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On the Global Uniqueness for the Einstein–Maxwell-Scalar Field System with a Cosmological Constant

Part 2. Structure of the Solutions and Stability of the Cauchy Horizon

João L. Costa[1,2], **Pedro M. Girão**[2], **José Natário**[2], **Jorge Drumond Silva**[2]

[1] ISCTE-Instituto Universitário de Lisboa, Lisbon, Portugal. E-mail: jlca@iscte.pt
[2] Center for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, Universidade de Lisboa, Lisbon, Portugal. E-mail: pgirao@math.ist.utl.pt; jnatar@math.ist.utl.pt; jsilva@math.ist.utl.pt

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Abstract: This paper is the second part of a trilogy dedicated to the following problem: given spherically symmetric characteristic initial data for the Einstein–Maxwell-scalar field system with a cosmological constant Λ, with the data on the outgoing initial null hypersurface given by a subextremal Reissner–Nordström black hole event horizon, study the future extendibility of the corresponding maximal globally hyperbolic development as a "suitably regular" Lorentzian manifold. In the first paper of this sequence (Costa et al., Class Quantum Gravity 32:015017, 2015), we established well posedness of the characteristic problem with general initial data. In this second paper, we generalize the results of Dafermos (Ann Math 158:875–928, 2003) on the stability of the radius function at the Cauchy horizon by including a cosmological constant. This requires a considerable deviation from the strategy followed in Dafermos (Ann Math 158:875–928, 2003), focusing on the level sets of the radius function instead of the red-shift and blue-shift regions. We also present new results on the global structure of the solution when the free data is not identically zero in a neighborhood of the origin. In the third and final paper (Costa et al., On the global uniqueness for the Einstein–Maxwell-scalar field system with a cosmological constant. Part 3. Mass inflation and extendibility of the solutions. arXiv:1406.7261, 2015), we will consider the issue of mass inflation and extendibility of solutions beyond the Cauchy horizon.

Contents

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1. Introduction

This paper is the second part of a trilogy dedicated to the following problem: given spherically symmetric characteristic initial data for the Einstein–Maxwell-scalar field system with a cosmological constant Λ, with the data on the outgoing initial null hypersurface given by a subextremal Reissner–Nordström black hole event horizon, and the remaining data otherwise free, study the future extendibility of the corresponding maximal globally hyperbolic development as a "suitably regular" Lorentzian manifold. We are motivated by the strong cosmic censorship conjecture and the question of determinism in general relativity. As explained in detail in the Introduction of Part 1, strong cosmic censorship is one of the most fundamental open problems in general relativity (see the classic monographs [3,10] and the discussions in [1,6,9] for the general context of this problem). Although significant developments have been achieved in the last five decades (from the initial heuristic works [14,15] to rigorous mathematical results [6–8]), including some recent encouraging progress (see [9,11,13] and references therein), a complete resolution of the conjecture at hand still seems out of reach. Nonetheless, the spherically symmetric self-gravitating scalar field model has provided considerable insight into the harder problem of vacuum collapse without symmetries [2]; this was explored in [12] to obtain the first promising steps towards understanding the stability of Cauchy horizons without symmetry assumptions.

In Part 1, we established the equivalence (under appropriate regularity conditions for the initial data) between the Einstein Eqs. (1)–(5) and the system of first order PDE (14)–(23). We proved existence, uniqueness and identified a breakdown criterion for solutions of this system (see Sect. 2).

In the current paper we are concerned with the structure of the solutions of the characteristic problem, and wish to address the question of existence and stability of the Cauchy horizon when the initial data is as above. This is intimately related to the issue of global uniqueness for the Einstein equations: it is the possibility of extension of solutions across this horizon that leads to the breakdown of global uniqueness and, in case the phenomenon persists for generic initial data, to the failure of the strong cosmic censorship conjecture.

As in [6], we introduce a certain generic element in the formulation of our problem by perturbing a subextremal Reissner–Nordström black hole (whose Cauchy horizon formation is archetypal) by arbitrary characteristic data along the ingoing null direction. The study of the conditions under which the solutions can be extended across the Cauchy horizon is left to Part 3.

We take many ideas from [6,7] and build on these works. In particular, we borrow the following three very important techniques. (i) The partition of the spacetime domain of the solution into four regions and the construction of a carefully chosen spacelike curve to separate the last two. (ii) The use of the Raychaudhuri equation in v to estimate $\frac{\nu}{1-\mu}$ at a larger v from its value at a smaller v. (iii) The use of BV estimates for the field.

Nonetheless, the introduction of a cosmological constant Λ causes a significant difference that requires deviation from the original strategies developed in [6,7]. Moreover, we introduce some technical simplifications and obtain sharper and more detailed estimates. These improvements will be crucial for our arguments in Part 3.

Our approach therefore has three main departures from the one of Dafermos:

(i) First, due to the presence of the cosmological constant Λ, the curves of constant shift, which are used in [6,7], are no longer necessarily spacelike for $\Lambda > 0$ large. This forces us to find an alternative approach; we have chosen to work with curves of constant r coordinate instead of working with curves of constant shift, which turns out to be a simpler approach. Furthermore, it allows us to treat the cases $\Lambda < 0$, $\Lambda = 0$ and $\Lambda > 0$ in a unified framework.

(ii) Second, we show that the Bondi coordinates (r, v) are the ones most adapted to estimating the growth of the fields as we progress away from the event horizon. Our approach starts by controlling the field $\frac{\zeta}{\nu}$ using (54). Although this is similar to (53), there is one distinction which makes all the difference. It consists of the fact that in the double integral in (53) the field $\frac{\zeta}{\nu}$ is multiplied by the function ν. When we pass to Bondi coordinates this function disappears, making a simple application of Gronwall's inequality, such as the one we present, possible. This would not work in the double null coordinate system (u, v).

(iii) Third, our estimates are not subordinate to the division of the solution spacetime into red shift, no shift and blue shift regions. Instead, we consider the regions $\{r \geq \check{r}_+\}$, $\{\check{r}_- \leq r \leq \check{r}_+\}$ and $\{r \leq \check{r}_-\}$, where $\check{r}_+$ is smaller than but sufficiently close to the radius r_+ of the Reissner–Nordström event horizon, and $\check{r}_-$ is bigger than but sufficiently close to the radius r_- of the Reissner–Nordström Cauchy horizon. These may be loosely thought of as red shift, no shift and blue shift regions of the background Reissner–Nordström solution, even though the shift factor is not small and indeed changes significantly from red to blue in the intermediate region.

Our first objective is to obtain good upper bounds for $-\lambda$ in the different regions of spacetime. These will enable us to show that the radius function r is bounded below by a positive constant. However, good estimates for $-\nu$ and the fields θ and ζ will also be essential in Part 3.

The main result of this paper is therefore

Theorem 1.1. *Consider the characteristic initial value problem for the first order system of PDE* (14)–(23) *with initial data* (24)–(25) *(so that* $\{0\} \times [0, \infty[$ *is the event horizon of a subextremal Reissner–Nordström solution with mass* $M > 0$*). Assume that* ζ_0 *is continuous and* $\zeta_0(0) = 0$*. Then there exists* $U > 0$ *such that the domain* $\mathcal{P}$ *of the (future) maximal development contains* $[0, U] \times [0, \infty[$*. Moreover,*

$$\inf_{[0,U]\times[0,\infty[} r > 0,$$

the limit

$$r(u, \infty) := \lim_{v\to\infty} r(u, v)$$

exists for all $u \in]0, U]$ *and*

$$\lim_{u\searrow 0} r(u, \infty) = r_-.$$

So, under the hypotheses of Theorem 1.1, the argument in [7, Section 11], shows that, as in the case when $\Lambda = 0$, the spacetime is extendible across the Cauchy horizon with a C^0 metric.

We also prove that only in the case of the Reissner–Nordström solution does the curve $\{r = r_-\}$ coincide with the Cauchy horizon. As soon as the initial data field is not identically zero, the curve $\{r = r_-\}$ is contained in $\mathcal{P}$ (Theorem 8.1). This is an interesting geometrical condition and it is conceptually relevant given the importance that we confer to the curves of constant r. We also prove that, in contrast with what happens with the Reissner–Nordström solution, the presence of any nonzero field immediately causes the integral $\int_0^\infty \kappa(u, v)\, dv$ to be finite for any $u > 0$ (Lemma 8.2). As a consequence, the affine parameter of any outgoing null geodesic inside the event horizon is finite at the Cauchy horizon (Corollary 8.3).

2. Framework and Some Results from Part 1

The spherically symmetric Einstein–Maxwell-scalar field system with a cosmological constant. Consider a spherically symmetric spacetime with metric

$$g = -\Omega^2(u, v)\, du dv + r^2(u, v)\, \sigma_{\mathbb{S}^2},$$

where $\sigma_{\mathbb{S}^2}$ is the round metric on the 2-sphere. The Einstein–Maxwell-scalar field system with a cosmological constant Λ and total electric charge $4\pi e$ reduces to the following system of equations: the wave equation for r,

$$\partial_u \partial_v r = \frac{\Omega^2}{2} \frac{1}{r^2} \left(\frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \varpi \right), \tag{1}$$

the wave equation for ϕ,

$$\partial_u \partial_v \phi = -\frac{\partial_u r\, \partial_v \phi + \partial_v r\, \partial_u \phi}{r}, \tag{2}$$

the Raychaudhuri equation in the u direction,

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -r \frac{(\partial_u \phi)^2}{\Omega^2}, \tag{3}$$

the Raychaudhuri equation in the v direction,

$$\partial_v \left(\frac{\partial_v r}{\Omega^2} \right) = -r \frac{(\partial_v \phi)^2}{\Omega^2}, \tag{4}$$

and the wave equation for $\ln \Omega$,

$$\partial_v \partial_u \ln \Omega = -\partial_u \phi\, \partial_v \phi - \frac{\Omega^2 e^2}{2r^4} + \frac{\Omega^2}{4r^2} + \frac{\partial_u r\, \partial_v r}{r^2}. \tag{5}$$

The first order system. Given r, ϕ and Ω, solutions of the Einstein equations, let

$$\nu := \partial_u r \tag{6}$$
$$\lambda := \partial_v r, \tag{7}$$
$$\varpi := \frac{e^2}{2r} + \frac{r}{2} - \frac{\Lambda}{6}r^3 + \frac{2r}{\Omega^2}\nu\lambda, \tag{8}$$
$$\mu := \frac{2\varpi}{r} - \frac{e^2}{r^2} + \frac{\Lambda}{3}r^2, \tag{9}$$
$$\theta := r\partial_v\phi, \tag{10}$$
$$\zeta := r\partial_u\phi \tag{11}$$

and

$$\kappa := \frac{\lambda}{1-\mu}. \tag{12}$$

Notice that we may rewrite (8) as

$$\Omega^2 = -\frac{4\nu\lambda}{1-\mu} = -4\nu\kappa. \tag{13}$$

The Einstein equations imply the first order system for $(r, \nu, \lambda, \varpi, \theta, \zeta, \kappa)$

$$\partial_u r = \nu, \tag{14}$$
$$\partial_v r = \lambda, \tag{15}$$
$$\partial_u \lambda = \nu\kappa\partial_r(1-\mu), \tag{16}$$
$$\partial_v \nu = \nu\kappa\partial_r(1-\mu), \tag{17}$$
$$\partial_u \varpi = \frac{1}{2}(1-\mu)\left(\frac{\zeta}{\nu}\right)^2\nu, \tag{18}$$
$$\partial_v \varpi = \frac{1}{2}\frac{\theta^2}{\kappa}, \tag{19}$$
$$\partial_u \theta = -\frac{\zeta\lambda}{r}, \tag{20}$$
$$\partial_v \zeta = -\frac{\theta\nu}{r}, \tag{21}$$
$$\partial_u \kappa = \kappa\nu\frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2, \tag{22}$$

with the restriction

$$\lambda = \kappa(1-\mu). \tag{23}$$

Under appropriate regularity conditions for the initial data, the system of first order PDE (14)–(23) also implies the Einstein Eqs. (1)–(5).

Initial data. In Part 1 we study well posedness of the first order system for general initial data. In this paper we take the initial data on the outgoing null direction v to be the data on the event horizon of a subextremal Reissner–Nordström solution with mass M. The initial data on the ingoing null direction u is free. More precisely, we choose

$$\begin{cases} r(u,0) = r_0(u) = r_+ - u, \\ \nu(u,0) = \nu_0(u) = -1, \\ \zeta(u,0) = \zeta_0(u), \end{cases} \qquad \text{for } u \in [0,U], \tag{24}$$

$$\begin{cases} \lambda(0,v) = \lambda_0(v) = 0, \\ \varpi(0,v) = \varpi_0(v) = M, \\ \theta(0,v) = \theta_0(v) = 0, \\ \kappa(0,v) = \kappa_0(v) = 1, \end{cases} \qquad \text{for } v \in [0,\infty[. \tag{25}$$

Here $r_+ > 0$ is the radius of the event horizon. We assume ζ_0 is continuous and $\zeta_0(0) = 0$.

Well posedness of the first order system. Theorem 4.4 of Part 1, for the initial data above, reads:

Theorem 2.1. *The characteristic initial value problem* (14)–(23)*, with initial conditions* (24) *and* (25)*, where* ζ_0 *is continuous and* $\zeta_0(0) = 0$*, has a unique solution defined on a maximal past set* $\mathcal{P}$ *containing a neighborhood of* $[0,U] \times \{0\} \cup \{0\} \times [0,\infty[$.

Remark 2.2. Notice that the initial data (24) and (25) satisfies the regularity condition (h4) in Part 1 (that is, ν_0, λ_0 and κ_0 are C^1). Therefore the solution of the characteristic initial value problem (14)–(23) corresponds to a classical solution of the Einstein Eqs. (1)–(5).

Breakdown criterion. Theorem 5.4 of Part 1, for the initial data above, reads:

Theorem 2.3. *Suppose that* $(r, \nu, \lambda, \varpi, \theta, \zeta, \kappa)$ *is the maximal solution of the characteristic initial value problem* (14)–(23)*, with initial conditions* (24) *and* (25)*. If* (U', V') *is a point on the boundary of* $\mathcal{P}$ *with* $0 < U' < U$ *and* $V' > 0$*, then for all sequences* (u_n, v_n) *in* $\mathcal{P}$ *converging to* (U', V')*, we have*

$$r(u_n, v_n) \to 0 \quad \text{and} \quad \varpi(u_n, v_n) \to \infty.$$

Reissner–Nordström solution. For comparison purposes, we notice that the Reissner–Nordström solution (with a cosmological constant), obtained from the initial data $\zeta_0(u) = 0$, corresponds to

$$\lambda = 1 - \mu, \tag{26}$$

$$\nu = -\frac{1-\mu}{(1-\mu)(\,\cdot\,,0)}, \tag{27}$$

$$\varpi = \varpi_0, \tag{28}$$

$$\kappa = 1, \tag{29}$$

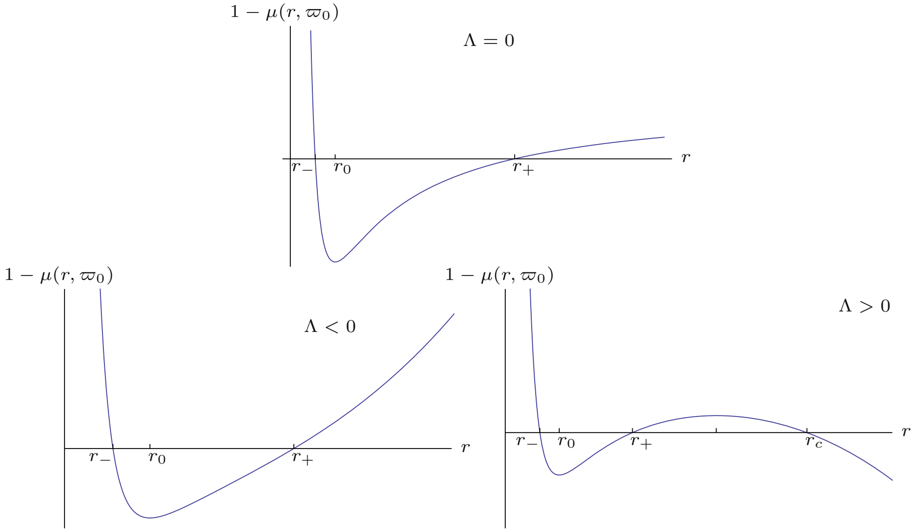
$$\zeta = \theta = 0. \tag{30}$$

3. Preliminaries on the Analysis of the Solution

We now take the initial data on the v axis to be the data on the event horizon of a subextremal Reissner–Nordström solution with mass $M > 0$. So, we choose initial data as in (24)–(25) with $\zeta_0(0) = 0$. Moreover, we assume ζ_0 to be continuous. Since in this case the function ϖ_0 is constant equal to M, we also denote M by ϖ_0. In particular, when $\Lambda < 0$, which corresponds to the Reissner–Nordström anti-de Sitter solution, and when $\Lambda = 0$, which corresponds to the Reissner–Nordström solution, we assume that

$$r \mapsto (1-\mu)(r, \varpi_0) = 1 - \frac{2\varpi_0}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2$$

has two zeros $r_-(\varpi_0) = r_- < r_+ = r_+(\varpi_0)$. When $\Lambda > 0$, which corresponds to the Reissner–Nordström de Sitter solution, we assume that $r \mapsto (1-\mu)(r, \varpi_0)$ has three zeros $r_-(\varpi_0) = r_- < r_+ = r_+(\varpi_0) < r_c = r_c(\varpi_0)$.



We define η to be the function

$$\eta = \frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \varpi.$$

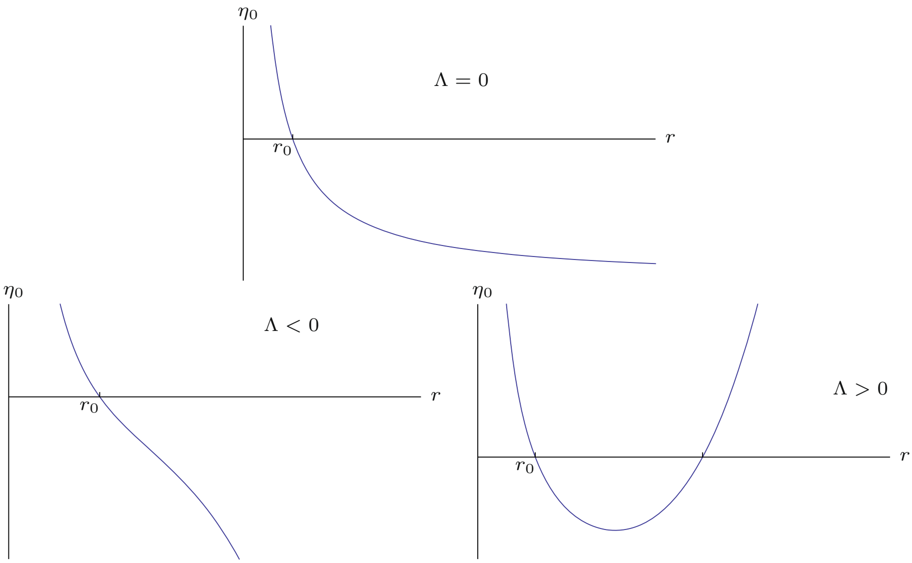
The functions $(r, \varpi) \mapsto \eta(r, \varpi)$ and $(r, \varpi) \mapsto (1-\mu)(r, \varpi)$ are related by

$$\eta = -\frac{r^2}{2} \partial_r (1-\mu). \tag{31}$$

We define the function $\eta_0 : \mathbb{R}^+ \to \mathbb{R}$ by

$$\eta_0(r) = \frac{e^2}{r} + \frac{\Lambda}{3} r^3 - \varpi_0.$$

We will repeatedly use the fact that $\eta(r, \varpi) \leq \eta_0(r)$ (see Lemma 3.1). If $\Lambda \leq 0$, then $\eta_0' < 0$. So η_0 is strictly decreasing and has precisely one zero. The zero is located between r_- and r_+. If $\Lambda > 0$, then η_0'' is positive, so η_0 is strictly convex and has precisely two zeros: one zero is located between r_- and r_+ and the other zero is located between r_+ and r_c. We denote by r_0 the zero of η_0 between r_- and r_+ in both cases.



