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On the Aharonov–Bohm Operators with Varying Poles: The Boundary Behavior of Eigenvalues

Benedetta Noris[1], **Manon Nys**[2,3,4], **Susanna Terracini**[5]

[1] INdAM-COFUND Marie Curie Fellow, Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, via Bicocca degli Arcimboldi 8, 20126 Milan, Italy. E-mail: benedettanoris@gmail.com
[2] Fonds National de la Recherche Scientifique, FNRS, Brussels, Belgium. E-mail: manonys@gmail.com
[3] Département de Mathématiques, Université Libre de Bruxelles (ULB), Boulevard du triomphe, 1050 Brussels, Belgium
[4] Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, via Bicocca degli Arcimboldi 8, 20126 Milan, Italy
[5] Dipartimento di Matematica "Giuseppe Peano", Università di Torino, Via Carlo Alberto 10, 20123 Turin, Italy. E-mail: susanna.terracini@unito.it

Abstract: We consider a magnetic Schrödinger operator with magnetic field concentrated at one point (the pole) of a domain and half integer circulation, and we focus on the behavior of Dirichlet eigenvalues as functions of the pole. Although the magnetic field vanishes almost everywhere, it is well known that it affects the operator at the spectral level (the Aharonov–Bohm effect, Phys Rev (2) 115:485–491, 1959). Moreover, the numerical computations performed in (Bonnaillie-Noël et al., Anal PDE 7(6):1365–1395, 2014; Noris and Terracini, Indiana Univ Math J 59(4):1361–1403, 2010) show a rather complex behavior of the eigenvalues as the pole varies in a planar domain. In this paper, in continuation of the analysis started in (Bonnaillie-Noël et al., Anal PDE 7(6):1365–1395, 2014; Noris and Terracini, Indiana Univ Math J 59(4):1361–1403, 2010), we analyze the relation between the variation of the eigenvalue and the nodal structure of the associated eigenfunctions. We deal with planar domains with Dirichlet boundary conditions and we focus on the case when the singular pole approaches the boundary of the domain: then, the operator loses its singular character and the k-th magnetic eigenvalue converges to that of the standard Laplacian. We can predict both the rate of convergence and whether the convergence happens from above or from below, in relation with the number of nodal lines of the k-th eigenfunction of the Laplacian. The proof relies on the variational characterization of eigenvalues, together with a detailed asymptotic analysis of the eigenfunctions, based on an Almgren-type frequency formula for magnetic eigenfunctions and on the blow-up technique.

1. Introduction

In this paper we continue the analysis started in [11,31] concerning the behavior of the eigenvalues of the magnetic Schrödinger operator

$$(i\nabla + A_a)^2 = -\Delta + i\nabla \cdot A_a + 2iA_a \cdot \nabla + |A_a|^2 \tag{1.1}$$

as the pole $a \in \Omega$ moves inside of Ω and eventually hits the boundary $\partial\Omega$. Here $\Omega \subset \mathbb{R}^2$ is open, bounded and simply connected, and we impose zero boundary conditions on $\partial\Omega$. The magnetic potential A_a has the form

$$A_a(x) = \frac{1}{2}\left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right), \tag{1.2}$$

for $a = (a_1, a_2) \in \Omega$ and $x = (x_1, x_2) \in \Omega\backslash\{a\}$. Such magnetic vector potential is generated by an infinitesimally thin solenoid orthogonal to the plane, and the associated magnetic field is a π-multiple of the Dirac delta orthogonal to the plane. The resulting magnetic Schrödinger equation describes the motion of a non relativistic, spinless charged quantum particle constrained in a potential well and interacting with the point magnetic field $\nabla \times A_a$.

Though the magnetic field vanishes almost everywhere, its presence affects the spectrum of the operator, giving rise to the Aharonov–Bohm effect [3]. At first, as a consequence of the diamagnetic inequality, the first eigenvalue is raised by the presence of the magnetic potential (see [26,28,29]). This is not the case for higher eigenvalues, which show a more complex behavior. This has been numerically detected and depicted in [11, Figure 2]. In these figures, which are shown below, the angular sector of aperture $\pi/4$, $\Sigma_{\pi/4}$, is represented in dark thick line. Outside the angular sector are represented the eigenvalues λ_j of the Dirichlet-Laplacian on $\Sigma_{\pi/4}$ (which do not depend on a). We observe the convergence proved in Theorem 1.3:

$$\forall j \geq 1, \quad \lambda_j^a \to \lambda_j \quad \text{as } a \to \partial\Sigma_{\pi/4}.$$

Figure 1 also illustrates the boundary behavior of the eigenvalues for a domain with a piecewise C^∞ boundary; note that the functions $a \mapsto \lambda_j^a$ are regular except at points where the eigenvalue λ_j^a is not simple.

Such a rich structure calls for some theoretical explanation. The study of the behavior of the eigenvalues in connection with the nodal domains of the corresponding eigenfunctions is motivated by its relation to spectral minimal partitions, see [10,17,19–23]. It has been proved in [21] that, in dimension 2, if all the clustering points of a minimal partition have an even multiplicity, then the partition is nodal, i.e., it is the nodal partition of an eigenfunction. The results in [7–9,31] establish a strong relation between the minimal partitions having points with odd multiplicity and the nodal domains of Aharonov–Bohm eigenfunctions.

Two natural spaces to study the operator (1.1) are given by $\mathcal{D}_{A_a}^{1,2}(\Omega)$ and $H_{A_a}^1(\Omega)$, respectively with and without vanishing boundary conditions on $\partial\Omega$. The first space $\mathcal{D}_{A_a}^{1,2}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega\backslash\{a\}, \mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)} := \|(i\nabla + A_a)u\|_{L^2(\Omega)},$$

while the second space $H_{A_a}^1(\Omega)$ is defined as the completion of the set

$$\left\{u \in C^\infty(\Omega, \mathbb{C})\,,\ u \text{ vanishes in a neighborhood of } a\right\}$$

(a) $a \mapsto \lambda_1^a$ **(b)** $a \mapsto \lambda_2^a$ **(c)** $a \mapsto \lambda_3^a$

(d) $a \mapsto \lambda_4^a$ **(e)** $a \mapsto \lambda_5^a$ **(f)** $a \mapsto \lambda_6^a$

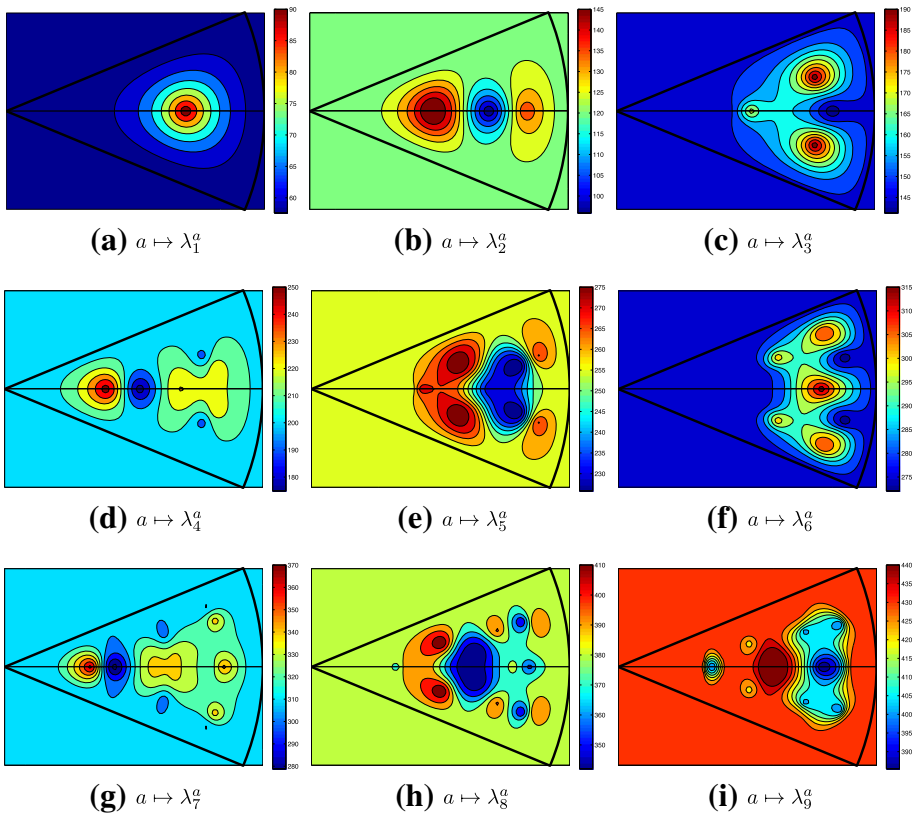**(g)** $a \mapsto \lambda_7^a$ **(h)** $a \mapsto \lambda_8^a$ **(i)** $a \mapsto \lambda_9^a$

Fig. 1. First nine eigenvalues of $(i\nabla + A_a)^2$ for a varying in $\Sigma_{\pi/4}$

with respect to the norm

$$\|u\|_{H^1_{A_a}(\Omega)} := \left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

We recall several properties of those spaces in Sect. 2.

The domain of the operator (1.1) will be one of the two previously defined function spaces, the operator, with values into its dual space, is the Riesz one and is obviously self-adjoint: for any pair of functions $u, v \in \mathcal{D}^{1,2}_{A_a}(\Omega)$, $(i\nabla + A_a)^2 u \in \left(\mathcal{D}^{1,2}_{A_a}(\Omega) \right)'$ (or respectively for $H^1_{A_a}(\Omega)$) and

$$\left\langle (i\nabla + A_a)^2 u, v \right\rangle := \int_\Omega (i\nabla + A_a)u \cdot \overline{(i\nabla + A_a)v}.$$

We study the following weighted eigenvalue problem

$$(i\nabla + A_a)^2 \varphi_k^a = \lambda_k^a \, p(x) \varphi_k^a, \quad \text{with} \quad \varphi_k^a \in \mathcal{D}^{1,2}_{A_a}(\Omega), \tag{1.3}$$

with p satisfying

$$p \in C^\infty(\overline{\Omega}), \quad p(x) > 0, \, x \in \Omega. \tag{1.4}$$

The weight has the role of incorporating the curvature of the domain when the boundary is locally flattened by a conformal change of variables. A bounded potential could be added as well. Throughout the paper, we will denote by λ_k^a, $k \in \mathbb{N}_0 = \{1, 2, \ldots\}$, the eigenvalues in (1.3) arranged in a real increasing sequence and counted with their multiplicity. We will reserve the notation λ_k, φ_k for the eigenvalues and eigenfunctions of the Laplacian with weight $p(x)$ in Ω and with zero boundary conditions (again increasing and counted with their multiplicity).

The main results obtained in [11] and [31] in connection with this topic concern the critical points of the map

$$a \in \Omega \mapsto \lambda_k^a.$$

In particular, in [11] the authors have analyzed the relation between such critical points and the nodal properties of the corresponding eigenfunctions. To this aim, let us recall the definition of order of vanishing of a function u at an interior point b.

Definition 1.1. (*Interior zero of order $h/2$*). Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$, $b \in \Omega$ and $h \in \mathbb{N}$.

(i) If h is even, we say that u has a zero of order $h/2$ at b if it is of class at least $C^{h/2}$ in a neighborhood of b and $u(b) = \cdots = D^{h/2-1}u(b) = 0$, while $D^{h/2}u(b) \neq 0$.

(ii) If h is odd, we say that u has a zero of order $h/2$ at b if $u(x^2 + b)$ has a zero of order h at $x = 0$ (here x^2 is the complex square).

Then, a link between the order of vanishing of the function $a \mapsto |\lambda_k^a - \lambda_k^b|$ can be established: indeed, the following result holds in case $p(x) \equiv 1$ (but it is valid for any p satisfying (1.4)).

Theorem 1.2. ([11,31]). *Let $\Omega \subset \mathbb{R}^2$ open, bounded and simply connected. Let $b \in \Omega$ and λ_k^b simple. φ_k^b has a zero of order $h/2$ at b, with $h \geq 3$ odd, if and only if b is a critical point of the map $a \mapsto \lambda_k^a$. Moreover, in this case we have*

$$|\lambda_k^a - \lambda_k^b| \leq C|a - b|^{(h+1)/2} \quad \text{as } a \to b, \tag{1.5}$$

for a constant $C > 0$ independent of a.

In this paper we address the study of the behavior of the magnetic eigenvalues as the pole approaches the boundary of the domain, starting from the observation that, since simply connectedness is restored when the pole lies on the boundary, there holds.

Theorem 1.3. ([11,27]). *Let $\Omega \subset \mathbb{R}^2$ open, bounded and simply connected. For every $k \in \mathbb{N}_0$ we have that $\lambda_k^a \to \lambda_k$ as $a \to \partial\Omega$.*

Our first aim is to obtain an expansion similar to (1.5) when the point b is on the boundary $\partial\Omega$. It is worthwhile to notice that the techniques that were developed in [11], based on the application of local inversion methods, hardly extend to the study of the behavior at the boundary. For this reason, we propose here a different and more efficient approach, based on the variational characterization of the eigenvalues.

Another phenomenon enlightened by the numerical simulations in [11] attracted our attention. The convergence $\lambda_k^a \to \lambda_k$ described in Theorem 1.3 can take place either from above or from below, depending on the value of k and on the position of the pole, see Fig. 4. Of course, by the diamagnetic inequality, $\lambda_1^a > \lambda_1$ for every $a \in \Omega$. A more detailed analysis suggests that the different behaviors are related to the position of the pole with respect to the nodal lines of φ_k. If the pole a moves from $\partial\Omega$ along a nodal
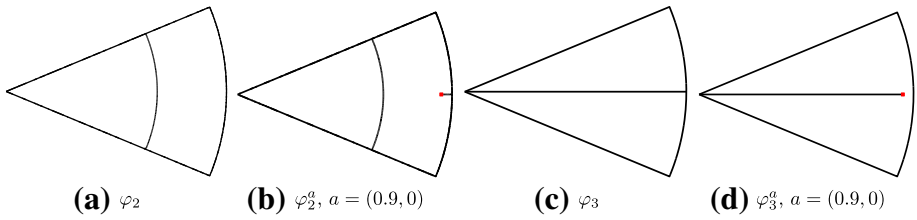
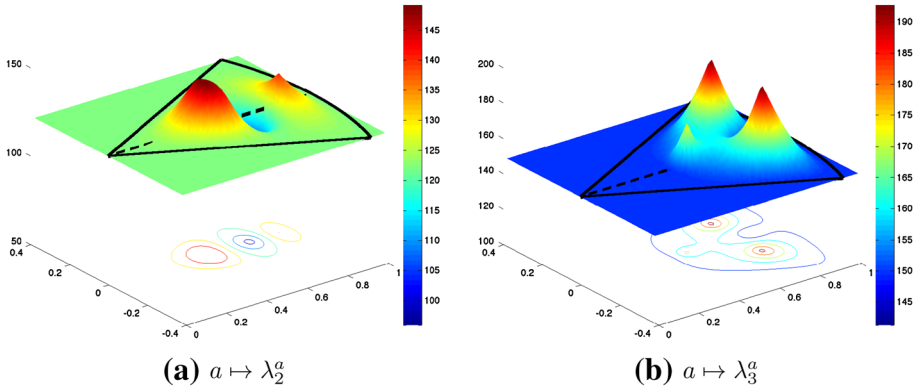**Fig. 2.** Nodal lines of second and third eigenfunctions in $\Sigma_{\pi/4}$

(a) $a \mapsto \lambda_2^a$ **(b)** $a \mapsto \lambda_3^a$

Fig. 3. 3-D Representation of the map $a \mapsto \lambda_j^a$ in $\Sigma_{\pi/4}$, for $j = 2, 3$

line of φ_k, then the nodal line of φ_k^a is shorter than the one of φ_k, locally for a close to the boundary, see Fig. 2c, d. This determines a decrease of the energy: we can observe in Figs. 3b, 4 that $\lambda_3^a < \lambda_3$ for a approaching the point $(1, 0) \in \partial\Omega$ along the symmetry axis of the angular sector. Conversely, if a moves from $\partial\Omega$ not on a nodal line of φ_k as in Figs. 2a, b, this creates a new nodal line in the magnetic eigenfunction and consequently an increase of the energy, as we can see in Figs. 3a, 4.

The second aim of this paper is to provide a theoretical justification of these facts. Before stating our main results, let us give the definition of order of vanishing at a boundary point $b \in \partial\Omega$ of a function u, with $u = 0$ on $\partial\Omega$. As we will see, this definition makes sense only if $\partial\Omega$ is sufficiently regular (see the case of conical singularities in Appendix A).

Definition 1.4. (*Boundary zero of order $h/2$*). Let $\Omega \subset \mathbb{R}^2$ open, bounded and of class C^∞. Let $u : \Omega \to \mathbb{C}$, $u = 0$ on $\partial\Omega$, $b \in \partial\Omega$ and h even.

We say that u has a zero of order $h/2$ at b if there exists a neighborhood $U(b)$ such that $u \in C^{h/2}(U(b) \cap \Omega)$ and $u(b) = \cdots = D^{h/2-1}u(b) = 0$ while $D^{h/2}u(b) \neq 0$ in $U(b) \cap \Omega$.

Note that, whereas for $b \in \Omega$ a zero of order $h/2$ corresponds to h arcs of nodal lines meeting at b, for $b \in \partial\Omega$ a zero of order $h/2$ corresponds to $h/2 - 1$ arcs of nodal lines meeting at b. This is due to the fact that we are considering zero boundary conditions. Our first main result is the following.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^2$ open, bounded, simply connected and of class C^∞. Let p satisfy (1.4). Suppose that $\lambda_{k-1} < \lambda_k$ and that there exists an eigenfunction φ_k associated*
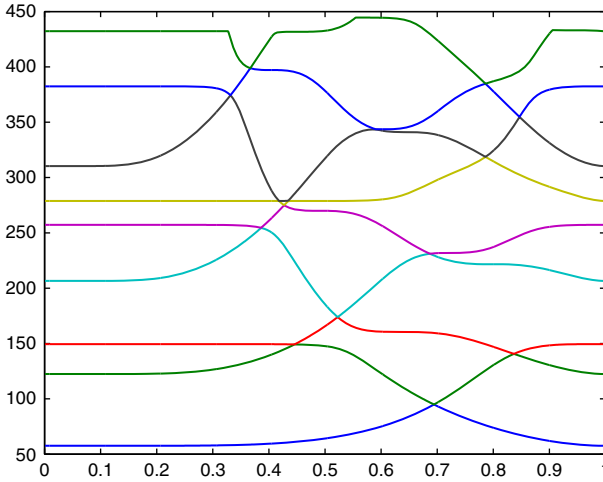
Fig. 4. $a \mapsto \lambda_j^a$, a belonging to the symmetry axis of $\Sigma_{\pi/4}$, for $j = 1, \ldots, 9$

to λ_k having a zero of order $h/2 \geq 2$ at $b \in \partial\Omega$, i.e., at least one piece of nodal line ending at b. Denote by Γ any such piece of nodal line. Then there exists $C > 0$, not depending on a, such that

$$\lambda_k^a \leq \lambda_k - C|a - b|^h \quad \text{for } a \in \Gamma, \ a \to b. \tag{1.6}$$

The study of the case when the pole approaches $\partial\Omega$ at a point where no nodal lines of φ_k end, requires additional work. The difficulty is that, in order to prove the opposite inequality with respect to (1.6), we need some information about the behavior of φ_k^a when a is close to the boundary. In this direction, we prove the uniqueness of the following limit profile.

Proposition 1.6. *Let* $e = (1, 0)$ *and let* ψ *be a solution in* $H^1_{A_e, loc}(\mathbb{R}^2_+)$ *to*

$$\begin{cases} (i\nabla + A_e)^2\psi = 0 & \mathbb{R}^2_+ \\ \psi = 0 & \{x_1 = 0\}, \end{cases} \tag{1.7}$$

satisfying the normalization condition

$$\lim_{r \to +\infty} \frac{r\|(i\nabla + A_e)\psi\|^2_{L^2(D_r^+(0))}}{\|\psi\|^2_{L^2(\partial D_r^+(0))}} = 1, \tag{1.8}$$

where $D_r^+(0) := D_r(0) \cap \{x_1 > 0\}$. *Then*

(i) ψ *is unique up to a multiplicative constant;*

(ii) for $r > 1$ *we have* $\psi(r, \theta) = Ce^{i\theta_e/2}\left(r\cos\theta - \dfrac{\beta}{\pi}\dfrac{\cos\theta}{r} + \displaystyle\sum_{n\geq 3, n\, odd} \dfrac{b_n}{r^n}\cos(n\theta)\right)$,

where $\beta > 0$ *is explicitly characterized in* (6.7) *and* $b_n \in \mathbb{R}$.

