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Supersymmetric $U(N)$ Chern–Simons-Matter Theory and Phase Transitions

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Abstract: We study $\mathcal{N} = 2$ supersymmetric $U(N)$ Chern–Simons with N_f fundamental and N_f antifundamental chiral multiplets of mass m in the parameter space spanned by (g, m, N, N_f), where g denotes the coupling constant. In particular, we analyze the matrix model description of its partition function, both at finite N using the method of orthogonal polynomials together with Mordell integrals and, at large N with fixed g, using the theory of Toeplitz determinants. We show for the massless case that there is an explicit realization of the Giveon–Kutasov duality. For finite N, with $N > N_f$, three regimes that exactly correspond to the known three large N phases of the theory are identified and characterized.

1. Introduction

In a classic paper [1], Mordell analyzed integrals of the type

$$I = \int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\, dt,$$

which were originally studied by Kronecker and Lerch in the late 1800s and, anticipating the comprehensive work by Mordell, they had also appeared in the study of the Riemann zeta function by Siegel and in the relationship with Mock theta functions by Ramanujan. This latter line of research is of much current interest after the work [2]. This integral also emerges in studies of unitary representations of extended superconformal algebras (see [3] and references therein).

In this paper we consider a quantum field theory where this integral also plays a central role and will show that it carries exact non-perturbative information on the quantum theory. The theory is $\mathcal{N} = 2$ supersymmetric $U(N)$ Chern–Simons (CS) with N_f fundamental and N_f antifundamental chiral multiplets of mass m. The partition function on $\mathbb{S}^3$ can be determined by localization techniques [4–7] and is given by

$$Z_{N_f}^{U(N)} = \int d^N\mu \frac{\prod_{i<j} 4\sinh^2(\frac{1}{2}(\mu_i - \mu_j))\, e^{-\frac{1}{2g}\sum_i \mu_i^2}}{\prod_i \left(4\cosh(\frac{1}{2}(\mu_i + m))\cosh(\frac{1}{2}(\mu_i - m))\right)^{N_f}}\,, \tag{1.1}$$

where $g = \frac{2\pi i}{k}$ with $k \in \mathbb{Z}$ the Chern–Simons level and $\mu_i/2\pi$ represent the eigenvalues of the scalar field σ belonging to the three dimensional $\mathcal{N} = 2$ vector multiplet. In (1.1) the radius R of the three-sphere has been set to one. It can be restored by rescaling $m \to mR$, $\mu_i \to \mu_i R$. The partition function is periodic in imaginary shifts of the mass, $Z(m + i2\pi n) = Z(m)$, for integer n.

Localization thus reduces the original functional integral to the (infinitely) simpler matrix integral (1.1). However, computing the remaining N integrations is not straightforward and requires the use of specific techniques. In the more general case, where the matter chiral multiplets have R-charge q and belong to the representation R of the gauge group, the matrix model (1.1) contains double sine functions [6,7]. A large number of works have been devoted to analyzing such a matrix model, albeit in a limited region of the parameter space (e.g. large N), whereas in this paper we focus on a more comprehensive analyzing of (1.1), which arises when $q = 1/2$ and $R = r \oplus \bar{r}$. The partition function (1.1) was calculated in [8] in the large N limit at fixed gN by exactly solving the saddle-point equations. The planar theory exhibits a number of interesting features. Non-trivial results emerge when the decompactification limit is taken by scaling the 't Hooft coupling $t = gN$ with the radius as $t = mR\lambda$, with fixed λ. An inspection of the saddle-point equations shows that this is the only possible self-consistent scaling that maintains matter multiplets in the theory (if, instead, t is fixed, then the decompactification limit just decouples matter multiplets). Thus the decompactification limit taken in [8] involves a strong coupling limit. Then, as λ is varied, the theory develops quantum phase transitions of the third order. The theory presents three phases when $0 < N_f < N$ and two phases for $N_f \geq N$. The different phases emerge as λ is increased from zero: the eigenvalue distribution starts flat and begins to extend around the origin until it hits $\pm m$. Upon further increasing of the coupling, the eigenvalues begin to accumulate at $\pm m$ and then, at some higher critical coupling, the boundary of the distribution overcomes $\pm m$ and continues extending gradually in the form of a flat distribution with delta function peaks at $\pm m$.

These phase transitions are very similar to phase transitions appearing in four-dimensional $\mathcal{N} = 2$ supersymmetric massive gauge theories [9–11]. Specifically, this case parallels phase transitions occurring in four-dimensional $\mathcal{N} = 2$ Super-QCD [10]. Recently, similar phase transitions were found in $\mathcal{N} = 2$ Chern–Simons theories with bifundamental matter, such as ABJM and generalizations [12]. Notably, they are the precise three-dimensional analog of the phase transitions occurring in the four-dimensional $\mathcal{N} = 2^*$ Super Yang–Mills theory [9,10].

The calculation of [8] showing the existence of phase transitions describes the CS theory with fundamental matter only in a special corner of parameter space. Therefore, it is of interest to explore the different physical and mathematical features of the theory in the complete parameter space spanned by $(g,\ m,\ N,\ N_f)$. To this aim, in this paper we will use different methods to determine the partition function of this theory first at finite N (by the method of orthogonal polynomials [13,14]), and then at large N with fixed g, by considering the unitary version of the matrix model, which allows the use of the theory of Toeplitz determinants [15,16].

The partition function can also be written, using the change of variables [8]

$$z_i = ce^{\mu_i}, \qquad c \equiv e^{g(N-N_f)}\,, \tag{1.2}$$

as

$$Z_{N_f}^{U(N)} = e^{-\frac{gN}{2}(N^2-N_f^2)} \int_{[0,\infty)^N} d^N z \prod_{i<j}(z_i - z_j)^2 \frac{e^{-\frac{1}{2g}\sum_i (\ln z_i)^2}}{\prod_i \left(1 + z_i \frac{e^m}{c}\right)^{N_f} \left(1 + z_i \frac{e^{-m}}{c}\right)^{N_f}}. \tag{1.3}$$

This ensemble can be formally viewed as a deformation, with logarithmic potentials, of the Stieltjes–Wigert ensemble whose associated orthogonal polynomials solve exactly [17] the Chern–Simons matrix model that describes pure $U(N)$ Chern–Simons theory on $\mathbb{S}^3$ [18]. Consideration of the orthogonal polynomial method as applied to the Hermitian ensemble (1.3) leads to the emergence of the Mordell integral as a crucial tool to obtain explicit analytical expressions for the partition function. This is developed in Sect. 2, following an introduction of the basic formalism of orthogonal polynomials in Sect. 2.1. This use of Mordell integrals not only allows one to obtain analytic expressions for the partition functions, but also provides a very explicit realization of the Giveon–Kutasov duality [19,20], as shown in detail in Sect. 2.7. In addition, the existence of such duality, together with the analytical method developed here, allows to obtain an explicit expression, of the finite-sum type, for the non-Abelian theory, as shown at the very end of Sect. 2.

In Sect. 3, we compute $Z_{N_f}^{U(N)}$ in a large g limit and with the mass m also scaling with g. In the large N calculations of [8], this limit was found to lead to phase transitions and we find here, for finite N, three regimes that are in exact correspondence to the three large N phases discussed above. Upon taking the large N limit, we will reproduce the free energies of each phase computed in [8].

Then, in Sects. 4 and 5, a complementary analysis of the matrix model is carried out by considering a unitary matrix model version of (1.1), in analogy to what occurs in pure Chern–Simons theory [21]. We show that the unitary matrix model can be written as

$$\begin{aligned}\widetilde{Z}_{N_f}^{U(N)} &= \left(\frac{g}{2\pi}\right)^{N/2} \int_{[0,2\pi]^N} \frac{d^N\mu}{(2\pi)^N} \prod_{j=1}^{N} \frac{\theta_3(e^{i\mu_j}, q)}{\left(4\cos(\frac{1}{2}(\mu_j + im))\cos(\frac{1}{2}(\mu_j - im))\right)^{N_f}} \\ &\quad \times \prod_{i<j} 4\sin^2(\frac{1}{2}(\mu_i - \mu_j)),\end{aligned} \tag{1.4}$$

where $\theta_3(e^{i\mu}, q)$ is a theta function, and proceed to analyze it using tools in the theory of Toeplitz determinants [15,16]. In particular, we give, using Szegö's theorem, a large N expression for the partition function when $g = 2\pi i/k$ is fixed, in contrast to the large N limit obtained in [8], which was taken keeping gN fixed. In addition, several properties of the matrix model are presented: (i) the existence of an equivalent matrix model, dual to (1.4) and (ii) the connection between (1.4) and supersymmetric versions of Schur polynomials. Both results are generalizations of properties that also hold for the matrix model that describes pure Chern–Simons theory [22–24].

In Sect. 5, we show that more refined results in the theory of Toeplitz determinants allow one to analyze the massless case [15,25], which is more delicate to handle than the massive one. In particular, we give an explicit expression for the partition function for strong-coupling and finite N and a large N expression for arbitrary coupling.

2. $U(N)$ Partition Function from Orthogonal Polynomials

2.1. Definitions and conventions. A set of functions $\{\phi_n\}$ satisfying

$$(\phi_n, \phi_m) = \int \phi_n(x)\phi_m(x)d\alpha(x) = \delta_{nm} \tag{2.1}$$

is said to be orthonormal.

From a set of functions $\{f_n\}$ with $n = 0, 1, 2, \ldots$, we can construct an orthogonal set $\{D_n^{(f)}\}$ as follows [13]

$$D_n^{(f)}(x) = \frac{1}{\mathsf{N_n}} \begin{vmatrix} (f_0, f_0) & (f_0, f_1) & \cdots & (f_0, f_n) \\ (f_1, f_0) & (f_1, f_1) & \cdots & (f_1, f_n) \\ \cdots & \cdots & \cdots & \cdots \\ (f_{n-1}, f_0) & (f_{n-1}, f_1) & \cdots & (f_{n-1}, f_n) \\ f_0(x) & f_1(x) & \cdots & f_n(x) \end{vmatrix} \tag{2.2}$$

with

$$\mathsf{N}_n = \begin{vmatrix} (f_0, f_0) & (f_0, f_1) & \cdots & (f_0, f_{n-1}) \\ (f_1, f_0) & (f_1, f_1) & \cdots & (f_1, f_n) \\ \cdots & \cdots & \cdots & \cdots \\ (f_{n-1}, f_0) & (f_{n-1}, f_1) & \cdots & (f_{n-1}, f_{n-1}) \end{vmatrix} \tag{2.3}$$

Here the factor N_n was chosen so that upon choosing $f_n(x) = x^n$, the polynomials $p_n(x) = D_n^{(f)}(x)$ have unit coefficient in its highest power $p_n(x) = x^n + \ldots$. We define h_n as

$$(p_n, p_m) = h_n \delta_{nm}.$$

$U(N)$ matrix models. An Hermitian matrix model has a Jacobian $\Delta^2(z) = \prod_{i<j}(z_i - z_j)^2$ arising from gauge fixing the $U(N)$ symmetry

$$Z = \int d^N z\, \Delta^2(z)\, e^{-\frac{1}{g}\sum_i V(z_i)}. \tag{2.4}$$

The factor Δ, known as Vandermonde determinant, can be written as

$$\Delta(z) = \begin{vmatrix} 1 & z_1 & (z_1)^2 & \cdots & (z_1)^{N-1} \\ 1 & z_2 & (z_2)^2 & \cdots & (z_2)^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z_N & (z_N)^2 & \cdots & (z_N)^{N-1} \end{vmatrix} \tag{2.5}$$

Choosing

$$d\alpha(z) = e^{-\frac{1}{g}V(z)}dz\ , \tag{2.6}$$

as measure, the matrix model orthogonal polynomials $p_n(z)$ satisfy

$$\int p_n(z)p_m(z)d\alpha(z) = h_n\delta_{nm}. \tag{2.7}$$

A $U(N)$ gauge theory requires the computation of the first N polynomials. Having the polynomials $p_n(z) = z^n + \ldots$, and rewriting the Vandermonde determinant (2.5) as

$$\Delta(z) = \begin{vmatrix} p_0(z_1) & p_1(z_1) & p_2(z_1) & \cdots & p_{N-1}(z_1) \\ p_0(z_2) & p_1(z_1) & p_2(z_2) & \cdots & p_{N-1}(z_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_0(z_N) & p_1(z_N) & p_2(z_N) & \cdots & p_{N-1}(z_N) \end{vmatrix} = \epsilon^{i_1 \ldots i_N} p_{i_1-1}(z_1) \cdots p_{i_N-1}(z_N).$$

The partition function (2.4) can then be computed as follows [14]:

$$\begin{aligned} Z &= \int d^N\alpha(z)\, \Delta^2(z) \\ &= \int d^N\alpha(z)\, \epsilon^{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} p_{i_1-1}(z_1) \cdots p_{i_N-1}(z_N) p_{j_1-1}(z_1) \cdots p_{j_N-1}(z_N) \\ &= \epsilon^{i_1 \ldots i_N} \epsilon^{j_1 \ldots j_N} \left(\int d\alpha(z_1)\, p_{i_1-1}(z_1) p_{j_1-1}(z_1) \right) \cdots \left(\int d\alpha(z_N)\, p_{i_N-1}(z_N) p_{j_N-1}(z_N) \right), \end{aligned}$$

i.e.