According to (16), we have $\partial_u \lambda(0, 0) = -\partial_r(1-\mu)(r_+, \varpi_0) < 0$. Since $\lambda(0, 0) = 0$, we may choose U small enough so that $\lambda(u, 0)$ is negative for $u \in]0, U]$. Again we denote by $\mathcal{P}$ the maximal past set where the solution of the characteristic initial value problem is defined. In Part 1 we saw that λ is negative on $\mathcal{P}\backslash\{0\} \times [0, \infty[$, and so, as κ is positive (from (22) and (25)), then $1 - \mu$ is negative on $\mathcal{P}\backslash\{0\} \times [0, \infty[$.

Using the above, we can thus particularize the result of Part 1 on signs and monotonicities to the case where the initial data is (24) and (25) as follows.

Lemma 3.1 (Sign and monotonicity). *Suppose that* $(r, \nu, \lambda, \varpi, \theta, \zeta, \kappa)$ *is the maximal solution of the characteristic initial value problem* (14)–(23)*, with initial conditions* (24) *and* (25)*. Then*:

- κ *is positive*;
- ν *is negative*;
- λ *is negative on* $\mathcal{P}\backslash\{0\} \times [0, \infty[$;
- $1 - \mu$ *is negative on* $\mathcal{P}\backslash\{0\} \times [0, \infty[$;
- r *is decreasing with both* u *and* v;
- ϖ *is nondecreasing with both* u *and* v.

Using (16) and (20), we obtain

$$\partial_u \frac{\theta}{\lambda} = -\frac{\zeta}{r} - \frac{\theta}{\lambda}\frac{\nu}{1-\mu}\partial_r(1-\mu), \tag{32}$$

and analogously, using (17) and (21),

$$\partial_v \frac{\zeta}{\nu} = -\frac{\theta}{r} - \frac{\zeta}{\nu}\frac{\lambda}{1-\mu}\partial_r(1-\mu). \tag{33}$$

Given $0 < \check{r} < r_+$, let us denote by $\Gamma_{\check{r}}$ the level set of the radius function

$$\Gamma_{\check{r}} := \{(u, v) \in \mathcal{P} : r(u, v) = \check{r}\}.$$

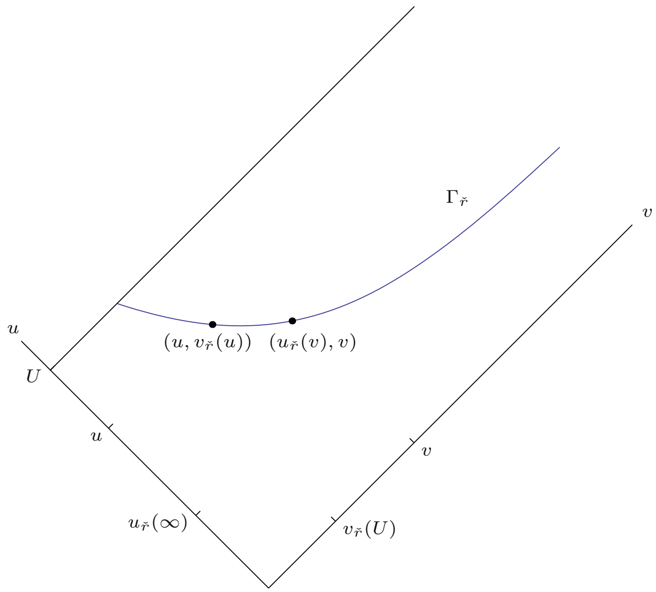
If nonempty, $\Gamma_{\check{r}}$ is a connected C^1 spacelike curve, since both ν and λ are negative on $\mathcal{P}\backslash\{0\} \times [0, \infty[$. Using the Implicit Function Theorem, the facts that $r(0, v) = r_+$, $r(u, 0) = r_+ - u$, the signs of ν and λ, and the breakdown criterion given in Theorem 2.3, one can show that $\Gamma_{\check{r}}$ can be parametrized by a C^1 function

$$v \mapsto (u_{\check{r}}(v), v),$$

whose domain is $[0, \infty[$ if $\check{r} \geq r_+ - U$, or an interval of the form $[v_{\check{r}}(U), \infty[$, for some $v_{\check{r}}(U) > 0$, if $\check{r} < r_+ - U$. Alternatively, $\Gamma_{\check{r}}$ can also be parametrized by a C^1 function

$$u \mapsto (u, v_{\check{r}}(u)),$$

whose domain is always an interval of the form $]u_{\check{r}}(\infty), \min\{r_+ - \check{r}, U\}]$, for some $u_{\check{r}}(\infty) \geq 0$. We prove below that if $\check{r} > r_-$, then $u_{\check{r}}(\infty) = 0$.



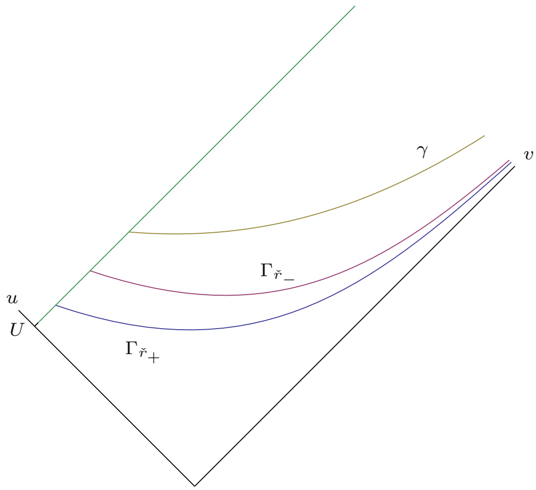
To analyze the solution we partition the domain into four regions (see figure below). We start by choosing $\check{r}_-$ and $\check{r}_+$ such that $r_- < \check{r}_- < r_0 < \check{r}_+ < r_+$. In Sect. 4 we treat the region $\check{r}_+ \leq r \leq r_+$. In Sect. 5 we consider the region $\check{r}_- \leq r \leq \check{r}_+$. In Sect. 6 we treat the region where (u, v) is such that

$$v_{\check{r}_-}(u) \leq v \leq (1 + \beta)\, v_{\check{r}_-}(u),$$

with $\beta > 0$ appropriately chosen (we will denote the curve $v = (1 + \beta)\, v_{\check{r}_-}(u)$ by γ). Finally, in Sect. 7 we consider the region where (u, v) is such that

$$v \geq (1 + \beta)\, v_{\check{r}_-}(u).$$

The reader should regard $\check{r}_-$, $\check{r}_+$ and β as fixed. Later, they will have to be carefully chosen for our arguments to go through.



The crucial step consists in estimating the fields $\frac{\theta}{\lambda}$ and $\frac{\zeta}{\nu}$. Once this is done, the other estimates follow easily. By integrating (32) and (33), we obtain

$$\begin{aligned}\frac{\theta}{\lambda}(u,v) &= \frac{\theta}{\lambda}(u_{\check{r}}(v),v)e^{-\int_{u_{\check{r}}(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},v)\,d\tilde{u}} \\ &\quad -\int_{u_{\check{r}}(v)}^{u}\frac{\zeta}{r}(\tilde{u},v)e^{-\int_{\tilde{u}}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\bar{u},v)\,d\bar{u}}\,d\tilde{u}, \end{aligned} \tag{34}$$

$$\begin{aligned}\frac{\zeta}{\nu}(u,v) &= \frac{\zeta}{\nu}(u,v_{\check{r}}(u))e^{-\int_{v_{\check{r}}(u)}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde{v})\,d\tilde{v}} \\ &\quad -\int_{v_{\check{r}}(u)}^{v}\frac{\theta}{r}(u,\tilde{v})e^{-\int_{\tilde{v}}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\bar{v})\,d\bar{v}}\,d\tilde{v}. \end{aligned} \tag{35}$$

Formula (34) is valid provided that $u_{\check{r}}(v)$ is defined and $u_{\check{r}}(v) \le u$, since the domain $\mathcal{P}$ is a past set; it also holds if we replace $u_{\check{r}}(v)$ by 0. Similarly, formula (35) is valid provided that $v_{\check{r}}(u)$ is defined and $v_{\check{r}}(u) \le v$; again it holds if we replace $v_{\check{r}}(u)$ by 0.

4. The Region $J^-(\Gamma_{\check{r}_+})$

Recall that $r_0 < \check{r}_+ < r_+$. In this section, we treat the region $\check{r}_+ \le r \le r_+$, that is, $J^-(\Gamma_{\check{r}_+})$.[1] Our first goal is to estimate (42) for $\frac{\zeta}{\nu}$. This will allow us to obtain the lower bound (43) for κ, which will then be used to improve estimate (42)–(46). Finally,

[1] Throughout this paper we follow the usual notations for the causal structure of the quotient Lorentzian manifold with coordinates (u, v) and time orientation such that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are future-pointing.

we successively bound $\frac{\theta}{\lambda}$, θ, ϖ, and use this to prove that the domain of $v_{\check{r}_+}(\cdot)$ is $]0, \min\{r_+ - \check{r}_+, U\}]$.

In this region, the solution with general ζ_0 can then be considered as a small perturbation of the Reissner–Nordström solution (26)–(30): ϖ is close to ϖ_0, κ is close to 1 and ζ, θ are close to 0. Besides, the smaller U is, the closer the approximation.

Substituting (34) in (35) [with both $u_{\check{r}}(v)$ and $v_{\check{r}}(u)$ replaced by 0], we get

$$\begin{aligned}\frac{\zeta}{\nu}(u,v) = {} & \frac{\zeta}{\nu}(u,0)e^{-\int_0^v\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v} \\ & + \int_0^v \frac{\theta}{\lambda}(0,\tilde v)e^{-\int_0^u\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde u,\tilde v)\,d\tilde u} \\ & \times\left[\frac{(-\lambda)}{r}\right](u,\tilde v)e^{-\int_{\tilde v}^v\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\bar v)\,d\bar v}\,d\tilde v \\ & + \int_0^v\left(\int_0^u\left[\frac{\zeta}{\nu}\frac{(-\nu)}{r}\right](\tilde u,\tilde v)e^{-\int_{\tilde u}^u\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\bar u,\tilde v)\,d\bar u}\,d\tilde u\right) \\ & \times\left[\frac{(-\lambda)}{r}\right](u,\tilde v)e^{-\int_{\tilde v}^v\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\bar v)\,d\bar v}\,d\tilde v.\end{aligned} \tag{36}$$

We make the change of coordinates

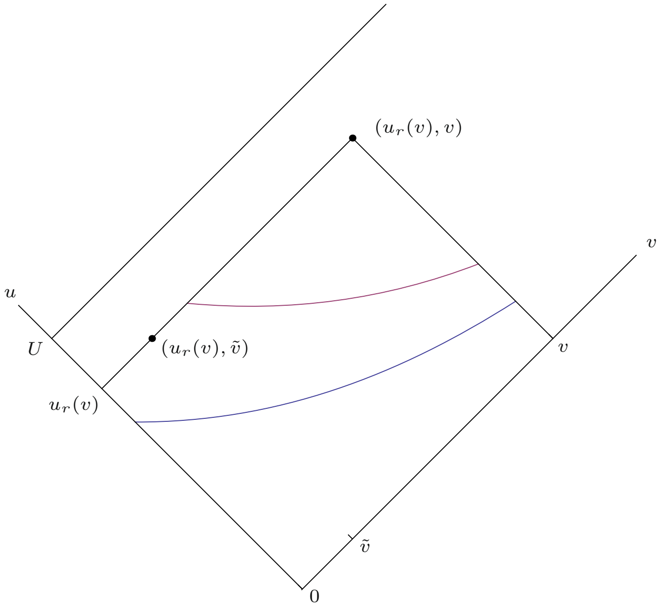
$$(u,v)\mapsto(r(u,v),v) \quad\Leftrightarrow\quad (r,v)\mapsto(u_r(v),v). \tag{37}$$

The coordinates (r, v) are called Bondi coordinates. We denote by $\widehat{\frac{\zeta}{\nu}}$ the function $\frac{\zeta}{\nu}$ written in these new coordinates, so that

$$\frac{\zeta}{\nu}(u,v) = \widehat{\frac{\zeta}{\nu}}(r(u,v),v) \quad\Leftrightarrow\quad \widehat{\frac{\zeta}{\nu}}(r,v) = \frac{\zeta}{\nu}(u_r(v),v).$$

The same notation will be used for other functions. In the new coordinates, Eq. (36) may be written

$$\begin{aligned}\widehat{\frac{\zeta}{\nu}}(r,v) = {} & \frac{\zeta}{\nu}(u_r(v),0)e^{-\int_0^v\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\tilde v)\,d\tilde v} \\ & + \int_0^v \widehat{\frac{\theta}{\lambda}}(r_+,\tilde v)e^{\int_{r(u_r(v),\tilde v)}^{r_+}\left[\frac{1}{\widehat{1-\mu}}\partial_r\widehat{(1-\mu)}\right](\tilde s,\tilde v)\,d\tilde s} \\ & \times\left[\frac{(-\lambda)(u_r(v),\tilde v)}{r(u_r(v),\tilde v)}\right]e^{-\int_{\tilde v}^v\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\bar v)\,d\bar v}\,d\tilde v \\ & + \int_0^v\left(\int_{r(u_r(v),\tilde v)}^{r_+}\frac{1}{\tilde s}\left[\widehat{\frac{\zeta}{\nu}}\right](\tilde s,\tilde v)e^{\int_{r(u_r(v),\tilde v)}^{\tilde s}\left[\frac{1}{\widehat{1-\mu}}\partial_r\widehat{(1-\mu)}\right](\bar s,\tilde v)\,d\bar s}\,d\tilde s\right) \\ & \times\left[\frac{(-\lambda)(u_r(v),\tilde v)}{r(u_r(v),\tilde v)}\right]e^{-\int_{\tilde v}^v\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\bar v)\,d\bar v}\,d\tilde v.\end{aligned} \tag{38}$$



We have $\theta(0, v) = 0$ and, from (20), $\partial_u\theta(0, v) = 0$, whereas $\lambda(0, v) = 0$ and, from (16), $\partial_u\lambda(0, v) < 0$. Writing

$$\frac{\theta}{\lambda}(u, v) = \frac{\int_0^u \partial_u\theta(\tilde{u}, v)d\tilde{u}}{\int_0^u \partial_u\lambda(\tilde{u}, v)d\tilde{u}},$$

it is easy to show that the function $\frac{\theta}{\lambda}$ can be extended as a continuous function to $\{0\} \times [0, \infty[$, with $\frac{\theta}{\lambda}(0, v) = 0$. Substituting this into (34) [again with $u_{\check{r}}(v)$ replaced by 0] yields

$$\frac{\theta}{\lambda}(u, v) = -\int_0^u \frac{\zeta}{r}(\tilde{u}, v)e^{-\int_{\tilde{u}}^u \left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\bar{u},v)\,d\bar{u}}\,d\tilde{u}.$$

We can rewrite this in the new coordinates as

$$\widehat{\frac{\theta}{\lambda}}(r, v) = \int_r^{r_+} \frac{1}{\tilde{s}}\left[\widehat{\frac{\zeta}{\nu}}\right](\tilde{s}, v)e^{\int_r^{\tilde{s}}\left[\frac{1}{1-\mu}\partial_r\widehat{(1-\mu)}\right](\bar{s},v)\,d\bar{s}}\,d\tilde{s}. \tag{39}$$

A key point is to bound the exponentials that appear in (38) and (39). As we go on, this will be done several times in different ways.

Lemma 4.1. *Assume that there exists $\alpha \geq 0$ such that, for $0 \leq \tilde{v} \leq v$, the following bounds hold*:

$$e^{-\int_{\tilde{v}}^v \left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\bar{v})\,d\bar{v}} \leq e^{-\alpha(v-\tilde{v})}$$

and

$$e^{\int_{r(u_r(v),\tilde v)}^{\tilde s}\left[\frac{1}{\widehat{1-\mu}}\partial_r\widehat{(1-\mu)}\right](\bar s,\tilde v)\,d\bar s}\le 1.$$

Then (38) *implies*

$$\left|\widehat{\frac{\zeta}{\nu}}\right|(r,v)\le e^{\frac{(r_+-r)^2}{rr_+}}\max_{u\in[0,u_r(v)]}|\zeta_0|(u)e^{-\alpha v}. \tag{40}$$

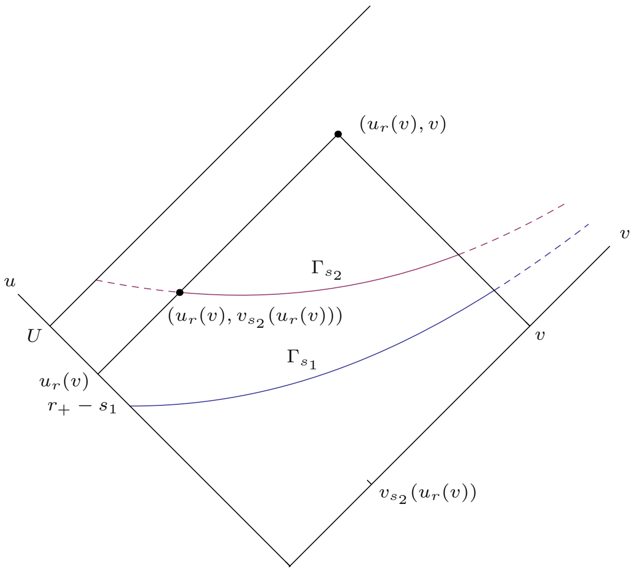
Proof. Combining (38) with $\widehat{\frac{\theta}{\lambda}}(r_+,v)\equiv 0$ and the bounds on the exponentials, we have

$$\left|\widehat{\frac{\zeta}{\nu}}\right|(r,v)\le|\zeta_0|(u_r(v))e^{-\alpha v}+\int_0^v\int_{r(u_r(v),\tilde v)}^{r_+}\frac{1}{\tilde s}\left[\widehat{\frac{\zeta}{\nu}}\right](\tilde s,\tilde v)\,d\tilde s \tag{41}$$
$$\times\left[\frac{(-\lambda)(u_r(v),\tilde v)}{r(u_r(v),\tilde v)}\right]e^{-\alpha(v-\tilde v)}\,d\tilde v.$$

For $r\le s<r_+$, define

$$\mathcal{Z}^\alpha_{(r,v)}(s)=\begin{cases}\max_{\tilde v\in[0,v]}\left\{e^{\alpha\tilde v}\left|\widehat{\frac{\zeta}{\nu}}\right|(s,\tilde v)\right\} & \text{if } r_+-u_r(v)\le s<r_+,\\ \max_{\tilde v\in[v_s(u_r(v)),v]}\left\{e^{\alpha\tilde v}\left|\widehat{\frac{\zeta}{\nu}}\right|(s,\tilde v)\right\} & \text{if } r\le s\le r_+-u_r(v).\end{cases}$$

Here the maximum is taken over the projection of $J^-(u_r(v),v)\cap\Gamma_s$ on the v-axis (see the figure below).