Once the uniqueness is established, we can prove that this limit profile provides a good description of the asymptotic behavior of φ_k^a. In the following theorem, we locally flatten the boundary of Ω near the boundary point through a conformal transformation (see Sect. 3 for more details), which allows us to work on half-balls. Then we perform a normalized blow up, which converges to the previous profile on the half-space in the limit.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^2$ open, bounded, simply connected and of class C^∞. Let p satisfy (1.4). Suppose that φ_k has a zero of order 1 at $b \in \partial\Omega$ (no nodal lines ending at b).*

Let Φ be a conformal map such that $\Phi^{-1} \in C^\infty(\overline{\Omega})$, $\Phi^{-1}(b) = 0$ and, for some small $r > 0$,

$$\Phi^{-1}(\Omega) \cap D_r(0) = \{x \in D_r(0) : x_1 > 0\} =: D_r^+(0).$$

Then there exists $K > 1$ such that, denoting by

$$a' = (a_1', a_2') = \Phi^{-1}(a), \qquad \psi_k^a(y) = \frac{\sqrt{Ka_1'}}{\|\varphi_k^a \circ \Phi\|_{L^2(\partial D_{Ka_1'}^+(0,a_2'))}} \varphi_k^a(\Phi(a_1' y_1, a_1' y_2 + a_2'))$$

we have

$$\psi_k^a \to \psi \quad in \ H^1_{A_e,loc}(\mathbb{R}^2_+) \ as \ a \to b,$$

where $e = (1, 0)$ and ψ is the unique solution to (1.7)–(1.8) with multiplicative constant C given explicitly in (7.34).

The previous two results are obtained by exploiting an Almgren-type frequency formula [5,6] for magnetic eigenfunctions, see Definition 5.1. This tool has been introduced in the context of magnetic operators in [13] to obtain, among other results, sharp regularity results for Aharonov–Bohm eigenfunctions.

The asymptotic analysis above allows us to prove the last main result of the paper.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^2$ open, bounded, simply connected and of class C^∞. Let p satisfy (1.4). Suppose that λ_k is simple and that φ_k has a zero of order 1 at $b \in \partial\Omega$ (no nodal lines ending at b). Then there exists $C > 0$, not depending on a, such that*

$$\lambda_k^a \geq \lambda_k + C \, (dist(a, \partial\Omega))^2 \quad as \ a \to b. \tag{1.9}$$

Remark 1.9. (i) Theorem 1.8 establishes a local diamagnetic spectral inequality when the pole approaches the boundary away from the nodal lines of the eigenfunction.

(ii) The exact behavior as $a \to b \in \partial\Omega$, b being the endpoint of one or more nodal lines of φ_k, but a not belonging to any such nodal line, remains an open problem.

(iii) The assumption of regularity of Ω can be weakened: it is enough to have $\partial\Omega \in C^{2,\gamma}$ for some $\gamma > 0$, see Remark 3.2.

(iv) If Ω presents a conical singularity, estimates (1.6) and (1.9) do not hold at the vertex of the cone. This can be observed in the numerical simulations: we see in Fig. 4 that the curve $a \mapsto \lambda_k^a$ is flat as a approaches the acute angle of the angular sector. Concerning for example λ_2^a, we see that relation (1.9) does not hold, despite the absence of nodal lines of φ_2. We treat this topic in Appendix A. If we particularize Theorem A.1 to the case of the second eigenvalue in $\Sigma_{\pi/4}$, we obtain for example

$$\lambda_2^a \geq \lambda_2 + C|a|^8,$$

as a moves along the angle bisector.

The paper is organized as follows. In Sect. 2 we define two function spaces related with the magnetic operator, $\mathcal{D}_{A_a}^{1,2}(\Omega)$ and $H_{A_a}^1(\Omega)$, respectively of functions with and without zero boundary conditions on $\partial\Omega$. We also recall a Hardy-type inequality and the asymptotic behavior of the eigenfunctions φ_k and φ_k^a around zeros of order $h/2$. In Sect. 3 we use a Riemann mapping theorem in order to flatten locally the boundary $\partial\Omega$ around the point 0. By doing this, we recover an equation similar to (1.3), the new weight verifying the same assumptions as the old one, see (1.4), thanks to the regularity hypothesis on $\partial\Omega$. Then, we prove some Poincaré-type inequalities in half-balls. In Sect. 4 we prove Theorem 1.5. In Sect. 5 we introduce the Almgren's function for the eigenfunctions φ_k^a and we study some properties of this object. In Sect. 6 we prove Proposition 1.6, and in Sect. 7 we prove Theorem 1.8. In Appendix A we treat the case of a Lipschitz domain with isolated conical singularities. Finally, in Appendix B we prove an estimate on the Green's function for a perturbed Laplace operator.

2. Preliminaries

We recall here some properties of the two functional spaces defined in the Introduction. As proved for example in [31, Lemma 2.1], we have the equivalent characterizations

$$\mathcal{D}_{A_a}^{1,2}(\Omega) = \left\{ u \in H_0^1(\Omega) \ : \ \frac{u}{|x-a|} \in L^2(\Omega) \right\}$$

and

$$H_{A_a}^1(\Omega) = \left\{ u \in H^1(\Omega) \ : \ \frac{u}{|x-a|} \in L^2(\Omega) \right\},$$

and moreover we have that $\mathcal{D}_{A_a}^{1,2}(\Omega)$ (respectively $H_{A_a}^1(\Omega)$) is continuously embedded in $H_0^1(\Omega)$ (respectively $H^1(\Omega)$) : there exists a constant $C > 0$ such that for every $u \in \mathcal{D}_{A_a}^{1,2}(\Omega)$ and $u \in H_{A_a}^1(\Omega)$ we have

$$\|u\|_{H_0^1(\Omega)} \le C \|u\|_{\mathcal{D}_{A_a}^{1,2}(\Omega)} \quad \text{and} \quad \|u\|_{H^1(\Omega)} \le C \|u\|_{H_{A_a}^1(\Omega)}. \tag{2.1}$$

This is proved by making use of a Hardy-type inequality, which was obtained by Laptev and Weidl [26] for functions in $\mathcal{D}_{A_a}^{1,2}(\Omega)$ and has been extended to functions in $H_{A_a}^1(\Omega)$ in [28, Lemma 7.4] (see also [29]). If Ω is simply connected and of class C^∞, there exists a constant $C > 0$ such that for every $u \in H_{A_a}^1(\Omega)$ the following holds

$$\left\| \frac{u}{|x-a|} \right\|_{L^2(\Omega)} \le C \|(i\nabla + A_a)u\|_{L^2(\Omega)}. \tag{2.2}$$

The constant C in (2.1) and (2.2) depends only on the circulation of the magnetic potential A_a and remains finite whenever the circulation is not an integer.

As a consequence of the continuous embedding, we have the following.

Lemma 2.1. *Let Im be the compact immersion of $\mathcal{D}_{A_a}^{1,2}(\Omega)$ into $(\mathcal{D}_{A_a}^{1,2}(\Omega))'$. Then, the operator $((i\nabla + A_a)^2)^{-1} \circ Im : \mathcal{D}_{A_a}^{1,2}(\Omega) \to \mathcal{D}_{A_a}^{1,2}(\Omega)$ is compact.*

As $((i\nabla + A_a)^2)^{-1}$ is also self-adjoint and positive in $(\mathcal{D}^{1,2}_{A_a}(\Omega))'$, we deduce that the spectrum of $(i\nabla + A_a)^2$ in $\mathcal{D}^{1,2}_{A_a}(\Omega)$ consists of a diverging sequences of real positive eigenvalues, having finite multiplicity.

We now recall a result proved in [18] concerning the magnetic operator (1.1) with half-integer circulation, which is fundamental for our analysis. The eigenfunctions of such operator coincide, up to a complex phase, with the antisymmetric eigenfunctions of the Laplace–Beltrami operator (real valued) on the twofold covering manifold of Ω. The complex phase is more precisely $e^{i\theta_a/2}$, θ_a being the angle of the polar coordinates centered at a. Notice that this is a multivalued function in Ω. The twofold covering manifold is the Riemann surface associated to the complex square root, so that $e^{i\theta_a/2}$ is continuous therein. We prefer to state the result in the following form, taken from [31], where a projection is applied from the twofold covering manifold into some flat, bounded domain of $\mathbb{R}^2$.

Lemma 2.2. ([18, Lemma 3.3], [31, Lemma 3.14]). *Suppose that A_a has the form* (1.2). *Then the function*

$$e^{-i\theta(y)}\varphi^a_j(y^2 + a) \quad \text{defined in } \{y \in \mathbb{C} : \ y^2 + a \in \Omega\},$$

(here θ is the angle of the polar coordinates around 0) is real valued and solves the following equation on its domain

$$-\Delta(e^{-i\theta(y)}\varphi^a_j(y^2 + a)) = \lambda^a_j p'(y)e^{-i\theta(y)}\varphi^a_j(y^2 + a), \qquad p'(y) = 4|y|^2 p(y^2 + a).$$

The projection allows us to recover the continuity of the phase thanks to the relation $e^{i\theta_a(y^2+a)/2} = e^{i\theta(y)}$. As a consequence of this result, the magnetic eigenfunctions behave, up to a complex phase, as the real laplacian eigenfunctions far from the singular point a. The behavior near the singularity is, up to a complex phase, that of the square root of an elliptic eigenfunction.

Since we are interested in the shape of the nodal lines, let us first recall the known results concerning the elliptic eigenfunctions in the plane (see also [4]).

Proposition 2.3. ([16, equation (5")], [21, Theorem 2.1]). *Let $\Omega \subset \mathbb{R}^2$ open, bounded, simply connected and of class C^∞. Let p satisfy* (1.4). *If φ_j has a zero of order $h/2$ at $0 \in \overline{\Omega}$, then h is even and we have*

$$\varphi_j(r, \theta) = r^{h/2}\left[c_h \cos\left(\frac{h}{2}\theta\right) + d_h \sin\left(\frac{h}{2}\theta\right)\right] + g(r, \theta),$$

with $x = re^{i\theta} \in \Omega$, $c^2_h + d^2_h \neq 0$ and

$$\lim_{r\to 0} \frac{\|g(r, \cdot)\|_{C^1(\partial D_r)}}{r^{h/2}} = 0. \tag{2.3}$$

In addition, there is a positive radius R such that

  (i) *if $0 \in \Omega$ then $(\varphi_j)^{-1}(\{0\}) \cap D_R(0)$ consists of h arcs of class C^∞, whose tangent lines divide the disk into h equal sectors;*
  (ii) *if $0 \in \partial\Omega$ then $(\varphi_j)^{-1}(\{0\}) \cap D_R(0) \cap \Omega$ consists of $h/2 - 1$ arcs of class C^∞, whose tangent lines divide the half disk into $h/2$ equal sectors.*

The behavior near the boundary at point (ii) above can be deduced from point (i). Indeed, since the boundary is regular, it can be locally rectified as described in Lemma 3.1 below. By performing an odd extension, we transform the boundary point into an interior point of type (i).

We summarize below the local properties of the magnetic eigenfunction near the pole. The proofs can be found in [13, Theorem 1.3], [18, Theorem 2.1] and [31, Theorem 1.5].

Proposition 2.4. *There exists an odd integer $h \geq 1$ such that φ_j^a has a zero of order $h/2$ at a. Moreover, the following asymptotic expansion holds near a*

$$\varphi_j^a(|x-a|, \theta_a) = e^{i\frac{\theta_a}{2}} |x-a|^{h/2} \left[c_h(a) \cos\left(h\frac{\theta_a}{2}\right) + d_h(a) \sin\left(h\frac{\theta_a}{2}\right) \right] + g(|x-a|, \theta_a) \tag{2.4}$$

where $x - a = |x-a|e^{i\theta_a}$, $c_h(a)^2 + d_h(a)^2 \neq 0$ and g satisfies (2.3). In addition, there is a positive radius R such that $(\varphi_j^a)^{-1}(\{0\}) \cap D_R(a)$ consists of h arcs of class C^∞. If $h \geq 3$ then the tangent lines to the arcs at the point a divide the disk into h equal sectors.

3. Equation on a Domain with Locally Rectified Boundary

The local analysis near $0 \in \partial\Omega$ is easier if the boundary is locally flat at 0, i.e. if there exists $r > 0$ such that

$$\Omega \cap D_r(0) = \{x \in D_r(0) : x_1 > 0\} =: D_r^+(0). \tag{3.1}$$

If $\partial\Omega$ is sufficiently regular, we can locally rectify it without altering the nature of the problem, as we show in the following lemma. This is not the case when $\partial\Omega$ presents an angle, as for the angular sector presented in the Introduction, see Appendix A.

Lemma 3.1. *Let $\Omega, \Omega' \subset \mathbb{C}$ be open, bounded and simply connected domains with C^∞ boundary. There exist a conformal transformation $\Phi : \Omega' \to \Omega$, $\Phi \in C^\infty(\overline{\Omega'})$ and a function $\chi \in C^\infty(\overline{\Omega'})$ such that, letting $w_k^{a'} = e^{-i\chi}(\varphi_k^a \circ \Phi)$, we have*

$$(i\nabla + A_{a'})^2 w_k^{a'} = \lambda_k^a p'(x) w_k^{a'}, \quad w_k^{a'} \in \mathcal{D}_{A_{a'}}^{1,2}(\Omega'),$$

where $\Phi(a') = a$ and p' satisfies (1.4).

Proof. The existence of a regular conformal map is ensured by the Riemann Mapping Theorem. We note that Φ is regular up to the boundary thanks to the assumption that Ω, Ω' have C^∞ boundary.

First define $v_k^{a'} = \varphi_k^a \circ \Phi$. This function solves

$$(i\nabla + B_{a'})^2 v_k^{a'} = \lambda_k^a p'(x) v_k^{a'},$$

where the weight is given by $p'(x) = |\Phi'|^2 p \circ \Phi(x)$, $B_{a'} = (\Phi'^t \cdot A_a) \circ \Phi$ and where Φ' is the matrix of the derivatives of Φ. Indeed, remember that the magnetic potential A_a is defined as $A_a = \nabla\theta_a/2$ almost everywhere in Ω. Under the conformal transformation, the magnetic potential will transform as

$$B_{a'} = \frac{\nabla(\theta_a \circ \Phi)}{2} = (\Phi'^t \cdot A_a) \circ \Phi.$$

Next, we can verify explicitly that this new magnetic potential $B_{a'}$ has the same circulation than A_a. Indeed, if we consider a closed path γ, winding once around the point a' in Ω', we obtain

$$\frac{1}{2\pi} \oint_{x \equiv \gamma} B^t_{a'}(x)|_{x=\gamma(t)} \cdot \gamma'(t) \, dt = \frac{1}{2\pi} \oint_{x \equiv \gamma} (A^t_a \cdot \Phi')(\Phi(x))|_{x=\gamma(t)} \cdot \gamma'(t) \, dt$$

$$= \frac{1}{2\pi} \oint_{y \equiv \sigma} (A^t_a \cdot \Phi')(y)|_{y=\sigma(t)} \cdot \Phi'^{-1}\sigma'(t) \, dt$$

$$= \frac{1}{2\pi} \oint_{y \equiv \sigma} A^t_a(y)|_{y=\sigma(t)} \cdot \sigma'(t) \, dt = \frac{1}{2},$$

where we used the change of variable $y = \Phi(x)$. The new path $\sigma = \Phi \circ \gamma$ is a closed path winding once around a in Ω and its derivative is $\gamma' = \Phi'^{-1} \cdot \sigma'$. Moreover, we can verify that $\nabla \times B_{a'} = 0$ everywhere in $\Omega' \backslash \{a'\}$. Then, there exists $\chi \in C^\infty(\overline{\Omega'})$ such that $B_{a'} = A_{a'} + \nabla \chi$, where $A_{a'}$ has the form given in (1.2).

Finally, we can gauge away the term $\nabla \chi$ by letting $w^{a'}_k = e^{-i\chi} v^{a'}_k$.   □

As a consequence, by simply renaming the weight $p(x)$ in Eq. (1.3), we will work in a new domain satisfying (3.1).

Remark 3.2. In fact, in what follows, it will be enough to have a weight p of class C^1. Thanks to [25, Theorem 5.2.4], to this aim it is sufficient to assume $\partial\Omega \in C^{2,\gamma}$ for some positive γ. This ensures that Φ in the previous lemma is $C^2(\overline{\Omega'})$, so that the transformed weight $p'(x) = |\Phi'|^2 p \circ \Phi(x)$ is $C^1(\overline{\Omega'})$.

In Sects. 5 and 7, we will study the behavior of the eigenfunctions φ^a_j as $a = (a_1, a_2)$ approaches $0 \in \partial\Omega$. As we will see later, the significant parameter will be the distance of a from the boundary $\partial\Omega$. Such distance is a_1 if $\partial\Omega$ is locally flat at 0 and $|a|$ is sufficiently small. We will perform the analysis in half balls centered at $(0, a_2)$; all the future estimates will be independent from a_2. To this aim, it will be useful to have some general inequalities for functions $u \in H^1_{A_a}(D^+_r(0, a_2))$ with $u = 0$ on $\{x_1 = 0\}$, $a \in D^+_r(0, a_2)$. To simplify the notation, we write

$$\pi(a) = (0, a_2),$$

π corresponding then to the projection onto the x_2-axis, so that

$$D^+_r(\pi(a)) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - a_2)^2 < r^2, \ x_1 > 0\}.$$

Lemma 3.3. (Poincaré inequality). *Let $a \in D^+_r(\pi(a))$. For all $u \in H^1_{A_a}(D^+_r(\pi(a)))$, with $u = 0$ on $\{x_1 = 0\}$, the following inequality is verified*

$$\frac{1}{r^2} \int_{D^+_r(\pi(a))} |u|^2 \, dx \leq \frac{1}{r} \int_{\partial D^+_r(\pi(a))} |u|^2 \, d\sigma + \int_{D^+_r(\pi(a))} |(i\nabla + A_a)u|^2 \, dx. \qquad (3.2)$$

Proof. By explicit calculation we see that, for every $u \in H^1_{A_a}(D^+_r(\pi(a)))$, we have

$$\nabla\cdot(|u|^2(x-\pi(a))) = 2Re\left(iu\,\overline{(i\nabla + A_b)u} \cdot (x - \pi(a))\right) + 2|u|^2 \qquad \text{a.e. } x \in D^+_r(\pi(a)).$$

$$(3.3)$$

Then

$$\int_{D_r^+(\pi(a))} |u|^2 \, \mathrm{d}x = -Re\left(i \int_{D_r^+} u \, \overline{(i\nabla + A_a)u \cdot (x - \pi(a))} \, \mathrm{d}x\right) + \frac{r}{2} \int_{\partial D_r^+(\pi(a))} |u|^2 \, \mathrm{d}\sigma$$

$$\leq \frac{1}{2} \int_{D_r^+(\pi(a))} |u|^2 \, \mathrm{d}x + \frac{r^2}{2} \int_{D_r^+(\pi(a))} |(i\nabla + A_a)u|^2 \, \mathrm{d}x$$

$$+ \frac{r}{2} \int_{\partial D_r^+(\pi(a))} |u|^2 \, \mathrm{d}\sigma,$$

where we used the Young inequalities and the fact that $(x - \pi(a)) = r\nu$ on $\partial D_r^+(\pi(a))$. This proves the statement. $\quad\square$

Similarly, we can prove the similar Poincaré inequality for all functions v in $H^1(D_r^+(0), \mathbb{R})$ with zero boundary conditions on $\{x_1 = 0\}$,

$$\frac{1}{r^2} \int_{D_r^+(0)} |v|^2 \, \mathrm{d}x \leq \frac{1}{r} \int_{\partial D_r^+(0)} |v|^2 \, \mathrm{d}\sigma + \int_{D_r^+(0)} |\nabla v|^2 \, \mathrm{d}x. \tag{3.4}$$

