KKS}(x, g_1 g_2) + L_{g_1 g_2}\theta(x) + L_{g_1 g_2} A_r(x) - R_{g_1 g_2} r_0 \\
&= L_{g_1}(\pi_{KKS}(x, g_2) + \theta(x, g_2)) + L_{g_1}(L_{g_2} A_r(x) - R_{g_2} r_0) + R_{g_2}(L_{g_1} r_0 - R_{g_1} r_0) \\
&= g_{1*}(\pi_r(a)) + a_*(\pi_G(g_1)),
\end{aligned} \tag{24}$$

where $R_g(L_g)$ denotes the right(left) translation from the identity element to g. □

By Poisson reduction, the H-invariant Poisson tensor π_r on $\mathfrak{h}^* \times G$ induces a Poisson structure π_s on $(\mathfrak{h}^* \times G)/H = G \times_H \mathfrak{h}^*$. Since the G-action and the H-action on $\mathfrak{h}^* \times G$ commute, $(\mathfrak{h}^* \times G)/H$ carries an induced left G action, $G \times (\mathfrak{h}^* \times G)/H \to (\mathfrak{h}^* \times G)/H$. Furthermore, we have

Theorem 2.17. *$((\mathfrak{h}^* \times G)/H, \pi_s)$ is a Poisson (G, π_G)-space.*

The above theorem is a consequence of the following general proposition.

Proposition 2.18. *Let (M, π_M) be a Poisson (G, π_G)-space. Suppose a Lie group H acts freely on M, commuting with the G action, such that π_M is H-invariant. Then the reduced Poisson manifold $(M/H, \pi_{red})$ is a Poisson (G, π_G)-space with respect to the induced G-action.*

Proof. By definition, for any $g \in G$ and $a \in \mathfrak{h}^* \times G$, we have

$$\pi_r(g \cdot a) = g_* \pi_r(a) + a_*(\pi_G(g)). \tag{25}$$

By quotienting the H action on the two sides, we see that the left action map of G on $(\mathfrak{h}^* \times G)/H$ is a Poisson map. $\square$

The result in this subsection gives a geometric interpretation of Lu's construction of Poisson homogeneous spaces from dynamical r-matrices in [14].

2.3. Generalized dynamical r-matrices associated with conjugacy classes. Given a conjugacy class $\mathcal{C}$ in G, let us identify the tangent space $T_g\mathcal{C}$ at g with $\mathfrak{g}_g^\perp$, where $\mathfrak{g}_g$ is the centralizer of g and $\mathfrak{g}_g^\perp$ its complement. The operator $\mathrm{Ad}_g - 1$ is invertible on $\mathfrak{g}_g^\perp$. Thus we get a linear operator

$$\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}|\mathfrak{g}_g^\perp := (\frac{\mathrm{Ad}_g + 1}{\mathrm{Ad}_g - 1}) \circ \mathrm{Pr}_{\mathfrak{g}_g^\perp} : \mathfrak{g} \to \mathfrak{g},$$

where $\mathrm{Pr}_{\mathfrak{g}_g^\perp}$ is the projection of $\mathfrak{g}$ on $\mathfrak{g}_g^\perp$. Let $\sum_{a\in I} R_a \wedge L_a$ be a bivector field on G (take the convention in Example 2.6). Then we have

Proposition 2.19. (Proposition 3.4, [1]) $\sum_{a\in I} R_a \wedge L_a = \sum_{a,b\in I} \frac{1}{2}(\frac{\mathrm{Ad}_g+1}{\mathrm{Ad}_g-1}|\mathfrak{g}_g^\perp)_{ab} X_a \wedge X_b$ *as bivector fields on G, where* $\{e_a\}_{a\in I}$ *is a basis of* $\mathfrak{g}$ *and* $X_a = L_a - R_a$ *for any* $e_a \in \mathfrak{g}$.

Let $(G \times G, \pi_{G^2})$ be the quasi-Poisson G-manifold given in Example 2.11. As a result of the above proposition, for any conjugacy classes $\mathcal{C}_1$, $\mathcal{C}_2$, $\pi_{G^2}|_{\mathcal{C}_1\times\mathcal{C}_2}$ is tangent to $\mathcal{C}_1 \times \mathcal{C}_2$, i.e., $(\mathcal{C}_1 \times \mathcal{C}_2, \pi_{G^2}|_{\mathcal{C}_1\times\mathcal{C}_2})$ is a quasi-Poisson manifold with respect to the diagonal conjugation G-action. Assume that the G-action is locally free on some open subset of $\mathcal{C}_1 \times \mathcal{C}_2$, which is ture for most interesting cases, for example when G is semisimple and $\mathcal{C}_1, \mathcal{C}_2$ are generic. We will choose a cross-section and study the associated generalized dynamical r-matrix. To do this, let $T \subset G$ be a maximal torus. Let $p \in \mathcal{C}_1 \cap T$ and G_p the centralizer of p with $\mathfrak{g}_p = \mathrm{Lie}(G_p)$. For any some open subset of $\mathcal{C}_2$ where the conjugation G_p-action is locally free, we then choose a cross-section $U \subset \mathcal{C}_2$. Thus $\{p\} \times U$ is a cross-section of the G-action on $\mathcal{C}_1 \times \mathcal{C}_2$. Following Theorem 2.7, associated to the choice of $\{p\} \times U$, there is a dynamical r-matrix $(\pi_{p\times U}, \theta, r)$. To write it down explicitly, we introduce a function $H \in C^\infty(U, \mathrm{End}(\mathfrak{g}))$ as follows. For each point $x \in U$, let $\mathfrak{g}'_x$ be the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{g}'_x = \{e \in \mathfrak{g} \mid \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\exp(te)} x \in T_x U\}.$$