Note that $\mathcal{Z}^\alpha_{(r,v)}(r)=e^{\alpha v}\left|\widehat{\frac{\zeta}{\nu}}\right|(r,v)$. Inequality (41) implies

$$\begin{aligned}\mathcal{Z}^{\alpha}_{(r,v)}(r) &\leq |\zeta_0|(u_r(v)) + \int_0^v \int_{r(u_r(v),\tilde v)}^{r_+} \mathcal{Z}^{\alpha}_{(r,v)}(\tilde s)\, d\tilde s \Big[\frac{(-\lambda)(u_r(v),\tilde v)}{[r(u_r(v),\tilde v)]^2}\Big] d\tilde v \\ &\leq \max_{s\in[r,r_+]} |\zeta_0|(u_s(v)) + \int_r^{r_+} \mathcal{Z}^{\alpha}_{(r,v)}(\tilde s)\, d\tilde s \int_0^v \Big[\frac{(-\lambda)(u_r(v),\tilde v)}{[r(u_r(v),\tilde v)]^2}\Big] d\tilde v \\ &\leq \max_{u\in[0,u_r(v)]} |\zeta_0|(u) + \Big(\frac{1}{r} - \frac{1}{r_+ - u_r(v)}\Big) \int_r^{r_+} \mathcal{Z}^{\alpha}_{(r,v)}(\tilde s)\, d\tilde s.\end{aligned}$$

We still consider $r \leq s < r_+$. Let $\tilde v \in [0, v]$ if $r_+ - u_r(v) \leq s < r_+$, and $\tilde v \in [v_s(u_r(v)), v]$ if $r \leq s \leq r_+ - u_r(v)$. In this way $(u_s(\tilde v), \tilde v) \in J^-(u_r(v), v)$. In the same way one can show that

$$e^{\alpha\tilde v}\Big|\widehat{\frac{\zeta}{\nu}}\Big|(s,\tilde v) \leq \max_{u\in[0,u_s(v)]} |\zeta_0|(u) + \Big(\frac{1}{s} - \frac{1}{r_+ - u_s(v)}\Big)\int_s^{r_+} \mathcal{Z}^{\alpha}_{(r,v)}(\tilde s)\, d\tilde s$$

because $J^-(u_s(\tilde v), \tilde v) \cap \Gamma_{\tilde s} \subset J^-(u_r(v), v) \cap \Gamma_{\tilde s}$, for $s \leq \tilde s < r_+$, and so $\mathcal{Z}^{\alpha}_{(s,\tilde v)}(\tilde s) \leq \mathcal{Z}^{\alpha}_{(r,v)}(\tilde s)$. Since $u_s(v) \leq u_r(v)$ for $r \leq s < r_+$, we have

$$\mathcal{Z}^{\alpha}_{(r,v)}(s) \leq \max_{u\in[0,u_r(v)]} |\zeta_0|(u) + \Big(\frac{1}{r} - \frac{1}{r_+}\Big)\int_s^{r_+} \mathcal{Z}^{\alpha}_{(r,v)}(\tilde s)\, d\tilde s.$$

Using Gronwall's inequality, we get

$$\mathcal{Z}^{\alpha}_{(r,v)}(r) \leq e^{\frac{(r_+-r)^2}{rr_+}} \max_{u\in[0,u_r(v)]} |\zeta_0|(u).$$

This establishes (40). □

Lemma 4.2. *Let $r_0 \leq r < r_+$ and $v > 0$. Then*

$$\Big|\widehat{\frac{\zeta}{\nu}}\Big|(r,v) \leq e^{\frac{(r_+-r)^2}{rr_+}} \max_{u\in[0,u_r(v)]} |\zeta_0|(u). \tag{42}$$

Proof. We bound the exponentials in (38). From (31), the definition of η and $\varpi \geq \varpi_0$,

$$-\partial_r(1-\mu) = \frac{2\eta}{r^2} \leq \frac{2\eta_0}{r^2} = -\partial_r(1-\mu)(r,\varpi_0) \leq 0.$$

Therefore in the region $J^-(\Gamma_{r_0})$ the exponentials are bounded by 1. Applying Lemma 4.1 with $\alpha = 0$ we obtain (40) with $\alpha = 0$, which is precisely (42). □

According to (42), the function $\frac{\zeta}{\nu}$ is bounded in the region $J^-(\Gamma_{r_0})$, say by $\hat\delta$. From (22),

$$\begin{aligned}\kappa(u,v) &= e^{\int_0^u \left(\frac{\zeta^2}{r\nu}\right)(\tilde u,v)\, d\tilde u} \\ &\geq e^{\hat\delta^2 \int_0^u \left(\frac{\nu}{r}\right)(\tilde u,v)\, d\tilde u} \\ &\geq \left(\frac{r_0}{r_+}\right)^{\hat\delta^2}.\end{aligned} \tag{43}$$

We recall from Part 1 that Eqs. (15), (17), (19) and (23) imply

$$\partial_v\left(\frac{1-\mu}{\nu}\right) = -\frac{\theta^2}{\nu r\kappa}, \tag{44}$$

which is the Raychaudhuri equation in the v direction. We also recall that the integrated form of (18) is

$$\varpi(u,v) = \varpi_0(v)e^{-\int_0^u \left(\frac{\zeta^2}{r\nu}\right)(u',v)\,du'} + \int_0^u e^{-\int_s^u \frac{\zeta^2}{r\nu}(u',v)\,du'} \left(\frac{1}{2}\left(1+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2\right)\frac{\zeta^2}{\nu}\right)(s,v)\,ds. \tag{45}$$

These will be used in the proof of the following result.

Proposition 4.3. *Let $r_0 < \check{r}_+ \le r < r_+$ and $v > 0$. Then there exists $\alpha > 0$ (given by (50) below) such that*

$$\left|\widehat{\frac{\zeta}{\nu}}\right|(r,v) \le e^{\frac{(r_+-r)^2}{rr_+}} \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}, \tag{46}$$

$$\left|\widehat{\frac{\theta}{\lambda}}\right|(r,v) \le \hat{C}_r \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}, \tag{47}$$

$$|\hat{\theta}|(r,v) \le C \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}. \tag{48}$$

For $(u,v)\in J^-(\Gamma_{\check{r}_+})$, and U sufficiently small, we have

$$\varpi_0 \le \varpi(u,v) \le \varpi_0 + C\left(\sup_{\tilde{u}\in[0,u]} |\zeta_0|(\tilde{u})\right)^2. \tag{49}$$

Moreover, the curve $\Gamma_{\check{r}_+}$ intersects every line of constant u provided that $0 < u \le \min\{r_+ - \check{r}_+, U\}$. Therefore, $u_{\check{r}_+}(\infty) = 0$.

Proof. In $J^-(\Gamma_{\check{r}_+})$, we have $\partial_r(1-\mu)(r,\varpi_0) \ge \min_{r\in[\check{r}_+,r_+]} \partial_r(1-\mu)(r,\varpi_0) > 0$ and

$$\begin{aligned} -\frac{\lambda}{1-\mu}\partial_r(1-\mu) &\le -\kappa\,\partial_r(1-\mu)(r,\varpi_0) \\ &\le -\inf_{J^-(\Gamma_{\check{r}_+})} \kappa \times \partial_r(1-\mu)(r,\varpi_0) \\ &\le -\left(\frac{\check{r}_+}{r_+}\right)^{\hat{\delta}^2} \min_{r\in[\check{r}_+,r_+]} \partial_r(1-\mu)(r,\varpi_0) \\ &=: -\alpha < 0, \end{aligned} \tag{50}$$

where we have used (43) (with $\check{r}_+$ instead of r_0). Thus, we can improve the bounds on the exponentials in (38) that involve integrals in v as follows:

$$e^{-\int_{\tilde{v}}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\bar{v})\,d\bar{v}} \le e^{-\alpha(v-\tilde{v})}.$$

Since

$$\frac{1}{\widehat{1-\mu}}\partial_r(\widehat{1-\mu}) \le 0,$$

as before, we have

$$e^{\int_{r(u_r(v),\tilde{v})}^{\tilde{s}}\left[\frac{1}{\widehat{1-\mu}}\partial_r(\widehat{1-\mu})\right](\bar{s},\tilde{v})\,d\bar{s}} \le 1.$$

We apply Lemma 4.1 again, this time with a positive α, to get (46).

Now we may use (39) and (46) to obtain

$$\left|\frac{\widehat{\theta}}{\lambda}\right|(r,v) \leq e^{\frac{(r_+-r)^2}{rr_+}} \ln\left(\frac{r_+}{r}\right) \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}$$
$$= \hat{C}_r \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}.$$

In order to bound ϖ in $J^-(\Gamma_{\check{r}_+})$, we note that

$$-\int_0^u \left(\frac{\zeta^2}{r\nu}\right)(\tilde{u},v)\,d\tilde{u} \leq C^2\left(\sup_{\tilde{u}\in[0,u]} |\zeta_0|(\tilde{u})\right)^2 \ln\left(\frac{r_+}{\check{r}_+}\right),$$
$$\left|1+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2\right| \leq 1+\frac{e^2}{\check{r}_+^2}+\frac{|\Lambda|}{3}r_+^2$$

and

$$-\int_0^u \nu(\tilde{u},v)\,d\tilde{u} = r_+ - r(u,v) \leq r_+ - \check{r}_+.$$

From (45), we conclude that

$$\varpi(u,v) \leq \varpi_0 e^{C\left(\sup_{\tilde{u}\in[0,u]}|\zeta_0|(\tilde{u})\right)^2} + C\left(\sup_{\tilde{u}\in[0,u]} |\zeta_0|(\tilde{u})\right)^2.$$

Inequality (49) follows from $e^x \leq 1+2x$, for small x, since ζ_0 is continuous and $\zeta_0(0)=0$.

Given that $\kappa \leq 1$, we have $(1-\mu) \leq \lambda$. Moreover, since ϖ is bounded in the region $J^-(\Gamma_{\check{r}_+})$, $1-\mu$ is bounded from below, and so λ is also bounded from below. Hence (47) implies (48).

Let $0 < u \leq \min\{r_+ - \check{r}_+, U\}$. We claim that

$$\sup\left\{v \in [0,\infty[: (u,v) \in J^-(\Gamma_{\check{r}_+})\right\} < \infty. \tag{51}$$

To see this, first note that (17) shows that $v \mapsto \nu(u,v)$ is decreasing in $J^-(\Gamma_{\check{r}_+})$, as $\partial_r(1-\mu) \geq 0$ for $r_0 \leq r \leq r_+$ (recall that $\eta(r,\varpi) \leq \eta_0(r)$). Then (44) shows $v \mapsto (1-\mu)(u,v)$ is also decreasing in $J^-(\Gamma_{\check{r}_+})$. Thus, as long as v is such that $(u,v) \in J^-(\Gamma_{\check{r}_+})$, we have $(1-\mu)(u,v) \leq (1-\mu)(u,0) < 0$. Combining the previous inequalities with (43), we get

$$\lambda(u,v) \leq \left(\frac{\check{r}_+}{r_+}\right)^{\hat{\delta}^2} (1-\mu)(u,0) < 0.$$

Finally, if (51) did not hold for a given u, we would have

$$r(u,v) = r(u,0) + \int_0^v \lambda(u,v')\,dv'$$
$$\leq r(u,0) + \left(\frac{\check{r}_+}{r_+}\right)^{\hat{\delta}^2} (1-\mu)(u,0)\,v \to -\infty,$$

as $v \to \infty$, which is a contradiction. This establishes the claim. □

5. The Region $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$

In this section, we treat the region $\check{r}_- \le r \le \check{r}_+$. Recall that we assume that $r_- < \check{r}_- < r_0 < \check{r}_+ < r_+$. By decreasing $\check{r}_-$, if necessary, we will also assume that

$$-(1-\mu)(\check{r}_-, \varpi_0) \le -(1-\mu)(\check{r}_+, \varpi_0). \tag{52}$$

In Sect. (5.1), we obtain estimates (55) and (56) for $\frac{\zeta}{\nu}$ and $\frac{\theta}{\lambda}$, which will allow us to obtain the lower bound (66) for κ, the upper bound (67) for ϖ, and to prove that the domain of $v_{\check{r}_-}(\,\cdot\,)$ is $]0, \min\{r_+ - \check{r}_-, U\}]$. In Sect. (5.2), we obtain upper and lower bounds for λ and ν, as well as more information about the region $\check{r}_- \le r \le \check{r}_+$. In Sect. (5.3), we use the results from the previous subsection to improve the estimates on $\frac{\zeta}{\nu}$ and $\frac{\theta}{\lambda}$ to (86) and (91). We also obtain the bound (92) for θ.

As in the previous section, the solution with general ζ_0 is qualitatively still a small perturbation of the Reissner–Nordström solution (26)–(30): ϖ, κ, ζ and θ remain close to ϖ_0, 1 and 0, respectively. Moreover, λ is bounded from below by a negative constant, and away from zero by a constant depending on $\check{r}_+$ and $\check{r}_-$, as is also the case in the Reissner–Nordström solution (see Eq. (26)). Likewise, ν has a similar behavior to its Reissner–Nordström counterpart (see Eq. (27)): when multiplied by u, ν behaves essentially like λ.

5.1. First estimates. By reducing $U > 0$, if necessary, we can assume $U \le r_+ - \check{r}_+$. We turn our attention to the region $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$. Substituting (34) in (35) with $\check{r} = \check{r}_+$, we get

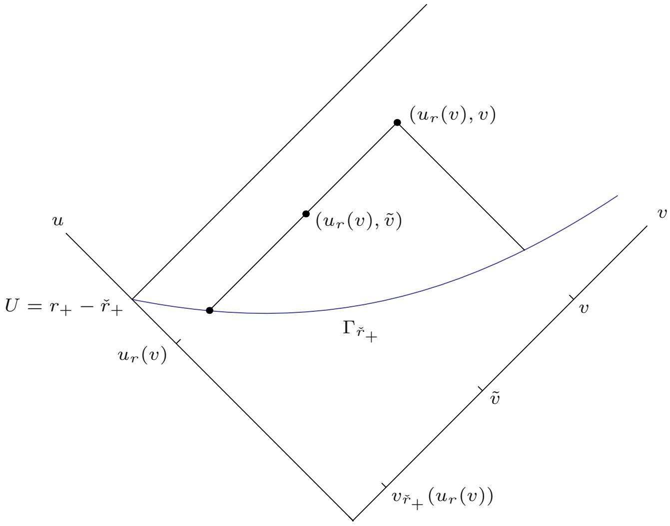
$$\begin{aligned}
\frac{\zeta}{\nu}(u,v) &= \frac{\zeta}{\nu}(u, v_{\check{r}_+}(u)) e^{-\int_{v_{\check{r}_+}(u)}^{v} \left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde{v})\,d\tilde{v}} \\
&\quad + \int_{v_{\check{r}_+}(u)}^{v} \frac{\theta}{\lambda}(u_{\check{r}_+}(\tilde{v}), \tilde{v}) e^{-\int_{u_{\check{r}_+}(\tilde{v})}^{u} \left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},\tilde{v})\,d\tilde{u}} \\
&\quad \times \left[\frac{(-\lambda)}{r}\right](u,\tilde{v}) e^{-\int_{\tilde{v}}^{v} \left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\bar{v})\,d\bar{v}}\,d\tilde{v} \\
&\quad + \int_{v_{\check{r}_+}(u)}^{v} \left(\int_{u_{\check{r}_+}(\tilde{v})}^{u} \left[\frac{\zeta}{\nu}\frac{(-\nu)}{r}\right](\tilde{u},\tilde{v}) e^{-\int_{\tilde{u}}^{u} \left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\bar{u},\tilde{v})\,d\bar{u}}\,d\tilde{u} \right) \\
&\quad \times \left[\frac{(-\lambda)}{r}\right](u,\tilde{v}) e^{-\int_{\tilde{v}}^{v} \left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\bar{v})\,d\bar{v}}\,d\tilde{v}.
\end{aligned} \tag{53}$$

We make the change of coordinates (37). Then, Eq. (53) may be written

$$\begin{aligned}
\widehat{\frac{\zeta}{\nu}}(r,v) &= \widehat{\frac{\zeta}{\nu}}(\check{r}_+, v_{\check{r}_+}(u_r(v))) e^{-\int_{v_{\check{r}_+}(u_r(v))}^{v} \left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\tilde{v})\,d\tilde{v}} \\
&\quad + \int_{v_{\check{r}_+}(u_r(v))}^{v} \widehat{\frac{\theta}{\lambda}}(\check{r}_+, \tilde{v}) e^{\int_{r(u_r(v),\tilde{v})}^{\check{r}_+} \left[\frac{1}{1-\mu}\partial_r\widehat{(1-\mu)}\right](\tilde{s},\tilde{v})\,d\tilde{s}} \\
&\quad \times \left[\frac{(-\lambda)(u_r(v),\tilde{v})}{r(u_r(v),\tilde{v})}\right] e^{-\int_{\tilde{v}}^{v} \left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\bar{v})\,d\bar{v}}\,d\tilde{v}
\end{aligned} \tag{54}$$

$$+\int_{v_{\check r_+}(u_r(v))}^{v}\left(\int_{r(u_r(v),\tilde v)}^{\check r_+}\left[\frac{\widehat{\zeta}}{\nu}\frac{1}{\tilde s}\right](\tilde s,\tilde v)e^{\int_{r(u_r(v),\tilde v)}^{\tilde s}\left[\frac{1}{\widehat{1-\mu}}\partial_r\widehat{(1-\mu)}\right](\bar s,\tilde v)\,d\bar s}\,d\tilde s\right)$$
$$\times\left[\frac{(-\lambda)(u_r(v),\tilde v)}{r(u_r(v),\tilde v)}\right]e^{-\int_{\tilde v}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u_r(v),\bar v)\,d\bar v}\,d\tilde v.$$

For (r,v) such that $(u_r(v),v)\in J^-(\Gamma_{\check r_-})\cap J^+(\Gamma_{\check r_+})$, $v_{\check r_+}(u_r(v))$ is well defined because $U\le r_+-\check r_+$.



