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Critical Two-Point Function of the 4-Dimensional Weakly Self-Avoiding Walk

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Abstract: We prove $|x|^{-2}$ decay of the critical two-point function for the continuous-time weakly self-avoiding walk on $\mathbb{Z}^d$, in the upper critical dimension $d = 4$. This is a statement that the critical exponent η exists and is equal to zero. Results of this nature have been proved previously for dimensions $d \geq 5$ using the lace expansion, but the lace expansion does not apply when $d = 4$. The proof is based on a rigorous renormalisation group analysis of an exact representation of the continuous-time weakly self-avoiding walk as a supersymmetric field theory. Much of the analysis applies more widely and has been carried out in a previous paper, where an asymptotic formula for the susceptibility is obtained. Here, we show how observables can be incorporated into the analysis to obtain a pointwise asymptotic formula for the critical two-point function. This involves perturbative calculations similar to those familiar in the physics literature, but with error terms controlled rigorously.

1. Main Result

1.1. Introduction. The critical behaviour of the self-avoiding walk depends on the spatial dimension d. The upper critical dimension is 4, and for $d \geq 5$ the lace expansion has been used to prove that the asymptotic behaviour is Gaussian [20,29–31,43]. In particular, for the *strictly* self-avoiding walk in dimensions $d \geq 5$, the critical two-point function has $|x|^{-(d-2+\eta)}$ decay with critical exponent $\eta = 0$, both for spread-out walks [21,30] and for the nearest-neighbour walk [29]. For $d = 3$, the problem remains completely unsolved from a mathematical point of view, but numerical and other evidence provides convincing evidence that the behaviour is not Gaussian. In particular, numerical values of the critical exponents γ and ν [22,42], together with Fisher's relation $\gamma = (2 - \eta)\nu$, indicate that the critical two-point function has approximate decay $|x|^{-1.031}$ for $d = 3$. For $d = 2$, the critical two-point function is predicted to decay as $|x|^{-5/24}$ [40], and recent work suggests that the scaling behaviour is described by $\mathrm{SLE}_{8/3}$ [37], but neither has been proved. The case of $d = 1$ is of interest for weakly self-avoiding walk, where a fairly complete understanding has been obtained [33]. More about mathematical results for self-avoiding walk can be found in [8,38].

In the present paper, we prove that the critical two-point function of the continuous-time weakly self-avoiding walk is asymptotic to a multiple of $|x|^{-2}$ as $|x| \to \infty$, in dimension $d = 4$. This is a statement that the critical exponent η exists and is equal to zero. The proof is based on a rigorous renormalisation group method; a summary of the method and proof is given in [12]. Early indications of the critical nature of the dimension $d = 4$ were given in [3,9], following proofs of triviality of ϕ^4 field theory above dimension 4 [2,25].

Logarithmic corrections to scaling are common in statistical mechanical models at the upper critical dimension, and are predicted for the susceptibility and correlation length and several other interesting quantities [10,35,45], but not for the leading decay of the critical two-point function of the 4-dimensional self-avoiding walk. In [5], it is proved that the susceptibility of the 4-dimensional weakly self-avoiding walk does have a logarithmic correction to scaling, with exponent $\frac{1}{4}$. We now extend the methods of [5] to study the critical two-point function.

We use an integral representation to rewrite the two-point function of the continuous-time weakly self-avoiding walk as the two-point function of a supersymmetric field theory, and apply a rigorous renormalisation group argument to analyse the field theory.

Our proof involves an extension of the ideas and structure developed in [5], and to avoid repetition we refer below frequently to [5] for ideas and notation that apply without modification to our present purpose. A feature present here but not in [5] is the use of a complex observable field σ; this requires aspects of [6,17–19] concerning observables that were not used in [5]. A similar extension was used to study correlations of the dipole gas in [23].

Our general approach applies more widely. In [44], it has been extended to prove existence of logarithmic corrections to scaling for 4-dimensional critical networks of weakly self-avoiding walks, and for critical correlation functions of the 4-dimensional n-component $|\varphi|^4$ spin model.

1.2. Main result. We now define the two-point function for continuous time weakly self-avoiding walk, and state our main result. Fix a dimension $d > 0$. Let X be the stochastic process on $\mathbb{Z}^d$ with right-continuous sample paths, that takes its steps at the times of the events of a rate-$2d$ Poisson process. Steps are independent both of the Poisson process and of all other steps, and are taken uniformly at random to one of the $2d$ nearest neighbours of the current position. Let E_a denote the corresponding expectation for the process started at $X(0) = a$. The *local time* at x up to time T is defined by $L_{x,T} = \int_0^T \mathbb{1}_{X(s)=x} ds$, and the *self-intersection local time* up to time T is $I(T) = \sum_{x\in\mathbb{Z}^d} L_{x,T}^2$. The continuous-time weakly self-avoiding walk *two-point function* is then defined by

$$G_{g,\nu}(a,b) = \int_0^\infty E_a\left(e^{-gI(T)} \mathbb{1}_{X(T)=b}\right) e^{-\nu T} dT, \tag{1.1}$$

where $g > 0$, and ν is a parameter (possibly negative) chosen so that the integral converges. By translation invariance, $G_{g,\nu}(a,b)$ only depends on a, b via $a - b$. For $d = 4$, the continuous-time weakly self-avoiding walk is identical to the lattice Edwards model; see [38, Section 10.1].

In (1.1), self-intersections are suppressed by the factor $e^{-gI(T)}$. In the limit $g \to \infty$, if ν is simultaneously sent to $-\infty$ in a suitable g-dependent manner, it is known that the limit of the two-point function (1.1) is a multiple of the two-point function of the standard discrete-time strictly self-avoiding walk [13]. The model defined by (1.1) is predicted to be in the same universality class as the strictly self-avoiding walk for all $g > 0$. Our analysis is restricted to small $g > 0$.

The *susceptibility* is defined by

$$\chi_g(\nu) = \sum_{b \in \mathbb{Z}^d} G_{g,\nu}(a,b), \tag{1.2}$$

and the critical value $\nu_c(g)$ is defined by $\nu_c(g) = \inf\{\nu \in \mathbb{R} : \chi_g(\nu) < \infty\}$. It is proved in [5, Lemma A.1] that $\nu_c = \nu_c(g,d) \in (-\infty, 0]$ for all $g > 0$ and $d > 0$, and that moreover

$$\chi_g(\nu) < \infty \quad \text{if and only if} \quad \nu > \nu_c. \tag{1.3}$$

Moreover, it is shown in [5, Theorem 1.2] that for $d = 4$, as $g \downarrow 0$,

$$\nu_c(g) = -\mathsf{a} g(1 + O(g)), \tag{1.4}$$

where the positive constant a is given by $\mathsf{a} = -2\Delta_{00}^{-1}$. In particular, (1.4) implies that $\nu_c(g) < 0$ for small positive g.

Our main result is the following theorem which gives the decay of the critical two-point function in dimension 4, for sufficiently small g.

Theorem 1.1. *Let $d = 4$. There exists $\delta > 0$ such that for each $g \in (0, \delta)$ there exists $c(g) = (2\pi)^{-2}(1 + O(g))$ such that as $|a - b| \to \infty$,*

$$G_{g,\nu_c(g)}(a,b) = \frac{c(g)}{|a-b|^2}\left(1 + O\left(\frac{1}{\log|a-b|}\right)\right). \tag{1.5}$$

In [12], an extension of Theorem 1.1 states that the critical two-point function has decay $|a-b|^{2-d}$ for all dimensions $d \geq 4$, but [12] provides only a sketch of proof. Our principal interest is the critical dimension $d = 4$, and we provide the details of the proof for $d = 4$ here. The restriction to $d = 4$ avoids additional complications required to handle general high dimensions. We intend to return to the general case in a future publication.

We define the Laplacian Δ on $\mathbb{Z}^d$ by $(\Delta f)_x = \sum_{e:|e|=1}(f_{x+e} - f_x)$. For $g = 0$ and $\nu \geq 0$, the two-point function is given by $G_{0,\nu}(a,b) = (-\Delta + \nu)^{-1}_{ab}$, and $\nu_c(0) = 0$. Theorem 1.1 proves that for $d = 4$ and small positive g, the critical two-point function has the same $|a-b|^{-2}$ decay as the lattice Green function $-\Delta^{-1}_{ab}$ on $\mathbb{Z}^4$. In contrast, for $\nu > \nu_c(g)$, $G_{g,\nu}(a,b)$ decays exponentially as $|a-b| \to \infty$; an elementary proof is sketched below in Proposition 2.1.

In [5, Section 4], it is shown that the susceptibility of the weakly self-avoiding walk is *equal* to the susceptibility of the simple random walk with renormalised diffusion constant (field strength) $1 + z_0$ and killing rate (mass) m^2, with z_0 and m^2 functions of g, ε with $\varepsilon = \nu - \nu_c(g)$. More precisely,

$$\chi_g(\nu) = \frac{1 + z_0}{m^2}, \tag{1.6}$$

with $z_0 = O(g)$ and with m^2 asymptotic to a multiple of $\varepsilon(\log \varepsilon^{-1})^{-1/4}$ as $\varepsilon \downarrow 0$. In the proof of Theorem 1.1 we extend this correspondence and prove that (with $z_0 = z_0(g, 0)$)

$$G_{g,\nu_c(g)}(a, b) \sim (1 + z_0)(-\Delta_{\mathbb{Z}^d})^{-1}_{ab} \quad (|a - b| \to \infty), \tag{1.7}$$

which shows that the critical interacting two-point function is asymptotic to the critical non-interacting two-point function, with the *same* renormalised diffusion constant.

The proof of Theorem 1.1 uses a supersymmetric integral representation for the two-point function, which requires us to work first in finite volume and with $\nu > \nu_c$. Because of this, our analysis initially stays slightly away from the critical point. A related issue is that the Laplacian annihilates constants in finite volume, and hence is not invertible without the addition of some mass term. Ultimately, we first take the infinite volume limit with $\nu > \nu_c$, and then let $\nu \downarrow \nu_c$.

A variant of the 4-dimensional Edwards model was analysed in [34] using a renormalisation group method. Although this variant is not a model of walks taking steps in a lattice, it is presumably in the same universality class as the 4-dimensional self-avoiding walk, and the results of [34] are of a similar nature to ours. For the 4-dimensional φ^4 model, related results were obtained via block spin renormalisation in [26–28,32], and via partially renormalised phase space expansion in [24]. Our method is applied to the n-component $|\varphi|^4$ model in [7,44].

2. Integral Representation for the Two-Point Function

The proof of Theorem 1.1 is based on an integral representation for a finite volume approximation of the two-point function. To discuss this, we first show how the two-point function can be approximated by a two-point function on a finite torus.

2.1. Finite volume approximation. Let $L \geq 3$ and $N \geq 1$ be integers, and let $\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ denote the discrete torus of side L^N. We are ultimately interested in the limit $N \to \infty$, and regard Λ_N as a finite volume approximation to $\mathbb{Z}^d$. It is convenient at times to consider Λ_N to be a box (approximately) centred at the origin in $\mathbb{Z}^d$, without identifying opposite sides to create the torus. For fixed $a, b \in \mathbb{Z}^d$, we can then regard a, b as points in Λ_N provided N is large enough, and we make this identification throughout the paper. In particular, we tacitly assume that N is sufficiently large to contain given a, b.

For $a, b \in \Lambda_N$, let

$$G_{N,g,\nu}(a, b) = \int_0^\infty E_a^{\Lambda_N}\left(e^{-gI(T)} \mathbb{1}_{X(T)=b}\right) e^{-\nu T} dT, \tag{2.1}$$

where $E_a^{\Lambda_N}$ denotes the continuous-time simple random walk on the torus Λ_N, started from the point a. By the Cauchy–Schwarz inequality, $T = \sum_{x \in \Lambda} L_T^x \leq (|\Lambda| I(T))^{1/2}$ and hence $I(T) \geq T^2/|\Lambda|$, from which we conclude that the integral (2.1) is finite for all $g > 0$ and $\nu \in \mathbb{R}$. The following proposition shows that the infinite volume two-point function (1.1) can be approximated by the finite volume two-point function (2.1), and that it is possible to study the critical two-point function on $\mathbb{Z}^d$ in the double limit in which first $N \to \infty$ and then $\nu \downarrow \nu_c$.

