



Gluing Nekrasov Partition Functions

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Abstract: In this paper we summarise the localisation calculation of 5D super Yang-Mills on simply connected toric Sasaki-Einstein (SE) manifolds. We show how various aspects of the computation, including the equivariant index, the asymptotic behaviour and the factorisation property are governed by the combinatorial data of the toric geometry. We prove that the perturbative partition function on a simply connected SE manifold corresponding to an n -gon toric diagram factorises to n copies of perturbative part (zero instanton sector) of the Nekrasov partition function. This leads us to conjecture a prescription for the computation of the complete partition function, by gluing n copies of the full Nekrasov partition functions. This work is a generalisation of some earlier computation carried out on $Y^{p,q}$ manifolds, whose moment map cone has a quadrangle base and our result is valid for manifolds whose moment map cones have pentagon base, hexagon base, etc. The algorithm we used for dealing with general cones may also be of independent interest.

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1. Introduction

Starting from Pestun’s work [1] there has been an explosion in the applications of localisation technique for supersymmetric gauge theories in diverse dimensions. The calculations were mainly concerned with the evaluation of partition functions and the expectation values of the supersymmetric Wilson loops on (squashed) S^d and on $S^d \times S^1$, while other geometries were not investigated in detail. However, in order to understand the geometrical properties of partition functions, it is important to perform calculations on more general geometries. Five dimensional supersymmetric gauge theories on SE manifolds offer us this possibility, and this is the subject of this paper.

In order to be able to localise 5D supersymmetric Yang-Mills theory we need at least two supersymmetries. Indeed we can construct the supersymmetric gauge theory on any simply connected Sasaki-Einstein manifold and the theory preserves two supersymmetries. In particular there exist very nice examples of such manifolds, toric Sasaki-Einstein manifolds (their cones are toric Calabi-Yau manifolds). The goal of this work is to present the uniform treatment of localisation calculation for perturbative partition function of 5D supersymmetric Yang-Mills on any simply connected toric Sasaki-Einstein manifolds (for the earlier related work in 4D see [2]). Every such manifold is described in terms of an n -gon toric diagram and topologically corresponds to $(n - 3)(S^2 \times S^3)$ which is $(n - 3)$ -fold connected sums of $S^2 \times S^3$ (see Proposition 11.4.3 in [3] or Corollary 5.4 in [4]) and they are known as the Smale manifolds. This work is a natural continuation and generalisation of the previous calculations for $Y^{p,q}$ -spaces [5,6].

Let us summarise our main results. Let X be a simply connected toric SE manifold (we will give a brief review in Sect. 2 of some features of such manifolds), with moment map cone $C_\mu(X)$ defined by

$$C_\mu(X) = \{ \vec{r} \in \mathbb{R}^3 \mid \vec{r} \cdot \vec{v}_i \geq 0, i = 1, \dots, n \},$$

where \vec{v}_i are the inward pointing normals of the n faces of this cone; for example, see Fig. 1. The SE condition also implies that there exists a primitive vector $\vec{\xi}$, such that

$$\vec{\xi} \cdot \vec{v}_i = 1, \quad \forall i, \tag{1.1}$$

known as the 1-Gorenstein condition. Up to an $SL(3, \mathbb{Z})$ rotation, we can make $\vec{\xi} = [1, 0, 0]$, we will use this convention throughout the paper.

Next let \vec{r} be a three vector parameterising the Reeb vector field, satisfying the dual cone condition (see the Eq. (2.2)). The perturbative partition function of 5D SYM with a hypermultiplet of mass m and representation \underline{R} on X is given by the matrix model integral

$$Z^{pert} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r}{8\gamma_M} \text{Tr}[a^2]} \cdot \frac{\det'_{adj} S_3^X(ia; \vec{r})}{\det_{\underline{R}} S_3^X(ia + im + r^1/2; \vec{r})}, \tag{1.2}$$

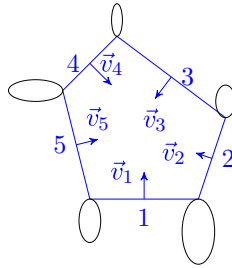


Fig. 1. The polygon base of a polytope cone. Over the interior of the polygon there is a T^3 fibre, but over the faces the T^3 degenerates into T^2 , which further degenerates over the vertices to S^1 , drawn as the *circles* in the figure. These *circles* are the only generic closed Reeb orbits

where a is the Coulomb branch parameter, integrated over \mathfrak{t} , the Cartan subalgebra. Here we have defined a generalised triple sine associated to X

$$S_3^X(x; \vec{r}) = \prod_{\vec{m} \in C_\mu(X) \cap \mathbb{Z}^3} (\vec{m} \cdot \vec{r} + x) (\vec{m} \cdot \vec{r} + \vec{\xi} \cdot \vec{r} - x), \tag{1.3}$$

where $\vec{\xi}$ is defined in (1.1), if we take $\vec{\xi} = [1, 0, 0]$ as above, then $\vec{\xi} \cdot \vec{r}$ is simply r^1 , the first component of \vec{r} . The product is taken over integer points inside the cone $C_\mu(X)$. Once we have computed the answer (1.2), we may allow \vec{r} to have complex components. Keeping the real part of \vec{r} within the dual cone, but giving it a generic imaginary part, we can then factorise the above partition function into

$$Z^{pert} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r}{g_Y^2 M} \varrho \text{Tr}[a^2] + B_{vec}(ia) + B_{hyp}(ia)} \cdot \frac{\prod_{i=1}^n \det'_{adj} (e^{-a\beta_i} | e^{i\beta_i \epsilon_i}, e^{i\beta_i \epsilon'_i})_\infty}{\prod_{i=1}^n \det_{\underline{R}} (e^{-(a+m^*)\beta_i} | e^{i\beta_i \epsilon_i}, e^{i\beta_i \epsilon'_i})_\infty}, \tag{1.4}$$

where $m^* = m - \frac{iR^1}{2}$.

We now explain the notations. The index i labels the n closed Reeb orbits in X . Each such orbit has circumference β_i and the special function $(e^{-a\beta_i} | e^{i\beta_i \epsilon_i}, e^{i\beta_i \epsilon'_i})_\infty$, defined in Appendix A by (A.1), is the perturbative part of the Nekrasov partition function on $\mathbb{C}^2 \times S^1$ with equivariant parameters ϵ_i and ϵ'_i . The Nekrasov partition function [7, 8] is defined as counting of states on $\mathbb{C}^2 \times S^1$

$$Z_{\mathbb{C}^2 \times S^1}^{full} = \text{Tr}_{\mathcal{H}} \left((-1)^{2(j_L + j_R)} e^{-\beta H - i(\epsilon - \epsilon') J_L^3 - i(\epsilon + \epsilon') J_R^3 - i(\epsilon + \epsilon') J^3} \right),$$

where H is the Hamiltonian, j_L and j_R generates the rotations of the first and second factor of \mathbb{C} in $\mathbb{C}^2 \times S^1$, and J_i^3 is the generator of the R-symmetry group $SU(2)$. The quantity m^* is the effective mass $m^* = m - iR^1/2$, and r^1 here comes from the combination $\vec{r} \cdot \vec{\xi}$ and the choice $\vec{\xi} = [1, 0, 0]$. Finally the quantities $\beta, \epsilon, \epsilon'$ are defined as follows. Let i label the corner of the intersection of faces i and $i + 1$ in Fig. 1, and choose \vec{n} such that $\det[\vec{v}_i, \vec{v}_{i+1}, \vec{n}] = 1$, then

$$\frac{\beta_i}{2\pi} = \det[\vec{v}_i, \vec{v}_{i+1}, \vec{r}]^{-1}, \quad \epsilon_i = \det[\vec{r}, \vec{v}_{i+1}, \vec{n}], \quad \epsilon'_i = \det[\vec{v}_i, \vec{r}, \vec{n}]. \tag{1.5}$$

It is important to stress that the identification of parameters ϵ, ϵ' is not unique, one may always add to ϵ, ϵ' integer multiples of $2\pi\beta^{-1}$. The terms $B_{vec}(x)$ and $B_{hyp}(x)$ are polynomials defined in Appendix A by (4.22) and (4.23).

The above manner of presenting the factorisation for $Y^{p,q}$ was used in [6], but it has one drawback, namely the piece we call the perturbative Nekrasov partition function, in particular, the denominator of (1.4), is not manifestly symmetric under exchange $\underline{R} \rightarrow \bar{\underline{R}}$, namely under $a + m \rightarrow -a - m$, and it only becomes so when combined with the piece B_{hyp} . However, this symmetry is expected since the denominator of (1.2) does possess this symmetry. The reason that the Nekrasov partition function lacks this symmetry is that in the trace, one must let $\epsilon \in \epsilon'$ be complex in order to define the index as a formal power series. In doing so the matter fields of representations \underline{R} and $\bar{\underline{R}}$ are treated unequally leading to the lack of symmetry. However, we can follow the work [9] and factorise also the Bernoulli pieces B_{vec}, B_{hyp} and make the symmetry manifest. So a second way of presenting the factorisation is

$$Z^{pert} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r}{g_{YM}^2} \varrho \text{Tr}_f[a^2]} \cdot \frac{\prod_{i=1}^n (\det'_{adj} (e^{-\beta_i a} | e^{i\beta_i \epsilon_i}, e^{i\beta_i \epsilon'_i})_{\infty} (a \rightarrow -iR^1 - a))^{1/2}}{\prod_{i=1}^n (\det_{\underline{R}} (e^{-\beta_i (a+m-iR^1/2)} | e^{i\beta_i \epsilon_i}, e^{i\beta_i \epsilon'_i})_{\infty} (a+m \rightarrow -a-m))^{1/2}}. \tag{1.6}$$

In this way, the partition function is presented as the product of n blocks, each of which corresponds to a copy of the partition function associated to $\mathbb{C}^2 \times S^1$, for further investigations of the properties of these blocks see [9, 10]. At this point it is natural to conjecture that the full partition function on X is given by the same gluing of n -copies of the full Nekrasov partition functions.

The paper is organised as follows: in Sect. 2 we give an overview of the 5D toric SE manifolds, with emphasis on how to read off the geometry from the toric data. In Sect. 3 we present the derivation of the full perturbative partition function for any toric simply connected SE manifolds. We explain that the answer can be written in two equivalent ways, either using the restricted lattice or using the cone description. The result is given in terms of some new special function that is a generalisation of the triple sine function. Section 4 contains the detailed technical proof of the factorisation of the perturbative partition function in terms of Nekrasov’s partition functions on $\mathbb{C}^2 \times S^1$. In Sect. 6 we summarise our paper and we conjecture the full nonperturbative answer, which contains instantons. We also point out some puzzles and open problems in that section. The paper is supplemented by two appendices. In Appendix A we collect some basic facts and conventions of the special functions. We also prove a property of a special function that we used in the main text. In Appendix B we make some comments on the description of the good cone condition.

2. Toric Sasaki-Einstein Manifolds

In this section we briefly review some background material concerning the 5D toric Sasaki-Einstein geometry. In particular we concentrate on how one may read off from the toric diagram information about the geometry. The reader may find similar review in [6] and for more detailed exposition one may consult [3, 4].

Take a manifold X and consider its metric cone $C(X) = X \times \mathbb{R}^{\geq 0}$ with metric $G = d\tau^2 + \tau^2 g_X$, with τ being the coordinate of $\mathbb{R}^{\geq 0}$. If $C(X)$ is Kähler, then one

says that X is a Sasaki manifold, if further $C(X)$ is Calabi-Yau, then X is said to be Sasaki-Einstein (SE). In particular, its Ricci tensor satisfies

$$R_{mn} = 4g_{mn}$$

for dimension 5.

Given a Sasaki manifold, one has the metric contact structure, with the Reeb vector field \vec{r} and contact 1-form κ given by

$$R = J(\tau\partial_\tau), \quad \kappa = i(\bar{\partial} - \partial) \log \tau, \tag{2.1}$$

where J is the complex structure over $C(X)$. If there is an effective, holomorphic and Hamiltonian action of the torus T^3 on the metric cone $C(X)$, and the Reeb vector field is a linear combination of the torus action, then one says that X is toric. Our main examples S^5 , $Y^{p,q}$ -spaces discovered in [11] and $L^{a,b,c}$ -spaces discovered in [12] are all toric SE manifolds. Next we turn to the toric description of these examples and more general toric SE manifolds.

Let $\vec{\mu}$ be the moment map for the three torus actions, then due to the cone structure on $C(X)$, the image of $\vec{\mu}$ will also be a cone in \mathbb{R}^3 , denoted as $C_\mu(X)$. From the cone one can read off almost all information of the manifold, in fact, it is was shown in [13] by Lerman, extending the well-known Delzant construction [14], that from a given good cone (definition to come shortly), one can reconstruct the manifold itself. One will see an inkling of how this is done in Sect. 3.2.

Lerman termed a cone to be good¹ if at each intersection of its two adjacent faces F_i and F_{i+1} , their inward pointing normals $\vec{v}_i, \vec{v}_{i+1} \in \mathbb{Z}^3$ can be completed into a basis of \mathbb{Z}^3 . That is, there exists a third vector \vec{n} such that $\det[\vec{v}_i, \vec{v}_{i+1}, \vec{n}] = 1$. A useful way of viewing the manifold $C(X)$ is the following: away from the boundary of the moment map cone $C_\mu(X)$, one has the torus fibration $T^3 \rightarrow C(X)|_{C_\mu(X)^\circ} \rightarrow C_\mu(X)^\circ$, where $^\circ$ means the interior. While at face i , the particular torus as singled out by \vec{v}_i degenerates.