Because U is a cross-section of the conjugation G_p action, we have a decomposition $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{g}'_x$. Then we define $H_x \in \mathrm{End}(\mathfrak{g})$ to be the projection of $\mathfrak{g}$ on $\mathfrak{g}'_x$. Furthermore, associated to each $e_a \in \mathfrak{g}$, we define a function $H(e_a) \in C^\infty(U, \mathfrak{g})$ and a vector field $H(X_a) \in \Gamma(TU)$ by

$$H(X_a)|_x := \frac{d}{dt}|_{t=0} \mathrm{Ad}_{\exp(tH_x(e_a))} x, \ \forall x \in U. \tag{26}$$

Theorem 2.20. *The generalized dynamical r-matrix $(\pi_{p\times U}, \theta, r)$ induced from the reduction of the quasi-Poisson tensor π_{G^2} on $\mathcal{C}_1 \times \mathcal{C}_2$ takes the form*

$$\pi_{p\times U} = \frac{1}{2}\sum_{a,b}((\frac{\mathrm{Ad}_p+1}{\mathrm{Ad}_p-1}|_{\mathfrak{g}_p^\perp})_{ab} + \frac{\mathrm{Ad}_x+1}{\mathrm{Ad}_x-1}|\mathfrak{g}_x^\perp)_{ab})H(X_a)\wedge H(X_b), \tag{27}$$

$$\begin{aligned}\theta &= \frac{1}{2}\sum_a H(X_a)\otimes e_a + \frac{1}{2}(\frac{\mathrm{Ad}_p+1}{\mathrm{Ad}_p-1}|_{\mathfrak{g}_p^\perp})_{ab}H(X_a)\otimes H(e_b)\\ &+\frac{1}{2}\sum_{a,b}(\frac{\mathrm{Ad}_x+1}{\mathrm{Ad}_x-1}|\mathfrak{g}_x^\perp)_{ab}H(X_a)\otimes(H(e_b)-e_b),\end{aligned} \tag{28}$$

$$\begin{aligned}r &= \frac{1}{2}\sum_a e_a\wedge H(e_a) + \frac{1}{2}(\frac{\mathrm{Ad}_p+1}{\mathrm{Ad}_p-1}|_{\mathfrak{g}_p^\perp})_{ab}H(e_a)\wedge H(e_b)\\ &+\frac{1}{2}\sum_{a,b}(\frac{\mathrm{Ad}_x+1}{\mathrm{Ad}_x-1}|\mathfrak{g}_x^\perp)_{ab}(e_a-H(e_a))\wedge(e_b-H(e_b)).\end{aligned} \tag{29}$$

Proof. Note that for the cross-section $p\times U$ of the G_p action on $\mathcal{C}_1\times\mathcal{C}_2$, the triple $(\pi_{p\times U},\theta,r)$ corresponds to the decomposition of $\pi_{G^2}|_{p\times U}$ with respect to TU and the complement $\rho(\mathfrak{g})|_U$ generated by the diagonal G-action, where $\rho(e_a) = X_a^1 + X_a^2$ for any $e_a\in\mathfrak{g}$. On the other hand, from Example 2.11 and Proposition 2.19, we have at any point $(p,x)\in\{p\}\times U$

$$\pi_{G^2} = \frac{1}{4}\sum_{a,b}(\frac{\mathrm{Ad}_p+1}{\mathrm{Ad}_p-1}|\mathfrak{g}_p^\perp)_{ab}X_a^1\wedge X_b^1 + \frac{1}{4}\sum_{a,b}(\frac{\mathrm{Ad}_x+1}{\mathrm{Ad}_x-1}|\mathfrak{g}_x^\perp)_{ab}X_a^2\wedge X_b^2 + \sum_a X_a^1\wedge X_a^2. \tag{30}$$

With the help of this expression, we just need to compute the decomposition of $X_a^i \in \Gamma(T(\mathcal{C}_1\times\mathcal{C}_2)|_U)$ along the two directions TU and $\rho(\mathfrak{g})|_U$. After a direct computation, we get the following decompositions:

$$X_a^1|_{p\times U} = -H(X_a) + \rho(H(e_a))|_{p\times U}, \tag{31}$$
$$X_a^2|_{p\times U} = H(X_a) + \rho(e_a - H(e_a))|_{p\times U}, \tag{32}$$

where $\rho(H(e_a))\in\Gamma(T(\mathcal{C}_1\times\mathcal{C}_2))$ is given by $\rho(H(e_a))|_{y,x} = (X^1_{H_x(e_a)} + X^2_{H_x(e_a)})|_{y,x}$ for all $(y,x)\in\mathcal{C}_1\times\mathcal{C}_2$. To be precise, by the definition of H, $e_a - H(e_a)\in C^\infty(U,\mathfrak{g}_p)$ where $\mathfrak{g}_p$ is the Lie algebra of the isotropic group G_p, so we get $X_a^1 = X^1_{H(e_a)}$ when restricts to $p\times U$. Thus we have

$$-H(X_a) + \rho(H(e_a))|_{p\times U} = -H(X_a) + X^1_{H(e_a)}|_{p\times U} + H(X_a) = X_a^1|_{p\times U}.$$

A similar calculation gives the Eq. (32).

In the end, we get the expression of the triple $(\pi_{p\times U},\theta,r)$ by plugging (31) and (32) in the expression of $\pi_{G^2}|_{p\times U}$. □

We refer to the generalized dynamical r-matrices associated to two conjugacy classes in G as moduli space generalized dynamical r-matrices.