Proposition 2.1. *Let $d > 0$, $g > 0$, and $\nu > \nu_c(g)$. Then $G_\nu(a,b)$ decays exponentially in $|a-b|$, and*

$$G_{g,\nu}(a,b) = \lim_{N\to\infty} G_{N,g,\nu}(a,b). \tag{2.2}$$

Also, for all $\nu \ge \nu_c(g)$,

$$G_{g,\nu}(a,b) = \lim_{\nu'\downarrow\nu} \lim_{N\to\infty} G_{N,g,\nu'}(a,b). \tag{2.3}$$

Proof. We fix $g > 0$ and drop it from the notation. Once we prove (2.2), then (2.3) follows because, by monotone convergence, $G_{g,\nu}(a,b)$ is right continuous for $\nu \ge \nu_c(g)$.

Let $c_T(a,b) = E_a(e^{-gI(T)}\mathbb{1}_{X(T)=b})$ and $c_{N,T}(a,b) = E_a^{\Lambda_N}(e^{-gI(T)}\mathbb{1}_{X(T)=b})$. Fix $\nu > \nu' > \nu_c$ and $S > 0$. By the triangle inequality,

$$|\,G_\nu(a,b) - G_{N,\nu}(a,b)\,| \le \int_0^S |c_T(a,b) - c_{N,T}(a,b)| e^{-\nu T}\, dT + \int_S^\infty c_T(a,b) e^{-\nu T}\, dT + \int_S^\infty c_{N,T}(a,b) e^{-\nu T}\, dT. \tag{2.4}$$

For the analysis of the right-hand side of (2.4), we define $\chi_N(\nu) = \sum_{b\in\Lambda} G_{N,\nu}(a,b)$, and recall from [5, Lemma 2.1] that $\chi_N(\nu) \le \chi(\nu)$. From this it follows that

$$\limsup_{N\to\infty} G_{N,\nu'}(a,b) \le \limsup_{N\to\infty} \chi_N(\nu') \le \chi(\nu') < \infty. \tag{2.5}$$

Let $\delta = \nu - \nu' > 0$. Then

$$\int_S^\infty c_T(a,b) e^{-\nu T}\, dT \le e^{-\delta S} G_{\nu'}(a,b), \tag{2.6}$$

$$\limsup_{N\to\infty} \int_S^\infty c_{N,T}(a,b) e^{-\nu T}\, dT \le e^{-\delta S} \limsup_{N\to\infty} G_{N,\nu'}(a,b). \tag{2.7}$$

This shows that the last two terms in (2.4) can be made as small as desired, uniformly in N, by choosing S large.

To estimate the first contribution, let $(Y_t)_{t\ge 0}$ be a rate-$2d$ Poisson process with corresponding probability measure P. Since contributions to the difference $|c_T - c_{N,T}|$ only arise from walks that reach the inner boundary $\partial\Lambda$ of the torus (identified with a subset of $\mathbb{Z}^d$ so that it does have a boundary), for any $0 \le T \le S$ we have

$$\begin{aligned} |c_T(a,b) - c_{N,T}(a,b)| &\le E_a\left(e^{-gI(T)}\mathbb{1}_{\{X([0,T])\cap\partial\Lambda\ne\varnothing\}}\right) \\ &\quad + E_a^\Lambda\left(e^{-gI(T)}\mathbb{1}_{\{X([0,T])\cap\partial\Lambda\ne\varnothing\}}\right) \\ &\le P_a\left\{X\big([0,T]\big)\cap\partial\Lambda\ne\varnothing\right\} + P_a^\Lambda\left\{X\big([0,T]\big)\cap\partial\Lambda\ne\varnothing\right\} \\ &\le 2P\{Y_T \ge \operatorname{diam}(\Lambda)\} \le 2P\{Y_S \ge \operatorname{diam}(\Lambda)\}, \end{aligned} \tag{2.8}$$

and the right-hand side goes to zero as $N \to \infty$ with S fixed. By this estimate, the first integral in (2.4) converges to 0 as $N \to \infty$, for any fixed S. This completes the proof of (2.2) and hence of (2.3).

We do not use the exponential decay in this paper, and its proof is standard, so we only sketch the argument. Given any $\alpha > 0$, we write $x = b - a$ and make the division

$$G_\nu(a,b) = \int_0^{\alpha|x|} c_T(a,b) e^{-\nu T} dT + \int_{\alpha|x|}^\infty c_T(a,b) e^{-\nu T} dT. \tag{2.9}$$

For the second integral on the right-hand side, we set $c_T = \sum_{b\in\mathbb{Z}^d} c_T(a,b)$ and use $c_T(a,b) \le c_T$. It can be shown that c_T obeys $c_{S+T} \le c_S c_T$ and that this implies $c_T^{1/T} \to e^{\nu_c}$ as $T \to \infty$, from which we see that $c_T(a,b) \le C_\epsilon e^{(\nu_c+\epsilon)T}$ for any $\epsilon > 0$. This gives exponential decay in x for the second integral. For the first integral, we recall the Chernoff estimate for the Poisson distribution, in the form that if X is Poisson with mean λ and $k > \lambda$, then $P(X > k) \le e^{-\lambda}(e\lambda/k)^k$. Since a walk can travel from a to b in time T only if the number of steps taken is at least x, it follows from the Chernoff bound that if $2d\alpha < 1$ and $T \le \alpha|x|$ then

$$c_T(a,b) \le P(Y_T \ge |x|) \le e^{-2dT}(2dTe)^{|x|}|x|^{-|x|} \le (2de\alpha)^{|x|}. \tag{2.10}$$

By choosing α sufficiently small depending on ν (recall that $\nu < 0$ is possible), we see that the first term on the right-hand side of (2.9) also exhibits exponential decay in x.

□

2.2. Integral representation. We use the supersymmetric integral representation for the two-point function discussed in detail in [5, Section 3]. We refer to that discussion for further details, notation, and definitions, and here we recall the minimum needed for our present purposes.

In terms of the complex boson field $\phi, \bar\phi$ and conjugate fermion fields $\psi, \bar\psi$ introduced in [5, Section 3], and using the same notation, for $x \in \Lambda$ we define the differential forms

$$\tau_x = \phi_x \bar\phi_x + \psi_x \wedge \bar\psi_x, \tag{2.11}$$

$$\tau_{\Delta,x} = \frac{1}{2}\Big(\phi_x(-\Delta\bar\phi)_x + (-\Delta\phi)_x\bar\phi_x + \psi_x \wedge (-\Delta\bar\psi)_x + (-\Delta\psi)_x \wedge \bar\psi_x\Big), \tag{2.12}$$

where $\Delta = \Delta_\Lambda$ is the lattice Laplacian on Λ given by $\Delta\phi_x = \sum_{y:|y-x|=1}(\phi_y - \phi_x)$. Here $\wedge$ denotes the wedge product; we drop the wedge from the notation subsequently with the understanding that forms are always multiplied using this anti-commutative product.

Let $\mathbb{E}_C$ denote the Gaussian super-expectation with covariance matrix C, as defined in [5, Definition 3.2]. In [5, (4.7)–(4.8)], it is shown that for $N < \infty$, $g > 0$, $\nu \in \mathbb{R}$, $m^2 > 0$, and $z_0 > -1$,

$$G_{N,g,\nu}(a,b) = (1+z_0)\mathbb{E}_C\left(e^{-U_0(\Lambda)}\bar\phi_a\phi_b\right), \tag{2.13}$$

where $C = (-\Delta + m^2)^{-1}$,

$$U_0(\Lambda) = \sum_{x\in\Lambda}\big(g_0\tau_x^2 + \nu_0\tau_x + z_0\tau_{\Delta,x}\big), \tag{2.14}$$

and

$$g_0 = g(1+z_0)^2, \qquad \nu_0 = (1+z_0)\nu - m^2. \tag{2.15}$$

The identity (2.13) is a rewriting of an identity from [11,15] that was inspired by [39,41]; see also [16, Theorem 5.1] for a self-contained proof.

In [5], we also use (2.13), but there write V instead of U. In the present paper, we use V for an extension of U that incorporates also an *observable field*, discussed next.

2.3. Observable field. We introduce an external field $\sigma \in \mathbb{C}$ and define

$$V_0(\Lambda) = U_0(\Lambda) - \sigma\bar{\phi}_a - \bar{\sigma}\phi_b. \tag{2.16}$$

We refer to σ as the *observable field*. Then we can compute the two-point function using the identity

$$G_{N,g,\nu}(a,b) = (1+z_0)\frac{\partial^2}{\partial\sigma\partial\bar{\sigma}}\Big|_0 \mathbb{E}_C e^{-V_0(\Lambda)}, \tag{2.17}$$

which follows from (2.13). To prove Theorem 1.1, we analyse the derivative of the Gaussian super-expectation on the right-hand side of (2.17), without making further reference to its connection with self-avoiding walks.

An external field is also employed to analyse the susceptibility in [5, Section 4.1], but in a different way. There the external field is a test function $J : \Lambda \to \mathbb{R}$, and $U_0(\Lambda)$ becomes replaced by $U_0(\Lambda) - \sum_{x\in\Lambda}(J_x\bar{\phi}_x + \bar{J}_x\phi_x)$. In [5] the interest is in the *constant* external field $J_x = 1$ for all $x \in \Lambda$, and the macroscopic regularity of this test function is important. Here, in contrast, (2.16) corresponds to setting $J_x = \sigma\mathbb{1}_{x=a}$ and $\bar{J}_x = \bar{\sigma}\mathbb{1}_{x=b}$ (so the two are not precisely complex conjugates). To work with such a *singular* external field, we use a different analysis based on ideas prepared in [17–19]. It would be desirable to allow all coupling constants to be spatially varying, not just the external field. This extension has been achieved for hierarchical models in [1].

Our attention to the dependence on the external field is quite limited: we only wish to compute the derivative (2.17), and as such we make no use of any functional dependence on $\sigma, \bar{\sigma}$ beyond expansion to second order, i.e., including terms of order $1, \sigma, \bar{\sigma}, \sigma\bar{\sigma}$. We formalise this notion by identifying quantities with the same expansion to second order, as follows. Recall the space $\mathcal{N}$ of even differential forms introduced in [5, Section 3.1], which we now denote instead by $\mathcal{N}^{\varnothing}$. As in [5, (3.5)], an element of $\mathcal{N}^{\varnothing}$ has the form

$$\sum_{x,y} F_{x,y}(\phi,\bar{\phi})\psi^x\bar{\psi}^y. \tag{2.18}$$

We extend this notion by now allowing the coefficients $F_{x,y}$ to be functions of the external field $\sigma, \bar{\sigma}$ as well as of the boson field $\phi, \bar{\phi}$. Let $\mathcal{N}$ be the resulting algebra of differential forms. Let $\mathcal{I}$ denote the ideal in $\mathcal{N}$ consisting of those elements of $\mathcal{N}$ whose expansion to second order in the external field is zero. The quotient algebra $\mathcal{N}/\mathcal{I}$ then has the direct sum decomposition

$$\mathcal{N}/\mathcal{I} = \mathcal{N}^{\varnothing} \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab}, \tag{2.19}$$

where elements of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$ are respectively given by elements of $\mathcal{N}^{\varnothing}$ multiplied by σ, by $\bar{\sigma}$, and by $\sigma\bar{\sigma}$. For example, $\phi_x\bar{\phi}_y\psi_x\bar{\psi}_x \in \mathcal{N}^{\varnothing}$, and $\sigma\bar{\phi}_x \in \mathcal{N}^a$. There are canonical projections $\pi_\alpha : \mathcal{N} \to \mathcal{N}^\alpha$ for $\alpha \in \{\varnothing, a, b, ab\}$. We use the abbreviation $\pi_* = 1 - \pi_\varnothing = \pi_a + \pi_b + \pi_{ab}$. The quotient space is used also in [17–19], e.g., around [17, (1.60)]. Since we have no further use of $\mathcal{N}$, to simplify the notation we henceforth write simply $\mathcal{N}$ instead of $\mathcal{N}/\mathcal{I}$. As functions of the external field, elements of $\mathcal{N}$ are then polynomials of degree at most 2, by definition. For example, we identify $e^{\sigma\bar\phi_a+\bar\sigma\phi_b}$ and $1+\sigma\bar\phi_a+\bar\sigma\phi_b+\sigma\bar\sigma\bar\phi_a\phi_b$, as both are elements of the same equivalence class in the quotient space.

3. Renormalisation Group Map

In this section, we sketch only the most important ingredients of our renormalisation group method from [5,6,18,19]. A more detailed introduction is given in [5] (see also [7,12]).

