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Uniqueness in Calderón's Problem for Conductivities with Unbounded Gradient

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Abstract: We prove uniqueness in the inverse conductivity problem for uniformly elliptic conductivities in $W^{s,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is Lipschitz, $3 \le n \le 6$, and s and p are such that $W^{s,p}(\Omega) \not\subset W^{1,\infty}(\Omega)$. In particular, we obtain uniqueness for conductivities in $W^{1,n}(\Omega)$ $(n = 3, 4)$. This improves on the result of the author and Tataru, who assumed that the conductivity is Lipschitz.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. To a positive real-valued function γ on Ω with $0 < c < \gamma < c^{-1}$ we associate an elliptic operator L_γ in divergence form:

$$L_\gamma u := \operatorname{div}(\gamma \nabla u).$$

Given $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution u_f to the Dirichlet problem

$$\begin{aligned} L_\gamma u_f &= 0 \quad \text{in } \Omega \\ u_f\big|_{\partial\Omega} &= f, \end{aligned}$$

and we define the Dirichlet-to-Neumann map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ formally by

$$\Lambda_\gamma(f) := \gamma \frac{\partial u_f}{\partial \nu}\bigg|_{\partial\Omega},$$

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where ∂/∂_ν is the outward normal derivative at the boundary. By identifying $H^{1/2}(\partial\Omega)$ with the quotient $H^1(\Omega)/H_0^1(\Omega)$, we can interpret this definition in a weak sense as follows: if $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = g$, then

$$\langle \Lambda_\gamma(f), g\rangle := \int_\Omega \gamma \nabla u_f \cdot \overline{\nabla v}\, dx,$$

where the notation $\langle\cdot,\cdot\rangle$ indicates the pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Our main theorem is that the map $\gamma \mapsto \Lambda_\gamma$ is injective for certain γ:

Theorem 1.1. *The map $\gamma \mapsto \Lambda_\gamma$ is injective for $\gamma \in W^{s,p}(\Omega)$, where*

$$(s,p) = \begin{cases} (1,n) & n = 3,4 \\ (1+(1-\theta)(\frac{1}{2}-\frac{2}{n}), \frac{n}{1-\theta}) & n = 5,6 \end{cases}$$

and $\theta \in [0,1)$.

This problem was introduced by Calderón, who proved uniqueness in [Cal80] for the linearized problem. The basic approach to this problem in this paper is the method introduced by Sylvester and Uhlmann [SU87], where they proved uniqueness for $n \geq 3$ and $\gamma \in C^2$ based on ideas in [Cal80]. It is of interest to determine how much of this regularity condition can be relaxed. Uniqueness is known to fail (at least in the anistropic problem) for conductivities that are sufficiently singular, as was shown in [GLU03,KSVW08].

For $n \geq 3$, the regularity assumption in [SU87] was relaxed to $\gamma \in C^{3/2+}$ by Brown [Bro96], to $C^{3/2}$ by [PPU03], to $W^{3/2,2n+}$ in [Bro03], and to C^1 conductivities or Lipschitz conductivities close to the identity in [HT13]. Recently, the smallness condition for Lipschitz conductivities was removed in [CR14]. In [NS14], uniqueness was shown in three dimensions for conductivities in $W^{3/2+,2}$.

In two dimensions, the low-regularity theory is fairly well-understood. There are essentially sharp results, even for anisotropic conductivities. In particular, uniqueness holds for γ in L^∞, which is invariant under the scaling associated to the Dirichlet problem. The methods in the plane are somewhat different, and we refer the reader to [ALP11] and references therein.

There are some reasons to doubt that uniqueness holds in higher dimensions for conductivities with less than one derivative. Calderón's problem seems to be closely related to unique continuation; in particular, most of the progress in both of these problems involves Carleman estimates. Unique continuation in the plane holds for elliptic operators in divergence form when the coefficients are merely bounded [Ale92]; in higher dimensions, however, this is only known for Lipschitz coefficients [AKS62]. Furthermore, there are counterexamples to unique continuation for elliptic equations where the coefficients are C^α with any $\alpha < 1$ [Pli63,Mil73,Man98].

The conductivity equation $\operatorname{div}(\gamma\nabla u) = 0$ is equivalent to

$$(\Delta + A\cdot\nabla)u = 0,$$

where $A = \nabla \log\gamma$. Unique continuation holds for this equation as long as $A \in L^n$ [Wol92,KT01]. Brown [Bro03] conjectured uniqueness in the inverse conductivity problem for $\gamma \in W^{1,n}$. We verify that this conjecture holds in dimensions three and four.

One can also study the closely-related problem of determining a Schrödinger potential q from the Cauchy data associated with the operator $-\Delta + q$. In this setting Lavine and

Nachman used the L^p Carleman estimates of [KRS87] to show that the Cauchy data determine $q \in L^{n/2}$ (see also [Cha90,DSFKSU09,NS14] for similar results). These L^p Carleman estimates are the starting point for our analysis.

It was shown in [SU87] that the inverse conductivity problem reduces to the inverse problem for $-\Delta + q$, where $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. One step in this reduction is to show that the map Λ_γ determines γ and its normal derivatives at the boundary. In [KV84], Kohn and Vogelius established that for smooth conductivities, the map $\gamma \to \Lambda_\gamma$ determines the values of γ and all of its derivatives on $\partial\Omega$. This boundary determination result holds in much greater generality [Ale90,SU88]. In particular, Brown [Bro13] showed that the boundary values of a $W^{1,1}$ conductivity are determined by the Dirichlet-to-Neumann map. This improvement will be a crucial ingredient in this paper.

The key idea in [SU87] is that if γ_i are such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then

$$\int_\Omega (q_1 - q_2)\, u_1\, u_2\, dx = 0,$$

where the u_i are arbitrary solutions to the Schrödinger equation $(-\Delta + q)u_i = 0$ in Ω. It follows that one way to show that the potentials q_1 and q_2 coincide is to produce enough solutions to the corresponding Schrödinger equations that their products are dense in some sense. This idea goes back to the original paper of Calderón [Cal80]. In [SU87], Sylvester and Uhlmann proved a uniqueness result for C^2 conductivities by constructing complex geometrical optics (CGO) solutions of the form $u_i = e^{x\cdot\zeta_i}(1 + \psi_i)$. Here the $\zeta_i \in \mathbb{C}^n$ are chosen so that $\zeta_i \cdot \zeta_i = 0$, so that $e^{x\cdot\zeta_i}$ is harmonic, and $e^{x\cdot\zeta_1}e^{x\cdot\zeta_2} = e^{ix\cdot k}$ for some fixed frequency $k \in \mathbb{R}^n$. In three or more dimensions, these conditions allow for an infinite family of pairs ζ_1, ζ_2 with $|\zeta_i| \to \infty$. The remainders ψ_i decay to zero in a suitable sense as $|\zeta_i| \to \infty$, so that the product $u_1 u_2$ converges to $e^{ix\cdot k}$. Since k is arbitrary, uniqueness follows from Fourier inversion.

To construct these CGO solutions, fix $\zeta \in \mathbb{C}^n$ such that $\zeta \cdot \zeta = 0$, and note that $e^{-x\cdot\zeta}\Delta(e^{x\cdot\zeta}\psi) = (\Delta + 2\zeta\cdot\nabla)\psi$. Then $u = e^{x\cdot\zeta}(1+\psi)$ solves $\Delta u = qu$ if

$$\Delta_\zeta \psi := \Delta\psi + 2\zeta\cdot\nabla\psi = q(1+\psi). \tag{1}$$

Let m_q be the map sending ψ to $q\psi$. We will treat this equation perturbatively, by viewing $\Delta_\zeta - m_q$ as a perturbation of Δ_ζ. The operator Δ_ζ has a right inverse defined by

$$\widehat{\Delta_\zeta^{-1} f}(\xi) = p_\zeta(\xi)^{-1}\hat{f}(\xi),$$

where

$$p_\zeta(\xi) = -|\xi|^2 + 2i\zeta\cdot\xi.$$

We take ζ of the form $\tau(e_1 - ie_2)$, where e_1, e_2 are orthogonal unit vectors. Then Δ_ζ is characteristic on a codimension-2 sphere Σ_ζ, with

$$\Sigma_\zeta = \{\xi : \xi\cdot e_1 = 0, |\xi - \tau e_2| = \tau\}.$$

To construct a solution to (1), we show that m_q is a perturbation of Δ_ζ with respect to an appropriate norm.

It was observed in [PPU03] that it is possible to construct CGO solutions for C^1 conductivities using Picard iteration in Sobolev spaces. However, because the inhomogeneity in (1) is not bounded in the correct space, one cannot control these solutions.