Let us take $G = \mathrm{SU}(2)$ for a concrete example. Let

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix},\quad e_2 = \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix},\quad e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$$

be a basis of su(2) and $\mathcal{C} \subset \mathrm{SU}(2)$ the conjugacy class through p= $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Then $\mathcal{C}$ can be identified with the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and an element in $\mathcal{C}$ takes the form $\begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}$. The diagonal matrix $\begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix}$ acts on $\mathcal{C} = S^2 \in \mathbb{R}^3$ by rotation with respect to x-axis, i.e., $e^{i\beta} \circ (x, y, z) = (x, e^{2i\beta} y, e^{2i\beta} z)$. For this S^1 action, we choose a simple cross-section

$$U := \{(x, y, z) \mid -1 < x < 1,\ y = 0,\ z > 0\} \subset \mathcal{C} = S^2 \in \mathbb{R}^3.$$

We parameterize U by introducing α such that $x = \sin\alpha$ and $z = \cos\alpha$. Then the isotropic subspace of $\mathfrak{g}$ at a point $\alpha \in U \subset \mathfrak{g}$ is defined by

$$\mathfrak{g}'_\alpha = \{A \in \mathfrak{g} \mid \frac{d}{dt}|_{t=0} \mathrm{e}^{tA} \alpha \mathrm{e}^{-tA} \in T_\alpha U\}.$$

A direct calculation gives

Proposition 2.21. *In terms of* e_1, e_2, e_3*,* $\mathfrak{g}'_\alpha = \mathrm{Span}\{e_1 + \tan\alpha e_2, e_3\}$.

It follows that the function $H \in C^\infty(U, \mathrm{End}(\mathfrak{g}))$ is given by

$$H(e_1) = 0, \quad H(e_2) = \cot\alpha e_1 + e_2, \quad H(e_3) = e_3.$$

The corresponding vector fields on U generated by adjoint action are given by

$$H(X_1) = 0, \quad H(X_2) = 0, \quad H(X_3) = \frac{\partial}{\partial\alpha}.$$

Another straightforward computation shows that $\mathfrak{g}_\alpha^\perp = \mathrm{Span}\{\tan\alpha e_1 - e_2, e_3\}$, where $\mathfrak{g}_\alpha$ is the Lie subalgebra of the stabilizer of $\mathfrak{g}$ at $\alpha \in U$. However, we have

$$(\mathrm{Ad}_\alpha + 1)e_3 = (\mathrm{Ad}_\alpha + 1)(\tan\alpha e_1 - e_2) = 0.$$

It indicates that $(\frac{\mathrm{Ad}_\alpha+1}{\mathrm{Ad}_\alpha-1}|\mathfrak{g}_\alpha^\perp) : \mathfrak{g}_\alpha^\perp \to \mathfrak{g}_\alpha^\perp$ is a zero transformation. Eventually, by Theorem 2.20, the dynamical r-matrix associated to the local section $p \times U$ of $(\mathcal{C} \times \mathcal{C})/G$ takes the form of

$$r = \tan\alpha e_1 \wedge e_2, \quad \theta = \frac{\partial}{\partial\alpha} \otimes e_3. \tag{33}$$

2.4. Gauge transformations of generalized classical dynamical r-matrices. Let G be a Lie group and $\Theta = g^{-1}dg$ the Cartan one form. Let $r : U \to \mathfrak{g} \wedge \mathfrak{g}$ be a generalized dynamical r-matrix coupled with a Poisson manifold (U, π_U) via $\theta \in \Gamma(TU \otimes \mathfrak{g})$ w.r.t some Ω.

Definition 2.22. We define the gauge transformation of a smooth map $\sigma : U \to G$ on (π_U, θ, r) by

$$r^\sigma := \mathrm{Ad}_\sigma \otimes \mathrm{Ad}_\sigma (r + \widehat{\langle \sigma^*\Theta, \theta\rangle} + \langle \pi_U, \sigma^*\Theta \wedge \sigma^*\Theta\rangle), \tag{34}$$

$$\theta^\sigma := \mathrm{Ad}_\sigma (\theta + \langle \pi_U, \sigma^*\Theta\rangle), \tag{35}$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between forms and (muti)vector fields, and $\widehat{\langle \sigma^*\Theta, \theta\rangle} : U \to \mathfrak{g} \wedge \mathfrak{g}$ is the skew-symmetrization of $\langle \sigma^*\Theta, \theta\rangle$.

Proposition 2.23. *r^σ is a generalized classical dynamical r-matrix coupled with U via θ^σ with respect to Ω.*

Proof. Following Theorem 2.8, given the dynamical r-matrix (π_U, θ, r), we can construct a right invariant bivector $\pi := \pi_U + \rho^L(\hat{\theta}) + \rho^L(r)$ on $U \times G$ such that $[\pi, \pi] = -2\rho(\Omega)$. We denote the graph of the map $\sigma : U \to G$ by $U' \subset U \times G$. Then π is a right G-invariant bivector fields and U' is a cross-section of the right G action. Following the argument in Theorem 2.7, associated to π and U' there exists a dynamical r-matrix $(\pi_{U'}, \theta_{U'}, r_{U'})$ w.r.t Ω. Let us take the isomorphism $F : U \to U'$; $F(x) = (x, \sigma(x)) \in U'$ for any $x \in U$. A straightforward calculation shows that

$$F_*\pi_U = \pi_{U'}, \quad F^*\theta^\sharp_{U'} = \theta^{\sigma\sharp} \quad \text{and} \quad r_{U'} \circ F = r^\sigma.$$

It means that if we identify U' with U by F, the triple $(\pi_{U'}, \theta_{U'}, r_{U'})$ becomes $(\pi_U, \theta^\sigma, r^\sigma)$. It finishes the proof. □

The geometric meaning of gauge transformations of generalized dynamical r-matrices is illuminated in the proof of Proposition 2.23. Another interpretation is as follows. Let (M, π_M) be a (quasi-)Poisson G-manifold and (π_U, θ, r) be a dynamical r-matrix with respect to Ω. Given a gauge transformation $\sigma : U \to G$, we define a diffeomorphism from $U \times M$ to itself by

$$\sigma \cdot (x, p) = (x, \sigma(x) \cdot p), \ \forall p \in G. \tag{36}$$

Proposition 2.24. *Following Theorem 2.12, let $(U \times M, \pi_r)$ and $(U \times M, \pi_{r^\sigma})$ be the Poisson manifolds associated to (π_U, θ, r) and $(\pi_U, \theta^\sigma, r^\sigma)$ respectively. Then we have*

$$\{F \circ \sigma, G \circ \sigma\}_r = \{F, G\}_{r^\sigma} \circ \sigma,$$

for any $F, G \in C^\infty(U \times G)$.