The Reeb vector field is by definition a linear combination of the three torus actions, so one can represent r as a 3-vector \vec{r} . The actual manifold X can be obtained by restricting $C(X)$ to the plane $\vec{y} \cdot \vec{r} = 1/2$, and we shall call the intersection

$$\{\vec{y} \in C_\mu(X) | \vec{y} \cdot \vec{r} = 1/2\} = B_\mu(X),$$

where B stands for 'base'. This base is a compact polygon iff the 3-vector \vec{r} is within the dual cone

$$\vec{r} = \sum_{i=1}^n \lambda_i \vec{v}_i, \quad \lambda_i > 0, \quad \forall i. \tag{2.2}$$

This condition also appears later as the condition for the partition function to converge. From this discussion, one may similarly view X as a torus fibration over $B_\mu(X)^\circ$ and again at the boundary of $B_\mu(X)$, different tori degenerates. An immediate consequence of this view is that the fundamental group of X can be computed as

$$\pi_1(X) \sim \mathbb{Z}^3 / \text{span}_{\mathbb{Z}}(\vec{v}_1, \dots, \vec{v}_n). \tag{2.3}$$

The meaning of this formula is clear: only those tori that cannot be written as a linear combination of \vec{v}_i are not contractible. As a technical remark, if X is simply connected,

¹ The original formulation is slightly different from the one given here, and since the equivalence does not seem obvious to us, we provide a short proof in the Appendix B.

then it implies that the matrix $[\vec{v}_1, \dots, \vec{v}_n]$ can be completed into an $SL(n, \mathbb{Z})$ matrix. Indeed, up to right multiplying by an $SL(n, \mathbb{Z})$ matrix, one can put $\vec{v}_{1,2,3}$ into $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$, and the rest is clear.

Furthermore, if \vec{r} is generic, then the orbit of the Reeb vector field is not closed, except at the corners of $B_\mu(X)$, where only one S^1 is acting non-trivially. When restricted to a neighbourhood of a corner, the manifold X is a solid torus, i.e. diffeomorphic to $S^1 \times \mathbb{C}^2$, where S^1 is the closed Reeb orbit over the corner point, for example see Fig. 1. But the solid torus is twisted, as one completes a cycle along S^1 , the two planes also rotate by some angles. The central message of this paper is that to compute the partition function, one need only include one copy of the Nekrasov instanton partition function for each closed Reeb orbit, where the twisting parameters appear as the equivariant parameters of the Nekrasov partition function.

Let us focus now on the neighbourhood of one of the corners of, say, the intersection of face i and face $i + 1$, let \vec{n} be an integer-entry 3-vector such that $\det[\vec{n}, \vec{v}_i, \vec{v}_{i+1}] = 1$ (the existence of \vec{n} is a consequence of the moment map cone being good). One can then decompose the Reeb vector as a linear combination of $\vec{n}, \vec{v}_i, \vec{v}_{i+1}$, that is one decompose the Reeb into one $U(1)$ that remains non-degenerate at the corner, which gives the closed Reeb orbit there, plus two more that degenerate at the same corner, giving the twisting parameter of the solid torus. This reasoning leads to the formulae (1.5) for the circumference and twisting parameters.

The Calabi-Yau condition can also be phrased in terms of the data of the cone. Assuming that the number of faces is larger than 3, then it turns out that if there exists an integer vector $\vec{\xi}$ such that $\vec{\xi} \cdot \vec{v}_i = 1, \forall i$, then $C(X)$ is Calabi-Yau. In fact, it is convenient to choose a basis of the 3-tori so that the first component of \vec{v}_i is 1 for all i and then $\vec{\xi} = [1, 0, 0]$. This property plays a pivotal role in our calculation, in that it allows us to perform a summation within the cone $C_\mu(X)$.

Next we give some examples, first the $Y^{p,q}$ space treated in [6], one chooses the four normals to be

$$\vec{v}_1 = [1, 0, 0], \quad \vec{v}_2 = [1, -1, 0], \quad \vec{v}_3 = [1, -2, -p + q], \quad \vec{v}_4 = [1, -1, -p], \quad (2.4)$$

where $p > q > 1$ and $\gcd(p, q) = 1$.

A generalisation to the $Y^{p,q}$ space is the $L^{a,b,c}$ space, with $d = a + b - c > 0$ and $\gcd(a, c) = \gcd(a, d) = \gcd(b, c) = \gcd(b, d) = 1$. The four normals are

$$\vec{v}_1 = [1, c, -bn], \quad \vec{v}_2 = [1, a, bm], \quad \vec{v}_3 = [1, 0, 1], \quad \vec{v}_4 = [1, 0, 0], \quad (2.5)$$

where m, n are chosen so that $mc + na = 1$. The metric cone $C(L^{a,b,c})$ can be constructed as Kähler quotient of \mathbb{C}^4 with $U(1)$ with the charges $(a, b, -c, -a - b + c)$.

As an example of a pentagon toric cone, one has the so called $X^{p,q}, p > q > 0$ space, whose normals are

$$\begin{aligned} \vec{v}_1 &= [1, 0, 0], \quad \vec{v}_2 = [1, 1, 0], \quad \vec{v}_3 = [1, 0, p], \\ \vec{v}_4 &= [1, -1, p + q], \quad \vec{v}_5 = [1, -1, p + q - 1]. \end{aligned}$$

The metric cone of this space can be constructed from the Kähler quotient of \mathbb{C}^5 with respect to two $U(1)$'s of charge $[1, 0, -1, p, -p]$ and $[1, -1, 1, q - 1, -q]$.

One can also use the following alternative description of $C(X)$ as Kähler quotient [31]

$$C(X) = \mathbb{C}^n // (U(1)^{n-3} \times \Lambda), \quad (2.6)$$

where Λ is some discrete abelian group, and the $U(1)$'s act on \mathbb{C}^n with the charges $\vec{Q}_a = (Q_a^1, \dots, Q_a^i, \dots, Q_a^n)$, $a = 1, \dots, n - 3$. These charges are determined from the inward normals \vec{v}_i according to the equation

$$\sum_{i=1}^n \vec{v}_i Q_a^i = 0.$$

The CY condition translates to

$$\sum_{i=1}^n Q_a^i = 0. \tag{2.7}$$

If X is a simply connected SE manifold (which is the only case we consider in this paper) then $\Lambda = 0$.

The Kähler reduction picture is at times more suitable for index computation. And for the construction of examples, it is also easier to start from \mathbb{C}^n and postulate $n - 3$ charges Q satisfying (2.7). To ensure the simply connectedness, one demands that one can pick vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3 \in \mathbb{Z}^n$ such that $A = [\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{Q}_1, \dots, \vec{Q}_{n-3}]$ forms an $SL(n, \mathbb{Z})$ matrix. Then the inward normals of the original cone $\vec{v}_1, \dots, \vec{v}_n$ are recovered as the first 3 rows of A^{-1} (but not necessarily in the correct order). And then one necessarily has $\text{span}(\vec{v}_1, \dots, \vec{v}_n) = \mathbb{Z}^3$, i.e. one gets a simply connected manifold.

The above two-way relation mirrors the Delzant and Lerman constructions [13, 14], that is, by embedding a cone in \mathbb{R}^3 into \mathbb{R}^n as the intersection of $n - 3$ hyperplanes (whose normals are the \vec{Q}_a 's), one can present the original manifold as a Kähler quotient of \mathbb{C}^n .

In what follows we concentrate only on simply connected SE toric manifolds which topologically correspond to $(n - 3)(S^2 \times S^3)$, namely $(n - 3)$ connected sum of $S^2 \times S^3$. We will make a few comments about non-simply connected SE manifolds in the last Sect. 6.

3. Localisation of 5D SYM

In this section we sketch briefly the actual localisation calculation. Our presentation is the generalisation of the previous works [5, 15, 16] to the case of general simply connected toric SE manifolds. We also discuss two different representations of the answer.

3.1. Localisation calculation. In [17] the SYM theory coupled to matter on the round S^5 was written down. Due to the SE structure over S^5 , one can find a pair of normalised Killing spinors $\xi_{1,2}$, such that the bilinear $\xi_1 \Gamma^m \xi_2$ is proportional to the Reeb vector field R^m on S^5 . The two Killing spinors will pick out a particular susy charge called δ that satisfies the key relation

$$\delta^2 = -i L_R + G, \quad \text{for the vector multiplet} \tag{3.1}$$

$$\delta^2 = -i L_R^s + G, \quad \text{for the hypermultiplet} \tag{3.2}$$

where G stands for gauge transformation and $L_R (L_R^s)$ is the (spinor) Lie derivative.

It turns out that a change of variables (which again involves the Killing spinors) allows us to formulate the vector multiplet in terms of differential forms, and the only

feature that is required from the geometry is the metric contact structure. This was called the twisted SYM in [15], and the susy complex was called the cohomological complex. Using the algebra (3.1), the path integral localises onto the so called contact instanton configurations, and one needs to integrate over the Gaussian fluctuations around such configurations. To calculate the full partition function from first principles appears to be hard at the moment. However, the expansion around zero connection configuration is doable and one obtains the perturbative partition function as a matrix model. Furthermore, since the actual SE metric is not required once we pass to the cohomological complex formulation, we can consider the partition function for the deformed Reeb vector field, i.e., the squashed five sphere. Equivalently, one can turn on extra background gauge fields and put the original SYM theory directly on a squashed S^5 and perform the computation from there, see [18, 19] (see also [20–22]), but the work load is considerably heavier this way around. For the hypermultiplet, one would need in principle the SE metric, however, once the result is obtained, it is obvious how to generalise it to the squashed sphere.

Much of the story can be repeated for an infinite class of simply connected SE manifolds $Y^{p,q}$. The simply connectedness is there to ensure that the zero instanton configuration actually corresponds to the trivial connection. The calculation was completed in [5] for the $Y^{p,q}$ manifolds. The main technical aspect of the calculation, the computation of an equivariant index, relies on using the known index structure on $S^3 \times S^3$, and imposing a lattice constraint, as we shall review shortly. This calculation carries over to the $L^{a,b,c}$ as a straightforward generalisation. But for toric SE manifolds with a more complicated moment map cone, the method used there gets cumbersome, and it is more systematic to employ the fixed point theorem [23], presented in the appendix of [5].

Now our goal is to generalise the result from [5] to any simply connected toric SE manifold. Following the logic presented in [5, 15, 16] for any simply connected SE manifold the perturbative partition function of $N = 1$ SYM with a hypermultiplet in representation \underline{R} and mass m is written as the superdeterminant of the two operators in (3.1) and (3.2), taken over the $\Omega_H^{0,\cdot}$ -complex²

$$Z^{pert} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r}{g_{YM}^2} \varrho \text{Tr}[a^2]} \cdot \frac{\det'_{adj} \text{sdet}_{\Omega_H^{0,\cdot}}(-irL_R - ia)}{\det_{\underline{R}} \text{sdet}_{\Omega_H^{0,\cdot}}(-irL_R^s - ia - im)}, \tag{3.3}$$

where r is a parameter controlling the overall size of X , ϱ is the squashed volume of X normalised against $\text{Vol}_{S^5} = \pi^3$. The actual non-trivial calculation is centred around the explicit evaluation of superdeterminants in (3.3).

Sasaki manifolds have a transverse Kähler structure, that is, one can write the 5D metric as

$$g = \kappa \otimes \kappa + g_H$$

with g_H being a local Kähler metric, see Sect. 1.2 of [4]. Thus one has the complex of horizontal $(0, i)$ forms, with $i = 0, 1, 2$, which we denoted as $\Omega_H^{0,\cdot}$. The differential for this complex is obtained by projecting the de Rham differential to its component that increase the degree $(0, i) \rightarrow (0, i + 1)$. We call this differential the transverse Dolbeault differential $\bar{\partial}_H$. The cohomology of this complex is called the Kohn-Rossi

² In writing this expression we skipped a few technical steps, in particular, the integral of a within the Lie-algebra of the gauge group is written as the integral over Cartan times a determinant factor, which combines with the contribution from the ghost sector to give this neat expression (3.3).

(KR) cohomology. The various fields in the vector-, hypermultiplet can be reduced to the horizontal $(0, i)$ forms using Fierz identity, and fit nicely into the $\Omega_H^{0,\cdot}$ complex. This is why the localisation computation reduces to the super determinants in (3.3), see either [15] or [16] for details of the reduction.

In our setting of toric SE manifolds, the operator $\bar{\partial}_H$ is a transversally elliptic operator and one can evaluate the super determinant in (3.3) by employing the index theorem for such operators, as was done in [5]. But here we shall use the approach of Schmude [24], see also [25,26].

As is standard for localisation, only those modes that are in the KR cohomology (which we denote simply as $H^{0,\cdot}$) make a net contribution to the superdeterminant, so the final answer is

$$Z^{pert} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r}{g_{YM}^2} \text{Tr}[a^2]} \times \frac{\det'_{adj} \text{sdet}_{H^{0,\cdot}}(-irL_R - ia)}{\det_{\underline{R}} \text{sdet}_{H^{0,\cdot}}(-irL_R^s - ia - im)}. \tag{3.4}$$

In writing Z^{pert} , we ignore some possible (a independent) phases coming from the determinant factors.