A simplified explanation of the problem is as follows: in problems involving Carleman estimates, it is most natural to work with Sobolev spaces depending on a large parameter τ (or a small parameter in the semiclassical notation). In our problem τ is proportional to $|\zeta|$. The quantity τ is thought of as equivalent to a derivative ∇. In view of this correspondence, we define the Sobolev space $H^s_\tau(\mathbb{R}^n)$ by

$$\|u\|_{H^s_\tau} := \|(-\Delta + \tau^2)^{s/2} u\|_{L^2}.$$

For the operator Δ_ζ^{-1} the following Carleman estimate holds [ST09]:

$$\|u\|_{H^1_\tau} \lesssim \tau \|\Delta_\zeta u\|_{H^{-1}_\tau},$$

where u is supported in some fixed compact set. This means that once we account for the physical space localization in the problem, the operator Δ_ζ^{-1} (heuristically speaking) maps H^{-1}_τ to H^1_τ with constant τ. On the other hand, we have $q = \Delta \frac{1}{2} \log \gamma + l.o.t.$

$$|\langle m_q u, v\rangle_{L^2}| \lesssim \|\nabla(\log\gamma)\nabla(u\overline{v})\|_{L^1} + l.o.t. \tag{2}$$

$$\lesssim \|\gamma\|_{W^{1,\infty}}(\|\nabla u\|_{L^2}\|v\|_{L^2} + \|u\|_{L^2}\|\nabla v\|_{L^2}) + \cdots \tag{3}$$

$$\lesssim \tau^{-1}\|\gamma\|_{W^{1,\infty}}\|u\|_{H^1_\tau}\|v\|_{H^1_\tau}. \tag{4}$$

By duality, this implies that m_q maps H^1_τ back to H^{-1}_τ with constant τ^{-1}. This means that the composition $m_q\Delta_\zeta^{-1}$ is bounded, and there is some hope of construction CGO solutions perturbatively (see [KU14] for this type of analysis). We are trying to solve $(\Delta_\zeta - m_q)u = q$, and we certainly have $q \in H^{-1}_\tau$. However, because we lost a factor of τ in the Carleman estimate, our solution only satisfies an estimate of the form $\|\psi\|_{H^1_\tau} \leq \tau\|q\|_{H^{-1}_\tau}$. This is bad, because ψ is supposed to be small in H^1_τ.

In [NS14], this problem with Sobolev spaces is circumvented by showing that the first iterate in the solution procedure is bounded on average. This avoids the use of specialized spaces at the expense of requiring slightly more differentiability.

In [HT13], Tataru and the author dealt with this problem using specialised function spaces. These are inspired by the $X^{s,b}$ spaces of Bourgain [Bou93], and were used in the context of Carleman estimates by [Tat96]. Define the space $\dot{X}^b_\zeta$ by the norm

$$\|u\|_{\dot{X}^b_\zeta} = \||p_\zeta(\xi)|^b \hat{u}(\xi)\|_{L^2},$$

where $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta\cdot\xi$ is the symbol of Δ_ζ. In this paper, we take $b = \pm 1/2$. It is easy to see that $\|\Delta_\zeta^{-1}\|_{\dot{X}^{-1/2}_\zeta \to \dot{X}^{1/2}_\zeta} = 1$. We will also make use of the inhomogeneous spaces X^b_ζ with norm

$$\|u\|_{X^b_\zeta} = \|(|\zeta| + |p_\zeta(\xi)|)^b \hat{u}(\xi)\|_{L^2}.$$

The map m_q satisfies

$$\|m_q\|_{\dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta} \lesssim \|\gamma\|_{\mathrm{Lip}}, \tag{5}$$

and we may solve (1) perturbatively as before. This bound follows from (4), the easy estimate

$$\|u\|_{H^1_\tau} \lesssim \tau^{1/2}\|u\|_{X^{1/2}_\zeta}, \tag{6}$$

and the fact that for localized u, the $X_\zeta^{1/2}$ and $\dot{X}_\zeta^{1/2}$ norms are equivalent.

Solving in this way gives a CGO solution ψ with $\|\psi\|_{\dot{X}_\zeta^{1/2}} \lesssim \|q\|_{\dot{X}_\zeta^{-1/2}}$. Unfortunately, the best bound on $\|q\|_{\dot{X}_\zeta^{-1/2}}$ is $\tau^{1/2}\|\gamma\|_{H^1}$. This means that the $\dot{X}_\zeta^{1/2}$ norm of CGO solutions might grow like $\tau^{1/2}$ as $\tau \to \infty$.

By an averaging argument, however, $\|q\|_{\dot{X}_\zeta^{-1/2}}$ is bounded for a large set of ζ (which may depend on q) as long as $\gamma \in H^1$. This is because the $\dot{X}^{-1/2}$ norm is only large when $\hat{q}$ concentrates near Σ_ζ. As we vary ζ, the characteristic set Σ_ζ varies through a family of growing codimension 2 spheres, and $\hat{q}$ cannot concentrate near all of them. In particular, the estimate $\|q\|_{\dot{X}_\zeta^{-1/2}} \lesssim \|\gamma\|_{H^1}$ holds on average. Once this is established, uniqueness for $\gamma \in C^1$ follows from the standard arguments.

If $\nabla\gamma$ is unbounded, then we need to replace the H_τ^1 norm on the left hand side of (6) with an L^p norm, where $p > 2$. We can obtain such an estimate using the methods of [KRS87], which essentially give[1]

$$\|u\|_{2n/(n-2)} \lesssim \|u\|_{\dot{X}_\zeta^{1/2}}. \tag{7}$$

This puts u in a better L^p space, but at the cost of a factor of τ. Since there is no such room in (5), it seems that such a bound does not hold for $\gamma \in W^{1,p}$ if $p < \infty$.

To do better, we need a refined version of (7). If $\hat{u}_\mu$ is supported in the region $\{\xi : d(\xi, \Sigma_\zeta) \sim \mu\}$, then we can replace (7) by

$$\|u_\mu\|_{2n/(n-2)} \lesssim (\mu/\tau)^{1/n}\|u_\mu\|_{\dot{X}_\zeta^{1/2}}. \tag{8}$$

and (6) by

$$\|u_\mu\|_2 \lesssim (\mu\tau)^{-1/2}\|u_\mu\|_{X_\zeta^{1/2}}. \tag{9}$$

Assume we are given v_ν satisfying a similar condition, with $\mu \le \nu$. Define $f = \nabla \log \gamma$. Then

$$\left|\int (\nabla f)u_\mu\overline{v}_\nu\right| \lesssim \|\nabla f\|_n(\mu/\tau)^{1/n}(\nu\tau)^{-1/2}\|u_\mu\|_{X_\zeta^{1/2}}\|v_\nu\|_{X_\zeta^{1/2}}.$$

Now we exploit the fact that the Fourier transform of $u_\mu v_\nu$ is supported in $\{\xi : |\xi| \lesssim \tau, |\xi \cdot e_1| \lesssim \nu\}$. By orthogonality, we may restrict f to this region, so that the above becomes

$$\left|\int (\nabla f)u_\mu\overline{v}_\nu\right| \lesssim \|D^{1/2-1/n}D_1^{1/n-1/2}f\|_n\|u_\mu\|_{X_\zeta^{1/2}}\|v_\nu\|_{X_\zeta^{1/2}}.$$

where D and D_1 are operators with symbols $|\xi|$ and $|\xi \cdot e_1|$, respectively. An argument along these lines gives an estimate of the form

$$\|m_{\nabla f}\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}} \lesssim \|D^{1/2-1/n}D_1^{1/n-1/2}f\|_n.$$

Although we have lost $1/2 - 1/n$ derivatives in this estimate, this is counterbalanced by a gain of $1/2 - 1/n$ derivatives in the e_1 direction. This gain is useless if the Fourier support of f concentrates near the plane perpendicular to e_1. However, we expect that

[1] The author would like to thank Russell Brown for pointing this out.

this behavior does not occur on average, and we can take advantage of this by exploiting our freedom in choosing ζ.

It is easiest to average over all choices of $e_1 \in S^{n-1}$. In L^2 we have

$$\int_{e_1 \in S^{n-1}} \|D^\alpha D_1^{-\alpha} f\|_2^2 \, d\sigma(e_1) \lesssim \|f\|_2.$$

for $\alpha < 1/2$. Heuristically, we can interpolate this with the trivial observation that $\sup_{e_1 \in S^{n-1}} \|f\|_\infty \lesssim \|f\|_\infty$ to obtain

$$\left(\int_{e_1 \in S^{n-1}} \|D^\beta D_1^{-\beta} f\|_p^p \, d\sigma(e_1) \right)^{1/p} \lesssim \|f\|_p,$$

when $\beta < 1/p$. In three dimensions, we have $1/2 - 1/3 < 1/3$, and we find that $\|m_{\nabla f}\|_{X_\zeta^{1/2} \to X_\zeta^{-1/2}}$ is bounded on average. In four dimensions, we have $1/2 - 1/4 = 1/4$. This causes a logarithmic divergence, which turns out to be harmless. For $n \geq 5$, however, we do not have a way to avoid losing derivatives.