2.5. Generalized classical dynamical r-matrices and Poisson groupoids. In this subsection, we discuss the geometric interpretation of the generalized CDYB equation. Recall that in [7], Etingof and Varchenko found a geometric interpretation of the CDYB equation that generalizes Drinfeld's interpretation of the CYB equation in terms of Poisson-Lie groups. Namely, they constructed a so called dynamical Poisson-Lie groupoid structure on the direct product manifold $\eta^* \times G \times \eta^*$, where η is a Lie subalgebra of $\mathfrak{g}$. The CYBE can be viewed as the special case of the generalized CYBE (see Example 2.2). An observation here is that $\eta^* \times G \times \eta^*$ is the Lie groupoid integrating the Lie algebroid $T\eta^* \oplus \mathfrak{g}$. Furthermore, the Poisson structure on $\eta^* \times G \times \eta^*$ induces a Lie bialgebroid structure on $T\eta^* \oplus \mathfrak{g}$. Similarly, in the case of the generalized dynamical r-matrix, we have the following theorem. Let M be a manifold, $\mathfrak{g}$ be a Lie algebra and $(TM \oplus \mathfrak{g}, [\cdot,\cdot]_L)$ be a Lie algebroid with the anchor map given by the projection to TM, the bracket given by

$$[X + A, Y + B]_L = [X, Y] + L_X B - L_Y A + [A, B]_{\mathfrak{g}}, \tag{37}$$

for all $X, Y \in \Gamma(TM)$ and $A, B \in \Gamma(M \times \mathfrak{g})$.

Theorem 2.25. (Theorem 4.5, [13]) *A solution r of the generalized DYBE coupled with (M,π) via θ induces a coboundary Lie bialgebroid structure $(TM \oplus \mathfrak{g}, d_*)$ where the differential $d_* : \Gamma(\wedge^\bullet(TM \oplus \mathfrak{g})) \to \Gamma(\wedge^{\bullet+1}(TM \oplus \mathfrak{g}))$ corresponding to the Lie algebroid structure on $T^*M \oplus \mathfrak{g}^*$ is of the form*

$$d_* = [\pi_M + \hat{\theta} + r, \cdot]_L,$$

where $[\cdot,\cdot]_L$ is the Schouten bracket on $\wedge^\bullet(TM \oplus \mathfrak{g})$.

Similarly, a solution of the generalized Poisson-Lie DYBE coupled with (M, π_M) via θ gives a Lie bialgebroid $(TM \oplus \mathfrak{g}, d_*)$, where the differential $d_* = \delta + [\pi_M + \hat{\theta} + r, \cdot]$. According to the theory of integration of Lie bialgebroids in [16], we have

Corollary 2.26. *Associated to a generalized classical dynamical r-matrix coupled with (M,π) via θ, there is a Poisson groupoid structure on $\mathcal{G} = M \times G \times M$ whose tangent Lie bialgebroid is $(TM \oplus \mathfrak{g}, d_*)$.*

Thus the Poisson groupoid $M \times G \times M$ gives a geometric interpretation of the generalized DYBE that generalizes Drinfeld's interpretation of the CYBE in terms of Poisson-Lie groups.

A smooth manifold M is called $\mathcal{G}$-space for a Lie groupoid $(\mathcal{G} \rightrightarrows P, s, t)$ if there are two smooth maps, the moment map and the action map, $J : M \to P$ and

$$\alpha : \mathcal{G} \times_J M = \{(x,m) \in \mathcal{G} \times M \mid t(x) = J(m)\} \to M$$

such that, writing $\alpha(x,m) = x \cdot m$, for all compatible $x, y \in \mathcal{G}$ and $m \in M$,

(i) $J(x \cdot m) = s(x)$;
(ii) $(x \cdot y) \cdot m = x \cdot (y \cdot m)$;
(iii) $J(m) \cdot m = m$.

Now suppose that M is a $\mathcal{G}$-space. The action of $\mathcal{G}$ on M is a Poisson action if its graph $\{(x, m, x \cdot m) \mid t(x) = J(m)\}$ is a coisotropic submanifold of $\mathcal{G} \times M \times \overline{M}$ [17]. Then M is called a Poisson $\mathcal{G}$-space.

Associated to a dynamical r-matrix r coupled with (M, π_M) via θ with respect to the Cartan 3-tensor and a quasi-Poisson G-space N, we have a Poisson groupoid $\mathcal{G} = M \times G \times M$ (Corollary 2.26) and a Poisson manifold $(M \times N, \pi = \pi_M + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)$ (by Theorem 2.9). Furthermore, there is a natural action of the groupoid $\mathcal{G} \rightrightarrows M$ on $M \times N$,

$$(x, g, y) \cdot (y, p) = (x, g \cdot p)$$

for all $x, y \in M$, $g \in G$ and $p \in N$. This is a Lie groupoid action with respect to the moment map $J : M \times N \to M$ given by the natural projection. An observation here is that this action is a Poisson action.

Theorem 2.27. *The $\mathcal{G}$-space $(M \times N, \pi)$ is a Poisson $\mathcal{G}$-space.*

To prove this theorem, we need the following results.