$$Z = N! \prod_{i=1}^{N} h_i. \tag{2.8}$$

Alternatively, the partition function Z can be determined from the formula (2.5) for Δ. This gives

$$Z = N!\, \mathsf{N}_N. \tag{2.9}$$

2.2. The "Mordell" ensemble. In the present case, we need to compute

$$\begin{aligned} (\mathsf{f}_i, \mathsf{f}_j) &= \int d\alpha(z)\, z^{i+j} \\ &= \int_0^\infty dz \frac{z^{i+j}}{\left(1 + z\frac{e^{+m}}{c}\right)^{N_f} \left(1 + z\frac{e^{-m}}{c}\right)^{N_f}} e^{-\frac{1}{2g}(\ln z)^2} \\ &= c^{i+j+1} e^{-\frac{1}{2g}(\ln c)^2} \int_{-\infty}^{\infty} d\mu \frac{e^{\mu(i+j+1+N_f-N)}}{(1+e^{\mu+m})^{N_f} \left(1+e^{\mu-m}\right)^{N_f}} e^{-\frac{1}{2g}\mu^2}. \end{aligned} \tag{2.10}$$

We have called $z = c\, e^{\mu}$ and $i, j = 0, 1, \ldots, N-1$. The second line reduces to the integral appearing in the Stieltjes–Wigert ensemble by formally regarding c as an independent parameter and taking $c \to \infty$. This is only a formal connection because here c depends on g. In particular, note that

$$c^{i+j+1} e^{-\frac{1}{2g}(\ln c)^2} = e^{g(N-N_f)\left(i+j+1-\frac{1}{2}(N-N_f)\right)}. \tag{2.11}$$

The partition function is thus given by

$$Z_{N_f}^{U(N)} = N!\, e^{-\frac{gN}{2}(N^2-N_f^2)} \det(\mathsf{f}_i, \mathsf{f}_j). \tag{2.12}$$

2.3. Case $N_f = 1$. When $N_f = 1$ we can use

$$\frac{1}{(1+ax)(1+bx)} = \frac{1}{a-b}\left(\frac{a}{1+ax} - \frac{b}{1+bx}\right)$$

to obtain

$$(\mathsf{f}_i, \mathsf{f}_j) = e^{\frac{g}{2}(N^2-1)}\, e^{\ell g(N-1)}\, \frac{I(\ell, m) - I(\ell, -m)}{2\sinh m}\ , \tag{2.13}$$

where the function I is given by

$$I(\ell, m) = \int_{-\infty}^{\infty} d\mu \frac{e^{(\ell+1)\mu+m}}{1+e^{\mu+m}} e^{-\frac{1}{2g}\mu^2}\ , \tag{2.14}$$

and the integer $\ell = i + j + 1 - N$ runs from $\ell = 1 - N, \ldots, N - 1$. Note that the first exponential factor in (2.13) cancels a similar one in (2.12) upon taking the determinant.

The integral (2.14) is a particular case of a Mordell integral [1], which can in general be evaluated in terms of expressions involving infinite sums. In special cases, the Mordell integrals simplify to finite Gauss sums, as we shall discuss below and in Sect. 2.5. For generic values of the parameters, the integral (2.14) can also be given in terms of infinite sums of error functions,

$$\begin{aligned} I(\ell, m) &= \int_{-\infty}^{-m} d\mu \frac{e^{(\ell+1)\mu+m}}{1+e^{\mu+m}} e^{-\frac{1}{2g}\mu^2} + \int_{-m}^{\infty} d\mu \frac{e^{(\ell+1)\mu+m}}{1+e^{\mu+m}} e^{-\frac{1}{2g}\mu^2} \\ &= \sqrt{\frac{\pi g}{2}} \sum_{n=0}^{\infty} (-1)^n e^{m(n+1)} e^{\frac{g}{2}(n+\ell+1)^2} \mathsf{erfc}\left(\frac{g(\ell+n+1)+m}{\sqrt{2g}}\right) \\ &\quad + \sqrt{\frac{\pi g}{2}} \sum_{n=0}^{\infty} (-1)^n e^{-mn} e^{\frac{g}{2}(n-\ell)^2} \mathsf{erfc}\left(\frac{g(n-\ell)-m}{\sqrt{2g}}\right), \end{aligned} \tag{2.15}$$

where $\mathsf{erfc}(x) = 1 - \mathsf{erf}(x)$ denotes the complementary error function and we have used $\mathsf{erf}(-x) = -\mathsf{erf}(x)$.

2.4. Case $N_f = 1$ and $m = gp$, $p \in \mathbb{N}$. In this particular case equation (2.15) dramatically simplifies and one obtains

$$I(\ell, g\, p) = \begin{cases} \sqrt{\frac{\pi g}{2}}\, e^{-\frac{gp}{2}(p+2\ell)} \sum_{n=0}^{2(p+\ell)} (-1)^n e^{\frac{g}{2}(p+\ell-n)^2}\ , & p + \ell \geq 0 \\ \sqrt{\frac{\pi g}{2}}\, e^{-\frac{gp}{2}(p+2\ell)} \sum_{n=0}^{-2(p+\ell+1)} (-1)^n e^{\frac{g}{2}(p+\ell+n+1)^2}\ , & p + \ell \leq -1 \end{cases} \tag{2.16}$$

This formula permits the calculation of $Z^{U(N)}$ in terms of elementary functions, using (2.12) (in this section, $N_f = 1$). In what follows we give examples for gauge groups $U(1)$, $U(2)$ and $U(3)$.

$U(1)$ gauge group:

$$Z_p^{U(1)} = \frac{\sqrt{2\pi g}\; e^{gp-\frac{gp^2}{2}}}{e^{2gp}-1} \sum_{n=0}^{2p-1} (-1)^n e^{\frac{1}{2}g(p-n)^2}. \tag{2.17}$$

In particular,

$$Z_{p=1}^{U(1)} = \frac{\sqrt{2\pi g}\; e^{\frac{g}{2}}}{\left(e^{\frac{g}{2}}+1\right)\left(e^g+1\right)}$$

$$Z_{p=2}^{U(1)} = \frac{\sqrt{2\pi g}\;\left(e^{\frac{3g}{2}}+e^g+e^{\frac{g}{2}}-1\right)}{\left(e^{\frac{g}{2}}+1\right)\left(e^g+1\right)\left(e^{2g}+1\right)}$$

$$Z_{p=3}^{U(1)} = \frac{\sqrt{2\pi g}\; e^{-3g/2}\;\left(e^{3g}+e^{\frac{3g}{2}}-2e^{\frac{g}{2}}+1\right)}{\left(e^{\frac{3g}{2}}+1\right)\left(e^{3g}+1\right)}.$$

Note that the potential pole at $g=0$ in (2.17) cancels against a zero of the numerator.

$U(2)$ gauge group:

$$Z_{p=1}^{U(2)} = \frac{g\pi\, e^{-g}(e^{\frac{g}{2}}-1)(e^g+2e^{\frac{g}{2}}-1)}{(e^{\frac{g}{2}}+1)(e^g+1)}$$

$$Z_{p=2}^{U(2)} = \frac{g\pi\, e^{-4g}(e^{\frac{g}{2}}-1)(2e^{\frac{3g}{2}}+e^g-1)(2e^{5g/2}+e^{2g}-2e^{\frac{g}{2}}+1)}{(e^{\frac{g}{2}}+1)(e^g+1)(e^{2g}+1)}$$

$$Z_{p=3}^{U(2)} = \frac{g\pi e^{-9g}(e^{\frac{g}{2}}-1)(2e^{3g}-e^g-e^{\frac{g}{2}}+1)(2e^{5g}+2e^{\frac{9g}{2}}-2e^{\frac{7g}{2}}+e^{3g}-2e^{2g}+2e^{\frac{g}{2}}-1)}{(e^{\frac{3g}{2}}+1)(e^{3g}+1)}$$

$U(3)$ gauge group:

$$Z_{p=1}^{U(3)} = \frac{3\sqrt{2}\,\pi^{\frac{3}{2}} g^{\frac{3}{2}}\, e^{-2g}\left(e^{\frac{g}{2}}-1\right)^3}{e^g+1}\left(e^{\frac{3g}{2}}+e^g+e^{\frac{g}{2}}-1\right)$$

$$Z_{p=2}^{U(3)} = \frac{3\sqrt{2}\,\pi^{\frac{3}{2}} g^{\frac{3}{2}}\, e^{-6g}\left(e^{\frac{g}{2}}-1\right)^3}{(e^g+1)\left(e^{2g}+1\right)}\left(e^{2g}+2e^{\frac{3g}{2}}-1\right)$$
$$\times\left(e^{\frac{7g}{2}}+e^{3g}+e^{\frac{5g}{2}}+e^{2g}-e^{\frac{3g}{2}}-e^g-e^{\frac{g}{2}}+1\right)$$

$$Z_{p=3}^{U(3)} = \frac{3\sqrt{2}\,\pi^{\frac{3}{2}} g^{\frac{3}{2}}\, e^{-12g}\left(e^{\frac{g}{2}}-1\right)^3\left(2e^{\frac{7g}{2}}+e^{3g}-e^{\frac{5g}{2}}-e^{\frac{3g}{2}}-e^{\frac{g}{2}}+1\right)}{(e^g+1)\left(e^g-e^{\frac{g}{2}}+1\right)\left(e^{2g}-e^g+1\right)}$$
$$\times\left(2e^{6g}+e^{\frac{11g}{2}}+e^{5g}+e^{\frac{9g}{2}}-e^{4g}-e^{\frac{7g}{2}}-e^{3g}-e^{\frac{5g}{2}}-e^{2g}+e^{\frac{3g}{2}}+e^g+e^{\frac{g}{2}}-1\right)$$

It is interesting to interpret these results in terms of the quantized CS coupling k using $g=2\pi i/k$. In this case the mass, $m=2\pi ip/k$ is imaginary and the partition function

(1.1) depends only on p mod k. From the expressions above we see that for $N = 1, 2, 3$ the partition function $Z_p^{U(N)}$ has singularities for particular values of k. For example, $Z_{p=3}^{U(2)}$ has singularities at $k = 1, 2, 3, 6$. In general, the partition function is regular for $k > 2p$. The singularity at $k = 2p$ arises because in this case $m = i\pi$ and the integrand in the partition function (1.1) acquires a pole on the integration region. The analytic continuation to imaginary g is therefore only justified for $k > 2p$, for $k < 2p$ the above expressions cease to be valid. In the following section we will give general expressions valid for any integer k.

2.5. Calculation of Z in terms of Mordell integrals. The basic integral I (2.14) that is used to construct the orthogonal polynomials has been computed by Mordell [1] for general parameters. In general, it is given in terms of infinite sums. However, in a specific case it assumes the form of a Gauss's finite sum. Mordell gives the remarkable formulas[1]

$$\int_{-\infty}^{\infty} dt\, \frac{e^{-i\pi \frac{a}{b} t^2 - 2\pi t x}}{e^{2\pi t} - 1} = G_-(a, b, x)\,, \qquad \int_{-\infty}^{\infty} dt\, \frac{e^{i\pi \frac{a}{b} t^2 - 2\pi t x}}{e^{2\pi t} - 1} = G_+(a, b, x), \tag{2.18}$$

$$G_-(a, b, x) \equiv \frac{1}{e^{i\pi b(2x-a)} - 1}\left(\sqrt{\frac{-ib}{a}} \sum_{r=0}^{a-1} e^{-i\pi \frac{b}{a}(x-r)^2} + i \sum_{s=1}^{b} e^{i\pi s(2x+s\frac{a}{b})}\right), \tag{2.19}$$

$$G_+(a, b, x) \equiv \frac{1}{e^{i\pi b(2x-a)} - 1}\left(-\sqrt{\frac{ib}{a}} \sum_{r=1}^{a} e^{i\pi \frac{b}{a}(x+r)^2} + i \sum_{s=0}^{b-1} e^{i\pi s(2x-s\frac{a}{b})}\right), \tag{2.20}$$

where a, b are any positive integers, the square root should be understood as having positive real part and the integration contour is deformed to the lower (upper) half plane to avoid the singularity in G_-(G_+). Notice that $G_\pm$ depend only on the ratio a/b, even though it is not manifest in the expressions.