3.1. Progressive Gaussian integration. We use decompositions of the covariances $C = (-\Delta_{\Lambda_N}+m^2)^{-1}$ and $(-\Delta_{\mathbb{Z}^4}+m^2)^{-1}$ for the torus and $\mathbb{Z}^4$, respectively, as discussed in [5, Section 5.1], and we use the same notation as in [5]. These decompositions take the form

$$(-\Delta_{\mathbb{Z}^4}+m^2)^{-1} = \sum_{j=1}^{\infty} C_j \qquad (m^2\in[0,\delta)), \tag{3.1}$$

$$C = (-\Delta_{\Lambda_N}+m^2)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N} \qquad (m^2\in(0,\delta)), \tag{3.2}$$

where the covariance $C_{N,N}$ is special because of the effect of the torus. The particular finite-range decomposition we use is developed in [4,14], with properties given in [6]. The finite-range condition is the statement that $C_{j;x,y}=0$ when $|x-y|\geq \frac{1}{2}L^j$; this condition is important for results we use from [18,19]. As discussed in [5, Section 5.1], the Gaussian super-expectation of $F\in\mathcal{N}$ can be carried out progressively, via the identity

$$\mathbb{E}_C\theta F = \big(\mathbb{E}_{C_{N,N}}\theta\circ\mathbb{E}_{C_{N-1}}\theta\circ\cdots\circ\mathbb{E}_{C_1}\theta\big)F. \tag{3.3}$$

The external field $\sigma,\bar\sigma$ is treated as a constant by the super-expectation. To compute $\mathbb{E}_C e^{-V_0(\Lambda)}$ of (2.17), we use (3.3), and define

$$Z_0 = e^{-V_0(\Lambda)}, \quad Z_{j+1} = \mathbb{E}_{C_{j+1}}\theta Z_j \quad (j<N). \tag{3.4}$$

For $j+1=N$, we interpret the convolution $\mathbb{E}_{C_{j+1}}\theta$ as the convolution $\mathbb{E}_{C_{N,N}}\theta$, i.e., the last covariance is taken to be the one appropriate for the torus Λ_N. Then the desired expectation is given by $Z_N^0(0)$, where the superscript 0 denotes projection onto the degree-0 part of the differential form (i.e., the fermion field is set to 0) and the argument 0 means that the boson field is evaluated at $\phi=0$. Thus we are led to study the recursion $Z_j\mapsto Z_{j+1}$. By (2.17), the two-point function is given by

$$G_{N,g,\nu}(a,b) = (1+z_0)Z^0_{N;\sigma\bar\sigma}(0), \tag{3.5}$$

where $F_{\sigma\bar\sigma}\in\mathcal{N}^{\varnothing}$ denotes the coefficient of $\sigma\bar\sigma$ in $F\in\mathcal{N}$, i.e., $\pi_{ab}Z_N^0 = \sigma\bar\sigma Z^0_{N;\sigma\bar\sigma}$.

3.2. The interaction functional. Let $\mathcal{Q}^{(0)}$ and $\mathcal{Q}^{(1)}$ respectively denote the vector space of local polynomials of the form

$$V^{(0)} = g\tau^2 + \nu\tau + z\tau_\Delta - \lambda_a \mathbb{1}_a \sigma\bar{\phi} - \lambda_b \mathbb{1}_b \bar{\sigma}\phi, \tag{3.6}$$

$$V^{(1)} = V^{(0)} - \tfrac{1}{2}\sigma\bar{\sigma}(q_a \mathbb{1}_a + q_b \mathbb{1}_b), \tag{3.7}$$

where $g, \nu, z \in \mathbb{R}$, $\lambda_a, \lambda_b, q_a, q_b \in \mathbb{C}$, and the indicator functions are defined by the Kronecker delta $\mathbb{1}_{a,x} = \delta_{a,x}$. (We believe that in fact only real coupling constants $\lambda_a, \lambda_b, q_a, q_b$ are required, but we did not prove this and it costs us nothing to permit complex coupling constants.) The terms involving σ are referred to as *observables*, while the terms involving τ^2, τ, and τ_Δ are *bulk terms*. We frequently identify elements of $\mathcal{Q}^{(0)}$ and $\mathcal{Q}^{(1)}$ as sequences $V^{(0)} = (g, \nu, z, \lambda_a, \lambda_b)$, $V^{(1)} = (g, \nu, z, \lambda_a, \lambda_b, q_a, q_b)$, and typically write $U = \pi_\varnothing V = (g, \nu, z)$.

Recall from [5, Section 5.3] the set $\mathcal{B}_j$ of scale-j blocks, and the set $\mathcal{P}_j$ of scale-j polymers in Λ. We also recall from [5, Section 5.4] the interaction functional $I_j : \mathcal{Q}^{(0)} \times \mathcal{P}_j \to \mathcal{N}$ defined for $B \in \mathcal{B}_j$, $X \in \mathcal{P}_j$, and $V \in \mathcal{Q}^{(0)}$ by

$$I_j(V, B) = e^{-V(B)}(1 + W_j(V, B)), \qquad I_j(V, X) = \prod_{B \in \mathcal{B}_j} I_j(V, B), \tag{3.8}$$

where W_j is an explicit quadratic function of V defined in [6]. In particular, $W_0 = 0$. We often write simply $I_j(X) = I_j(V, X)$. By (3.8), $I_0(V, X) = e^{-V(X)}$ for all $X \subset \Lambda$, with $V(X) = \sum_{x \in X} V_x$.

Motivation for the definition (3.8) is given in [6, Section 2]. In the present paper, we do not give the details of the definitions of W_j and I_j since we do not need them here. They are, however, important in [6,18,19] and we rely on results from those references. The V domain of I_j is larger here than in [5], due to the presence of observables, but the larger domain is permitted and present in the analysis of [6,18,19].

3.3. Renormalisation group coordinates. Given $F_1, F_2 : \mathcal{P}_j \to \mathcal{N}$, we define the *circle product* $F_1 \circ F_2 : \mathcal{P}_j \to \mathcal{N}$ by

$$(F_1 \circ F_2)(Y) = \sum_{X \in \mathcal{P}_j : X \subset Y} F_1(X) F_2(Y \backslash X) \qquad (Y \in \mathcal{P}_j). \tag{3.9}$$

The terms $X = \varnothing$ and $X = \Lambda$ are included in the summation on the right-hand side, and we demand that all functions $F : \mathcal{P}_j \to \mathcal{N}$ obey $F(\varnothing) = 1$. The circle product depends on the scale j, is associative, and is also commutative due to our restriction in $\mathcal{N}$ to forms of even degree. Its identity element is $\mathbb{1}_\varnothing$, defined by $\mathbb{1}_\varnothing(X) = 1$ if X is empty, and otherwise $\mathbb{1}_\varnothing(X) = 0$.

In the definition of I_0 we set $V = V_0$, with V_0 defined in (2.16), so that $I_0(X) = I_0(V_0, X) = e^{-V_0(X)}$ for all $X \subset \Lambda$. Let $K_0 : \mathcal{P}_0 \to \mathcal{N}$ be defined by $K_0 = \mathbb{1}_\varnothing$, and set $q_0 = 0$. Then $Z_0 = I_0(V_0, \Lambda)$ of (3.4) is also given by

$$Z_0 = I_0(\Lambda) = e^{q_0 \sigma\bar{\sigma}}(I_0 \circ K_0)(\Lambda). \tag{3.10}$$

Our strategy is to define $q_j \in \mathbb{C}$, $V_j \in \mathcal{Q}^{(0)}$, $K_j : \mathcal{P}_j \to \mathcal{N}$, and set $I_j = I_j(V_j)$, so as to maintain this form as

$$Z_j = e^{q_j \sigma\bar{\sigma}}(I_j \circ K_j)(\Lambda) \qquad (0 \le j \le N) \tag{3.11}$$

in the recursion $Z_j \mapsto Z_{j+1} = \mathbb{E}_{C_{j+1}}\theta Z_j$ of (3.4), with the initial condition given by (3.10). At the final scale $j = N$, the only two polymers are the single block $\Lambda = \Lambda_N$ and the empty set $\varnothing$, and since $I_j(\varnothing) = K_j(\varnothing) = 1$, by assumption, (3.11) simply reads

$$Z_N = e^{q_N \sigma\bar\sigma}(I_N \circ K_N)(\Lambda) = e^{q_N \sigma\bar\sigma}(I_N(\Lambda) + K_N(\Lambda)). \tag{3.12}$$

If we set $\delta q_{j+1} = q_{j+1} - q_j$, then (3.11) can equivalently be written as

$$\mathbb{E}_{C_{j+1}}\theta(I_j \circ K_j)(\Lambda) = e^{\delta q_{j+1}\sigma\bar\sigma}(I_{j+1} \circ K_{j+1})(\Lambda). \tag{3.13}$$

In view of (3.13), and since I_j is determined by V_j, we are led to study the *renormalisation group map*

$$(V_j, K_j) \mapsto (\delta q_{j+1}, V_{j+1}, K_{j+1}). \tag{3.14}$$

The coupling constants of $V_j \in \mathcal{Q}^{(0)}$ are written as $g_j, \nu_j, z_j, \lambda_{a,j}, \lambda_{b,j}$. Ultimately we express the two-point function in terms of the sequence (q_j), so this sequence is fundamentally important in the proof of Theorem 1.1. Our construction creates δq_j as the average

$$\delta q_j = \frac{1}{2}(\delta q_{a,j} + \delta q_{b,j}) \tag{3.15}$$

of two sequences $\delta q_{a,j}$ and $\delta q_{b,j}$ (see [19, (1.50)]).

3.4. Renormalisation group map. To implement the above strategy, given suitable $V_j \in \mathcal{Q}^{(0)}$ and $K_j : \mathcal{P}_j \to \mathcal{N}$, we define $\delta q_{j+1} \in \mathbb{C}$, $V_{j+1} \in \mathcal{Q}^{(0)}$ and $K_{j+1} : \mathcal{P}_{j+1} \to \mathcal{N}$ in such a way that

$$\begin{aligned} Z_{j+1} &= \mathbb{E}_{C_{j+1}}\theta Z_j = e^{q_j\sigma\bar\sigma}\mathbb{E}_{C_{j+1}}\theta(I_j \circ K_j)(\Lambda) \\ &= e^{q_{j+1}\sigma\bar\sigma}(I_{j+1} \circ K_{j+1})(\Lambda) \quad (j < N). \end{aligned} \tag{3.16}$$

Thus (3.11) does retain its form under progressive integration. We use the explicit choice for the renormalisation group map (3.14) that is given in [19], from now on. This choice achieves (3.16) for fixed $j < N$, assuming that (V_j, K_j) is in an appropriate domain, and it provides good estimates for $(\delta q_{j+1}, V_{j+1}, K_{j+1})$.

To simplify the notation, we set $V_+ = (\delta q_+, V_+^{(0)}) \in \mathcal{Q}^{(1)}$ and write (3.14) as $(V, K) \mapsto (V_+, K_+)$. We typically drop subscripts j and write $+$ in place of $j + 1$, also leave the dependence of the maps on the mass parameter m^2 of the covariance $(-\Delta + m^2)^{-1}$ implicit.

3.5. Bulk flow. By [19, (1.68)], the renormalisation group map has the property

$$\pi_\varnothing V_+(V, K) = V_+(\pi_\varnothing V, \pi_\varnothing K), \quad \pi_\varnothing K_+(V, K) = K_+(\pi_\varnothing V, \pi_\varnothing K). \tag{3.17}$$

Thus, under (3.14), the *bulk coordinates* $(\pi_\varnothing V_j, \pi_\varnothing K_j)$ satisfy a closed evolution equation of their own. We denote its evolution map by $(V_+^\varnothing, K_+^\varnothing)$ and write $U = \pi_\varnothing V$. Then (3.14) reduces to

$$(U_{j+1}, \pi_\varnothing K_{j+1}) = (V_+^\varnothing(U_j, \pi_\varnothing K_j), K_+^\varnothing(U_j, \pi_\varnothing K_j)). \tag{3.18}$$

The construction of a critical global renormalisation group flow of the bulk coordinates (3.18) is achieved in [5]. Namely, there is a construction of $(U_j, \pi_\varnothing K_j)$ for $0 \le j \le N$ such that (3.18) holds for all $0 \le j \le N$. This construction provides detailed information about the sequence U_j, and good estimates on $\pi_\varnothing K_j$, sufficient for studying the infinite volume limit at the critical point. In Sect. 4, we use this bulk flow to study observables.

It is convenient to change perspective on which variables are independent. The weakly self-avoiding walk has parameters g, ν. In (2.14), additional parameters m^2, g_0, ν_0, z_0 were introduced. For the moment we consider these as independent variables and do not consider g, ν directly. The relation between m^2, g_0, ν_0, z_0 and the original parameters g, ν is addressed in Sect. 3.6.