Lemma 2.28. (Theorem 3.3, [12]) *Let $\mathcal{G}$ be a Poisson groupoid with its tangent Lie bialgebroid (A, A^*). Then a Poisson manifold (M,π) is a Poisson $\mathcal{G}-$space if and only if the vector bundle morphism from T^*M to A^*, the dual of the infinitesimal action map, is a Lie algebroid morphism.*

Lemma 2.29. (Theorem 3.1, [12]) *Let A^* be a Lie algebroid over P, and (M,π) a Poisson manifold. Then, for the cotangent Lie algebroid T^*M induced by the Poisson structure, a vector bundle morphism $\Phi : T^*M \to A^*$ over $J : M \to P$ is a Lie algebroid morphism if and only if the following two conditions hold:*

(i) $H_{J^*f} = -\Phi^*(d_*f),\ \forall f \in C^\infty(P)$;
(ii) $L_{\Phi^*(X)}\pi = -\Phi^*(d_*S),\ \forall S \in \Gamma(A)$,

where H_{J^*f} denotes the Hamiltonian vector field on M, which is defined by $H_g h = \pi(dg, dh)$ for all $g, h \in C^\infty(M)$. The differential d_* comes from the Lie algebroid structure on A^*.

Proof of Theorem 2.27. In our case, the Poisson manifold is $(M\times N, \pi = \pi_M + \rho_N(\hat{\theta}) + \rho_N(r) + \pi_N)$ and the Lie bialgebroid is $(TM \oplus \mathfrak{g}, d_* = [\pi_M + \hat{\theta} + r, \cdot])$. So by Lemmas 2.28 and 2.29, we just need to prove $H_{J^*f} = -F(d_*f),\ \forall f \in C^\infty(M)$ and $L_{F(S)}\pi_M = -F(d_*S),\ \forall S \in \Gamma(TM \otimes \mathfrak{g})$, where the bundle map $F : TM \oplus \mathfrak{g} \to T(M\times N)$ is the infinitesimal action of $M\times G\times M$ on $M\times N$, explicitly given by $F(X+e) = X + \rho(e)$ for $X \in \Gamma(TM)$ and $e \in \mathfrak{g}$.

(1) Following the expression of the Poisson tensor π on $M \times N$ given in Theorem 2.9, we have that for all $f \in C^\infty(M)$,

$$H_{J^*f} = \pi_M^*(df) + \rho(\theta^*(df)),$$

where $J : M \times N \to M$ is the natural projection map. On the other hand, by the definition of the differential d_*,

$$d_*f = [\pi_M, f] + [\hat{\theta}, f] + [r, f].$$

Note that $[r, f] = 0$, $F([\pi_M, f]) = [\pi_M, f]$, and $F([\hat{\theta}, f]) = F(-\theta^*(df)) = -\rho(\theta^*(df))$. Therefore,

$$H_{J^*f} = -[\pi_M + \rho(\hat{\theta}) + \rho(r), f] = -F(d_*f).$$

(2) Set $S = X + e \in \Gamma(TM \oplus \mathfrak{g})$, then

$$d_*S = [\pi_M + \hat{\theta} + r, X + e] = [\pi_M + \hat{\theta} + r, X] + [\pi_M + \hat{\theta} + r, e].$$

By the definition of $F : \wedge^*(TM \oplus \mathfrak{g}) \to T(M \times N)$, we see that the map F and the Schouten-bracket $[\cdot,\cdot]_L$ on $\wedge^*(TM \otimes \mathfrak{g})$ commute. As a result,

$$F([\pi_M + \hat{\theta} + r, X]) = [F(\pi_M) + F(\hat{\theta}) + F(r), F(X)] = [\pi, X].$$

Similarly,

$$F([\pi_M + \hat{\theta} + r, e]) = [F(\pi_M) + F(\hat{\theta}) + F(r), F(e)] = [\pi, \rho(e)].$$

Eventually, we get $-F(d_*S) = L_{F(S)}\pi$. This finishes the proof. $\square$

3. Generalized Dynamical r-Matrices and Moduli Spaces of Flat Connections on Surfaces

3.1. Poisson and quasi-Poisson structures on the moduli spaces of flat connections on surfaces. In [3], Atiyah and Bott introduced canonical symplectic structures on the moduli spaces of flat G-connections on oriented surfaces. A convenient finite dimensional description of these moduli spaces is as follows. Let $\Sigma_{g,n}$ be an oriented surface of genus g with n punctures and $\{\mathcal{C}_i\}_{i=1,\ldots,n}$ a set of conjugacy classes of G. Then the moduli space of flat G-connections on $\Sigma_{g,n}$ is given by the character variety, i.e., the space of group homomorphisms $h : \pi_1(\Sigma_{g,n}) \to G$ that map the homotopy equivalence class of a loop around the ith puncture to the associated conjugacy class $\mathcal{C}_i \subset G$. Two such group homomorphisms describe gauge-equivalent connections if and only if they are related by conjugation with an element of G. This implies that the moduli space of flat G-connections on $\Sigma_{g,n}$ is given by

$$X_{G,\mathcal{C}}(\Sigma_{g,n}) = \mathrm{Hom}_{\mathcal{C}_1,\ldots,\mathcal{C}_n}(\pi_1(\Sigma_{g,n}, G))/G = \{h \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G) \mid h(m_i) \in \mathcal{C}_i\}/G,$$

where G acts by conjugation, $\mathcal{C}$ denotes the choice of the set of conjugacy classes, and $m_i \in \pi_1(\Sigma_{g,n})$ corresponds to the loop around ith puncture. By characterising the group homomorphisms in terms of the images of the generators of $\pi_1(S_{g,n})$, it is the set