Lemma 3.4. *Let $a \in D_r^+(\pi(a))$. For $u \in H^1_{A_a}(D_r^+(\pi(a)))$, with $u = 0$ on $\{x_1 = 0\}$, the following holds*

$$\frac{1}{r} \int_{\partial D_r^+(\pi(a))} |u|^2 \, \mathrm{d}\sigma \leq \int_{D_r^+(\pi(a))} |(i\nabla + A_a)u|^2 \, \mathrm{d}x. \tag{3.5}$$

Proof. We will prove the following statement: for all $v \in H^1(D_r^+(\pi(a)), \mathbb{R})$, with $v = 0$ on $\{x_1 = 0\}$, we have

$$\frac{1}{r} \int_{\partial D_r^+(\pi(a))} |v|^2 \, \mathrm{d}\sigma \leq \int_{D_r^+(\pi(a))} |\nabla v|^2 \, \mathrm{d}x. \tag{3.6}$$

The lemma follows from it by taking $v = |u|$ and by applying the diamagnetic inequality. It is sufficient to prove (3.6) in the ball $D_1^+(0)$ since the general case can be recovered by performing a translation and a dilation. Let

$$\beta = \inf\left\{\int_{D_1^+(0)} |\nabla w|^2 \, \mathrm{d}x : \int_{\partial D_1^+(0)} w^2 \, \mathrm{d}\sigma = 1, \, w \in H^1(D_1^+(0), \mathbb{R}), \, w = 0 \text{ on } \{x_1 = 0\}\right\}.$$

Let w_k be a minimizing sequence. Then $\sup_k \int_{D_1^+(0)} |\nabla w_k|^2 \, \mathrm{d}x \leq C$ and by the Poincaré inequality (3.4) we have $\sup_k \|w_k\|_{H^1(D_1^+(0), \mathbb{R})} \leq C'$. Hence there exists a function $\bar{w} \in H^1(D_1^+(0), \mathbb{R})$ such that, up to a subsequence,

$$w_k \rightharpoonup \bar{w} \text{ in } H^1(D_1^+(0), \mathbb{R}) \quad \text{and} \quad w_k \to \bar{w} \text{ in } L^2(D_1^+(0), \mathbb{R})$$

by the compact injection. Using the lower semi-continuity of the H^1-norm we obtain

$$\int_{D_1^+(0)} |\nabla \bar{w}|^2 \, \mathrm{d}x \leq \liminf_{k \to +\infty} \int_{D_1^+(0)} |\nabla w_k|^2 \, \mathrm{d}x = \beta.$$

By the compact embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ (see e.g. [2]) we also have $\int_{\partial D_1^+(0)} |\bar{w}|^2 \, d\sigma$ $= 1$. Therefore $\bar{w}$ is a minimizer and solves the associated Euler-Lagrange equation

$$
\begin{cases}
-\Delta \bar{w} = 0 & D_1^+(0) \\
\dfrac{\partial \bar{w}}{\partial \nu} = \beta \bar{w} & \partial D_1^+(0) \\
\bar{w} = 0 & x_1 = 0.
\end{cases}
$$

If we decompose the boundary trace in Fourier series $\bar{w}(1, \theta) = \sum_{k \text{ odd}} \alpha_k \cos(k\theta)$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\bar{w}(r, \theta) = \sum_{k \text{ odd}} \alpha_k r^k \cos(k\theta)$. The boundary conditions imply $k\alpha_k = \beta\alpha_k$ for every $k \geq 1$. We deduce that $\bar{w}(r, \theta) = \alpha_\beta r^\beta e^{i\beta\theta}$ for some integer $\beta \geq 0$. Since $\bar{w}$ can not vanish because of the constraint, the infimum is assumed by $\beta = 1$ and $\bar{w} = \sqrt{\frac{2}{\pi}} x_1$. $\quad\square$

4. Pole Approaching the Boundary on a Nodal Line of φ_k

In this section we prove Theorem 1.5. We use some of the ideas introduced in [11, Section 6]. As we already mentioned, we modify the argument therein both in order to avoid the use of local inversion methods and in order to prove that the convergence $\lambda_k^a \to \lambda_k$ takes place from below.

We adopt the standard notation $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x \to x_0} f(x)/g(x)$ is zero, $f(x) = O(g(x))$ as $x \to x_0$ if $\limsup_{x \to x_0} |f(x)/g(x)|$ is finite, $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x \to x_0} f(x)/g(x)$ is finite and different from zero.

Lemma 4.1. *Let $\lambda > 0$ and p satisfy* (1.4). *Let $a \to 0$ so that $\frac{a}{|a|} \to e \notin \{x_1 = 0\}$. Consider the following set of equations in the parameter a*

$$
\begin{cases}
(i\nabla + A_a)^2 v = \lambda p(x)v & \text{in } D_{2|a|}^+(0) \\
v = 0 & \text{on } \{x_1 = 0\} \\
v = e^{i\frac{\theta_a}{2}} \left\{ |a|^{1+n} f + g(2|a|, \cdot) \right\} & \text{on } \partial D_{2|a|}^+(0) \cap \mathbb{R}_+^2,
\end{cases}
\tag{4.1}
$$

where $f, g(2|a|, \cdot) \in H^1(\partial D_{2|a|}^+(0) \cap \mathbb{R}_+^2)$ are real valued, vanish at $-\pi/2$ and at $\pi/2$, $f \not\equiv 0$ and, for some $n \in \mathbb{N}_0$,

$$
\lim_{|a| \to 0} \frac{\|g(2|a|, \cdot)\|_{H^1(\partial D_{2|a|}^+(0) \cap \mathbb{R}_+^2)}}{|a|^{1+n}} = 0.
$$

Then for $|a|$ sufficiently small there exists a unique solution of (4.1) *in $H_{A_a}^1(D_{2|a|}^+(0) \cap \mathbb{R}_+^2)$, which moreover satisfies*

$$
\|v\|_{L^2(\partial D_{2|a|}^+(0))} \sim |a|^{\frac{3}{2}+n}, \quad \|v\|_{L^2(D_{2|a|}^+(0))} \sim |a|^{2+n},
$$

$$
\|(i\nabla + A_a)v \cdot v\|_{L^2(\partial D_{2|a|}^+(0) \cap \mathbb{R}_+^2)} \sim |a|^{\frac{1}{2}+n}, \quad \left| \int_{\partial D_{2|a|}^+(0)} (i\nabla + A_a)v \cdot vv \, d\sigma \right| \sim |a|^{2n+2}.
$$

Proof. Notice that the boundary trace is continuous on $\partial D_{2|a|}^+(0)$. Indeed, we can choose θ_a to be discontinuous on the segment joining a with the origin, so that $e^{i\theta_a/2}$ restricted to the boundary is discontinuous only at the origin, where the boundary trace vanishes. The existence and uniqueness of the minimizer follow plainly by the fact that the quadratic form

$$\int_{D_{2|a|}^+(0)} \left[|(i\nabla + A_a)v|^2 - \lambda p(x)|v|^2 \right] dx \tag{4.2}$$

is coercive for $|a|$ sufficiently small.

The estimate on the $L^2(\partial D_{2|a|}^+(0))$-norm is immediate. In order to prove the remaining estimates we will make use of some computations contained in the proof of [11, Lemma 6.1].

Let Φ_a be a family of conformal transformations in the parameter a satisfying the properties

$$\Phi_a^{-1} \in C(\overline{D_2^+(0)}), \quad \Phi_a^{-1}(D_2^+(0)) = D_2^+(0), \quad \text{and} \quad \Phi_a^{-1}\left(\frac{a}{|a|}\right) = e.$$

Since $a/|a| \to e$, we have that Φ_a is a small perturbation of the identity. Moreover, thanks to the fact that $e \notin \{x_1 = 0\}$, we obtain that

$$|\Phi_a'| \leq C \text{ uniformly in } a. \tag{4.3}$$

Making use of such maps Φ_a, we decompose v into the sum of two functions

$$v(x) = |a|^{1+n} z_1\left(\Phi_a^{-1}\left(\frac{x}{|a|}\right)\right) + |a|^{1+n} z_2\left(\Phi_a^{-1}\left(\frac{x}{|a|}\right)\right),$$

where

$$(i\nabla + A_e)^2 z_1 = 0 \text{ in } D_2^+(0), \qquad z_1 = |a|^{-(1+n)} v(|a|\Phi_a) \text{ on } \partial D_2^+(0),$$
$$(i\nabla + A_e)^2 z_2 = \lambda |a|^2 p(|a|\Phi_a) |\Phi_a'|^2 (z_1 + z_2) \text{ in } D_2^+(0), \qquad z_2 = 0 \text{ on } \partial D_2^+(0).$$

These transformed equations have the advantage that both the domain of validity, $D_2^+(0)$, and the position of the singularity, e, are fixed. The dependence on the parameter a has been transferred to the coefficients of the equation (and is still visible on the boundary trace).

Notice that $e^{i\theta_a(|a|\Phi_a)/2} = e^{i\chi} e^{i\theta_e/2}$ for some regular function χ and let

$$\tilde{\Omega} = \left\{ y \in \mathbb{C} : y^2 + e \in D_2^+(0) \right\}$$

be the double covering, which does not depend on a. By applying Lemma 2.2 we see that the new functions

$$\tilde{z}_i(y) = e^{-i\frac{\theta_e}{2} - i\chi} z_i(y^2 + e)$$

are real valued and solve the following elliptic equations in $\tilde{\Omega}$:

$$-\Delta \tilde{z}_1 = 0 \text{ in } \tilde{\Omega}, \qquad \tilde{z}_1 = e^{-i\frac{\theta_e}{2} - i\chi} |a|^{-(1+n)} v(|a|\Phi_a(y^2 + e)) \text{ on } \partial\tilde{\Omega},$$

which is the same as the relation above (6.3) in [11], and

$$-\Delta \tilde{z}_2 = \lambda |a|^2 p(|a|\Phi_a(y^2 + e)) |\Phi_a'(y^2 + e)|^2 (\tilde{z}_1 + \tilde{z}_2) \text{ in } \tilde{\Omega}, \qquad \tilde{z}_2 = 0 \text{ on } \partial\tilde{\Omega},$$

which is the same as (6.4) in [11]. Thanks to the bound (4.3), one can proceed exactly as in the proof of [11, Lemma 6.1] and show that $\tilde{z}_2$ provides a negligible contribution in the sense that

$$\|\tilde{z}_2\|_{H^1(\tilde{\Omega})} + \|\nabla \tilde{z}_2 \cdot \nu\|_{L^2(\partial\tilde{\Omega})} \leq C|a|^2,$$

while the contribution of $\tilde{z}_1$ is the following

$$c_1 \leq \|\tilde{z}_1\|_{H^1(\tilde{\Omega})} \leq c_2, \qquad c_1 \leq \|\nabla \tilde{z}_1 \cdot \nu\|_{L^2(\partial\tilde{\Omega})} \leq c_2,$$

for positive constants c_1, c_2 not depending on a. This also provides

$$c_1 \leq \left| \int_{\partial\tilde{\Omega}} \nabla(\tilde{z}_1 + \tilde{z}_2) \cdot \nu(\tilde{z}_1 + \tilde{z}_2) \, d\sigma \right| \leq c_2.$$

By performing all the inverse changes of variables and going back to the original domain $D^+_{2|a|}(0)$, we obtain the statement. □

Lemma 4.2. *Let* $k \geq 2$ *and let* $M = M(|a|) = (m_{ij})$ *be a* $k \times k$ *hermitian matrix depending in a smooth way on the parameter* $|a|$. *Suppose that there exist* $n \in \mathbb{N}_0$ *and* $C_k > 0$ *such that, as* $|a| \to 0$, *we have*

$$m_{ii} = \lambda_i + O(|a|^2), \; i = 1, \ldots, k-1, \quad m_{kk} = \lambda_k - C_k|a|^{2n+2} + o(|a|^{2n+2}),$$
$$m_{ij} = O(|a|^2), \; i \neq j, \; i, j = 1, \ldots, k-1, \quad m_{ij} = O(|a|^{n+2}), \; i \neq j \text{ and } i = k \text{ or } j = k.$$

If $\lambda_{k-1} < \lambda_k$, *then the greatest eigenvalue of* M *satisfies*

$$\lambda_{max}(M) = \lambda_k - C_k|a|^{2n+2} + o(|a|^{2n+2}) \text{ as } a \to 0.$$

Proof. In order to estimate the eigenvalues of M, we compute the determinant of the matrix $B = M - t Id = (b_{ij})$. We have

$$\det(B) = \sum_{\sigma \in P_k} \text{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)}, \tag{4.4}$$

where σ is a permutation of the set $\{1, \ldots, k\}$, P_k is the set of all such permutations, $\text{sgn}(\sigma)$ is the sign of σ and $\sigma(i)$ is the image of the element $i \in \{1, \ldots, k\}$ under the action of σ. We recall that a fixed point of σ is an element i such that $\sigma(i) = i$. We define, for $r = 0, \ldots, k$,

$$P_{k,r} = \{\sigma \in P_k : \sigma \text{ has exactly } r \text{ fixed points}\}.$$

Notice that $P_{k,k} = \{Id\}$ and $P_{k,k-1} = \emptyset$. We split the sum in (4.4) in the following way:

$$\det(B) = \prod_{i=1}^{k} b_{ii} + \sum_{r=1}^{k-2} \sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k)=k}} \text{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)} + \sum_{r=0}^{k-2} \sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k) \neq k}} \text{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)}.$$

Due to the specific form of M we can estimate each piece as follows

$$\prod_{i=1}^{k} b_{ii} = (\lambda_k - C_k |a|^{2n+2} + o(|a|^{2n+2}) - t) \prod_{i=1}^{k-1} (\lambda_i + O(|a|^2) - t),$$

$$\sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k)=k}} \text{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)} = (\lambda_k - C_k |a|^{2n+2} - t) O\left(|a|^{2(k-r)}\right) Q_{r-1}(t, |a|^2),$$

$$r = 1, \ldots, k-2,$$

$$\sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k) \neq k}} \text{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)} = O\left(|a|^{2(n+2)}\right) O\left(|a|^{2(k-r-2)}\right) Q_r(t, |a|^2), \quad r = 0, \ldots, k-2,$$

where $Q_r(t, |a|^2)$ denotes a polynomial of degree r in the variable t, which depends on $|a|$ with terms of order $O(|a|^2)$. More explicitly, $Q_r(t, |a|^2)$ is given by the sum over any possible choice of r numbers in the set $\{1, \ldots, k-1\}$ of a product of r terms of the form $(\lambda_i + O(|a|^2) - t)$, for some $i \neq k$. We can also define $Q_{k-1}(t, |a|^2) = \prod_{i=1}^{k-1}(\lambda_i + O(|a|^2) - t)$ and remark that

$$Q_{k-1}(\lambda_k, 0) = \prod_{i=1}^{k-1}(\lambda_i - \lambda_k) \neq 0, \tag{4.5}$$

where we used the assumption that $\lambda_{k-1} < \lambda_k$.

Let $\varepsilon = |a|^2$. We rewrite the determinant in terms of ε, obtaining

$$\det(B) = (\lambda_k - C_k \varepsilon^{n+1} + o(\varepsilon^{n+1}) - t) \left\{ Q_{k-1}(t, \varepsilon) + \sum_{r=1}^{k-2} O\left(\varepsilon^{k-r}\right) Q_{r-1}(t, \varepsilon) \right\}$$

$$+ \sum_{r=0}^{k-2} O\left(\varepsilon^{n+k-r}\right) Q_r(t, \varepsilon) =: f(t, \varepsilon).$$

The assumptions of the implicit function theorem hold for $f(t, \varepsilon)$ at the point $(\lambda_k, 0)$. Indeed, $f(\lambda_k, 0) = 0$, f is at least C^{n+1} in a neighbourhood of $(\lambda_k, 0)$, and $\frac{\partial f}{\partial t}(\lambda_k, 0) = -Q_{k-1}(\lambda_k, 0) \neq 0$ thanks to (4.5). Then there exists a function $\lambda(\varepsilon) \in C^{n+1}$, defined in a neighbourhood of $\varepsilon = 0$, such that $f(\lambda(\varepsilon), \varepsilon) = 0$.

Let us first differentiate this relation with respect to ε and estimate it in $(\lambda_k, 0)$

$$\frac{\partial f}{\partial t}(\lambda_k, 0) \lambda'(0) + \frac{\partial f}{\partial \varepsilon}(\lambda_k, 0) = 0.$$

Since $n \geq 1$, $\frac{\partial f}{\partial \varepsilon}(\lambda_k, 0) = 0$ and we conclude that $\lambda'(0) = 0$. We can differentiate $n+1$ times the identity $f(\lambda(\varepsilon), \varepsilon) = 0$ and each time use the relations obtained in the previous step. We have

$$\frac{\partial f}{\partial t}(\lambda_k, 0) \lambda^{(j)}(0) + \frac{\partial^j f}{\partial \varepsilon^j}(\lambda_k, 0) = 0, \quad j = 1, \ldots, n+1.$$

Thanks to the fact that $n + k - r > n + 1$ for all $r = 0, \ldots, k - 2$, and using (4.5), we deduce

$$\frac{\partial^j f}{\partial \varepsilon^j}(\lambda_k, 0) = 0, \; j = 1, \ldots, n, \qquad \frac{\partial^{n+1} f}{\partial \varepsilon^{n+1}}(\lambda_k, 0) = -C_k(n+1)! Q_{k-1}(\lambda_k, 0) \neq 0.$$

Then

$$\lambda^{(j)}(0) = 0, \; j = 1, \ldots, n, \qquad \lambda^{(n+1)}(0) = -C_k(n+1)!$$

and we conclude that $\lambda(\varepsilon) = \lambda_k - C_k \varepsilon^{n+1} + o(\varepsilon^{n+1})$ as $\varepsilon \to 0$. $\quad\square$

Proof of Theorem 1.5. We can assume without loss of generality that $b = 0$ and moreover, by Lemma 3.1, that Ω satisfies (3.1). Let φ_k have a zero of order $h/2 \geq 2$ at 0, corresponding to

$$n = \frac{h}{2} - 1 \geq 1 \tag{4.6}$$

arcs of nodal line ending at 0. Denote by Γ any such piece of nodal line and let $a \in \Gamma$. We shall take advantage of the min-max characterization of the eigenvalues, which we exploit by constructing suitable finite dimensional spaces of competitor functions.

Step 1. Construction of the space of competitors. As shown in [11, Lemma 4.1], we can choose the discontinuity of θ_a on the piece of Γ connecting a with the origin, so that

$$\frac{\nabla \theta_a}{2} = A_a \quad \text{globally in } \Omega \backslash D^+_{|a|}(0).$$

For $i = 1, \ldots, k$ we define

$$v_i^{ext}(x) = e^{i \frac{\theta_a}{2}(x)} \varphi_i(x), \qquad x \in \Omega \backslash D^+_{2|a|}(0). \tag{4.7}$$

Since $e^{i \frac{\theta_a}{2}}$ is univalued and regular in $\Omega \backslash D^+_{|a|}(0)$, the gauge invariance implies

$$(i \nabla + A_a)^2 v_i^{ext} = \lambda_i p(x) v_i^{ext} \quad \text{in } \Omega \backslash D^+_{2|a|}(0). \tag{4.8}$$

In the interior of the small disk we take the solution of the magnetic equation in $H^1_{A_a}(D^+_{2|a|}(0))$, having the same boundary trace, that is, for $i = 1, \ldots, k$,

$$(i \nabla + A_a)^2 v_i^{int} = \lambda_i p(x) v_i^{int} \quad \text{in } D^+_{2|a|}(0), \qquad v_i^{int} = e^{i \frac{\theta_a}{2}} \varphi_i \quad \text{on } \partial D^+_{2|a|}(0). \tag{4.9}$$

By uniqueness, v_i^{int} can also be characterized as the function which achieves

$$\inf \left\{ \int_{D^+_{2|a|}(0)} \left[|(i\nabla + A_a)v|^2 - \lambda_i p(x)|v|^2 \right] dx \; : \; v \in H^1_{A_a}(D^+_{2|a|}(0)), \right.$$
$$\left. v = e^{i \frac{\theta_a}{2}} \varphi_i \text{ on } \partial D^+_{2|a|}(0) \right\}. \tag{4.10}$$

Though v_i^{int} and v_i^{ext} solve the same equation on the respective domains, the competitor functions defined as

$$v_i = \begin{cases} v_i^{int} & D^+_{2|a|}(0) \\ v_i^{ext} & \Omega \backslash D^+_{2|a|}(0) \end{cases} \tag{4.11}$$

do not solve the equation in Ω. Indeed, we have, for all $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$,

$$\int_{\Omega} \left[(i\nabla + A_a)v_i \cdot \overline{(i\nabla + A_a)\phi} - \lambda_i \, p(x)v_i \overline{\phi} \right] \mathrm{d}x$$

$$= i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v^{ext}_i - v^{int}_i) \cdot v\overline{\phi} \, \mathrm{d}\sigma, \qquad (4.12)$$

where we used the formula of integration by parts, (4.8) and (4.9).