To state the result about the bulk flow, let $\bar{g}_j$ be the (m^2, g_0)-dependent sequence determined by $\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2$ with $\bar{g}_0 = g_0$ and with $\beta_j = \beta_j(m^2)$ defined in [5, (6.5)]. We also recall the sequence χ_j defined in [5, (6.7)], but its precise definition is not important for our present needs. It obeys $0 \le \chi_j \le 1$, eventually decays exponentially when $m^2 > 0$, and is identically equal to 1 when $m^2 = 0$. Also, by [5, Proposition 6.1] and [5, (8.22)] respectively,

$$\chi_j \bar{g}_j \le O\left(\frac{g_0}{1 + g_0 j}\right) \quad \text{uniformly in } (m^2, g_0) \in [0, \delta)^2, \tag{3.19}$$

$$\sum_{k=j}^{\infty} \chi_k \bar{g}_k^2 = O(\chi_j \bar{g}_j). \tag{3.20}$$

Without multiplication by χ_j, the sequence $\bar{g}_j$ converges to 0 when $m^2 = 0$ but not when $m^2 > 0$. (To apply (2.3), in which the limit $\nu \downarrow \nu_c$ follows the limit $N \to \infty$, we do consider limits $j \to \infty$ with $m^2 > 0$, corresponding to $\nu > \nu_c$, to prove Theorem 1.1.)

The following theorem is a reduced version of [5, Proposition (8.1)]. Some of its notation is explained after the statement.

Theorem 3.1. *Let $d = 4$ and let $\delta > 0$ be sufficiently small. There exist $M > 0$ and an infinite sequence of functions $U_j = (g_j^c, \nu_j^c, z_j^c)$ of $(m^2, g_0) \in [0, \delta)^2$, independent of $N \in \mathbb{N}$, such that:*

(i) *assuming $\sigma = 0$, given $N \in \mathbb{N}$, for initial conditions $U_0 = (g_0, \nu_0^c, z_0^c)$ with $g_0 \in (0, \delta)$, $K_0 = \mathbb{1}_\varnothing$, and with mass $m^2 \in [0, \delta)$, a flow $(U_j, K_j) \in \mathbb{D}_j^\varnothing$ exists such that (3.18) holds for all $j + 1 < N$, and given $m^2 \in [\delta L^{-2(N-1)}, \delta)$, also for $j + 1 = N$. Then, in particular,*

$$\|K_j\|_{\mathcal{W}_j} = \|\pi_\varnothing K_j\|_{\mathcal{W}_j} \le M \chi_j \bar{g}_j^3 \quad (j \le N) \tag{3.21}$$

and $g_j^c = O(\bar{g}_j)$. In addition, $z_j^c = O(\chi_j \bar{g}_j)$ and $\nu_j = O(\chi_j L^{-2j} \bar{g}_j)$.

(ii) *z_0^c, ν_0^c are continuous in $(m^2, g_0) \in [0, \delta)^2$.*

The definition of the $\mathcal{W}_j$ norm in (3.21) is discussed at length in [19], and we do not repeat the details here, as we now only need the fact that (3.21) with $j = N$ implies that

$$|\pi_\varnothing K_N^0(\Lambda, 0)| \le M \chi_N \bar{g}_N^3, \tag{3.22}$$

uniformly in $m^2 \in [\delta L^{-2(N-1)}, \delta)$, as a consequence of [19, (1.64)].

The $\mathcal{W}_j = \mathcal{W}_j(\tilde{s})$ norm depends on a parameter $\tilde{s} = (\tilde{m}^2, \tilde{g}) \in [0, \delta) \times (0, \delta)$. Its significance is discussed in [5, Section 6.3]. In particular, useful choices of this parameter depend on the scale j, as well as on approximate values of the mass parameter m^2 of the covariance and the coupling constant g_j. Throughout the paper, we use the convention that when the parameter $\tilde{s}$ is omitted, it is given by $\tilde{s} = s_j = (m^2, \tilde{g}_j(m^2, g_0))$. Here $\tilde{m}^2 = m^2$ is the mass parameter of the covariance, and $\tilde{g} = \tilde{g}_j$ is defined in terms of the initial condition g_0 by

$$\tilde{g}_j = \tilde{g}_j(m^2, g_0) = \bar{g}_j(0, g_0)\mathbb{1}_{j \le j_m} + \bar{g}_{j_m}(0, g_0)\mathbb{1}_{j > j_m}, \tag{3.23}$$

where the *mass scale* j_m is the smallest integer j such that $L^{2j} m^2 \ge 1$. By [5, Lemma 7.4],

$$\tilde{g}_j = \bar{g}_j + O(\bar{g}_j^2), \tag{3.24}$$

so the two sequences are the same to leading order. However, $\tilde{g}_j$ is more convenient for aspects of the analysis in [5].

The domain $\mathbb{D}_j^\varnothing = \mathbb{D}_j^\varnothing(\tilde{s})$ also depends on $\tilde{s}$ (with the same convention when the parameter is omitted) and is defined as follows. For the universal constant $C_{\mathcal{D}} \ge 2$ determined in [5], for $j < N$,

$$\mathbb{D}_j^\varnothing(\tilde{s}) = \{(g, \nu, z) \in \mathbb{R}^3 : C_{\mathcal{D}}^{-1}\tilde{g} < g < C_{\mathcal{D}}\tilde{g},\ L^{2j}|\nu|, |z| \le C_{\mathcal{D}}\tilde{g}\} \times B_{\mathcal{W}_j^\varnothing}(\alpha \tilde{\chi}_j \tilde{g}^3). \tag{3.25}$$

The first factor is the important stability domain defined in [18, (1.55)], restricted to the bulk coordinates and real scalars. In the second factor, $B_X(a)$ denotes the open ball of radius a centred at the origin of the Banach space X, and α is fixed in [5]; it can be taken to be $4M$ where M is the constant of Theorem 3.1. Compared to [19], we have replaced $\chi^{3/2}$ by χ for notational convenience. The space $\mathcal{W}^\varnothing$ is the restriction of $\mathcal{W}$ to elements $K \in \mathcal{W}$ with $\pi_* K(X) = 0$ for all polymers X. Since the renormalisation group acts triangularly, by (3.17), the distinction between $\mathcal{W}$ and $\mathcal{W}^\varnothing$ is unimportant for the bulk flow, and $\mathcal{W}^\varnothing$ is denoted by $\mathcal{W}$ in [5].

3.6. Change of variables. Theorem 3.1 is stated in terms of the parameters m^2, g_0, rather than the parameters g, ν of the weakly self-avoiding walk. The following proposition, proved in [5, Proposition 4.2(ii)], relates these sets of parameters via the functions z_0^c, ν_0^c of Theorem 3.1 and (2.15).

Proposition 3.2. *Let $d = 4$ and let $\delta_1 > 0$ be sufficiently small. There exists a function $[0, \delta_1)^2 \to [0, \delta)^2$, written $(g, \varepsilon) \mapsto (\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon))$, such that (2.15) holds with $\nu = \nu_c(g) + \varepsilon$, if $z_0 = z_0^c(\tilde{m}^2, \tilde{g}_0)$ and $\nu_0 = \nu_0^c(\tilde{m}^2, \tilde{g}_0)$. The functions $\tilde{m}, \tilde{g}_0$ are right-continuous as $\varepsilon \downarrow 0$, with $\tilde{m}^2(g, 0) = 0$, and $\tilde{m}^2(g, \varepsilon) > 0$ if $\varepsilon > 0$.*

We also write

$$\tilde{z}_0(g, \varepsilon) = z_0^c(\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon)), \qquad \tilde{\nu}_0(g, \varepsilon) = \nu_0^c(\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon)). \tag{3.26}$$

The functions $\tilde{z}_0, \tilde{\nu}_0$ are right-continuous as $\varepsilon \downarrow 0$. For the problem without observables, considered in [5], we analysed the sequence Z_j by choosing variables as follows. First, starting from (g, ν), Proposition 3.2 gives us $(\tilde{m}^2, \tilde{g}_0)$, and then Theorem 3.1 gives us an initial condition $U_0 = (\tilde{g}_0, \tilde{z}_0, \tilde{\nu}_0)$ for which there exists a global bulk flow of the renormalisation group map. In the next section, we extend this to include observables.

4. Observable Flow

It follows from Proposition 2.1 and (3.5) that

$$G_{g,\nu_c}(a,b) = \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty} G_{N,g,\nu_c+\varepsilon}(a,b) = \lim_{\varepsilon\downarrow 0}\Big((1+z_0)\lim_{N\to\infty} Z^0_{N;\sigma\bar\sigma}(0)\Big), \tag{4.1}$$

provided the parameters (m^2, g_0, ν_0, z_0) implicit on the right-hand side obey (2.15) with $\nu = \nu_c(g)+\varepsilon$. To analyse (4.1) via the renormalisation group flow, our remaining task is to supplement the bulk flow of Theorem 3.1 with the flow of the observable coupling constants $\lambda_{a,j}, \lambda_{b,j}, q_{a,j}, q_{b,j}$ and of the observable part $\pi_* K_j$ of K_j. In other words, we extend Theorem 3.1 to the case of nonzero σ. This is truly an extension, in the sense that the bulk flow needs no modification because the equations for $\lambda_{a,j}, \lambda_{b,j}, q_{a,j}, q_{b,j}, \pi_* K_j$ depend on but do not appear in the flow of $(g_j, z_j, \nu_j, \pi_\varnothing K_j)$ which corresponds to $\sigma = 0$, by (3.17). With the estimates provided by [19], we will prove Theorem 1.1 using the kind of perturbative calculations familiar in the physics literature, in a mathematically rigorous manner.

4.1. Perturbative flow of observables.

Definition 4.1. *Given $a, b \in \Lambda$, the* coalescence scale *j_{ab} is defined by*

$$j_{ab} = \big\lfloor \log_L(2|a-b|)\big\rfloor. \tag{4.2}$$

The coalescence scale is related to the finite-range property of the covariance decomposition mentioned in Sect. 3.1, namely that $C_{j;x,y} = 0$ if $|x-y| \ge \frac{1}{2}L^j$. Thus j_{ab} is such that $C_{j_{ab};a,b} = 0$, but $C_{j_{ab}+1;a,b}$ need not be zero. By definition, $L^{-2j_{ab}}$ is bounded above and below by multiples of $|a-b|^{-2}$, in fact $L^{j_{ab}} \le 2|a-b|$.

In [6], the flow of the coupling constants in V is computed at a perturbative level. The perturbative flow is without control of errors uniformly in the volume, and we address the uniform control below. The perturbative flow is determined by a map $V = (g,\nu,z,\lambda_a,\lambda_b,q_a,q_b) \mapsto V_{\rm pt} = (g_{\rm pt},\nu_{\rm pt},z_{\rm pt},\lambda_{a,\rm pt},\lambda_{b,\rm pt},q_{a,\rm pt},q_{b,\rm pt})$; here we are only interested in λ, q. The perturbative flow of λ, q is reported in [6, (3.34)–(3.35)] as the scale-dependent map $V \mapsto (\lambda_{\rm pt}, q_{\rm pt})$ given, for $x = a, b$, by

$$\lambda_{x,\rm pt} = \begin{cases} (1-\delta[\nu w^{(1)}])\lambda_x & (j+1 < j_{ab}) \\ \lambda & (j+1 \ge j_{ab}), \end{cases} \tag{4.3}$$

$$q_{x,\rm pt} = q_x + \lambda_a\lambda_b\, C_{j+1;a,b}. \tag{4.4}$$

In (4.3)–(4.4), j refers to the scale of the initial V, with $(\lambda_{\rm pt}, q_{\rm pt})$ being scale-$(j+1)$ objects. Also, $w^{(1)} = w^{(1)}_j = \sum_{x\in\Lambda}\sum_{i=1}^{j} C_{i;0,x}$, and

$$\delta[\nu w^{(1)}] = \nu^+ w^{(1)}_{j+1} - \nu w^{(1)}_j \quad \text{with } \nu^+ = \nu + 2gC_{j+1;0,0}. \tag{4.5}$$

The coalescence scale j_{ab} has the property that $q_{\rm pt} = 0$ if $q = 0$ for $j \le j_{ab}$ because the factor $C_{j+1;a,b}$ on the right-hand side of (4.4) is zero when $j+1 \le j_{ab}$. The considerations that lead to the stopping of the flow of λ at the coalescence scale in (4.3) are discussed in [6, Section 3.2].

As discussed above (3.13), it is convenient to express the renormalisation group map in terms of δq rather than q. For this, we identify elements $V \in \mathcal{Q}^{(0)}$ with elements of $\mathcal{Q}^{(1)}$ having $q_a = q_b = 0$, and, when $V \in \mathcal{Q}^{(0)}$ we write $\delta q_{\rm pt}$ instead of $q_{\rm pt}$.