$$\begin{aligned}\{(M_1, \ldots, M_n, A_1, B_1, \ldots, A_g, B_g) \in G^{n+2g} | M_i \in \mathcal{C}_i, [B_g, A_g]\\ \cdots [B_1, A_1] \cdot M_n \cdots M_1 = 1\}/G,\end{aligned}$$

where the quotient is taken with respect to the diagonal action of G on G^{n+2g}. The smooth part of this space carries a natural symplectic structure [11]. In the Fock-Rosly approach [10], an explicit description of the symplectic structure is obtained by Poisson reduction of a Poisson structure on the enlarged ambient space G^{n+2g}. To write down the Poisson tensor on this enlarged space, let us introduce two natural operators ∇_R, $\nabla_L \in \Gamma(TG \otimes \mathfrak{g}^*)$ which are given for all $A \in \mathfrak{g}$, $p \in G$ by

$$\langle \nabla_R, A\rangle f(p) := \frac{d}{dt}|_{t=0} f(p\mathrm{e}^{-tA}), \tag{38}$$

$$\langle \nabla_L, A\rangle f(p) := \frac{d}{dt}|_{t=0} f(\mathrm{e}^{tA}p). \tag{39}$$

Then we define $2(n+2g)$ covariant differential operators in the following way:

$$\begin{aligned}\nabla_{2i-1} &= \nabla_R^{M_i}, \;\; \nabla_{2i} = \nabla_L^{M_i} \;\; for\ i = 1, \ldots, n;\\ \nabla_{n+4i-3} &= \nabla_R^{A_i}, \;\; \nabla_{n+4i-1} = \nabla_L^{A_i} \;\; for\ i = 1, \ldots, g;\\ \nabla_{n+4i-2} &= \nabla_R^{B_i}, \;\; \nabla_{n+4i-1} = \nabla_L^{B_i} \;\; for\ i = 1, \ldots, g.\end{aligned} \tag{40}$$

Definition 3.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $r \in \mathfrak{g} \otimes \mathfrak{g}$, the corresponding the Fock-Rosly bivector $B_r^{n,g} \in \Gamma(\wedge^2(TG^{n+2g}))$ is defined by

$$B_r^{n,g}(df, dh) := \frac{1}{2}\sum_i \langle r, \nabla_i f \wedge \nabla_i h\rangle + \sum_{i<j}\langle r, \nabla_i f \wedge \nabla_j h\rangle. \tag{41}$$

Theorem 3.2. [10] *Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate Ad-invariant symmetric bilinear form. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{42}$$

then $B_r^{n,g}$ defines a Poisson structure on G^{n+2g}. Furthermore, when the symmetric part κ of r dual to the bilinear form on $\mathfrak{g}$, this Poisson structure induces the canonical symplectic structure on the moduli space of flat G-connections on $\Sigma_{g,n}$.

From the expression of $B_r^{n,g}$, we see that the Poisson bracket of two functions on G^{n+2g} depends only on the symmetric component of r if one of the two functions is invariant under the diagonal action of G on G^{n+2g}. Thus we can use the symmetric part κ of r and reduction procedure to describe the Poisson structure on the quotient space G^{n+2g}/G. Notice that the bivector $B_\kappa^{g,n}$ given in (41) is the part of $B_r^{g,n}$ which only depends on κ. It turns out that $B_\kappa^{g,n}$ coincides with the quasi-Poisson bivector on the fusion product $G \circledast \cdots \circledast G \circledast D(G) \circledast \cdots \circledast D(G)$ (n copies of G and g coies of $D(G)$, where G and $D(G)$ are the quasi-Poisson manifolds given in Examples 2.6 and 2.11 respectively. If we restrict to a set of conjugacy classes $\{\mathcal{C}_i\}_{i=1,\dots,n}$, then it gives a way to describe the standard symplectic structure on $X_{G,\mathcal{C}}(\Sigma_{n,g}) = \{h \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}, G) | h(m_i) \in \mathcal{C}_i\}/G$ by using quasi-Poisson geometry.

Theorem 3.3. [1] *Consider the quasi-Poisson manifold*

$$P_{g,n} = \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_n \circledast D(G) \circledast \cdots \circledast D(G),$$

where $\mathcal{C}_1,\dots,\mathcal{C}_n$ are conjugacy classes of G. Then the quasi-Poisson reductions of $P_{g,n}$ are isomorphic to the moduli spaces of flat G-connections on $\Sigma_{g,n}$ with the Atiyah-Bott symplectic form.

3.2. GCDYB equations and moduli spaces of flat connections on surfaces. In this subsection, we will combine the discussion in previous sections and give our main result which describes the canonical symplectic structure on the moduli spaces of flat connections on surfaces by using generalized dynamical r-matrices.

Following Theorem 3.3, the symplectic structure on $X_{G,\mathcal{C}}(\Sigma_{g,n})$ is given by the reduction of the quasi-Poisson structure $B_\kappa^{g,n}$ on $P_{g,n}$ with respect to the simultaneous conjugation action of G.