Step 2. Estimates on the single competitor functions. By Proposition 2.3, φ_i has the following behavior on $\partial D^+_{2|a|}(0)$, for $i < k$,

$$\varphi_i|_{\partial D^+_{2|a|}(0)} = |a|c_1 \cos\theta + o(|a|), \quad \text{as } a \to 0,$$

with c_1 eventually 0, whereas for φ_k we have

$$\varphi_k|_{\partial D^+_{2|a|}(0)} = |a|^{1+n} \left(c_{1+n} \cos[(1+n)\theta] + d_{1+n} \sin[(1+n)\theta] \right) + o(|a|^{1+n}), \quad \text{as } a \to 0,$$

with $c_{1+n} \neq 0$, $d_{1+n} = 0$ if n is even and $c_{1+n} = 0$, $d_{1+n} \neq 0$ if n is odd. Since a belongs to one of the nodal lines of φ_k, Γ, and the tangents to the nodal lines divide π into equal angles, we have that $a/|a| \to e \notin \{x_1 = 0\}$ and we recover the property (4.3). Hence Lemma 4.1 applies providing the following estimates

$$\|v^{int}_i\|_{L^2(\partial D^+_{2|a|}(0))} = O(|a|^{\frac{3}{2}}), \quad \|v^{int}_i\|_{L^2(D^+_{2|a|}(0))} = O(|a|^2),$$

$$\|(i\nabla + A_a)v^{int}_i \cdot v\|_{L^2(\partial D^+_{2|a|}(0)\cap\Omega)} = O(|a|^{\frac{1}{2}}), \qquad (4.13)$$

for $i = 1, \ldots, k - 1$, and

$$\|v^{int}_k\|_{L^2(\partial D^+_{2|a|}(0))} \sim |a|^{\frac{3}{2}+n}, \quad \|v^{int}_k\|_{L^2(D^+_{2|a|}(0))} \sim |a|^{2+n}, \qquad (4.14)$$

$$\|(i\nabla + A_a)v^{int}_k \cdot v\|_{L^2(\partial D^+_{2|a|}(0)\cap\Omega)} \sim |a|^{\frac{1}{2}+n}, \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)v^{int}_k \cdot v v^{int}_k \, \mathrm{d}\sigma \sim |a|^{2n+2}. \qquad (4.15)$$

Step 3. We claim that there exists a constant $C_k > 0$ such that

$$i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v^{ext}_k - v^{int}_k) \cdot v\overline{v}_k \, \mathrm{d}\sigma = -C_k |a|^{2n+2} + o(|a|^{2n+2}). \qquad (4.16)$$

The asymptotic behavior is consequence of (4.15). Let us prove that the quantity we want to estimate is negative. To this aim, we extend the function $v^{ext}_k = e^{i\frac{\theta_a}{2}}\varphi_k$ to all Ω. Such extension is continuous in $D^+_{2|a|}(0)$, since φ_k vanishes on Γ and θ_a is regular outside Γ, and solves

$$\begin{cases} (i\nabla + A_a)^2 v^{ext}_k = \lambda_k \, p(x)v^{ext}_k & \Omega \setminus \Gamma \\ v^{ext}_k = 0 & \partial\Omega. \end{cases}$$

Since $v^{ext}_k = 0$ on Γ, we can test this equation by v^{ext}_k itself in $D^+_{2|a|}(0)$ and apply the formula of integration by parts to obtain

$$i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)v^{ext}_k \cdot v\overline{v}_k \, \mathrm{d}\sigma = -\int_{D^+_{2|a|}(0)} \left[|(i\nabla + A_a)v^{ext}_k|^2 - \lambda_k \, p(x)|v^{ext}_k|^2 \right] \mathrm{d}x.$$

On the other hand, by testing (4.9) by v_k^{int} we obtain the same expression for v_k^{int}. By subtracting the two equalities, and recalling the characterization of v_k^{int} in (4.10), we obtain that the boundary integral in (4.16) is negative.

Step 4. Estimate of the eigenvalue. Let

$$F_k = \left\{ \Phi = \sum_{i=1}^{k} \alpha_i v_i \; : \; \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k \right\} \subset \mathcal{D}^{1,2}_{A_a}(\Omega),$$

where v_i are the competitor functions defined in (4.11). By (4.13), (4.14) we have, for $i \neq j$,

$$\left| \int_\Omega p(x) v_i \bar{v}_j \, dx \right| = \left| \int_\Omega p(x) \varphi_i \bar{\varphi}_j \, dx + \int_{D^+_{2|a|}(0)} p(x) (v_i^{int} \bar{v}_j^{int} - \varphi_i \bar{\varphi}_j) \, dx \right| \le C|a|^4$$

(the last estimate improves to $|a|^{4+n}$ in case $i = k$ or $j = k$). Therefore F_k is a k-dimensional subspace of $\mathcal{D}^{1,2}_{A_a}(\Omega)$ for $|a|$ sufficiently small and we have

$$\lambda_k^a \le \sup_{\Phi \in F_k} \frac{\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)}}{\int_\Omega p(x)|\Phi|^2 \, dx}.$$

Relation (4.12) provides

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)}$$
$$= \sum_{i,j=1}^{k} \alpha_i \overline{\alpha_j} \left\{ \lambda_i \int_\Omega p(x) v_i \bar{v}_j \, dx + i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v_i^{ext} - v_i^{int}) \cdot \nu \bar{v}_j \, d\sigma \right\}.$$

Thus we can write

$$\lambda_k^a \le \sup_{\alpha \in \mathbb{C}^k} \frac{\overline{\alpha}^T M \alpha}{\overline{\alpha}^T N \alpha} = \lambda_{max}(N^{-1}M),$$

where $\overline{\alpha}^T$ denotes the transposed and the complex conjugation of the vector α, $\lambda_{max}(\cdot)$ is the largest eigenvalue of a matrix and M, N are by construction $k \times k$ hermitian matrices with entries

$$m_{ij} = \lambda_i \int_\Omega p(x) v_i \bar{v}_j \, dx + i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v_i^{ext} - v_i^{int}) \cdot \nu \bar{v}_j \, d\sigma,$$

$$n_{ij} = \int_\Omega p(x) v_i \bar{v}_j \, dx.$$

By exploiting (4.13)–(4.15), we see that M has the form in Lemma 4.2 and

$$N = \begin{pmatrix} 1 + O(|a|^4) & O(|a|^4) & O(|a|^{n+4}) \\ O(|a|^4) & \ddots & \vdots \\ & 1 + O(|a|^4) & O(|a|^{n+4}) \\ O(|a|^{n+4}) & \dots & O(|a|^{n+4}) & 1 + O(|a|^{2n+4}) \end{pmatrix}.$$

By writing $N = Id + \mathcal{E}(|a|)$ we have $N^{-1} = \sum_{j=0}^{\infty} (-1)^j \mathcal{E}(|a|)^j \sim Id - \mathcal{E}(|a|)$ as $a \to 0$, so that N^{-1} has the same form as N. Therefore $N^{-1}M$ has the same form as M and we can apply Lemma 4.2 obtaining

$$\lambda_k^a \le \lambda_k - C_k |a|^{2n+2} + o(|a|^{2n+2}).$$

The result follows recalling that $2n + 2 = h$.   □

5. Frequency Formula for Magnetic Eigenfunctions at Boundary Points

Throughout this section we assume that Ω is regular and that $p(x)$ satisfies (1.4). Given a pole $b = (b_1, b_2) \in \mathbb{R}^2$, we recall the following notation from Sect. 3

$$\pi(b) = (0, b_2), \quad D_r^+(\pi(b)) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - b_2)^2 < r^2, \ x_1 > 0\}.$$

We define a Almgren-type frequency function in $D_r^+(\pi(b))$ as follows.

Definition 5.1. Let $b \in \mathbb{C}$, $r > 0$ such that $b \in D_r^+(\pi(b))$ and $u \in H_{A_b}^1(D_r^+(\pi(b)))$ with $u = 0$ on $\{x_1 = 0\}$. Let $\lambda \in \mathbb{R}$, and $p(x)$ satisfy (1.4) in $D_r^+(\pi(b))$. We define

$$E(u, r, \pi(b), \lambda, A_b) = \int_{D_r^+(\pi(b))} \left(|(i\nabla + A_b)u|^2 - \lambda p(x)|u|^2 \right) dx, \quad (5.1)$$

$$H(u, r, \pi(b)) = \frac{1}{r} \int_{\partial D_r^+(\pi(b))} |u|^2 \, d\sigma, \quad (5.2)$$

and the frequency function

$$N(u, r, \pi(b), \lambda, A_b) = \frac{E(u, r, \pi(b), \lambda, A_b)}{H(u, r, \pi(b))}. \quad (5.3)$$

In the notation above we keep track of all the parameters involved, apart from the weight p, since we will need to let them change from section to section. The weight is not explicitly mentioned because it does not play a significant role, as long as it satisfies (1.4).

In particular, in this section we will estimate the frequency function for $u = \varphi_k^a$ and $\lambda = \lambda_k^a$. We shall omit the index k since we will work with a fixed k from now up to Sect. 7.4. By Lemma 3.1, we can assume that $\partial\Omega$ is locally flat near the origin, so that we consider the following equation

$$\begin{cases} (i\nabla + A_a)^2 \varphi^a = \lambda^a p(x) \varphi^a & D_{2r_0}^+(0) \\ \varphi^a = 0 & \{x_1 = 0\} \\ \varphi^a \in H_{A_a}^1(D_{2r_0}^+(0)). \end{cases} \quad (5.4)$$

Here r_0 is chosen such that

$$r_0 < (2\lambda^a \|p\|_{L^\infty})^{-1/2} \text{ for } |a| \text{ sufficiently small}, \quad (5.5)$$

which is possible due to the fact that p is bounded and that $\lambda^a \to \lambda$ as $a \to 0$, as recalled in Theorem 1.3. For $r < r_0$ and $|a| < r$ we have that $D_r^+(\pi(a)) \subset D_{2r_0}^+(0)$ so that, for such r and a, the frequency function for solutions of (5.4) is well defined.

5.1. Estimates on $H(\varphi^a, r, \pi(a))$. We can compute the derivative of H with respect to r similarly to the standard frequency function for non-magnetic eigenfunctions. In the following $a = (a_1, a_2)$.

Lemma 5.2. *If φ^a is a solution of* (5.4), *for $a_1 < r < r_0$ we have*

$$\frac{d}{dr} H(\varphi^a, r, \pi(a)) = -\frac{2i}{r} \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot v \overline{\varphi^a} \, d\sigma = \frac{2}{r} E(\varphi^a, r, \pi(a), \lambda^a, A_a).$$

$$(5.6)$$

Proof. By the change of variables $y = (x - \pi(a))/r$ we have

$$H(\varphi^a, r, \pi(a)) = \frac{1}{r} \int_{\partial D_r^+(\pi(a))} |\varphi^a|^2(x) \, d\sigma(x) = \int_{\partial D_1^+(0)} |\varphi^a|^2(ry + \pi(a)) \, d\sigma(y).$$

By taking the derivative with respect to r we obtain

$$\begin{aligned}
\frac{d}{dr} H(\varphi^a, r, \pi(a)) &= 2Re \int_{\partial D_1^+(0)} \nabla\varphi^a(ry + \pi(a)) \cdot y \, \overline{\varphi^a(ry + \pi(a))} \, d\sigma(y) \\
&= \frac{2}{r} Re \int_{\partial D_r^+(\pi(a))} \nabla\varphi^a \cdot v \, \overline{\varphi^a} \, d\sigma(x) \\
&= \frac{2}{r} Re \left\{ -i \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot v \overline{\varphi^a} \, d\sigma \right\},
\end{aligned}$$

where we used the fact that $Re(-i|\varphi^a|^2 A_a \cdot v) = 0$. On the other hand, by testing Eq. (5.4) by φ^a and integrating by parts, we see that

$$E(\varphi^a, r, \pi(a), \lambda^a, A_a) = -i \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot v \, \overline{\varphi^a} \, d\sigma \quad \in \mathbb{R}, \qquad (5.7)$$

which concludes the proof. □

We can prove the following estimate.

Lemma 5.3. *Let φ^a be a solution of* (5.4) *and r_0 be as in* (5.5). *If $a_1 < r_1 < r_2 < r_0$ then*

$$\frac{H(\varphi^a, r_2, \pi(a))}{H(\varphi^a, r_1, \pi(a))} \geq e^{-Cr_0^2} \left(\frac{r_2}{r_1}\right)^2. \qquad (5.8)$$

If $|a|$ is sufficiently small, we can choose $C = 4\lambda\|p\|_{L^\infty}$, where λ is the limit of λ^a as $a \to 0$.

Proof. By combining Lemmas 3.3 and 3.4 we see that

$$\frac{1}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx \leq 2 \int_{D_r^+(\pi(a))} |(i\nabla + A_a)\varphi^a|^2 \, dx. \qquad (5.9)$$

Next we apply Lemma 5.2 and, in order, the inequalities (5.9) and (3.5) in the following way

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} H(\varphi^a, r, \pi(a)) &= \frac{2}{r} \int_{D_r^+(\pi(a))} (|(i\nabla + A_a)\varphi^a|^2 - \lambda^a p(x)|\varphi^a|^2)\, \mathrm{d}x \\
&\geq \frac{2}{r}(1 - 2\lambda^a \|p\|_{L^\infty} r^2) \int_{D_r^+(\pi(a))} |(i\nabla + A_a)\varphi^a|^2 \mathrm{d}x \\
&\geq \frac{2}{r}(1 - 2\lambda^a \|p\|_{L^\infty} r^2) H(\varphi^a, r, \pi(a))).
\end{aligned}
$$

Integrating the last inequality between r_1 and r_2 we obtain

$$
\begin{aligned}
\log\left(\frac{H(\varphi^a, r_2, \pi(a))}{H(\varphi^a, r_1, \pi(a))}\right) &\geq \log\left(\frac{r_2}{r_1}\right)^2 - 2\lambda^a \|p\|_{L^\infty}(r_2^2 - r_1^2) \\
&\geq \log\left(\frac{r_2}{r_1}\right)^2 - 2\lambda^a \|p\|_{L^\infty} r_0^2.
\end{aligned}
$$

Taking the exponential of both sides and recalling that $\lambda^a \to \lambda$ we obtain the statement.
$\square$

Remark 5.4. Relation (5.9) in the previous proof shows that $E(\varphi^a, r, \pi(a), \lambda^a, A_a) \geq 0$ for $r < r_0$.

5.2. Estimates on $E(\varphi^a, r, \pi(a), \lambda^a, A_a)$. We will need the following Pohozaev-type identity for the solution φ^a of (5.4). Also compare with [13, Section 4].

Lemma 5.5. (Pohozaev-type identity). *If φ^a is the solution of (5.4), the following identity is valid for $a_1 < r < r_0$*

$$
\frac{r}{2} \int_{\partial D_r^+(\pi(a))} \left\{ |(i\nabla + A_a)\varphi^a|^2 - 2|(i\nabla + A_a)\varphi^a \cdot \nu|^2 - \lambda^a p|\varphi^a|^2 \right\} \mathrm{d}\sigma
$$
$$
+ \lambda^a \int_{D_r^+(\pi(a))} |\varphi^a|^2 \left(p + \frac{\nabla p \cdot (x - \pi(a))}{2} \right) \mathrm{d}x + M_a = 0, \tag{5.10}
$$

where

$$
M_a = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon(a)} \left\{ Re\left[(i\nabla + A_a)\varphi^a \cdot \nu \, \overline{(i\nabla + A_a)\varphi^a \cdot (x - \pi(a))} \right] \right.
$$
$$
\left. - \frac{1}{2} |(i\nabla + A_a)\varphi^a|^2 (x - \pi(a)) \cdot \nu \right\} \mathrm{d}\sigma. \tag{5.11}
$$

Proof. We test the Eq. (5.4) with the vector field $\xi = (i\nabla + A_a)\varphi^a \cdot (x - \pi(a))$ in $D_r^+(\pi(a)) \setminus D_\varepsilon(a)$. We need to remove a small ball around the singularity because $\nabla\varphi^a$ may be singular at a (it is singular in the case that φ^a has a zero of order $1/2$ at a). Multiplying by i and taking the real part we obtain

$$
Re\left\{ i \int_{D_r^+(\pi(a)) \setminus D_\varepsilon(a)} (i\nabla + A_a)^2 \varphi^a \, \overline{\xi} \, \mathrm{d}x \right\} = Re\left\{ i\lambda^a \int_{D_r^+(\pi(a)) \setminus D_\varepsilon(a)} \varphi^a p(x) \overline{\xi} \, \mathrm{d}x \right\}.
$$

$$
\tag{5.12}
$$

Similarly to (3.3), the following identity with the weight holds

$$\frac{1}{2}\nabla \cdot \left(p|\varphi^a|^2 (x - \pi(a)) \right) = |\varphi^a|^2 \left(p + \frac{\nabla p \cdot (x - \pi(a))}{2} \right) + Re\left(ip\varphi^a \overline{\xi} \right).$$

It allows to rewrite the right hand side of (5.12) as

$$\frac{\lambda^a}{2}\int_{\partial(D_r^+(\pi(a))\setminus D_\varepsilon(a))} p|\varphi^a|^2 (x - \pi(a)) \cdot \nu \, d\sigma$$

$$-\lambda^a \int_{D_r^+(\pi(a))\setminus D_\varepsilon(a)} |\varphi^a|^2 \left(p + \frac{\nabla p \cdot (x - \pi(a))}{2} \right) dx.$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$\frac{\lambda^a r}{2}\int_{\partial D_r^+(\pi(a))} p|\varphi^a|^2 \, d\sigma - \lambda^a \int_{D_r^+(\pi(a))} |\varphi^a|^2 \left(p + \frac{\nabla p \cdot (x - \pi(a))}{2} \right) dx.$$

The integral on $\partial D_\varepsilon(a)$ vanishes as $\varepsilon \to 0$ because $|\varphi^a|$ behaves at least like $\varepsilon^{1/2}$ on $\partial D_\varepsilon(a)$ by (2.4). Next we look at the left-hand side of (5.12). Integrating by parts and using the identity

$$Re\left(i(i\nabla + A_a)\varphi^a \cdot \overline{(i\nabla + A_a)\xi} \right) = \frac{1}{2}\nabla \cdot \left(|(i\nabla + A_a)\varphi^a|^2 (x - \pi(a)) \right) \quad \text{a.e. } x \in D_r^+,$$

we rewrite it as

$$Re\left\{ i \int_{D_r^+(\pi(a))\setminus D_\varepsilon(a)} (i\nabla + A_a)\varphi^a \cdot \overline{(i\nabla + A_a)\xi} \, dx - \int_{\partial(D_r^+(\pi(a))\setminus D_\varepsilon(a))} (i\nabla + A_a)\varphi^a \cdot \nu\overline{\xi} \, d\sigma \right\}$$

$$= \int_{\partial(D_r^+(\pi(a))\setminus D_\varepsilon(a))} \left\{ \frac{1}{2}|(i\nabla + A_a)\varphi^a|^2 (x - \pi(a)) \cdot \nu - Re[(i\nabla + A_a)\varphi^a \cdot \nu\overline{\xi}] \right\} d\sigma$$

$$= r \int_{\partial D_r^+(\pi(a))} \left\{ \frac{1}{2}|(i\nabla + A_a)\varphi^a|^2 - |(i\nabla + A_a)\varphi^a \cdot \nu|^2 \right\} d\sigma$$

$$+ \int_{\partial D_\varepsilon(a)} \left\{ Re[(i\nabla + A_a)\varphi^a \cdot \nu\overline{\xi}] - \frac{1}{2}|(i\nabla + A_a)\varphi^a|^2 (x - \pi(a)) \cdot \nu \right\} d\sigma.$$

By taking the limit as $\varepsilon \to 0$ and by combining the two contributions in (5.12) we obtain the result.   □

This identity allows to compute the derivative of $E(\varphi^a, r, \pi(a), \lambda^a, A_a)$ with respect to r.