Lemma 5.1. *Let $\check r_-\le r\le\check r_+$. Then*

$$\left|\frac{\widehat{\zeta}}{\nu}\right|(r,v)\le\tilde C\max_{u\in[0,u_r(v)]}|\zeta_0|(u), \tag{55}$$
$$\left|\frac{\widehat{\theta}}{\lambda}\right|(r,v)\le C\max_{u\in[0,u_r(v)]}|\zeta_0|(u). \tag{56}$$

Proof. From (52) we have

$$\begin{aligned}(1-\mu)(r,\varpi)&\le(1-\mu)(r,\varpi_0)\\&\le\max\left\{(1-\mu)(\check r_-,\varpi_0),(1-\mu)(\check r_+,\varpi_0)\right\}\\&=(1-\mu)(\check r_-,\varpi_0)\end{aligned} \tag{57}$$

and

$$\frac{\partial_r(1-\mu)}{1-\mu} = \frac{2\eta/r^2}{-(1-\mu)} \le \frac{2\eta_0/r^2}{-(1-\mu)} \le \frac{2\eta_0(\check r_-)/\check r_-^2}{-(1-\mu)}$$
$$\le \frac{2\eta_0(\check r_-)/\check r_-^2}{-(1-\mu)(r,\varpi_0)} \le \frac{2\eta_0(\check r_-)/\check r_-^2}{-(1-\mu)(\check r_-,\varpi_0)} =: c_{\check r_-}.$$

(For the second inequality, see the graph of η_0 in Sect. 3.) Each of the five exponentials in (54) is bounded by

$$e^{c_{\check r_-}(\check r_+-\check r_-)} =: C. \tag{58}$$

Hence, for (r, v) such that $(u_r(v), v) \in J^-(\Gamma_{\check r_-}) \cap J^+(\Gamma_{\check r_+})$, we have from (54)

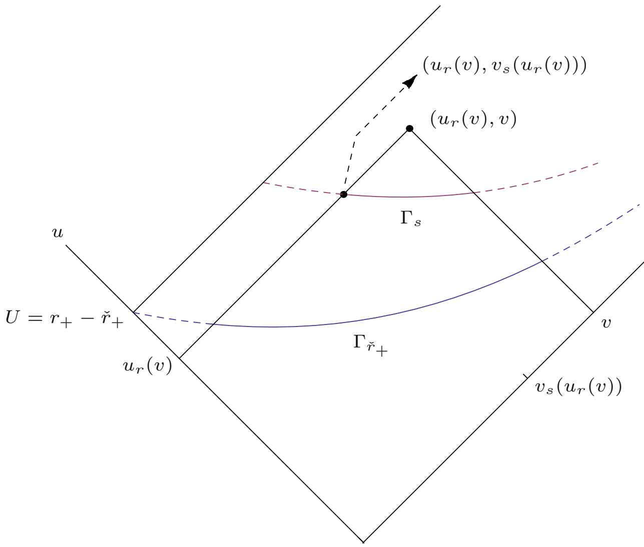
$$\begin{aligned}\left|\frac{\widehat{\zeta}}{\nu}\right|(r,v) \le C\left|\frac{\widehat{\zeta}}{\nu}\right|(\check r_+, v_{\check r_+}(u_r(v))) \\ + C^2\int_{v_{\check r_+}(u_r(v))}^{v}\left|\frac{\widehat{\theta}}{\lambda}\right|(\check r_+,\tilde v)\Big[\frac{(-\lambda)(u_r(v),\tilde v)}{r(u_r(v),\tilde v)}\Big]d\tilde v \\ + C^2\int_{v_{\check r_+}(u_r(v))}^{v}\int_{r(u_r(v),\tilde v)}^{\check r_+}\left|\frac{\widehat{\zeta}}{\nu}\right|(\tilde s,\tilde v)\,d\tilde s\Big[\frac{(-\lambda)(u_r(v),\tilde v)}{[r(u_r(v),\tilde v)]^2}\Big]d\tilde v.\end{aligned} \tag{59}$$

For $r \le s \le \check r_+$, define

$$\mathcal{Z}_{(r,v)}(s) = \max_{\tilde v\in[v_s(u_r(v)),v]}\left|\frac{\widehat{\zeta}}{\nu}\right|(s,\tilde v) \tag{60}$$

and

$$\mathcal{T}_{(r,v)}(\check r_+) = \max_{\tilde v\in[v_{\check r_+}(u_r(v)),v]}\left|\frac{\widehat{\theta}}{\lambda}\right|(\check r_+,\tilde v). \tag{61}$$



Recall that $[v_s(u_r(v)), v]$ is the projection of $J^-(u_r(v), v) \cap \Gamma_s$ on the v-axis.

Note that $\mathcal{Z}_{(r,v)}(r) = \left|\widehat{\frac{\zeta}{\nu}}\right|(r, v)$. Inequality (59) implies

$$\begin{aligned}
\mathcal{Z}_{(r,v)}(r) &\leq C\mathcal{Z}_{(r,v)}(\check{r}_+) \\
&\quad + C^2 \int_{v_{\check{r}_+}(u_r(v))}^{v} \mathcal{T}_{(r,v)}(\check{r}_+)\Big[\frac{(-\lambda)(u_r(v), \tilde{v})}{r(u_r(v), \tilde{v})}\Big]\, d\tilde{v} \\
&\quad + C^2 \int_{v_{\check{r}_+}(u_r(v))}^{v} \int_{r(u_r(v),\tilde{v})}^{\check{r}_+} \mathcal{Z}_{(r,v)}(\tilde{s})\, d\tilde{s}\Big[\frac{(-\lambda)(u_r(v), \tilde{v})}{[r(u_r(v), \tilde{v})]^2}\Big]\, d\tilde{v} \\
&\leq C\mathcal{Z}_{(r,v)}(\check{r}_+) + C^2 \ln\Big(\frac{\check{r}_+}{r}\Big)\mathcal{T}_{(r,v)}(\check{r}_+) + C^2\Big(\frac{1}{r} - \frac{1}{\check{r}_+}\Big)\int_r^{\check{r}_+} \mathcal{Z}_{(r,v)}(\tilde{s})\, d\tilde{s}.
\end{aligned}$$

Again consider $r \leq s \leq \check{r}_+$ and let $\tilde{v} \in [v_s(u_r(v)), v]$, so that $(u_s(\tilde{v}), \tilde{v}) \in J^-(u_r(v), v) \cap J^+(\Gamma_{\check{r}_+})$. In the same way one can show that

$$\left|\widehat{\frac{\zeta}{\nu}}\right|(s, \tilde{v}) \leq C\mathcal{Z}_{(r,v)}(\check{r}_+) + C^2 \ln\Big(\frac{\check{r}_+}{s}\Big)\mathcal{T}_{(r,v)}(\check{r}_+) + C^2\Big(\frac{1}{s} - \frac{1}{\check{r}_+}\Big)\int_s^{\check{r}_+} \mathcal{Z}_{(r,v)}(\tilde{s})\, d\tilde{s},$$

because $J^-(u_s(\tilde{v}), \tilde{v}) \cap \Gamma_{\check{r}_+} \subset J^-(u_r(v), v) \cap \Gamma_{\check{r}_+}$. Therefore,

$$\mathcal{Z}_{(r,v)}(s) \leq C\mathcal{Z}_{(r,v)}(\check{r}_+) + C^2 \ln\Big(\frac{\check{r}_+}{r}\Big)\mathcal{T}_{(r,v)}(\check{r}_+) + C^2\Big(\frac{1}{r} - \frac{1}{\check{r}_+}\Big)\int_s^{\check{r}_+} \mathcal{Z}_{(r,v)}(\tilde{s})\, d\tilde{s}.$$

Using Gronwall's inequality, we get

$$\mathcal{Z}_{(r,v)}(r) \leq C\Big[\mathcal{Z}_{(r,v)}(\check{r}_+) + C \ln\Big(\frac{\check{r}_+}{r}\Big)\mathcal{T}_{(r,v)}(\check{r}_+)\Big]e^{\frac{C^2(\check{r}_+ - r)^2}{r\check{r}_+}}. \tag{62}$$

To bound $\mathcal{Z}_{(r,v)}$ and $\mathcal{T}_{(r,v)}$, it is convenient at this point to use (42) and

$$\left|\widehat{\frac{\theta}{\lambda}}\right|(r, v) \leq \hat{C}_r \max_{u\in[0,u_r(v)]} |\zeta_0|(u) \tag{63}$$

(valid for $\check{r}_+ \leq r < r_+$), in spite of having the better estimates (46) and (47). Indeed, if these better estimates are used, the improvement is just $e^{-\alpha v_{\check{r}_+}(u_r(v))}$ (that is, an exponential factor computed over $\Gamma_{\check{r}_+}$ for the same value of u); to turn this into an exponential decay in v we must first obtain a more accurate control of the various quantities in the region $\check{r}_- \leq r \leq \check{r}_+$. Applying first the definition (60) and then (42), we have

$$\begin{aligned}
\mathcal{Z}_{(r,v)}(\check{r}_+) &= \max_{\tilde{v}\in[v_{\check{r}_+}(u_r(v)),v]} \left|\widehat{\frac{\zeta}{\nu}}\right|(\check{r}_+, \tilde{v}) \\
&\leq e^{\frac{(r_+ - \check{r}_+)^2}{\check{r}_+ r_+}} \max_{\tilde{v}\in[v_{\check{r}_+}(u_r(v)),v]} \max_{u\in[0,u_{\check{r}_+}(\tilde{v})]} |\zeta_0|(u) \\
&\leq e^{\frac{(r_+ - \check{r}_+)^2}{\check{r}_+ r_+}} \max_{u\in[0,u_r(v)]} |\zeta_0|(u),
\end{aligned} \tag{64}$$

because $u_{\check{r}_+}(\tilde{v}) \leq u_r(v)$. Applying first the definition (61) and then (63), we have

$$\begin{aligned}
\mathcal{T}_{(r,v)}(\check{r}_+) &\leq e^{\frac{(r_+ - \check{r}_+)^2}{\check{r}_+ r_+}} \ln\Big(\frac{r_+}{\check{r}_+}\Big) \max_{\tilde{v}\in[v_{\check{r}_+}(u_r(v)),v]} \max_{u\in[0,u_{\check{r}_+}(\tilde{v})]} |\zeta_0|(u) \\
&\leq \hat{C}_{\check{r}_+} \max_{u\in[0,u_r(v)]} |\zeta_0|(u).
\end{aligned} \tag{65}$$

We use (64) and (65) in (62). This yields (55).

Finally, writing (34) in the (r, v) coordinates (with $\check{r} = \check{r}_+$) gives

$$\frac{\widehat{\theta}}{\lambda}(r, v) = \frac{\widehat{\theta}}{\lambda}(\check{r}_+, v)e^{\int_r^{\check{r}_+}\left[\frac{1}{\widehat{1-\mu}}\partial_r(\widehat{1-\mu})\right](\tilde{s},v)\,d\tilde{s}}$$
$$+\int_r^{\check{r}_+}\left[\frac{\widehat{\zeta}}{\nu}\frac{1}{\tilde{s}}\right](\tilde{s}, v)e^{\int_r^{\tilde{s}}\left[\frac{1}{\widehat{1-\mu}}\partial_r(\widehat{1-\mu})\right](\bar{s},v)\,d\bar{s}}\,d\tilde{s}.$$

The exponentials are bounded by the constant C in (58). We use the estimates (63) and (55) to obtain

$$\begin{aligned}\left|\frac{\widehat{\theta}}{\lambda}\right|(r, v) &\leq C\left|\frac{\widehat{\theta}}{\lambda}\right|(\check{r}_+, v) + C\int_r^{\check{r}_+}\left[\left|\frac{\widehat{\zeta}}{\nu}\right|\frac{1}{\tilde{s}}\right](\tilde{s}, v)\,d\tilde{s}\\ &\leq C\hat{C}_{\check{r}_+}\max_{u\in[0,u_{\check{r}_+}(v)]}|\zeta_0|(u)\\ &\quad + C\tilde{C}\ln\left(\frac{\check{r}_+}{r}\right)\max_{u\in[0,u_r(v)]}|\zeta_0|(u)\\ &= C\max_{u\in[0,u_r(v)]}|\zeta_0|(u),\end{aligned}$$

which is (56). □

According to (42) and (55), the function $\frac{\zeta}{\nu}$ is bounded in the region $J^-(\Gamma_{\check{r}_-})$, let us say by $\hat{\delta}$. Arguing as in the deduction of (43), we obtain

$$\kappa(u, v) \geq \left(\frac{\check{r}_-}{r_+}\right)^{\hat{\delta}^2}. \tag{66}$$

Lemma 5.2. *For $(u, v) \in J^-(\Gamma_{\check{r}_-})$, and $U \leq r_+ - \check{r}_+$ sufficiently small, we have*

$$\varpi_0 \leq \varpi(u, v) \leq \varpi_0 + C\left(\sup_{\tilde{u}\in[0,u]}|\zeta_0|(\tilde{u})\right)^2. \tag{67}$$

The curve $\Gamma_{\check{r}_-}$ intersects every line of constant u. Therefore, $u_{\check{r}_-}(\infty) = 0$.

Proof. The proof of (67) is identical to the proof of (49).

Because ϖ is bounded, the function $1 - \mu$ is bounded below in $J^-(\Gamma_{\check{r}_-})$. Also, by (57), the function $1-\mu$ is bounded above in $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$ by $(1-\mu)(\check{r}_-, \varpi_0)$.

We claim that for each $0 < u \leq U$

$$\sup\left\{v \in [0, \infty[\,: (u, v) \in J^-(\Gamma_{\check{r}_-})\right\} < \infty. \tag{68}$$

The proof is similar to the proof of (51): since κ is bounded below by a positive constant and $1 - \mu$ is bounded above by a negative constant, λ is bounded above by a negative constant in $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$, say $-c_\lambda$. Then, as long as (u, v) belongs to $J^-(\Gamma_{\check{r}_-})$, we have the upper bound for $r(u, v)$ given by

$$r(u, v) \leq r_+ - u - c_\lambda v,$$

since $0 < u \leq U \leq r_+ - \check{r}_+$). Finally, if (68) did not hold for a given u, we would have $r(u, v) \to -\infty$ as $v \to \infty$, which is a contradiction. This proves the claim. □

5.2. Estimates for ν, λ and the region $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$.

Lemma 5.3. *In the region $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$, we have the following estimates from above and from below on λ and ν:*

$$-\tilde{C} \le \lambda \le -\tilde{c} \tag{69}$$

and

$$-\frac{\tilde{C}}{u} \le \nu \le -\frac{\tilde{c}}{u}, \tag{70}$$

where the constants $\tilde{c}$ and $\tilde{C}$ depend on $\check{r}_+$ and $\check{r}_-$.

Furthermore, if $0 < \delta < r_+ - r_0$ and $(u, v) \in \Gamma_{r_+-\delta}$ then

$$-C\, \partial_r(1-\mu)(r_+, \varpi_0)\, \delta \le \lambda(u, v) \le -c\, \partial_r(1-\mu)(r_+, \varpi_0)\, \delta \tag{71}$$

and

$$-C\frac{\delta}{u} \le \nu(u, v) \le -c\frac{\delta}{u}, \tag{72}$$

where the constants $0 < c < 1 < C$ may be chosen independently of δ. Given $\varepsilon > 0$ then $1-\varepsilon < c < 1$ and $1 < C < 1+\varepsilon$ for small enough δ.

Proof. From (22) we obtain (the Raychaudhuri equation)

$$\partial_u\left(\frac{\lambda}{1-\mu}\right) = \frac{\lambda}{1-\mu}\left(\frac{\zeta}{\nu}\right)^2 \frac{\nu}{r},$$

and from (44) we obtain (the Raychaudhuri equation)

$$\partial_v\left(\frac{\nu}{1-\mu}\right) = \frac{\nu}{1-\mu}\left(\frac{\theta}{\lambda}\right)^2 \frac{\lambda}{r}. \tag{73}$$

Let $\hat{\delta} > 0$. By decreasing U, if necessary, using (46), (47), (55) and (56), we have $\left|\frac{\theta}{\lambda}\right| < \hat{\delta}$ and $\left|\frac{\zeta}{\nu}\right| < \hat{\delta}$ in $J^-(\Gamma_{\check{r}_-})$. Since $\int_0^u \frac{\nu}{r}(\tilde{u}, v)\, d\tilde{u} = \ln\left(\frac{r(u,v)}{r_+}\right)$, $\int_0^v \frac{\lambda}{r}(u, \tilde{v})\, d\tilde{v} = \ln\left(\frac{r(u,v)}{r_+-u}\right)$, $\frac{\check{r}_-}{r_+} \le \frac{r(u,v)}{r_+} \le 1$ and $\frac{\check{r}_-}{r_+} \le \frac{r(u,v)}{r_+-u} \le 1$, for $(u, v) \in J^-(\Gamma_{\check{r}_-})$ we have

$$\left(\frac{\check{r}_-}{r_+}\right)^{\hat{\delta}^2} \le e^{\int_0^u \left((\frac{\zeta}{\nu})^2 \frac{\nu}{r}\right)(\tilde{u}, v)\, d\tilde{u}} \le 1, \tag{74}$$

$$\left(\frac{\check{r}_-}{r_+}\right)^{\hat{\delta}^2} \le e^{\int_0^v \left((\frac{\theta}{\lambda})^2 \frac{\lambda}{r}\right)(u, \tilde{v})\, d\tilde{v}} \le 1. \tag{75}$$

So, integrating the Raychaudhuri equations, we get

$$\left(\frac{\check{r}_-}{r_+}\right)^{\hat{\delta}^2} = \left(\frac{\check{r}_-}{r_+}\right)^{\hat{\delta}^2} \frac{\lambda}{1-\mu}(0, v) \le \frac{\lambda}{1-\mu}(u, v) \le \frac{\lambda}{1-\mu}(0, v) = 1 \tag{76}$$

[as $\kappa(0, v) = 1$], and

$$-\left(\frac{\check{r}_-}{r_+}\right)^{\hat{\delta}^2} \frac{1}{1-\mu}(u, 0) \le \frac{\nu}{1-\mu}(u, v) \le \frac{\nu}{1-\mu}(u, 0) = -\frac{1}{1-\mu}(u, 0). \tag{77}$$

To bound $(1-\mu)(u,0)$, using (16) and (22), we compute

$$\partial_u(1-\mu)=\partial_u\left(\frac{\lambda}{\kappa}\right)=\nu\partial_r(1-\mu)-(1-\mu)\frac{\nu}{r}\left(\frac{\zeta}{\nu}\right)^2. \tag{78}$$

At the point $(u,v)=(0,0)$ this yields

$$\partial_u(1-\mu)(0,0)=-\partial_r(1-\mu)(r_+,\varpi_0).$$

Fix $0<\varepsilon<1$. Since the function $u\mapsto(1-\mu)(u,0)$ is C^1, by decreasing U if necessary, we have

$$-\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1-\varepsilon}<\partial_u(1-\mu)(u,0)<-\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1+\varepsilon}$$

for $0\le u\le U$, and so

$$-\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1-\varepsilon}\,u<(1-\mu)(u,0)<-\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1+\varepsilon}\,u.$$

Using these inequalities in (77) immediately gives

$$\left(\frac{\check r_-}{r_+}\right)^{\hat\delta 2}\frac{1-\varepsilon}{\partial_r(1-\mu)(r_+,\varpi_0)\,u}\le\frac{\nu}{1-\mu}(u,v)\le\frac{1+\varepsilon}{\partial_r(1-\mu)(r_+,\varpi_0)\,u}. \tag{79}$$

To obtain bounds on λ and ν from (76) and (79), recall that, in accordance with (57), in the region $J^-(\Gamma_{\check r_-})\cap J^+(\Gamma_{\check r_+})$ the function $1-\mu$ is bounded above by a negative constant. On the other hand, the bounds we obtained earlier on ϖ in $J^-(\Gamma_{\check r_-})$ imply that $1-\mu$ is bounded below in $J^-(\Gamma_{\check r_-})$. In summary, there exist $\overline{c}$ and $\overline{C}$ such that

$$-\overline{C}\le 1-\mu\le-\overline{c}.$$

Therefore, from (76) and (79), in the region $J^-(\Gamma_{\check r_-})\cap J^+(\Gamma_{\check r_+})$, we get (69) and (70):

$$-\overline{C}\le\lambda\le-\overline{c}\left(\frac{\check r_-}{r_+}\right)^{\hat\delta 2},$$

$$-\overline{C}\frac{1+\varepsilon}{\partial_r(1-\mu)(r_+,\varpi_0)}\frac{1}{u}\le\nu\le-\overline{c}\left(\frac{\check r_-}{r_+}\right)^{\hat\delta 2}\frac{1-\varepsilon}{\partial_r(1-\mu)(r_+,\varpi_0)}\frac{1}{u}.$$

By decreasing $\overline{c}$ and increasing $\overline{C}$, if necessary, we can guarantee (69) and (70) hold, without having to further decrease U.