The averaging argument in this paper is somewhat different from the argument in [HT13]. There, k was taken to be fixed, and $\hat{q}(k)$ was determined by testing against CGO solutions with $\zeta \sim \tau(\eta_1 - i\eta_2)$, where η_1 and η_2 are perpendicular to the frequency k. This approach does not give control of $D_1^\beta D_1^{-\beta} f$, since averaging only over η_1 perpendicular to k is useless if $\hat{f}$ is concentrated along the k direction.

Instead of fixing k, we vary the triple $\{k, \eta_1, \eta_2\}$ over an small open set of triples of orthonormal vectors. This set is essentially parameterized by $\{Ue_1, Ue_2, Ue_3\}$, where the e_i denote fixed orthonormal vectors, and U is an orthogonal transformation. The idea is then to average the relevant quantities (which depend on U) with respect to the Haar measure on $O(n)$. Using all of the degrees of freedom in this way allows for an improvement in the estimate for m_q and clarifies the estimate for $\|q\|_{X_\zeta^{-1/2}}$. This idea comes from [NS14], where uniqueness is established in three dimensions for conductivities in $W^{3/2+,2}$. Remarkably, they showed that under this assumption the boundedness of m_q on average can be proven without taking advantage of the curvature of Σ_ζ. Using our framework, this corresponds (roughly speaking) to applying Bernstein's inequality to u_μ in the e_1 direction and Sobolev embedding to f in the other directions, to obtain

$$\begin{aligned}
|\int (\nabla f) u_\mu \overline{v}_\nu| &\lesssim \|\nabla f\|_{L^2_{e_1} L^\infty_{e_2,e_3}} \|u_\mu\|_{L^\infty_{e_1} L^2_{e_2,e_3}} \|v_\nu\|_{L^2} \\
&\lesssim \|\langle D\rangle^{2+} f\|_2 \, \mu^{1/2} \|u_\mu\|_2 \|v_\nu\|_2 \\
&\lesssim \nu^{-1/2} \tau^{-1} \|D^{2+} f\|_2 \|u_\mu\|_{X_\zeta^{1/2}} \|u_\nu\|_{X_\zeta^{1/2}}.
\end{aligned}$$

Taking f supported in $\{|\xi| \lesssim \tau, |\xi_1| \lesssim \nu\}$, we obtain

$$|\int (\nabla f) u_\mu \overline{v}_\nu| \lesssim \|D^{1/2} D_1^{-1/2} D^{1/2+} f\|_2 \|u_\mu\|_{X_\zeta^{1/2}} \|v_\nu\|_{X_\zeta^{1/2}},$$

and we can estimate $\|D^{1/2} D_1^{-1/2} D^{1/2+} f\|_2$ on average as before.

When $n \geq 7$, the situation is essentially the same, but there is a new technical difficulty. Since our methods are global in space, we need to extend the conductivities $\gamma_i \in W^{s,p}(\Omega)$ to some $\gamma_i \in W^{s,p}(\mathbb{R}^n)$ which agree outside of Ω. When $s \leq 1 + 1/p$, we can do this as long as $\gamma_1 = \gamma_2$ on the boundary. However, when $s > 1 + 1/p$, we also need $\partial_\nu \gamma_1 = \partial_\nu \gamma_2$ on the boundary, and there does not seem to be such a boundary identification result in the literature.

It is possible that one can relax the uniform ellipticity condition on γ. The natural condition to impose is then $\nabla \log \gamma \in L^n$, in which case $\log \gamma$ is only in BMO, which would correspond to the results of [ALP11] in the plane. We will not address this issue, as it introduces numerous technical difficulties. We note, however, that the assumption in Theorem 5.3 is of this type.

2. Notation

Let $\zeta = \tau(e_1 - ie_2)$, where $e_1, e_2 \in \mathbb{R}^n$ are orthogonal unit vectors. Define the conjugated Laplacian

$$\Delta_\zeta := e^{-x\cdot\zeta} \Delta e^{x\cdot\zeta},$$

a differential operator whose symbol is

$$p_\zeta(\xi) := -|\xi|^2 + 2i\zeta \cdot \xi.$$

This symbol vanishes simply on the characteristic set

$$\Sigma_\zeta := \{\xi : p_\zeta(\xi) = 0\} = \{\xi : \xi_1 = 0, |\xi - \tau e_2| = \tau\},$$

which is a sphere of codimension two. In fact, it is not hard to check that

$$|p_\zeta(\xi)| \sim \begin{cases} \tau d(\xi, \Sigma_\zeta) & d(\xi, \Sigma_\zeta) \le \tau/8 \\ \tau^2 + |\xi|^2 & d(\xi, \Sigma_\zeta) > \tau/8 \end{cases} \tag{10}$$

where $d(\xi, \Sigma_\zeta) \sim |\xi_1| + ||\xi - \tau e_2| - \tau|$ is the distance from ξ to Σ_ζ. We will refer to this distance as the *modulation*.

Define the Banach spaces $\dot{X}^b_\zeta$ and X^b_ζ with norms

$$\|u\|_{\dot{X}^b_\zeta} = \||p_\zeta(\xi)|^b \hat{u}\|_{L^2}$$
$$\|u\|_{X^b_\zeta} = \|(|p_\zeta(\xi)| + \tau)^b \hat{u}\|_{L^2}.$$

We will use the Greek letters λ, μ, ν to represent dyadic integers of the form 2^k, where $k \ge 0$. For $\lambda > 1$ we define E_λ to be the set of ξ with modulation comparable to λ:

$$E_\lambda := \{\xi : d(\xi, \Sigma_\zeta) \in (\lambda/2, \lambda]\}.$$

Similarly, for any λ we write

$$E_{\le\lambda} := \{\xi : d(\xi, \Sigma_\zeta) \le \lambda\}.$$

Since our problem is localized to a fixed compact set, the uncertainty principle implies that we need not distinguish frequencies which are separated on the unit scale. Therefore, by abuse of notation we will define $E_1 := E_{\le 1}$. Let m_λ denote the characteristic function of E_λ, and similarly for $m_{\le\lambda}$.

Let $Q_\lambda, Q_{\le\lambda}$ be the Fourier multipliers with symbols $m_\lambda, m_{\le\lambda}$. We will wish to distinguish the cases $\lambda \le \tau/8$ and $\lambda \gtrsim \tau/8$, so we define projections onto low and high modulation by

$$Q_l = \sum_{1 \le \lambda \le \tau/8} Q_\lambda$$
$$Q_h = \sum_{\lambda > \tau/8} Q_\lambda,$$

where the λ vary over dyadic integers. By (10), we have

$$\|Q_h u\|_{H^1_\tau} \lesssim \|u\|_{\dot{X}^{1/2}_\zeta}. \tag{11}$$

We will also need the standard Littlewood–Paley projections. For these we choose a smooth dyadic partition of unity, i.e. a function $\chi \in C_0^\infty(\mathbb{R})$ supported on $[1/2, 2]$ such that

$$1 = \sum_{k=-\infty}^{\infty} \chi(2^{-k}\rho)$$

for any $\rho > 0$. For a dyadic integer $\lambda > 1$, we set $\chi_\lambda(\xi) = \chi(|\xi|/\lambda)$, and by abuse of notation we again set $\chi_1 = \sum_{\lambda \le 1} \chi_\lambda$. The Littlewood–Paley projections P_λ are defined as the Fourier multipliers with symbols χ_λ. Given a direction $\omega \in S^{n-1}$, we can also define the Littlewood–Paley projections P_λ^ω in the ω direction using the Fourier multipliers $\chi(|\xi \cdot \omega|/\lambda)$.

3. Strichartz Estimates

Our goal in this section is to prove L^p estimates for functions in $\dot{X}^{1/2}_\zeta$. We follow [KRS87]. Since the symbol $p_\zeta(\xi)$ is characteristic on a sphere Σ_ζ, we begin with the Stein–Tomas restriction theorem.