Lemma 5.6. *If φ^a is a solution of* (5.4), *then for $a_1 < r < r_0$ we have*

$$\frac{d}{dr}E(\varphi^a, r, \pi(a), \lambda^a, A_a) = 2 \int_{\partial D_r^+(\pi(a))} |(i\nabla + A_a)\varphi^a \cdot \nu|^2 \, d\sigma$$

$$- \frac{\lambda^a}{r}\int_{D_r^+(\pi(a))} |\varphi^a|^2 (2p + \nabla p \cdot (x - \pi(a))) \, dx - \frac{2}{r}M_a,$$

$$\tag{5.13}$$

where M_a is defined in (5.11).

Proof. We have

$$\frac{d}{dr} E(\varphi^a, r, \pi(a), \lambda^a, A_a) = \int_{\partial D_r^+(\pi(a))} \left(|(i\nabla + A_a)\varphi^a|^2 - \lambda^a p|\varphi^a|^2 \right) d\sigma.$$

Then we use the Pohozaev identity (5.10) to conclude.   □

In what follows we will estimate the remainder M_a which appears in the derivative of E in Eq. (5.13).

Lemma 5.7. *Let* $v(y) = e^{-i\theta(y)}\varphi^a(a_1 y^2 + a)$, *defined in the set* $\{y : a_1 y^2 + a \in D_{2r_0}^+(0)\}$. *Then*

$$M_a = \pi Re \left\{ \left(\frac{\partial v(0)}{\partial y} \right)^2 \right\}. \tag{5.14}$$

Proof. First we shall prove that

$$M_a = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon(a)} Re[(i\nabla + A_a)\varphi^a \cdot v \overline{(i\nabla + A_a)\varphi^a \cdot (x - \pi(a))}] \, d\sigma = \frac{a_1 \pi}{4} (c_1^2 - d_1^2), \tag{5.15}$$

where $c_1 = c_1(a)$, $d_1 = d_1(a)$ are the coefficients appearing in the asymptotic expansion (2.4) of φ^a, and $a = (a_1, a_2)$. Indeed, by differentiating (2.4) we obtain a.e.

$$(i\nabla + A_a)\varphi^a = \frac{ie^{i\theta_a/2}}{2\sqrt{r_a}} \left(c_1 \cos \frac{\theta_a}{2} - d_1 \sin \frac{\theta_a}{2}, c_1 \sin \frac{\theta_a}{2} + d_1 \cos \frac{\theta_a}{2} \right)$$
$$+ o(r_a^{-1/2}) \quad \text{as } r_a \to 0,$$

and hence

$$|(i\nabla + A_a)\varphi^a|^2 = \frac{1}{4r_a}(c_1^2 + d_1^2) + o(r_a^{-1}) \quad \text{as } r_a \to 0.$$

Moreover notice that $x - \pi(a) = (a_1, 0) + \varepsilon(\cos\theta_a, \sin\theta_a)$ and $v = (\cos\theta_a, \sin\theta_a)$ on $\partial D_\varepsilon(a)$. Therefore

$$\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon(a)} |(i\nabla + A_a)\varphi^a|^2 (x - \pi(a)) \cdot v \, d\sigma$$
$$= \lim_{\varepsilon \to 0} \varepsilon \int_0^{2\pi} \left[\frac{1}{4\varepsilon}(c_1^2 + d_1^2) + o(\varepsilon^{-1}) \right] [a_1 \cos\theta_a + \varepsilon] \, d\theta_a = 0,$$

and we have estimated the second term in (5.11). By a direct calculation one estimates the first term in (5.11) and obtains (5.15).

Now, by changing variables in (2.4), we obtain the following expansion for v

$$v(r, \theta) = \sqrt{a_1} r (c_1 \cos\theta + d_1 \sin\theta) + o(r) \quad \text{as } r \to 0.$$

Hence we have $\frac{\partial v(0)}{\partial y} = \frac{\sqrt{a_1}}{2} (c_1 - id_1)$ and (5.14) follows by combining with (5.15).
□

From (5.15), we remark that the constant M_a is identically zero if the eigenfunction φ^a has a zero of order strictly greater than 1 at a.

Lemma 5.8. *There exists $C > 0$ not depending on a_1 such that*

$$\frac{|M_a|}{H(\varphi^a, 2a_1, \pi(a))} \leq C.$$

Proof. The quantity M_a is expressed in terms of v in Lemma 5.7. We also rewrite

$$H(\varphi^a, 2a_1, \pi(a)) = \int_\gamma v^2 |y| \, d\sigma, \tag{5.16}$$

where, letting $\Omega = \{y : a_1 y^2 + a \in D_{2a_1}^+(\pi(a))\}$, we have $\gamma = \partial \Omega$. By Lemma 2.2, v solves $-\Delta v = 4a_1^2 |y|^2 \tilde{p} \lambda^a v$ in Ω, where $\tilde{p}(y) = p(a_1 y^2 + a)$ has the same properties as $p(x)$. Since γ does not depend on a, Lemma B.1 applies, providing for a_1 sufficiently small the representation formula

$$v(x) = - \int_\gamma v(y) \, \partial_\nu G(x, y) \, d\sigma(y),$$

for $x \in \Omega$ and moreover,

$$\|\partial_{x_i} G(x, \cdot) - \partial_{x_i} \Phi(x, \cdot)\|_{W^{2,q}(\Omega)} \leq C a_1^2 \quad \text{for } 1 \leq q < 2.$$

Therefore we have, by the Hölder and traces inequalities (for the trace embedding, see for example [2, Theorem 5.36]) and the estimate above, taking for example $q = 4/3$, we have

$$|\partial_{x_i} v(0)|^2 = \left(\int_\gamma v \partial_\nu \partial_{x_i} G(0, y) \, d\sigma(y) \right)^2$$

$$\leq \int_\gamma v^2 \, d\sigma \int_\gamma |\partial_\nu \partial_{x_i} G(0, y)|^2 \, dy \leq C \int_\gamma v^2 |y| \, d\sigma.$$

Hence, by Lemma 5.7 and by (5.16), we have

$$\frac{|M_a|}{H(\varphi^a, 2a_1, \pi(a))} \leq C \frac{|\nabla v(0)|^2}{H(\varphi^a, 2a_1, \pi(a))} \leq C.$$

□

Lemma 5.9. *There exists $C > 0$ independent of a_1 such that*

$$\frac{|M_a|}{H(\varphi^a, ka_1, \pi(a))} \leq \frac{C}{k^2} \text{ for every } k > 2.$$

Proof. It is a straightforward consequence of Lemmas 5.3 and 5.8:

$$\frac{|M_a|}{H(\varphi^a, ka_1, \pi(a))} = \frac{|M_a|}{H(\varphi^a, 2a_1, \pi(a))} \cdot \frac{H(\varphi^a, 2a_1, \pi(a))}{H(\varphi^a, ka_1, \pi(a))} \leq \frac{C}{k^2}.$$

□

5.3. Estimates on $N(\varphi^a, r, \pi(a), \lambda^a, A_a)$. The function N may not be increasing, because of the remainder M_a which appears in the derivative of E in (5.13). Nonetheless, we can use the estimates proved in the previous paragraph to obtain a bound from below on the derivative of N.

Lemma 5.10. *Let φ^a be a solution of* (5.4) *and r_0 be as in* (5.5). *For $a_1 < r < r_0$ we have*

$$\frac{1}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx \le \frac{E(\varphi^a, r, \pi(a), \lambda^a, A_a) + H(\varphi^a, r, \pi(a))}{1 - Cr_0^2}. \tag{5.17}$$

If $|a|$ is sufficiently small, we can choose $C = 2\lambda \|p\|_{L^\infty}$, where λ is the limit of λ^a as $a \to 0$.

Proof. On the one hand, the Poincaré inequality (3.2) provides

$$\frac{1}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx - \lambda^a \int_{D_r^+(\pi(a))} p(x)|\varphi^a|^2 \, dx$$
$$\le E(\varphi^a, r, \pi(a), \lambda^a, A_a) + H(\varphi^a, r, \pi(a)).$$

On the other hand, if we take $r < r_0$, we obtain that

$$\frac{1 - r_0^2 \lambda^a \|p\|_{L^\infty}}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx \le \frac{1}{r^2}$$
$$\times \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx - \lambda^a \int_{D_r^+(\pi(a))} p(x)|\varphi^a|^2 \, dx.$$

The result follows by combining the previous two inequalities. $\square$

Lemma 5.11. *Let φ^a be a solution of* (5.4) *and r_0 be as in* (5.5). *For every $k > 1$, $a_1 < r_0/k$ and $ka_1 < r < r_0$ we have*

$$N(\varphi^a, r, \pi(a), \lambda^a, A_a) \le [N(\varphi^a, r_0, \pi(a), \lambda^a, A_a) + 1] \exp\left(\frac{Ce^{Cr_0^2}}{k^2} + \frac{Cr_0^2}{1 - Cr_0^2}\right) - 1, \tag{5.18}$$

with $C > 0$ independent from a_1, k, r, r_0.

Proof. Let for the moment $N = N(\varphi^a, r, \pi(a), \lambda^a, A_a)$ and analogously for H and E. We use Lemmas 5.2 and 5.6 to obtain, for $r > a_1$,

$$\frac{dN}{dr} = \frac{1}{H^2}\left\{ \frac{2}{r} \int_{\partial D_r^+(\pi(a))} |(i\nabla + A_a)\varphi^a \cdot \nu|^2 \, d\sigma \int_{\partial D_r^+(\pi(a))} |\varphi^a|^2 \, d\sigma \right.$$

$$\left. - \frac{2}{r}\left(i \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot \nu \, \overline{\varphi^a} \, d\sigma\right)^2 \right\}$$

$$- \frac{1}{r^2 H^2}\left\{ 2M_a + \lambda^a \int_{D_r^+(\pi(a))} |\varphi^a|^2(2p + \nabla p \cdot (x - \pi(a)))dx \right\} \int_{\partial D_r^+(\pi(a))} |\varphi^a|^2 \, d\sigma$$

$$\ge -\frac{1}{rH}\left\{ 2|M_a| + \lambda^a \|2p + \nabla p \cdot (x - \pi(a))\|_{L^\infty} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx \right\}. \tag{5.19}$$

In the last step we used the Schwarz inequality and the regularity assumption on p (1.4). Therefore we have

$$
\frac{\mathrm{d}}{\mathrm{d}r} \log(N+1)
$$

$$
\geq -\frac{1}{r(E+H)} \left\{ 2|M_a| + \lambda^a \|2p + \nabla p \cdot (x - \pi(a))\|_{L^\infty} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, \mathrm{d}x \right\}.
$$

(5.20)

We look at the first term in the right hand side of (5.20). By Lemma 5.3 we have

$$
\frac{H(\varphi^a, r, \pi(a))}{H(\varphi^a, ka_1, \pi(a))} \geq e^{-Cr_0^2} \frac{r^2}{(ka_1)^2}, \qquad ka_1 < r < r_0.
$$

This together with Remark 5.4 and Lemma 5.9 provides

$$
-\frac{|M_a|}{r(E+H)} \geq -\frac{|M_a|}{rH} \geq -\frac{e^{Cr_0^2}}{r^3} \frac{|M_a|(ka_1)^2}{H(\varphi^a, ka_1, \pi(a))} \geq -Ce^{Cr_0^2} \frac{a_1^2}{r^3}, \qquad ka_1 < r < r_0.
$$

Concerning the second term in the right hand side of (5.20), we apply Lemma 5.10 to obtain

$$
-\frac{1}{r(E+H)} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, \mathrm{d}x \geq -\frac{r}{1 - Cr_0^2}, \qquad ka_1 < r < r_0.
$$

Thus we have obtained

$$
\frac{\mathrm{d}}{\mathrm{d}r} \log(N+1) \geq -\frac{Ce^{Cr_0^2}a_1^2}{r^3} - \frac{Cr}{1 - Cr_0^2}, \qquad ka_1 < r < r_0.
$$

By integrating between r and r_0 we arrive at

$$
\log \frac{N(\varphi^a, r_0, \pi(a), \lambda^a, A_a) + 1}{N(\varphi^a, r, \pi(a), \lambda^a, A_a) + 1} \geq Ca_1^2 \left(\frac{e^{Cr_0^2}}{r_0^2} - \frac{e^{Cr_0^2}}{r^2} \right) - \frac{C}{1 - Cr_0^2}(r_0^2 - r^2)
$$

$$
\geq -Ce^{Cr_0^2} \frac{a_1^2}{r^2} - \frac{Cr_0^2}{1 - Cr_0^2} \geq -\frac{Ce^{Cr_0^2}}{k^2} - \frac{Cr_0^2}{1 - Cr_0^2},
$$

for $ka_1 < r < r_0$. The statement follows by exponentiating both terms. $\quad\square$

6. Proof of Proposition 1.6

Proof of Proposition 1.6. **Step 1.** Suppose by contradiction that there are two solutions ψ and $\tilde{\psi}$ to (1.7) in $H^1_{A_e, \mathrm{loc}}(\mathbb{R}^2_+)$, which do not differ by a multiplicative constant. By Proposition 2.4 we have

$$
\psi(|x - e|, \theta_e) = e^{i\frac{\theta_e}{2}} \sqrt{|x - e|} \left(c_1 \cos \frac{\theta_e}{2} + d_1 \sin \frac{\theta_e}{2} \right) + o(\sqrt{|x - e|}),
$$

$$
\tilde{\psi}(|x - e|, \theta_e) = e^{i\frac{\theta_e}{2}} \sqrt{|x - e|} \left(\tilde{c}_1 \cos \frac{\theta_e}{2} + \tilde{d}_1 \sin \frac{\theta_e}{2} \right) + o(\sqrt{|x - e|}),
$$

as $|x - e| \to 0$, $e = (1, 0)$. Suppose first that $c_1^2 + d_1^2 \neq 0$. We consider the linear combination $t\psi + \tilde{\psi}$ that we can write, thanks to the expressions above,

$$(t\psi + \tilde{\psi})(|x - e|, \theta_e) = e^{i\frac{\theta_e}{2}} \sqrt{|x - e|} \left[(tc_1 + \tilde{c}_1) \cos \frac{\theta_e}{2} + (td_1 + \tilde{d}_1) \sin \frac{\theta_e}{2} \right]$$
$$+ o(\sqrt{|x - e|}). \tag{6.1}$$

The parameter t is chosen in such a way that

$$M := \pi \frac{(tc_1 + \tilde{c}_1)^2 - (td_1 + \tilde{d}_1)^2}{4} = 0,$$

where M is the constant associated to $t\psi + \tilde{\psi}$, see (5.15). More explicitly we have $t = (\tilde{d}_1 - \tilde{c}_1)/(c_1 - d_1)$ if $c_1 \neq d_1$ and $t = -(\tilde{c}_1 + \tilde{d}_1)/(c_1 + d_1)$ if $c_1 \neq -d_1$. Exactly as in (5.19) we have for $r > 1$

$$\frac{\mathrm{d}}{\mathrm{d}r} N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \geq -\frac{2M}{r H(t\psi + \tilde{\psi}, r, 0)} = 0, \tag{6.2}$$

thanks to our choice of t, so that $N(t\psi + \tilde{\psi}, \cdot, 0, 0, A_e)$ is increasing.

We claim that

$$\lim_{r \to \infty} N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \leq 1. \tag{6.3}$$

Suppose by contradiction that there exist δ, $R_\delta > 0$ such that $N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \geq 1 + \delta$ for every $r > R_\delta$. Then, since $t\psi + \tilde{\psi}$ solves the equation, proceeding as in (5.6), we find

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(t\psi + \tilde{\psi}, r, 0) = \frac{2}{r} N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \geq \frac{2}{r}(1 + \delta), \qquad r > R_\delta. \tag{6.4}$$

Integrating between R_δ and r we obtain

$$H(t\psi + \tilde{\psi}, r, 0) \geq Cr^{2(1+\delta)}, \qquad r > R_\delta. \tag{6.5}$$

On the other hand, by assumption (1.8) we have (by eventually taking a larger R_δ)

$$N(\psi, r, 0, 0, A_e) < 1 + \frac{\delta}{2}, \quad N(\tilde{\psi}, r, 0, 0, A_e) < 1 + \frac{\delta}{2} \qquad r > R_\delta.$$

Proceeding as above, this implies $H(\psi, r, 0) + H(\tilde{\psi}, r, 0) \leq Cr^{2(1+\delta/2)}$ for $r > R_\delta$. Hence, by the Young inequality, we obtain

$$H(t\psi + \tilde{\psi}, r, 0) \leq 2 \left[H(t\psi, r, 0) + H(\tilde{\psi}, r, 0) \right] \leq Cr^{2(1+\delta/2)}, \qquad r > R_\delta,$$

which contradicts (6.5). Hence (6.3) is proved.

On the other hand, it is not difficult to see that, since $t\psi + \tilde{\psi}$ vanishes on $\{x_1 = 0\}$ but is not identically zero, we must have

$$N(t\psi + \tilde{\psi}, 0, 0, 0, A_e) \geq 1. \tag{6.6}$$

Indeed, suppose by contradiction that there exist $\varepsilon > 0$, $r_\varepsilon < 1$ such that

$$N(t\psi + \tilde{\psi}, r, 0, 0, A_e) < 1 - \varepsilon, \quad r < r_\varepsilon.$$

Using this inequality as we did in (6.4), and then integrating between r and r_ε, we arrive at

$$\frac{H(t\psi + \tilde\psi, r_\varepsilon, 0)}{H(t\psi + \tilde\psi, r, 0)} \leq \left(\frac{r_\varepsilon}{r}\right)^{2-2\varepsilon}.$$

Conversely, Lemmas 5.2 and 3.4 provide

$$\frac{d}{dr} H(t\psi + \tilde\psi, r, 0) = \frac{2}{r} \int_{D_r^+(0)} |(i\nabla + A_e)(t\psi + \tilde\psi)|^2 \, dx \geq \frac{2}{r} H(t\psi + \tilde\psi, r, 0),$$

and hence

$$\frac{H(t\psi + \tilde\psi, r_\varepsilon, 0)}{H(t\psi + \tilde\psi, r, 0)} \geq \left(\frac{r_\varepsilon}{r}\right)^2,$$

which contradicts the previous inequality for $r < r_\varepsilon$.

We conclude from (6.2), (6.3) and (6.6) that $N(t\psi + \tilde\psi, r, 0, 0, A_e) \equiv 1$ and, in turn, that $t\psi + \tilde\psi = e^{i\theta_e/2} r g(\theta)$, for some function g depending only on the angle. This contradicts the asymptotic behavior (6.1) of $t\psi + \tilde\psi$ at e. We have obtained uniqueness up to a multiplicative constant in case $c_1^2 + d_1^2 \neq 0$. If $c_1 = d_1 = 0$ then all the previous considerations apply with $\tilde\psi$ in place of $t\psi + \tilde\psi$ and we still obtain a contradiction.