Now suppose that $(u,v)\in\Gamma_{r_+-\delta}$. Then

$$\begin{aligned}(1-\mu)(u,v)&=(1-\mu)(r_+-\delta,\varpi)\\&\le(1-\mu)(r_+-\delta,\varpi_0)\\&\le-\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1+\varepsilon}\,\delta,\end{aligned}\tag{80}$$

where ε is any fixed positive number, provided that δ is sufficiently small. If δ is not small, then (80) also holds but with $1+\varepsilon$ replaced by a larger constant.

Using again (78),

$$(1-\mu)(u,v)=-\int_0^u e^{-\int_{\tilde u}^{u}\left(\frac{\nu}{r}\left(\frac{\zeta}{\nu}\right)^2\right)(\bar u,v)\,d\bar u}\left(\frac{2\nu}{r^2}\eta\right)(\tilde u,v)\,d\tilde u.$$

We take into account that

$$e^{-\int_{\bar{u}}^{u}\left(\frac{\nu}{r}\left(\frac{\zeta}{\nu}\right)^2\right)(\bar{u},v)\,d\bar{u}} \leq \left(\frac{r_+}{r_+-\delta}\right)^{\hat{\delta}^2}$$

and

$$\begin{aligned} -\frac{2\nu}{r^2}\eta &\geq 2\nu\left(-\frac{e^2}{r^3}-\frac{\Lambda}{3}r+\frac{\varpi_0}{r^2}\right)+2\nu\frac{\tilde{\delta}}{r^2} \\ &= \nu\partial_r(1-\mu)(r,\varpi_0)+2\nu\frac{\tilde{\delta}}{r^2} \end{aligned}$$

provided U is chosen small enough so that $\varpi \leq \varpi_0 + \tilde{\delta}$ in $J^-(\Gamma_{r_+-\delta})$. We get

$$\begin{aligned} (1-\mu)(u,v) &\geq \left(\frac{r_+}{r_+-\delta}\right)^{\hat{\delta}^2}\left((1-\mu)(r_+-\delta,\varpi_0)-\frac{2\tilde{\delta}\delta}{r_+(r_+-\delta)}\right) \\ &\geq -\left(\frac{r_+}{r_+-\delta}\right)^{\hat{\delta}^2}\left(\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1-\varepsilon}+\frac{4\tilde{\delta}}{r_+^2}\right)\delta, \end{aligned} \tag{81}$$

where $0<\varepsilon<1$, provided δ is sufficiently small. We notice that in the case under consideration the integration is done between r_+ and $r_+-\delta$ and so the left hand sides of (74) and (75) can be improved to $\left(\frac{r_+-\delta}{r_+}\right)^{\hat{\delta}^2}$. Estimates (76), (80) and (81) yield, for $\hat{\delta}\leq 1$,

$$\begin{aligned} -\left(\frac{r_+}{r_+-\delta}\right)\left(\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1-\varepsilon}+\frac{4\tilde{\delta}}{r_+^2}\right)\delta \leq \lambda \\ \leq -\left(\frac{r_+-\delta}{r_+}\right)\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{1+\varepsilon}\,\delta, \end{aligned} \tag{82}$$

whereas estimates (79), (80) and (81) yield, again for $\hat{\delta}\leq 1$,

$$\begin{aligned} -\left(\frac{r_+}{r_+-\delta}\right)\left(\frac{1+\varepsilon}{1-\varepsilon}+\frac{4\tilde{\delta}(1+\varepsilon)}{r_+^2\partial_r(1-\mu)(r_+,\varpi_0)}\right)\frac{\delta}{u} \leq \nu \\ \leq -\left(\frac{r_+-\delta}{r_+}\right)\frac{1-\varepsilon}{1+\varepsilon}\,\frac{\delta}{u}. \end{aligned} \tag{83}$$

Estimates (71) and (72) are established. Note that $u\leq\delta$ when $(u,v)\in\Gamma_{r_+-\delta}$. Since

$$c=c(\delta,\varepsilon,\tilde{\delta})=c(\delta,\varepsilon(U,\delta),\tilde{\delta}(U))=c(\delta,\varepsilon(U(\delta),\delta),\tilde{\delta}(U(\delta))),$$

and analogously for C, we see that c and C can be chosen arbitrarily close to one, provided that δ is sufficiently small. □

Lemma 5.4. *Let $\varepsilon>0$. If δ is sufficiently small, then for any point $(u,v)\in\Gamma_{r_+-\delta}$ we have*

$$\delta\, e^{-[\partial_r(1-\mu)(r_+,\varpi_0)+\varepsilon]\,v} \leq u \leq \delta\, e^{-[\partial_r(1-\mu)(r_+,\varpi_0)-\varepsilon]\,v}. \tag{84}$$

For any point $(u,v)\in J^-(\Gamma_{\check{r}_-})\cap J^+(\Gamma_{r_+-\delta})$ we have

$$\delta\, e^{-[\partial_r(1-\mu)(r_+,\varpi_0)+\varepsilon]\,v} \leq u \leq \delta\, e^{\frac{r_+}{c}}\, e^{-[\partial_r(1-\mu)(r_+,\varpi_0)-\varepsilon]\,v}. \tag{85}$$

Proof. Obviously, we have

$$r(u_{r_+-\delta}(v), v) = r_+ - \delta.$$

Since r is C^1 and ν does not vanish, $v \mapsto u_{r_+-\delta}(v)$ is C^1. Differentiating both sides of the last equality with respect to v we obtain

$$u'_{r_+-\delta}(v) = -\frac{\lambda(u_{r_+-\delta}(v), v)}{\nu(u_{r_+-\delta}(v), v)}.$$

Using (71) and (72), we have

$$-\frac{C}{c}\partial_r(1-\mu)(r_+, \varpi_0)u_{r_+-\delta}(v) \leq u'_{r_+-\delta}(v) \leq -\frac{c}{C}\partial_r(1-\mu)(r_+, \varpi_0)u_{r_+-\delta}(v).$$

Integrating the last inequalities between 0 and v, as $u_{r_+-\delta}(0) = \delta$, we have

$$\delta e^{-\frac{C}{c}\partial_r(1-\mu)(r_+,\varpi_0)v} \leq u_{r_+-\delta}(v) \leq \delta e^{-\frac{c}{C}\partial_r(1-\mu)(r_+,\varpi_0)v}.$$

This proves (84).

Let $(u, v) \in J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{r_+-\delta})$. Integrating (70) between $u_{r_+-\delta}(v)$ and u, we get

$$1 \leq \frac{u}{u_{r_+-\delta}(v)} \leq e^{\frac{r_+}{\tilde{c}}}.$$

Combining $v_{r_+-\delta}(u) \leq v$ with the first inequality in (84) applied at the point $(u, v_{r_+-\delta}(u))$,

$$\begin{aligned} u &\geq \delta\, e^{-[\partial_r(1-\mu)(r_+,\varpi_0)+\varepsilon]\, v_{r_+-\delta}(u)} \\ &\geq \delta\, e^{-[\partial_r(1-\mu)(r_+,\varpi_0)+\varepsilon]\, v}, \end{aligned}$$

and combining $u \leq e^{\frac{r_+}{\tilde{c}}} u_{r_+-\delta}(v)$ with the second inequality in (84) applied at the point $(u_{r_+-\delta}(v), v)$,

$$u \leq e^{\frac{r_+}{\tilde{c}}} u_{r_+-\delta}(v) \;\leq\; \delta\, e^{\frac{r_+}{\tilde{c}}} e^{-[\partial_r(1-\mu)(r_+,\varpi_0)-\varepsilon]\, v}.$$

□

5.3. Improved estimates.

Lemma 5.5. *Let $\check{r}_- \leq r \leq \check{r}_+$. Then*

$$\left|\widehat{\frac{\zeta}{\nu}}\right|(r, v) \leq \tilde{C}_{\check{r}_-} \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}. \tag{86}$$

Proof. Applying first the definition (60) and then (46),

$$\begin{aligned} \mathcal{Z}_{(r,v)}(\check{r}_+) &= \max_{\tilde{v}\in[v_{\check{r}_+}(u_r(v)),v]} \left|\widehat{\frac{\zeta}{\nu}}\right|(\check{r}_+, \tilde{v}) \\ &\leq e^{\frac{(r_+-\check{r}_+)^2}{\check{r}_+r_+}} \max_{\tilde{v}\in[v_{\check{r}_+}(u_r(v)),v]} \; \max_{u\in[0,u_{\check{r}_+}(\tilde{v})]} |\zeta_0|(u)e^{-\alpha\tilde{v}} \end{aligned}$$

$$\le e^{\frac{(r_+-\check r_+)^2}{\check r_+ r_+}} \max_{u\in[0,u_{\check r_+}(v_{\check r_+}(u_r(v)))]} |\zeta_0|(u)e^{-\alpha v_{\check r_+}(u_r(v))}$$

$$= e^{\frac{(r_+-\check r_+)^2}{\check r_+ r_+}} \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v_{\check r_+}(u_r(v))} \tag{87}$$

because $u_{\check r_+}(\tilde v) \ge u_{\check r_+}(v)$. Integrating (69) between $v_{\check r_+}(u_r(v))$ and v, we get

$$v - v_{\check r_+}(u_r(v)) \le \tfrac{\check r_+ - r}{\check c} =: c_{r,\check r_+} \le c_{\check r_-,\check r_+}. \tag{88}$$

This allows us to continue the estimate (87), to obtain

$$\mathcal{Z}_{(r,v)}(\check r_+) \le e^{\frac{(r_+-\check r_+)^2}{\check r_+ r_+}} e^{\alpha c_{\check r_-,\check r_+}} \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}. \tag{89}$$

Applying first the definition (61) and then (47), and repeating the computations that lead to (87) and (88),

$$\mathcal{T}_{(r,v)}(\check r_+) \le e^{\frac{(r_+-\check r_+)^2}{\check r_+ r_+}} \ln\left(\frac{r_+}{\check r_+}\right) \max_{\tilde v\in[v_{\check r_+}(u_r(v)),v]} \max_{u\in[0,u_{\check r_+}(\tilde v)]} |\zeta_0|(u)e^{-\alpha\tilde v}$$

$$\le e^{\frac{(r_+-\check r_+)^2}{\check r_+ r_+}} e^{\alpha c_{\check r_-,\check r_+}} \ln\left(\frac{r_+}{\check r_+}\right) \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}. \tag{90}$$

We use (89) and (90) in (62). This yields (86). □

Lemma 5.6. *Let $\check r_- \le r \le \check r_+$. Then*

$$\left|\frac{\widehat{\theta}}{\lambda}\right|(r,v) \le \overline{C} \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}, \tag{91}$$

$$|\hat\theta|(r,v) \le \check C \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}. \tag{92}$$

Proof. Just like inequality (56) was obtained from (63) (that is, Eq. (47) with $\alpha = 0$) and (55), inequality (91) will be obtained from (47) and (86). Writing (34) in the (r, v) coordinates,

$$\frac{\widehat{\theta}}{\lambda}(r,v) = \frac{\widehat{\theta}}{\lambda}(\check r_+,v)e^{\int_r^{\check r_+}\left[\frac{1}{1-\mu}\partial_r\widehat{(1-\mu)}\right](\tilde s,v)\,d\tilde s}$$
$$+\int_r^{\check r_+}\left[\frac{\widehat{\zeta}}{\nu}\frac{1}{\tilde s}\right](\tilde s,v)e^{\int_r^{\tilde s}\left[\frac{1}{1-\mu}\partial_r\widehat{(1-\mu)}\right](\bar s,v)\,d\bar s}\,d\tilde s.$$

The exponentials are bounded by the constant C in (58). We use the estimates (47) and (86) to obtain

$$\left|\frac{\widehat{\theta}}{\lambda}\right|(r,v) \le C\left|\frac{\widehat{\theta}}{\lambda}\right|(\check r_+,v) + C\int_r^{\check r_+}\left[\left|\frac{\widehat{\zeta}}{\nu}\right|\frac{1}{\tilde s}\right](\tilde s,v)\,d\tilde s$$
$$\le C\hat C_{\check r_+} \max_{u\in[0,u_{\check r_+}(v)]} |\zeta_0|(u)e^{-\alpha v}$$
$$+ C\tilde C_{\check r_-} \ln\left(\frac{\check r_+}{r}\right) \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}$$

$$= \overline{C} \max_{u\in[0,u_r(v)]} |\zeta_0|(u)e^{-\alpha v}.$$

Using (69), the function λ is bounded from below in $J^-(\Gamma_{\check{r}_-}) \cap J^+(\Gamma_{\check{r}_+})$. Hence (91) implies (92). □

Remark 5.7. For use in Part 3, we observe that (47) and (91) imply

$$\lim_{\substack{(u,v)\to(0,\infty)\\ (u,v)\in J^-(\Gamma_{\check{r}_-})}} \left|\frac{\theta}{\lambda}\right|(u,v) = 0. \tag{93}$$

6. The Region $J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$

In this section, we define a curve γ to the future of $\Gamma_{\check{r}_-}$. Our first aim is to obtain the bounds in Corollary 6.2, $r(u,v) \geq r_- - \frac{\varepsilon}{2}$ and $\varpi(u,v) \leq \varpi_0 + \frac{\varepsilon}{2}$, for $(u,v) \in J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$ with $u \leq U_\varepsilon$. In the process, we will bound $\int_{u_{\check{r}_-}(v)}^u \left[\left|\frac{\zeta}{\nu}\right||\zeta|\right](\tilde{u},v)\,d\tilde{u}$ (this is inequality (102)). Then we will obtain a lower bound on κ, as well as upper and lower bounds on λ and ν. Therefore this region, where r may already be below r_-, is still a small perturbation of the Reissner–Nordström solution.

We choose a positive number[2]

$$0 < \beta < \frac{1}{2}\left(\sqrt{1 - 8\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{\partial_r(1-\mu)(r_-,\varpi_0)}} - 1\right), \tag{94}$$

and define $\gamma = \gamma_{\check{r}_-,\beta}$ to be the curve parametrized by

$$u \mapsto \big(u, (1+\beta)\, v_{\check{r}_-}(u)\big), \tag{95}$$

for $u \in [0,U]$. Since the curve $\Gamma_{\check{r}_-}$ is spacelike, so is γ [$u \mapsto v_{\check{r}_-}(u)$ is strictly decreasing].

Lemma 6.1. *For each β satisfying (94) there exist $r_- < \overline{\check{r}_-} < r_0$ and $0 < \varepsilon_0 < r_-$ for which, whenever $\check{r}_-$ and ε are chosen satisfying $r_- < \check{r}_- \leq \overline{\check{r}_-}$ and $0 < \varepsilon \leq \varepsilon_0$, the following holds: there exists U_ε (depending on $\check{r}_-$ and ε) such that if $(u,v) \in J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$, with $0 < u \leq U_\varepsilon$, and*

$$r(u,v) \geq r_- - \varepsilon, \tag{96}$$

then

$$r(u,v) \geq r_- - \tfrac{\varepsilon}{2} \quad\text{and}\quad \varpi(u,v) \leq \varpi_0 + \tfrac{\varepsilon}{2}. \tag{97}$$

Corollary 6.2. *Suppose that β is given satisfying (94), and let $\overline{\check{r}_-}$ and ε_0 be as in the previous lemma. Fix $r_- < \check{r}_- \leq \overline{\check{r}_-}$ and $0 < \varepsilon < \varepsilon_0$. If $(u,v) \in J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$ with $0 < u \leq U_\varepsilon$, then*

$$r(u,v) \geq r_- - \tfrac{\varepsilon}{2} \quad\text{and}\quad \varpi(u,v) \leq \varpi_0 + \tfrac{\varepsilon}{2}. \tag{98}$$

[2] We always have $-\partial_r(1-\mu)(r_-,\varpi_0) > \partial_r(1-\mu)(r_+,\varpi_0)$ (see Appendix A of Part 3). So, in particular, we may choose $\beta = -\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{\partial_r(1-\mu)(r_-,\varpi_0)}$.