In our case, the integral (2.14) involves a denominator with positive relative sign between the two terms. This case can be easily obtained by a suitable contour deformation as explained in [1]. We find

$$\int_{-\infty}^{\infty} dt\, \frac{e^{-i\pi \frac{a}{b} t^2 - 2\pi t x}}{e^{2\pi t} - e^{2\pi i\lambda}} = e^{-i\pi(2\lambda + 2\lambda x - \frac{a}{b}\lambda^2)}\, G_-\big(a, b, x - \frac{a}{b}\lambda\big)\,, \tag{2.21}$$

$$\int_{-\infty}^{\infty} dt\, \frac{e^{i\pi \frac{a}{b} t^2 - 2\pi t x}}{e^{2\pi t} - e^{-2\pi i\lambda}} = e^{i\pi(2\lambda + 2\lambda x - \frac{a}{b}\lambda^2)}\, G_+\big(a, b, x - \frac{a}{b}\lambda\big), \tag{2.22}$$

where $0 \le \Re(\lambda) < 1$. In particular, for $\lambda = 1/2$, we get a denominator with positive relative sign. Consider now I (2.14) with $g = 2\pi i/k$, by performing a shift of integration variable $\mu + m \to \mu$ and then a rescaling $\mu \to 2\pi t$, we can put it into the form

$$I(\ell, m) = 2\pi\, e^{-m\ell + \frac{ikm^2}{4\pi}} \int_{-\infty}^{\infty} dt\, \frac{e^{i\pi k t^2 + 2\pi t(\ell+1) - itkm}}{e^{2\pi t} + 1}. \tag{2.23}$$

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[1] We correct two small typos on the RHS of (8.2) of [1] (corresponding to (2.20) above): there is no minus sign inside the square root on the first term, and the limits of summation in the second term must be shifted by 1.

Strikingly, thanks to the fact that k is an integer in CS theory, we can apply Mordell's formulas (2.21)–(2.22), which assume that a and b are positive integers. This implies a drastic simplification of the partition function, since, otherwise, for a generic real number k, the integral $I(\ell, m)$ would be given by a complicated expression involving infinite sums.

Thus, using (2.22) with $x = -\ell - 1 + ikm/2\pi$, $a = k$, $b = 1$ and $\lambda = 1/2$, for any positive integer k we have

$$I(\ell, m) = 2\pi\, e^{-i\pi(\ell+\frac{k}{4})}\, e^{-m(\ell+\frac{k}{2})+\frac{ikm^2}{4\pi}}\, G_+\big(k, 1, -\ell - 1 + i\frac{km}{2\pi} - \frac{k}{2}\big)\,, \tag{2.24}$$

with

$$G_+\big(k, 1, -\ell-1+i\frac{km}{2\pi}-\frac{k}{2}\big) = \frac{1}{e^{-km}-1}\left(-\sqrt{\frac{i}{k}}\sum_{r=1}^{k} e^{\frac{i\pi}{k}(r-\ell-1-\frac{k}{2}+i\frac{km}{2\pi})^2} + i\right) \tag{2.25}$$

For negative k, one must use the analog formula with G_-, namely

$$I(\ell, m) = 2\pi\, e^{i\pi(\ell-\frac{k}{4})}\, e^{-m(\ell-\frac{k}{2})+\frac{ikm^2}{4\pi}}\, G_-\big(-k, 1, -\ell - 1 + i\frac{km}{2\pi} + \frac{k}{2}\big)\,, \tag{2.26}$$

with

$$G_-\big(-k, 1, -\ell-1+i\frac{km}{2\pi}+\frac{k}{2}\big) = \frac{1}{1-e^{km}}\left(\sqrt{\frac{i}{k}}e^{km}\sum_{r=1}^{-k} e^{\frac{i\pi}{k}(r+\ell-\frac{k}{2}-i\frac{km}{2\pi})^2} + i\right) \tag{2.27}$$

We now apply these expressions to compute the $U(N)$ Chern–Simons matter partition function for arbitrary level and mass. We will make use of the generalized Gauss's sum identities

$$\frac{1}{\sqrt{ik}}\sum_{r=1}^{k} e^{\frac{i\pi}{k}(r-\ell-\frac{k}{2})^2} = 1, \qquad k > 0\,, \tag{2.28}$$

$$\frac{1}{\sqrt{ik}}\sum_{r=1}^{-k} e^{\frac{i\pi}{k}(r+\ell-\frac{k}{2})^2} = 1, \qquad k < 0\,. \tag{2.29}$$

valid for $\ell \in \mathbb{Z}$.

It is useful to compare (2.24), (2.26) with the formulas (2.16) for the case $m = gp$. The denominator in (2.25) becomes singular for $m = gp = 2\pi i p/k$ with integer p, however, also the numerator vanishes in virtue of (2.28).[2] By taking the limit $p \to$ integer in (2.24), and comparing with (2.16), we also find the remarkable identity

$$I(\ell, gp) = \frac{2\pi i}{k} e^{-i\pi(\ell+\frac{k}{4})}\, e^{-\frac{i\pi p}{k}(2\ell+k+p)}\left(p + \frac{k}{2}+\ell+1-\frac{1}{\sqrt{ik}}\sum_{r=1}^{k} r e^{\frac{i\pi}{k}\left(r-\frac{k}{2}-p-\ell-1\right)^2}\right) \tag{2.30}$$

[2] One can reverse the logic and use the fact that (2.23) [and therefore G_+ in (2.25)] is regular at $m = 2\pi i p/k$ to actually provide another proof of the Gauss's identity (2.28).

$$
= \begin{cases} \pi\sqrt{\frac{i}{k}}\, e^{-\frac{i\pi p}{k}(p+2\ell)} \sum_{n=0}^{2(p+\ell)} (-1)^n e^{\frac{i\pi}{2k}(p+\ell-n)^2}, & p+\ell \geq 0 \\ \pi\sqrt{\frac{i}{k}}\, e^{-\frac{i\pi p}{k}(p+2\ell)} \sum_{n=0}^{-2(p+\ell+1)} (-1)^n e^{\frac{i\pi}{k}(p+\ell+n+1)^2}, & p+\ell \leq -1 \end{cases}
$$

valid for $k > 0$, and which we verified case by case for various values of $p,\ k,\ \ell$.

Summarizing, the partition function can be computed in terms of simple formulas involving finite sums in two cases: (i) when $m = gp$ for arbitrary complex number g and integer p, or (ii) when $k = 2\pi i/g$ is an integer for arbitrary m. The identity (2.30) ensures that both approaches agree in the overlapping region of the parameters, that is, when both $k = 2\pi i/g$ and $p = m/g$ are integers.

In what follows, we give examples for the partition function for arbitrary m and different gauge groups. In all cases, $N_f = 1$.

$U(1)$ gauge group: In the abelian case, the partition function (2.12) reduces to

$$
Z_k^{U(1)} = \frac{I(0,m) - I(0,-m)}{2\sinh m} \tag{2.31}
$$

from (2.24), (2.26) we obtain

$$
Z_k^{U(1)} = \frac{2\pi\, e^{-m+\frac{ik(m-i\pi)^2}{4\pi}}}{(1-e^{-2m})(e^{km}-1)} \times\left(\sqrt{\frac{i}{k}}\sum_{r=1}^{k}\left(e^{\frac{i\pi}{k}\left(r-1-\frac{k}{2}-\frac{ikm}{2\pi}\right)^2} + e^{\frac{i\pi}{k}\left(r-1-\frac{k}{2}+\frac{ikm}{2\pi}\right)^2}\right) - 2i\right) \tag{2.32}
$$

for $k > 0$ and

$$
Z_k^{U(1)} = \frac{2\pi\, e^{-m+\frac{ik(m+i\pi)^2}{4\pi}}}{(1-e^{-2m})(e^{-km}-1)} \times\left(\sqrt{\frac{i}{k}}\sum_{r=1}^{-k}\left(e^{\frac{i\pi}{k}\left(r-\frac{k}{2}-\frac{ikm}{2\pi}\right)^2+km} + e^{\frac{i\pi}{k}\left(r-\frac{k}{2}+\frac{ikm}{2\pi}\right)^2-km}\right) + 2i\right) \tag{2.33}
$$

for $k < 0$. These formulas contain perturbative as well as non-perturbative terms. The perturbative terms arise from the weak-coupling expansion of factors $e^{\frac{i\pi}{k}(r-1)^2} = e^{\frac{g}{2}(r-1)^2}$, whereas non-perturbative terms are factors $e^{\frac{ik(m-i\pi)^2}{4\pi}} = e^{-\frac{(m-i\pi)^2}{2g}}$ and $e^{km} = e^{\frac{2\pi i m}{g}}$.

For particular values of k, we obtain

$$
Z_{(k=1)}^{U(1)} = \frac{2\pi\, e^m e^{i\frac{\pi}{4}}\left(e^m + 1 - 2e^{\frac{m}{2}+\frac{im^2}{4\pi}}\right)}{(e^m-1)^2\,(e^m+1)}
$$

$$
Z_{(k=2)}^{U(1)} = \frac{\sqrt{2}\pi\, e^{\frac{i\pi}{4}} e^m\left(e^{2m} + 1 - 2e^{m+\frac{i\pi}{2}} - 2\sqrt{2}\, e^{m+\frac{im^2}{2\pi}-\frac{i\pi}{4}}\right)}{\left(e^{2m}-1\right)^2}
$$

$U(2)$ gauge group:

$$Z^{U(2)}_{(k=1)} = \frac{8i\pi^2 e^{m+\frac{im^2}{2\pi}}\left(e^m + 1 - 2e^{\frac{m}{2}-\frac{im^2}{4\pi}}\right)}{(e^m-1)^2(e^m+1)} \tag{2.34}$$

$$Z^{U(2)}_{(k=2)} = \frac{8\pi^2 e^{2m}\left(e^{\frac{im^2}{2\pi}}-1\right)\left(e^{\frac{im^2}{2\pi}}+i\right)}{\left(e^{2m}-1\right)^2} \tag{2.35}$$

$U(3)$ gauge group:

$$Z^{U(3)}_{(k=1)} = 48\pi^3 e^{\frac{im^2}{2\pi}+\frac{3i\pi}{4}} \tag{2.36}$$

$$Z^{U(3)}_{(k=2)} = \frac{24\sqrt{2}\pi^3 e^{\frac{i\pi}{4}}\, e^{m+\frac{im^2}{\pi}}}{\left(e^{2m}-1\right)^2}\left(e^{2m}+2ie^m+1-2\sqrt{2}e^{i\frac{\pi}{4}}e^{m-\frac{im^2}{2\pi}}\right) \tag{2.37}$$

2.6. Massless theory. The partition function can also be computed in the massless limit. A convenient way to obtain this case is to consider (2.24),(2.26) and take the limit

$$\begin{aligned}&\lim_{m\to 0}\frac{I(\ell,m)-I(\ell,-m)}{2\sinh m}\\ &\quad = \frac{\pi(-1)^\ell}{k^{\frac{3}{2}}}\, e^{\frac{i\pi}{4}(1-k)}\sum_{n=0}^{k-1} e^{\frac{i\pi}{k}\left(n-\frac{k}{2}-\ell\right)^2}\left(\left(n-\frac{k}{2}\right)^2-\frac{ik}{2\pi}-\ell^2\right),\end{aligned} \tag{2.38}$$

valid for $k > 0$, and

$$\begin{aligned}&\lim_{m\to 0}\frac{I(\ell,m)-I(\ell,-m)}{2\sinh m}\\ &\quad = \frac{\pi(-1)^\ell}{(-k)^{\frac{3}{2}}}\, e^{-\frac{i\pi}{4}(1-k)}\sum_{n=1}^{-k} e^{\frac{i\pi}{k}\left(n-\frac{k}{2}+\ell\right)^2}\left(\left(n+\frac{k}{2}\right)^2-\frac{ik}{2\pi}-\ell^2\right),\end{aligned} \tag{2.39}$$

for $k < 0$.

Substituting these equations into (2.13), (2.12) we can obtain the partition function in the massless case for any $U(N)$ gauge group and $N_f = 1$.

As an example, we quote the case of $U(1)$ gauge theory:

$$\begin{aligned}Z^{U(1)}\Big|_{m=0} &= \frac{\pi}{k^{\frac{3}{2}}}e^{\frac{i\pi}{4}}\sum_{n=0}^{k-1}(-1)^n e^{\frac{i\pi}{k}n^2}\left(\left(n-\frac{k}{2}\right)^2-\frac{ik}{2\pi}\right)\\ &= \frac{1}{2}e^{-\frac{i\pi k}{4}}+\frac{\pi}{k^{\frac{3}{2}}}e^{\frac{i\pi}{4}}\sum_{n=0}^{k-1}(-1)^n e^{\frac{i\pi}{k}n^2}\left(n-\frac{k}{2}\right)^2, \quad k>0,\\ Z^{U(1)}\Big|_{m=0} &= \frac{\pi}{k^{\frac{3}{2}}}e^{\frac{5i\pi}{4}}\sum_{n=1}^{-k}(-1)^n e^{\frac{i\pi}{k}n^2}\left(\left(n+\frac{k}{2}\right)^2-\frac{ik}{2\pi}\right)\\ &= \frac{1}{2}e^{-\frac{i\pi k}{4}}+\frac{\pi}{k^{\frac{3}{2}}}e^{\frac{5i\pi}{4}}\sum_{n=1}^{-k}(-1)^n e^{\frac{i\pi}{k}n^2}\left(n+\frac{k}{2}\right)^2, \quad k<0,\end{aligned} \tag{2.40}$$

Generically, the partition function for arbitrary gauge group $U(N)$ in the massless limit can be obtained from (2.32)–(2.37) by taking the $m \to 0$ limit.

Finally, note that expressions involving finite sums of the Gauss type are typical of partition functions in finite quantum mechanics [26]. Thus, it would be interesting to see if the partition function above and also the massive one (2.32) can be naturally interpreted as $\mathrm{Tr}\left(e^{-\beta H}\right)$ over a finite-dimensional Hilbert space.