Step 2. We will use some ideas in [14], in particular Lemmas 2.4 and 2.9 (see also [1]). Let $Q_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \ x_2 > 0\}$ and let $\Gamma_1 = \{(x_1, 0) \in \mathbb{R}^2 : 0 < x_1 < 1\}$. We consider the following minimization problem

$$\frac{\beta}{2} = \inf \left\{ \int_{Q_1} |\nabla u|^2 \, dx : u \in \mathcal{D}^{1,2}(Q_1), \ u = 0 \text{ on } \{x_1 = 0\}, \ u = -x_1 \text{ on } \Gamma_1 \right\},$$
(6.7)

where we denote by $\mathcal{D}^{1,2}(Q_1)$ the closure of $C_0^\infty(Q_1)$ with respect to $\|\nabla u\|_{L^2(Q_1)}$. By standard variational methods the infimum is achieved by a unique function $w \in \mathcal{D}^{1,2}(Q_1)$ (see for example [32, Theorem 8.4]). Due to the symmetries of the problem, we can extend w to $\mathbb{R}_+^2$ in such a way that $w(x_1, -x_2) = w(x_1, x_2)$ and moreover w satisfies the following properties

$$-\Delta w = 0 \text{ in } \mathbb{R}_+^2 \backslash \Gamma_1, \ w = 0 \text{ on } \{x_1 = 0\}, \ w = -x_1 \text{ on } \Gamma_1, \quad \int_{\mathbb{R}_+^2} |\nabla w|^2 \, dx = \beta < \infty.$$

By the maximum principle we have $w < 0$ in $\mathbb{R}_+^2$. One can check that

$$\tilde\psi = e^{i\theta_e/2}(x_1 + w)$$

solves (1.7), by passing to the double covering as in Lemma 2.2. We aim at showing that $\psi = \tilde\psi$; thanks to step 1, it will be sufficient to prove that $\tilde\psi$ satisfies (1.8). Let $\tilde w$ be the Kelvin transform of w, that is $\tilde w(y) = w(y/|y|^2)$ for $|y| < 1$. Because w is harmonic outside of $D_1^+(0)$, $\tilde w$ is harmonic in $D_1^+(0)$ with zero boundary conditions on $\{y_1 = 0\}$. Moreover, $\int_{D_1^+(0)} |\nabla \tilde w|^2 \, dx = \int_{\mathbb{R}_+^2 \backslash D_1^+(0)} |\nabla w|^2 \, dx < \beta$, then $\tilde w$ has finite energy. Therefore $\tilde w$ is analytic in $D_1^+(0)$ and admits the following expansion in $D_1^+(0)$

$$\tilde w(y) = \sum_{n=1}^\infty Re(\tilde b_n y^n), \ \tilde b_n \in \mathbb{C}, \quad \text{so that} \quad w(x) = \sum_{n=1}^\infty Re\left(\tilde b_n \frac{x^n}{|x|^{2n}}\right) \quad \text{for } |x| > 1.$$

By passing to polar coordinates and taking into account the symmetries of w, we find

$$w(r, \theta) = \sum_{n \text{ odd}} \frac{b_n}{r^n} \cos(n\theta), \quad r > 1, \quad b_n \in \mathbb{R}, \text{ with } b_1 < 0. \tag{6.8}$$

Hence $w(r, \theta) = b_1 \cos \theta / r + O(r^{-3})$ as $r \to \infty$, and an explicit calculation provides

$$\lim_{r \to \infty} N(\tilde{\psi}, r, 0, 0, A_e) = 1 \quad \text{and hence} \quad \psi = \tilde{\psi} = e^{i\theta_e/2}(x_1 + w).$$

To conclude the proof of point (ii) it remains to show that $b_1 = -\beta/\pi$. By testing the equation $-\Delta w = 0$ in $\mathbb{R}_+^2 \backslash \Gamma_1$ by w we deduce

$$\beta = \int_{\mathbb{R}_+^2} |\nabla w|^2 \, dx = -2 \int_{\Gamma_1} x_1 \nabla w \cdot \nu \, d\sigma. \tag{6.9}$$

On the other hand, by testing the equation for w by x_1 in $D_R^+(0)$, $R > 1$, and the equation $\Delta x_1 = 0$ by w in $D_R^+(0)$ and subtracting we obtain

$$\int_{\partial D_R^+(0)} (w \nabla x_1 - x_1 \nabla w) \cdot \nu \, d\sigma - 2 \int_{\Gamma_1} x_1 \nabla w \cdot \nu \, d\sigma = 0. \tag{6.10}$$

By combining (6.8)–(6.10) we obtain

$$\begin{aligned}
\beta &= \lim_{R \to \infty} \int_{\partial D_R^+(0)} (x_1 \nabla w - w \nabla x_1) \cdot \nu \, d\sigma \\
&= \lim_{R \to \infty} \left\{ -\sum_{n \text{ odd}} \frac{(n+1)b_n}{R^{n-1}} \int_{-\pi/2}^{\pi/2} \cos(n\theta) \cos \theta \, d\theta \right\} \\
&= -2b_1 \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta = -\pi b_1
\end{aligned}$$

which concludes the proof. $\quad \square$

We can interpret β as the cost, in terms of energy, needed to impose that w vanishes on Γ_1, or equivalently as the energy cost of the nodal line of ψ.

7. Pole Approaching the Boundary not on a Nodal Line of φ_k

Let φ^a be a solution of (5.4). In this section we treat the case when $a \to 0$ and φ has a zero of order 1 at 0 (no nodal lines). In this case, if $\pi(a) = (0, a_2)$ is sufficiently close to 0, then φ has a zero of order 1 also at $\pi(a)$: there exists $\bar{a}_2 > 0$ such that, for $|\pi(a)| < \bar{a}_2$, we have

$$\begin{aligned}
\varphi(|x - \pi(a)|, \theta_{\pi(a)}) &= |x - \pi(a)| \\
&\quad \left(c_1(\pi(a)) \cos \theta_{\pi(a)} + d_1(\pi(a)) \sin \theta_{\pi(a)} \right) + O(|x - \pi(a)|^2),
\end{aligned} \tag{7.1}$$

as $|x - \pi(a)| \to 0$, with $c_1(\pi(a))^2 + d_1(\pi(a))^2 \neq 0$ and $x - \pi(a) = |x - \pi(a)|(\cos \theta_{\pi(a)}, \sin \theta_{\pi(a)})$. In the following, we keep the notation used in Sect. 5.

7.1. Estimates on the frequency function.

Lemma 7.1. *Let φ^a be a solution of* (5.4) *and suppose that φ has a zero of order 1 at* 0. *Let $|\pi(a)| = |a_2| < \bar{a}_2$ so that* (7.1) *holds. For every $\varepsilon > 0$, there exists $\tilde{r}_\varepsilon > 0$ such that for all $r_\varepsilon \leq \tilde{r}_\varepsilon$ there exists $\bar{a}_{1,\varepsilon,r_\varepsilon} > 0$ such that*

$$1 \leq N(\varphi^a, r_\varepsilon, \pi(a), \lambda^a, A_a) \leq 1 + \frac{\varepsilon}{2} \quad \text{for all } a_1 < \bar{a}_{1,\varepsilon,r_\varepsilon}. \tag{7.2}$$

Proof. The bound from below can be proved as in (6.6). Let us concentrate on the bound from above. Relation (7.1) implies that $1 \leq N(\varphi, r, \pi(a), \lambda, 0) \leq 1 + O(r)$ as $r \to 0$ (see for example [15, Corollary 2.2.4]). Let $\tilde{r}_\varepsilon$ be such that $N(\varphi, \tilde{r}_\varepsilon, \pi(a), \lambda, 0) \leq 1 + \varepsilon/8$. By the monotonicity property of the Almgren function for the eigenfunctions of the Laplacian (see for example [15, Corollary 3.1.2]), we have that $N(\varphi, r, \pi(a), \lambda, 0) \leq 1 + \varepsilon/4$ for every $r \leq \tilde{r}_\varepsilon$. Fix $0 < \varepsilon < \tilde{r}_\varepsilon$. Since we know from [11, Remark 4.4] that

$$\lambda^a \to \lambda \quad \text{and} \quad \|e^{-i\theta_a/2}\varphi^a - \varphi\|_{H^1(\Omega)} \to 0, \quad \text{as } a \to 0,$$

we deduce $|N(\varphi^a, r_\varepsilon, \pi(a), \lambda^a, A_a) - N(\varphi, r_\varepsilon, \pi(a), \lambda, 0)| \leq \varepsilon/4$ for $|\pi(a)| < \bar{a}_2$ and $a_1 < \bar{a}_{1,\varepsilon,r_\varepsilon}$. $\quad\square$

So far we have obtained an estimate on N for a fixed radius r_ε. Since N is not increasing, this is not sufficient to obtain the estimate for $r \to 0$. Nonetheless, we can provide a bound on N for r sufficiently far from the singularity. This is done by exploiting the estimates proved in the Sect. 5.

Lemma 7.2. *Let φ^a be a solution of* (5.4) *and suppose that φ has a zero of order 1 at 0. Let $|\pi(a)| < \bar{a}_2$ so that* (7.1) *holds. For every $\varepsilon > 0$ there exist $r_\varepsilon, \bar{a}_{1,\varepsilon} > 0$ and $k_\varepsilon > 1$ such that*

$$N(\varphi^a, r, \pi(a), \lambda^a, A_a) \leq 1 + \varepsilon \tag{7.3}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ and for every $k_\varepsilon a_1 < r < r_\varepsilon$, and

$$\frac{H(\varphi^a, r_2, \pi(a))}{H(\varphi^a, r_1, \pi(a))} \leq \left(\frac{r_2}{r_1}\right)^{2(1+\varepsilon)}. \tag{7.4}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ and $k_\varepsilon a_1 < r_1 < r_2 < r_\varepsilon$.

Proof. To prove the first inequality we combine the previous lemma with Lemma 5.11. In Lemma 5.11 we choose $r_0 = r_\varepsilon < \tilde{r}_\varepsilon$. For every $k > 1$, $a_1 < \min\{r_\varepsilon/k, \bar{a}_{1,\varepsilon,r_\varepsilon}\}$ and $ka_1 < r < r_\varepsilon$ we have

$$N(\varphi^a, r, \pi(a), \lambda^a, A_a) \leq \left(2 + \frac{\varepsilon}{2}\right) \exp\left(\frac{Ce^{Cr_\varepsilon^2}}{k^2} + \frac{2\lambda^a r_\varepsilon^2}{1 - \lambda^a r_\varepsilon^2}\right) - 1.$$

We can impose that the right hand side above is less than $1 + \varepsilon$ by choosing r_ε sufficiently small and $k = k_\varepsilon$ sufficiently large. Then we let $\bar{a}_{1,\varepsilon} < \min\{r_\varepsilon/k_\varepsilon, \bar{a}_{1,\varepsilon,r_\varepsilon}\}$.

Let us look at the second inequality. We deduce from Lemma 5.2 and from (7.3) that

$$\frac{d}{dr} \log H(\varphi^a, r, \pi(a)) = \frac{2}{r} N(\varphi^a, r, \pi(a), \lambda^a, A_a) \leq \frac{2(1+\varepsilon)}{r}$$

for $a_1 < \bar{a}_{1,\varepsilon}$ and $k_\varepsilon a_1 < r < r_\varepsilon$. Integrating between r_1 and r_2 we obtain the result. $\quad\square$

In Lemma 5.3, we obtained a superior bound on the function $H(\varphi^a, r, \pi(a))$. In the case where r is sufficiently far from the singularity, we can obtain an inferior bound on $H(\varphi^a, r, \pi(a))$, which can be improved with respect to (7.4). This is the object of the following lemma.

Lemma 7.3. *Let φ^a be a solution of* (5.4) *and suppose that φ has a zero of order 1 at 0. Let $|\pi(a)| < \bar{a}_2$ so that* (7.1) *holds.*

For every $K > \sqrt{\beta/\pi}$ (β defined in (6.7)*) there exists $\bar{a}_1 > 0$ (depending on K) and $C > 0$ such that*

$$H(\varphi^a, Ka_1, \pi(a)) \geq C(Ka_1)^2 \quad \text{for every } a_1 < \bar{a}_1.$$

Proof. We consider the function ψ which has been introduced in the statement of Proposition 1.6 (with abuse of notation we divide by the multiplicative constant C which is not relevant in this context). The rescaled function

$$\Phi(x) = a_1 \psi\left(\frac{x - \pi(a)}{a_1}\right)$$

satisfies

$$\begin{cases} (i\nabla + A_a)^2 \Phi = 0 & \mathbb{R}^2_+ \\ \Phi = 0 & \{x_1 = 0\}, \end{cases}$$

and the following expansion, where $x - \pi(a) = \rho\,(\cos\theta_{\pi(a)}, \sin\theta_{\pi(a)})$:

$$\Phi(\rho, \theta_{\pi(a)})$$
$$= e^{i\frac{\theta_a}{2}}\left(\rho\cos\theta_{\pi(a)} - \frac{\beta}{\pi}a_1^2\frac{\cos\theta_{\pi(a)}}{\rho} + \sum_{\substack{n\geq 3 \\ n \text{ odd}}} b_n a_1^{n+1}\frac{\cos(n\theta_{\pi(a)})}{\rho^n}\right) \quad \text{for } \rho > a_1.$$

$$(7.5)$$

By testing the equation satisfied by φ^a by Φ in $D_r^+(\pi(a))$ ($r > a_1$), we obtain:

$$\lambda^a\int_{D_r^+(\pi(a))} p(x)\varphi^a\overline{\Phi}\,dx = i\int_{\partial D_r^+(\pi(a))}\left\{(i\nabla + A_a)\varphi^a \cdot \nu\,\overline{\Phi} + \varphi^a\,\overline{(i\nabla + A_a)\Phi \cdot \nu}\right\}d\sigma.$$

$$(7.6)$$

Fix $K > \sqrt{\beta/\pi}$. For $\rho > a_1$ we also define

$$\Gamma(\rho, \theta_{\pi(a)}) = e^{i\frac{\theta_a}{2}}\left\{\left(K^2 - \frac{\beta}{\pi}\right)a_1^2\frac{\cos\theta_{\pi(a)}}{\rho} + \sum_{\substack{n\geq 3 \\ n \text{ odd}}} b_n a_1^{n+1}\frac{\cos(n\theta_{\pi(a)})}{\rho^n}\right\}, \quad (7.7)$$

so that

$$\begin{cases} (i\nabla + A_a)^2\Gamma = 0 & \mathbb{R}^2_+\setminus D_{a_1}^+(\pi(a)) \\ \Gamma = 0 & \{x_1 = 0\} \\ \Gamma = \Phi & \partial D_{Ka_1}^+(\pi(a)). \end{cases}$$

By testing the equation satisfied by φ^a by Γ in an annulus $(D_R^+ \setminus D_r^+)(\pi(a))$ $(R > r > a_1)$, we obtain:

$$\lambda^a \int_{(D_R^+ \setminus D_r^+)(\pi(a))} p(x)\varphi^a \overline{\Gamma} \, dx$$

$$= i \int_{\partial(D_R^+ \setminus D_r^+)(\pi(a))} \left\{ (i\nabla + A_a)\varphi^a \cdot \nu \, \overline{\Gamma} + \varphi^a \overline{(i\nabla + A_a)\Gamma \cdot \nu} \right\} \, d\sigma. \quad (7.8)$$

In Eqs. (7.6) and (7.8) we choose $r = Ka_1$ and $R > Ka_1$ to be fixed later. Adding the two equations we obtain

$$i \int_{\partial D_{Ka_1}^+(\pi(a))} \varphi^a \left\{ \overline{(i\nabla + A_a)\Phi \cdot \nu} - \overline{(i\nabla + A_a)\Gamma \cdot \nu} \right\} \, d\sigma$$

$$= \lambda^a \int_{D_{Ka_1}^+(\pi(a))} p(x)\varphi^a \overline{\Phi} \, dx + \lambda^a \int_{(D_R^+ \setminus D_{Ka_1}^+)(\pi(a))} p(x)\varphi^a \overline{\Gamma} \, dx$$

$$- i \int_{\partial D_R^+(\pi(a))} \left\{ (i\nabla + A_a)\varphi^a \cdot \nu \overline{\Gamma} + \varphi^a \overline{(i\nabla + A_a)\Gamma \cdot \nu} \right\} \, d\sigma. \quad (7.9)$$

Noticing that

$$(i\nabla + A_a)\Phi \cdot \nu \, |_{\partial D_{Ka_1}^+(\pi(a))} = i e^{i\frac{\theta_a}{2}} \left\{ \left(1 + \frac{\beta}{\pi K^2}\right) \cos\theta_{\pi(a)} - \sum_{\substack{n \geq 3 \\ n \text{ odd}}} \frac{nb_n}{K^{n+1}} \cos(n\theta_{\pi(a)}) \right\},$$

$$(i\nabla + A_a)\Gamma \cdot \nu \, |_{\partial D_{Ka_1}^+(\pi(a))} = -i e^{i\frac{\theta_a}{2}} \left\{ \left(1 - \frac{\beta}{\pi K^2}\right) \cos\theta_{\pi(a)} + \sum_{\substack{n \geq 3 \\ n \text{ odd}}} \frac{nb_n}{K^{n+1}} \cos(n\theta_{\pi(a)}) \right\},$$

we can estimate the left hand side of (7.9) from above as follows

$$\left| i \int_{\partial D_{Ka_1}^+(\pi(a))} \varphi^a \left\{ \overline{(i\nabla + A_a)\Phi \cdot \nu} - \overline{(i\nabla + A_a)\Gamma \cdot \nu} \right\} \, d\sigma \right|$$

$$= \left| \int_{\partial D_{Ka_1}^+(\pi(a))} \varphi^a e^{-i\frac{\theta_a}{2}} 2\cos\theta_{\pi(a)} \, d\sigma \right|$$

$$\leq 2\|\varphi^a\|_{L^2(\partial D_{Ka_1}^+(\pi(a)))} \|\cos\theta_{\pi(a)}\|_{L^2(\partial D_{Ka_1}^+(\pi(a)))} = Ka_1\sqrt{2\pi H(\varphi^a, Ka_1, \pi(a))}. \quad (7.10)$$

In what follows we will estimate the right hand side of (7.9). To this aim, recall that for every $r > 0$ it holds

$$\|e^{-i\frac{\theta_a}{2}}\varphi^a - \varphi\|_{C^\infty(\Omega \setminus D_r^+(0))} \to 0 \quad \text{as } a \to 0.$$

Moreover, φ satisfies (7.1). Hence we have

$$\varphi^a \, |_{\partial D_\rho^+(\pi(a))} = e^{i\frac{\theta_a}{2}} c\rho \cos\theta_{\pi(a)} + h(\rho, \theta_{\pi(a)}) + o_{a_1}(1), \quad \text{for every } \rho > a_1, \quad (7.11)$$

where $c \in \mathbb{R}$ and h satisfies (see (2.3))

$$\lim_{\rho \to 0} \frac{\|h(\rho, \cdot)\|_{C^1(\partial D_\rho^+(\pi(a)))}}{\rho} = 0. \tag{7.12}$$

Let's first look at the boundary term in the right hand side of (7.9). Taking into account that R is fixed and $a_1 \to 0$, we have

$$(i\nabla + A_a)\varphi^a \cdot \nu \, \overline{\Gamma} \,|_{\partial D_R^+(\pi(a))}$$
$$= i\left(K^2 - \frac{\beta}{\pi}\right)\frac{a_1^2}{R}\left\{c \cos^2 \theta_{\pi(a)} + \frac{\partial h}{\partial \rho}(R, \theta_{\pi(a)}) \cos \theta_{\pi(a)}\right\} + o(a_1^2),$$
$$\varphi^a \, \overline{(i\nabla + A_a)\Gamma \cdot \nu} \,|_{\partial D_R^+(\pi(a))}$$
$$= i\left(K^2 - \frac{\beta}{\pi}\right)\frac{a_1^2}{R}\left\{c \cos^2 \theta_{\pi(a)} + \frac{h(R, \theta_{\pi(a)})}{R} \cos \theta_{\pi(a)}\right\} + o(a_1^2),$$

so that

$$-i\int_{\partial D_R^+(\pi(a))}\left\{(i\nabla + A_a)\varphi^a \cdot \nu\overline{\Gamma} + \varphi^a\overline{(i\nabla + A_a)\Gamma \cdot \nu}\right\} d\sigma = c(\pi K^2 - \beta)a_1^2$$
$$+ \left(K^2 - \frac{\beta}{\pi}\right)\frac{a_1^2}{R}\int_{\partial D_R^+(\pi(a))}\left(\frac{h(R, \theta_{\pi(a)})}{R} + \frac{\partial h}{\partial \rho}(R, \theta_{\pi(a)})\right)\cos \theta_{\pi(a)} \, d\sigma + o(a_1^2)$$
$$\geq c(\pi K^2 - \beta)a_1^2 - Ca_1^2\left\|\frac{h(R, \cdot)}{R} + \frac{\partial h}{\partial \rho}(R, \cdot)\right\|_{L^\infty(\partial D_R^+(\pi(a)))} + o(a_1^2) \geq C'K^2 a_1^2, \tag{7.13}$$

for suitable C', $R > 0$ and a_1 sufficiently small, thanks to (7.12).