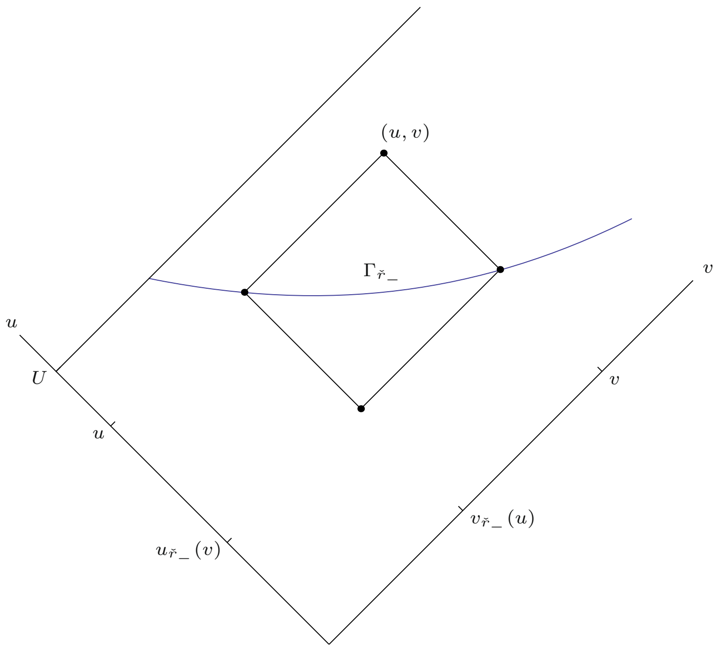
Proof. On $\Gamma_{\check r_-}$ we have $r = \check r_- > r_- > r_- - \frac{\varepsilon}{2}$. Suppose that there exists a point $(u, v) \in J^-(\gamma) \cap J^+(\Gamma_{\check r_-})$, with $0 < u \leq U_\varepsilon$, such that $r(u, v) < r_- - \frac{\varepsilon}{2}$. Then there exists a point $(\tilde u, v)$, with $0 < \tilde u < u \leq U_\varepsilon$, such that $r_- - \varepsilon \leq r(\tilde u, v) < r_- - \frac{\varepsilon}{2}$. The point $(\tilde u, v)$ belongs to $J^-(\gamma) \cap J^+(\Gamma_{\check r_-})$. Applying Lemma 6.1 at the point $(\tilde u, v)$, we reach a contradiction. The rest of the argument is immediate. □

Proof of Lemma 6.1. Let $(u, v) \in J^-(\gamma) \cap J^+(\Gamma_{\check r_-})$ such that (96) holds. Because of the monotonicity properties of r,

$$\min_{J^-(u,v)\cap J^+(\Gamma_{\check r_-})} r \geq r_- - \varepsilon.$$

According to Proposition 13.2 of [7] (this result depends only on Eqs. (20) and (21), and so does not depend on the presence of Λ), there exists a constant $\underline{C}$ (depending on $r_- - \varepsilon_0$) such that

$$\begin{aligned} &\int_{v_{\check r_-}(u)}^{v} |\theta|(u, \tilde v)\, d\tilde v + \int_{u_{\check r_-}(v)}^{u} |\zeta|(\tilde u, v)\, d\tilde u \\ &\leq \underline{C}\left(\int_{v_{\check r_-}(u)}^{v} |\theta|(u_{\check r_-}(v), \tilde v)\, d\tilde v + \int_{u_{\check r_-}(v)}^{u} |\zeta|(\tilde u, v_{\check r_-}(u))\, d\tilde u\right). \end{aligned} \tag{99}$$



The first integral on the right hand side of (99) can be estimated using (48), (92) and (95):

$$\begin{aligned}\int_{v_{\check r_-}(u)}^{v}|\theta|(u_{\check r_-}(v),\tilde v)\,d\tilde v&\le C\sup_{[0,u]}|\zeta_0|\int_{v_{\check r_-}(u)}^{v}e^{-\alpha\tilde v}\,d\tilde v\\&\le C\sup_{[0,u]}|\zeta_0|e^{-\alpha v_{\check r_-}(u)}\beta v_{\check r_-}(u)\\&\le C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta}v}\beta v\\&\le \tilde C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta^+}v},\end{aligned}$$

where we have used $v_{\check r_-}(u)=\frac{v_\gamma(u)}{1+\beta}\ge\frac{v}{1+\beta}$ and $v_{\check r_-}(u)\le v$, and denoted by β^+ a fixed number strictly greater than β. The second integral on the right hand side of (99) can be estimated using (46), (86) and (95):

$$\begin{aligned}\int_{u_{\check r_-}(v)}^{u}|\zeta|(\tilde u,v_{\check r_-}(u))\,d\tilde u&=\int_{u_{\check r_-}(v)}^{u}\left|\frac{\zeta}{\nu}\right|(-\nu)(\tilde u,v_{\check r_-}(u))\,d\tilde u\\&\le C(r_+-\check r_-)\sup_{[0,u]}|\zeta_0|e^{-\alpha v_{\check r_-}(u)}\\&\le \tilde C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta}v}.\end{aligned}$$

These lead to the following estimate for the left hand side of (99):

$$\int_{v_{\check r_-}(u)}^{v}|\theta|(u,\tilde v)\,d\tilde v+\int_{u_{\check r_-}(v)}^{u}|\zeta|(\tilde u,v)\,d\tilde u\le C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta^+}v}. \tag{100}$$

In order to use (35), note that, using $\eta(r,\varpi)\le\eta_0(r)$,

$$\begin{aligned}e^{-\int_{v_{\check r_-}(u)}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v}&\le e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta v_{\check r_-}(u)}\\&\le e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta v}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\frac{\zeta}{\nu}\right|(u,v)&\le\left|\frac{\zeta}{\nu}\right|(u,v_{\check r_-}(u))e^{-\int_{v_{\check r_-}(u)}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v}\\&\quad+\int_{v_{\check r_-}(u)}^{v}\frac{|\theta|}{r}(u,\bar v)e^{-\int_{\bar v}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v}\,d\bar v\\&\le C\sup_{[0,u]}|\zeta_0|e^{-\alpha v_{\check r_-}(u)}e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta v}\\&\quad+\frac{e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta v}}{r_--\varepsilon_0}\int_{v_{\check r_-}(u)}^{v}|\theta|(u,\bar v)\,d\bar v\\&\le C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta}v}e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta v}\\&\quad+\frac{e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta v}}{r_--\varepsilon_0}C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta^+}v}\\&\le C\sup_{[0,u]}|\zeta_0|e^{-\left(\frac{\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta\right)v}.\end{aligned} \tag{101}$$

Clearly, the right hand side of the last inequality also bounds $\max_{\tilde{u}\in[u_{\check{r}_-}(v),u]}\left|\frac{\zeta}{\nu}\right|(\tilde{u},v)$. In order to bound $\varpi(u,v)$, note that

$$\begin{aligned}
&\int_{u_{\check{r}_-}(v)}^{u}\left[\left|\frac{\zeta}{\nu}\right||\zeta|\right](\tilde{u},v)\,d\tilde{u}\\
&\quad\leq C\sup_{[0,u]}|\zeta_0|e^{-\left(\frac{\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta\right)v}\int_{u_{\check{r}_-}(v)}^{u}|\zeta|(\tilde{u},v)\,d\tilde{u}\\
&\quad\leq C\big(\sup_{[0,u]}|\zeta_0|\big)^2e^{-\left(\frac{2\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta\right)v}.
\end{aligned}\tag{102}$$

Using (45) and the last estimate, we get

$$\begin{aligned}
\varpi(u,v)&\leq\varpi(u_{\check{r}_-}(v),v)e^{\frac{1}{r_--\varepsilon_0}\int_{u_{\check{r}_-}(v)}^{u}\left[\left|\frac{\zeta}{\nu}\right||\zeta|\right](\tilde{u},v)\,d\tilde{u}}\\
&\quad+C\int_{u_{\check{r}_-}(v)}^{u}e^{\frac{1}{r_--\varepsilon_0}\int_s^u\left[\left|\frac{\zeta}{\nu}\right||\zeta|\right](\tilde{u},v)\,d\tilde{u}}\left[\left|\frac{\zeta}{\nu}\right||\zeta|\right](s,v)\,ds\\
&\leq\varpi(u_{\check{r}_-}(v),v)e^{C\left(\sup_{[0,u]}|\zeta_0|\right)^2e^{-\left(\frac{2\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta\right)v}}\\
&\quad+Ce^{C\left(\sup_{[0,u]}|\zeta_0|\right)^2e^{-\left(\frac{2\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta\right)v}}\\
&\quad\times\big(\sup_{[0,u]}|\zeta_0|\big)^2e^{-\left(\frac{2\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta\right)v}.
\end{aligned}$$

Let $\delta>0$. Using the definition of α in (50), the constant in the exponent

$$\frac{2\alpha}{1+\beta+\delta}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta\tag{103}$$

is positive for

$$\beta<\tfrac{1}{2}\left(\sqrt{(1+\delta)^2-8\frac{(\frac{\check{r}_+}{r_+})^{\hat{\delta}^2}\min_{r\in[\check{r}_+,r_+]}\partial_r(1-\mu)(r,\varpi_0)}{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)}}-(1+\delta)\right).\tag{104}$$

Now, the right hand side tends to

$$\tfrac{1}{2}\left(\sqrt{1-8\frac{\partial_r(1-\mu)(r_+,\varpi_0)}{\partial_r(1-\mu)(r_-,\varpi_0)}}-1\right)$$

as $(\check{r}_+,\varepsilon_0,\delta)\to(r_+,0,0)$. So, if β satisfies (94), we may choose $\check{r}_+$, ε_0 and δ such that (104) holds. Having done this, Eqs. (67) and (85) now imply that for each $0<\bar{\varepsilon}<\varepsilon_0$ there exists $\bar{U}_{\bar{\varepsilon}}>0$ such that

$$\varpi(u,v)\leq\varpi_0+\tfrac{\bar{\varepsilon}}{2},$$

provided that $u\leq\bar{U}_{\bar{\varepsilon}}$. Since $1-\mu$ is nonpositive and $1-\mu=(1-\mu)(r,\varpi_0)-\frac{2(\varpi-\varpi_0)}{r}$, we have

$$(1-\mu)(r(u,v),\varpi_0)\leq\frac{2(\varpi(u,v)-\varpi_0)}{r}\leq\frac{\bar{\varepsilon}}{r_--\varepsilon_0}.$$

Hence, by inspection of the graph of $(1-\mu)(r,\varpi_0)$, there exists $\bar\varepsilon_0$ such that for $0<\bar\varepsilon\leq\bar\varepsilon_0$, we have $r(u,v)>r_- - \frac{\varepsilon}{2}$ provided that $u\leq \bar U_{\bar\varepsilon}$. For $0<u\leq U_\varepsilon:=\min\{\bar U_{\bar\varepsilon_0},\bar U_\varepsilon\}$, both inequalities (97) hold. □

Remark 6.3. Given $\varepsilon>0$, we may choose U sufficiently small so that if $(u,v)\in J^-(\gamma)\cap J^+(\Gamma_{\check r_-})$, then

$$\kappa(u,v)\geq 1-\varepsilon. \tag{105}$$

This is a consequence of (66) and (102), since r is bounded away from zero.

Consider the reference subextremal Reissner–Nordström black hole with renormalized mass ϖ_0, charge parameter e and cosmological constant Λ. The next remark will turn out to be crucial in Part 3.

Remark 6.4. Suppose that there exist positive constants C and s such that $|\zeta_0(u)|\leq Cu^s$. Then, instead of choosing β according to (94), in Lemma 6.1 we may choose

$$0<\beta<\frac{1}{2}\left(\sqrt{1-8\frac{(1+s)\partial_r(1-\mu)(r_+,\varpi_0)}{\partial_r(1-\mu)(r_-,\varpi_0)}}-1\right). \tag{106}$$

Proof. Let $(u,v)\in J^-(\gamma)\cap J^+(\Gamma_{\check r_-})$. According to (85), we have

$$\begin{aligned} u&\leq Ce^{-[\partial_r(1-\mu)(r_+,\varpi_0)-\varepsilon]v_{\check r_-}(u)}\\ &\leq Ce^{-[\partial_r(1-\mu)(r_+,\varpi_0)-\varepsilon]\frac{v}{1+\beta}}\\ &\leq Ce^{-\partial_r(1-\mu)(r_+,\varpi_0)\frac{v}{1+\beta^+}}. \end{aligned} \tag{107}$$

Thus, the exponent in the upper bound for ϖ in (103) may be replaced by

$$\frac{2s\partial_r(1-\mu)(r_+,\varpi_0)}{1+\beta+\delta}+\frac{2\alpha}{1+\beta+\delta}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\beta.$$

This is positive for

$$\beta<\frac{1}{2}\left(\sqrt{(1+\delta)^2-8\frac{[(\frac{\check r_+}{r_+})^{\hat\delta^2}+s]\min_{r\in[\check r_+,r_+]}\partial_r(1-\mu)(r,\varpi_0)}{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)}}-(1+\delta)\right). \tag{108}$$

Given β satisfying (106), we can guarantee that it satisfies the condition above by choosing $(\check r_+,\varepsilon_0,\delta)$ sufficiently close to $(r_+,0,0)$. □

Corollary 6.5. *If, for example,* $|\zeta_0|(u)\leq e^{-1/u^2}$*, then instead of choosing* β *according to* (94)*, in* Lemma 6.1 *we may choose any positive* β*.*

Lemma 6.6. *Suppose that* β *is given satisfying* (94)*. Choose* $\check r_-$ *and* ε_0 *as in the statement of* Lemma 6.1*. Let* γ *be the curve parametrized by* (95)*. Let also* $\delta>0$*,* $\beta^-<\beta$ *and* $\beta^+>\beta$*. There exist constants,* $\tilde c$*,* $\tilde C$*,* $\bar c$ *and* $\overline{C}$*, such that for* $(u,v)\in\gamma$*, with* $0<u\leq U_{\varepsilon_0}$*, we have*

$$\tilde ce^{(1+\delta)\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\beta}{1+\beta}v} \tag{109}$$

$$\leq -\lambda(u,v)\leq$$

$$\tilde Ce^{(1-\delta)\partial_r(1-\mu)(\check r_-,\varpi_0)\frac{\beta}{1+\beta}v} \tag{110}$$

and

$$\bar cu^{-\frac{1+\beta^+}{1+\beta^-}\frac{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta-1}\leq-\nu(u,v)\leq\overline{C}u^{-\frac{1+\beta^-}{1+\beta^+}\frac{\partial_r(1-\mu)(\check r_-,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta-1}. \tag{111}$$

Proof. Let us first outline the proof. According to (16) and (17),

$$-\lambda(u,v) = -\lambda(u_{\check{r}_-}(v),v)e^{\int_{u_{\check{r}_-}(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},v)\,d\tilde{u}}, \tag{112}$$

$$-\nu(u,v) = -\nu(u,v_{\check{r}_-}(u))e^{\int_{v_{\check{r}_-}(u)}^{v}\left[\kappa\partial_r(1-\mu)\right](u,\tilde{v})\,d\tilde{v}}. \tag{113}$$

In this region we cannot proceed as was done in the previous section because we cannot guarantee $1-\mu$ is bounded away from zero. The idea now is to use these two equations to estimate λ and ν. For this we need to obtain lower and upper bounds for

$$\int_{u_{\check{r}_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde{u},v)\,d\tilde{u} \tag{114}$$

and

$$\int_{v_{\check{r}_-}(u)}^{v}\kappa(u,\tilde{v})\,d\tilde{v}, \tag{115}$$

when $(u,v)\in J^-(\gamma)\cap J^+(\Gamma_{\check{r}_-})$. The estimates for (115), and thus for ν, are easy to obtain. We estimate (114) by comparing it with

$$\int_{u_{\check{r}_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u}. \tag{116}$$

Using (73), we see that (114) is bounded above by (116). We can also bound (114) from below by (116), divided by $1+\varepsilon$, once we show that

$$e^{\int_{v_{\check{r}_-}(\tilde{u})}^{v}\left[\left|\frac{\theta}{\lambda}\right|\frac{|\theta|}{r}\right](\tilde{u},\tilde{v})\,d\tilde{v}}\leq 1+\varepsilon.$$

The estimates for $\frac{\theta}{\lambda}$ are obtained via (34) and via upper estimates for (114). To bound (116) we use the fact that the integrals of ν and λ along $\Gamma_{\check{r}_-}$ coincide.

We start the proof by differentiating the equation

$$r(\tilde{u},v_{\check{r}_-}(\tilde{u})) = \check{r}_-$$

with respect to $\tilde{u}$, obtaining

$$\nu(\tilde{u},v_{\check{r}_-}(\tilde{u}))+\lambda(\tilde{u},v_{\check{r}_-}(\tilde{u}))v_{\check{r}_-}'(\tilde{u}) = 0. \tag{117}$$

For $(u,v)\in J^+(\Gamma_{\check{r}_-})$, integrating (117) between $u_{\check{r}_-}(v)$ and u, we get

$$\int_{u_{\check{r}_-}(v)}^{u}\nu(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u}+\int_{u_{\check{r}_-}(v)}^{u}\lambda(\tilde{u},v_{\check{r}_-}(\tilde{u}))v_{\check{r}_-}'(\tilde{u})\,d\tilde{u} = 0.$$

By making the change of variables $\tilde{v}=v_{\check{r}_-}(\tilde{u})$, this last equation can be rewritten as

$$\int_{u_{\check{r}_-}(v)}^{u}\nu(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u}-\int_{v_{\check{r}_-}(u)}^{v}\lambda(u_{\check{r}_-}(\tilde{v}),\tilde{v})\,d\tilde{v} = 0, \tag{118}$$

as $v_{\check{r}_-}(u_{\check{r}_-}(v))=v$ and $\frac{d\tilde{v}}{d\tilde{u}}=v_{\check{r}_-}'(\tilde{u})$.