Reduction with respect to two punctures

We assume that there are at least two punctures on the surface $\Sigma_{g,n}$, ie., $n \geq 2$. If we choose a local cross-section of the G action on $X_{G,\mathcal{C}}(\Sigma_{g,n})$, then the reduced Poisson structure on this section is viewed to be a local model of the Poisson structure on $X_{G,\mathcal{C}}(\Sigma_{g,n})$. We proceed the reduction in a "minimal" way, i.e., imposing gauge fixing conditions on the first two punctures as follows. First, we think of $P_{g,n}$ as the fusion product of $(\mathcal{C}_1 \times \mathcal{C}_2, \pi_{G^2})$ and $(P_{g,n-2}, B_\kappa^{g,n-2})$, i.e., $P_{g,n} = (\mathcal{C}_1 \circledast \mathcal{C}_2) \circledast P_{g,n-2}$, where $P_{g,n-2} := \mathcal{C}_3 \circledast \cdots \circledast \mathcal{C}_n \circledast D(G) \circledast \cdots \circledast D(G)$. Then let U be any local cross-section of the diagonal conjugation action of G on $\mathcal{C}_1 \times \mathcal{C}_2$ and (π_U, θ, r) the associated moduli space generalized dynamical r-matrix. Finally, we have that $U \times P_{g,n-2}$ is a local cross-section of the G action on $X_{G,\mathcal{C}}(\Sigma_{g,n})$, and by Theorem 2.9, the reduced Poisson structure on it is given by

$$\pi_{\mathrm{red}} = \pi_U + \rho(\hat{\theta}) + \rho(r) + B_\kappa^{g,n-2},$$

where $\rho : \mathfrak{g} \to P_{g,n-2}$ is the infinitesimal action generated by simultaneous conjugation G action. Furthermore, a simple comparison shows that $\rho(r) = B_r^{g,n-2}$ as bivector fields on $U \times P_{g,n-2}$, where $B_r^{g,n-2}$ is the Fock-Rosly bivector field associated to $r : U \to \mathfrak{g}\wedge\mathfrak{g}$ given by (41) (depending on a parameter space U). As a result, $\rho(r) + B_\kappa^{g,n-2} = B_{r+\kappa}^{g,n-2}$, where r and κ can be seen as skew-symmetric and symmetric parts of an entire function $r + k \in C^\infty(U, \mathfrak{g} \otimes \mathfrak{g})$. Eventually, we obtain

Theorem 3.4. *The quasi-Poisson structure $B_\kappa^{g,n}$ on $P_{g,n}$ induces a Poisson bracket on $U \times \mathcal{C}_3 \cdots \times \mathcal{C}_n \times G^{2g}$, which is isomorphic to the Atiyah-Bott symplectic structure and takes the following form:*

(1) *For $f, g \in C^\infty(\mathcal{C}_3 \times \cdots \times \mathcal{C}_n \times G^{2g})$:*

$$\{f, g\} = B_{r+\kappa}^{n-2,g}(df, dg) \tag{43}$$

(2) *For $f \in C^\infty(\mathcal{C}_3 \times \cdots \times \mathcal{C}_n \times G^{2g})$ and $\phi, \varphi \in C^\infty(U)$:*

$$\{f, \phi\} = \rho(\hat{\theta})(df, d\phi) \tag{44}$$

$$\{\phi, \varphi\} = \pi_U(d\phi, d\varphi), \tag{45}$$

Note that the original Fock-Rosly bivector field $B_r^{n,g}$ on G^{n+2g} is associated to a classical r-matrix. Here we introduce a dynamical version of the Fock-Rosyly bivector field which is related to a dynamical r-matrix, and use it to give a new description of the Atiyah-Bott symplectic structure on the moduli space $X_{G,\mathcal{C}}(\Sigma_{g,n})$. One immediate consequence of this viewpoint is the following proposition due to Theorem 2.27. It indicates a Poisson groupoid symmetry of the moduli space $X_{G,\mathcal{C}}(\Sigma_{g,n})$.

Proposition 3.5. *Let $\mathcal{G} = U \times G \times U$ be the Poisson Lie groupoid associated to the moduli space dynamical r-matrix (π_U, r, θ), then $(U \times \mathcal{C}_3 \times \cdots \times \mathcal{C}_n \times G^{2g}, \{\cdot,\cdot\})$ carries a natural Poisson $\mathcal{G}$ action.*

Gauge fixing and classical dynamical r-matrices in ISO(2, 1)*-Chern-Simons theory*

In [15], Meusburger and Schönfeld obtained classical dynamical r-matrices by considering gauge fixing in ISO(2, 1)-Chern-Simons theory. Now, we interpret these classical dynamical r-matrices as moduli space dynamical r-matrices corresponding to the special case $G = \mathrm{ISO}(2, 1)$.

First, let us give the required notations. We denote by $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$, $e_2 = (0, 0, 1)$ the standard basis of $\mathbb{R}^3$. By ε_{abc} we denote the totally skew-symmetric tensor in three dimensions with the convention $\varepsilon_{012} = 1$. The indices of ε_{abc} are raised with the three-dimensional Minkowski metric $\eta = \mathrm{diag}(1, -1, -1)$.

The Poincaré group in 3-D is the semidirect product $\mathrm{ISO}(2, 1) = \mathrm{SO}_+(2, 1) \ltimes \mathbb{R}^3$ of the proper orthochronous Lorentz group $\mathrm{SO}_+(2, 1)$ and the translation group $\mathbb{R}^3$. The elements of ISO(2, 1) are parameterized as

$$(u, \mathbf{a}) = (u, 0) \cdot (1, -\mathbf{j}) = (u, -\mathrm{Ad}(u)\mathbf{j}) \ \textit{with}\ u \in \mathrm{SO}_+(2, 1),\ \mathbf{j}, \mathbf{a} \in \mathbb{R}^3.$$

The corresponding coordinate functions $\{j^a\}_{a=0,1,2}$ are given by

$$j^a : \mathrm{ISO}(2, 1) \to \mathbb{R}, \quad (u, -\mathrm{Ad}(u)\mathbf{q}) \to q^a.$$

Let $\{J_a\}_{a=0,1,2}$ be a basis of $\mathfrak{so}(2, 1)$ such that the Lie bracket takes the form $[J_a, J_b] = \varepsilon_{ab}^{\ \ c} J_c$. Hence a basis of the Lie algebra $\mathfrak{iso}(2, 1)$ is given by $\{J_a\}_{a=0,1,2}$ together with a basis $\{P_a\}_{a=0,1,2}$ of the abelian Lie algebra $\mathbb{R}^3$.