It was pointed out by Schmude that the KR cohomology can be reduced to $H^0(\mathcal{O}(C(X)))$, with $\mathcal{O}(C(X))$ being the sheaf of holomorphic functions on the metric cone of X . We will go over this argument here. Since the Reeb is Killing with respect to the metric, the operator L_R will commute with $\bar{\partial}_H$, and we can analyse the cohomology of $\bar{\partial}_H$ with definitive $-iL_R$ eigenvalues (this eigenvalue is in fact the R-charge). Now one can find a map relating the horizontal $(0, i)$ forms on X to those on the metric cone $C(X)$. Assuming X is embedded in $C(X)$ at $\tau = 1$, the Dolbeault differential $\bar{\partial}$ on $C(X)$ is related to $\bar{\partial}_H$ in local coordinates by

$$\bar{\partial} = \bar{\partial}_H + \frac{1}{2}(d \log \tau - i\kappa)(\tau \partial_\tau + i \partial_\theta), \tag{3.5}$$

where θ is the local coordinate such that ∂_θ is the Reeb vector, and we have used the relation (2.1) between the homothetic vector and the Reeb. Assuming that $\omega \in \Omega_H^{0,i}$ has eigenvalue ζ under $-iL_R$, then we can extend it to a form on the $C(X)$ as

$$ext : \omega \rightarrow \omega \tau^\zeta,$$

the extension makes sense since the point $\tau = 0$ is removed. Furthermore if ω is closed (exact) under $\bar{\partial}_H$ then $\omega \tau^\zeta$ is closed (exact) under $\bar{\partial}$, thus the extension induces a map of the corresponding cohomology. Conversely a $(0, i)$ -form on $C(X)$ can be restricted to X , the map ext composed with the restriction gives the identity map $res \circ ext = 1$. So we see that the induced map on cohomology induced by ext must be injective. This implies immediately that $H^{0,1}(X)$ is zero since $H^1(\mathcal{O}(C(X))) = 0$. For the zeroth cohomology $H^{0,0}(X)$, since there are no exact forms, we need only focus on the holomorphic functions on $C(X)$. If a nonzero function f is holomorphic on $C(X)$, its restriction to $\tau = 1$ is non-zero, as can be seen from (3.5) (for example, in f the modes of different powers in τ must have different $-iL_R$ -eigenvalue and hence cannot cancel out at $\tau = 1$). So we actually get a bijection

$$ext : H^{0,0}(X) \simeq H^0(\mathcal{O}(C(X))),$$

For the $(0, 2)$ forms, one can use the holomorphic volume form Ω on $C(X)$ to construct a pairing between $(0, 0)$ and $(0, 2)$ forms. Since Ω is a top holomorphic form, it is closed,

and its restriction to X (also denoted as Ω) is closed as well. The restriction of Ω has the property that $\kappa \wedge \Omega = 0$ and $\iota_R \Omega$ is in $\Omega_H^{2,0}(X)$. From these properties, we see that the integration

$$\langle f, \omega \rangle = \int_X \Omega f \omega, \quad f \in \Omega^{0,0}(X), \quad \omega \in \Omega_H^{0,2}(X)$$

is a non-degenerate paring. It is also a non-degenerate pairing between $H^{0,0}(X)$ and $H^{0,2}(X)$, to see this, let $f \in H^{0,0}(X)$, and $\omega = \bar{\partial}_H \zeta$, $\zeta \in \Omega_H^{0,1}$, then

$$\langle f, \bar{\partial}_H \zeta \rangle = \int_X \Omega f \bar{\partial}_H \zeta = \int_X \Omega f d\zeta = \int_X \Omega df \zeta = \int_X \Omega \bar{\partial}_H f \zeta = 0,$$

where $\kappa \wedge \Omega = 0$ and $\iota_R \Omega \in \Omega_H^{2,0}(X)$ is used. From these considerations $H^{0,2}(X) \simeq (H^{0,0}(X))^*$.

To summarise, to obtain the KR cohomology for our specific problem, it suffices to compute $H^0(\mathcal{O}(C(X)))$, i.e. the holomorphic functions on $C(X)$. But the latter object has a combinatorial description, one simply enumerates the integral points within the moment map cone (this follows almost directly from the definition of a toric Kähler manifold) and each such point gives a holomorphic function on $C(X)$. What is more, the three coordinates of these points give the charges of these functions under the three $U(1)$'s. In particular, one can also read off their L_R eigenvalue. To figure out the $U(1)$ charges of $H^{0,2}$ groups, one needs to get the charges of Ω . To do this, let $\vec{\xi}$ be the 3-vector such that $\vec{\xi} \cdot \vec{v}_i = 1$ (see (1.1)), then the charges of Ω are the 3-components of $\vec{\xi}$, which is also a standard fact of toric geometry. Then in particular, the R-charge of Ω (the $-iL_R$ eigenvalue) is $\vec{\xi} \cdot \vec{r}$. In all of the examples given earlier, this vector $\vec{\xi}$ is chosen to be $[1, 0, 0]$, and so the R-charge is R^1 .

With these preparation, one can write the superdeterminant in (3.4) as

$$\text{sdet}_{H^{0,\bullet}}(-irL_R + x) = \prod_{\vec{n} \in C_\mu(X) \cap \mathbb{Z}^3} (\vec{n} \cdot \vec{r} + x) (\vec{n} \cdot \vec{r} - x + \vec{\xi} \cdot \vec{r}) = S_3^X(x; \vec{r}), \quad (3.6)$$

where the second factor comes from $H^{0,2}$ and we have as usual discarded overall multiplicative constants. The superdeterminant of $-iL_R^s$ is similar, one makes a shift $x \rightarrow x + R^1/2$, which originates from expressing L_R^s in terms of L_R [5]. In (3.6) we defined a new special function $S_3^X(x; \vec{r})$ associated with the moment map cone of any 5D simply connected toric SE manifold X . This function is a generalisation of the usual triple sine function, since by taking $X = S^5$, whose moment map cone $C_\mu(S^5) = \mathbb{R}_{\geq 0}^3$, one recovers the definition of the standard triple sine function (A.6).

To summarise, the perturbative partition function of $N = 1$ supersymmetric Yang-Mills over a 5D simply connected toric SE manifold, with hypermultiplet in representation \underline{R} is given by

$$Z^{pert} = \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r}{g_{YM}^2} \text{Tr}[a^2]} \cdot \frac{\det'_{adj} S_3^X(ia; \vec{r})}{\det_{\underline{R}} S_3^X(ia + im + R^1/2; \vec{r})}, \quad (3.7)$$

where we have fixed $\vec{\xi} = [1, 0, 0]$. Let us make a couple of concluding remarks. In the setup of the supersymmetric Yang-Mills, especially for the hypermultiplet, we have

used the SE metric, and so in particular, the classical action evaluated at the localisation locus (the term in the exponent above) should be $-8r g_{YM}^{-2} \text{Vol}_{X_{SE}} \text{Tr}[a^2]$ with $\text{Vol}_{X_{SE}}$ computed with the SE metric. However the superdeterminant of the operator L_R may be computed for a Reeb being any combination of the three $U(1)$'s, provided \vec{r} is in the dual cone. These Reeb's do not give rise to an SE metric, and so we have also replaced the volume factor in the exponent by the squashed volume

$$\text{Vol}_X = \varrho \pi^3.$$

For a self-contained justification of this replacement, one should set up the supersymmetric Yang-Mills with a general Reeb, which then entails turning on an extra background connection to maintain supersymmetry. Alternatively we may adopt the cohomological complex as the starting point, as in [15], and then this classical term appears as $\int_X \kappa d\kappa^2 \text{Tr}[\sigma^2]$, which is a supersymmetry completion of the Chern-Simons like observable $\int_X \kappa F F$. Since the integral of $1/2\kappa d\kappa d\kappa$ leads to the squashed volume, it is natural to make the replacement as we did above.

Using these arguments the answer given above should be regarded as a general equivariant answer. This is valuable since the equivariant parameters that enter into the \vec{r} can tell us about how the geometry of the underlying manifold affect the partition function. In particular, since the result (3.7) only depends on the Reeb vector, which is the complex structure moduli of the Kähler cone $C(X)$, we can argue that the partition function is independent of the Kähler moduli. However, in our case, the independence of the Kähler moduli is only visible after the computation, yet one probably can use arguments similar to [27] and demonstrate this directly.

The independence of the partition function on the Kähler moduli is presumably what lies behind the factorisation phenomenon that we shall show presently. Namely, one can go to a point in the Kähler moduli where the closed Reeb orbits are widely separated, so that at each closed orbit, the local geometry is $\mathbb{C}^2 \times S^1$. Then for each closed Reeb orbit, one can insert a copy of the Nekrasov partition function on $\mathbb{C}^2 \times S^1$, which is computed in the Coulomb branch, i.e. the asymptotic value of the σ field is fixed. And the final complete answer is obtained by taking all these copies and integrating over their common Coulomb branch parameter. What we do next is an explicit check that this prescription is correct for the perturbative part of the partition function.

3.2. Relation between the restricted lattice and the cone descriptions. In this section we show that the original presentation of the partition function in [5] in terms of a constrained lattice is equivalent to the cone description given above. For those familiar with toric geometry, the equivalence is probably quite obvious and he may skip to the next section.

In [5] the superdeterminant for $Y^{p,q}$ is given in terms of a generalised triple sine function, which is defined through the ζ -function regularised infinite product on a lattice

$$\begin{aligned} \text{sdet}_{H^{0,\cdot}}(-irL_R + x) &= S_3^{\Lambda(p,q)}(x|\omega_1, \omega_2, \omega_3, \omega_4) \\ &= \prod_{(i,j,k,l) \in \Lambda_{(p,q)}^+} \left(i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4 + x \right) \left(i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4 + \sum \omega_i - x \right), \end{aligned} \tag{3.8}$$

where the lattice $\Lambda_{(p,q)}^+$ is defined as

$$\Lambda_{(p,q)}^+ = \{i, j, k, l \in \mathbb{Z}_{\geq 0} \mid i(p+q) + j(p-q) - kp - lp = 0\}, \tag{3.9}$$

and $\omega_1, \omega_2, \omega_3, \omega_4$ are equivariant parameters which are related to the Reeb vector as follows

$$R^1 = \sum \omega_i, \quad R^2 = -\omega_1 - \omega_2 - 2\omega_4, \quad R^3 = -p\omega_2 + (q-p)\omega_4, \tag{3.10}$$

If one replaces the constraint (3.9) for the lattice by

$$\Lambda_{(a,b,c)}^+ = \{i, j, k, l \in \mathbb{Z}_{\geq 0} \mid ia + jb - kc - l(a+b-c) = 0\}, \tag{3.11}$$

one obtains the generalised triple sine function $S_3^{\Lambda(a,b,c)}$ that gives the perturbative partition function for the $L^{a,b,c}$ manifolds. Next we shall see how to get these relations for a general toric SE manifold.

In general situation for any toric simply connected X we assume that we have a lattice of $\mathbb{Z}_{\geq 0}^n$, obeying $n-3 > 0$ constraints

$$\Lambda^+ = \{n_i \geq 0, i = 1, \dots, n \mid \sum_{i=1}^n Q_a^i n_i = 0, a = 1, \dots, n-3\}. \tag{3.12}$$

The charges Q_a^i are the same as in the description of $C(X)$ as Kähler quotient in (2.6). Introducing the squashing parameters $\vec{\omega} = (\omega_1, \dots, \omega_n)$ we define the generalised triple sine associated with the lattice Λ^+

$$S_3^\Lambda(x; \vec{\omega}) = \prod_{\vec{n} \in \Lambda^+} (\vec{n} \cdot \vec{\omega} + x)(\vec{n} \cdot \vec{\omega} - x + \sum_{i=1}^n \omega_i). \tag{3.13}$$

This was how the result was presented in [5], next we show that this is equivalent with the function S_3^X defined in (1.3), which is a more intrinsic description.

According to the discussion at the end of section 2, for X simply connected, we can pick the basis vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3 \in \mathbb{Z}^3$ such that $A = [\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{Q}_1, \dots, \vec{Q}_{n-3}]$ forms an $SL(n, \mathbb{Z})$ matrix. Apply A to the lattice Λ^+ , then the n conditions $n_i \geq 0$ in (3.12) turn into $\sum_{a=1}^3 v_i^a m_a \geq 0$, where v_i^a are the first 3 rows of A^{-1} . We denote these by $\vec{v}_1, \dots, \vec{v}_n$, i.e. \vec{v}_i are 3-vectors, and the conditions $\sum_{a=1}^3 v_i^a m_a \geq 0$ describes a cone inside \mathbb{R}^3 .

As an illustration, take the lattice (3.9), then \vec{Q} is the 4-vector $[-p-q, q-p, p, p]$. One can complete it into an $SL(4, \mathbb{Z})$ matrix

$$A = \begin{pmatrix} 0 & -1 & a & -p-q \\ 0 & 0 & -a-2b & q-p \\ 1 & 1 & b & p \\ 0 & 0 & b & p \end{pmatrix}, \quad (a+b)p + bq = 1.$$

Its inverse is

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & -2 \\ 0 & -p & 0 & q-p \\ 0 & b & 0 & a+2b \end{pmatrix},$$

and from the first three rows of A^{-1} one finds the four inward normals given in (2.4). Also the first three rows give the relation of the Reeb vector with ω_i as in (3.10).

Continuing with our manipulation of the lattice, we let $\vec{\omega}$ be an n -vector, by inserting AA^{-1} into $\vec{n} \cdot \vec{\omega}$, we see that the summation over the constrained lattice can be written as

$$\sum_{\Lambda^+} \vec{n} \cdot \vec{\omega} = \sum_{m_a \in C_\mu(X) \cap \mathbb{Z}^3} \sum_{a=1}^3 m_a (A^{-1}\vec{\omega})_a. \tag{3.14}$$

Thus we have proved the equality of the two products

$$\prod_{\vec{n} \in \Lambda^+} (\vec{n} \cdot \vec{\omega} + x) = \prod_{\vec{m} \in C_\mu(X) \cap \mathbb{Z}^3} (\vec{m} \cdot \vec{r} + x), \text{ where } r_a = (A^{-1}\vec{\omega})_a.$$

Also notice that since $\vec{\xi} \cdot \vec{v}_i = 1, \forall i$, and that $[\vec{v}_1, \dots, \vec{v}_n]$ constitutes the first three rows of A^{-1} , so the quantity $\vec{\xi} \cdot \vec{r}$ can be written as

$$\vec{\xi} \cdot \vec{r} = \sum_{a=1}^3 \xi_a r_a = \sum_{a=1}^3 \xi_a (A^{-1}\vec{\omega})_a = \sum_{i=1}^n \omega_i.$$

By comparing the definition (3.13) and (1.3) of $S_3^\Lambda(x, \vec{\omega})$ and $S_3^X(x, \vec{r})$, we get the equality

$$S_3^\Lambda(x; \vec{\omega}) = S_3^X(x; \vec{r}),$$

and also the equivalence between the constrained lattice presentation and the cone representation.