Theorem 3.1 [Ste93,Tom75]. *Suppose $p \ge (2d+2)/(d-1)$. Let σ denote the surface measure on S^{d-1}. Then*

$$\|\widehat{f\,d\sigma}\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(S^{d-1})}.$$

Let τS^{d-1} denote the sphere of radius τ. Given a set E we define its λ-neighborhood

$$N_\lambda(E) := \{\xi : d(\xi, E) \le \lambda\}.$$

We use the following rescaled and localized variant of the restriction theorem:

Corollary 3.2. *Let p be as above. Suppose that $\hat{g}$ is supported in $N_\lambda(\tau S^{d-1})$, where $\lambda \le \tau/8$. Then*

$$\|g\|_{L^p} \lesssim \lambda^{1/2} \tau^{(d-1)/2 - d/p} \|\hat{g}\|_{L^2(N_\lambda(\tau S^{d-1}))}$$

Proof. By Fourier inversion, we have

$$\begin{aligned} g(x) &= c_d \int \hat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= c_d \int_{\tau-\lambda}^{\tau+\lambda} \int_{S^{d-1}} \hat{g}(\rho\omega) e^{i\rho\langle x,\omega\rangle} \rho^{d-1}\, d\rho\, d\sigma \\ &= c_d \int_{\tau-\lambda}^{\tau+\lambda} \rho^{d-1} (\hat{g}(\rho\omega)\, d\sigma)^\vee(\rho x)\, d\rho. \end{aligned}$$

By Minkowski's inequality, the restriction theorem, Cauchy–Schwarz, and Plancherel this implies that

$$
\begin{aligned}
\|g\|_{L^p} &\lesssim \int_{\tau-\lambda}^{\tau+\lambda} \|(\hat{g}(\rho\omega)\,d\sigma)^\vee(\rho x)\|_{L^p(\mathbb{R}^d)}\,\rho^{d-1}\,d\rho \\
&\lesssim \int_{\tau-\lambda}^{\tau+\lambda} \rho^{-d/p}\|(\hat{g}(\rho\omega)\,d\sigma)^\vee(x)\|_{L^p(\mathbb{R}^d)}\,\rho^{d-1}\,d\rho \\
&\lesssim \tau^{-d/p}\int_{\tau-\lambda}^{\tau+\lambda} \|\hat{g}(\rho\omega)\|_{L^2(S^{d-1})}\,\rho^{d-1}\,d\rho \\
&\lesssim \tau^{-d/p}(\lambda\tau^{d-1})^{1/2}\left(\int_{\tau-\lambda}^{\tau+\lambda} \|\hat{g}(\rho\omega)\|^2_{L^2(S^{d-1})}\,\rho^{d-1}\,d\rho\right)^{1/2} \\
&= \lambda^{1/2}\tau^{(d-1)/2-d/p}\|\hat{g}\|_{L^2(N_\lambda(\tau S^{d-1}))}.
\end{aligned}
$$

□

We deduce the following Strichartz-type estimates

Lemma 3.3. *Let* $p = 2n/(n-2)$, $\lambda \le \tau/8$. *Then*[2]

$$\|Q_\lambda f\|_p \lesssim (\lambda/\tau)^{1/n}\|f\|_{X_\zeta^{1/2}}. \tag{12}$$

$$\|f\|_p \lesssim \|f\|_{X_\zeta^{1/2}}. \tag{13}$$

Proof. By a change of coordinates, we may assume $e_1 = (1, 0, \ldots, 0)$. We use the notation $\xi = (\xi_1, \xi')$.

For (12), write $g = Q_\lambda f$. Note that

$$E_\lambda \subset \{\xi : |\xi_1| \le c\lambda, ||\xi' - \tau e_2'| - \tau| \le c\lambda\}.$$

We can write $g = \phi_\lambda *_{x_1} g$, where $\phi_\lambda(x_1) = \lambda\phi(\lambda x_1)$ for some Schwartz ϕ and the convolution is taken in the x_1 variable only. By Minkowski's inequality and Young's inequality, we have

$$
\begin{aligned}
\|g\|_p &= \left\| \int \phi_\lambda(x_1 - y_1) g(y_1, x')\,dy_1 \right\|_p \\
&\lesssim \left\| \int |\phi_\lambda(x_1 - y_1)| \|g(y_1)\|_{L^p_{x'}}\,dy_1 \right\|_{L^p_{x_1}} \\
&\lesssim \lambda^{1/2-1/p}\|g\|_{L^2_{x_1}L^p_{x'}}.
\end{aligned}
$$

If we regard g as a function in the x' variable, we see that its Fourier transform lies in $N_{c\lambda}(\tau S^{n-2} + \tau e_2')$. By Corollary 3.2 and translation invariance, we have $\|g(x_1)\|_{L^p_{x'}} \lesssim \lambda^{1/2}\tau^{(n-2)/(2n)}\|\hat{g}(x_1)\|_{L^2_{x'}}$ for each x_1. It follows that

$$\|g\|_p \lesssim \lambda^{1/2}\lambda^{1/2-1/p}\tau^{1/2-1/n}\|\hat{g}\|_2 \lesssim \lambda^{1/n}\tau^{-1/n}\|f\|_{X_\zeta^{1/2}}.$$

For (13), we apply (12) near Σ_ζ and Sobolev embedding away from Σ_ζ. On E we have

[2] Strictly speaking, the $X_\zeta^{1/2}$ norm should be replaced with $\dot{X}_\zeta^{1/2}$, but this will not be important.

$$\begin{aligned}\|Q_l f\|_p &\lesssim \sum_{1\le\lambda\le\tau/8} \|Q_\lambda f\|_p \\ &\lesssim \sum_{1\le\lambda\le\tau/8} (\lambda/\tau)^{1/n}\|Q_\lambda f\|_{X_\zeta^{1/2}} \\ &\lesssim \left(\sum_{1\le\lambda\le\tau/8} \|Q_\lambda f\|^2_{X_\zeta^{1/2}}\right)^{1/2} \\ &\le \|f\|_{X_\zeta^{1/2}}.\end{aligned}$$

Away from E we have

$$\|Q_h f\|_p \lesssim \|Q_h f\|_{H^1} \lesssim \|f\|_{X_\zeta^{1/2}}$$

by (11). Combining these estimates gives the claimed inequality. □

4. Bilinear Estimates

Given a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, define the map $m_f : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by $m_f\phi := f\phi$. We would like to control $\|m_f\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}}$. By duality, this is equivalent to establishing a bilinear estimate of the form

$$|m_f(u,v)| \lesssim \|u\|_{X_\zeta^{1/2}}\|v\|_{X_\zeta^{1/2}},$$

where

$$m_f(u,v) = \langle m_f u, v\rangle.$$

Suppose that $f \in L^{n/2}$. By (13), we have

$$\begin{aligned}|m_f(u,v)| &\lesssim \|f\|_{n/2}\|u\|_{2n/(n-2)}\|v\|_{2n/(n-2)} && (14)\\ &\lesssim \|f\|_{n/2}\|u\|_{X_\zeta^{1/2}}\|v\|_{X_\zeta^{1/2}}. && (15)\end{aligned}$$

We also have

$$|m_f(u,v)| \lesssim \|f\|_\infty\|u\|_2\|v\|_2 \lesssim \tau^{-1}\|f\|_\infty\|u\|_{X_\zeta^{1/2}}\|v\|_{X_\zeta^{1/2}}. \tag{16}$$

A more difficult task is to control $m_{\nabla f}$. We record the main computation in the following lemma:

Lemma 4.1. *Let s, p, θ be as in Theorem 1.1. Let $1/q = 1/2 - 1/p$. There is some $\alpha > 0$ such for fixed $\lambda \le 100\tau$, we have*

$$\sum_{\substack{\mu\le\nu\le\tau/8\\ \nu<\lambda}} (\nu/\lambda)^{(1-\theta)/n}\|Q_\mu u\|_q\|Q_\nu v\|_2 \lesssim (\lambda/\tau)^\alpha\lambda^{s-2}\|u\|_{X_\zeta^{1/2}}\|v\|_{X_\zeta^{1/2}}, \tag{17}$$

and

$$\sum_{\substack{\mu\le\nu\le\tau/8\\ \lambda\le\nu<\tau/8}} \|Q_\mu u\|_q\|Q_\nu v\|_2 \lesssim \lambda^{-1}(\lambda/\tau)^\alpha\|u\|_{X_\zeta^{1/2}}\|v\|_{X_\zeta^{1/2}}. \tag{18}$$

Proof. We may interpolate (12) with the trivial estimate $\|Q_\mu u\|_2 \lesssim (\mu\tau)^{-1/2}\|Q_\mu u\|_{X_\zeta^{1/2}}$ to obtain

$$\|Q_\mu u\|_q \lesssim (\mu/\tau)^{(1-\theta)/n}(\mu\tau)^{-\theta/2}\|Q_\mu u\|_{X_\zeta^{1/2}}$$

where $1/q = 1/2 - 1/p$ and θ is such that $p = n/(1-\theta)$. Combining this with the trivial L^2 estimate for v, we obtain

$$\|Q_\mu u\|_q\|Q_\nu v\|_2 \lesssim B_{\mu,\nu}\|Q_\mu u\|_{X_\zeta^{1/2}}\|Q_\nu v\|_{X_\zeta^{1/2}}$$

where

$$B_{\mu,\nu} := \tau^{-(1-\theta)/n-\theta/2-1/2}\mu^{(1-\theta)/n-\theta/2}\nu^{-1/2}.$$

Set

$$\beta := \frac{1-\theta}{n} - \frac{\theta}{2}.$$

Suppose first that $\beta > 0$. When $\nu \geq \lambda$ we have

$$\begin{aligned} B_{\mu,\nu} &\lesssim \tau^{-(1-\theta)/n-\theta/2-1/2}\lambda^{(1-\theta)/n-\theta/2-1/2}(\mu/\nu)^\beta \\ &= \lambda^{-\theta-1}(\lambda/\tau)^{(1-\theta)/n+\theta/2+1/2}(\mu/\nu)^\beta. \end{aligned}$$

We take $\alpha \leq (1-\theta)/n+(1+\theta)/2$ and use the discrete Young's inequality to establish (18).