2.7. Giveon–Kutasov duality. In recent years there has been considerable interest in 3d Seiberg-like dualities [19,20]. Our analytical computations with Mordell integrals allow for an explicit check of such a duality, as we show in what follows, focussing on the massless case. The duality applies to the partition function of the type (1.1) which, written in the same variables and with the same prefactors as in [20] reads

$$\mathcal{Z}^{U(N)}_{N_f,k} = \frac{1}{N!}\int d^N\lambda \frac{\prod_{i<j} 4\sinh^2(\pi(\lambda_i-\lambda_j))\, e^{\pi i k\sum_i \lambda_i^2}}{\prod_i \left(4\cosh(\pi(\lambda_i+m))\cosh(\pi(\lambda_i-m))\right)^{N_f}}. \tag{2.41}$$

In [20] it is shown, in the context of localization and matrix models, that the Giveon–Kutasov duality of $U(N)$ $\mathcal{N}$=2 Chern–Simons-matter theories [19] also holds for $\mathcal{N}$=3 supersymmetry. More specifically, they find that[3]

$$\mathcal{Z}^{U(N_c)}_{N_f,k}(\eta) = e^{\mathrm{sgn}(k)\pi i\left(c_{|k|,N_f}-\eta^2\right)}\mathcal{Z}^{U(|k|+2N_f-N_c)}_{N_f,-k}(\eta), \tag{2.42}$$

where the l.h.s. denotes the partition function of a theory with N_c colors, N_f fundamental chiral multiplets, Chern–Simons level k, and a Fayet-Iliopoulos term η. The term $c_{|k|,N_f}$ is a phase.

In particular, the matrix model for the case of N_f fundamental chiral multiplets of mass m is considered in [20] and the duality checked for low values of N_f.

We will now show that our formulas are consistent with Giveon–Kutasov duality (2.42). In particular, for $N_c = N_f = 1$, in the massless case with $\eta = 0$, the duality (2.42) becomes

$$\mathcal{Z}^{U(1)}_{1,-k} = e^{i\pi\phi(k)}\mathcal{Z}^{U(|k|+1)}_{1,k}, \tag{2.43}$$

where $\phi(k)$ denotes a k-dependent phase. One can therefore use this duality to study large N_c limits in terms of a simple integral.

Using (2.40) we find (recalling that the variables in (2.41) and (1.1) are related by $2\pi\lambda = \mu$ and a $N!$ prefactor)

$$\begin{aligned}
\mathcal{Z}^{U(1)}_{1,-1} &= \frac{1}{8\pi}e^{\frac{i\pi}{4}}(2-i\pi)\\
\mathcal{Z}^{U(1)}_{1,-2} &= \frac{1}{8\pi}e^{\frac{i\pi}{2}}(2-(1+i)\pi)\\
\mathcal{Z}^{U(1)}_{1,-3} &= \frac{1}{72\pi}e^{\frac{i\pi}{4}}\left(18i+\left(3-8i\sqrt{3}\right)\pi\right)\\
\mathcal{Z}^{U(1)}_{1,-4} &= \frac{1}{8\pi}e^{\frac{i\pi}{2}}\left(2i+(1-2e^{\frac{i\pi}{4}})\pi\right)
\end{aligned}$$

[3] In this paper $N_f = 1$ denotes a pair of fundamental and anti-fundamental chiral multiplets, therefore there is a factor of 2 relative to the N_f of [20].

We need to compare now with the massless limit of (2.34) and (2.37), together with a couple of additional higher-rank cases, finding the highly non-trivial identities

$$\begin{aligned}
\lim_{m\to 0} \mathcal{Z}^{U(2)}_{1,1} &= -\frac{1}{8\pi}(2-i\pi) = e^{\frac{3i\pi}{4}}\mathcal{Z}^{U(1)}_{1,-1} \\
\lim_{m\to 0} \mathcal{Z}^{U(3)}_{1,2} &= -\frac{1}{8\pi}\left(2-(1+i)\,\pi\right) = e^{\frac{i\pi}{2}}\mathcal{Z}^{U(1)}_{1,-2} \\
\lim_{m\to 0} \mathcal{Z}^{U(4)}_{1,3} &= \frac{1}{72\pi}e^{\frac{4i\pi}{3}}\left(18i+\left(3-8i\sqrt{3}\right)\pi\right) = e^{-\frac{11i\pi}{12}}\mathcal{Z}^{U(1)}_{1,-3} \\
\lim_{m\to 0} \mathcal{Z}^{U(5)}_{1,4} &= -\frac{1}{8\pi}\left(2i+(1-2e^{\frac{i\pi}{4}})\pi\right) = e^{\frac{i\pi}{2}}\mathcal{Z}^{U(1)}_{1,-4}
\end{aligned}$$

Thus, the Giveon–Kutasov dualities are satisfied. From the above relations, we find the following general expression for the phase:

$$\phi(k) = \frac{1}{6}+\frac{1}{2}k+\frac{7}{12}k^2. \tag{2.44}$$

Like in [20], the phase depends quadratically on k. The quadratic ansatz is completely determined by the first three cases $U(2)$, $U(3)$, $U(4)$ in the above relations; the last case $U(5)$ is then satisfied identically. We have checked that the same formula for the phase holds for higher rank cases.

We stress that the computation of one side of the duality involves the determinant of an $N_c \times N_c$ matrix of integrals, in particular, a determinant of a 5×5 matrix in the last line, whereas on the other side we have a simple integral. More generally, using the duality, we have derived the formula

$$\mathcal{Z}^{U(N_c)}_{1,k}\Big|_{m=0} = e^{-i\pi\phi(k)}\left(\frac{1}{2}e^{\frac{i\pi k}{4}}+\frac{\pi}{k^{\frac{3}{2}}}e^{-\frac{i\pi}{4}}\sum_{n=1}^{k}(-1)^n e^{-\frac{i\pi}{k}n^2}\left(n-\frac{k}{2}\right)^2\right),\quad k=N_c-1. \tag{2.45}$$

3. Large Coupling g Limit and Phase Transitions at Large N

Our starting point is the basic integral (2.10) that is used to compute the determinant N_N

$$(\mathsf{f}_i,\mathsf{f}_j) = g\; e^{g(\ell+N)(N-N_f)}e^{-\frac{1}{2}g(N-N_f)^2}\;J_{ij}\;, \tag{3.1}$$

with

$$J_{ij} = \int_{-\infty}^{\infty}dx\,\frac{e^{-\frac{g}{2}(x^2-2x\ell)}}{\left(4\cosh\frac{1}{2}(gx+m)\cosh\frac{1}{2}(gx-m)\right)^{N_f}}, \tag{3.2}$$

where $\ell = i+j+1-N$ with $1-N\le \ell\le N-1$. The approximations below rely on the observation that when the coupling g is large, the main contributions to the integrals come from the saddle-point.

In this limit the hyperbolic cosine functions in the denominator can be replaced by exponential functions. We will consider the limit where m scales with g, i.e. $m=gp$, where p is an arbitrary positive real number. This implies $2\cosh\frac{g}{2}(x\pm p)\to \exp\frac{g}{2}|x\pm p|$. In the large N calculations of [8], this limit was found to lead to phase transitions. We now study the partition function of the Chern–Simons-matter (CSM) theory in the same limit but for any finite N, and arbitrary N_f. It should be noted that this limit is equivalent to a decompactification limit, since the three-sphere radius appear in the combination mR.

In this limit we thus have

$$J_{ij} \approx \int_{-\infty}^{-p} dx\, e^{-\frac{g}{2}(x^2-2x(\ell+N_f))} + e^{-gpN_f} \int_{-p}^{p} dx\, e^{-\frac{g}{2}(x^2-2x\ell)} + \int_{p}^{\infty} dx\, e^{-\frac{g}{2}(x^2-2x(\ell-N_f))}.$$

Which term is dominant depends on the interval where the saddle point lies. The saddle point at $x = \ell$ in the second term lies inside the interval $(-p, p)$ for $p > N - 1$ (as $|\ell| \le N - 1$). In this case J_{ij} is just given by the Gaussian integral of the second term. When $N - N_f - 1 < p \le N - 1$, the main contribution comes from the boundaries at $x = \pm p$. Finally, when $p \le N - N_f - 1$, the saddle points at $x = \ell \pm N_f$ in the first and third terms can lie on the intervals (p, ∞) or $(-\infty, -p)$, depending on the value of ℓ, in which case the main contributions come from the first or the third integral.

To keep the discussion general, we may compute analytically the integrals in terms of error functions. Computing the integrals, we obtain

$$J_{ij} \approx \sqrt{\frac{g\pi}{2}} \Bigg(e^{\frac{g}{2}\ell^2-mN_f} \Big(\mathrm{erf}(\frac{m+g\ell}{\sqrt{2g}}) + \mathrm{erf}(\frac{m-g\ell}{\sqrt{2g}}) \Big) + e^{\frac{g}{2}(\ell+N_f)^2}\, \mathrm{erfc}(\frac{m+g(\ell+N_f)}{\sqrt{2g}}) + e^{\frac{g}{2}(\ell-N_f)^2}\, \mathrm{erfc}(\frac{m+g(-\ell+N_f)}{\sqrt{2g}}) \Bigg). \quad (3.3)$$

In what follows we will make use of the asymptotic behavior for the error function $\mathrm{erf}(x)$ at large $|x|$

$$\mathrm{erf}(x) \approx \mathrm{sign}(x) - \frac{e^{-x^2}}{x\sqrt{\pi}}\,, \quad (3.4)$$

which implies

$$\mathrm{erfc}(x) = \begin{cases} \frac{e^{-x^2}}{x\sqrt{\pi}}\,, & \text{for } x > 0, \\ 2 + \frac{e^{-x^2}}{x\sqrt{\pi}}\,, & \text{for } x < 0. \end{cases}$$

The asymptotic behavior of the error functions in (3.3) depends crucially on the sign of their arguments. In turn, these depend on i, j and on the different parameters g, m, N, N_f. The strategy is to compute the determinant by keeping the dominant terms. Notice that for the special case $m = gp$ with integer p, the argument of the error functions may vanish for some $i,\ j$ and one has to use $\mathrm{erf}(0) = 0$ instead of the above asymptotic form.

We now discuss the behavior of the partition function as we increase the 't Hooft coupling gN from 0 to $gN \gg m$. Taking into account that $|\ell| \le (N - 1)$, we can distinguish three different regimes:

I. $0 < g < m/(N-1)$: as long as the 't Hooft coupling is bounded by the mass, the arguments of error functions will always be positive. Then, the dominant terms are those in the first line of (3.3), with the sum of the two error functions replaced by 2. We thus obtain

$$J_{ij} \approx \sqrt{2\pi g}\, e^{\frac{g}{2}\ell^2-mN_f}. \quad (3.5)$$

The partition function, for arbitrary N, N_f and mass m satisfying $m > (N-1)g$, to leading order in g results

$$Z_{N_f}^{U(N)} = N!\, e^{-\frac{t}{2}N^2(1-\zeta^2)} \det(\mathsf{f}_i, \mathsf{f}_j) \approx N!\,(2\pi g)^{N/2} e^{-mN_f N} e^{\frac{1}{6}gN(N^2-1)}. \quad (3.6)$$

Note that the matrix model has become a multiple of the strong coupling limit of the CS matrix model (see eqn 4.43 in [27])

$$Z_{N_f}^{U(N)} = e^{-mN_f N} Z_{CS}(\mathbb{S}^3)\,, \qquad Z_{CS}(\mathbb{S}^3) \approx N!(2\pi g)^{N/2} e^{\frac{1}{6}gN(N^2-1)}. \quad (3.7)$$

In the strong coupling limit, the non-trivial Vandermonde term in $Z_{CS}(\mathbb{S}^3)$ simplifies, i.e. $\sinh\left(\left(\mu_i - \mu_j\right)/2\right)$ is "bosonized" to $\exp\left(\left|\mu_i - \mu_j\right|/2\right)$. Therefore, the matrix model for Phase I is simplified to

$$Z = e^{-mN_f N} \int d^N\mu \prod_{i<j} \exp\left(|\mu_i - \mu_j|\right) e^{-\frac{1}{2g}\sum_i \mu_i^2}.$$

One can check that the formula (3.6) exactly reproduces the $U(1),\ U(2),\ U(3)$ cases of Sect. 2.4. For $U(3)$ and $U(2)$, the condition $m > (N-1)g$ is satisfied for $p \geq 3$ and $p \geq 2$ respectively; for $U(1)$, it is always satisfied. The formula (3.6) then arises by keeping the leading exponentials in the formulas for $U(1),\ U(2),\ U(3)$ of Sect. 2.4.