We may bound the integral of λ along $\Gamma_{\check{r}_-}$ in terms of the integral of κ along $\Gamma_{\check{r}_-}$ in the following way:

$$-\max_{\Gamma_{\check{r}_-}}(1-\mu)\int_{v_{\check{r}_-}(u)}^{v}\kappa(u_{\check{r}_-}(\tilde{v}),\tilde{v})\,d\tilde{v} \tag{119}$$

$$\leq -\int_{v_{\check{r}_-}(u)}^{v}\lambda(u_{\check{r}_-}(\tilde{v}),\tilde{v})\,d\tilde{v}\leq$$

$$-\min_{\Gamma_{\check{r}_-}}(1-\mu)\int_{v_{\check{r}_-}(u)}^{v}\kappa(u_{\check{r}_-}(\tilde{v}),\tilde{v})\,d\tilde{v}. \tag{120}$$

Analogously, we may bound the integral of ν along $\Gamma_{\check{r}_-}$ in terms of the integral of $\frac{\nu}{1-\mu}$ along $\Gamma_{\check{r}_-}$ in the following way:

$$-\max_{\Gamma_{\check{r}_-}}(1-\mu)\int_{u_{\check{r}_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u} \tag{121}$$

$$\leq -\int_{u_{\check{r}_-}(v)}^{u}\nu(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u}\leq$$

$$-\min_{\Gamma_{\check{r}_-}}(1-\mu)\int_{u_{\check{r}_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u}. \tag{122}$$

Let now $(u,v)\in J^-(\gamma)\cap J^+(\Gamma_{\check{r}_-})$. Using successively (73), (121), (118) and (120), we get

$$\begin{aligned}
&\int_{u_{\check{r}_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde{u},v)\,d\tilde{u}\\
&\quad\leq\int_{u_{\check{r}_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u}\\
&\quad\leq\frac{1}{-\max_{\Gamma_{\check{r}_-}}(1-\mu)}\int_{u_{\check{r}_-}(v)}^{u}-\nu(\tilde{u},v_{\check{r}_-}(\tilde{u}))\,d\tilde{u}\\
&\quad=\frac{1}{-\max_{\Gamma_{\check{r}_-}}(1-\mu)}\int_{v_{\check{r}_-}(u)}^{v}-\lambda(u_{\check{r}_-}(\tilde{v}),\tilde{v})\,d\tilde{v}\\
&\quad\leq\frac{\min_{\Gamma_{\check{r}_-}}(1-\mu)}{\max_{\Gamma_{\check{r}_-}}(1-\mu)}\int_{v_{\check{r}_-}(u)}^{v}\kappa(u_{\check{r}_-}(\tilde{v}),\tilde{v})\,d\tilde{v} &(123)\\
&\quad\leq\frac{\min_{\Gamma_{\check{r}_-}}(1-\mu)}{\max_{\Gamma_{\check{r}_-}}(1-\mu)}\beta v_{\check{r}_-}(u) &(124)\\
&\quad\leq\frac{\min_{\Gamma_{\check{r}_-}}(1-\mu)}{\max_{\Gamma_{\check{r}_-}}(1-\mu)}\beta v.
\end{aligned}$$

We can now bound the field $\frac{\theta}{\lambda}$ for $(u,v)\in J^-(\gamma)\cap J^+(\Gamma_{\check{r}_-})$. Using (34),

$$\begin{aligned}
\left|\frac{\theta}{\lambda}\right|(u,v)\leq{}&\left|\frac{\theta}{\lambda}\right|(u_{\check{r}_-}(v),v)e^{-\int_{u_{\check{r}_-}(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},v)\,d\tilde{u}}\\
&+\int_{u_{\check{r}_-}(v)}^{u}\frac{|\zeta|}{r}(\bar{u},v)e^{-\int_{\bar{u}}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},v)\,d\tilde{u}}\,d\bar{u}. &(125)
\end{aligned}$$

We can bound the exponentials in (125) by

$$\begin{aligned}
&e^{-\int_{\bar u}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde u,v)\,d\tilde u}\\
&\quad\le e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\int_{\bar u}^{u}\left[\frac{\nu}{1-\mu}\right](\tilde u,v)\,d\tilde u}\\
&\quad\le e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\int_{u_{\check r_-}(v)}^{u}\left[\frac{\nu}{1-\mu}\right](\tilde u,v)\,d\tilde u}\\
&\quad\le e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\beta v}.
\end{aligned}$$

Combining this inequality with (91), (98) and (100), leads to

$$\begin{aligned}
\left|\frac{\theta}{\lambda}\right|(u,v)&\le\left(C\sup_{[0,u]}|\zeta_0|e^{-\alpha v}+C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta^+}v}\right)\\
&\quad\times e^{-\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\beta v}\\
&\le C\sup_{[0,u]}|\zeta_0|e^{-\left(\frac{\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\beta\right)v}.
\end{aligned}\tag{126}$$

We consider the two possible cases. Suppose first that the exponent in (126) is nonpositive. Then, from (100) we get

$$\begin{aligned}
\int_{v_{\check r_-}(u)}^{v}\left[\left|\frac{\theta}{\lambda}\right||\theta|\right](u,\tilde v)\,d\tilde v&\le C\int_{v_{\check r_-}(u)}^{v}|\theta|(u,\tilde v)\,d\tilde v\\
&\le C\sup_{[0,u]}|\zeta_0|e^{-\frac{\alpha}{1+\beta^+}v}.
\end{aligned}$$

Suppose now the exponent in (126) is positive. Using (126) and (100) again,

$$\begin{aligned}
&\int_{v_{\check r_-}(u)}^{v}\left[\left|\frac{\theta}{\lambda}\right||\theta|\right](u,\tilde v)\,d\tilde v\\
&\quad\le C\sup_{[0,u]}|\zeta_0|e^{-\left(\frac{\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\beta\right)v}\int_{v_{\check r_-}(u)}^{v}|\theta|(u,\tilde v)\,d\tilde v\\
&\quad\le C(\sup_{[0,u]}|\zeta_0|)^2e^{-\left(\frac{2\alpha}{1+\beta^+}+\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\beta\right)v}.
\end{aligned}$$

Therefore, in either case, given $\varepsilon>0$ we may choose U sufficiently small so that if $(u,v)\in J^-(\gamma)\cap J^+(\Gamma_{\check r_-})$, then

$$e^{\frac{1}{r_--\varepsilon_0}\int_{v_{\check r_-}(\bar u)}^{v}\left[\left|\frac{\theta}{\lambda}\right||\theta|\right](\bar u,\tilde v)\,d\tilde v}\le 1+\varepsilon,\tag{127}$$

for $\bar u\in[u_{\check r_-}(v),u]$.

Next we use (73), (121), (122) and (127). We may bound the integral of ν along $\Gamma_{\check r_-}$ in terms of the integral of $\frac{\nu}{1-\mu}$ on the segment $\left[u_{\check r_-}(v), u\right] \times \{v\}$ in the following way:

$$
\begin{aligned}
&-\max_{\Gamma_{\check r_-}}(1-\mu)\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u, v)\,d\tilde u \qquad (128)\\
&\quad\le -\max_{\Gamma_{\check r_-}}(1-\mu)\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u, v_{\check r_-}(\tilde u))\,d\tilde u\\
&\quad\le -\int_{u_{\check r_-}(v)}^{u}\nu(\tilde u, v_{\check r_-}(\tilde u))\,d\tilde u \le\\
&\qquad -\min_{\Gamma_{\check r_-}}(1-\mu)\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u, v_{\check r_-}(\tilde u))\,d\tilde u \le\\
&\qquad -(1+\varepsilon)\min_{\Gamma_{\check r_-}}(1-\mu)\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u, v)\,d\tilde u. \qquad (129)
\end{aligned}
$$

Now we consider $(u, v) \in \gamma$. In (124) we obtained an upper bound for $\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u, v)\,d\tilde u$. Now we use (129) to obtain a lower bound for this quantity. Applying successively (129), (118), (119), and (105),

$$
\begin{aligned}
&\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u, v)\,d\tilde u\\
&\quad\ge \tfrac{1}{-(1+\varepsilon)\min_{\Gamma_{\check r_-}}(1-\mu)}\int_{u_{\check r_-}(v)}^{u}-\nu(\tilde u, v_{\check r_-}(\tilde u))\,d\tilde u\\
&\quad= \tfrac{1}{-(1+\varepsilon)\min_{\Gamma_{\check r_-}}(1-\mu)}\int_{v_{\check r_-}(u)}^{v}-\lambda(u_{\check r_-}(\tilde v), \tilde v)\,d\tilde v\\
&\quad\ge \tfrac{\max_{\Gamma_{\check r_-}}(1-\mu)}{(1+\varepsilon)\min_{\Gamma_{\check r_-}}(1-\mu)}\int_{v_{\check r_-}(u)}^{v}\kappa(u_{\check r_-}(\tilde v), \tilde v)\,d\tilde v \qquad (130)\\
&\quad\ge \tfrac{(1-\varepsilon)\max_{\Gamma_{\check r_-}}(1-\mu)}{(1+\varepsilon)\min_{\Gamma_{\check r_-}}(1-\mu)}\beta v_{\check r_-}(u)\\
&\quad= \tfrac{(1-\varepsilon)}{(1+\varepsilon)}\tfrac{\max_{\Gamma_{\check r_-}}(1-\mu)}{\min_{\Gamma_{\check r_-}}(1-\mu)}\tfrac{\beta}{1+\beta}v.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&e^{\int_{u_{\check r_-}(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde u,v)\,d\tilde u}\\
&\quad\le e^{\left[\max_{J^-(\gamma)\cap J^+(\Gamma_{\check r_-})}\partial_r(1-\mu)\right]\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u,v)\,d\tilde u}\\
&\quad\le e^{\left[\max_{J^-(\gamma)\cap J^+(\Gamma_{\check r_-})}\partial_r(1-\mu)\right]\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{\max_{\Gamma_{\check r_-}}(1-\mu)}{\min_{\Gamma_{\check r_-}}(1-\mu)}\frac{\beta}{1+\beta}v}\\
&\quad\le e^{\left[\partial_r(1-\mu)(\check r_-,\varpi_0)+\max_{J^-(\gamma)\cap J^+(\Gamma_{\check r_-})}\frac{2(\varpi-\varpi_0)}{r^2}\right]\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{\max_{\Gamma_{\check r_-}}(1-\mu)}{\min_{\Gamma_{\check r_-}}(1-\mu)}\frac{\beta}{1+\beta}v}\\
&\quad\le e^{\left[\partial_r(1-\mu)(\check r_-,\varpi_0)+\frac{\varepsilon}{(r_--\varepsilon_0)^2}\right]\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{\max_{\Gamma_{\check r_-}}(1-\mu)}{\min_{\Gamma_{\check r_-}}(1-\mu)}\frac{\beta}{1+\beta}v}. \qquad (131)
\end{aligned}
$$

On the other hand, using (124),

$$
\begin{aligned}
&e^{\int_{u_{\check r_-}(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde u,v)\,d\tilde u}\\
&\quad\geq e^{\left[\min_{J^-(\gamma)\cap J^+(\Gamma_{\check r_-})}\partial_r(1-\mu)\right]\int_{u_{\check r_-}(v)}^{u}\frac{\nu}{1-\mu}(\tilde u,v)\,d\tilde u}\\
&\quad\geq e^{\left[\min_{J^-(\gamma)\cap J^+(\Gamma_{\check r_-})}\partial_r(1-\mu)\right]\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\frac{\beta}{1+\beta}v}\\
&\quad\geq e^{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\frac{\beta}{1+\beta}v}.
\end{aligned}
\tag{132}
$$

We continue assuming $(u,v)\in\gamma$. Taking into account (69), estimate (131) allows us to obtain an upper bound for $-\lambda(u,v)$,

$$
\begin{aligned}
-\lambda(u,v)&=-\lambda(u_{\check r_-}(v),v)e^{\int_{u_{\check r_-}(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde u,v)\,d\tilde u}\\
&\leq Ce^{\frac{(1-\varepsilon)}{(1+\varepsilon)}\frac{\max_{\Gamma_{\check r_-}}(1-\mu)}{\min_{\Gamma_{\check r_-}}(1-\mu)}\left[\partial_r(1-\mu)(\check r_-,\varpi_0)+\frac{\varepsilon}{(r_--\varepsilon_0)^2}\right]\frac{\beta}{1+\beta}v}\\
&\leq \tilde C e^{(1-\delta)\partial_r(1-\mu)(\check r_-,\varpi_0)\frac{\beta}{1+\beta}v},
\end{aligned}
$$

and estimate (132) allows us to obtain a lower bound for $-\lambda(u,v)$,

$$
\begin{aligned}
-\lambda(u,v)&=-\lambda(u_{\check r_-}(v),v)e^{\int_{u_{\check r_-}(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde u,v)\,d\tilde u}\\
&\geq ce^{\frac{\min_{\Gamma_{\check r_-}}(1-\mu)}{\max_{\Gamma_{\check r_-}}(1-\mu)}\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\beta}{1+\beta}v}\\
&\geq \tilde c e^{(1+\delta)\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\beta}{1+\beta}v}.
\end{aligned}
$$

Next, we turn to the estimates on ν. Let, again, $(u,v)\in\gamma$. Using (105),

$$
(1-\varepsilon)\tfrac{\beta}{1+\beta}v\leq\int_{v_{\check r_-}(u)}^{v}\kappa(u,\tilde v)\,d\tilde v\leq\tfrac{\beta}{1+\beta}v.
$$

These two inequalities imply

$$
\begin{aligned}
&e^{\int_{v_{\check r_-}(u)}^{v}[\kappa\partial_r(1-\mu)](u,\tilde v)\,d\tilde v}\\
&\quad\leq e^{\left[\max_{J^-(\gamma)\cap J^+(\Gamma_{\check r_-})}\partial_r(1-\mu)\right]\int_{v_{\check r_-}(u)}^{v}\kappa(u,\tilde v)\,d\tilde v}\\
&\quad\leq e^{(1-\varepsilon)\left[\partial_r(1-\mu)(\check r_-,\varpi_0)+\frac{\varepsilon}{(r_--\varepsilon_0)^2}\right]\frac{\beta}{1+\beta}v}
\end{aligned}
\tag{133}
$$

and

$$
\begin{aligned}
&e^{\int_{v_{\check r_-}(u)}^{v}[\kappa\partial_r(1-\mu)](u,\tilde v)\,d\tilde v}\\
&\quad\geq e^{\left[\min_{J^-(\gamma)\cap J^+(\Gamma_{\check r_-})}\partial_r(1-\mu)\right]\int_{v_{\check r_-}(u)}^{v}\kappa(u,\tilde v)\,d\tilde v}\\
&\quad\geq e^{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\beta}{1+\beta}v}.
\end{aligned}
\tag{134}
$$

We note that according to (85) we have

$$\begin{aligned} ce^{-\partial_r(1-\mu)(r_+,\varpi_0)\frac{v}{1+\beta^-}} &\leq ce^{-[\partial_r(1-\mu)(r_+,\varpi_0)+\tilde{\varepsilon}]\frac{v}{1+\beta}} \qquad (135)\\ &= ce^{-[\partial_r(1-\mu)(r_+,\varpi_0)+\tilde{\varepsilon}]v_{\check{r}_-}(u)}\\ &\leq u \leq Ce^{-[\partial_r(1-\mu)(r_+,\varpi_0)-\tilde{\varepsilon}]v_{\check{r}_-}(u)}\\ &= Ce^{-[\partial_r(1-\mu)(r_+,\varpi_0)-\tilde{\varepsilon}]\frac{v}{1+\beta}}\\ &\leq Ce^{-\partial_r(1-\mu)(r_+,\varpi_0)\frac{v}{1+\beta^+}}, \qquad (136) \end{aligned}$$

as $(u, v) \in \gamma$. (The bound (136) is actually valid in $J^-(\gamma) \cap J^+(\Gamma_{\check{r}_-})$, see (107).) Recalling (113) and (70), and using (133) and (135),

$$\begin{aligned} -\nu(u,v) &= -\nu(u, v_{\check{r}_-}(u))e^{\int_{v_{\check{r}_-}(u)}^{v}[\kappa\partial_r(1-\mu)](u,\tilde{v})\,d\tilde{v}}\\ &\leq \frac{C}{u}e^{(1-\varepsilon)\left[\partial_r(1-\mu)(\check{r}_-,\varpi_0)+\frac{\varepsilon}{(r_--\varepsilon_0)^2}\right]\frac{\beta}{1+\beta}v}\\ &\leq \frac{C}{u}e^{\partial_r(1-\mu)(\check{r}_-,\varpi_0)\frac{\beta}{1+\beta^+}v}\\ &\leq \overline{C}u^{-\frac{1+\beta^-}{1+\beta^+}\frac{\partial_r(1-\mu)(\check{r}_-,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta-1}, \end{aligned}$$

whereas using (134) and (136),

$$\begin{aligned} -\nu(u,v) &= -\nu(u, v_{\check{r}_-}(u))e^{\int_{v_{\check{r}_-}(u)}^{v}[\kappa\partial_r(1-\mu)](u,\tilde{v})\,d\tilde{v}}\\ &\geq \frac{c}{u}e^{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)\frac{\beta}{1+\beta}v}\\ &\geq \overline{c}u^{-\frac{1+\beta^+}{1+\beta^-}\frac{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta-1}. \end{aligned}$$

□

Remark 6.7. Since $-\partial_r(1-\mu)(r_-, \varpi_0) > \partial_r(1-\mu)(r_+, \varpi_0)$ (see Appendix A of Part 3), we can make our choice of β and other parameters $(\check{r}_-, \varepsilon_0, U)$ so that

$$-\frac{1+\beta^-}{1+\beta^+}\frac{\partial_r(1-\mu)(\check{r}_-,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta - 1 > 0$$

and

$$-\frac{1+\beta^+}{1+\beta^-}\frac{\partial_r(1-\mu)(r_--\varepsilon_0,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta - 1 > 0.$$

Having done so, for (u, v) on the curve γ, we obtain

$$cu^{s_2} \leq -\nu(u,v) \leq Cu^{s_1},$$

with $0 < s_1 < s_2$.

7. The Region $J^+(\gamma)$

Using (112) and (113), we wish to obtain upper bounds for $-\lambda$ and for $-\nu$ in the future of γ while r is greater than or equal to $r_- - \varepsilon$. To do so, we partition this set into two regions, one where the mass is close to ϖ_0 and another one where the mass is not close to ϖ_0. In the former case $\partial_r(1-\mu) < 0$ and in the latter case $\frac{\partial_r(1-\mu)}{1-\mu}$ is bounded. This information is used to bound the exponentials that appear in (112) and (113).

Here the solution with general ζ_0 departs qualitatively from the Reissner–Nordström solution (26)–(30), but the radius function remains bounded away from zero, and approaches r_- as $u \to 0$. This shows that the existence of a Cauchy horizon is a stable property when ζ_0 is perturbed away from zero.