4.2. A single renormalisation group step. Now we consider the renormalisation group map

$$(V, K) \mapsto (V_+^{(1)}, K_+) = (\delta q_+, V_+, K_+), \tag{4.6}$$

which pertains not only to the bulk, but also to the observable coupling constants as well as $\pi_* K = (\pi_a K, \pi_b K, \pi_{ab} K)$. To state the estimates we require from [19] for the map (4.6), we recall (3.25), and define similarly

$$\begin{aligned}\mathbb{D}_j(\tilde{s}) = \{&(g, \nu, z, \lambda_a, \lambda_b) \in \mathbb{R}^3 \times \mathbb{C}^2 : C_{\mathcal{D}}^{-1}\tilde{g} < g < C_{\mathcal{D}}\tilde{g},\ L^{2j}|\nu|, |z| \le C_{\mathcal{D}}\tilde{g},\\ &|\lambda_a|, |\lambda_b| \le C_{\mathcal{D}}\} \times B_{\mathcal{W}_j}(\alpha\tilde{\chi}_j\tilde{g}^3).\end{aligned} \tag{4.7}$$

The first factor is the same as [19, (1.55)], but restricted to real values. Compared to $\mathbb{D}^{\varnothing}$ of (3.25), the coupling constants λ_a, λ_b are included in $\mathbb{D}$ of (4.7). Also, the Banach spaces $\mathcal{W}_j = \mathcal{W}_j(\tilde{s})$ now pertain to K with components in $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$; these spaces are discussed in detail in [19, Sections 1.6–1.7]. The domain $\mathbb{D}^{\varnothing}$ is obtained by projecting both factors in the definition (4.7) by the appropriate definitions of $\pi_\varnothing$ on $\mathcal{Q}^{(0)}$ and $\mathcal{W}_j$ separately.

A j-dependent norm on $\mathcal{Q}^{(1)}$ is defined by

$$\|V\|_{\mathcal{Q}} = \max\{|g|, L^{2j}|\nu_j|, |z_j|, \ell_j\ell_{\sigma,j}|\lambda_a|, \ell_j\ell_{\sigma,j}|\lambda_b|, \ell_{\sigma,j}^2|q_a|, \ell_{\sigma,j}^2|q_b|\} \tag{4.8}$$

where

$$\ell_j = \ell_0 L^{-j}, \quad \ell_{\sigma,j} = 2^{(j-j_{ab})_+} L^{(j\wedge j_{ab})}\tilde{g}. \tag{4.9}$$

The significance of the weights $\ell_j, \ell_{\sigma,j}$ is explained in [18, Remark 3.3]; the constant $\ell_0 > 0$ is determined in [18, (1.73)] and is of no direct importance here.

The following theorem concerns a single renormalisation group step (3.14), with observables. It is a reduced version of the main result of [19], combining the relevant parts of [19, Theorems 1.10–1.11, 1.13] into a single statement. Such a result was used in [5, Theorem 6.3], but now observables are included in V and K. In fact, only the observable part of the statement is of interest here—the bulk flow is independent and has already been analysed—but it is convenient to state the theorem in its general form, applying to both bulk and observables simultaneously. The bounds on derivatives provided by [19] are not stated in Theorem 4.2 as they are not needed here.

The map $V_+^{(1)} = (\delta q_+, V_+)$ is a perturbation of the map V_{pt} discussed in Sect. 4.1, and it is convenient to describe it in terms of the difference

$$R_+(V, K) = V_+^{(1)}(V, K) - V_{\mathrm{pt}}(V). \tag{4.10}$$

Thus R_+ is an element of $\mathcal{Q}^{(1)}$ with components for all seven of the coupling constants $(g, \nu, z, \lambda_a, \lambda_b, \delta q_a, \delta q_b)$, and δq is defined by $\delta q = \frac{1}{2}(\delta q_a + \delta q_b)$. As in [5], considerable care is required to express the continuity of the maps R_+, K_+ in the mass parameter m^2, and we define the intervals

$$\mathbb{I}_j = \begin{cases}[0, \delta] & (j < N)\\ [\delta L^{-2(N-1)}, \delta] & (j = N),\end{cases} \tag{4.11}$$

and, for $\tilde{m}^2 \in \mathbb{I}_j$,

$$\tilde{\mathbb{I}}_j = \tilde{\mathbb{I}}_j(\tilde{m}^2) = \begin{cases} [\frac{1}{2}\tilde{m}^2, 2\tilde{m}^2] \cap \mathbb{I}_j & (\tilde{m}^2 \neq 0) \\ [0, L^{-2(j-1)}] \cap \mathbb{I}_j & (\tilde{m}^2 = 0). \end{cases} \tag{4.12}$$

For the statement of the theorem, we write $\tilde{s} = (\tilde{m}^2, \tilde{g})$ and $\tilde{s}_+ = (\tilde{m}^2, \tilde{g}_+)$. We assume $\frac{1}{2}\tilde{g}_+ \leq \tilde{g} \leq 2\tilde{g}_+$ and write $\tilde{\chi} = \chi_j(\tilde{m}^2)$. We subsequently use the explicit choice $\tilde{s} = s_j$ and $\tilde{s}_+ = s_{j+1}$, discussed in Sect. 3.5, and the choice of α mentioned below (3.25). Then in particular $\tilde{\chi} = \chi_j$.

Theorem 4.2. *Let $d = 4$. Let $C_{\mathcal{D}}$ and L be sufficiently large. There exist $M > 0$ and $\delta > 0$ such that for $\tilde{g} \in (0, \delta)$ and $\tilde{m}^2 \in \mathbb{I}_+$, and with the domain $\mathbb{D}$ defined using any $\alpha > M$, the maps*

$$R_+ : \mathbb{D}(\tilde{s}) \times \tilde{\mathbb{I}}_+(\tilde{m}^2) \to \mathcal{Q}^{(1)}, \quad K_+ : \mathbb{D}(\tilde{s}) \times \tilde{\mathbb{I}}_+(\tilde{m}^2) \to \mathcal{W}_+(\tilde{s}_+) \tag{4.13}$$

are analytic in (V, K), and satisfy the estimates

$$\|R_+\|_{\mathcal{Q}} \leq M\tilde{\chi}\tilde{g}_+^3, \qquad \|K_+\|_{\mathcal{W}_+} \leq M\tilde{\chi}\tilde{g}_+^3. \tag{4.14}$$

In addition, R_+, K_+ are jointly continuous in all arguments m^2, V, K.

In a precise and non-trivial sense, Theorem 4.2 shows that the error to the perturbative calculation of Sect. 4.1 is of third-order in the coupling constants. However, unlike the bulk coupling constants, which remain small, the observables are not small, e.g., $\lambda_0 = 1$, and this is compensated by the weights in (4.9).

In the remainder of the paper, we write

$$f \prec g \text{ when there is a } C > 0 \text{ such that } f \leq Cg; \tag{4.15}$$

the constant C is always uniform in g, ε and the scale j but may depend on L.

For $x = a, b$, let $R_+^{\lambda_x}$ denote the coupling constant corresponding to λ_x in R_+, and similarly for $R_+^{q_x}$. In [19, Proposition 1.14], it is shown that for $(V, K) \in \mathbb{D}_j$ and $x = a, b$,

$$|R_+^{\lambda_x}| \prec \chi_j \tilde{g}_j^2 \mathbb{1}_{j<j_{ab}}, \tag{4.16}$$

$$|R_+^{q_x}| \prec |a-b|^{-2}\chi_j 4^{-(j-j_{ab})}\tilde{g}_j \mathbb{1}_{j\geq j_{ab}}. \tag{4.17}$$

The perturbative contribution $\lambda_{\mathrm{pt},x}$ to the observable coupling constant is independent of $x = a, b$, as is apparent from (4.3). However, the paving of the torus Λ by blocks breaks translation invariance, and this allows λ_x to have non-perturbative contributions that depend on the relative positions of $x = a, b$ within blocks. Nevertheless, our main result Theorem 1.1 does not depend on the positions of a, b in the initial paving of Λ by blocks.

4.3. Observable flow. The achievement of Theorem 4.2 is to show that if (V_j, K_j) lies in the domain $\mathbb{D}_j$, then we have good control of $\lambda_{x,j+1}, q_{x,j+1}$ and also the observable part of K_{j+1} (whose bulk part has been controlled along with the bulk coupling constants already in Theorem 3.1). The following proposition links scales together via an inductive argument to conclude that (V_j, K_j) remains in $\mathbb{D}_j$ for *all* $j \le N$.

In particular, this requires that the bulk flow is well-defined for all $j \le N$. For this, we recall that, given the parameters $(m^2, g_0) \in [0, \delta)^2$, the critical initial conditions for the global existence of the bulk renormalisation group flow are given by

$$U_0 = U_0^c = (g_0, z_0^c(m^2, g_0), \nu_0^c(m^2, g_0)), \tag{4.18}$$

by Theorem 3.1. We also recall the corresponding sequence $U_j(m^2, g_0)$.

According to [19, (1.69)], in the presence of observables, (3.17) is supplemented by the statement that, for $x = a$ or $x = b$,

$$\begin{aligned} &\text{if } \pi_x V = 0 \quad \text{and} \quad \pi_x K(X) = 0 \quad \text{for all } X \in \mathcal{P} \text{ then} \\ &\pi_x V_+ = \pi_{ab} V_+ = 0 \quad \text{and} \quad \pi_x K_+(U) = \pi_{ab} K_+(U) = 0 \quad \text{for all } U \in \mathcal{P}_+, \end{aligned} \tag{4.19}$$

and, in addition, $\lambda_{a,+}$ is independent of each of λ_b, $\pi_b K$, and $\pi_{ab} K$, and the same is true with a, b interchanged.

As a consequence, using Theorem 4.2, the next proposition shows that the flow with observables, and with initial conditions

$$\pi_\varnothing V_0 = U_0^c, \quad \lambda_{x,0} \in \{0, 1\}, \quad q_{x,0} = 0, \quad (x = a, b) \tag{4.20}$$

exists for all $j \le N$. Note that we permit one or both of $\lambda_{x,0}$ to equal zero, and in this case we regard the observable at x as being absent, so the concept of coalescence becomes vacuous. We therefore use the convention that

$$j_{ab} = \infty \quad \text{if } \lambda_{a,0} = 0 \text{ or } \lambda_{b,0} = 0. \tag{4.21}$$

Proposition 4.3. *Let $\lambda_{x,0} \in \{0, 1\}$ and $q_{x,0} = 0$ for $x = a, b$.*

(i) *For $(m^2, g_0) \in [\delta L^{-2(N-1)}, \delta) \times (0, \delta)$, there is a choice of $(q_{a,j}, q_{b,j}, V_j, K_j)$ such that (3.16) holds for $0 \le j \le N$. This choice is such that $\pi_\varnothing V_j = U_j(m^2, g_0)$. If $\lambda_{x,0} = 0$ then $\lambda_{x,j} = 0$ for all $0 \le j \le N$, whereas if $\lambda_{x,0} = 1$ then*

$$\lambda_{x,j} = \begin{cases} (1 + \nu_j w_j^{(1)})^{-1} \Big(1 + \sum_{k=0}^{j-1} \check{v}_{\lambda_x,k}\Big) & (j + 1 < j_{ab}) \\ \lambda_{j_{ab}-1} & (j + 1 \ge j_{ab}). \end{cases} \tag{4.22}$$

If $\lambda_{x,0} = 0$ for one or both of $x = a, b$ then $q_{a,j} = q_{b,j} = 0$ for all $0 \le j \le N$, whereas if $\lambda_{a,0} = \lambda_{b,0} = 1$ then, for $x = a, b$,

$$q_{x,j} = \sum_{i=j_{ab}-1}^{j-1} \left(\lambda_{a,j_{ab}-1} \lambda_{b,j_{ab}-1} C_{i+1;a,b} + v_{q_x,i}\right). \tag{4.23}$$

For $\lambda_{x,0} \in \{0, 1\}$,

$$\|K_j\|_{\mathcal{W}_j} \le M \chi_j \tilde{g}_j^3. \tag{4.24}$$

In the above estimates, M is the constant appearing in (4.14), and $\check{v}_{\lambda_x,j}, v_{q_x,j} \in \mathbb{C}$ obey, uniformly in $(m^2, g_0) \in [0, \delta)^2$,

$$|\check{v}_{\lambda_x,j}| \prec \chi_j \tilde{g}_j^2 \mathbb{1}_{j<j_{ab}}, \qquad |v_{q_x,j}| \prec |a - b|^{-2} \chi_j 4^{-(j-j_{ab})} \tilde{g}_j \mathbb{1}_{j \ge j_{ab}}. \tag{4.25}$$

(ii) *For $j \le N$, each of $\lambda_{x,j}, \delta q_{x,j}, q_{x,j}$ is independent of N, in the sense that, e.g., $q_{x,1}, \dots, q_{x,N}$ have the same values on Λ_N as on a larger torus $\Lambda_{N'}$ with $N' > N$. In addition, each is defined as a continuous function of $(m^2, g_0) \in [0, \delta)^2$. Finally, $\lambda_{a,j}$ is independent of $\lambda_{b,0}$, and $\lambda_{b,j}$ is independent of $\lambda_{a,0}$.*

Proof. To simplify the notation, we drop the labels $x = a, b$ from λ, q when their role is insignificant.