The moduli space of flat G-connections $X_{G,\mathcal{C}}(\Sigma_{g,n})$ can be viewed as a constrained system in the sense of Dirac [4]. In this spirit, the moduli space is obtained from $P_{n,g} = \mathcal{C}_1 \times \cdots \mathcal{C}_n \times D(G) \times \cdots \times D(G)$ by imposing a group-valued constraint that arises from the defining relation of the fundamental group $\pi_1(\Sigma_{g,n})$. In the case of $G = \mathrm{ISO}(2,1)$, the group-valued constraint is a set of six first constraints in the Dirac gauge fixing formalism for the Fock-Rosly Poisson tensor on $P_{n,g}$. The associated gauge transformations which they generate via the Poisson bracket are given by the diagonal action of ISO(2, 1) on $\mathrm{ISO}(2,1)^{n+2g}$.

A choice of gauge fixing conditions for the constraints is investigated in [15]. These gauge fixing conditions implement the quotient by ISO(2, 1) and restrict the first two components of all points $(M_1, \dots, B_g) \in \Sigma = C^{-1}(0)$ in such a way that M_1, M_2 are determined uniquely by two real parameters ψ and α given in terms of the components of the product $M_2 \cdot M_1 = (u_{12}, -Ad(u_{12})\mathbf{j}_{12})$ as

$$\psi = f(\mathrm{Tr}(u_{12})), \quad \alpha = g(\mathrm{Tr}(u_{12}))\mathrm{Tr}(j_{12}^a J_a \cdot u_{12}) + h(\mathrm{Tr}(u_{12})), \tag{46}$$

where $f, g \in C^\infty(\mathbb{R})$ are arbitrary diffeomorphisms and $h \in C^\infty(R)$. This allows us to identify the constraint surface $\Sigma = C^{-1}(0)$ with a subset of $\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g}$, where the $\mathbb{R}^2$ is parameterized by (ψ, α) and $\mathrm{ISO}(2,1)^{n-2+2g}$ by $(M_3, \dots, B_g)$.

Let us pose the Dirac gauge fixing constraints in such form. By Theorem 4.5 in [15], there exist maps

$$\mathbf{q}_\psi,\ \mathbf{q}_\alpha,\ \mathbf{q}_\delta,\ m : \mathbb{R}^2 \to \mathbb{R}^3,\ V : \mathbb{R} \to \mathrm{Mat}(3, \mathbb{R})$$

such that the associated Dirac bracket is given in terms of them. On the other hand, the Dirac gauge fixing is equivalent to choose a cross-section of the ISO(2, 1) action on $\mathcal{C}_1 \times \mathcal{C}_2$, which is the locus of the constraint functions. Therefore by Theorem 3.4, associated to this cross-section, there is a moduli space dynamical r-matrix. It interprets the origin of the dynamical r-matrices found in [15], which are given in our framework by the following propostion.

Proposition 3.6. *The moduli space dynamical classical r-matrix (π, θ, r) corresponding to the Dirac gauge fixing procedure is given by*

$$\pi = 0, \quad \theta = q_\alpha^a \frac{\partial}{\partial\alpha} \otimes J_a + q_\psi^a \frac{\partial}{\partial\psi} \otimes P_a + q_\delta^a \frac{\partial}{\partial\alpha} \otimes P_a, \tag{47}$$

$$r = -V^{bc}(\psi)(P_b \otimes J^c - J^c \otimes P_b) + \varepsilon^{bcd} m_d(\psi, \alpha) P_b \otimes P_c. \tag{48}$$

Moreover, the induced Poisson bracket takes the following form:
for any $f, g \in C^\infty(\mathrm{ISO}(2,1)^{n-2+2g})$,

$$\{\alpha, \psi\} = 0, \quad \{\alpha, f\} = \rho(\hat{\theta})(d\alpha, df), \{f, g\} = B_{r+\kappa}^{n-2,g}(df, dg) \tag{49}$$

where $\kappa = P_a \otimes J^a$.

Given a map $\sigma : \mathbb{R}^2 \to \mathrm{ISO}(2,1)$, let us consider the smooth map

$$\Phi^\sigma : \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g} \to \mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g},$$
$$(\psi, \alpha, M_3, \dots, B_g) \mapsto (\psi, \alpha, \mathrm{Ad}_\sigma M_3, \dots, \mathrm{Ad}_\sigma B_g).$$

As a consequence of Proposition 2.24, we have

Corollary 3.7. *Let* $\{\cdot,\cdot\}$ *be the bracket given in* (49) *with respect to* (θ, r). *Then for all* $F, G \in C^\infty(\mathbb{R}^2 \times \mathrm{ISO}(2,1)^{n-2+2g})$,

$$\{F \circ \Phi^\sigma, G \circ \Phi^\sigma\} = \{F, G\}^\sigma \circ \Phi^\sigma, \tag{50}$$

where $\{\cdot,\cdot\}^\sigma_D$ *is the bracket given in* (49) *with respect to* $(\theta^\sigma, r^\sigma)$, *the gauge transformation of* $\sigma : \mathbb{R}^2 \to \mathrm{ISO}(2,1)$ *on* (θ, r).

Particularly, the map $\sigma = (g, -Ad(g)\mathbf{t})$ satisfying $\partial_\alpha g = \partial_\alpha^2 \mathbf{t} = 0$ is called dynamical Poincaré transformation in [15]. Dynamical r-matrices from different gauge fixing conditions subject to extra conditions given in [15] are related by dynamical Poincaré transformations.

A standard set of dynamical r-matrices from the Dirac gauge fixing in ISO(2, 1)-Chern-Simons theory is given explicitly in [15]. This set of solutions corresponds to special gauge fixing condition which is motivated by its direct physical interpretation in the application to the Chern-Simons formulation of (2 + 1)-gravity.

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