Next we shall work with a general good cone that corresponds to a 5D simply connected toric SE manifold. Assume that the moment map cone has $n \geq 4$ faces, and that the normals are chosen so that their first component is 1.

The perturbative partition is given in (3.7) and our central task is to evaluate the two products

$$I : \prod_{\vec{m} \in C_\mu(X) \cap \mathbb{Z}^3} (\vec{m} \cdot \vec{r} + x), \tag{3.15}$$

$$II : \prod_{\vec{m} \in C_\mu(X) \cap \mathbb{Z}^3} (\vec{m} \cdot \vec{r} - x + r^1). \tag{3.16}$$

4. Derivation of Factorisation

4.1. *Conversion to the triple sine functions.* Since the real part of the Reeb vector \vec{r} is assumed to be within the dual cone, and that x has a small but positive real part, the real part of the factors in (3.15) is bounded away from zero and tends to infinity, so one can use ζ -function regularisation to make sense of the infinite product. Bearing this in mind, one can treat the infinite product at its face value, and do the usual manipulations.

The product or summation over the integral points within the cone is investigated in [28] through subdividing the cone into smaller portions. We will use similar strategies that work for any cone that gives rise to simply connected toric SE manifolds. We fix the inward normals of the cone to be $\vec{v}_i = [1, -\vec{L}_i], i = 1, \dots, n$ for some two vectors $\vec{L}_i = [L_i^2, L_i^3]$.

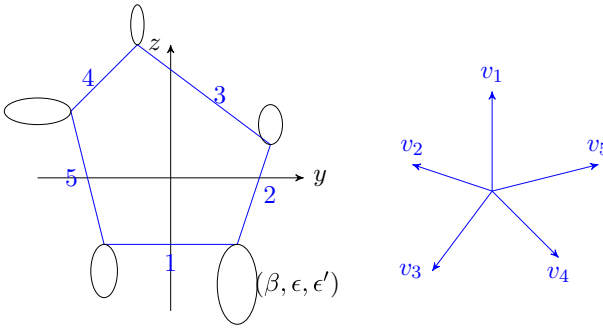


Fig. 2. The polytope cone, projected onto the plane $y = 1$, depending on the specific case, one of the faces may move off to infinity, that its two neighbouring faces turn parallel. The circles represent the closed Reeb orbits. The right panel is the inward pointing normals of the cone

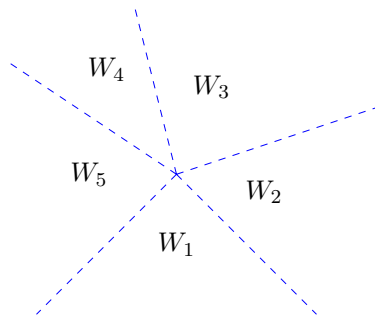


Fig. 3. The division of the m_2 - m_3 plane, each W corresponds to a face of the moment map cone

From the constraint $\vec{v}_i \cdot \vec{m} \geq 0$, the limit of m_1 is $\infty > m_1 \geq L_i^2 m_2 + L_i^3 m_3$, which changes as i increments. So we need to divide the m_2 - m_3 plane into n (5, in the case of Fig. 2) wedges, one for each edge, and we get the picture of Fig. 3. So in W_i the lower limit of m_1 is $m_1 \geq L_i^2 m_2 + L_i^3 m_3$.

The product within each wedge reads

$$\begin{aligned}
 I|_{W_i} &= \prod_{(m_2, m_3) \in W_i} \prod_{m_1 \geq L_i^2 m_2 + L_i^3 m_3} (\vec{m} \cdot \vec{R} + x) \\
 &= \prod_{(m_2, m_3) \in W_i} \prod_{m_1 \geq 0} \left((R^2 + R^1 L_i^2) m_2 + (R^3 + R^1 L_i^3) m_3 + R^1 m_1 + x \right). \quad (4.1)
 \end{aligned}$$

We will denote by \tilde{r}_i the 2-vector

$$\tilde{r}_i = (R^2 + R^1 L_i^2, R^3 + R^1 L_i^3), \quad (4.2)$$

which changes from one wedge to another.

The product over m_1 is now straightforward, and we have reduced the problem to the following. Consider two lines in \mathbb{R}^2 with rational slopes that bound W_i , how do we perform the summation (or the product, all the same) of the weight $\vec{\xi} \cdot \vec{n} = \xi_1 n_1 + \xi_2 n_2$ over the integral points between these two lines? We assume that the normals of the two lines are v_1, v_2 , which are primitive integer 2-vectors, see Fig. 4. Then we have the sum

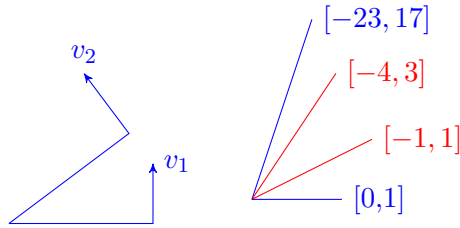


Fig. 4. Sum between the two blue lines, depicted in the left panel. One can add more lines in between, as in the right panel. The numbers label the normal of each line. The slopes of the lines are not drawn to scale (colour figure online)

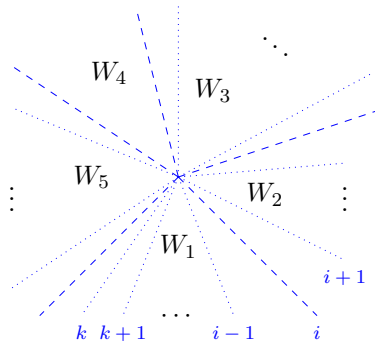


Fig. 5. Further division of the m_2 - m_3 plane, by adding lines. The normals of all lines are pointing counter-clockwise

$$\sum_{\vec{n} \cdot v_1 \geq 0; \vec{n} \cdot v_2 \leq 0} \vec{\xi} \cdot \vec{n}.$$

The strategy is to add more lines between the two given lines, so that the two normals of each pair of neighbouring lines form an $SL(2, \mathbb{Z})$ matrix, then one can, by an $SL(2, \mathbb{Z})$ matrix, transform the two lines into the x - and y -axis, in which situation the sum would be simple. Surely, one cannot know how many lines one would need to add, but so long as the process contains only finite number of steps, which we show next, the lack of explicitness need not hinder us.

Without loss of generality, one can assume $v_1 = [0, 1]$, i.e. the first line is the x -axis (by applying an $SL(2, \mathbb{Z})$ transformation, since v_1 is primitive). For definiteness, we also assume $v_2 = [-p, q]$ with $\gcd(p, q) = 1$, $p, q > 0$, the other possibilities can be treated entirely similarly, see Fig. 4. One simply observes that given two numbers $p, q > 0$ coprime, one can find $s, t > 0$ such that $pt - qs = 1$ and that $p > s, q > t$. The proof is a simple exercise and is left for the reader, otherwise consult [29]. Then it is easy to see that $p/q > s/t$ so the new line has a smaller slope. Further $\det[-s, t; -p, q] = 1$, which is part of we set out to achieve. One can continue this process, since the size s, t as well as the slope decreases each time, the process will stop after finitely many steps. It is not at all important to know exactly about the lines added, so long as they exist.

We will now further subdivide each wedge of Fig. 3 using the algorithm described above, and get Fig. 5. We denote by \vec{u}_k the normals (counterclockwise pointing) of all the lines.

Now we have myriads of wedges over which we need to do the sum, as an example we consider first the product of (4.1) from (and including) the line k up to (but excluding) the line $k + 1$

$$I|_{[k,k+1]} = \prod_{\vec{n} \cdot \vec{u}_k \geq 0, \vec{n} \cdot \vec{u}_{k+1} < 0} \prod_{m \geq 0} (\tilde{r}_1 \cdot \vec{n} + R^1 m + x), \tag{4.3}$$

where \tilde{r} is defined in (4.2). Since by assumption $\det[\vec{u}_k, \vec{u}_{k+1}] = 1$, the product is simply

$$\begin{aligned} I|_{[k,k+1]} &= \prod_{\vec{n} \cdot \vec{u}_k \geq 0, \vec{n} \cdot \vec{u}_{k+1} < 0} \prod_{m \geq 0} \left((\vec{n} \cdot \vec{u}_k)(\tilde{r}_1 \times \vec{u}_{k+1}) - (\vec{n} \cdot \vec{u}_{k+1})(\tilde{r}_1 \times \vec{u}_k) + R^1 m + x \right) \\ &= \prod_{m^{1,2,3} \geq 0} \left(m^2(\tilde{r}_1 \times \vec{u}_{k+1}) + m^3(\tilde{r}_1 \times \vec{u}_k) + m^1 R^1 + x + (\tilde{r}_1 \times \vec{u}_k) \right) \\ &= \Gamma_3(x + (\tilde{r}_1 \times \vec{u}_k) | \tilde{r}_1 \times \vec{u}_{k+1}, \tilde{r}_1 \times \vec{u}_k, R^1)^{-1}, \end{aligned}$$

where we use the short hand notation $\vec{u} \times \vec{v} = \det[u, v]$ for 2-vectors.

Before we do the product of the second factor in (3.16), we need to make a technical remark. From the way all the dividing lines are chosen, one has

$$\vec{L}_1 + \vec{u}_i = \vec{L}_2. \tag{4.4}$$

To see this, note the line $[y, z]$ separating W_1 and W_2 satisfies $[y, z] \cdot (\vec{L}_1 - \vec{L}_2) = 0$, so its normal \vec{u}_i is parallel to $\vec{L}_1 - \vec{L}_2$. From the goodness of the cone, the 2-vector $\vec{L}_1 - \vec{L}_2$ is primitive, thus $\vec{u}_i = \pm(\vec{L}_1 - \vec{L}_2)$, and a little more thought would reveal the right sign. This relation holds for every line that separates two wedges W_l and W_{l+1} .

The previous observation has the consequence that

$$\tilde{r}_1 \times \vec{u}_i = \tilde{r}_2 \times \vec{u}_i, \tag{4.5}$$

and thus it does not matter if one includes the contribution along the line i in W_1 or W_2 . Now for the second product (3.16), we use this freedom to perform the product from (but excluding) the line k up to (and including) the line $k + 1$

$$\begin{aligned} II|_{(k,k+1]} &= \prod_{\vec{n} \cdot \vec{u}_k > 0, \vec{n} \cdot \vec{u}_{k+1} \leq 0} \prod_{m \geq 0} (\tilde{r}_1 \cdot \vec{n} + R^1 m + R^1 - x) \\ &= \prod_{m^{1,2,3} \geq 0} \left(m^2(\tilde{r}_1 \times \vec{u}_{k+1}) + m^3(\tilde{r}_1 \times \vec{u}_k) + m^1 R^1 + R^1 - x + (\tilde{r}_1 \times \vec{u}_{k+1}) \right) \\ &= \Gamma_3(-x + R^1 + (\tilde{r}_1 \times \vec{u}_{k+1}) | \tilde{r}_1 \times \vec{u}_{k+1}, \tilde{r}_1 \times \vec{u}_k, R^1)^{-1}. \end{aligned}$$

Now one can combine $I|_{[k,k+1]}$ and $II|_{(k,k+1]}$, one gets the triple sine function

$$I|_{[k,k+1]} \times II|_{(k,k+1]} = S_3(x + (\tilde{r}_1 \times \vec{u}_k) | \tilde{r}_1 \times \vec{u}_{k+1}, \tilde{r}_1 \times \vec{u}_k, R^1). \tag{4.6}$$

Note that in this way of dividing the cone, one always misses the points in (4.3) with $\vec{n} = 0, m \geq 0$, but this can be done easily, and one gets

$$I_0 = \prod_{m \geq 0} (R^1 m + x) \tag{4.7}$$

and for the factor II

$$II_0 = \prod_{m \geq 1} (R^1 m - x). \tag{4.8}$$

These two terms give

$$I_0 \cdot II_0 \sim \sin\left(\frac{\pi x}{R^1}\right) \sim e^{-\pi i \frac{x}{R^1}} (1 - e^{2\pi i \frac{x}{R^1}}). \tag{4.9}$$

where \sim means up to an overall multiplicative constant. For the hypermultiplet, instead of (4.9), we will get

$$e^{\pi i \frac{(x+im+R^1/2)}{R^1}} (1 - e^{2\pi i \frac{(x+im+R^1/2)}{R^1}})^{-1}. \tag{4.10}$$

Later on, the second factors of (4.9) and (4.10) will cancel against terms coming from the S_3 function, while the first will be combined with the Bernoulli factors.