(i) As a preliminary step, we introduce a change of variables that diagonalises the evolution of λ to linear order in V. For (V_j, K_j), we write $\lambda_{\rm pt} = \lambda_{\rm pt}(V_j)$ and $v_{\lambda,j} = R^\lambda_{j+1}(V_j, K_j)$. Then the λ-component of (4.6) can be written as $\lambda_{j+1} = \lambda_{\rm pt} + v_{\lambda,j}$. We define

$$\check{\lambda}_j = \lambda_j(1 + \nu_j w_j^{(1)}). \tag{4.26}$$

By (4.3), the recursion for $\check{\lambda}_j$ can then be written as

$$\check{\lambda}_{j+1} = \check{\lambda}_j + \check{v}_{\lambda,j} \tag{4.27}$$

with

$$\check{v}_{\lambda,j} = (\nu_{j+1} - \nu_j^+)\lambda_j w_{j+1}^{(1)} + v_{\lambda,j}(1 + \nu_{j+1} w_{j+1}^{(1)}) - \delta_j[\nu w^{(1)}]\lambda_j \nu_{j+1} w_{j+1}^{(1)}. \tag{4.28}$$

The solution to (4.27) with initial condition $\lambda_0 = 1$ is $\check{\lambda}_j = 1 + \sum_{k=0}^{j-1} \check{v}_{\lambda,k}$, and hence

$$\lambda_j = (1 + \nu_j w_j^{(1)})^{-1}\Big(1 + \sum_{k=0}^{j-1} \check{v}_{\lambda,k}\Big). \tag{4.29}$$

By (4.4) and (4.10), and with $v_{q,j} = R^q_{j+1}(V_j, K_j)$, δq_j is simply given by

$$\delta q_{j+1} = \delta q_{\rm pt} + v_{q,j} = \lambda_{a,j}\lambda_{b,j} C_{j+1;a,b} + v_{q,j}. \tag{4.30}$$

Now we can prove (4.22)–(4.25) by induction on j, with induction hypothesis:

IH_j: for all $k \le j$, $(V_k, K_k) \in \mathbb{D}_k$, (4.22)–(4.25) hold with j replaced by k.

By direct verification, IH_0 holds (with $\check{v}_{\lambda,-1} = v_{q,-1} = 0$).

We assume IH_j and show that it implies IH_{j+1}. By IH_j and the bound (4.14) of Theorem 4.2, K_{j+1} obeys (4.24). In particular, this estimate implies $K_{j+1} \in B_{\mathcal{W}_j}(\alpha \chi_j \tilde{g}_j)$.

By (3.17)–(3.18), $\pi_\varnothing V_j = U_j$ for all j, and by Theorem 3.1, U satisfies the bounds required for $\pi_\varnothing V$ in the definiton of $\mathbb{D}$. Therefore, to verify $(V_{j+1}, K_{j+1}) \in \mathbb{D}_{j+1}$, it suffices to show $|\lambda_{j+1}| \le C_{\mathcal{D}}$.

By (4.10), (4.3), and (4.16), $\lambda_j = \lambda_{j_{ab}-1}$ for all $j \ge j_{ab}$, so we assume that $j < j_{ab} - 1$. To estimate $\check{v}_{\lambda,j}$, we use the fact that $|\lambda_j| \le C_{\mathcal{D}}$ by assumption, and $|\nu_j| \prec L^{-2j}\chi_j \tilde{g}_j$ by Theorem 3.1. We apply [6, Lemma 6.2] and [19, (1.80)] to see that $w_j^{(1)} \prec L^{2j}$ and $|\nu_{j+1} - \nu_j^+| \prec L^{-2j}\chi_j \tilde{g}_j^2$, and also $|\nu_j| w_j^{(1)} \prec L^{-2j}\chi_j \tilde{g}_j w_j^{(1)} \prec \chi_j \tilde{g}_j$. The factor $v_{\lambda,j}$ is bounded via (4.16), and the last term on the right-hand side of (4.28) is similarly bounded (without any need for cancellation in the δ term). We conclude that $|\check{v}_{\lambda,j}| \prec \chi_j \tilde{g}_j^2$, as required. With (3.20), this leads to $|\lambda_{j+1}| = 1 + O(g_0) \le C_{\mathcal{D}}$ (since we have assumed above (3.25) that $C_{\mathcal{D}} \ge 2$). This establishes that λ_{j+1} obeys the condition required in the definition of the domain $\mathbb{D}_{j+1}$, and all necessary properties for λ_{j+1} have been established.

By (4.30) and (4.17) with IH_j, (4.23) holds with $v_{q,j}$ obeying (4.25). This advances the induction and completes the proof of part (i). The claim that $\lambda_{x,0} = 0$ implies $q_{a,j} = q_{b,j} = 0$ for all $0 \le j \le N$ also follows by induction and (4.19).

(ii) The N-independence of $\lambda_j, \delta q_j$ follows exactly as in the proof of [5, Proposition 8.1], so we only sketch the argument. By (4.3)–(4.4), λ_{pt} and δq_{pt} are independent of N. Moreover, by [19, Proposition 1.18(i)], $R_+(V, K)$ is independent of N provided that V is independent of N and that the family K has Property $\mathbb{Z}^d$ defined in [19]. That the renormalisation group map preserves Property $\mathbb{Z}^d$ for K is shown in [19, Proposition 1.17].

To show that $\lambda_j, \delta q_j$ (and thus also q_j) are continuous as functions of $(m^2, g_0) \in [0, \delta)^2$, assuming that $V_0 = V_0^c(m^2, g_0)$, we can proceed exactly as in [5, Section 8.2]. The definition of *continuous functions of the renormalisation group coordinates at scale-j*, provided by [5, Definition 8.2] for the bulk coordinates, applies literally also to the renormalisation group coordinates with observables. By Theorem 4.2 for R_+ and [6, Lemma 6.2] for V_{pt}, both of $\lambda_+, \delta q_+$ are continuous functions of the renormalisation group coordinates at scale-j. By [5, Proposition 8.3], which also applies literally with observables, we conclude continuity of $\lambda_j, \delta q_j$ for all j.

Finally, it follows inductively from (4.19) and the statement below (4.19) that $\lambda_{a,j}$ is independent of $\lambda_{b,0}$, and vice versa, as required. This completes the proof. □

In the following lemma, we denote the derivative of $Z_N^0(\phi, \bar\phi)$ with respect to $\bar\phi$, in the direction of a test function $J : \Lambda \to \mathbb{C}$, as $D_{\bar\phi} Z_N^0(\phi, \bar\phi; J) = \frac{d}{dt} Z_N^0(\phi, \bar\phi + tJ)|_0$. Let 1 denote the constant test function $1_x = 1$ for all $x \in \Lambda$. We systematically use subscripts σ or $\sigma\bar\sigma$ to denote the coefficient of σ or $\sigma\bar\sigma$ in $F \in \mathcal{N}$, under the decomposition (2.19). For example, we write $K_{N;\sigma\bar\sigma}(\Lambda) = \frac{1}{\sigma\bar\sigma}\pi_{ab} K_N(\Lambda)$.

Lemma 4.4. *The flow of Proposition 4.3 obeys*

$$\lambda_{a,N} = D_{\bar\phi} Z_{N;\sigma}^0(0, 0; 1) - D_{\bar\phi} W_{N;\sigma}^0(\Lambda; 0, 0; 1) - D_{\bar\phi} K_{N;\sigma}^0(\Lambda; 0, 0; 1). \tag{4.31}$$

Proof. As in [5, (8.13)],

$$Z_N^0 = I_N^0(\Lambda) + K_N^0(\Lambda) = e^{-V_N^0(\Lambda)}(1 + W_N^0(\Lambda)) + K_N^0(\Lambda). \tag{4.32}$$

Therefore, since $\pi_a(FG) = (\pi_a F)(\pi_\varnothing G) + (\pi_\varnothing F)(\pi_a G)$, and since

$$\pi_\varnothing(e^{-V_N^0(\Lambda)}) = e^{-U_N^0(\Lambda)}, \quad \pi_a(e^{-V_N^0(\Lambda)}) = \sigma \lambda_{a,N} \bar\phi_a, \tag{4.33}$$

we obtain

$$Z_{N;\sigma}^0 = \lambda_{a,N} \bar\phi_a e^{-U_N^0(\Lambda)}(1 + W_N^{0,\varnothing}(\Lambda)) + e^{-U_N^0(\Lambda)} W_{N;\sigma}^0(\Lambda) + K_{N;\sigma}^0(\Lambda). \tag{4.34}$$

Differentiating with respect to $\bar\phi$ at $(\phi, \bar\phi) = (0, 0)$, we obtain

$$D_{\bar\phi} Z_{N;\sigma}^0(0, 0; 1) = \lambda_{a,N} + D_{\bar\phi} W_{N;\sigma}^0(\Lambda; 0, 0; 1) + D_{\bar\phi} K_{N;\sigma}^0(\Lambda; 0, 0; 1), \tag{4.35}$$

where we used $e^{-U_N^0(\Lambda;0,0)} = 1$ and the fact that $W_N^{0,\varnothing}(\Lambda; 0, 0) = 0$ since $W_N^{0,\varnothing}$ is a polynomial in ϕ with no monomials of degree below two. □

The W and K terms in the statement of Lemma 4.4 are estimated using the following lemma.

Lemma 4.5. *The flow of Proposition 4.3 obeys, uniformly in $m^2 \in [\delta L^{-2(N-1)}, \delta)$:*

$$\left|K^0_{N;\sigma\bar\sigma}(\Lambda;0,0)\right| \prec \frac{1}{4^{(N-j_{ab})_+}}\frac{1}{|a-b|^2}\chi_N\bar g_N, \tag{4.36}$$

$$|D_{\bar\phi}K^0_{N;\sigma}(\Lambda;0,0;1)| \prec \chi_N\bar g_N^2\left(\frac{L}{2}\right)^{(N-j_{ab})_+}, \tag{4.37}$$

$$|D_{\bar\phi}W^0_{N;\sigma}(\Lambda;0,0;1)| \prec \chi_N\bar g_N\left(\frac{L}{2}\right)^{(N-j_{ab})_+}. \tag{4.38}$$

Proof. Recall the definitions of the $\mathcal{Q}$ norm from (4.8), and the definitions of the $T_{0,j}(\ell_j)$ and $\Phi_j(\ell_j)$ norms from [5, Section 6.3].

Recall from [17, (1.61)] that in the T_0 norm each occurrence of σ or $\bar\sigma$ gives rise to a weight

$$\ell_{\sigma,j} = 2^{(j-j_{ab})_+}L^{(j\wedge j_{ab})}\tilde g_j. \tag{4.39}$$

There is therefore a factor $\ell^2_{\sigma,j}$ inside the norm of $\pi_{ab}K_j$. In particular,

$$|K^0_{N;\sigma\bar\sigma}(\Lambda;0,0)| \le \ell^{-2}_{\sigma,j}\|K_N(\Lambda)\|_{T_{0,N}(\ell_N)}. \tag{4.40}$$

We apply [19, (1.62)], which uses this fact, and which implies that the bound

$$|K^0_{N;\sigma\bar\sigma}(\Lambda;0,0)| \le \ell^{-2}_{\sigma,j}\|K_N(\Lambda)\|_{T_{0,N}(\ell_N)} \le \ell^{-2}_{\sigma,j}\|K_N\|_{\mathcal{W}_N} \prec 4^{-(N-j_{ab})_+}L^{-2j_{ab}}\chi_N\tilde g_N \tag{4.41}$$

holds uniformly in $m^2 \in [\delta L^{-2(N-1)}, \delta)$. As mentioned below Definition 4.1, $L^{j_{ab}}$ and $|a-b|$ are comparable. With (4.24) and (3.24), this shows that (4.36) holds.