II. $m/(N-1) \leq g < m/(N-1-N_f)$, with $N_f < N$. In this case, the arguments of the two error functions in the second line of (3.3) are always positive and can be replaced by their asymptotic form $\frac{e^{-x^2}}{x\sqrt{\pi}}$. However, the sign in the argument of the error functions in the first line of (3.3) can be positive, negative or zero, depending on the value of $i+j$. Writing $m = gp$, it can be zero when $N-p-1$ is an even number. As a result, the expression for Z is more involved.
When $N-p-1$ is not an even number, we find

$$Z_{N_f}^{U(N)} = N!N_f^{2\beta+2}(2\pi g)^{\frac{N}{2}-1-\beta} e^S \prod_{j=0}^{\beta} \frac{1}{(N-1-p-2j)^2(1+2j-N+N_f+p)^2}, \quad (3.8)$$

with

$$\beta = \left[\frac{1}{2}(N-p-1)\right],$$

$$S = \frac{1}{6}g\Big(N\left(12\beta(\beta+2)+6p\left(2\beta+2-N_f\right)+11\right)+N^3-6(\beta+1)N^2 -2(\beta+1)\left(4\beta(\beta+2)+3p^2+6(\beta+1)p+3\right)\Big),$$

where "[...]" denotes integer part. Here p is any positive real number in the interval $N-1-N_f < p \leq N-1$, but with the only condition that $N-p-1$ is not even. If $N_f \geq N$, then this regime II extends to arbitrary low values of $m = gp$.

When $N-p-1$ is even we find

$$Z_{N_f}^{U(N)} = \frac{1}{4}\, N!\, N_f^{2\beta}(2\pi g)^{\frac{N}{2}-\beta} e^{S'} \prod_{j=0}^{\beta-1} \frac{1}{(N-1-p-2j)^2(1+2j-N+N_f+p)^2}, \quad (3.9)$$

with

$$m = gp, \qquad \beta = \left[\frac{1}{2}(N-p-1)\right] = \frac{1}{2}(N-p-1),$$
$$S' = -gpNN_f + \frac{1}{6}g\left(3N^2 + p^2 - 3Np - 1\right).$$

This formula can be compared with the formulas given in the $U(2)$, $U(3)$ case in Sect. 2.4, for $p = 1$ and $p = 2$ respectively–so that the condition $g(N-1-N_f) < m \leq g(N-1)$ is satisfied. Keeping the leading exponential in g, one checks that (3.9) is exactly reproduced.

III. $m/(N-1-N_f) \leq g$. This regime exists only when $N_f < N$. Now the arguments of all error functions in (3.3) may be either positive or negative according to the value of $i + j$ (or 0, for special values of m and i, j). As a result, Z is complicated also in this case. For a generic $m = gp$, $0 < p \leq N-1-N_f$, we obtain

$$Z_{N_f}^{U(N)} = N!\, N_f^{2\beta-2\gamma} (2\pi g)^{\frac{N}{2}-\beta+\gamma}\, e^I \prod_{j=\gamma+1}^{\beta} \frac{1}{(N-1-p-2j)^2(1+2j-N+N_f+p)^2}\,, \tag{3.10}$$

with

$$m = gp\,, \qquad \beta = \left[\frac{1}{2}(N-p-1)\right], \quad \gamma = \left[\frac{1}{2}(N-p-1-N_f)\right], \tag{3.11}$$
$$\begin{aligned} I = \frac{1}{6}g\Big(&8\gamma^3 + 2\gamma\left(-6N\left(N_f+p+2\right) + 6(p+2)N_f + 3N_f^2 + 3N^2\right.\\ &\left.+3p^2 + 12p + 11\right)\\ &+2\left(6(p+1)N_f + 3N_f^2 - \beta\left(4\beta^2 + 12\beta + 3p^2 + 6(\beta+2)p + 11\right)\right)\\ &+N\left(12\beta^2 - 6(p+2)N_f + 12\beta(p+2) - 1\right) - 12\gamma^2\left(-N_f + N - p - 2\right)\\ &+N^3 - 6\beta N^2\Big). \end{aligned}$$

If N_f is an even number, then $\gamma = \beta - N_f/2$ and the expression for I simplifies:

$$\begin{aligned} I = \frac{1}{6}g\Big(&-3N^2N_f - N_f\left(3pN_f + N_f^2 + 3p^2 - 1\right)\\ &+N\left(3N_f^2 - 1\right) + N^3\Big), \qquad N_f \text{ even.} \end{aligned}$$

Similar simplifications can be made for N_f odd, leading to formulas which depend on whether N is even or odd. There are simplifications also for integer p.

The above three regimes correspond to the three large N phases found in [8]. Adopting the same definition of free energy as in [8],

$$F_{N_f}^{U(N)} \equiv -\frac{1}{N^2} \ln Z_{N_f}^{U(N)}\,,$$

we can now compare the free energies computed in [8] for the three different phases. We take the same Veneziano limit as in [8]: $N \to \infty$, with

$$t \equiv gN\,, \qquad \zeta \equiv \frac{N_f}{N}$$

fixed.

Phase I) $m > g(N-1)$ case. we now find

$$F_{N_f}^{U(N)} = \frac{1}{N^2}\left(-\ln N! - \frac{N}{2}\ln(2\pi g) + N^2\zeta m - \frac{1}{6}t(N^2-1)\right)$$
$$\longrightarrow \frac{1}{6}\big(6\zeta m - t\big).$$

This exactly matches eq. (3.10) of [8] (in [8], $\lambda \equiv t/m$).

Phase II) $g(N-1-N_f) < m \le g(N-1)$. The leading order $O(N^2)$ contribution in $\ln Z$ comes from the exponent S. Replacing β by $(N-p)/2$, and restoring m by $p \to m/g = N/\lambda$, we find

$$F_{N_f}^{U(N)} \approx -\frac{1}{N^2}\,S = \frac{m}{6\lambda^2}\big(3(2\zeta-1)\lambda^2 + 3\lambda - 1\big) + O(1/N)\,,$$

which exactly matches the free energy in the intermediate regime of [8].

Phase III) $0 < m \le g(N-1-N_f)$. The order $O(N^2)$ contribution in $\ln Z$ now comes from I. Recall $p \to N/\lambda$. At large N, we can replace $\beta \to (N-p)/2$, $\gamma \to (N-p-N_f)/2$. We then find

$$F_{N_f}^{U(N)} = \frac{m}{6\lambda}\big((\zeta-1)^3\lambda^2 + 3\zeta^2\lambda + 3\zeta\big) + O(1/N).$$

This exactly matches (3.12) of [8].

As pointed out in [8], the above free energies exhibit discontinuities in the third derivative with respect to λ. As in the four-dimensional case [9–11], the discontinuities occur due to resonances produced by extra massless particles appearing in the spectrum. In the presence of a vev for the scalar field σ of the vector multiplet, the chiral multiplet masses are proportional to $|\mu_i \pm m|$. In the large N limit, the matrix integral is determined by a saddle-point where eigenvalues are distributed continuously in some interval $(-A, A)$ [8]. Therefore, extra massless chiral multiplets contribute to the saddle point when A is greater or equal m. This is the case for $m < gN$, thus producing the discontinuous behavior in the transition from phase I to phase II. The transition to phase III—occurring only for $N_f < N$—seems to be caused by a different effect: by the time m becomes lower than $g(N-N_f)$, there is a saturation of N_f eigenvalues located at $\pm m$. In the present context, the origin of the three regimes can be understood from the changing behavior of matrix element J_{ij} in the three different intervals, as described above. It is also worth stressing that for finite N the eigenvalue distribution is not continuous; the average separation of μ_i eigenvalues is of $O(1/N)$ and this is the typical value of a light mass in the spectrum. Thus there are no sharp resonance effects in this case, unless N is very large.

In conclusion, we have computed the same large t, m limit that in [8] led to phase transitions, but now for arbitrary (finite) N. Expressions (3.6), (3.9), (3.10) for $Z_{N_f}^{U(N)}$ apply to any value of N and N_f, even low values such as $N=1$ or $N=2$, they only involve the limit $g \gg 1$, with m scaling with g as $m = gp$ and fixed positive real p.

4. Unitary Matrix Model Formulation and Large N

We will now analyze a unitary version of the matrix model, in which the eigenvalues of the matrix model lie on $\mathbb{S}^1$. For pure Chern–Simons theory on $\mathbb{S}^3$ one can indistinctly use the Hermitian matrix model or the unitary matrix model [21]. By considering the unitary version of the matrix model one can employ then tools from the theory of Toeplitz determinants and also establish relationships with symmetric functions/polynomials. These relationships parallel the ones existing for pure Chern–Simons theory, where they are known to describe some of the connections between Chern–Simons theory and 2d Yang–Mills theories [24].

Thus, in addition to computing large N free energies for both the massive and massless cases, we shall establish some mathematical properties involving supersymmetric versions of Schur polynomials which parallel results for the pure $U(N)$ Chern–Simons theory on $\mathbb{S}^3$.

4.1. Toeplitz determinants and Szegö theorem. We begin with a reminder on unitary matrix models through discussion of their equivalent formulation in terms of Toeplitz determinants and their computation employing Szegö's theorem. These tools have already been used in gauge theory in [22–24,28].

Let $f(z)$ be a complex-valued function on $\mathbb{C}$ with Laurent series expansion $f(z) = \sum_{k\in\mathbb{Z}} f_k\, z^k$, and let $T_N(f) = (f_{i-j})_{i,j=1,\ldots,N}$ be the associated Toeplitz operator of dimension N and symbol f. By the Heine–Szegö identity, the corresponding Toeplitz determinant is the partition function of a $U(N)$ unitary matrix model

$$Z_N[f] := \det T_N(f) = \int_{[0,2\pi)^N} \frac{d^N\phi}{(2\pi)^N} \prod_{l<k} \left| e^{i\phi_l} - e^{i\phi_k} \right|^2 \prod_{j=1}^{N} f(e^{i\phi_j}). \tag{4.1}$$

Notice that the symbol of the Toeplitz determinant is the weight function of the matrix model and recall that one typically writes $f\left(e^{i\phi}\right) = \exp\left(-V(e^{i\phi})\right)$ and $V(e^{i\phi})$ is the potential of the matrix model. Let $[\ln f]_k$, $k \in \mathbb{Z}$ denote the coefficients in the Fourier series expansion on the unit circle $\mathbb{S}^1$ of the logarithm of the symbol

$$\ln f(z) = \sum_{k=-\infty}^{\infty} [\ln f]_k\, z^k \ ,$$

and suppose that they obey the absolute summability conditions

$$\sum_{k=-\infty}^{\infty} \left| [\ln f]_k \right| < \infty \qquad \text{and} \qquad \sum_{k=-\infty}^{\infty} k \left| [\ln f]_k \right|^2 < \infty.$$

Let $\widehat{G}(f) = \exp([\ln f]_0)$ denote the geometric mean of the symbol f. Then the strong Szegö limit theorem for Toeplitz determinants states [16]

$$\lim_{N\to\infty} \frac{\det T_N(f)}{\widehat{G}(f)} = \exp\Big(\sum_{k=1}^{\infty} k\ [\ln f]_k\ [\ln f]_{-k} \Big). \tag{4.2}$$

Thus the theorem gives an expression for the large N limit of the partition function (or free energy) of the matrix model in terms of the Fourier coefficients of the potential. One practical advantage of using directly this theorem is that one does not need to study the density of states in the large N (with a saddle-point approximation) in order to compute the free energy.

We shall focus on the two types of symbol functions that describe pure and supersymmetric Chern–Simons theory with massive fundamental matter. The relevant symbol in pure Chern–Simons theory is [22,24].

$$\varphi(z) = \prod_{i=1}^{r} (1 - x_i z)^{-1} (1 - y_i z^{-1})^{-1}. \tag{4.3}$$

The result in [29] shows that this symbol is dual to the symbol

$$\widetilde{\varphi}(z) = \prod_{i=1}^{r} (1 + x_i z)(1 + y_i z^{-1}) \tag{4.4}$$

and the corresponding Toeplitz determinants are identical $\det_N(\varphi) = \det_N(\widetilde{\varphi})$ (see also [21,22]). Notice that the principal specialization[4] $x_i = y_i = q^{i-1/2}$ of the latter directly gives the unitary matrix model with potential

$$\exp(-V_1(e^{i\theta})) = \lim_{r\to\infty} \widetilde{\varphi}(z; x_i = q^{i-1/2}, y_i = q^{i-1/2}) = \frac{\theta_3(e^{i\theta}, q)}{(q;q)_\infty}, \tag{4.5}$$

with the theta function given by

$$\theta_3(e^{i\theta}, q) = \sum_{n=-\infty}^{\infty} q^{n^2/2} e^{in\theta} = \prod\nolimits_{j=1}^{\infty} \left(1 - q^j\right) \left(1 + q^{j-\frac{1}{2}} e^{i\theta}\right) \left(1 + q^{j-\frac{1}{2}} e^{-i\theta}\right), \tag{4.6}$$

and $(q;q)_\infty = \prod_{j=1}^{\infty} \left(1 - q^j\right)$ is the q-Pochhammer symbol.[5] On the other hand, the principal specialization of the first symbol (4.3) gives

$$\exp(-V_2(e^{i\theta})) = \lim_{r\to\infty} \varphi(z; x_i = q^{i-1/2}, y_i = q^{i-1/2}) = (q;q)_\infty\, \theta_3(-e^{i\theta}, q)^{-1}. \tag{4.7}$$

Both matrix models have the same partition function, which is essentially the $U(N)$ Chern–Simons partition function on $\mathbb{S}^3$. It holds that in the case $r = N$:

$$\begin{aligned} Z &= \int_{(0,2\pi]^N} \frac{d^N\mu}{(2\pi)^N} \prod_{i<j} 4\sin^2\left(\frac{1}{2}(\mu_i - \mu_j)\right) \prod_{j=1}^{N} \varphi(e^{i\mu_j}) \\ &= \int_{(0,2\pi]^N} \frac{d^N\mu}{(2\pi)^N} \prod_{i<j} 4\sin^2\left(\frac{1}{2}(\mu_i - \mu_j)\right) \prod_{j=1}^{N} \widetilde{\varphi}(e^{i\mu_j}) \\ &= \prod_{i,j}^{N} \frac{1}{1 - x_i y_j}. \end{aligned} \tag{4.8}$$

The final Cauchy–Binet expression for the two equivalent matrix models in (4.8) follows from Gessel's and Baxter's identities [30], which are spelled out in detail in the Appendix. From (4.8) and when $x_i = y_i = q^{i-1/2}$ the matrix models above have a partition function

$$Z = \prod_{j=1}^{N-1} \frac{1}{\left(1 - q^j\right)^j}. \tag{4.9}$$

[4] In our case, $q = e^{-g} = \exp(-2\pi i/k)$. That is, there is no shift $k \to k + N$ as happens in pure Chern–Simons theory.