4.2. Factorisation of the Triple Sines. We will use the factorisation formula for the triple sine [30]

$$S_3(z|\omega_1, \omega_2, \omega_3) = e^{-\frac{\pi i}{6} B_{3,3}(x|\omega_1, \omega_2, \omega_3)} (e^{2\pi i z/\omega_2}; e^{2\pi i \omega_1/\omega_2}, e^{2\pi i \omega_3/\omega_2})_\infty \times (e^{2\pi i z/\omega_1}; e^{2\pi i \omega_3/\omega_1}, e^{2\pi i \omega_2/\omega_1})_\infty (e^{2\pi i z/\omega_3}; e^{2\pi i \omega_1/\omega_3}, e^{2\pi i \omega_2/\omega_3})_\infty, \tag{4.11}$$

and in what follows we will write $(x|y, z)$ instead of $(e^{2\pi i x}; e^{2\pi i y}, e^{2\pi i z})_\infty$.

Now the expression (4.6) can be factorised

$$\begin{aligned} (4.6) &= B \cdot \left(\frac{x + \tilde{r}_1 \times \vec{u}_k}{R^1} \middle| \frac{\tilde{r}_1 \times \vec{u}_k}{R^1}, \frac{\tilde{r}_1 \times \vec{u}_{k+1}}{R^1} \right) \\ &\quad \left(\frac{x + \tilde{r}_1 \times \vec{u}_k}{\tilde{r}_1 \times \vec{u}_k} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_k}, \frac{\tilde{r}_1 \times \vec{u}_{k+1}}{\tilde{r}_1 \times \vec{u}_k} \right) \left(\frac{x + \tilde{r} \times \vec{u}_k}{\tilde{r}_1 \times \vec{u}_{k+1}} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_{k+1}}, \frac{\tilde{r}_1 \times \vec{u}_k}{\tilde{r}_1 \times \vec{u}_{k+1}} \right) \\ &= B \cdot \left(\frac{x}{R^1} \middle| -\frac{\tilde{r}_1 \times \vec{u}_k}{R^1}, \frac{\tilde{r}_1 \times \vec{u}_{k+1}}{R^1} \right)^{-1} \\ &\quad \left(\frac{x}{\tilde{r}_1 \times \vec{u}_k} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_k}, \frac{\tilde{r}_1 \times \vec{u}_{k+1}}{\tilde{r}_1 \times \vec{u}_k} \right) \left(\frac{x}{\tilde{r}_1 \times \vec{u}_{k+1}} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_{k+1}}, -\frac{\tilde{r}_1 \times \vec{u}_k}{\tilde{r}_1 \times \vec{u}_{k+1}} \right)^{-1}, \end{aligned} \tag{4.12}$$

where B is the Bernoulli polynomial that we shall collect in Sect. 4.3 and we have also used (A.2). One can also use the factorisation in (A.10), then the Bernoulli polynomials do not occur.

The second factor of the first line of (4.12) can be simplified into

$$\prod_k \left(\frac{x}{R^1} \middle| -\frac{R^\perp \times \vec{u}_k}{R^1}, \frac{R^\perp \times \vec{u}_{k+1}}{R^1} \right)^{-1},$$

where R^\perp is the second and third component of \vec{r} , i.e. $R^\perp = \tilde{r}_i - R^1 \vec{L}_i = [R^2, R^3]$. This manipulation is justified by using the periodicity of $(-|-, -)$. In Appendix A.2 this product is shown to be

$$\prod_k \left(\frac{x}{R^1} \middle| -\frac{R^\perp \times \vec{u}_k}{R^1}, \frac{R^\perp \times \vec{u}_{k+1}}{R^1} \right)^{-1} = (1 - \exp\left(\frac{2\pi i x}{R^1}\right))^{-1}. \tag{4.13}$$

This factor will cancel the second factor in (4.9) (or (4.10) in the case of hypermultiplet).

In the rest of this section, we focus on the second line of (4.12), which will give us a copy of the Nekrasov partition function for each corner of the moment map cone. For every three neighbouring lines, say, $k - 1, k$ and $k + 1$ that are in the same wedge W_1 , we will get the contribution

$$\left(\frac{x}{\tilde{r}_1 \times \vec{u}_k} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_k}, \frac{\tilde{r}_1 \times \vec{u}_{k+1}}{\tilde{r}_1 \times \vec{u}_k}\right) \left(\frac{x}{\tilde{r}_1 \times \vec{u}_k} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_k}, -\frac{\tilde{r}_1 \times \vec{u}_{k-1}}{\tilde{r}_1 \times \vec{u}_k}\right)^{-1}.$$

Here we make the observation that since $\vec{u}_{k-1} \times \vec{u}_k = \vec{u}_k \times \vec{u}_{k+1} = 1$, one has

$$\vec{u}_{k-1} + \vec{u}_{k+1} = \mathbb{Z}\vec{u}_k. \tag{4.14}$$

Consequently $\tilde{r}_1 \times \vec{u}_{k+1} + \tilde{r}_1 \times \vec{u}_{k-1} = \mathbb{Z}\tilde{r}_1 \times \vec{u}_k$, and the above combination cancels by using the periodicity of the special function $(-|- , -)$.

In contrast, take three lines as $i - 1, i$ and $i + 1$ with i straddling two wedges W_1, W_2 , then one gets instead the contribution

$$\star = \left(\frac{x}{\tilde{r}_2 \times \vec{u}_i} \middle| \frac{R^1}{\tilde{r}_2 \times \vec{u}_i}, \frac{\tilde{r}_2 \times \vec{u}_{i+1}}{\tilde{r}_2 \times \vec{u}_i}\right) \left(\frac{x}{\tilde{r}_1 \times \vec{u}_i} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_i}, -\frac{\tilde{r}_1 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}\right)^{-1}.$$

One uses then (4.14) and (4.5) to get

$$\star = \left(\frac{x}{\tilde{r}_1 \times \vec{u}_i} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_i}, -\frac{\tilde{r}_2 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}\right) \left(\frac{x}{\tilde{r}_1 \times \vec{u}_i} \middle| \frac{R^1}{\tilde{r}_1 \times \vec{u}_i}, -\frac{\tilde{r}_1 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}\right)^{-1},$$

and that

$$-\frac{\tilde{r}_2 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i} = -\frac{(\tilde{r}_1 + R^1\vec{u}_i) \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i} = -\frac{\tilde{r}_1 \times \vec{u}_{i-1} - R^1}{\tilde{r}_1 \times \vec{u}_i}. \tag{4.15}$$

Now one invokes (A.4) and combine the two factors of \star

$$\star = \left(\frac{x}{\tilde{r}_1 \times \vec{u}_i} \middle| -\frac{\tilde{r}_2 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}, \frac{\tilde{r}_1 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}\right) = \left(\frac{x}{\tilde{r}_1 \times \vec{u}_i} \middle| \frac{\tilde{r}_2 \times \vec{u}_{i+1}}{\tilde{r}_1 \times \vec{u}_i}, \frac{\tilde{r}_1 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}\right). \tag{4.16}$$

To conclude, apart from the Bernoulli polynomials, the partition function receives a contribution of (4.16), for every corner of the moment map cone. If one were to use the factorisation (A.10), then the second factor there combines in a similar fashion into

$$\begin{aligned} \star' &= \left(-\frac{x}{\tilde{r}_1 \times \vec{u}_i} \middle| \frac{\tilde{r}_1 \times \vec{u}_{i-1} - R^1}{\tilde{r}_1 \times \vec{u}_i}, -\frac{\tilde{r}_1 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}\right) \\ &= \left(\frac{R^1 - x}{\tilde{r}_1 \times \vec{u}_i} \middle| -\frac{\tilde{r}_1 \times \vec{u}_{i-1} - R^1}{\tilde{r}_1 \times \vec{u}_i}, \frac{\tilde{r}_1 \times \vec{u}_{i-1}}{\tilde{r}_1 \times \vec{u}_i}\right). \end{aligned} \tag{4.17}$$

The same manipulation applies to the hypermultiplet, one needs only replace in the above formulae $x \rightarrow x + R^1/2 + im$.

Next we will show that this factor is the perturbative Nekrasov partition function on $S^1 \times \mathbb{C}^2$. Since the wedges W correspond to the faces of the moment map cone,

one observes that if the normals to face 1 and 2 are \vec{v} and \vec{v}' , i.e. $\vec{v} = [1, -\vec{L}_1]$ and $\vec{v}' = [1, -\vec{L}_2]$, then

$$\det[\vec{v}, \vec{v}', \vec{R}] = \det \begin{pmatrix} 1 & 1 & R^1 \\ -L_1^2 & -L_2^2 & R^2 \\ -L_1^3 & -L_2^3 & R^3 \end{pmatrix} = \tilde{R}_1 \times \vec{u}_i.$$

Thus one recognizes the quantity $\tilde{R}_1 \times \vec{u}_i$ as the inverse circumference $2\pi/\beta$ of the Reeb orbit above the corner at the intersection of face 1 and 2 (see Fig. 2).

For the equivariant parameters, let $\vec{n} = [0, -\vec{u}_{i+1}]$, one observes that

$$\det[\vec{v}, \vec{v}', \vec{n}] = \det \begin{pmatrix} 1 & 1 & 0 \\ -\vec{L}_1 & -\vec{L}_2 & -\vec{u}_{i+1} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & 0 \\ \vec{u}_i & -\vec{L}_2 & -\vec{u}_{i+1} \end{pmatrix} = 1.$$

Then from the recipe (1.5) for ϵ, ϵ' , one gets

$$\begin{aligned} \epsilon &= \det[\vec{n}, \vec{R}, \vec{v}'] = \tilde{R}_2 \times \vec{u}_{i+1}, \\ \epsilon' &= \det[\vec{v}, \vec{R}, \vec{n}] = -\tilde{R}_1 \times \vec{u}_{i+1} = \tilde{R}_1 \times \vec{u}_{i-1} + \mathbb{Z}\tilde{R}_1 \times \vec{u}_i. \end{aligned}$$

From this we see that the partition function receives one copy of the perturbative Nekrasov partition function for each corner of the toric moment cone, or for each closed Reeb orbit, with the expected equivariant parameters

$$\star = \left(\frac{\beta}{2\pi}x \middle| \frac{\beta}{2\pi}\epsilon, \frac{\beta}{2\pi}\epsilon'\right), \quad \star' = \left(\frac{\beta}{2\pi}(R^1 - x) \middle| \frac{\beta}{2\pi}\epsilon, \frac{\beta}{2\pi}\epsilon'\right).$$

If we adopt the second factorisation of the triple sine (A.10), we will get the following

$$Z^{pert} = \int_t da e^{-\frac{8\pi^3 r}{s_{YM}} \text{Tr}[a^2]} \cdot \frac{\prod_{i=1}^n \left(\det'_{adj} \left(i \frac{\beta_i}{2\pi} a \middle| \frac{\beta_i}{2\pi} \epsilon_i, \frac{\beta_i}{2\pi} \epsilon'_i\right) (a \rightarrow -iR^1 - a)\right)^{1/2}}{\prod_{i=1}^n \left(\det_{\underline{R}} \left(i \frac{\beta_i}{2\pi} (a+m-iR^1/2) \middle| \frac{\beta_i}{2\pi} \epsilon_i, \frac{\beta_i}{2\pi} \epsilon'_i\right) (a+m \rightarrow -a-m)\right)^{1/2}} \tag{4.18}$$

where the index i runs over all the n closed Reeb orbits. This way of writing the factorization, though involving a square root, is manifestly symmetric under $\underline{R} \rightarrow \bar{\underline{R}}$.

4.3. Collection of the Bernoulli Polynomials. In this section we collect the Bernoulli polynomials left over from (4.12). The Bernoulli polynomial $B_{3,3}$ is defined in (A.8). From the contribution from line k to line $k + 1$, one receives

$$\begin{aligned} &-\frac{\pi i}{6} B_{3,3}(x + (\tilde{R}_1 \times \vec{u}_k) \middle| \tilde{R}_1 \times \vec{u}_{k+1}, \tilde{R}_1 \times \vec{u}_k, R^1) \\ &= \frac{\pi i}{6} B_{3,3}(x \middle| \tilde{R}_1 \times \vec{u}_{k+1}, -\tilde{R}_1 \times \vec{u}_k, R^1), \end{aligned} \tag{4.19}$$

where (A.9) is used.

We collect the x^3 term first

$$\text{coef of } x^3 = \frac{\pi i}{6} \frac{1}{R^1(\tilde{R}_1 \times \vec{u}_{k+1})(-\tilde{R}_1 \times \vec{u}_k)}.$$

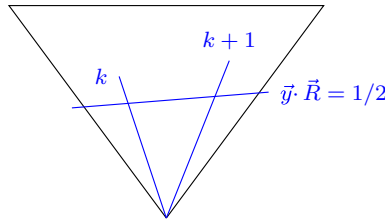


Fig. 6. The *big triangle* is the face 1, and we are interested in the area enclosed on face 1 by three planes: $\vec{y} \cdot [0, \vec{u}_k] = 0$, $\vec{y} \cdot [0, \vec{u}_{k+1}] = 0$ and $\vec{y} \cdot \vec{R} = 1/2$

The right hand side is actually proportional to the area of the part of a face (face 1 in this particular instance, see Fig. 2) bounded by the three planes $\vec{y} \cdot [0, \vec{u}_k] = 0$, $\vec{y} \cdot [0, \vec{u}_{k+1}] = 0$ and $\vec{y} \cdot \vec{R} = 1/2$, $\vec{y} \in \mathbb{R}^3$, see Fig. 6.

Indeed, the area is given by the expression

$$A = \frac{1}{8} \frac{|w_2| \det[w_1, w_3, w_2]}{\det[w_1, w_2, \vec{R}] \cdot \det[w_3, w_2, \vec{R}]},$$

where $w_1 = [0, \vec{u}_k]$, $w_2 = [1, -\vec{L}_1]$, $w_3 = [0, \vec{u}_{k+1}]$.