Concerning the integral in the annulus in (7.9), we replace (7.7) and (7.11) to obtain

$$\left|\int_{(D_R^+ \setminus D_{Ka_1}^+)(\pi(a))} p(x)\varphi^a\overline{\Gamma} \, dx\right| \leq \|p\|_{L^\infty}\left|c\frac{\pi}{4}\left(K^2 - \frac{\beta}{\pi}\right)a_1^2 R^2\right|$$
$$+ \|p\|_{L^\infty}\|h\|_{L^\infty}\int_{Ka_1}^R\int_{\partial D_\rho^+(\pi(a))}\left|\sum_{\substack{n \geq 3 \\ n \text{ odd}}} b_n a_1^{n+1}\frac{\cos(n\theta_{\pi(a)})}{\rho^n}\right| d\sigma \, d\rho + o(a_1^2)$$
$$\leq C\left\{a_1^2 K^2 R^2 + \sum_{\substack{n \geq 3 \\ n \text{ odd}}}|b_n|a_1^{n+1}\left|\frac{1}{R^{n-2}} - \frac{1}{(Ka_1)^{n-2}}\right|\right\} + o(a_1^2) \leq Ca_1^2 K^2 R^2 + o(a_1^2), \tag{7.14}$$

since R, K are fixed while $a_1 \to 0$.

In order to estimate the last term, we apply Lemma 3.3, the equation satisfied by Φ and the expansion (7.5) with $\rho = Ka_1 > a_1$, as follows

$$\|\Phi\|_{L^2(D_{Ka_1}^+(\pi(a)))}^2 \leq Ka_1\int_{\partial D_{Ka_1}^+(\pi(a))}|\Phi|^2 \, d\sigma + (Ka_1)^2\int_{D_{Ka_1}^+(\pi(a))}|(i\nabla + A_a)\Phi|^2 \, dx$$
$$= Ka_1\int_{\partial D_{Ka_1}^+(\pi(a))}|\Phi|^2 \, d\sigma - i(Ka_1)^2\int_{\partial D_{Ka_1}^+(\pi(a))}(i\nabla + A_a)\Phi \cdot \nu\overline{\Phi} \, d\sigma = O(a_1^4).$$

In a similar way

$$\|\varphi^a\|^2_{L^2(D^+_{Ka_1}(\pi(a)))}$$

$$\leq Ka_1 \int_{\partial D^+_{Ka_1}(\pi(a))} |\varphi^a|^2 \, d\sigma + (Ka_1)^2 \left\{ \lambda^a \int_{D^+_{Ka_1}(\pi(a))} p(x)|\varphi^a|^2 \, dx \right.$$

$$\left. -i \int_{\partial D^+_{Ka_1}(\pi(a))} (i\nabla + A_a)\varphi^a \cdot \nu \overline{\varphi^a} \, d\sigma \right\},$$

so that, using (7.11),

$$(1 - \lambda^a \|p\|_{L^\infty}(Ka_1)^2)\|\varphi^a\|^2_{L^2(D^+_{Ka_1}(\pi(a)))} = O(a_1^4).$$

The Hölder inequality provides

$$\left| \int_{D^+_{Ka_1}(\pi(a))} \varphi^a \overline{\Phi} \, dx \right| \leq \|\varphi^a\|_{L^2(D^+_{Ka_1}(\pi(a)))} \|\Phi\|_{L^2(D^+_{Ka_1}(\pi(a)))} = O(a_1^4). \qquad (7.15)$$

By combining (7.9), (7.10), (7.13), (7.14) and (7.15), we obtain

$$Ka_1\sqrt{2\pi H(\varphi^a, Ka_1, \pi(a))} \geq C(Ka_1)^2 - C'(Ka_1R)^2 + o(a_1^2) \geq C''(Ka_1)^2,$$

for a suitable choice of R and for a_1 sufficiently small, and hence the thesis. $\quad\square$

Lemma 5.3 and 7.3 allow us to say that $H(\varphi^a, Ka_1, \pi(a)) = O\left((Ka_1)^2\right)$ for $K > \max\{\beta/\pi, 1\}$ and $a_1 < \bar{a}_1$ ($\bar{a}_1$ defined in Lemma 7.3).

7.2. Normalized blow-up at the pole.
In order to analyze the behavior of φ^a near a (for $|a|$ close to 0), we perform a normalized blow-up of the function near the pole. For a fixed $\varepsilon > 0$, let

$$r_\varepsilon, \bar{a}_{1,\varepsilon}, k_\varepsilon \text{ be as in Lemma 7.2.} \qquad (7.16)$$

We define

$$\psi^a(y) = \frac{\varphi^a(a_1 y + \pi(a))}{\sqrt{H(\varphi^a, k_\varepsilon a_1, \pi(a))}}, \quad y \in D^+_{R_0}(0), \ R_0 = \frac{r_0}{a_1}. \qquad (7.17)$$

Note that these are the functions which appear in the statement of Theorem 1.7 (with $K = k_\varepsilon$) and that they are singular at $e = (0, 1)$, independently of a. We also remark that ψ^a solves the problem

$$\begin{cases} (i\nabla + A_e)^2\psi^a = \lambda^a a_1^2 \hat{p}(y)\psi^a & D^+_{R_0}(0) \\ \psi^a = 0 & \{y_1 = 0\}, \end{cases} \qquad (7.18)$$

where $\hat{p}(y) = p(a_1 y + \pi(a))$.

A direct calculation provides the following relations between the frequency function for φ^a and that for ψ^a

$$E(\psi^a, R, 0, \lambda^a a_1^2, A_e) = \frac{E(\varphi^a, Ra_1, \pi(a), \lambda^a, A_a)}{H(\varphi^a, k_\varepsilon a_1, \pi(a))}, \tag{7.19}$$

$$H(\psi^a, R, 0) = \frac{H(\varphi^a, Ra_1, \pi(a))}{H(\varphi^a, k_\varepsilon a_1, \pi(a))}, \tag{7.20}$$

$$N(\psi^a, R, 0, \lambda^a a_1^2, A_e) = N(\varphi^a, Ra_1, \pi(a), \lambda^a, A_a), \tag{7.21}$$

for $R > 1$. Here, with an abuse of notation, the frequency function for φ^a contains the weight $p(x)$, while in the frequency function for ψ^a appears $\hat{p}(y)$ due to the change of variables in the integral. This has no influence in the calculations, since both p and $\hat{p}$ satisfy (1.4).

We will show that the boundedness of the Almgren's function implies the convergence of the blow-up sequence as $a_1 \to 0$. To this aim, notice that Lemma 7.2 and relations (7.19)–(7.21) provide the following bounds.

Lemma 7.4. *Given $\varepsilon > 0$, take the same assumptions and notations of Lemma 7.2. Let ψ^a be as in (7.17). Then*

$$N(\psi^a, R, 0, \lambda^a a_1^2, A_e) \leq 1 + \varepsilon \tag{7.22}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ and $k_\varepsilon < R < r_\varepsilon/a_1$, and

$$\frac{H(\psi^a, R_2, 0)}{H(\psi^a, R_1, 0)} \leq \left(\frac{R_2}{R_1}\right)^{2(1+\varepsilon)}. \tag{7.23}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ and $k_\varepsilon < R_1 < R_2 < r_\varepsilon/a_1$.

Lemma 7.5. *Given $\varepsilon > 0$, take the same assumptions and notations of Lemma 7.2. Let ψ^a be as in (7.17). For every $R > k_\varepsilon$, there exists a constant $C(\varepsilon, R) > 0$ such that*

$$\|\psi^a\|_{H^1_{A_e}(D_R^+(0))} \leq C(\varepsilon, R) \quad \text{for every } a_1 < \min\left\{\frac{r_\varepsilon}{R}, \bar{a}_{1,\varepsilon}\right\}. \tag{7.24}$$

Proof. Relation (7.23) and our choice of the normalization provide

$$H(\psi^a, R, 0) = \frac{H(\psi^a, R, 0)}{H(\psi^a, k_\varepsilon, 0)} \leq \left(\frac{R}{k_\varepsilon}\right)^{2(1+\varepsilon)} \leq C(\varepsilon)R^{2(1+\varepsilon)}. \tag{7.25}$$

This, together with the definition of N and (7.22), implies

$$E(\psi^a, R, 0, \lambda^a a_1^2, A_e) = N(\psi^a, R, 0, \lambda^a a_1^2, A_e)H(\psi^a, R, 0) \leq C(\varepsilon)R^{2(1+\varepsilon)}.$$

Both relations hold for $R > k_\varepsilon$ and $a_1 < \min\{r_\varepsilon/R, \bar{a}_{1,\varepsilon}\}$. Then

$$\int_{D_R^+(0)} |(i\nabla + A_e)\psi^a|^2 \, dy \leq C(\varepsilon)R^{2(1+\varepsilon)} + \lambda^a a_1^2 \int_{D_R^+(0)} \hat{p}(y)|\psi^a|^2 \, dy$$

$$\leq C(\varepsilon)R^{2(1+\varepsilon)} + \lambda^a a_1^2 \|p\|_{L^\infty} R^2 \left(H(\psi^a, R, 0) + \int_{D_R^+(0)} |(i\nabla + A_e)\psi^a|^2 \, dy\right)$$

$$\leq C(\varepsilon)R^{2(1+\varepsilon)} + \lambda^a \|p\|_{L^\infty} r_\varepsilon^2 C(\varepsilon)R^{2(1+\varepsilon)} + \lambda^a \|p\|_{L^\infty} r_\varepsilon^2 \int_{D_R^+(0)} |(i\nabla + A_e)\psi^a|^2 \, dy.$$

At the second line we used the Poincaré inequality (3.2), at the third line we used (7.25) and the fact that $R \leq r_\varepsilon / a_1$. Then, thanks to (5.5), we have

$$\int_{D_R^+(0)} |(i\nabla + A_e)\psi^a|^2 \, dy \leq C(\varepsilon) R^{2(1+\varepsilon)} \frac{1 + \lambda^a \|p\|_{L^\infty} r_\varepsilon^2}{1 - \lambda^a \|p\|_{L^\infty} r_\varepsilon^2}.$$

We look then at the second part of the norm. Using Poincaré inequality (3.2), we obtain

$$\int_{D_R^+(0)} |\psi^a|^2 \, dy \leq R^2 H(\psi^a, R, 0) + R^2 \int_{D_R^+(0)} |(i\nabla + A_e)\psi^a|^2 \, dy \leq C(\varepsilon, R),$$

where we used the previous inequality and (7.25). Finally, we combine the two contributions and obtain a constant depending only on R and ε. $\square$

Lemma 7.6. *Given $\varepsilon > 0$, take the same assumptions and notations of Lemma 7.2. Let ψ^a be as in (7.17). There exists $\psi \in H^1_{A_e, loc}(\mathbb{R}^2_+)$, $\psi \not\equiv 0$, such that for every $R > k_\varepsilon$ we have, up to a subsequence, $\psi^a \to \psi$ in $H^1_{A_e}(D_R^+(0))$ as $|a| \to 0$. Moreover, ψ solves*

$$\begin{cases} (i\nabla + A_e)^2 \psi = 0 & \mathbb{R}^2_+ \\ \psi = 0 & \{y_1 = 0\}. \end{cases} \tag{7.26}$$

Proof. By Lemma 7.5, there exists ψ such that, up to a subsequence, $\psi^a \rightharpoonup \psi$ in $H^1_{A_e}(D_R^+(0))$ and $\psi^a \to \psi$ in $L^2(D_R^+(0))$ as $|a| \to 0$. Due to the compactness of the trace embedding, we have $\int_{\partial D_{k_\varepsilon}^+(0)} |\psi|^2 \, d\sigma = k_\varepsilon$, so that $\psi \not\equiv 0$. For every $R > k_\varepsilon$ and for every test function $\phi \in C_0^\infty(D_R^+(0) \backslash \{e\})$, we have

$$\int_{D_R^+(0)} (i\nabla + A_e)\psi^a \cdot \overline{(i\nabla + A_e)\phi} \, dy = \lambda^a a_1^2 \int_{D_R^+(0)} \hat{p}(y)\psi^a \bar{\phi} \, dy.$$

By the weak convergence in $H^1_{A_e}(D_R^+(0))$, the first term converges

$$\int_{D_R^+(0)} (i\nabla + A_e)\psi^a \cdot \overline{(i\nabla + A_e)\phi} \, dy \to \int_{D_R^+(0)} (i\nabla + A_e)\psi \cdot \overline{(i\nabla + A_e)\phi} \, dy.$$

We estimate the second term as follows by means of Lemma 7.5

$$\left| \lambda^a a_1^2 \int_{D_R^+(0)} \hat{p}(y)\psi^a \bar{\phi} \, dy \right| \leq \lambda^a a_1^2 \|p\|_{L^\infty} \|\phi\|_{L^2(D_R^+(0))} \|\psi^a\|_{L^2(D_R^+(0))}$$

$$\leq C a_1^2 \|\psi^a\|_{H^1_{A_e}(D_R^+(0))} \leq C(\varepsilon, R) a_1^2 \to 0,$$

so that ψ solves the limit equation (7.26). In order to prove the strong convergence, we consider the equation satisfied by $\psi^a - \psi$. We have

$$(i\nabla + A_e)^2(\psi^a - \psi) = \lambda^a a_1^2 \hat{p}(y)\psi^a \quad \text{in } D_R^+(0).$$

By Lemma 7.5 and the Sobolev embeddings, the right hand side above converges to 0 in $L^p(D_R^+(0))$ for every $p < \infty$ as $|a| \to 0$. The Kato inequality

$$-\Delta |\psi^a - \psi| \leq |(i\nabla + A_e)^2(\psi^a - \psi)|$$

(see for example [24]) and the standard regularity theory for elliptic equations, imply that $|\psi^a - \psi| \to 0$ in $W^{2,p}(D_R^+(0))$ for every $p < \infty$ as $|a| \to 0$. This in turn implies that the convergence is $C_{loc}^{1,\tau}(D_R^+(0) \backslash \{e\})$ for every $\tau \in (0, 1)$ and $H^1(D_r^+(0))$. $\square$

As a consequence of the strong convergence and of Lemma 7.4, we deduce the following.

Lemma 7.7. *Let ψ be defined in Lemma 7.6. We have*

$$N(\psi, R, 0, 0, A_e) \leq 1 + \varepsilon \quad \text{for every } R > k_\varepsilon, \tag{7.27}$$

$$\frac{H(\psi, R_2, 0)}{H(\psi, R_1, 0)} \leq \left(\frac{R_2}{R_1}\right)^{2(1+\varepsilon)} \quad \text{for every } k_\varepsilon < R_1 < R_2. \tag{7.28}$$

Lemma 7.8. *Let ψ be defined in Lemma 7.6. There exists $d \in [0, +\infty]$ such that*

$$\lim_{R \to +\infty} N(\psi, R, 0, 0, A_e) = d.$$

Proof. Reasoning as in (5.19) we find

$$\frac{\mathrm{d}}{\mathrm{d}R} N(\psi, R, 0, 0, A_e) \geq -\frac{2|M|}{RH(\psi, R, 0)},$$

where M is now the constant

$$M = \lim_{\varepsilon \to 0} Re \int_{\partial D_\varepsilon(e)} (i\nabla + A_e)\psi \cdot \nu \overline{(i\nabla + A_e)\psi \cdot y} \, \mathrm{d}\sigma.$$

We can prove as in Lemma 5.3 that

$$\frac{H(\psi, R, 0)}{H(\psi, k_\varepsilon, 0)} \geq \left(\frac{R}{k_\varepsilon}\right)^2,$$

for $R > k_\varepsilon$. Recalling that $H(\psi, k_\varepsilon, 0) = 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R} N(\psi, R, 0, 0, A_e) \geq -\frac{Ck_\varepsilon^2}{R^3},$$

for a positive constant C. Let us show that this implies the existence of the limit. Let for the moment $N(R) = N(\psi, R, 0, 0, A_e)$. Integrating the last inequality in (R_1, R_2), with $k_\varepsilon < R_1 < R_2$, we obtain

$$N(R_2) - N(R_1) \geq Ck_\varepsilon^2 \left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right). \tag{7.29}$$

If $d = +\infty$ there is nothing to prove. Otherwise, we claim that $N(R)$ is bounded. Indeed, $d \neq \infty$ implies the existence of $K > 0$ and of a sequence $R_n \to \infty$ such that $N(R_n) < K$ for every n, so that for R sufficiently large and $R_n > R$ we have, by (7.29)

$$N(R) \leq N(R_n) - Ck_\varepsilon^2 \left(\frac{1}{R_n^2} - \frac{1}{R^2}\right) \leq K + o(1) \quad \text{as } R \to \infty,$$

so that N is bounded. Suppose by contradiction that $N(R)$ does not admit limit $d \in [0, \infty)$. Then for every $\delta > 0$ there exists a sequence $R_n \to \infty$ such that $|N(R_n) - N(R_{n+1})| \geq \delta$. The case $N(R_n) \geq N(R_{n+1}) + \delta$ contradicts (7.29) if R_n is great enough, the case $N(R_{n+1}) \geq N(R_n) + \delta$ contradicts the fact that N is bounded. $\quad\square$

In the next subsection we will prove that $d = 1$.

7.3. Proof of Theorem 1.7. In order to study the behavior of the limit function ψ at infinity, we perform a rescaling (blow-down) on the independent variable by a factor R. As before, using the boundedness of the Almgren's frequence of ψ, we prove the convergence of the blow-down function as $R \to \infty$. Moreover, we will prove that the limit function is an homogeneous harmonic function of degree 1. Then, this aims us to conclude that sufficiently far from the singularity ψ behaves like an harmonic function, up to a complex phase. More specifically, we prove that this function ψ verifies the conditions of Proposition 1.6.

Lemma 7.9. *Let ψ be the function introduced in Lemma 7.6. We define*

$$w_R(x) = \frac{\psi(Rx)}{\sqrt{H(\psi, R, 0)}}. \tag{7.30}$$

For every $r > 0$ there exists a constant $C(\varepsilon, r)$ such that $\|w_R\|_{H^1_{A_{e/R}}(D_r^+(0))} \leq C(\varepsilon, r)$ for every $R > k_\varepsilon$.

Proof. For $r > 1$ and $R > k_\varepsilon$ we have

$$N(w_R, r, 0, 0, A_{e/R}) = N(\psi, rR, 0, 0, A_e) \leq 1 + \varepsilon \quad \text{and}$$

$$H(w_R, r, 0) = \frac{H(\psi, rR, 0)}{H(\psi, R, 0)} \leq r^{2(1+\varepsilon)},$$

by Lemma 7.7. By combining the two, we obtain

$$E(w_R, r, 0, 0, A_{e/R}) \leq (1 + \varepsilon) r^{2(1+\varepsilon)}$$

for every $r > 1$ and $R > k_\varepsilon$. As a consequence, using Lemma 3.3 we estimate

$$\|w_R\|_{H^1_{A_{e/R}}(D_r^+(0))} \leq (1 + r^2) E(w_R, r, 0, 0, A_{e/R}) + r^2 H(w_R, r, 0) \leq C(\varepsilon, r)$$

for $R > k_\varepsilon$. $\square$

Lemma 7.10. *Let w_R be defined in (7.30). There exists $w \in H^1_{loc}(\mathbb{R}^2_+)$, $w \not\equiv 0$, such that $e^{-i\theta_{e/R}/2} w_R \rightharpoonup w$ in $H^1_{loc}(\mathbb{R}^2_+)$. In addition, w is harmonic in $\mathbb{R}^2_+$ with zero boundary conditions and, for almost every $r > 0$, we have*

$$\lim_{R \to \infty} E(w_R, r, 0, 0, A_{e/R}) = E(w, r, 0, 0, 0). \tag{7.31}$$

Proof. Fix $r > 0$. By (2.1) and Lemma 7.9, there exists a constant $C(\varepsilon, r) > 0$ (not depending on R) such that

$$\|w_R\|_{H^1(D_r^+(0))} \leq C\|w_R\|_{H^1_{A_{e/R}}(D_r^+(0))} \leq C(\varepsilon, r).$$

In order to check that the constant C in the previous inequality does not depend on the position of the singularity e/R, one can extend functions in $H^1_{A_{e/R}}(D_r^+(0))$ which vanish on $\{x_1 = 0\}$ trivially to functions belonging to $H^1_{A_{e/R}}(D_r(0))$, and then proceed as in the proof of [28, Lemma 7.4]. Hence there exists $\tilde{w} \in H^1(D_r^+(0))$ such that $w_R \rightharpoonup \tilde{w}$ in $H^1(D_r^+(0))$ and $w_R \to \tilde{w}$ in $L^2(D_r^+(0))$, as $R \to +\infty$. Since $H(w_R, 1, 0) = 1$ for every R, the trace embeddings provide $\tilde{w} \not\equiv 0$.