[5] Obviously, just a nomenclature and not a symbol of a Toeplitz determinant, like (4.3) or (4.4).

To obtain the full Chern–Simons partition function the two matrix models have to be endowed with the right normalization, which is given by the first factor in the r.h.s. of (4.6), which is missing in both symbols, namely the q-Pochhammer symbol in its finite version $(q;q)_N$. Indeed, multiplying the weight function of the two matrix models in (4.8) by $(q;q)_N$ gives a numerical pre-factor $\left((q;q)_N\right)^N$, which manifestly transforms (4.9) into the Chern–Simons partition function:

$$Z_{CS}\left(\mathbb{S}^3\right) = \prod_{j=1}^{N-1}\left(1-q^j\right)^{N-j}.$$

In principle, Cauchy identity holds when the matrix model is infinite-dimensional but for this symbol it also holds for the finite case [30,31]. We show this explicitly in the Appendix, together with the fact that Szegö's theorem actually corresponds to the Cauchy identity when the latter is written in terms of Miwa variables.

4.2. Unitary matrix model and large N. Let us first write down the trigonometric version for our model corresponding to supersymmetric CS theory with massive fundamental matter:

$$\widetilde{Z}_{N_f}^{U(N)} = \int_{[-\infty,\infty]^N}\frac{d^N\mu}{(2\pi)^N}\frac{e^{-\frac{1}{2g}\sum_i\mu_i^2}\prod_{i<j}4\sin^2(\frac{1}{2}(\mu_i-\mu_j))}{\prod_i\left(4\cos(\frac{1}{2}(\mu_i+im))\cos(\frac{1}{2}(\mu_i-im))\right)^{N_f}}, \tag{4.10}$$

making the range of integration compact, as with the pure CS matrix model [21] brings the Gaussian factor into a theta function. Let us see this explicitly by making the range of integration compact in which case the weight function is rewritten as follows

$$\begin{aligned}
&\int_{[-\infty,\infty]^N}\prod_{j=1}^{N}\frac{\mathrm{e}^{-\frac{1}{2g}\sum_{j=1}^N\mu_j^2}}{\left(4\cos(\frac{1}{2}(\mu_j+im))\cos(\frac{1}{2}(\mu_j-im))\right)^{N_f}}\frac{d\mu_j}{2\pi}\prod_{i<j}4\sin^2(\frac{1}{2}(\mu_i-\mu_j))\\
&=\frac{g^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\int_{[0,2\pi]^N}\prod_{j=1}^{N}\frac{\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\frac{g}{2}n^2+in\mu_j}}{\left(4\cos(\frac{1}{2}(\mu_j+im))\cos(\frac{1}{2}(\mu_j-im))\right)^{N_f}}\frac{d\mu_j}{2\pi}\prod_{i<j}4\sin^2(\frac{1}{2}(\mu_i-\mu_j))\\
&=\frac{g^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}}\int_{[0,2\pi]^N}\prod_{j=1}^{N}\frac{\theta_3(e^{i\mu_j},q)}{\left(4\cos(\frac{1}{2}(\mu_j+im))\cos(\frac{1}{2}(\mu_j-im))\right)^{N_f}}\frac{d\mu_j}{2\pi}\prod_{i<j}4\sin^2(\frac{1}{2}(\mu_i-\mu_j)),
\end{aligned}$$

where the first equality comes out by expressing the integral over $[-\infty,\infty]$ as an infinite sum of integrals over $[0,2\pi]$ while taking into account the periodicity of the trigonometric functions in the integrand and the identity

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\beta(u+2\pi n)^2} = \frac{1}{\sqrt{4\pi\beta}}\sum_{n=-\infty}^{\infty}e^{-n^2/(4\beta)}e^{inu}, \tag{4.11}$$

which follows from Poisson resummation. This allows to make the identification, in the last equality above, with the theta function (4.6), giving

$$\widetilde{Z}_{N_f}^{U(N)} = \left(\frac{g}{2\pi}\right)^{N/2}\int_{(0,2\pi]^N}\frac{d^N\mu}{(2\pi)^N}\frac{\prod_j\theta_3(e^{i\mu_j},q)\prod_{i<j}4\sin^2(\frac{1}{2}(\mu_i-\mu_j))}{\prod_i\left(4\cos(\frac{1}{2}(\mu_i+im))\cos(\frac{1}{2}(\mu_i-im))\right)^{N_f}}. \tag{4.12}$$

The denominator can be conveniently factorized as

$$4\cos(\frac{1}{2}(\mu + im))\cos(\frac{1}{2}(\mu - im)) = e^{m}\left(1 + e^{-i\mu}e^{-m}\right)\left(1 + e^{i\mu}e^{-m}\right). \tag{4.13}$$

Hence, we can study the problem from the point of view of Toeplitz determinants, having to study the symbol:

$$\varphi_{\mathrm{CSM}}(z) = \frac{\theta_3(z,q)}{e^{mN_f}\left(1 + e^{-m}/z\right)^{N_f}\left(1 + e^{-m}z\right)^{N_f}}. \tag{4.14}$$

As we shall see below, this type of symbol emerges when studying supersymmetric Schur polynomials [32], in the same way the pure CS matrix model is related to Schur polynomials [23,24]. We will also show below, in (4.23), that it exists a dual symbol which gives the same partition function.

4.2.1. Large N limit of the model using Szegö's theorem. Computation of the Fourier coefficients $[\ln\varphi(z)]_k$ and $[\ln\varphi(z)]_{-k}$ corresponding to (4.14) and application of Szeg ö's theorem (4.2) gives

$$\widetilde{Z}_{N_f}^{U(N)} = \left(\frac{g}{2\pi}\right)^{N/2}\frac{e^{-NN_f|m|}}{\left(1 - e^{-2|m|}\right)^{N_f^2}}\prod_{j=1}^{\infty}\left(1 - q^j\right)^{N-j}\left(1 - q^{j-\frac{1}{2}}e^{-|m|}\right)^{2N_f} \text{ for } N \to \infty. \tag{4.15}$$

Note that this is different from the large N limit obtained in [8], which was taken keeping gN fixed. Here $g = 2\pi i/k$ is fixed.

If we further take the limit of $g \to \infty$ with m/g fixed, then (4.15) reproduces to the expression (3.6) corresponding to phase I.[6] The other phases II and III cannot be recovered because in the unitary model $|e^{\pm i\mu}e^{-m}|$ is always < 1 and hence in the large m limit the product of cosine functions in (4.13) just reduces to e^m. As a result, (4.12) becomes proportional to the CS matrix partition function model, as in (3.7).

4.3. Supersymmetric Schur polynomials. The mathematical structure involving Schur polynomials and relating Chern–Simons theory to 2d Yang–Mills theory and its q-deformation [24,33], also appears in our model but with supersymmetric Schur polynomials [32]. This suggests a relationship between supersymmetric Chern–Simons theory with massive fundamental matter and the zero area limit of a supersymmetric version of the combinatorial Migdal–Witten description of 2d Yang–Mills theory[7] on $\mathbb{S}^2$. This is similar to the relationship between refined Chern–Simons theory and refined q-deformed 2d Yang–Mills theory [24,36] and to the link found between the superconformal index, which is a twisted supersymmetric partition function of an $\mathcal{N} = 2$ superconformal field theory on $\mathbb{S}^3 \times \mathbb{S}^1$, and the zero area limit of q-deformed 2d Yang–Mills theory [37].

[6] In the present section the normalization of the partition function differs by a factor of $(2\pi)^N$ from the previous one.

[7] Two dimensional Yang–Mills theory with a supergroup symmetry, such as $U(m|n)$, does not seem to have been previously studied in the literature. Its extension by substitution of dimensions with superdimensions might be possible since the supersymmetric Schur polynomials are known to be characters of both typical and atypical representations [34]. In addition, the related Chern–Simons theory has been extended to the supergroup setting [35].

We also find, as we shall see below, a connection of this type but with a supersymmetric q-deformed version of the dimensions of the zero area 2d Yang–Mills theory on $\mathbb{S}^2$.

We consider the analogue of the expression (4.8) involving supersymmetric Schur polynomials $\mathrm{HS}_\lambda(x|z)$ [32], which naturally emerges in the representation theory of Lie superalgebras. In particular, they are characters of irreducible covariant and contravariant tensor representations of $\mathfrak{gl}(m|n)$ while Schur polynomials are well-known to be characters in $\mathfrak{gl}(m)$. In our setting, we will have $m = N$ and $n = N_f$. The polynomials are defined by [32]

$$\mathrm{HS}_\lambda(x|z) = \sum_{\mu,\nu} N_{\mu\nu}{}^{\lambda}\, s_\mu(x)\, s_{\nu'}(z) \tag{4.16}$$

where $s_\lambda(x)$ are Schur polynomials [38], λ, μ and ν denote representations, indexed by partitions which are characterized by a sequence of ordered positive numbers, such as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. The partition ν' is the conjugate to ν and the coefficients $N_{\mu\nu}{}^{\lambda} \in \mathbb{Z}_{\geq 0}$ are the Littlewood–Richardson coefficients defined by expressing the ring structure on the space of symmetric polynomials in the basis of Schur functions as [38]

$$s_\mu(x)\, s_\nu(x) = \sum_\lambda N_{\mu\nu}{}^{\lambda} s_\lambda(x)\;, \tag{4.17}$$

where the sum is over partitions λ of size $|\mu| + |\nu|$ [39], see Appendix A for definitions. The Cauchy–Binet identity is now [32] (see also [39,40])

$$\sum_\lambda \mathrm{HS}_\lambda(x|z)\, \mathrm{HS}_\lambda(y|w) = \prod_{i,j\geq 1} \frac{(1 + x_i\, w_j)\,(1 + y_i\, z_j)}{(1 - x_i\, y_j)\,(1 - z_i\, w_j)}\;, \tag{4.18}$$

which we note is symmetric under interchange $(x, y) \leftrightarrow (z, w)$. We point out that while the sums in (4.16) and (4.18) are formally over all representations, the size of the partitions that are summed over is bounded in terms of the number of variables in the symmetric polynomials, due to the fact that a Schur polynomial is identically 0 if the length of its partition is larger than the number of its variables [38]; see again Appendix A for details. An analogous sum to (4.18) but with an explicit bound on the size of the first row of λ admits a unitary matrix model description [40]

$$\begin{aligned}&\sum_{\lambda,\lambda_1\leq N} \mathrm{HS}_\lambda(x_1,\ldots,x_{k_1}|z_1,\ldots,z_{l_1})\,\mathrm{HS}_\lambda(y_1,\ldots,y_{k_2}|w_1,\ldots,w_{l_2})\\ &= \int_{[0,2\pi)^N} \prod_{i=1}^{N} \frac{d\phi_i}{2\pi}\, \frac{\prod_{j=1}^{k_1}\left(1 + x_j\, e^{i\phi_i}\right)\prod_{j=1}^{k_2}\left(1 + y_j\, e^{-i\phi_i}\right)}{\prod_{j=1}^{l_1}\left(1 - z_j\, e^{i\phi_i}\right)\prod_{j=1}^{l_2}\left(1 - w_j\, e^{-i\phi_i}\right)} \prod_{i<j} 4\sin^2\!\left(\frac{1}{2}(\phi_i - \phi_j)\right).\end{aligned} \tag{4.19}$$