Working this out, we have

$$A = \frac{1}{8} \frac{|[1, -\vec{L}_1]|}{(\vec{r}_1 \times \vec{u}_k)(\vec{r}_1 \times \vec{u}_{k+1})}.$$

Coming back to the coefficient of x^3 , summing over k we get

$$\text{coef of } x^3 = -\frac{4\pi i}{3R^1} \sum_{i=1}^n \frac{1}{|\vec{v}_i|} A_i,$$

where A_i is the area of face i topped off by the plane $\vec{y} \cdot \vec{R} = 1/2$, and i runs over all faces.

We collect the x^2 term next

$$\begin{aligned} \text{coef of } x^2 &= -\frac{\pi i}{4} \frac{R^1 + \vec{r}_1 \times (\vec{u}_{k+1} - \vec{u}_k)}{R^1 (\vec{r}_1 \times \vec{u}_{k+1})(-\vec{r}_1 \times \vec{u}_k)} \\ &= \frac{\pi i}{4} \left(\frac{1}{(\vec{r}_1 \times \vec{u}_{k+1})(\vec{r}_1 \times \vec{u}_k)} + \frac{1}{R^1 (\vec{r}_1 \times \vec{u}_k)} - \frac{1}{R^1 (\vec{r}_1 \times \vec{u}_{k+1})} \right). \end{aligned}$$

The last two terms will drop once we sum over all k (using again (4.5)). and summing over all k , one gets

$$\text{coef of } x^2 = 2\pi i \sum_{i=1}^n \frac{1}{|v_i|} A_i.$$

For the x^1 term, we get (remembering the extra contribution from (4.13) and (4.10))

$$\text{coef of } x^1 = \frac{\pi i}{12} \left(\frac{\omega_1}{\omega_2 \omega_3} + \frac{1}{\omega_1} \left(\frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} + 3 \right) + 3 \left(\frac{1}{\omega_3} + \frac{1}{\omega_2} \right) \right) - \frac{\pi i}{R^1},$$

where $\omega_1 = R^1$, $\omega_2 = \tilde{r}_1 \times \vec{u}_{k+1}$ and $\omega_3 = -\tilde{r}_1 \times \vec{u}_k$. Taking the sum over k , the last term will drop, and the first term has been dealt with above. For the middle term, we only need to investigate the following

$$\sum_k \left(-3 + \frac{\tilde{r}_1 \times \vec{u}_{k+1}}{\tilde{r}_1 \times \vec{u}_k} + \frac{\tilde{r}_1 \times \vec{u}_k}{\tilde{r}_1 \times \vec{u}_{k+1}} \right).$$

Using (4.14),

$$\frac{\tilde{r}_1 \times (\vec{u}_{k-1} + \vec{u}_{k+1})}{\tilde{r}_1 \times \vec{u}_k} \in \mathbb{Z}$$

if $k - 1, k$ and $k + 1$ are in the same wedge. Otherwise, if k separates W_1 and W_2 one gets

$$\frac{\tilde{r}_2 \times \vec{u}_{k+1}}{\tilde{r}_2 \times \vec{u}_k} + \frac{\tilde{r}_1 \times \vec{u}_{k-1}}{\tilde{r}_1 \times \vec{u}_k} = \frac{-\tilde{r}_2 \times \vec{u}_{k-1} + \tilde{r}_1 \times \vec{u}_{k-1}}{\tilde{r}_1 \times \vec{u}_k} + \mathbb{Z} = \frac{R^1}{\det[\vec{r}, \vec{v}_1, \vec{v}_2]} + \mathbb{Z},$$

where (4.15) is used. And one recognizes the last combination as proportional to the circumference of the closed Reeb orbits at the corner of the intersection of faces 1 and 2. In total the x^1 term gives

$$\text{coef of } x^1 = \frac{\pi i}{12} \left(-8R^1 \sum_{i=1}^n \frac{1}{|\vec{v}_i|} A_i - \frac{1}{2\pi} \sum_{i=1}^n \beta_i - \frac{c}{R^1} - \frac{12}{R^1} \right),$$

where the undetermined integer is named c and it will be shown to be -12 at the end of this section.

Finally we come to the x^0 term. One might wonder why do we bother with this since it is just a constant and we have been discarding constants all along, but the point is that the same type of terms appearing here will appear in the asymptotic behaviour of the partition function where they will be important. The $B_{3,3}$ has the constant term

$$\text{coef of } x^0 = -\frac{\pi i}{24} \left(3 + \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} + \frac{\omega_3}{\omega_1} + \frac{\omega_1}{\omega_3} \right),$$

with the same ω 's as above.

Taking the sum over k one gets

$$\frac{\pi i}{24} \left(\frac{R^1}{2\pi} \sum_{i=1}^n \beta_i + c \right).$$

To summarise the collection of Bernoulli polynomials gives (with $c = -12$)

$$\pi i \left(-\frac{4x^3}{3R^1} + 2x^2 - \frac{2}{3}R^1 x \right) \sum_{i=1}^n \frac{1}{|\vec{v}_i|} A_i + \pi i \left(-\frac{1}{12}x + \frac{1}{24}R^1 \right) \frac{1}{2\pi} \sum_{i=1}^n \beta_i - \frac{\pi i}{2}.$$

As an aside when the cone corresponds to a CY toric manifold, which is the case we are dealing with, one can write the sum of volume of faces above as the volume of the

manifold X . One uses the fact that the end points of the normals \vec{v}_i lie on a hyperplane, the sum of the volume of the faces above can be written as

$$\sum_{i=1}^n \frac{1}{|\vec{v}_i|} A_i = 6R^1 \text{vol}_{\Delta_R^{1/2}} = \frac{R^1}{(2\pi)^3} \text{vol}_X, \tag{4.20}$$

where $\Delta_R^{1/2}$ is the intersection $C_\mu(X) \cap \{\vec{r} \in \mathbb{R}^3 | \vec{r} \cdot \vec{r} \leq 1/2\}$. The above relation is derived in [31], it was also shown in that paper that

$$\int_{C(X)^1} R_{C(X)} = (2R^1 - 6) \text{vol}_X, \tag{4.21}$$

where $R_{C(X)}$ is the Ricci scalar of the metric cone $C(X)$, and $C(X)^1$ is the metric cone cut off at $r \leq 1$, see Sect. 2 for notations.

To apply this result to the vector multiplet, one can discard the odd powers of x , since $x = i\langle a, \lambda \rangle$, $a \in \mathfrak{t}$ and λ runs over all the roots, so the odd powers of x cancel out. We get

$$B_{vec}(x) = 2\pi i x^2 \sum_{i=1}^n \frac{1}{|\vec{v}_i|} A_i + \frac{\pi i}{24} \frac{R^1}{2\pi} \sum_{i=1}^n \beta_i - \frac{i\pi}{2}. \tag{4.22}$$

For a hypermultiplet with mass m , one gets

$$\begin{aligned} B_{hyp}(x) &= \frac{4\pi i}{3} \left(\frac{1}{R^1} (x + im)^3 - \frac{1}{4} R^1 (x + im) \right) \sum_{i=1}^n \frac{1}{|\vec{v}_i|} A_i \\ &\quad + \frac{\pi i}{12} (x + im) \frac{1}{2\pi} \sum_{i=1}^n \beta_i + \frac{\pi i}{2}. \end{aligned} \tag{4.23}$$

We will now prove $c = -12$ (see page 44 [29]). First, one needs to establish that given a subdivision of the plane, the number c is unchanged if one inserts further lines. To see this, let $\vec{v}_{i-2}, \vec{v}_{i-1}, \vec{v}_i$ and \vec{v}_{i+1} be the normals to four consecutive lines such that $\vec{v}_k \times \vec{v}_{k+1} = 1$, $k = i - 2, \dots, i$, and we can assume that $\vec{v}_{i-1} = [-1, 0]$ and $\vec{v}_i = [0, 1]$. We insert a fifth line between $i - 1$ and i , with normal \vec{u} , then one must have $\vec{v}_{i-1} + \vec{v}_i = \vec{u} = [-1, 1]$. Doing this would change c by

$$\delta c = \frac{\tilde{R} \times (\vec{u} - \vec{v}_{i-1})}{\tilde{R} \times \vec{v}_i} - 3 + \frac{\tilde{R} \times (\vec{v}_{i+1} + \vec{v}_{i-1})}{\tilde{R} \times \vec{u}} + \frac{\tilde{R} \times (\vec{u} - \vec{v}_i)}{\tilde{R} \times \vec{v}_{i-1}} = 1 - 3 + 1 + 1 = 0.$$

One can go further and establish that c does not change if we add k redundant lines in between $i - 1$ and i . To see this, if one of the k lines we add has normal $\vec{u} = [-1, 1]$, then since $\vec{u} \times \vec{v}_{i-1} = \vec{v}_i \times \vec{u} = 1$, and there are fewer lines between either \vec{u}, \vec{v}_{i-1} or \vec{u}, \vec{v}_i , and the proof follows from an induction. Next we show that such a line can always be found among the k lines. Assume first that all k lines are between \vec{u} and \vec{v}_{i-1} (resp. \vec{v}_i), then the last (resp. first) of these lines must have normal \vec{u} , and we are finished. In the remaining case, that is, there are lines between \vec{u}, \vec{v}_{i-1} as well as between \vec{u}, \vec{v}_i . Assume that none of the k lines have normal \vec{u} , then the two lines right next to it must have primitive normals $[-a, b]$ and $[-c, d]$ with $a, b, c, d > 0$ and $a > b \geq 1, d > c \geq 1$, then $\det[-c, d; -a, b] = -bc + ad > 1$ and we get a contradiction.

It is also easy to check that for the case of three standard lines with normals $[0, 1]$, $[-1, 0]$ and $[1, -1]$, then $c = -12$. With this understanding, now given any subdivision problem consisting of a set of n lines with normals $\vec{v}_i, i = 1, \dots, n$, one add to the list three standard lines with the above normals. It does not matter if one of the original lines happen to coincide with the standard lines, but for definiteness, let us assume otherwise. Now one can follow the subdivision algorithm to add more lines to the list of $n + 3$ lines. This subdivision certainly solves the subdivision problem of the original list of n lines, but it also can be viewed as adding redundant lines to the set of three standard lines, and hence $c = -12$ from the above argument.

5. The Asymptotic Behaviour and Large N

Using the method of subdividing the moment map cone, we can now give a general formula for the asymptotic behaviour, expressed in terms of the geometrical data from the moment map cone.

In the two products of (3.15) and (3.16), we give x a small real part and send its imaginary part to infinity. As usual, the infinite product is taken under the zeta-function regularisation

$$\log I = -\frac{\partial}{\partial s} \frac{1}{\Gamma(s)} \int_0^\infty \sum_{\vec{m} \in C_\mu(X) \cap \mathbb{Z}^3} e^{-(\vec{m} \cdot \vec{R} + x)t} t^{s-1} dt \Big|_{s=0},$$

and $\log II$ is obtained by replacing $x = R^1 - x$. The summation will now be done as in the earlier sections by dividing $C_\mu(X)$. In the i^{th} wedge between line k and $k + 1$, one gets (see 4.3 and Fig. 5 for the explanation of the notation)

$$\begin{aligned} \log I \Big|_{[k,k+1]} &= -\frac{\partial}{\partial s} \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(x + \tilde{R}_1 \times \vec{u}_k)t}}{(1 - e^{-\tilde{R}_1 \times \vec{u}_{k+1}t})(1 - e^{-\tilde{R}_1 \times \vec{u}_k t})(1 - e^{-R^1 t})} t^{s-1} dt \Big|_{s=0}, \\ \log II \Big|_{(k,k+1]} &= -\frac{\partial}{\partial s} \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(R^1 - x + \tilde{R}_1 \times \vec{u}_{k+1})t}}{(1 - e^{-\tilde{R}_1 \times \vec{u}_{k+1}t})(1 - e^{-\tilde{R}_1 \times \vec{u}_k t})(1 - e^{-R^1 t})} t^{s-1} dt \Big|_{s=0}. \end{aligned}$$

The large $\text{Im } x$ behaviour is then given by taking the Laurent series of the denominator at $t = 0$ up to t^0 and then performing the integral. The details can be found in Sect. 6 of [5], here we just give the result

$$-\log I \Big|_{[k,k+1]} - \log II \Big|_{(k,k+1]} = \frac{i\pi}{6} \text{sgn}(\text{Im } x) B_{3,3}(x|\omega_1, -\omega_2, \omega_3).$$

where $\omega_1 = R^1, \omega_2 = \tilde{R}_1 \times \vec{u}_k$ and $\omega_3 = \tilde{R}_1 \times \vec{u}_{k+1}$.