By definition of the $T_{0,j}(\ell_j)$ norm, for any $F \in \mathcal{N}^{\varnothing}$ and any test function $J : \Lambda \to \mathbb{C}$,

$$|D_{\bar\phi}F^0(0,0;J)| \le \|F\|_{T_{0,N}(\ell_N)}\|J\|_{\Phi_N(\ell_N)}. \tag{4.42}$$

By definition, $\|1\|_{\Phi_N(\ell_N)} = \ell_N^{-1}$ (see [5, (8.55)]). With (4.42), this gives

$$|D_{\bar\phi}K^0_{N;\sigma}(\Lambda;0,0;1)| \le \ell^{-1}_{\sigma,N}\|K_N(\Lambda)\|_{T_{0,N}(\ell_N)}\|1\|_{\Phi_N(\ell_N)} = \ell^{-1}_{\sigma,N}\ell_N^{-1}\|K_N\|_{\mathcal{W}_N}. \tag{4.43}$$

With (4.9) and (4.24), this proves (4.37). Finally, by [18, Proposition 4.1],

$$\|W_N(\Lambda)\|_{T_{0,N}} \prec \chi_N g_N^2, \tag{4.44}$$

and (4.38) then follows as in (4.43). □

The next two lemmas apply Proposition 4.3 to study limits of the sequences $\lambda_{x,j}$, $q_{x,j}$. By Proposition 4.3(ii), $q_{x,j}$ is independent of N (assuming that N is larger than j_{ab}), and $\lambda_{x,j}$ is independent of j_{ab} and N if $j < j_{ab} \le N$, and we can therefore define sequences $\lambda^*_{x,j}$ for all $j \in \mathbb{N}_0$, with $\lambda^*_{x,0} = 1$, such that $\lambda_{x,j} = \lambda^*_{x,j}$ for $j < j_{ab}$. The sequence $\lambda^*_{a,j}$ is independent of $\lambda_{b,0}$, and vice versa. By definition,

$$\lambda_{x,j} = \lambda^*_{x,j\wedge(j_{ab}-1)}, \tag{4.45}$$

and

$$\lambda_{a,j} = \lambda^*_{a,j} \quad \text{for all } j \le N \text{ when } \lambda_{b,0} = 0. \tag{4.46}$$

We make the dependence on (m^2, g_0) explicit by writing $\lambda_{x,j} = \lambda_{x,j}(m^2, g_0)$ and $q_{x,j} = q_{x,j}(m^2, g_0)$.

Lemma 4.6. *For* $(m^2, g_0) \in [0,\delta)^2$, *for* $x = a$ *or* $x = b$, *for* $\lambda_{x,0} = 1$, *and for* $j \in \mathbb{N}_0$,

$$|1 - \lambda^*_j(m^2, g_0)| \prec \chi_j \bar{g}_j. \tag{4.47}$$

In particular,

$$|1 - \lambda_{x,j_{ab}-1}(m^2, g_0)| \prec \chi_{j_{ab}} \bar{g}_{j_{ab}}. \tag{4.48}$$

Proof. By Proposition 4.3, the flow of λ_a is independent of the choice of $\lambda_{b,0}$, and vice versa. We give the proof for the case $x = a$, and the same argument applies to $x = b$.

We choose the initial conditions $(\lambda_{a,0}, \lambda_{b,0}) = (1, 0)$. As discussed above Proposition 4.3, in this case we have $j_{ab} = N$. By Lemma 4.4,

$$\lambda^*_{a,N} = D_{\bar\phi} Z^0_{N;\sigma}(0,0;1) - D_{\bar\phi} W^0_{N;\sigma}(\Lambda;0,0;1) - D_{\bar\phi} K^0_{N;\sigma}(\Lambda;0,0;1). \tag{4.49}$$

By Lemma 4.5, this gives

$$\lambda^*_{a,N} = D_{\bar\phi} Z^0_{N;\sigma}(0,0;1) + O(\chi_N g_N). \tag{4.50}$$

The limit of the first term on the right-hand side of (4.50), as $N \to \infty$, can be evaluated exactly, as follows. Let C be the covariance defined in (3.2). Recall from [5, (4.23)] that, for any external field $J : \Lambda \to \mathbb{C}$,

$$\Sigma_a(J, \bar J) = \mathbb{E}_C\left(e^{-V_0(\Lambda)+(J,\bar\phi)+(\bar J,\phi)}\right) = e^{(J,C\bar J)} Z^0_N(CJ, C\bar J), \tag{4.51}$$

where the superscript 0 denotes projection onto the degree-0 part of the form Z_N. As opposed to [5], we include the observable term $\sigma\bar\phi_a$ in V_0 and Z_N here, and we emphasise this by writing Σ_a instead of Σ; the potential V_0 without observable terms is again denoted by U_0. Each side of (4.51) has a decomposition as in (2.19), and we equate the coefficients of σ in the components in $\mathcal{N}^a$ to obtain

$$\mathbb{E}_C\left(e^{-U_0(\Lambda)+(J,\bar\phi)+(\bar J,\phi)}\bar\phi_a\right) = e^{(J,C\bar J)} Z^0_{N;\sigma}(CJ, C\bar J). \tag{4.52}$$

Let 1 be the constant test function $1_x = 1$ for all $x \in \Lambda$. Then $C1 = m^{-2}1$. Differentiation of (4.52) at $(0,0)$ with respect to J, in direction 1, gives

$$\sum_x \mathbb{E}_C\left(e^{-U_0(\Lambda)}\phi_x\bar\phi_a\right) = D_{\bar\phi} Z^{0;a}_N(0,0;C1) = m^{-2} D_{\bar\phi} Z^0_{N;\sigma}(0,0;1). \tag{4.53}$$

By translation invariance of $\mathbb{E}_C$ and U_0, the left-hand side is independent of $a \in \Lambda$. In fact, it is equal to $\hat{\chi}_N$ defined in [5, (4.9)], which converges to m^{-2} as $N \to \infty$, by [5, Theorem 4.1]. Therefore,

$$\lim_{N\to\infty} D_{\bar{\phi}} Z^0_{N;\sigma}(0, 0; 1) = 1. \tag{4.54}$$

We then apply Lemma 4.5 (with $(N - j_{ab})_+ = 0$), together with $\chi_N g_N \to 0$, to conclude from (4.54) that the right-hand side of (4.50) tends to 1. On the other hand, by (4.22), together with (4.25) and the estimate $|\nu_j| w_j^{(1)} \prec \chi_j \tilde{g}_j$ used in the proof of Proposition 4.3,

$$\lim_{N\to\infty} \lambda^*_{a,N} = 1 + \sum_{k=0}^{\infty} \check{v}_{\lambda_x,k}, \qquad \text{so} \quad \sum_{k=0}^{\infty} \check{v}_{\lambda_x,k} = 0. \tag{4.55}$$

(Note that the convergence of the sum in (4.55) is guaranteed by (4.25) and (3.20).) Finally, by (4.22) and (3.20), uniformly in (m^2, g_0) we have

$$\lambda^*_j - 1 = -\nu_j w_j^{(1)} \lambda_j - \sum_{k=j}^{\infty} \check{v}_{\lambda,k} = O(\chi_j \bar{g}_j), \tag{4.56}$$

and the proof is complete. □

Lemma 4.7. *For $(m^2, g_0) \in [0, \delta)^2$ and $x = a, b$, the limit*

$$q_{x,\infty}(m^2, g_0) = \lim_{j\to\infty} q_{x,j}(m^2, g_0), \tag{4.57}$$

exists, is continuous, and, as $|a - b| \to \infty$,

$$q_{x,\infty}(0, g_0) = (-\Delta_{\mathbb{Z}^4})^{-1}_{ab} \left(1 + O\left(\frac{1}{\log|a - b|}\right)\right). \tag{4.58}$$

Proof. We again drop the labels $x = a, b$ from λ, q when their role is insignificant.

By (4.23),

$$q_j = \sum_{i=j_{ab}-1}^{j-1} \left(\lambda_{a,j_{ab}-1}\lambda_{b,j_{ab}-1}\, C_{i+1;a,b} + v_{q,i}\right). \tag{4.59}$$

Since $C_{i+1;a,b} = 0$ for $i < j_{ab}$, we can restore the scales $i < j_{ab}$ to the sum in the first term on the right-hand side. In the limit $j \to \infty$, we obtain the complete finite-range decomposition for the inverse Laplacian on $\mathbb{Z}^4$ as in (3.1),

$$\sum_{i=j_{ab}-1}^{\infty} C_{i+1;a,b} = \sum_{i=0}^{\infty} C_{i+1;a,b} = (-\Delta_{\mathbb{Z}^4} + m^2)^{-1}_{ab}. \tag{4.60}$$

The dependence of $\lambda_{x,j_{ab}-1}$ is continuous in $[0, \delta)^2$ by Proposition 4.3, and $(-\Delta_{\mathbb{Z}^4} + m^2)^{-1}_{ab}$ is continuous in $m^2 \in [0, \delta)$. By Proposition 4.3, in the limit $j \to \infty$ the sum of $v_{q,i}$ on the right-hand side of (4.59) is a uniformly convergent sum of terms that are continuous. Therefore the sum $q_\infty(m^2, g_0)$ is also continuous, and

$$q_\infty = \lambda_{a,j_{ab}-1}\lambda_{b,j_{ab}-1}(-\Delta_{\mathbb{Z}^4} + m^2)^{-1}_{ab} + \sum_{i=j_{ab}}^{\infty} v_{q,i}. \tag{4.61}$$

By (4.17),

$$\sum_{i=j_{ab}}^{\infty} |v_{q,i}| \prec |a-b|^{-2} \sum_{i=j_{ab}}^{\infty} 4^{-(i-j_{ab})}\chi_i \tilde{g}_i \prec |a-b|^{-2}\chi_{j_{ab}}\bar{g}_{j_{ab}}. \tag{4.62}$$

Therefore,

$$\left|q_\infty - \lambda_{a,j_{ab}-1}\lambda_{b,j_{ab}-1}(-\Delta_{\mathbb{Z}^4} + m^2)^{-1}_{ab}\right| \prec \chi_{j_{ab}}\bar{g}_{j_{ab}}|a-b|^{-2}. \tag{4.63}$$

By Lemma 4.6,

$$|1 - \lambda_{a,j_{ab}-1}\lambda_{b,j_{ab}-1}| \prec \chi_{j_{ab}}\bar{g}_{j_{ab}}. \tag{4.64}$$

With (4.63) and $|(-\Delta_{\mathbb{Z}^4} + m^2)^{-1}_{ab}| \prec |a-b|^{-2}$, this gives

$$\left|q_\infty - (-\Delta_{\mathbb{Z}^4} + m^2)^{-1}_{ab}\right| \prec \chi_{j_{ab}}\bar{g}_{j_{ab}}|a-b|^{-2}, \tag{4.65}$$

uniformly in (m^2, g_0). By (3.19), $\chi_{j_{ab}}\bar{g}_{j_{ab}} \prec j_{ab}^{-1} \prec (\log|a-b|)^{-1}$. In particular, the limit $q_\infty(0, g_0)$ obeys (4.58). □

4.4. Proof of main result. We now prove Theorem 1.1. In addition to the study of q_j, which provides the leading contribution, this requires the estimate (4.36) on $\pi_{ab}K_N$.