This is an extension of Gessel identity, quoted in Appendix A, to the case of supersymmetric Schur polynomials. Notice that our unitary matrix model, (4.12) above, is of this type since a principal specialization of the x and y set of variables $x_i = y_i = q^{i-1/2}$ $(i = 1, \ldots, N)$ and the semiclassical limit of a principal specialization of the z and w variables, namely $z_j = w_j = -e^{-m}$ $(j = 1, \ldots, N_f)$ gives for the r.h.s. of (4.19)

$$\int_{(0,2\pi]^N} \frac{d^N\mu}{(2\pi)^N}\, \frac{\prod_{i<j} 4\sin^2(\frac{1}{2}(\mu_i - \mu_j))\, \prod_j \theta_3^{(N)}(e^{i\mu_j}, q)}{\prod_i \left(4\cos(\frac{1}{2}(\mu_i + im))\cos(\frac{1}{2}(\mu_i - im))\right)^{N_f}},$$

where $\theta_3^{(N)}(e^{i\mu}, q)$ denotes a truncated theta function [41]

$$\theta_3^{(N)}(z,q) = \sum_{n=-N}^{N} \begin{bmatrix} 2N \\ n+N \end{bmatrix}_q q^{n^2/2} z^n = \left(\sqrt{q}\, z\,; q\right)_N \left(\sqrt{q}\, z^{-1}\,;\, q\right)_N .$$

If we consider the sum over all representations λ in (4.19) (i.e. without the restriction $\lambda_1 \le N$) as in a 2d Yang–Mills theory and take $k_1 \to \infty$ and $k_2 \to \infty$ (while keeping $l_1 = l_2 = N_f$) then we have

$$\begin{aligned}
&\sum_\lambda e^{-2m|\lambda|} \mathrm{sdim}_q^2 \lambda \\
&= \int_{(0,2\pi]^\infty} \prod_{k=1}^{\infty} \frac{\prod_{j=1}^{\infty}\left(1+q^{j-\frac{1}{2}}\mathrm{e}^{i\mu_k}\right)\left(1+q^{j-\frac{1}{2}}\mathrm{e}^{-i\mu_k}\right)}{\left(1+e^{-m}\mathrm{e}^{i\mu_k}\right)^{N_f}\left(1+e^{-m}\mathrm{e}^{-i\mu_k}\right)^{N_f}} \frac{d\mu_k}{2\pi} \prod_{i<j} 4\sin^2\left(\frac{1}{2}(\mu_i-\mu_j)\right) \\
&= \lim_{N\to\infty} \widehat{Z}_{N_f}^{U(N)} = \frac{1}{\left(1-e^{-2|m|}\right)^{N_f^2}} \prod_{j=1}^{N} (1-q^j)^{-j} (1-q^{j-\frac{1}{2}} e^{-|m|})^{2N_f}, \qquad (4.20)
\end{aligned}$$

where $\widehat{Z}_{N_f}^{U(N)} \equiv (2\pi/g)^{N/2} e^{NN_f|m|} \widetilde{Z}_{N_f}^{U(N)} / ((q;q)_\infty)^N$ and we have defined the "supersymmetric half-q-deformed" dimensions[8]

$$\mathrm{sdim}_q \lambda \equiv \mathrm{HS}_\lambda(q^{1/2}, q^{3/2}, \ldots | -1, \ldots, -1).$$

Notice that in (4.20), as usual with this type of description, we do not have the full CS partition function part and a $((q;q)_\infty)^N$ piece has to be added. Likewise, the factor $e^{-NN_f|m|}$ in (4.15) also does not appear in the r.h.s. of (4.20) because such a term comes from the numerical pre-factor in (4.14) and the symbol that arises from writing the l.h.s. of (4.20) as a Toeplitz determinant only gives the z-dependent part of (4.14) without numerical pre-factors.

As we shall see below, the massless case can also be analyzed with an extension of Szegö's theorem. We collect here, for comparison with (4.20), the ensuing result

$$\begin{aligned}
&\sum_\lambda \mathrm{sdim}_q^2 \lambda \\
&= \int_{(0,2\pi]^\infty} \prod_{k=1}^{\infty} \frac{\prod_{j=1}^{\infty}\left(1+q^{j-\frac{1}{2}}\mathrm{e}^{i\mu_k}\right)\left(1+q^{j-\frac{1}{2}}\mathrm{e}^{-i\mu_k}\right)}{\left(1+e^{-m}\mathrm{e}^{i\mu_k}\right)^{N_f}\left(1+e^{-m}\mathrm{e}^{-i\mu_k}\right)^{N_f}} \frac{d\mu_k}{2\pi} \prod_{i<j} 4\sin^2\left(\frac{1}{2}(\mu_i-\mu_j)\right) \\
&= \lim_{N\to\infty} \widehat{Z}_{N_f}^{U(N)}(m=0) = \frac{G^2(1+N_f)}{G(1+2N_f)} N^{N_f^2} \prod_{j=1}^{N} (1-q^j)^{-j} (1-q^{j-\frac{1}{2}})^{2N_f},
\end{aligned}$$

where G is the Barnes G-function [42], whose main definition and properties are collected in the Appendix.

[8] These dimensions can be seen as a q-deformation of the t-dimensions in [34]. In addition, the ring of q-superdimensions and its appearance in $U(m|n)$ Chern–Simons theory were studied in [35]. It remains to be analyzed whether the definition in [35] is also given by a specialization of the supersymmetric Schur polynomial.

4.4. Dual symbol. Let us prove that there is also a dual symbol. This gives an alternative, equivalent matrix model description, in analogy to the case of pure Chern–Simons theory on $\mathbb{S}^3$ [23,24,31]. In addition, it also justifies the use of the Fisher–Hartwig formalism below, for the massless case. We have mentioned above the duality between the symbols (4.3) and (4.4). This result is ultimately due to the existence and equivalence of the Jacobi–Trudi formula and its dual (also known as Nagelsbach–Kostka formula) [38]

$$s_\lambda\left(x_1,\ldots,x_N\right)=\det\left(h_{\lambda_i+j-i}\right)_{i,j=1}^N=\det\left(e_{\lambda_i'+j-i}\right)_{i,j=1}^N,$$

where h_λ and e_λ are homogeneous and elementary symmetric polynomials [38], respectively. These determinantal expressions can also be interpreted as another definition of Schur polynomials, alternative to the one given in Appendix A. The same result holds for the supersymmetric Schur polynomials, replacing the homogeneous and elementary symmetric functions with its supersymmetric counterparts [34]

$$\mathrm{HS}_\lambda(x|z)=\det\left(h_{\lambda_i+j-i}(x|z)\right)_{i,j=1}^N=\det\left(e_{\lambda_i'+j-i}(x|z)\right)_{i,j=1}^N, \tag{4.21}$$

where the generating function of the supersymmetric homogenous and elementary symmetric functions is now [34]

$$\sum_{r\geq 0}h_r(x|z)t^r=\frac{\prod_{j=1}^{l_1}\left(1+z_jt\right)}{\prod_{i=1}^{k_1}\left(1-x_jt\right)}\quad\text{and}\quad\sum_{r\geq 0}e_r(x|z)t^r=\frac{\prod_{i=1}^{k_1}\left(1+x_jt\right)}{\prod_{j=1}^{l_1}\left(1-z_jt\right)}. \tag{4.22}$$

Hence, it immediately holds that the symbol and its dual are

$$\varphi(z)=\frac{\prod_{j=1}^{k_1}\left(1+x_jz\right)\prod_{j=1}^{k_2}\left(1+y_j/z\right)}{\prod_{j=1}^{l_1}\left(1-z_jz\right)\prod_{j=1}^{l_2}\left(1-w_j/z\right)},$$
$$\widetilde{\varphi}(z)=\frac{\prod_{j=1}^{l_1}\left(1+z_jz\right)\prod_{j=1}^{l_2}\left(1+w_j/z\right)}{\prod_{j=1}^{k_1}\left(1-x_jz\right)\prod_{j=1}^{k_2}\left(1-y_j/z\right)}.$$

After the principal specialization $x_j=y_j=q^{j-1/2}$ and $z_j=w_j=e^{-m}$ with $l_1=l_2=N_f$ and $k_1=k_2\to\infty$ we have that

$$\varphi(z)=\frac{\theta_3(z,q)}{\left(1+e^{-m}/z\right)^{N_f}\left(1+e^{-m}z\right)^{N_f}}\quad\text{and}\quad\widetilde{\varphi}(z)=\frac{\left(1-e^{-m}/z\right)^{N_f}\left(1-e^{-m}z\right)^{N_f}}{\theta_3(-z,q)}, \tag{4.23}$$

where $\varphi(z)=e^{mN_f}\varphi_{\mathrm{CSM}}(z)$ [recall (4.14)]. The numerical factor e^{mN_f} does not appear in any of the symbols in (4.23) because it comes out of the relationship (4.13). Notice also how consideration of Szegö's theorem confirms that the determinant for both cases in (4.23) coincides. It is also worth mentioning that, while the two symbols give the same partition function, if one studies Wilson loops in a representation λ, then $\langle W_\lambda\rangle_{\varphi(z)}=\langle W_{\lambda'}\rangle_{\widetilde{\varphi}(z)}$. This is shown explicitly for the pure Chern–Simons case in [31] and the same proof again follows here with the use of (4.21) and (4.22) instead of their non-supersymmetric versions. Alternatively, notice that it holds that $\mathrm{HS}_\lambda(x|z)=\mathrm{HS}_{\lambda'}(z|x)$.

5. Massless Case

While the massive case can be analyzed with the strong Szegö theorem and with generalized Cauchy identities, the massless case develops a Fisher–Hartwig singularity [15,25]. This is the only particular case of our problem where the Cauchy identity and Szegö's theorem is not directly applicable since the situation where the symbol of the Toeplitz determinant (weight function of the unitary matrix model) has a zero/singularity on $\mathbb{S}^1$ is well-known to require Fisher–Hartwig (FH) asymptotics [15,25], which refines the strong Szegö theorem.

5.1. $g = \infty$ limit case. In this particular case, we do not have the Gaussian/theta function part and we end up with a Toeplitz determinant whose symbol has just one FH singularity. The Cauchy identity diverges in this case because it corresponds to the specialization $x_i = y_i = 1$ for $i = 1, \ldots, N$.

This corresponds to the absence of a Chern–Simons term, a case which has been studied in [43] but for large N and in the setting of a more general matter content, where the matrix model is characterized by double sine functions. In the Appendix A of [20], the massive case (with different masses) without Chern–Simons term is also studied, and their resulting formula is nothing else but the Cauchy determinant. As explained above, the massless case is outside the domain of convergence of such formula.

Taking into account the duality between symbols discussed above we can directly use the result in [44,45], which computes the matrix model (4.1) for finite N for a symbol $\phi(z) = (1-z)^{\alpha}(1-z^{-1})^{\beta}$, giving the result

$$\det T_N(\phi) = G(N+1)\frac{G(\alpha+\beta+N+1)}{G(\alpha+\beta+1)}\frac{G(\alpha+1)}{G(\alpha+N+1)}\frac{G(\beta+1)}{G(\beta+N+1)}, \tag{5.1}$$

where $G(z)$ is again Barnes G-function. Then, if $\alpha = \beta = N_f$, then $\det T_N(\phi) = \widehat{Z}_{N_f}^{U(N)}$ and we have

$$\widehat{Z}_{N_f}^{U(N)}(m=0, g=\infty) = G(N+1)\frac{G(2N_f+N+1)}{G(2N_f+1)}\frac{G^2(N_f+1)}{G^2(N_f+N+1)}. \tag{5.2}$$

We note that consideration of Selberg integral also leads to (5.2) [15]. Notice that this is in principle very different from the massive case, which is given by Cauchy identity, even for N finite:

$$Z_N = \prod_{i,j=1}^{N_f}\frac{1}{1-x_i y_j} = \frac{1}{\left(1-e^{-2|m|}\right)^{N_f^2}} \quad \text{valid for} \quad N \geq N_f.$$

The large N limit of (5.2) is very well-known

$$\widehat{Z}_{N_f}^{U(N)}(m=0, g=\infty) = \frac{G^2(1+N_f)}{G(1+2N_f)}N^{N_f^2} \quad \text{for} \quad N \to \infty.$$

This will be a piece of the large N result of the massless case with g finite, as we shall see below. Note that this $g = \infty$ limit for the massless case cannot be connected with the $g = \infty$ limit of Sect. 3, where we assumed that m is also large and scales with g.