Suppose now that $\nu < \lambda$. When $n = 3$, we set $\theta = 0$,

$$\begin{aligned} (\nu/\lambda)^{1/3}B_{\mu,\nu} &= (\nu/\lambda)^{1/3}\tau^{-5/6}\mu^{1/3}\nu^{-1/2} \\ &= (\mu/\nu)^{1/3}(\nu/\tau)^{1/6}(\lambda/\tau)^{2/3}\lambda^{-1}. \end{aligned}$$

By Young's inequality we have (17) for $\alpha \leq 2/3$. When $n = 4$ we take θ to be zero and obtain

$$\begin{aligned} (\nu/\lambda)^{1/4}B_{\mu,\nu} &= (\nu/\lambda)^{1/4}\tau^{-3/4}\mu^{1/4}\nu^{-1/2} \\ &= (\mu/\nu)^{1/4}(\lambda/\tau)^{3/4}\lambda^{-1}. \end{aligned}$$

Applying Young's inequality we have (17) for $\alpha \leq 3/4$.

When $n > 4$, we have

$$(\nu/\lambda)^{(1-\theta)/n}B_{\mu,\nu} = (\mu/\nu)^\beta\nu^{-1/2+2(1-\theta)/n-\theta/2}\lambda^{s-2}(\lambda/\tau)^{(1-\theta)/n+\theta/2+1/2}.$$

In this case we have (17) for $\alpha \leq (1-\theta)/n + \theta/2 + 1/2$

In higher dimensions, we also want to consider the case $(1-\theta)/n - \theta/2 \leq 0$. For $\nu \geq \lambda$ we have

$$\begin{aligned} B_{\mu,\nu} &\leq \mu^\beta\lambda^{-1/2}\tau^{-(1-\theta)/n-\theta/2-1/2} \\ &\lesssim \lambda^{-1}\tau^{-(1-\theta)/n-\theta/2}. \end{aligned}$$

Then we have (18) for $\alpha < (1-\theta)/n+\theta/2$, since there are only $\sim \log\tau$ possible values of μ, ν.

For $\lambda \geq \nu$ we have

$$(\nu/\lambda)^{(1-\theta)/n}B_{\mu,\nu} \lesssim \nu^{(1-\theta)/n-1/2}\lambda^{-2(1-\theta)/n-\theta/2-1/2}(\lambda/\tau)^{(1-\theta)/n+\theta/2+1/2}.$$

Thus we have (17) for $\alpha \leq (1-\theta)/n + \theta/2 + 1/2$. □

Let P_λ denote the Littlewood–Paley projections, and let P_μ^1 denote the Littlewood–Paley projections in the e_1 direction. Then

Lemma 4.2. *Let s, p be as in Theorem 1.1. Then for any $f \in W^{s-1,p}(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$,*

$$\|m_{\nabla f}\|_{X_\zeta^{1/2} \to X_\zeta^{-1/2}} \lesssim \|f\|_n + \sup_{\nu \le \lambda \le 100\tau} (\lambda/\tau)^\beta (\lambda/\nu)^{1/p} \lambda^{s-1} \|P_\lambda P^1_{\le 8\nu} f\|_p,$$

where $\beta > 0$.

Proof. Write

$$m_{\nabla f}(u, v) = m_{\nabla f}(Q_h u, Q_h v) + m_{\nabla f}(Q_h u, Q_l v) + m_{\nabla f}(Q_l u, Q_h v) + m_{\nabla f}(Q_l u, Q_l v).$$

We can treat all but the last term using (11) and (13). Integrating by parts,

$$\begin{aligned} |m_{\nabla f}(Q_h u, Q_h v)| &\lesssim \|f\|_n \|Q_h \nabla u\|_2 \|Q_h v\|_{2n/(n-2)} + \|f\|_n \|Q_h u\|_{2n/(n-2)} \|Q_h \nabla v\|_2 \\ &\lesssim \|f\|_n \|u\|_{X_\zeta^{1/2}} \|v\|_{X_\zeta^{1/2}}. \end{aligned}$$

Since $Q_l v$ is supported in $|\xi| \lesssim \tau$,

$$\begin{aligned} |m_{\nabla f}(Q_h u, Q_l v)| &\lesssim \|f\|_n \|Q_h \nabla u\|_2 \|Q_l v\|_{2n/(n-2)} + \|f\|_n \|Q_h u\|_2 \|Q_l \nabla v\|_{2n/(n-2)} \\ &\lesssim \|f\|_n \|Q_h u\|_{H^1_\tau} \|Q_l v\|_{2n/(n-2)} \\ &\lesssim \|f\|_n \|u\|_{X_\zeta^{1/2}} \|v\|_{X_\zeta^{1/2}}. \end{aligned}$$

It remains to estimate $m_{\nabla f}(Q_l u, Q_l v)$. We have

$$m_{\nabla f}(Q_l u, Q_l v) = \sum_{\mu,\nu,\lambda} \int (\nabla P_\lambda f)\, Q_\mu u\, \overline{Q_\nu v}\, dx. \tag{19}$$

Suppose $\mu \le \nu$ (the case $\mu > \nu$ is identical). Because $Q_\mu u\, \overline{Q_\nu v}$ has Fourier support in $\{\xi : |\xi_1| \le 2\nu\}$, Plancherel's theorem and Hölder's inequality give

$$\begin{aligned} \left| \int (\nabla P_\lambda f)\, Q_\mu u\, \overline{Q_\nu v}\, dx \right| &= \left| \int P^1_{\le 8\nu} (\nabla P_\lambda f)\, Q_\mu u\, \overline{Q_\nu v}\, dx \right| \\ &\lesssim \|P^1_{\le 8\nu} \nabla P_\lambda f\|_p \|Q_\mu u\|_q \|Q_\nu v\|_2. \end{aligned}$$

Furthermore, since $Q_\mu u\, \overline{Q_\nu v}$ has Fourier support in $\{|\xi| \lesssim 100\tau\}$, we can assume $\lambda \le 100\tau$ in this sum. Applying Lemma 4.1, we get

$$\begin{aligned} |m_{\nabla f}(Q_l u, Q_l v)| &\lesssim \sum_{\substack{\nu \ge \lambda \\ \mu \le \nu}} \|\nabla P_\lambda f\|_p \|Q_\mu u\|_q \|Q_\nu v\|_2 \\ &\quad + \sum_{\substack{\nu < \lambda \le 100\tau \\ \mu \le \nu}} (\lambda/\nu)^{1/p} (\nu/\lambda)^{1/p} \|\nabla P_\lambda P^1_{\le 8\nu} f\|_p \|Q_\mu u\|_q \|Q_\nu v\|_2 \\ &\lesssim \sum_{\lambda \le 100\tau} \{(\lambda/\tau)^\alpha \lambda^{-1} \|\nabla P_\lambda f\|_p \\ &\quad + \sup_{\nu \le \lambda} (\lambda/\tau)^\alpha (\lambda/\nu)^{1/p} \lambda^{s-2} \|\nabla P_\lambda P^1_{\le 8\nu} f\|_p\} \\ &\quad \times \|u\|_{X_\zeta^{1/2}} \|v\|_{X_\zeta^{1/2}} \\ &\lesssim (\|f\|_p + \sup_{\nu \le \lambda \le 100\tau} (\lambda/\tau)^{\alpha/2} (\lambda/\nu)^{1/p} \lambda^{s-1} \|P_\lambda P^1_{\le 8\nu} f\|_p) \\ &\quad \times \|u\|_{X_\zeta^{1/2}} \|v\|_{X_\zeta^{1/2}}. \end{aligned}$$

□

5. Averaging

Given any vector $\omega \in S^{n-1}$, we define P_μ^ω to be Littlewood–Paley projection in ω direction. Let μ denote Haar measure on $O(n)$, normalized so that if σ is the usual spherical measure on S^{n-1} and $f : S^{n-1} \to \mathbb{R}$ is integrable, then for any $\theta \in S^{n-1}$ we have

$$\int_{O(n)} f(U\theta)\, d\mu(U) = \int_{S^{n-1}} f(\omega)\, d\sigma(\omega). \tag{20}$$

Lemma 5.1. *Suppose $p \in [2, \infty]$. Let $f \in L^p(\mathbb{R}^n)$. For $U \in O(n)$ and $\nu \le \lambda$, define*

$$A_{\lambda,\nu}(U) = (\lambda/\nu)^{1/p} \| P_\lambda P_{\le \nu}^{Ue_1} f\|_p.$$

Then

$$\|A_{\lambda,\nu}\|_{L^p(O(n))} \lesssim \|f\|_p.$$

Proof. We define an operator T mapping functions on $\mathbb{R}^n$ to functions on $O(n) \times \mathbb{R}^n$ by

$$Tf(U, x) = P_\lambda P_{\le \nu}^{Ue_1} f(x).$$

The lemma asserts that this operator is bounded from $L^p(\mathbb{R}^n)$ to $L^p(O(n) \times \mathbb{R}^n)$. By interpolation, it suffices to establish this at the endpoints $p = 2$ and $p = \infty$.