Let $w = e^{-i\theta_0/2}\tilde{w}$. In order to prove that w is harmonic, notice first that $(i\nabla + A_{e/R})^2 w_R = 0$ in $\mathbb{R}^2_+$ for every R. Given a test function $\phi \in C_0^\infty(D_r^+(0))$, let R be so large that $e/R \notin \text{supp}\{\phi\}$. Consequently we have

$$-\Delta(e^{-i\theta_{e/R}/2}w_R) = 0 \tag{7.32}$$

in $\text{supp}\{\phi\}$. This implies, using the weak convergence,

$$0 = \int_{D_r^+(0)} \nabla(e^{-i\theta_{e/R}/2}w_R) \cdot \nabla\phi \, dx \to \int_{D_r^+(0)} \nabla w \cdot \nabla\phi \, dx \quad \text{as } R \to \infty,$$

so that w is harmonic in $D_r^+(0)$.

To prove the last part of the statement, fix two concentric semi-annuli $\mathcal{A}_1 \subset \mathcal{A}_2$, centered at the origin and having positive distance from it. Let η be a cut-off function which is 1 in $\mathcal{A}_1$ and vanishes outside $\mathcal{A}_2$. For R sufficiently large, we have that (7.32) holds in $\mathcal{A}_2$. By testing the equation satisfied by $e^{-i\theta_{e/R}/2}w_R - w$ by $(e^{-i\theta_{e/R}/2}w_R - w)\eta$ in $\mathcal{A}_2$, we obtain

$$\int_{\mathcal{A}_1} |\nabla(e^{-i\theta_{e/R}/2}w_R - w)|^2 \, dx$$
$$\leq \left| \int_{\mathcal{A}_2} \nabla(e^{-i\theta_{e/R}/2}w_R - w)\nabla\eta(e^{-i\theta_{e/R}/2}w_R - w) \, dx \right| \to 0,$$

which tends to 0 as $R \to \infty$ by the weak convergence. This implies that

$$\int_{\partial D_\rho^+(0)} \left(|\nabla(e^{-i\theta_{e/R}/2}w_R - w)|^2 + |e^{-i\theta_{e/R}/2}w_R - w|^2 \right) d\sigma \to 0, \tag{7.33}$$

for almost every ρ such that $\partial D_\rho^+(0) \subset \mathcal{A}_1$, as $R \to +\infty$.

Finally, we use integration by parts as follows (the second equality is well defined provided for $R > 1/r$)

$$|E(w_R, r, 0, 0, A_{e/R}) - E(w, r, 0, 0, 0)|$$
$$\leq \int_{\partial D_r^+(0)} \left| -i(i\nabla + A_{e/R})w_R \cdot \nu\bar{w}_R - \nabla w \cdot \nu w \right| d\sigma$$
$$= \int_{\partial D_r^+(0)} \left| \nabla(e^{-i\theta_{e/R}/2}w_R) \cdot \nu \overline{e^{-i\theta_{e/R}/2}w_R} - \nabla w \cdot \nu w \right| d\sigma \to 0,$$

where the convergence to 0 comes from (7.33) for almost every $r > 0$. $\square$

End of the proof of Theorem 1.7. By combining (7.31) and Lemma 7.8 we obtain, for almost every $r > 0$,

$$N(w, r, 0, 0, 0) = \lim_{R\to\infty} N(w_R, r, 0, 0, A_{e/R}) = \lim_{R\to\infty} N(\psi, rR, 0, 0, A_e) = d$$

(recall that ψ was introduced in Lemma 7.6). Since $N(w, \cdot, 0, 0, 0)$ is continuous, it is constant. Since we proved in the previous lemma that w is harmonic with zero boundary conditions on $\{x_1 = 0\}$, we deduce from standard arguments (see for example [30, Proposition 3.9]) that $w(r, \theta) = Cr^d \cos(d\theta)$, for some $d \in \mathbb{N}_0$ odd. Comparing with

(7.27), taking for example $\varepsilon = 1/2$, we conclude that $d = 1$. In conclusion, by Proposition 1.6, ψ solves (1.7)–(1.8). Then, $\psi = Ce^{i\theta_e/2}\left(r\cos\theta - \frac{\beta}{\pi}\frac{\cos\theta}{r} + O(r^{-3})\right)$ for $r > 1$. Moreover, since $H(\psi, k_\varepsilon, 0) = 1$, the constant C is given by

$$C^2 = \frac{1}{\frac{k_\varepsilon^2\pi}{2} - \beta + O(\frac{1}{k_\varepsilon^2})}. \tag{7.34}$$

$\square$

7.4. Proof of Theorem 1.8.

We can assume without loss of generality that $b = 0$ and moreover, by Lemma 3.1, that Ω satisfies (3.1). Let φ_k have a zero of order 1 at $0 \in \partial\Omega$, meaning that there are no nodal lines of φ_k ending at 0. Let $K > \sqrt{\beta/\pi}$ large be such that the statement of Theorem 1.7 holds. We proceed similarly to the proof of Theorem 1.5.

For $i = 1, \ldots, k$ let

$$v_i^{ext} = e^{-i\frac{\theta_a}{2}}\varphi_i^a \quad \text{in } \Omega\backslash D_{Ka_1}(\pi(a)).$$

For a_1 sufficiently small, v_i^{int} is defined as the unique function which achieves

$$\inf\left\{\int_{D_{Ka_1}^+(\pi(a))}\left(|\nabla v|^2 - \lambda_i^a p(x)|v|^2\right)\,\mathrm{d}x : v \in H^1(D_{Ka_1}^+(\pi(a))),\right.$$
$$\left. \times\, v = v_i^{ext} \text{ on } \partial D_{Ka_1}^+(\pi(a))\right\}.$$

We let $v_i = v_i^{int}$ in $D_{Ka_1}^+(\pi(a))$, $v_i = v_i^{ext}$ in $\Omega\backslash D_{Ka_1}(\pi(a))$. Notice that estimate (4.13) holds in this case for every $1 \leq i \leq k$ since φ_k has no nodal line at 0. We take

$$F_k = \left\{\Phi = \sum_{i=1}^k \alpha_i v_i : \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k\right\} \subset H_0^1(\Omega),$$

so that

$$\lambda_k \leq \sup_{\Phi\in F_k}\frac{\|\nabla\Phi\|_{L^2(\Omega)}^2}{\int_\Omega p(x)|\Phi|^2\mathrm{d}x} = \sup_{\alpha\in\mathbb{R}^k}\frac{\alpha^T M\alpha}{\alpha^T N\alpha} = \lambda_{max}(N^{-1}M). \tag{7.35}$$

Here $\lambda_{max}(\cdot)$ is the largest eigenvalue of a matrix and M, N are $k \times k$ matrices with entries

$$m_{ij} = \int_\Omega \nabla v_i \cdot \nabla v_j\,\mathrm{d}x = \lambda_i^a\int_\Omega p(x)v_i v_j\,\mathrm{d}x + \int_{\partial D_{Ka_1}^+(\pi(a))}\nabla(v_i^{int} - v_i^{ext})\cdot vv_j\,\mathrm{d}\sigma,$$

$$n_{ij} = \int_\Omega p(x)v_i v_j\,\mathrm{d}x.$$

Let us estimate $m_{k,k}$. We perform the following change of variables in order to work with the function ψ_k^a defined in (7.17)

$$f_k^{a,ext}(y) = e^{-i\theta_e/2}\psi_k^a(y) = \frac{v_k^{ext}(a_1 y + \pi(a))}{\sqrt{H(\varphi_k^a, Ka_1, \pi(a))}}, \quad f_k^{a,int}(y) = \frac{v_k^{int}(a_1 y + \pi(a))}{\sqrt{H(\varphi_k^a, Ka_1, \pi(a))}}.$$

By Theorem 1.7 we have that $f_k^{a,ext} \to e^{-i\theta_e/2}\psi_k$ in $H^1(D_K^+(0))$ as $a \to 0$. Moreover, we have $f_k^{a,int} \to f_k^{int}$ in $H^1(D_K^+(0))$ as $a \to 0$ and $-\Delta f_k^{int} = 0$ in $D_K^+(0)$, $f_K^{int} = e^{-i\theta_e/2}\psi_k$ on $\partial D_K^+(0)$. From Proposition 1.6 (ii), we deduce the following behavior of the harmonic extension f_k^{int}

$$f_k^{int}(r,\theta) = C\left(1 - \frac{\beta}{\pi K^2}\right) r\cos\theta + C \sum_{\substack{n \geq 3 \\ n \text{ odd}}} \frac{b_n}{K^{2n}} \cos(n\theta)\, r^n, \quad r < K,$$

for b_n as in Proposition 1.6, (ii), and C given by (7.34) with $K = k_\varepsilon$. Therefore we have

$$\int_{\partial D_{Ka_1}^+(\pi(a))} \nabla(v_k^{int} - v_k^{ext}) \cdot \nu v_k \, d\sigma$$

$$= H(\varphi_k^a, Ka_1, \pi(a)) \int_{\partial D_K^+(0)} \nabla(f_k^{a,int} - f_k^{a,ext}) \cdot \nu f_k^{ext} \, d\sigma$$

$$= H(\varphi_k^a, Ka_1, \pi(a)) C^2 \left\{-\beta + O(K^{-2}) + o_{a_1}(1)\right\} = -C_k a_1^2$$

for some $C_k > 0$ as soon as K is sufficiently large and a_1 is sufficiently small, where in the last step we used Lemma 7.3.

We deduce that the matrices M and N appearing in (7.35) have the following form

$$M = \begin{pmatrix} \lambda_1^a + O(a_1^2) & & & O(a_1^2) \\ & \ddots & & & O(a_1^2) \\ O(a_1^2) & & \lambda_{k-1}^a + O(a_1^2) & \\ & O(a_1^2) & & \lambda_k^a - C_k a_1^2 + o(a_1^2) \end{pmatrix}$$

$$N = \begin{pmatrix} 1 + O(a_1^4) & O(a_1^4) & \\ & \ddots & \\ O(a_1^4) & & 1 + O(a_1^4) \end{pmatrix}.$$

Since λ_k is simple, proceeding similarly to Lemma 4.2, we obtain

$$\lambda_k \leq \lambda_k^a - C_k a_1^2 + o(a_1^2), \tag{7.36}$$

which concludes the proof. Indeed, $N^{-1}M$ has the same form as M. When looking for the eigenvalues of this matrix we search the t such that

$$(\lambda_k^a - C_k a_1^2 - t) Q_{k-1}(t, a_1^2) + a_1^4 Q_{k-2}(t, a_1^2) = 0,$$

where

$$Q_{k-1}(t, a_1^2) = \prod_{i=1}^{k-1} (\lambda_i^a + O(a_1^2) - t)$$

and $Q_{k-2}(t, a_1^2)$ is a polynomial of degree $k - 2$ in the variable t, which depends on a_1 with terms of order $O(a_1^2)$. We set $\varepsilon = a_1^2$ and we apply the implicit function theorem to

$$f(\varepsilon, x, t) = (x - t) Q_{k-1}(t, \varepsilon) + \varepsilon^2 Q_{k-2}(t, \varepsilon)$$

at the point $(0, \lambda_k^a - C_k a_1^2, \lambda_k^a - C_k a_1^2)$. We see that at any point $(0, \bar{x}, \bar{x})$ we have

$$\frac{\partial f}{\partial t}(0, \bar{x}, \bar{x}) = -Q_{k-1}(\bar{x}, 0) = -\prod_{i=1}^{k-1}(\lambda_i - \bar{x}),$$

so that the implicit function theorem applies at any point $(0, \bar{x}, \bar{x})$ such that $\bar{x} \neq \lambda_i$ for every $i = 1, \ldots, k - 1$, and we have $t(\varepsilon, \bar{x}) = \bar{x} + o(\varepsilon)$ in a neighborhood of $(\varepsilon, x) = (0, \bar{x})$. Taking $\bar{x} = \lambda_k^a - C_k a_1^2$ we obtain (7.36).

Appendix A. Domains with Conical Singularities

The proofs of the main theorems can be partially adapted to the case when Ω presents isolated conical singularities, as in the numerical simulations which appear in the Introduction. The results are qualitatively the same as for the smooth domain, but the rate of convergence of the eigenvalues depends on the aperture of the cone. We can interpret this fact in the following way: the zero boundary conditions on an acute angle of $\partial\Omega$ play the same role as the nodal lines of the eigenfunction. The tighter is the angle, the faster is the convergence.

Consider the following conical domain of aperture α, for some $0 < \alpha < \pi$

$$\Omega = \left\{ (r, \theta) : r \in (0, 1), \ \theta \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \right\}. \tag{A.1}$$

The counterpart of Theorem 1.5 holds.

Theorem A.1. *Let Ω be as in (A.1) and let p satisfy (1.4). Suppose that $\lambda_{k-1} < \lambda_k$ and that there exists an eigenfunction φ_k associated to λ_k having a zero of order $h/2 \geq 2$ at the origin. Then there exists $C > 0$, not depending on a, such that*

$$\lambda_k^a \leq \lambda_k - C|a|^{h\frac{\pi}{\alpha}} \quad \text{for } a \to 0 \text{ along a nodal line of } \varphi_k.$$

As for the analogous of Theorem 1.8, we can prescribe the behavior of the eigenvalues only in case the pole approaches the vertex of the cone along the angle bisector. This restriction is related to the open problem presented in Remark 1.9 (i).

Theorem A.2. *Let Ω be as in (A.1) and let p satisfy (1.4). Suppose that λ_k is simple and that φ_k has a zero of order 1 at the origin. Then there exists $C > 0$, not depending on a, such that*

$$\lambda_k^a \geq \lambda_k + Ca_1^{2\frac{\pi}{\alpha}} \quad \text{for } a = (a_1, 0), \ a_1 \to 0.$$

The strategy of proof consists in applying the conformal map $x^{\frac{\pi}{\alpha}}$, so that the conical domain is transformed into the regular half ball $D_1^+(0)$. We end up with a singular equation of the following type

$$(i\nabla + A_a)^2 \varphi_k^a = \lambda_k^a \left(\frac{\alpha}{\pi}\right)^2 \frac{p(x)}{|x|^{2-\frac{2\alpha}{\pi}}} \varphi_k^a \quad \text{in } D_1^+(0).$$

The singular potential $|x|^{-2+\frac{2\alpha}{\pi}}$ belongs to the Kato class, which allows to adapt the proofs of the previous sections. In particular, the following Hardy inequality holds: for every $\varepsilon > 0$ there exists a positive constant C such that

$$\frac{C}{r^\varepsilon} \int_{D_r(0)} \frac{|u|^2}{|x|^{2-\varepsilon}} \, dx \leq \int_{D_r(0)} |\nabla|u||^2 \, dx + \frac{1}{r} \int_{\partial D_r(0)} |u|^2 \, d\sigma, \qquad (A.2)$$

for every $u \in H^1(D_r(0), \mathbb{C})$ and for every $r > 0$ (see [33]). By combining with the diamagnetic inequality

$$\int_{D_r(0)} |\nabla|u||^2 \, dx \leq \int_{D_r(0)} |(i\nabla + A_a)u|^2 \, dx,$$

we obtain the counterpart of the Poincaré inequality (3.2).

Concerning Proposition 2.3, its validity in case of a singular potential belonging to the Kato class is stated in [13, Theorem 1.3].

Appendix B. Green's Function for a Perturbation of the Laplacian

Lemma B.1. *Consider the set of equations (depending on the parameter ε) $-\Delta f = \varepsilon c(x) f$ in $\Omega \subset \mathbb{R}^2$ bounded, with $c \in L^\infty(\Omega)$. For ε sufficiently small there exists a Green's function $G(x, y)$ such that the following representation formula holds for $x \in \Omega$*

$$f(x) = - \int_{\partial\Omega} f \, \partial_\nu G(x, \cdot) \, d\sigma(y).$$

Moreover, for every $1 \leq p < \infty$ there exists C independent from ε such that we have

$$\|\partial_{x_i} G(x, \cdot) - \partial_{x_i} \Phi(x, \cdot)\|_{W^{1,p}(\partial\Omega)} \leq C\varepsilon,$$

for $x \in \Omega$, where Φ is the Green function of the Laplacian with homogeneous Dirichlet boundary conditions in Ω.

Proof. We define the Green function as $G(x, y) = \Gamma(y - x) + L(x, y)$, where $\Gamma(x) = -\frac{1}{2\pi} \log|x|$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$ and $L(x, \cdot)$ solves, for $x \in \Omega$,

$$\begin{cases} -\Delta L(x, y) - \varepsilon c(y) L(x, y) = \varepsilon c(y) \Gamma(y - x) & y \in \Omega \\ L(x, y) = -\Gamma(y - x) & y \in \partial\Omega. \end{cases}$$

Notice that this equation admits a solution for ε small because the quadratic form

$$\int_\Omega (|\nabla v|^2 - \varepsilon c(x) v^2) \, dx \qquad (B.1)$$

is coercive for $v \in H_0^1(\Omega)$, and moreover $\Gamma \in L^2(\Omega)$.

The validity of the representation formula is standard. Indeed, the following identity holds (see for example [12], equation (25) in paragraph 2.2.4)

$$f(x) = - \int_\Omega \Gamma(y - x) \Delta f(y) \, dy + \int_{\partial\Omega} (\Gamma(y - x) \partial_\nu f(y) - f(y) \partial_\nu \Gamma(y - x)) \, d\sigma(y).$$

By using the Green formula

$$\int_{\Omega} (\Delta L \, f - L \Delta f) \, \mathrm{d}y = \int_{\partial \Omega} (\partial_\nu L \, f - L \partial_\nu \, f) \, \mathrm{d}\sigma(y)$$

and the equation satisfied by $L(x, \cdot)$, we obtain the representation formula for f.

In order to estimate $\partial_{x_i}(G - \Phi)$, we write $\Phi(x, y) = \Gamma(y - x) + H(x, y)$, with

$$\begin{cases} -\Delta H(x, y) = 0 & y \in \Omega \\ H(x, y) = -\Gamma(y - x) & y \in \partial \Omega, \end{cases}$$

so that $\partial_{x_i}(G - \Phi) = \partial_{x_i}(L - H) =: u$ solves

$$\begin{cases} -\Delta u - \varepsilon c(y) u = \varepsilon c(y) \partial_{x_i} \Phi(x, \cdot) & y \in \Omega \\ u = 0 & y \in \partial \Omega. \end{cases}$$

We apply Poincaré inequality and the positivity of the quadratic form in (B.1) as follows

$$\begin{aligned} \|u\|_{H^1(\Omega)} &\le C \|\nabla u\|_{L^2(\Omega)} \\ &\le C \left(\int_{\Omega} (|\nabla u|^2 - \varepsilon c(y) u^2) \, \mathrm{d}y \right)^{1/2} \\ &= C \left(\int_{\Omega} \varepsilon c(y) \partial_{x_i} \Phi(x, y) u \, \mathrm{d}y \right)^{1/2}. \end{aligned}$$

Since $\partial_{x_i} \Phi(x, \cdot) \in L^q(\Omega)$ for $1 \le q < 2$, we can apply the Hölder inequality and the Sobolev embedding to obtain

$$\|u\|_{H^1(\Omega)} \le C \varepsilon^{1/2} \left(\|\partial_{x_i} \Phi\|_{L^{3/2}(\Omega)} \|u\|_{L^3(\Omega)} \right)^{1/2} \le C \varepsilon^{1/2} \|u\|_{H^1(\Omega)}^{1/2}.$$

Finally, using again the Sobolev embeddings and a bootstrap argument, we obtain that $u \in W^{2,q}(\Omega)$ for every $1 \le q < 2$ and

$$\|u\|_{W^{2,q}(\Omega)} \le C \varepsilon. \tag{B.2}$$

$\square$

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