To apply this result to the vector multiplet, one can discard the even powers of x , since $x = i\langle a, \lambda \rangle, a \in \mathfrak{t}$ and we shall be summing over all the roots λ , so the even powers of x cancel out. We are left with

$$\begin{aligned} & -(\log I \Big|_{[k,k+1]} + \log II \Big|_{(k,k+1]}) \Big|_{vec} \\ &= \frac{i\pi \text{sgn}(\text{Im } x)}{12\omega_1\omega_2\omega_3} \left(2x^3 + x(\omega_1^2 + \omega_2^2 + \omega_3^2 - 3\omega_1\omega_2 - 3\omega_2\omega_3 + 3\omega_3\omega_1) \right). \end{aligned}$$

The assemblage of these contributions from all wedges is entirely similar to the treatment of the Bernoulli polynomials in Sect. 4.3, we get

$$\begin{aligned}
 & -(\log I + \log II)|_{vec} \\
 & = i\pi \operatorname{sgn}(\operatorname{Im} x) \left(\left(\frac{x^3}{3R^1} + \frac{R^1 x}{6} \right) \sum_i \frac{4}{|v_i|} A_i + \frac{x}{12} \left(\frac{1}{2\pi} \sum_i \beta_i + \frac{c}{R^1} \right) \right).
 \end{aligned}$$

The integer $c = -12$ was introduced in the previous section. One must not forget the contribution from the factors of (4.9), which gives

$$-\frac{i\pi}{R^1} \operatorname{sgn}(\operatorname{Im} x) x. \tag{5.1}$$

and the total asymptotic behaviour from the vector multiplet is

$$V_v^{asympt}(x) = -i\pi \operatorname{sgn}(\operatorname{Im} x) \left(\left(\frac{x^3}{3R^1} + \frac{R^1 x}{6} \right) \sum_i \frac{4}{|v_i|} A_i + \frac{x}{12} \frac{1}{2\pi} \sum_i \beta_i \right). \tag{5.2}$$

For the hypermultiplet $x = \langle \sigma, \mu \rangle$, but the weights of a general representation may not be symmetric. Also remembering the shift $x \rightarrow x + R^1/2$, one gets

$$\begin{aligned}
 & -(\log I|_{|k,k+1} + \log II|_{(k,k+1)})|_{hyp} = \frac{i\pi \operatorname{sgn}(\operatorname{Im} x)}{72\omega_1\omega_2\omega_3} \left(12x^3 + 18x^2(\omega_2 - \omega_3) \right. \\
 & \quad \left. - 3x(\omega_1^2 - 2\omega_2^2 - 2\omega_3^2 + 6\omega_2\omega_3) - 3\omega_2\omega_3(\omega_2 - \omega_3) - \frac{3}{2}\omega_1^2(\omega_2 - \omega_3) \right).
 \end{aligned}$$

Now as we assemble the contributions from all wedges, the even powers of x drop again

$$\begin{aligned}
 & -(\log I + \log II)|_{hyp} \\
 & = i\pi \operatorname{sgn}(\operatorname{Im} x) \left(\left(\frac{x^3}{3R^1} - \frac{R^1 x}{12} \right) \sum_i \frac{4}{|v_i|} A_i + \frac{x}{12} \left(\frac{1}{2\pi} \sum_i \beta_i + \frac{c}{R^1} \right) \right),
 \end{aligned}$$

The factors of (4.9) gives a similar contribution as in (5.1), and in total the asymptotic behaviour from the hypermultiplet is

$$V_h^{asympt}(x) = i\pi \operatorname{sgn}(\operatorname{Im} x) \left(\left(\frac{x^3}{3R^1} - \frac{R^1 x}{12} \right) \sum_i \frac{4}{|v_i|} A_i + \frac{x}{12} \frac{1}{2\pi} \sum_i \beta_i \right). \tag{5.3}$$

To summarise, asymptotically, the matrix model integral is given by

$$Z^{pert} \sim \int_{\mathfrak{t}} da e^{-\frac{8\pi^3 r}{g_{YM}^2} \varrho \operatorname{Tr}[a^2]} \cdot e^{\operatorname{Tr}_{adj} V_v^{asympt}(ia)} \cdot e^{\operatorname{Tr}_R V_h^{asympt}(ia)}, \tag{5.4}$$

with $V_{v,h}^{asympt}$ given in (5.2) and (5.3). This seems a better way of presenting the asymptotic behaviour of the potential than the way it was done in [5], since the role played by the geometry is more transparent now.

Using these asymptotics and following the analysis from [32] we get the free energy at the large N limit for the vector multiplet coupled to a hypermultiplet in adjoint with mass m

$$F = -\log Z = -\frac{g_{YM}^2 N^3}{96\pi r} \varrho \left(\frac{1}{4} (R^1)^2 + m^2 \right)^2.$$

for a squashed toric SE manifold. To go to the SE metric, one only needs to set $r^1 = 3$ [31]. The result is identical to that of the theory on S^5 up to a volume factor ϱ as expected.

6. Summary

In this paper we have derived the full perturbative partition function for the SYM coupled to hypermultiplets on any 5D toric simply connected SE manifold X . We have calculated the equivariant answer, which keeps track of three $U(1)$ isometries on X . The actual 5D calculation can be reduced to the counting of holomorphic functions on the corresponding CY cone $C(X)$. Thus it is very natural to ask if there is anything deep in this relation to 6D counting besides being a mere technical trick. It will be extremely interesting to construct an intrinsically 6D theory that will do the same counting. Another natural question is if the contact instantons (localisation locus for 5D theory) has a natural lift to 6D. Somehow it is conceivable that the counting of contact instantons on X also reduces to some counting problems on $C(X)$.

Another important result of this paper is the factorization property of the full perturbative answer on X into copies of perturbative Nekrasov partition functions on $\mathbb{C}^2 \times S^1$, with the twisting parameters controlled by the toric data of X . It is natural to conjecture that the full partition function on X is given by gluing the copies of the full Nekrasov partition function with the same set of twisting data as in the perturbative sector; however, a constructive proof of this conjecture from the first principle is beyond us so far.

One is also led naturally to interpret the Nekrasov partition function associated to $\mathbb{C}^2 \times S^1$ as some kind of wave function associated to the theory quantized on this space, analogous to the picture in 3d [33]. However this picture is obscured by our lack of understanding of surgery on 5-manifolds, it seems a very interesting direction to pursue, especially in light of a recent attempt in applying localisation to manifolds with boundary [34].

A puzzle that we do not completely clarify is the following. While proving the factorisation, we have studied the special function S_3^X depending on X through its toric data. When X is simply connected, the zero instanton localisation locus consists of just the zero connection, and the answer is given in terms of S_3^X . When X is not simply connected, one does not a priori have a Killing spinor. Moreover, we would need to take into account all non-trivial flat connections to produce the complete perturbative partition function. From physical considerations, one expects that the contribution of all the flat connections together should factorise, but not individually. However, our proof of the factorisability of S_3^X does not require the simply connectedness. One possible explanation is that the contribution from the zero connection is special and factorises all by itself, but we find this less appealing. Another explanation could be that in the absence of simply connectedness, the 1-Gorenstein condition (1.1) only ensures $c_1(K) = 0$, where K is the canonical bundle, but does not exclude the possibility of K being a flat but non-trivial bundle. In this case, Schmude’s method in computing the super determinant has to be modified since it relies on using the holomorphic volume form. We plan to investigate the localisation for non-simply connected manifolds and the corresponding factorisation properties in a separate paper.

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A. Special Functions

A.1. Definitions of special functions. The special function $(x|a_1, \dots, a_n)_\infty$ was introduced in [30]. It is defined differently in different domains

$$(x|a_1, \dots, a_n)_\infty = \prod_{i_1, \dots, i_n \geq 0} (1 - xa_1^{i_1} \dots a_{k-1}^{i_{k-1}} a_k^{-(i_k+1)} \dots a_n^{-(i_n+1)})^{-(-1)^{n-k}},$$

$$|a_1| < 1, \dots, |a_{k-1}| < 1, |a_k| > 1, \dots, |a_n| > 1. \tag{A.1}$$

This function is symmetric in the n arguments a_i , but it is not defined if any $|a_i| = 1$. These functions enjoy the property

$$(x|a_1, \dots, a_r)_\infty = \frac{1}{(a_j^{-1}x|a_1, \dots, a_j^{-1}, \dots, a_r)_\infty}. \tag{A.2}$$

Often we will use the short hand

$$(e^{2\pi iz} | e^{2\pi i\omega_1}, \dots, e^{2\pi i\omega_n})_\infty = (z|\omega_1, \dots, \omega_n). \tag{A.3}$$

One needs to remember that when using the latter notation, the function is periodic under shift by an integer of any of the arguments.

Lemma A.1.

$$\frac{(x|a, b)_\infty}{(x|a, ab)_\infty} = (x|b^{-1}, ab)_\infty^{-1}. \tag{A.4}$$

Proof. We prove the lemma case by case, first let $|a| < 1$ and $|b| < 1$, then

$$\begin{aligned} \frac{(x|a, b)_\infty}{(x|a, ab)_\infty} &= \frac{\prod_{i, j \geq 0} (1 - xa^i b^j)}{\prod_{i, j \geq 0} (1 - xa^i (ab)^j)} = \prod_{i \geq 0, j > i} (1 - xa^i b^j) \\ &= \prod_{i, j \geq 0} (1 - xb(ab)^i b^j) = (xb|ab, b)_\infty = (x|ab, b^{-1})_\infty^{-1}. \end{aligned}$$

If instead $|a| < 1, |b| > 1$ but $|ab| < 1$, then

$$\begin{aligned} \frac{(x|a, b)_\infty}{(x|a, ab)_\infty} &= \frac{1}{\prod_{i, j \geq 0} (1 - xa^i b^{-j-1}) \prod_{i, j \geq 0} (1 - xa^i (ab)^j)} \\ &= \frac{1}{\prod_{i \geq 0, j \leq i} (1 - xa^i b^j)} = \frac{1}{\prod_{i, j \geq 0} (1 - xb^{-i} (ab)^j)} \\ &= (xb|b, ab)_\infty = (x|b^{-1}, ab)_\infty^{-1}. \end{aligned}$$

But if $|ab| > 1$

$$\begin{aligned} \frac{(x|a, b)_\infty}{(x|a, ab)_\infty} &= \frac{\prod_{i, j \geq 0} (1 - xa^i (ab)^{-j-1})}{\prod_{i, j \geq 0} (1 - xa^i b^{-j-1})} = \prod_{j \geq 0, -j-1 \leq i < 0} (1 - xa^i b^{-j-1}) \\ &= \prod_{k, l \geq 0} (1 - xa^{-k-1} b^{k+l+1}) = (xb|ab, b)_\infty = (x|ab, b^{-1})_\infty^{-1}. \end{aligned}$$

By switching the role of a, b, ab one can obtain the other cases \square

We will also make use of the multiple Gamma function, defined as a ζ -regulated product

$$\Gamma_r = \prod_{n_1, \dots, n_r=0}^{\infty} (n_1\omega_1 + \dots + n_r\omega_r + x)^{-1}, \tag{A.5}$$

the domain of definition is that all $\omega_i \in \mathbb{C}$ should lie on the same side of some straight line through the origin and $x \in \mathbb{C}$.

The multiple sine function is defined as

$$S_r(x|\omega_1, \dots, \omega_r) = \Gamma_r(x|\omega_1, \dots, \omega_r)^{-1} \Gamma_r(\sum_{i=1}^r \omega_i - x|\omega_1, \dots, \omega_r)^{(-1)^r}. \tag{A.6}$$

The multiple sine function has an important factorisation property, see property 5 in [30], we shall only give the the case $r = 3$

$$S_3(x|\omega_1, \dots, \omega_r) = e^{-\frac{\pi i}{6} B_{3,3}(x|\omega_1, \dots, \omega_3)} (e^{2\pi i \frac{x}{\omega_1}} | e^{2\pi i \frac{\omega_2}{\omega_1}}, e^{2\pi i \frac{\omega_3}{\omega_1}})_{\infty} (e^{2\pi i \frac{x}{\omega_2}} | e^{2\pi i \frac{\omega_1}{\omega_2}}, e^{2\pi i \frac{\omega_3}{\omega_2}})_{\infty} (e^{2\pi i \frac{x}{\omega_3}} | e^{2\pi i \frac{\omega_1}{\omega_3}}, e^{2\pi i \frac{\omega_2}{\omega_3}})_{\infty}, \tag{A.7}$$

or one may have the factorisation

$$S_3(x|\omega_1, \dots, \omega_r) = e^{\frac{\pi i}{6} B_{3,3}(x|\omega_1, \dots, \omega_3)} (e^{-2\pi i \frac{x}{\omega_1}} | e^{-2\pi i \frac{\omega_2}{\omega_1}}, e^{-2\pi i \frac{\omega_3}{\omega_1}})_{\infty} (e^{-2\pi i \frac{x}{\omega_2}} | e^{-2\pi i \frac{\omega_1}{\omega_2}}, e^{-2\pi i \frac{\omega_3}{\omega_2}})_{\infty} (e^{-2\pi i \frac{x}{\omega_3}} | e^{-2\pi i \frac{\omega_1}{\omega_3}}, e^{-2\pi i \frac{\omega_2}{\omega_3}})_{\infty}.$$

where $B_{3,3}$ is the Bernoulli polynomial defined as

$$B_{3,3}(z|\omega_1, \omega_2, \omega_3) = \frac{z^3}{\omega_1\omega_2\omega_3} - \frac{3}{2} \frac{\omega_1 + \omega_2 + \omega_3}{\omega_1\omega_2\omega_3} z^2 + \frac{\omega_1^2 + \omega_2^2 + \omega_3^2 + 3\omega_1\omega_2 + 3\omega_2\omega_3 + 3\omega_3\omega_1}{2\omega_1\omega_2\omega_3} z - \frac{(\omega_1 + \omega_2 + \omega_3)(\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1)}{4\omega_1\omega_2\omega_3}. \tag{A.8}$$

These polynomials satisfy

$$B_{3,3}(z + \omega_2|\omega_1, \omega_2, \omega_3) = B_{3,3}(z|\omega_1, -\omega_2, \omega_3). \tag{A.9}$$

By comparing the two equivalent factorisations, one gets

$$e^{\frac{\pi i}{3} B_{3,3}(x|\omega_1, \dots, \omega_3)} = \frac{(e^{2\pi i \frac{x}{\omega_1}} | e^{2\pi i \frac{\omega_2}{\omega_1}}, e^{2\pi i \frac{\omega_3}{\omega_1}})_{\infty}}{(e^{-2\pi i \frac{x}{\omega_1}} | e^{-2\pi i \frac{\omega_2}{\omega_1}}, e^{-2\pi i \frac{\omega_3}{\omega_1}})_{\infty}} \cdot (\text{cyc perm in } \omega_{1,2,3}).$$