When $p = \infty$ this is just the fact that the Littlewood–Paley projections are bounded on L^∞.

When $p = 2$ we use Plancherel's theorem and Fubini.

$$\begin{aligned}\|Tf\|_{L^2}^2 &\sim \int_{O(n)} \int_{\mathbb{R}^n} |\phi(\xi/\lambda)\chi(\xi \cdot (Ue_1)/\nu)\hat{f}(\xi)|^2\, d\xi\, d\mu(U) \\ &\le \left(\sup_\xi \int_{O(n)} |\phi(\xi/\lambda)\chi(\xi \cdot (Ue_1)/\nu)|^2\, d\mu(U)\right) \|f\|_2^2.\end{aligned}$$

Here ϕ is supported on an annulus, and χ is supported on an interval. We estimate the last integral using (20) and spherical coordinates:

$$\begin{aligned}\int_{O(n)} |\phi(\xi/\lambda)\chi(\xi \cdot (Ue_1)/\nu)|^2\, d\mu(U) &\lesssim \sup_{|\xi|\sim\lambda} \int_{S^{n-1}} |\chi(|\xi|\omega \cdot e_1/\nu)|^2\, d\sigma(\omega) \\ &\lesssim \sup_{|\xi|\sim\lambda} \int_0^\pi |\chi(|\xi|\cos\theta/\nu)|^2 \sin(\theta)^{n-2}\, d\theta \\ &\lesssim \sup_{|\xi|\sim\lambda} \int_{-1}^1 \chi(|\xi|u/\nu)\, du \\ &\lesssim \sup_{|\xi|\sim\lambda} \frac{\nu}{|\xi|} \\ &\lesssim \frac{\nu}{\lambda}.\end{aligned}$$

This shows that

$$\|Tf\|_2 \lesssim (\nu/\lambda)^{1/2} \|f\|_2,$$

which completes the proof. □

Define $\zeta(\tau, U) = \tau U(e_1 - ie_2)$. Our next lemma establishes that $\|q\|_{X^{-1/2}_{\zeta(\tau,U)}}$ is small on average. This is implied by [HT13, Lemma 3.1], but we give a simpler proof here, based on [NS14]:

Lemma 5.2. *If $f \in \dot{H}^{-1}$, then*

$$M^{-1}\int_M^{2M}\int_{O(n)} \|f\|^2_{\dot{X}^{-1/2}_{\zeta(\tau,U)}}\, d\mu(U)\, d\tau \lesssim \|P_{\geq 100M} f\|^2_{\dot{H}^{-1}} + M^{-1}\|P_{<100M} f\|^2_{\dot{H}^{-1/2}}.$$

Proof. This is true if f is supported at frequencies $|\xi| \geq 100M$, because there we have $|p_\zeta(\xi)| \geq |\xi|^2$. Thus we may assume that f is supported at frequencies $|\xi| \lesssim M$, where we have $|p_\zeta(\xi)| \gtrsim 2\tau|\xi \cdot (Ue_1)| + |-|\xi|^2 + 2\tau\xi \cdot (Ue_2)|$. Here we use Plancherel and the identity $U^T = U^{-1}$ and estimate as in Lemma 5.1 by

$$\||\nabla|^{-1/2} f\|_2^2 \sup_{|\xi|\leq 100M} \frac{|\xi|}{M}\int_M^{2M}\int_{O(n)} (2\tau|(U^{-1}\xi)\cdot e_1| + |-|\xi|^2 + 2\tau(U^{-1}\xi)\cdot e_2|)^{-1}\, d\mu(U)\, d\tau.$$

By (20), the quantity inside the supremum is given by

$$\frac{1}{M}\int_M^{2M}\int_{S^{n-1}} (2\tau|\omega\cdot e_1| + |-|\xi| + 2\tau\omega\cdot e_2|)^{-1}\, d\sigma(\omega)\, d\tau.$$

We view (τ, ω) as polar coordinates and change variables to $u = \tau\omega$. Then in the region $\tau \in [M, 2M]$ the volume element du is bounded below by $M^{n-1}\, d\sigma(\omega)\, d\tau$, so this integral is bounded by

$$\frac{1}{M^n}\int_{|u|\in[M,2M]} (2|u_1| + |-|\xi| + 2u_2|)^{-1}\, du.$$

Writing $v = (u_1, u_2)$, and integrating over the remaining variables, we bound by

$$\frac{1}{M^n} M^{n-2}\int_{B(0,2M)} (2|v_1| + |-|\xi| + 2v_2|)^{-1}\, dv \leq \frac{1}{M^2}\int_{B(0,2M)} |v|^{-1}\, dv \sim \frac{1}{M}.$$

□

We summarize our estimates so far in the following

Theorem 5.3. *Let s, p be as in Theorem 1.1, and let γ be a positive real-valued function on $\mathbb{R}^n$ such that $\nabla \log \gamma \in W^{s-1,p}$ and $\gamma = 1$ outside of a large ball B. For $q = \gamma^{-1/2}\Delta\gamma^{1/2}$, we have*

$$M^{-1}\int_{M/2}^{2M}\int_{O(n)} \|q\|^2_{X^{-1/2}_{\zeta(\tau,U)}}\, d\mu(U)\, d\tau \to 0. \tag{21}$$

Furthermore,

$$\sup_{\tau\in[M/2,2M]} \|m_q\|_{X^{1/2}_{\zeta(\tau,U)} \to X^{-1/2}_{\zeta(\tau,U)}} \leq C_M + A_M(U), \tag{22}$$

where $C_M \to 0$ as $M \to \infty$ and

$$\sum_{k>2} k^{-1}\|A_{2^k}\|^p_{L^p(O(n))} < \infty. \tag{23}$$

Proof. First, we write

$$\gamma^{-1/2}\Delta\gamma^{1/2} = \tfrac{1}{2}\Delta\log\gamma + \tfrac{1}{4}|\nabla\log\gamma|^2 = \sum_i \nabla_i f_i + h,$$

where $f_i \in W^{s-1,p}$ and $h \in L^{p/2}$.

We decompose each term into a good part and a bad part. Let $\phi_\epsilon = \epsilon^{-n}\phi(x/\epsilon)$, where ϕ is a C_0^∞ function supported on the unit ball and $\int\phi = 1$. Define $f_\epsilon = f * \phi_\epsilon$.

By (16), we have

$$\begin{aligned}\|m_{\nabla f_\epsilon}\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}} + \|m_{h_\epsilon}\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}} &\lesssim \tau^{-1}(\|\nabla f_\epsilon\|_\infty + \|h_\epsilon\|_\infty)\\ &\lesssim \tau^{-1}\epsilon^{-2}(\|f\|_n + \|h\|_{n/2}).\end{aligned}$$

We also have

$$\begin{aligned}\|\nabla f_\epsilon\|_{X_\zeta^{-1/2}} &\lesssim \tau^{-1/2}\|\nabla f_\epsilon\|_2\\ &\lesssim \tau^{-1/2}\epsilon^{-1}\|f\|_2\\ &\lesssim \tau^{-1/2}\epsilon^{-1}\|f\|_n,\end{aligned}$$

since $n > 2$ and f is compactly supported. For $n \geq 4$ we have

$$\begin{aligned}\|h\|_{X_\zeta^{-1/2}} &\lesssim \tau^{-1/2}\|h\|_2\\ &\lesssim \tau^{-1/2}\|h\|_{n/2},\end{aligned} \tag{24}$$

and for $n = 3$ we have

$$\begin{aligned}\|h_\epsilon\|_{X_\zeta^{-1/2}} &\lesssim \tau^{-1/2}\|h_\epsilon\|_2\\ &\lesssim \tau^{-1/2}\epsilon^{-1/2}\|h\|_{3/2}.\end{aligned}$$

Taking $\epsilon = M^{-1/4}$, we find that if we replace q with q_ϵ then the left hand sides of (21) and (22) vanish as $\tau \to \infty$.