5.2. Large N. We can keep the Gaussian/theta function part and use the result on Fisher–Hartwig (FH) asymptotics, which is a generalization of Szegö's result [25]. Note that above we used instead an exact result for finite N. The symbols of FH class have the following form [15]

$$f(z) = e^{V(z)} z^{\sum_{j=0}^{m} \beta_j} \prod_{j=0}^{m} |z - z_j|^{2\alpha_j} g_{z_j, \beta_j}(z) z_j^{-\beta_j}, \qquad z = e^{i\theta}, \quad 0 \leq \theta < 2\pi, \tag{5.3}$$

for some $m = 0, 1, 2, \ldots,$ with

$$z_j = e^{i\theta_j}, \qquad j = 0, 1, \ldots, m, \qquad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < 2\pi, \tag{5.4}$$

$$g_{z_j \beta_j}(z) \equiv g_{\beta_j}(z) = e^{i\pi\beta_j} \text{ for } 0 \leq \arg z < \theta_j \ (e^{-i\pi\beta_j} \text{ otherwise}). \tag{5.5}$$

$$\Re\left(\alpha_j\right) > -\frac{1}{2}, \qquad \beta_j \in \mathbb{C}, \qquad j = 0, 1, \ldots, m, \tag{5.6}$$

and $V(e^{i\theta})$ is a sufficiently smooth function on $\mathbb{S}^1$. Here the condition on $\Re\left(\alpha_j\right)$ guarantees integrability. Note that a FH singularity at z_j, $j = 1, \ldots, m$, consists of a root-type singularity

$$|z - z_j|^{2\alpha_j} = |2 \sin \frac{\theta - \theta_j}{2}|^{2\alpha_j} \tag{5.7}$$

and a jump singularity $z^{\beta_j} g_{\beta_j}(z)$ at z_j (note that $z^{\beta_j} g_{\beta_j}(z)$ is continuous at $z = 1$ for $j \neq 0$). Notice that the symbol $\widetilde{\varphi}(z)$ in (4.23) is of this type with $m = 0$ and hence with only one FH singularity of the root-type, because $\alpha_0 = N_f$ and $\beta_0 = 0$. The asymptotic form of $\det T_N(f)$ for the general symbol above is,

$$\begin{aligned} \det T_N(f) &= E(e^V, \alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_m, \theta_0, \ldots, \theta_m)\, n^{\sum_{j=0}^{m}(\alpha_j^2 - \beta_j^2)} e^{NV_0}(1 + o(1)), \\ V_0 &= \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) d\theta \end{aligned} \tag{5.8}$$

as $N \to \infty$. For a Fisher–Hartwig symbol, in addition we have

$$\begin{aligned} &E(e^V, \alpha_0, \ldots, \alpha_m, \ \theta_0, \ldots, \theta_m) \\ &\quad = E(e^V) \prod_{0 \leq j < k \leq m} |e^{i\theta_j} - e^{i\theta_k}|^{-2\alpha_j\alpha_k} \prod_{j=0}^{m} e^{-\alpha_j \widehat{V}(e^{i\theta_j})} \times \prod_{j=0}^{m} E_{\alpha_j} \end{aligned} \tag{5.9}$$

where

$$E(e^V) = e^{\sum_{k=1}^{\infty} k V_k V_{-k}}, \qquad V_k = \text{ Fourier coefficient of } V(e^{i\theta}), \tag{5.10}$$

$$\widehat{V}(e^{i\theta_j}) = V(e^{i\theta_j}) - V_0 \tag{5.11}$$

$$E_{\alpha_j} = G^2(1 + \alpha_j)/G(1 + 2\alpha_j), \tag{5.12}$$

Notice that the term (5.10) is the content of Szegö's theorem, the rest therefore extends it with additional contributions. Note also that (5.12) is essentially the large N limit of the finite N result above (5.1). Since we only have one FH singularity ($j = 0$), the product term in (5.9) does not contribute. Taking into account that our symbol is (4.23) with $m = 0$ then $\alpha_0 = N_f$, $\beta_0 = 0$ and $z_0 = 1$ ($\theta_0 = 0$). Therefore, we obtain

$$\widetilde{Z}_{N_f}^{U(N)}(m=0) = \left(\frac{g}{2\pi}\right)^{N/2} \frac{G^2(1+N_f)}{G(1+2N_f)} N^{N_f^2} \prod_{j=1}^{\infty} \left(1-q^j\right)^{N-j} \left(1-q^{j-\frac{1}{2}}\right)^{2N_f} \text{ for } N \to \infty. \tag{5.13}$$

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Appendix A. Mathematical Identities

We collect here a number of mathematical identities and results used through the text. A partition is a finite sequence of nonnegative integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Associated to every partition is a Young diagram with λ_i squares in the i-th row and the rows are understood to be aligned on the left. There is a unique n such that $\lambda_n > 0$ but $\lambda_{n+1} = 0$ and this $n = l(\lambda)$ is the length of λ. The number $|\lambda| = \sum_i \lambda_i$ is called the size of λ and we denote by λ' the conjugate partition to λ. Schur polynomials $s_\lambda(x)$ are $|\lambda|$-th homogeneous symmetric polynomials, if λ is any partition of length n we define [38]

$$s_\lambda(x_1, \ldots, x_n) := \frac{\det\left(x_j^{\lambda_k+n-k}\right)_{j,k=1}^n}{\det\left(x_j^{n-k}\right)_{j,k=1}^n}.$$

We shall summarize now a number relationships between Schur polynomials and Toeplitz determinants (equivalently, unitary matrix models [22–24]). We begin with two classical results by Gessel and Baxter which are relevant in Sect. 4.

A.1 Gessel and Baxter identities. We first quote Gessel's formula for the product of Schur polynomials in terms of a Toeplitz determinant, which reads [46]

$$\sum_{\lambda\,;l(\lambda)\leq N} s_\lambda\,(x)\; s_\lambda\,(y) = \det(A_{i-j})_{i,j=1}^N, \tag{A.1}$$

where

$$A_i = A_i(x,y) = \sum_{l=0}^{\infty} \mathfrak{h}_{l+i}(x)\,\mathfrak{h}_l(y), \tag{A.2}$$

are the Fourier coefficients of the symbol (entries of the Toeplitz matrix) and $\mathfrak{h}_r(x)$ is the r-th homogeneous symmetric function, characterized by its generating function $\sum_{r\geq 0}\mathfrak{h}_r t^r = \prod_{j\geq 1}\left(1-x_j\, t\right)^{-1}$. The symbol of the Toeplitz determinant is then [29,46]

$$\varphi\left(z\right)=\sum_{i=-\infty}^{\infty} A_i(x,y)\, z^i=\prod_{j\geq 1}\left(1-y_j\, z^{-1}\right)^{-1}\left(1-x_j\, z\right)^{-1}. \tag{A.3}$$

The dual version is [29]

$$\sum_{\lambda\,:\,\lambda_1\leq N} s_\lambda\left(x\right)\, s_\lambda\left(y\right)=\det(\widetilde{A}_{i-j})_{i,j=1}^N, \tag{A.4}$$

where the Fourier coefficients are now in terms of elementary symmetric functions and the symbol is

$$\widetilde{\varphi}\left(z\right)=\sum_{i=-\infty}^{\infty} \widetilde{A}_i(x,y)\, z^i=\prod_{j\geq 1}\left(1+y_j\, z^{-1}\right)\left(1+x_j\, z\right). \tag{A.5}$$

Notice that the restriction in the sum over representations in (A.1) is a bound on the size of the first column whereas in (A.4) the first row is bounded by N. When the sum is not restricted, the Toeplitz determinants (equivalently, the unitary matrix models) are in principle infinite-dimensional and the Cauchy–Binet identity holds:

$$\sum_\lambda s_\lambda\left(x_1,\ldots,x_p\right)\, s_\lambda\left(y_1,\ldots,y_q\right)=\prod_{i=1}^p\prod_{j=1}^q\frac{1}{1-x_i y_j},$$

where the products goes from 1 to the number of x and y variables. Schur polynomials satisfy the property $s_\lambda(x_1,\ldots,x_n)=0$ if $l(\lambda)>n$ [38], which implies a truncation of the sum for a finite number of variables of the Schur polynomials. Thus, the sum on the l.h.s. is effectively over all partitions λ of length$\leq\min(p,q)$. This result is translated into an statement for Toeplitz determinants by the following Lemma:

Lemma 1 (Baxter [30, Lemma 7.4]). *Let $D_n(\sigma)$ denote the determinant of a Toeplitz matrix $n\times n$ and symbol σ, then*

$$D_n\left(\sigma\right)=\Pi_{i,j}\left(1-\alpha_i\beta_j\right)^{-1},$$

where the symbol is, specifically $\sigma\left(z\right)=\Pi_{i=1}^k\left(1-\alpha_i z\right)\Pi_{i=1}^m\left(1-\beta_j z^{-1}\right)$. This result is valid for $n\geq\max(k,m)$ and independent of n.

Notice again that Cauchy–Binet identity only says that $\lim_{n\to\infty}D_{n-1}(\sigma)=\Pi_{i,j}(1-\alpha_i\beta_j)^{-1}$, but for this symbol, the determinant will give the same result for any finite size, from infinite size, down to the number of product terms in the symbol. This result leads to the last identity in (4.8).

A.2 Cauchy–Binet identity and Szegö's theorem. Notice that, at least in our context, the statement of Szegö's theorem is equivalent to the Cauchy–Binet formula (4.8) when the latter is written in Miwa variables [47]

$$\sum_{\lambda} s_\lambda(x)\, s_\lambda(y) = \exp\left(\sum_{k\geq 1} k\, m_k\, t_k\right), \tag{A.6}$$

where

$$m_k = \frac{1}{k}\sum_{i\geq 1} x_i^k \qquad \text{and} \qquad t_k = \frac{1}{k}\sum_{i\geq 1} y_i^k$$

are power sums of the sets of variables x and y. Hence, the construction of Miwa variables is equivalent to the computation of the moments of the logarithm of the symbol, which is the potential of the matrix model (coming from the Taylor expansion of a logarithm).

A.3 Barnes G-function. The Barnes G-function [42] is a double-Gamma function that can be for example defined with the functional equation

$$G(z+1) = \Gamma(z)\, G(z)$$

with normalization $G(1) = 1$. Its asymptotic expansion is especially useful

$$\ln G(t+a+1) = \frac{1}{12} - \ln A - \frac{3t^2}{4} - at + \frac{t+a}{2}\ln(2\pi) + \left(\frac{t^2}{2} + at + \frac{a^2}{2} - \frac{1}{12}\right)\ln t + o(t^{-1}), \quad \text{as } t \to \infty.$$

Appendix B. Moment Problem and Discretization of the Matrix Model

In this paper, we have studied a one matrix model with potential

$$V(z) = \frac{1}{2g}\ln^2 z + N_f \ln\left(1 + z_i\frac{e^m}{c}\right)\left(1 + z_i\frac{e^{-m}}{c}\right), \tag{B.1}$$

where $z \in (0,\infty)$. Thus, the confining properties are those of the Stieltjes–Wigert potential [17] since, for large z, the first term in (B.1) dominates. Therefore, we expect the model to be associated to an undetermined moment problem, as happens with the Stieltjes–Wigert matrix model [17,24,48]. This means that there are infinitely many deformations of the measure (2.6) with identical orthogonal polynomials $p_n(z)$ and therefore identical (2.7). In consequence, every matrix model constructed from such a measure possesses the same partition function.

This is demonstrated by considering Krein's proposition [17], which gives a sufficient condition for a moment problem to be undetermined. The condition is for the weight function $\omega(z) = \exp(-V(z))$ to satisfy

$$-\int_0^\infty \frac{\ln\omega(z)}{(1+z)}\frac{dz}{\sqrt{z}} < \infty.$$

The integral converges for our potential (B.1), as it happens in the pure Stieltjes–Wigert case [17], and hence the moment problem associated is undetermined. Alternatively, this can be seen even more explicitly by following Stieltjes directly [17], by showing

$$\int_0^\infty z^k e^{-\frac{1}{2g}\ln^2 z+N_f\ln\left(1+z_i\frac{e^m}{c}\right)\left(1+z_i\frac{e^{-m}}{c}\right)}\sin\left(2\pi\ln z/\ln q\right)=0,$$

which follows, as happens in the case $N_f=0$, by the change of variables $v=-(k+1)/2+\ln z$, the periodicity of sin(.), and the fact that sin is an odd function. Thus for any $\theta\in[-1,1]$ the weight function $\omega_\theta(z)=\omega(z)(1+\theta\sin(2\pi\ln z/\ln q))$, where $\omega(z)=\exp(-V(z))$ and $V(z)$ is (B.1), has the same positive integer moments as $\omega(z)$ and therefore the corresponding (infinitely many) matrix models have the same partition function.

The set of all solutions to an indeterminate moment problem always includes discrete measures (the so-called canonical solutions of a moment problem are discrete measures), which implies that there is a discrete matrix model equivalent to the continuous one. In the case of the Stieltjes–Wigert matrix model, the discrete matrix model is known explicitly since the discrete measure with the same moments as $e^{-\frac{1}{2g}\ln^2 z}$ is known to be $M(q)\sum_{n\in\mathbb{Z}}q^{n^2/2+n}\delta(x-q^n)$ with $M(q)$ a suitable constant (see [24,48] and references therein). The analogous result for (B.1) is not known and not immediate to obtain. Hence to find the explicit form of the discrete matrix model which is equivalent to (1.3) and, after the change of variables, to (1.1), is an open problem.

Notice however that a straightforward discretization of the Mordell integral gives already very good results for large coupling constant g since it is known that, for the integral [49]

$$\varphi(g,c,z)=\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2g}(x-z)^2}}{e^{cx}+1}dx,$$

the straightforward discretization which is standard trapezoidal quadrature with step h, that is

$$\varphi(g,c,z)=h\sum_{k=-\infty}^{\infty}\frac{e^{-\frac{1}{2g}(kh-z)^2}}{e^{ckh}+1}+\mathcal{E}(h),$$

has an error term which is bounded by

$$|\mathcal{E}(h)|\le 2\exp(\pi^2/8c^2g-\pi^2/hc)\text{ for }g>\frac{h}{4c},$$
$$|\mathcal{E}(h)|\le 2\exp(-2\pi^2g/h^2)\text{ for }g\le\frac{h}{4c}.$$

The contour integral result in [49] allows for the generalization to the case corresponding to $N_f>1$.

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