So one may also write the factorisation as

$$S_3(x|\omega_1, \dots, \omega_r) = \left((e^{2\pi i \frac{x}{\omega_1}} | e^{2\pi i \frac{\omega_2}{\omega_1}}, e^{2\pi i \frac{\omega_3}{\omega_1}})_{\infty} \cdot (e^{-2\pi i \frac{x}{\omega_1}} | e^{-2\pi i \frac{\omega_2}{\omega_1}}, e^{-2\pi i \frac{\omega_3}{\omega_1}})_{\infty} \right)^{1/2} (\text{cyc perm in } \omega_{1,2,3}) \tag{A.10}$$

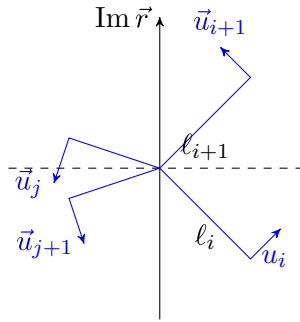


Fig. 7. The black line is $\text{Im } \vec{r}$, for the lines ℓ_j above the dotted line, its normal satisfies $\text{Im } \vec{r} \times \vec{u}_j > 0$

without the Bernoulli polynomial but at the cost of having a square root. One can use (A.2) to rewrite the above as

$$S_3(x|\omega_1, \dots, \omega_r) = \left((e^{2\pi i \frac{x}{\omega_1}} | e^{2\pi i \frac{\omega_2}{\omega_1}}, e^{2\pi i \frac{\omega_3}{\omega_1}})_\infty \cdot (x \rightarrow -x + \omega_2 + \omega_3) \right)^{1/2} \cdot (\text{cyc perm in } \omega_{1,2,3}).$$

A.2. *A lemma concerning the special function.* In this section we prove a useful identity, which may be of independent interest. To recapitulate the problem, one divides a 2-plane into a number of wedges with separating lines ℓ_i of rational slope. Assume that the normals (counter clockwise pointing) of every two neighbouring lines form an $SL(2, \mathbb{Z})$ basis, i.e. $\det[\vec{u}_i, \vec{u}_{i+1}] = \vec{u}_i \times \vec{u}_{i+1} = 1$ for all i . Let \vec{r} be a generic 2-vector in the sense that its imaginary part has irrational slope. We will prove

$$\prod_k (x|\vec{r} \times \vec{u}_k, -\vec{r} \times \vec{u}_{k+1}) = 1 - e^{2\pi i x}. \tag{A.11}$$

First, we remind the reader that we are using the short hand (A.3). Moreover, one has $\text{Im}(\vec{r} \times \vec{u}_i) \neq 0$ for all i , so the special function above is well defined. The following is a direct proof, but it is also possible to prove this identity using (A.4) plus an induction similar to the one used when proving $c = -12$ at the end of Sect. 4, which we leave to the reader.

Proof. Consider the line on \mathbb{R}^2 perpendicular to $\text{Im } \vec{r}$, since $\text{Im } \vec{r}$ is chosen generic, this line does not land on any integral points. We will only be interested in the four \vec{u} 's next to this line, see Fig. 7.

Let \vec{w}, \vec{v} be the normals of two lines (ordered counterclockwise), and assume first $\text{Im } \vec{r} \cdot \vec{w} > 0, \text{Im } \vec{r} \cdot \vec{v} > 0$, consider the infinite product

$$\begin{aligned} P^{++} &= \prod_{\vec{n} \cdot \vec{w} > 0; \vec{n} \cdot \vec{v} \leq 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r})) \\ &= \prod_{\vec{n} \cdot \vec{w} > 0; \vec{n} \cdot \vec{v} \leq 0} (1 - e^{2\pi i x} \exp 2\pi i ((\vec{n} \cdot \vec{w})(\vec{r} \times \vec{v}) - (\vec{n} \cdot \vec{v})(\vec{r} \times \vec{w}))) \\ &= \prod_{i, j \geq 0} (1 - e^{2\pi i x} \exp 2\pi i ((j+1)(\vec{r} \times \vec{v}) + i(\vec{r} \times \vec{w}))) = (x|\vec{r} \times \vec{w}, -\vec{r} \times \vec{v})^{-1}, \end{aligned}$$

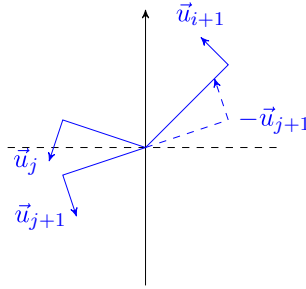


Fig. 8. One flips \vec{u}_{j+1} , and the product is now between $-\vec{u}_{j+1}$ and \vec{u}_j

Similarly for $\text{Im } \vec{r} \times \vec{w} < 0, \text{Im } \vec{r} \times \vec{v} < 0,$

$$P^{--} = \prod_{\vec{n} \cdot \vec{w} \geq 0; \vec{n} \cdot \vec{v} < 0} (1 - e^{2\pi i x} \exp 2\pi i (-\vec{n} \cdot \vec{r})) = (x|\vec{r} \times u, -\vec{r} \times v)^{-1},$$

and for $\text{Im } \vec{r} \times \vec{w} > 0, \text{Im } \vec{r} \times \vec{v} < 0,$

$$P^{+-} = \prod_{\vec{n} \cdot \vec{w} \leq 0; \vec{n} \cdot \vec{v} \leq 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r})) = (x|\vec{r} \times u, -\vec{r} \times v),$$

and finally if $\text{Im } \vec{r} \times \vec{w} < 0, \text{Im } \vec{r} \times \vec{v} > 0$

$$P^{-+} = \prod_{\vec{n} \cdot \vec{w} > 0; \vec{n} \cdot \vec{v} > 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r})) = (x|\vec{r} \times \vec{w}, -\vec{r} \times \vec{v}).$$

With these preparations, we can finish the proof. The product from $(x|\vec{r} \times \vec{u}_{i+1}, -\vec{r} \times \vec{u}_{i+2})$ to $(x|\vec{r} \times \vec{u}_{j-1}, -\vec{r} \times \vec{u}_j)$ can be combined into a single product

$$P_{[\vec{u}_{i+1}, \vec{u}_j]} = \prod_{\vec{n} \cdot \vec{u}_{i+1} > 0; \vec{n} \cdot \vec{u}_j \leq 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r}))^{-1}.$$

Similarly the factors from $(x|\vec{r} \times \vec{u}_{j+1}, -\vec{r} \times \vec{u}_{j+2})$ to $(x|\vec{r} \times \vec{u}_{i-1}, -\vec{r} \times \vec{u}_i)$

$$P_{[\vec{u}_{j+1}, \vec{u}_i]} = \prod_{\vec{n} \cdot \vec{u}_{j+1} \geq 0; \vec{n} \cdot \vec{u}_i < 0} (1 - e^{2\pi i x} \exp 2\pi i (-\vec{n} \cdot \vec{r}))^{-1}.$$

The factor $(x|e^{2\pi i(\vec{r} \times v_j)}, e^{-2\pi i(\vec{r} \times v_{j+1})})$ can be written as

$$\begin{aligned} P_{[-\vec{u}_{j+1}, \vec{u}_j]} &= \prod_{\vec{n} \cdot \vec{u}_j \leq 0; \vec{n} \cdot \vec{u}_{j+1} \leq 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r})) \\ &= \prod_{\vec{n} \cdot \vec{u}_j \leq 0; \vec{n} \cdot (-\vec{u}_{j+1}) \geq 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r})), \end{aligned}$$

the situation is depicted as in Fig. 8, that is, one flips \vec{u}_{j+1} so that both \vec{u}_j and $-\vec{u}_{j+1}$ stay above the dotted line.

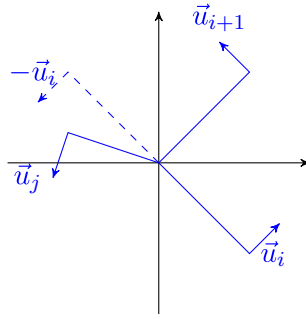


Fig. 9. One flips \vec{u}_i , and the product is now between \vec{u}_{i+1} and $-\vec{u}_i$

Then the combination

$$P(\vec{u}_{i+1}, \vec{u}_j) P[-\vec{u}_{j+1}, \vec{u}_j] = \prod_{n \cdot (-\vec{u}_{j+1}) \geq 0; n \cdot \vec{u}_{i+1} \leq 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r})) = P[-\vec{u}_{j+1}, \vec{u}_{i+1}]. \tag{A.12}$$

For the remaining factor $(x | e^{2\pi i (\vec{r} \times \vec{u}_i)}, e^{-2\pi i (\vec{r} \times \vec{u}_{i+1})})$, consider the Fig. 9 and we get the contribution

$$\begin{aligned} P(\vec{u}_{i+1}, -\vec{u}_i) &= \prod_{\vec{n} \cdot (-\vec{u}_i) < 0; \vec{n} \cdot \vec{u}_{i+1} > 0} (1 - x \exp 2\pi i (\vec{n} \cdot \vec{r})) \\ &= \prod_{\vec{n} \cdot \vec{u}_i < 0; \vec{n} \cdot (-\vec{u}_{i+1}) > 0} (1 - e^{2\pi i x} \exp 2\pi i (-\vec{n} \cdot \vec{r})). \end{aligned}$$

So the combination

$$\begin{aligned} P[\vec{u}_{j+1}, \vec{u}_i] P(\vec{u}_{i+1}, -\vec{u}_i) &= \prod_{\vec{n} \cdot \vec{u}_{j+1} \geq 0; \vec{n} \cdot (-\vec{u}_{i+1}) \leq 0} (1 - e^{2\pi i x} \exp 2\pi i (-\vec{n} \cdot \vec{r}))^{-1} \cdot (1 - e^{2\pi i x}) \\ &= \prod_{\vec{n} \cdot (-\vec{u}_{j+1}) \geq 0; \vec{n} \cdot \vec{u}_{i+1} \leq 0} (1 - e^{2\pi i x} \exp 2\pi i (\vec{n} \cdot \vec{r}))^{-1} \cdot (1 - e^{2\pi i x}). \end{aligned}$$

Note that when one combines the two sums in two wedges, extra care is needed for the origin, this is the reason one has an extra $(1 - e^{2\pi i x})$ factor above. What we get here cancels the $P[-\vec{u}_{j+1}, \vec{u}_{i+1}]$ term from (A.12), leaving us with the factor $(1 - e^{2\pi i x})$. We have proved the cancellation assuming the particular arrangement of the four lines $\vec{u}_{i,i+1} \vec{u}_{j,j+1}$ as in Fig. 7, if they are arranged in a different relative position, the proof still goes through with only minor modifications \square

B. A More Convenient Formulation of the Good Cone Condition

The original goodness condition of a cone given by Lerman is the following, at every codimension- k face, the k -normals $\vec{v}_{i_1}, \dots, \vec{v}_{i_k}$ satisfies

$$\text{span}_{\mathbb{R}} \langle \vec{v}_{i_1}, \dots, \vec{v}_{i_k} \rangle \cap \mathbb{Z}^m = \text{span}_{\mathbb{Z}} \langle \vec{v}_{i_1}, \dots, \vec{v}_{i_k} \rangle. \tag{B.1}$$

This condition is equivalent to saying that $\{\vec{v}_{i_1}, \dots, \vec{v}_{i_k}\}$ can be completed into an $SL(m, \mathbb{Z})$ -matrix. To see this, it is enough to consider $m = 3$.

At a codimension 1 face, we just have one normal, call it \vec{v} . For (B.1) to be true \vec{v} must be primitive. This is also sufficient, indeed, suppose $\vec{v} = [p, q, r]$, $\gcd(p, q, r) = 1$, there exist two integers s, t such that $sq - tp = \gcd(p, q)$ (s, t can be found using Euclid's algorithm). Consider the $SL(3, \mathbb{Z})$ matrix

$$A = \begin{pmatrix} \bar{q} & -\bar{p} & 0 \\ -t & s & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{p} = p/\gcd(p, q), \quad \bar{q} = q/\gcd(p, q).$$

Clearly $A\vec{v} = [0, \gcd(p, q), r]$. Now since $\gcd(\gcd(p, q), r) = 1$, one can find another $SL(3, \mathbb{Z})$ matrix A' such that $A'A\vec{v} = [0, 0, 1]$. Hence $A'A\vec{v}$ satisfies (B.1), and so \vec{v} also does. From this argument, we also see that \vec{v} can be completed into an $SL(3, \mathbb{Z})$ matrix. The above argument is quite a useful one, we restate it as, for any vector \vec{v} of dimension m , one can always find an $SL(m, \mathbb{Z})$ matrix A so that $A\vec{v} = [\gcd(\vec{v}), 0, \dots, 0]$.

Now proceed to the codimension 2 face, which is the intersection of two codimension 1 faces with primitive normals \vec{u}, \vec{v} . One can find an $SL(3, \mathbb{Z})$ matrix to put \vec{v} into $[0, 0, 1]$, denote by $w = A\vec{u}$. The span of $A\vec{u}, A\vec{v}$ is the same as the span of $[0, 0, 1]$ and $[w^1, w^2, 0]$, showing that $\gcd(w^1, w^2) = 1$ if (B.1) is to be satisfied. Then another $SL(3, \mathbb{Z})$ transformation can put $[\vec{u}, \vec{v}]$ into

$$[\vec{u}, \vec{v}] \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ * & 1 \end{pmatrix},$$

which can obviously be completed into an $SL(3, \mathbb{Z})$ matrix.

With minor modifications, the proof extends to higher dimensions as well.

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