It remains to treat the bad part $q - q_\epsilon$. Let $g = f - f_\epsilon$, and define

$$A(\tau, U) = \|m_{\nabla g}\|_{X^{1/2}_{\zeta(\tau,U)}\to X^{-1/2}_{\zeta(\tau,U)}}.$$

Using Lemma 4.2, we have

$$\sup_{\tau\in[M/2,2M]} A(\tau, U) \lesssim \|g\|_{L^p} + \left(\sum_{1\leq\nu\leq\lambda\leq 4M}[(\lambda/M)^\beta\lambda^{s-1}A_{\lambda,\nu}(U)]^p\right)^{1/p},$$

where $A_{\lambda,\nu}(U) = (\lambda/\nu)^{1/p}\|P_\lambda P^{Ue_1}_{\leq 8\nu} g\|_{L^p}$. As $M \to \infty$, we have $\epsilon = M^{-1/4} \to 0$, so $\|g\|_{L^p} \to 0$. We take $A_M(U)$ to be the second term on the right hand side of this inequality, which is clearly a measurable function on $O(n)$. Now, $P_\lambda g = P_\lambda P_{\sim\lambda} g$, where $P_{\sim\lambda} g = \sum_{\lambda/16\leq\mu\leq 16\lambda} P_\mu g$. Applying Lemma 5.1, we have

$$\|A_M(U)\|^p_{L^p(O(n))} \lesssim \sum_{1\le \nu\le\lambda\le M/4} [(\lambda/M)^\beta \lambda^{s-1}\|P_{\sim\lambda} g\|_{L^p}]^p$$
$$\lesssim \log M \sum_{1\le\lambda\le M/4} [(\lambda/M)^\beta \lambda^{s-1}\|P_{\sim\lambda} f\|_{L^p}]^p.$$

We control this quantity by taking a weighted sum over dyadic integers M, as in [NS14]. Namely, we have

$$\sum_{M\ge 2} (\log M)^{-1} \|A_M(U)\|^p_{L^p(O(n))} \lesssim \sum_\lambda \sum_{M\ge 4\lambda} (\lambda/M)^{\beta p}[\lambda^{s-1}\|P_{\sim\lambda} f\|_{L^p}]^p$$
$$\lesssim \sum_\lambda [\lambda^{s-1}\|P_{\sim\lambda} f\|_{L^p}]^p.$$

The last term is controlled by $\|f\|_{W^{s-1,p}}$ as a consequence of the Littlewood–Paley square function estimate. Thus we obtain (23).

By Lemma 5.2, we have

$$M^{-1}\int_{M/2}^{2M}\int_{O(n)} \|\nabla g\|^2 \, d\mu(U)\, d\tau \lesssim \|g\|^2_{L^2} \lesssim \|g\|^2_{L^p} \to 0.$$

Next we treat $h - h_\epsilon$. When $n \ge 4$ we have $\|h-h_\epsilon\|_{X_\zeta^{-1/2}} \to 0$ by (24). When $n = 3$, we have

$$\|h - h_\epsilon\|_{X_\zeta^{-1/2}} \lesssim \|h - h_\epsilon\|_{H^{-1/2}}$$
$$\lesssim \|h - h_\epsilon\|_{3/2}$$
$$\to 0$$

by Sobolev embedding. Finally, by (15) we have

$$\|m_{h-h_\epsilon}\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}} \lesssim \|h-h_\epsilon\|_{n/2} \to 0.$$

□

6. Localization

Because our problem is localized to a compact set, the uncertainty principle implies that the $X_\zeta^{1/2}$ norm is equivalent to the $\dot{X}_\zeta^{1/2}$ norm. To make this precise, we state the following

Lemma 6.1 [HT13]. *Let ϕ be a fixed Schwartz function. Then*

$$\|\phi u\|_{\dot{X}_\zeta^{-1/2}} \lesssim_\phi \|u\|_{X_\zeta^{-1/2}} \tag{25}$$
$$\|\phi u\|_{X_\zeta^{1/2}} \lesssim_\phi \|u\|_{\dot{X}_\zeta^{1/2}}, \tag{26}$$

where the constants depend on the seminorms $\|x^\alpha \nabla^\beta \phi\|_\infty$.

In particular, we have

Lemma 6.2. *Suppose that q is compactly supported. Then*

$$\|m_q\|_{\dot{X}_\zeta^{1/2}\to\dot{X}_\zeta^{-1/2}} \lesssim \|m_q\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}}. \tag{27}$$

Proof. Let ϕ be a Schwartz function that it equal to one on the support of q. Then

$$\begin{aligned}|\langle m_q u, v\rangle| &= |\langle m_q\phi u, \phi v\rangle|\\ &\lesssim \|m_q\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}}\|\phi u\|_{X_\zeta^{1/2}}\|\phi v\|_{X_\zeta^{1/2}}\\ &\lesssim \|m_q\|_{X_\zeta^{1/2}\to X_\zeta^{-1/2}}\|u\|_{\dot{X}_\zeta^{1/2}}\|v\|_{\dot{X}_\zeta^{1/2}}.\end{aligned}$$

□

We record the following useful fact:

Lemma 6.3. *Suppose $\zeta, \tilde{\zeta} \in \mathbb{C}^n$ satisfy $\zeta\cdot\zeta = \tilde{\zeta}\cdot\tilde{\zeta} = 0$. Then*

$$\|u\|_{X_\zeta^b} \lesssim (1+|\zeta-\tilde{\zeta}|)^{|b|}\|u\|_{X_{\tilde{\zeta}}^b}.$$

Proof. We have

$$\begin{aligned}|p_\zeta| &\le |p_{\tilde{\zeta}}| + 2|(\zeta-\tilde{\zeta})\cdot\xi|\\ &\le |p_{\tilde{\zeta}}| + 2|\zeta-\tilde{\zeta}||\xi|\\ &\lesssim (1+|\zeta-\tilde{\zeta}|)(|p_{\tilde{\zeta}}|+\tau)\end{aligned}$$

by (10). □

7. Proof of the Main Theorem

We summarize some known results which allow us to extend the γ_i to all of $\mathbb{R}^n$. First we transfer the problem to the interior, as in [SU87].

Lemma 7.1. *Suppose $n \ge 3$. Let $\gamma_1, \gamma_2 \in W^{1,n}(\mathbb{R}^n)$ be functions such that $0 < c \le \gamma_i \le c^{-1}$ for some c. If $\gamma_1 = \gamma_2$ outside Ω and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then for $q_j = \Delta\sqrt{\gamma_j}/\sqrt{\gamma_j}$, we have*

$$\langle q_1, v_1v_2\rangle = \langle q_2, v_1v_2\rangle$$

when each v_j is a solution in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ to $\Delta v_j - q_jv_j = 0$.

Proof. See [Bro03]. □

The following argument is apparently due to Alessandrini. It amounts to the fact that $q_1 = q_2$ implies that the function $\log\gamma_1 - \log\gamma_2$ solves the Dirichlet problem $\operatorname{div}\sqrt{g_1g_2}\nabla u = 0$ with $u = 0$ at infinity. See [SU87,Bro96,Bro03].

Lemma 7.2. *Let γ_i, q_i be as in Lemma 7.1, and suppose that $q_1 = q_2$ in the sense of distributions. Then $\gamma_1 = \gamma_2$.*

Proof. First, have $q_i \in H^{-1}(\mathbb{R}^n)$ for each i. To see this we note that

$$\begin{aligned}\|q\|_{H^{-1}} &= \|\tfrac{1}{2}\Delta \log \gamma + \tfrac{1}{2}|\nabla \log \gamma|^2\|_{H^{-1}} \\ &\lesssim \|\nabla \log \gamma\|_2 + \||\nabla \log \gamma|^2\|_{H^{-1}} \\ &\lesssim \|\nabla \log \gamma\|_n + \|\nabla \log \gamma\|_{4n/(n+2)}^2 \\ &\lesssim \|\nabla \log \gamma\|_n + \|\nabla \log \gamma\|_n\end{aligned}$$

by Sobolev embedding and Hölder's inequality. It follows that we may test $q_1 - q_2$ against the function $g_1 g_2(\log g_1 - \log g_2) \in H^1(\mathbb{R}^n)$, where $g_i = \sqrt{\gamma_i}$. This gives

$$\begin{aligned}0 &= \int [\nabla g_1 \cdot \nabla(g_2(\log g_1 - \log g_2)) - \nabla g_2 \cdot \nabla(g_1(\log g_1 - \log g_2))]\, dx \\ &= \int (g_2 \nabla g_1 - g_1 \nabla g_2) \cdot \nabla(\log g_1 - \log g_2)\, dx \\ &= \int g_1 g_2 |\nabla(\log g_1 - \log g_2)|^2\, dx,\end{aligned}$$

which implies that $g_1 = g_2$. □

Now we apply the boundary determination result of [Bro13], which implies

Theorem 7.3. *Suppose that* $0 < c < \gamma_i < c^{-1}$. *If* $\gamma_i \in W^{1,1}(\Omega)$ *and* $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, *then* $\gamma_1 = \gamma_2$ *on* $\partial\Omega$.