Lemma 7.1. *Let* $0 < \varepsilon_0 < r_-$. *There exists* $0 < \varepsilon \leq \varepsilon_0$ *such that for* $(u, v) \in \{r > r_- - \varepsilon\} \cap J^+(\gamma)$ *we have*

$$-\lambda(u, v) \leq Ce^{(1-\delta)\partial_r(1-\mu)(\check{r}_-,\varpi_0)\frac{\beta}{1+\beta}v}, \tag{137}$$

$$-\nu(u, v) \leq Cu^{-\frac{1+\beta^-}{1+\beta^+}\frac{\partial_r(1-\mu)(\check{r}_-,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta-1}. \tag{138}$$

Proof. We recall that on γ the function r is bounded above by $\check{r}_-$ and that

$$\eta = \eta_0 + \varpi_0 - \varpi.$$

The minimum of η_0 in the interval $[r_- - \varepsilon_0, \check{r}_-]$ is positive, since $\eta_0(\check{r}_-) > 0$. If

$$\varpi < \varpi_0 + \min_{r\in[r_- -\varepsilon_0,\check{r}_-]} \eta_0(r) \tag{139}$$

then clearly

$$\eta > 0. \tag{140}$$

On the other hand, if

$$\varpi \geq \varpi_0 + \min_{r\in[r_- -\varepsilon_0,\check{r}_-]} \eta_0(r) \tag{141}$$

then, for $r \in [r_- - \varepsilon, \check{r}_-]$,

$$(1-\mu)(r, \varpi) \leq (1-\mu)(r, \varpi_0) - \frac{2\min_{r\in[r_- -\varepsilon_0,\check{r}_-]} \eta_0(r)}{\check{r}_-},$$

where we used

$$(1-\mu)(r, \varpi) = (1-\mu)(r, \varpi_0) + \frac{2(\varpi_0 - \varpi)}{r}.$$

Choosing $0 < \varepsilon \leq \varepsilon_0$ such that

$$\max_{r\in[r_- -\varepsilon,\check{r}_-]} (1-\mu)(r, \varpi_0) \leq \frac{\min_{r\in[r_- -\varepsilon_0,\check{r}_-]} \eta_0(r)}{\check{r}_-}$$

we have

$$(1-\mu)(r, \varpi) \leq -\frac{\min_{r\in[r_- -\varepsilon_0,\check{r}_-]} \eta_0(r)}{\check{r}_-} < 0. \tag{142}$$

In case (139) we have (recall (31))

$$\frac{\nu}{1-\mu}\partial_r(1-\mu) < 0 \quad \text{and} \quad \frac{\lambda}{1-\mu}\partial_r(1-\mu) < 0.$$

In case (141), the absolute value of

$$-\frac{1}{1-\mu}\partial_r(1-\mu)$$

is bounded, say by C. Indeed, this is a consequence of two facts: (i) the denominators $1-\mu$ and r are bounded away from zero (we recall η also has a denominator equal to r); (ii) the equality

$$\lim_{\varpi\to+\infty} -\frac{1}{1-\mu}\partial_r(1-\mu) = \frac{1}{r}. \tag{143}$$

We define

$$\Pi_v = \Big\{u \in]0, U]\colon (u, v) \in \{r > r_- - \varepsilon\} \cap J^+(\gamma)$$
$$\text{and } \varpi(u, v) < \varpi_0 + \min_{r\in[r_- - \varepsilon_0, \check{r}_-]} \eta_0(r)\Big\},$$
$$\Pi^v = \Big\{u \in]0, U]\colon (u, v) \in \{r > r_- - \varepsilon\} \cap J^+(\gamma)$$
$$\text{and } \varpi(u, v) \geq \varpi_0 + \min_{r\in[r_- - \varepsilon_0, \check{r}_-]} \eta_0(r)\Big\},$$
$$\tilde{\Pi}_u = \Big\{v \in]0, \infty[\colon (u, v) \in \{r > r_- - \varepsilon\} \cap J^+(\gamma)$$
$$\text{and } \varpi(u, v) < \varpi_0 + \min_{r\in[r_- - \varepsilon_0, \check{r}_-]} \eta_0(r)\Big\}$$

and

$$\tilde{\Pi}^u = \Big\{v \in]0, \infty[\colon (u, v) \in \{r > r_- - \varepsilon\} \cap J^+(\gamma)$$
$$\text{and } \varpi(u, v) \geq \varpi_0 + \min_{r\in[r_- - \varepsilon_0, \check{r}_-]} \eta_0(r)\Big\}.$$

In order to estimate λ, we observe that

$$\begin{aligned}
&e^{\int_{u_\gamma(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},v)\,d\tilde{u}}\\
&\quad= e^{\int_{\tilde{u}\in[u_\gamma(v),u]\cap\Pi_v}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},v)\,d\tilde{u}}\\
&\qquad\times e^{\int_{\tilde{u}\in[u_\gamma(v),u]\cap\Pi^v}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde{u},v)\,d\tilde{u}}\\
&\quad\leq 1\times e^{C\int_{\tilde{u}\in[u_\gamma(v),u]\cap\Pi^v}(-\nu)(\tilde{u},v)\,d\tilde{u}}\\
&\quad\leq 1\times e^{C(\check{r}_- -(r_- -\varepsilon))} =: \hat{C}.
\end{aligned}$$

Similarly, to estimate ν we note that

$$
\begin{aligned}
& e^{\int_{v_\gamma(u)}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v} \\
&\quad = e^{\int_{\tilde v\in[v_\gamma(u),v]\cap\tilde\Pi_u}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v} \\
&\qquad \times e^{\int_{\tilde v\in[v_\gamma(u),v]\cap\tilde\Pi^u}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v} \\
&\quad \leq 1\times e^{C\int_{\tilde v\in[v_\gamma(u),v]\cap\tilde\Pi^u}(-\lambda)(u,\tilde v)\,d\tilde v} \\
&\quad \leq 1\times e^{C(\check r_- -(r_- -\varepsilon))} = \hat C.
\end{aligned}
$$

In conclusion, let $(u,v)\in\{r>r_- -\varepsilon\}\cap J^+(\Gamma_{\check r_-})$. Using (112) and (110), we have

$$
\begin{aligned}
-\lambda(u,v) &= -\lambda(u_\gamma(v),v)e^{\int_{u_\gamma(v)}^{u}\left[\frac{\nu}{1-\mu}\partial_r(1-\mu)\right](\tilde u,v)\,d\tilde u} \\
&\leq \hat C\tilde C e^{(1-\delta)\partial_r(1-\mu)(\check r_-,\varpi_0)\frac{\beta}{1+\beta}v}.
\end{aligned} \tag{144}
$$

Similarly, using (113) and (111), we have

$$
\begin{aligned}
-\nu(u,v) &= -\nu(u,v_\gamma(u))e^{\int_{v_\gamma(u)}^{v}\left[\frac{\lambda}{1-\mu}\partial_r(1-\mu)\right](u,\tilde v)\,d\tilde v} \\
&\leq \hat C\overline{C}u^{-\frac{1+\beta^-}{1+\beta^+}\frac{\partial_r(1-\mu)(\check r_-,\varpi_0)}{\partial_r(1-\mu)(r_+,\varpi_0)}\beta-1}.
\end{aligned}
$$

□

Lemma 7.2. *Let $\delta>0$. There exists $\tilde U_\delta$ such that for $(u,v)\in J^+(\gamma)$ with $u<\tilde U_\delta$, we have*

$$
r(u,v)>r_- -\delta.
$$

Proof. We denote by $\overline{\varepsilon}$ the value of ε that is provided in Lemma 7.1. Let $\delta>0$. Without loss of generality, we assume that δ is less than or equal to $\overline{\varepsilon}$. Choose the value of ε in Corollary 6.2 equal to δ. This determines an U_ε as in the statement of that corollary. Let $(u,v)\in J^+(\gamma)$ with $u\leq U_\varepsilon$. Then

$$
r(u,v_\gamma(u))\geq r_- -\frac{\delta}{2}\quad\text{and}\quad r(u_\gamma(v),v)\geq r_- -\frac{\delta}{2}
$$

because $u_\gamma(v)\leq u$. Here

$$
u\mapsto(u,v_\gamma(u))\quad\text{and}\quad v\mapsto(u_\gamma(v),v)
$$

are parameterizations of the spacelike curve γ. Integrating (138), we obtain

$$
-\int_{u_\gamma(v)}^{u}\frac{\partial r}{\partial u}(s,v)\,ds\leq\int_{u_\gamma(v)}^{u}Cs^{p-1}\,ds, \tag{145}
$$

for a positive p. This estimate is valid for $(u,v)\in\{r>r_- -\overline{\varepsilon}\}\cap J^+(\gamma)$. It yields

$$
\begin{aligned}
r(u,v) &\geq r(u_\gamma(v),v)-\frac{C}{p}(u^p-(u_\gamma(v))^p) \\
&\geq r_- -\frac{\delta}{2}-\frac{C}{p}u^p\ >\ r_- -\delta,
\end{aligned} \tag{146}
$$

provided $u < \min\left\{U_\varepsilon, \sqrt[p]{\frac{\delta p}{2C}}\right\} =: \tilde{U}_\delta$. Since δ is less than or equal to $\overline{\varepsilon}$ and $\gamma \subset \{r > r_- - \overline{\varepsilon}\}$, if $(u, v) \in J^+(\gamma)$ and $u < \tilde{U}_\delta$, then $(u, v) \in \{r > r_- - \overline{\varepsilon}\}$ and the estimate (145) does indeed apply.

Alternatively, we can obtain (146) integrating (137):

$$-\int_{v_\gamma(u)}^{v} \frac{\partial r}{\partial v}(u, s)\, ds \leq \int_{v_\gamma(u)}^{v} Ce^{-qs}\, ds,$$

for a positive q. This yields

$$\begin{aligned} r(u, v) &\geq r(u, v_\gamma(u)) - \frac{C}{q}\left(e^{-qv_\gamma(u)} - e^{-qv}\right) \\ &\geq r_- - \frac{\delta}{2} - \frac{C}{q}e^{-qv_\gamma(u)} \\ &\geq r_- - \frac{\delta}{2} - \tilde{C}u^{\tilde{q}}, \end{aligned}$$

for a positive $\tilde{q}$, according to (135). For $u < \min\left\{U_\varepsilon, \sqrt[\tilde{q}]{\frac{\delta}{2\tilde{C}}}\right\}$ we obtain, once more,

$$r(u, v) > r_- - \delta.$$

□

Corollary 7.3. *If* $\delta < r_-$ *then* $\mathcal{P}$ *contains* $[0, \tilde{U}_\delta] \times [0, \infty[$. *Moreover, estimates* (137) *and* (138) *hold on* $J^+(\gamma)$.

Due to the monotonicity of $r(u, \cdot)$ for each fixed u, we may define

$$r(u, \infty) = \lim_{v\to\infty} r(u, v).$$

As $r(u_2, v) < r(u_1, v)$ for $u_2 > u_1$, we have that $r(\cdot, \infty)$ is nonincreasing.

Corollary 7.4. *We have*

$$\lim_{u\searrow 0} r(u, \infty) = r_-. \tag{147}$$

The previous two corollaries prove Theorem 1.1. The argument in [7, Section 11], shows that, as in the case when $\Lambda = 0$, the spacetime is then extendible across the Cauchy horizon with C^0 metric.

8. Two Effects of Any Nonzero Field

This section contains two results concerning the structure of the solutions with general ζ_0. Theorem 8.1 asserts that only in the case of the Reissner–Nordström solution does the curve Γ_{r_-} coincide with the Cauchy horizon: if the field ζ_0 is not identically zero, then the curve Γ_{r_-} is contained in $\mathcal{P}$.

Lemma 8.2 states that, in contrast with what happens with the Reissner–Nordström solution, and perhaps unexpectedly, the presence of a nonzero field immediately causes the integral $\int_0^\infty \kappa(u, v)\, dv$ to be finite for any $u > 0$. This implies that the affine parameter of any outgoing null geodesic inside the event horizon is finite at the Cauchy horizon.

For each $u > 0$, we define

$$\varpi(u, \infty) = \lim_{v \nearrow +\infty} \varpi(u, v).$$

This limit exists, and $u \mapsto \varpi(u, \infty)$ is an increasing function.

Theorem 8.1. *Suppose that there exists a positive sequence* (u_n) *converging to* 0 *such that* $\zeta_0(u_n) \neq 0$. *Then* $r(u, \infty) < r_-$ *for all* $u \in]0, U]$.

Proof. The proof is by contradiction. Assume that $r(\bar{u}, \infty) = r_-$ for some $\bar{u} \in]0, U]$. Then $r(u, \infty) = r_-$ for all $u \in]0, \bar{u}]$. Let $0 < \delta < u \leq \bar{u}$. Clearly,

$$r(u, v) = r(\delta, v) + \int_\delta^u \nu(s, v)\, ds.$$

Fatou's lemma implies that

$$\liminf_{v \to \infty} \int_\delta^u -\nu(s, v)\, ds \geq \int_\delta^u \liminf_{v \to \infty} -\nu(s, v)\, ds.$$

So,

$$\begin{aligned} r_- = \lim_{v \to \infty} r(u, v) &= \lim_{v \to \infty} r(\delta, v) - \lim_{v \to \infty} \int_\delta^u -\nu(s, v)\, ds \\ &= r_- - \liminf_{v \to \infty} \int_\delta^u -\nu(s, v)\, ds \\ &\leq r_- - \int_\delta^u \liminf_{v \to \infty} -\nu(s, v)\, ds. \end{aligned} \tag{148}$$

Since δ is arbitrary, this inequality implies that $\liminf_{v \to \infty} -\nu(u, v)$ is equal to zero for almost all $u \in]0, \bar{u}]$. However, we will now show that, under the hypothesis on ζ_0, $\liminf_{v \to \infty} -\nu(u, v)$ cannot be zero for any positive u if $r(u, \infty) \equiv r_-$.

First, assume that $\varpi(u, \infty) = \infty$ for a certain u. Then, using (143),

$$\lim_{v \to \infty} \frac{\partial_r(1 - \mu)}{1 - \mu}(u, v) = -\frac{1}{r_-} < 0.$$

We may choose $V = V(u) > 0$ such that $\frac{\partial_r(1-\mu)}{1-\mu}(u, v) < 0$ for $v > V$. Integrating (17), for $v > V$,

$$\begin{aligned} -\nu(u, v) &= -\nu(u, V) e^{\int_V^v \left[\frac{\partial_r(1-\mu)}{1-\mu}\lambda\right](u, \tilde{v})\, d\tilde{v}} \\ &\geq -\nu(u, V) > 0. \end{aligned}$$

Thus, for such a u, it is impossible for $\liminf_{v \to \infty} -\nu(u, v)$ to be equal to zero.

Now assume $\varpi(u, \infty) < \infty$. The hypothesis on ζ_0 and (18) imply that $\varpi(u, 0) > \varpi_0$ for each $u > 0$, and so $\varpi(u, \infty) > \varpi_0$ for each $u > 0$. Then,

$$(1 - \mu)(u, \infty) = (1 - \mu)(r_-, \varpi(u, \infty)) < (1 - \mu)(r_-, \varpi_0) = 0.$$

We may choose $V = V(u) > 0$ such that $-(1-\mu)(u, v) \geq C(u) > 0$ for $v > V$. Hence, integrating (19), for $v > V$,

$$\begin{aligned}\varpi(u, v) &= \varpi(u, V) + \frac{1}{2}\int_V^v \left[-(1-\mu)\frac{\theta^2}{-\lambda}\right](u, v)\, dv \\ &\geq \varpi(u, V) + \frac{C(u)}{2}\int_V^v \left[\frac{\theta^2}{-\lambda}\right](u, v)\, dv.\end{aligned}$$

Since $\varpi(u, \infty) < \infty$, letting v tend to $+\infty$, we conclude

$$\int_V^\infty \left[\frac{\theta^2}{-\lambda}\right](u, v)\, dv < \infty.$$

Finally, integrating (73) starting from V, we see that $\frac{\nu(u,\infty)}{(1-\mu)(u,\infty)} > 0$. Since $(1-\mu)(u, \infty) < 0$, once again we conclude that $\liminf_{v\to\infty} -\nu(u, v) = -\nu(u, \infty) > 0$. □

Lemma 8.2. *Suppose that there exists a positive sequence* (u_n) *converging to* 0 *such that* $\zeta_0(u_n) \neq 0$. *Then*

$$\int_0^\infty \kappa(u, v)\, dv < \infty \quad \text{for all } u > 0. \tag{149}$$

Proof. We claim that for some decreasing sequence (u_n) converging to 0,

$$(1-\mu)(u_n, \infty) < 0.$$

To prove our claim, we consider three cases.

Case 1. If $\varpi(u, \infty) = \infty$ for each $u > 0$ then $(1-\mu)(u, \infty) = -\infty$.
Case 2. If $\lim_{u\searrow 0} \varpi(u, \infty) > \varpi_0$ then, using Corollary 7.4,

$$\lim_{u\searrow 0}(1-\mu)(u, \infty) = (1-\mu)(r_-, \lim_{u\searrow 0} \varpi(u, \infty)) < (1-\mu)(r_-, \varpi_0) = 0.$$

Case 3. Suppose that $\lim_{u\searrow 0} \varpi(u, \infty) = \varpi_0$. For sufficiently small u and $(u, v) \in J^+(\Gamma_{\check{r}_-})$, we have

$$\eta(u, v) \geq 0$$

(see (140)). So, we may define $\nu(u, \infty) = \lim_{v\nearrow+\infty} \nu(u, v)$. By Lebesgue's monotone convergence theorem, we have

$$r(u, \infty) = r(\delta, \infty) + \int_\delta^u \nu(s, \infty)\, ds. \tag{150}$$

Note that different convergence theorems have to be used in (148) and (150). If $\nu(u, \infty)$ were zero almost everywhere, then $r(u, \infty)$ would be a constant. If the constant were r_- we would be contradicting Theorem 8.1. If the constant were smaller than r_- we would be contradicting Lemma 7.2. We conclude there must exist a sequence $u_n \searrow 0$ such that $\nu(u_n, \infty) < 0$. Integrating (73), we get

$$\frac{\nu(u, \infty)}{(1-\mu)(u, \infty)} \leq \frac{\nu(u, 0)}{(1-\mu)(u, 0)} < \infty.$$

Therefore, $(1-\mu)(u_n,\infty)<0$. This proves our claim.

For any fixed index n, there exists a v_n such that

$$(1-\mu)(u_n,v)<\frac{1}{2}(1-\mu)(u_n,\infty)=:-\frac{1}{c_n},$$

for $v\geq v_n$. It follows that

$$\kappa(u_n,v)\leq c_n(-\lambda(u_n,v)),\quad \text{for } v\geq v_n.$$

Using the estimate (137) for $-\lambda$, we have

$$\int_{v_n}^{\infty}\kappa(u_n,v)\,dv<\infty.$$

Hence $\int_0^\infty \kappa(u_n,v)\,dv<\infty$. Recalling that $u\mapsto\kappa(u,v)$ is nonincreasing, we get (149). □

Corollary 8.3. *Let $u>0$. Consider an outgoing null geodesic $t\mapsto(u,v(t))$ for $(\mathcal{M},g)$, with g given by*

$$g=-\Omega^2(u,v)\,dudv+r^2(u,v)\,\sigma_{\mathbb{S}^2}.$$

Then $v^{-1}(\infty)<\infty$, i.e. the affine parameter is finite at the Cauchy horizon.

Proof. The function $v(\,\cdot\,)$ satisfies

$$\ddot{v}+\Gamma^v_{vv}(u,v)\,\dot{v}^2=0, \tag{151}$$

where the Christoffel symbol Γ^v_{vv} is given by

$$\Gamma^v_{vv}=\partial_v\ln\Omega^2.$$

So, we may rewrite (151) as

$$\frac{\ddot{v}}{\dot{v}}=-\partial_t(\ln\Omega^2)(u,v).$$

We integrate both sides of this equation to obtain

$$\ln\dot{v}+\ln c=-\ln\Omega^2(u,v),$$

with $c>0$, or

$$\frac{dt}{dv}=c\,\Omega^2(u,v).$$

Integrating both sides of the previous equation once again, the affine parameter t is given by

$$t=v^{-1}(0)+c\int_0^v\Omega^2(u,\bar{v})\,d\bar{v}=v^{-1}(0)-4c\int_0^v(\nu\kappa)(u,\bar{v})\,d\bar{v}.$$

If ζ_0 vanishes in a neighborhood of the origin, the solution corresponds to the Reissner–Nordström solution. The function κ is identically 1 and, using (73), $\frac{\nu}{1-\mu} = C(u)$, with $C(u)$ a positive function of u. Thus, $\nu = C(u)(1-\mu) = C(u)\lambda$ and

$$\int_0^\infty \Omega^2(u, \bar{v})\, d\bar{v} = -4cC(u) \int_0^\infty \lambda(u, \bar{v})\, d\bar{v} = 4cC(u)(r_+ - u - r_-) < \infty.$$

On the other hand, suppose that there exists a positive sequence (u_n) converging to 0 such that $\zeta_0(u_n) \neq 0$. Then, since ν is continuous, it satisfies the bound (138) for large v, and (149) holds. So we also have

$$\int_0^\infty \Omega^2(u, \bar{v})\, d\bar{v} < \infty.$$

□

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