Proof of Theorem 1.1. For small $g, \varepsilon > 0$, set $\nu = \nu_c(g) + \varepsilon$, and let $(m^2, g_0, \nu_0, z_0) = (\tilde{m}^2, \tilde{g}_0, \tilde{\nu}_0, \tilde{z}_0)$ be the functions of (g, ε) given by Proposition 3.2. Since $z_0 = \tilde{z}_0(g, \varepsilon) \to \tilde{z}_0(g, 0)$ as $\varepsilon \downarrow 0$, (4.1) gives

$$G_{g,\nu_c}(a, b) = \big(1 + \tilde{z}_0(g, 0)\big) \lim_{\varepsilon\downarrow 0} \lim_{N\to\infty} Z^0_{N;\sigma\bar\sigma}(0). \tag{4.66}$$

The arguments 0 on the right-hand side mean that the fields ϕ, ψ are to be set to zero in I_N, K_N. Thus $I^0_N(\Lambda, 0) = 1$, and for $K^0_N(\Lambda, 0)$ only dependence on $\sigma, \bar\sigma$ remains. From (3.12) we obtain

$$Z^0_N(0) = e^{q_N\sigma\bar\sigma}(I^0_N(\Lambda, 0) + K^0_N(\Lambda, 0)) = e^{q_N\sigma\bar\sigma}(1 + K^0_N(\Lambda, 0)), \tag{4.67}$$

with $q_N = \frac{1}{2}(q_{a,N} + q_{b,N})$ as in (3.15). Equating the coefficients of $\sigma\bar\sigma$ on both sides gives

$$Z^0_{N;\sigma\bar\sigma}(0) = q_N\big(1 + \pi_\varnothing K^0_N(\Lambda, 0)\big) + K^0_{N;\sigma\bar\sigma}(\Lambda, 0). \tag{4.68}$$

Since $\varepsilon > 0$ by assumption, it follows that $m^2 > 0$, by Proposition 3.2. Therefore, for N sufficiently large, the bounds (3.22) and (4.36) hold. In particular, by (3.19),

$$\lim_{N\to\infty} \pi_\varnothing K^0_N(\Lambda, 0) = 0, \qquad \lim_{N\to\infty} K^0_{N;\sigma\bar\sigma}(\Lambda, 0) = 0, \tag{4.69}$$

and therefore

$$\lim_{N\to\infty} Z^0_{N;\sigma\bar\sigma}(0) = \lim_{N\to\infty} q_N = q_\infty = \frac{1}{2}(q_{a,\infty} + q_{b,\infty}). \tag{4.70}$$

With (4.66) and Lemma 4.7, this gives

$$\begin{aligned} G_{g,\nu_c}(a,b) &= (1+\tilde z_0(g,0)) \lim_{\varepsilon\downarrow 0} q_\infty \\ &= \big(1+\tilde z_0(g,0)\big)(-\Delta_{\mathbb{Z}^4})^{-1}_{ab}\left(1 + O\left(\frac{1}{\log|a-b|}\right)\right). \end{aligned} \tag{4.71}$$

It is a standard fact that $(-\Delta_{\mathbb{Z}^4})^{-1}_{ab} = (2\pi)^{-2}|a-b|^{-2}(1+O(|a-b|^{-2}))$ (see, e.g., [36]—the different constant $(2\pi)^{-2}$ takes into account our definition of the Laplacian). Since $\tilde z_0(g,0) = O(g)$, the proof is complete. (Although our analysis allows q_j to become complex, the left-hand side of (4.71) is real by definition, so the right-hand side is as well.) □

Remark 4.8. The proof of (4.71) used the fact, proved in Lemma 4.6, that $\lambda^*_{x,j} \to 1$ as $j \to \infty$. The fact that this limit is exactly equal to 1 (without $O(g)$ error, as one might expect) is intimately related to the interpretation of $\lim_{j\to\infty}(1+z_0)^{1/2}\lambda_{x,j}$ as *field strength renormalisation*. Without using $\lambda^*_{x,\infty} = 1$, the above proof would show that the two-point function is asymptotic to $(1+z_0)\lambda^*_{a,\infty}\lambda^*_{b,\infty}(-\Delta)^{-1}_{ab}$. Thus, $\lambda^*_{x,\infty} = 1$ means that the field strength renormalisation is given by $(1+z_0)^{1/2}$ only. This was anticipated already in [5, Section 4], when we split the original potential $V_{g,\nu,1}$ into an effective free field with field strength $(1+z_0)^{1/2}$ and mass m, and a perturbation.

Remark 4.9. Note that

$$G_{g,\nu_c}(a,b) = (1+\tilde z_0(g,0))q_\infty \tag{4.72}$$

is an *equality*, and not merely an asymptotic formula. As such, it contains all information about the two-point function, including not just the leading asymptotic behaviour but also all higher-order corrections.

Remark 4.10. Equations (4.39) and (4.36) provide corrections to [12, (109)–(111)], which contain erroneous powers of $\bar g_N$ in the upper bounds. In [12, (109), (111)], the $\bar g_N^3$ in the upper bound should be $\bar g_N$, and in [12, (110)] a factor $\bar g_N$ is missing on the right-hand side (it is present in (4.39)). The above proof shows that the correct powers here remain sufficient to prove (4.71).

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References

1. Abdesselam, A., Chandra, A., Guadagni, G.: Rigorous quantum field theory functional integrals over the p-adics I: Anomalous dimensions. (2013) (Preprint)
2. Aizenman, M.: Geometric analysis of φ^4 fields and Ising models. Parts I and II. Commun. Math. Phys. **86**, 1–48 (1982)
3. Aragão de Carvalho, C., Caracciolo, S., Fröhlich, J.: Polymers and $g|\phi|^4$ theory in four dimensions. Nucl. Phys. B **215**(FS7), 209–248 (1983)
4. Bauerschmidt, R.: A simple method for finite range decomposition of quadratic forms and Gaussian fields. Probab. Theory Relat. Fields **157**, 817–845 (2013)
5. Bauerschmidt, R., Brydges, D.C., Slade, G.: Logarithmic correction for the susceptibility of the 4-dimensional weakly self-avoiding walk: a renormalisation group analysis. Commun. Math. Phys. (To appear)
6. Bauerschmidt, R., Brydges, D.C., Slade, G.: A renormalisation group method. III. Perturbative analysis. J. Stat. Phys. **159**, 492–529 (2015)
7. Bauerschmidt, R., Brydges, D.C., Slade, G.: Scaling limits and critical behaviour of the 4-dimensional n-component $|\varphi|^4$ spin model. J. Stat. Phys. **157**, 692–742 (2014)
8. Bauerschmidt, R., Duminil-Copin, H., Goodman, J., Slade, G.: Lectures on self-avoiding walks. In: Ellwood, D., Newman, C., Sidoravicius, V., Werner, W. (eds.) Probability and Statistical Physics in Two and More Dimensions, Clay Mathematics Proceedings, vol. 15, Am. Math. Soc., Providence, RI, pp. 395–467 (2012)
9. Bovier, A., Felder, G., Fröhlich, J.: On the critical properties of the Edwards and the self-avoiding walk model of polymer chains. Nucl. Phys. B **230**(FS10), 119–147 (1984)
10. Brézin, E., Le Guillou, J.C., Zinn-Justin, J.: Approach to scaling in renormalized perturbation theory. Phys. Rev. D **8**, 2418–2430 (1973)
11. Brydges, D., Evans, S.N., Imbrie, J.Z.: Self-avoiding walk on a hierarchical lattice in four dimensions. Ann. Probab. **20**, 82–124 (1992)
12. Brydges, D., Slade, G.: Renormalisation group analysis of weakly self-avoiding walk in dimensions four and higher. In: Bhatia, R., et al. (eds.) Proceedings of the International Congress of Mathematicians, Hyderabad 2010, pp. 2232–2257, Singapore. World Scientific. (2011)
13. Brydges, D.C., Dahlqvist, A., Slade, G.: The strong interaction limit of continuous-time weakly self-avoiding walk. In: Deuschel, J.-D., Gentz, B., König, W., von Renesse, M., Scheutzow, M., Schmock, U. (eds.) Probability in Complex Physical Systems: In Honour of Erwin Bolthausen and Jürgen Gärtner. Springer Proceedings in Mathematics, vol. 11, pp. 275–287. Springer, Berlin (2012)
14. Brydges, D.C., Guadagni, G., Mitter, P.K.: Finite range decomposition of Gaussian processes. J. Stat. Phys. **115**, 415–449 (2004)
15. Brydges, D.C., Imbrie, J.Z.: Green's function for a hierarchical self-avoiding walk in four dimensions. Commun. Math. Phys. **239**, 549–584 (2003)
16. Brydges, D.C., Imbrie, J.Z., Slade, G.: Functional integral representations for self-avoiding walk. Probab. Surv. **6**, 34–61 (2009)
17. Brydges, D.C., Slade, G.: A renormalisation group method. II. Approximation by local polynomials. J. Stat. Phys. **159**, 461–491 (2015)
18. Brydges, D.C., Slade, G.: A renormalisation group method. IV. Stability analysis. J. Stat. Phys. **159**, 530–588 (2015)
19. Brydges, D.C., Slade, G.: A renormalisation group method. V. A single renormalisation group step. J. Stat. Phys. **159**, 589–667 (2015)
20. Brydges, D.C., Spencer, T.: Self-avoiding walk in 5 or more dimensions. Commun. Math. Phys. **97**, 125–148 (1985)
21. Chen, L.-C., Sakai, A.: Critical two-point functions for long-range statistical-mechanical models in high dimensions. Ann. Probab. **43**, 639-–681 (2015)
22. Clisby, N.: Accurate estimate of the critical exponent ν for self-avoiding walks via a fast implementation of the pivot algorithm. Phys. Rev. Lett. **104**, 055702 (2010)
23. Dimock, J., Hurd, T.R.: A renormalization group analysis of correlation functions for the dipole gas. J. Stat. Phys. **66**, 1277–1318 (1992)
24. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: Construction and Borel summability of infrared Φ_4^4 by a phase space expansion. Commun. Math. Phys. **109**, 437–480 (1987)
25. Fröhlich, J.: On the triviality of φ_d^4 theories and the approach to the critical point in $d \geq 4$ dimensions. Nucl. Phys. B **200**(FS4), 281–296 (1982)
26. Gawędzki, K., Kupiainen, A.: Massless lattice φ_4^4 theory: Rigorous control of a renormalizable asymptotically free model. Commun. Math. Phys. **99**, 199–252 (1985)

27. Gawędzki, K., Kupiainen, A.: Asymptotic freedom beyond perturbation theory. In: Osterwalder, K., Stora, R. (eds.) Critical Phenomena, Random Systems, Gauge Theories, Amsterdam (1986) (North-Holland. Les Houches 1984)
28. Hara, T.: A rigorous control of logarithmic corrections in four dimensional φ^4 spin systems. I. Trajectory of effective Hamiltonians. J. Stat. Phys. **47**, 57–98 (1987)
29. Hara, T.: Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. Ann. Probab. **36**, 530–593 (2008)
30. Hara, T., van der Hofstad, R., Slade, G.: Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. Ann. Probab. **31**, 349–408 (2003)
31. Hara, T., Slade, G.: Self-avoiding walk in five or more dimensions. I. The critical behaviour. Commun. Math. Phys. **147**, 101–136 (1992)
32. Hara, T., Tasaki, H.: A rigorous control of logarithmic corrections in four dimensional φ^4 spin systems. II. Critical behaviour of susceptibility and correlation length. J. Stat. Phys. **47**, 99–121 (1987)
33. den Hollander, F.: Random Polymers. Lecture Notes in Mathematics vol. 1974. Ecole d'Eté de Probabilités de Saint–Flour. Springer, Berlin, XXXVII–2007 (2009)
34. Iagolnitzer, D., Magnen, J.: Polymers in a weak random potential in dimension four: rigorous renormalization group analysis. Commun. Math. Phys. **162**, 85–121 (1994)
35. Larkin, A.I., Khmel'Nitskiĭ, D.E.: Phase transition in uniaxial ferroelectrics. Soviet Phys. JETP **29**, 1123–1128 (1969). (English translation of Zh. Eksp. Teor. Fiz. **56**, 2087–2098 (1969))
36. Lawler, G.F.: Intersections of Random Walks. Birkhäuser, Boston (1991)
37. Lawler, G.F., Schramm, O., Werner, W.: On the scaling limit of planar self-avoiding walk. Proc. Symp. Pure Math. **72**, 339–364 (2004)
38. Madras, N., Slade, G.: The Self-Avoiding Walk. Birkhäuser, Boston (1993)
39. McKane, A.J.: Reformulation of $n \to 0$ models using anticommuting scalar fields. Phys. Lett. A **76**, 22–24 (1980)
40. Nienhuis, B.: Exact critical exponents of the $O(n)$ models in two dimensions. Phys. Rev. Lett. **49**, 1062–1065 (1982)
41. Parisi, G., Sourlas, N.: Self-avoiding walk and supersymmetry. J. Phys. Lett. **41**, L403–L406 (1980)
42. Schram, R.D., Barkema, G.T., Bisseling, R.H.: Exact enumeration of self-avoiding walks. J. Stat. Mech. P06019 (2011)
43. Slade, G.: The Lace Expansion and its Applications. Lecture Notes in Mathematics vol. 1879. Ecole d'Eté de Probabilités de Saint–Flour XXXIV–2004 Springer, Berlin (2006)
44. Slade, G., Tomberg, A.: Critical correlation functions for the 4-dimensional weakly self-avoiding walk and n-component $|\varphi|^4$ model. (2014) (Preprint). arXiv:1412.2668
45. Wegner, F.J., Riedel, E.K.: Logarithmic corrections to the molecular-field behavior of critical and tricritical systems. Phys. Rev. B **7**, 248–256 (1973)

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