Proof of Theorem 1.1. By Theorem 7.3, we have $\gamma_1 = \gamma_2$ on $\partial\Omega$. Our assumptions imply that $s - 1/p \leq 1$. Thus by [Mar87] we may extend the γ_i to functions in $W^{s,p}$ such that $\gamma_1 = \gamma_2$ outside of Ω. By Lemma 7.1, this implies that

$$\langle q_1, v_1 v_2\rangle = \langle q_2, v_1 v_2\rangle \tag{28}$$

when each v_j is a solution in $H^1_{\text{loc}}(\mathbb{R}^n)$ to $\Delta v_j - q_j v_j = 0$.

Fix $r > 0$ and three orthonormal vectors $\{e_1, e_2, e_3\}$, and define

$$\begin{aligned}\zeta_1(\tau, U) &= \tau U(e_1 - ie_2) \\ \zeta_2(\tau, U) &= -\zeta_1(\tau, U) \\ \tilde{\zeta}_1(\tau, U) &:= \tau U e_1 + i(rUe_3 - \sqrt{\tau^2 - r^2}Ue_2) \\ \tilde{\zeta}_2(\tau, U) &:= -\tau U e_1 + i(rUe_3 + \sqrt{\tau^2 - r^2}Ue_2).\end{aligned}$$

In what follows, all of inequalities will implicitly depend on r. For example, we have $|\zeta_i - \tilde{\zeta}_i| \lesssim 1$. In particular, by Lemma 6.3, the spaces $X^b_{\zeta_i}$ and $X^b_{\tilde{\zeta}_i}$ have equivalent norms.

Now let

$$F(\tau, U) = \sum_i \|m_{q_i}\|^p_{X^{1/2}_{\zeta_i(\tau,U)} \to X^{-1/2}_{\zeta_i(\tau,U)}} + \sum_{i,j} \|q_i\|^2_{X^{-1/2}_{\zeta_j(\tau,U)}}.$$

By Theorem 5.3 and the fact that $\sum_{k>1} k^{-1} = \infty$, we have

$$\liminf_{M\to\infty} M^{-1} \int_M^{2M} \int_{O(n)} F(\tau, U)\, d\mu(U)\, d\tau = 0. \tag{29}$$

Now we use an argument from [NS14, Section 3.3] to select τ and U. Their first observation is that if $\epsilon > 0$ and $B_\epsilon = \{U \in O(n) \colon \|U - I\| < \epsilon\}$, then by simply restricting (29) we have

$$\liminf_{M\to\infty} M^{-1}\mu(B_\epsilon)^{-1} \int_M^{2M} \int_{B_\epsilon} F(\tau, U)\, d\mu(U)\, d\tau = 0.$$

Thus we may choose, for a sequence of $M = M_l$ such that $M_l \to \infty$, some $\tau = \tau_{\epsilon,l} \in [M_l, 2M_l]$, $U = U_{\epsilon,l} \in B_\epsilon$ and $\delta = \delta_{\epsilon,l} > 0$ such that

$$\sum_i \|m_{q_i}\|_{X^{1/2}_{\zeta_i(\tau,U)} \to X^{-1/2}_{\zeta_i(\tau,U)}} + \sum_{i,j} \|q_i\|_{X^{-1/2}_{\zeta_j(\tau,U)}} \le \delta \tag{30}$$

where $\delta_{\epsilon,l} \to 0$ as $l \to \infty$.

By (27), we have

$$\|m_{q_i}\|_{\dot{X}^{1/2}_{\tilde{\zeta}_i(\tau,U)} \to \dot{X}^{-1/2}_{\tilde{\zeta}_i(\tau,U)}} \lesssim \|m_{q_i}\|_{X^{1/2}_{\tilde{\zeta}_i(\tau,U)} \to X^{-1/2}_{\tilde{\zeta}_i(\tau,U)}}.$$

It follows that,

$$\|m_{q_i}\|_{\dot{X}^{1/2}_{\tilde{\zeta}_i(\tau,U)} \to \dot{X}^{-1/2}_{\tilde{\zeta}_i(\tau,U)}} \lesssim \delta_{\epsilon,l}.$$

Since $\delta_{\epsilon,l} \to 0$ as $l \to \infty$, we can choose l large enough that the left hand side is less than $1/2$. Since $\|\Delta_\zeta^{-1}\|_{\dot{X}^{-1/2}_\zeta \to \dot{X}^{1/2}_\zeta} = 1$ for any ζ, we can use the contraction mapping principle to construct solutions $\psi_i \in \dot{X}^{1/2}_{\tilde{\zeta}_i(\tau,U)}$ to the equations $(\Delta_{\tilde{\zeta}_i(\tau,U)} - m_{q_i})\psi_i = q_i$, satisfying

$$\|\psi_i\|_{\dot{X}^{1/2}_{\tilde{\zeta}_i(\tau,U)}} \lesssim \|q\|_{\dot{X}^{-1/2}_{\tilde{\zeta}_i(\tau,U)}}.$$

Note that by (6), such a solution lies in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$. This implies that the corresponding solution $v_i = e^{x\cdot\tilde{\zeta}_i(\tau,U)}(1 + \psi_i)$ to the Schrödinger equation $(\Delta - q_i)v_i$ lies in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ as well.

Let $k = 2rUe_3$. By (28),

$$\begin{aligned} 0 &= \langle q_1 - q_2, e^{ik\cdot x}(1 + \psi_1)(1 + \psi_2)\rangle \\ &= \langle q_1 - q_2, e^{ik\cdot x}\rangle + \langle q_1 - q_2, e^{ik\cdot x}\psi_1\psi_2\rangle + \langle q_1 - q_2, e^{ik\cdot x}(\psi_1 + \psi_2)\rangle. \end{aligned}$$

We need to show that the second and third terms are small. Let ϕ be a Schwartz function that is equal to one on the support of q. Then

$$\begin{aligned} |\langle q_1, e^{ik\cdot x}\psi_1\psi_2\rangle| &= |\langle m_{q_1} e^{-ik\cdot x}\overline{\psi}_2, \psi_1\rangle| \\ &\lesssim \|e^{-ik\cdot x}\phi\overline{\psi}_2\|_{X^{1/2}_{\zeta_1(\tau,U)}} \|\phi\psi_1\|_{X^{1/2}_{\zeta_1(\tau,U)}} \\ &= \|e^{ik\cdot x}\phi\psi_2\|_{X^{1/2}_{\zeta_2(\tau,U)}} \|\phi\psi_1\|_{X^{1/2}_{\zeta_1(\tau,U)}} \\ &\lesssim \|\psi_2\|_{\dot{X}^{1/2}_{\tilde{\zeta}_2(\tau,U)}} \|\psi_1\|_{\dot{X}^{1/2}_{\tilde{\zeta}_1(\tau,U)}} \\ &\lesssim \|q_2\|_{X^{-1/2}_{\zeta_2(\tau,U)}} \|q_1\|_{X^{-1/2}_{\zeta_1(\tau,U)}}, \end{aligned}$$

since the seminorms of $e^{-ik\cdot x}\phi$ are bounded with a bound depending only on r. We can bound the q_2 term in the same way. On the other hand, we have

$$\begin{aligned}|\langle q_i, e^{ik\cdot x}\psi_1\rangle| &\lesssim \|q_i\|_{X^{-1/2}_{\zeta_1(\tau,U)}} \|\psi_1\|_{\dot{X}^{1/2}_{\tilde{\zeta}_1(\tau,U)}} \\ &\lesssim \|q_i\|_{X^{-1/2}_{\zeta_1(\tau,U)}} \|q_1\|_{X^{-1/2}_{\zeta_1(\tau,U)}}\end{aligned}$$

by duality of $\dot{X}^{1/2}_{\zeta_1(\tau,U)}$ and $\dot{X}^{-1/2}_{\zeta_1(\tau,U)}$. The terms with ψ_2 are the same. In summary, we obtain

$$|(\hat{q}_1 - \hat{q}_2)(2rUe_3)| \lesssim \sum_{1\le i,j,k,l\le 2} \|q_i\|_{X^{-1/2}_{\zeta_j(\tau,U)}} \|q_k\|_{X^{-1/2}_{\zeta_l(\tau,U)}} \lesssim \delta^2 \tag{31}$$

by (30).

To finish the proof, we again follow [NS14]. Since B_ϵ is compact, we may pass to a subsequence such that $U_{\epsilon,l} \to U_\epsilon$ for some $U_\epsilon \in B_\epsilon$. Since the $\hat{q}_i$ are continuous, we may pass to the limit in (31) to obtain

$$|\hat{q}_1 - \hat{q}_2|(2rU_\epsilon e_3) \lesssim \lim_{l\to\infty} \delta^2_{\epsilon,l} = 0.$$

Note that by construction, we have $U_\epsilon \to I$ as $\epsilon \to 0$. Thus, by taking limits again, we obtain $(\hat{q}_1 - \hat{q}_2)(2re_3) = 0$. Since $e_3 \in S^{n-1}$ and r were arbitrary, this means that $\hat{q}_1 - \hat{q}_2 = 0$.

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