Commun. Math. Phys. 338, 367–391 (2015)
Digital Object Identifier (DOI) 10.1007/s00220-015-2347-3

Communications in Mathematical Physics

Well-Posedness for the Cauchy Problem for a System of Semirelativistic Equations

Kazumasa Fujiwara[1], **Shuji Machihara**[2], **Tohru Ozawa**[3]

[1] Department of Pure and Applied Physics, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan. E-mail: k-fujiwara@asagi.waseda.jp
[2] Faculty of Science, Saitama Univeristy, 255 Shimo-Okubo, Saitama 338-8570, Japan. E-mail: machihar@rimath.saitama-u.ac.jp
[3] Department of Applied Physics, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan. E-mail: txozawa@waseda.jp

Received: 5 August 2014 / Accepted: 21 December 2014
Published online: 9 April 2015 –

Abstract: The local well-posedness for the Cauchy problem of a system of semirelativistic equations in one space dimension is shown in the Sobolev space H^s of order $s \geq 0$. We apply the standard contraction mapping theorem by using Bourgain type spaces $X^{s,b}$. We also use an auxiliary space for the solution in $L^2 = H^0$. We give the global well-posedness by this conservation law and the argument of the persistence of regularity.

1. Introduction

We study the local and global well-posedness of the following Cauchy problem for a system of semirelativistic equations

$$\begin{cases} i\partial_t u + \sqrt{m^2 - \Delta}\, u = \lambda \overline{u} v, \\ i\partial_t v - \sqrt{M^2 - \Delta}\, v = \mu u^2, \\ (u(0), v(0)) = (u_0, v_0), \end{cases} \tag{1.1}$$

where u, v are complex-valued functions of $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\partial_t = \partial/\partial t$, $m, M \in \mathbb{R}$, $\lambda, \mu \in \mathbb{C}$, $\Delta = \partial_x^2 = (\partial/\partial x)^2$ is the Laplacian in $\mathbb{R}$, and $\overline{u}$ is the complex conjugate of u. Such a semirelativistic equation is regarded as a model of relativistic quantum mechanics. The system (1.1) is a model of a couple of relativistic quantum particles with a quadratic interaction. We recall that semirelativistic equations with Hartree type nonlinearity are regarded as models of Boson stars, see [4,6,13] and the references therein.

There are some papers with regard to the single equation case of (1.1). Borgna and Rial studied the Cauchy problem for a single semirelativistic equation with cubic nonlinearity in [1] and they proved the existence of local solutions in H^s with $s > 1/2$, where $H^s = (1 - \Delta)^{-s/2} L^2(\mathbb{R})$ is the usual Sobolev space. The method of proof depends essentially on the uniform control $H^s \hookrightarrow L^\infty$, that is just Sobolev embedding.

Krieger, Lenzmann, and Raphaël studied the same problem in the massless case in [12, Appendix D] and they proved the existence of local solutions in $H^{1/2}$ by using compactness and Vladimirov type arguments [15,16,19]. We remark that the Sobolev embedding $H^{1/2} \hookrightarrow L^\infty$ fails.

In this paper, we study to what extent the Cauchy problem for a semirelativistic equation with power-type nonlinearity is well-posed in Sobolev spaces of low order, by which the uniform control breaks down. We study (1.1) as the Cauchy problem with a quadratic nonlinearity, one of the simplest power-type nonlinearity. We add a remark that Strichartz type estimates are not sufficient for a contraction argument for Cauchy problems for semirelativistic equations with power-type nonlinearity with positive exponent unless the uniform control by H^s norm is available. The situation in which the Strichartz type estimates are not sufficient to study the well-posedness for a Cauchy problem with respect to the regularity threshold happens in the case of nonlinear Dirac equations in one space dimension [3,14]. To our knowledge, the well-posedness of Cauchy problems for semirelativistic equations with power-type nonlinearity with positive exponent have not been studied unless the uniform control by H^s norm is available, where H^s is not the corresponding energy space. We also remark that neither can we apply the Delgado–Candy trick which is the special technique for the Dirac equation in one dimension in which the solution is divided into a free solution part and uniform bounded part. This technique depends on the algebraic structure of the Dirac equation which the semirelativistic equation does not have. We refer to [2,14] for the details of the Delgado–Candy trick. Whereas, in the case where $m = M = 0$, the equations of (1.1) have space-time dilation invariance under the scaling

$$u_\rho(t,x) = \rho u(\rho t, \rho x), \quad v_\rho(t,x) = \rho v(\rho t, \rho x)$$

for $\rho \geq 0$. Then, $\dot{H}^{-1/2} \times \dot{H}^{-1/2}$ norm is scaling invariant. We say $H^{-1/2} \times H^{-1/2}$ is critical in the sense of dilation. It is well-known that the Cauchy problems for evolution equations are expected to be well-posed in Sobolev spaces above the scaling-critical regularity. So we expect the well-posedness of (1.1) can hold in Sobolev spaces with lower regularity than the $H^{1/2}$ previously studied.

We apply the Bourgain method to study (1.1) in H^s with $s \leq 1/2$. This method does not use the Strichartz type estimate. This method also has been applied to the Dirac equation and the Dirac–Klein–Gordon equation in those cases [5,7,18].

Our results are the following.

Theorem 1. *Let $s \geq 0$ and let $(u_0, v_0) \in H^s \times H^s$. Then there exists $T > 0$ and a unique pair of solutions $(u, v) \in C([0, T), H^s \times H^s)$ to (1.1). For this pair of solutions, we define the maximal existence time of solutions $T(s)$ as*

$$T(s) = T(u_0, v_0, s) = \sup \left\{ T > 0\, ; \sup_{0<t<T} \left(\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \right) < \infty \right\}. \tag{1.2}$$

Then $T(s) = T(0)$.

Remark 1. The definition of the maximal existence time (1.2) makes sense because of the blow-up alternative argument. For any pair of initial data $(u_0, v_0) \in H^s \times H^s$, it is obvious $T(s) \leq T(0)$. Theorem 1 implies the persistence of regularity $T(s) \geq T(0)$.

Theorem 2. *Let λ and μ satisfy $\lambda = c\overline{\mu}$ with some constant $c > 0$. Let $s \geq 0$. Then the solutions of Theorem 1 extend globally and satisfy $(u, v) \in C(\mathbb{R}; H^s \times H^s) \cap L^\infty(\mathbb{R}; L^2 \times L^2)$.*

We prove Theorem 1 by a contraction argument based on the Bourgain norm in $X^{s,b}$. We also use the auxiliary norm in Y^s defined below especially for the case $s = 0$. We give a bilinear estimate by means of those norms, which is applied to the arguments of the well-posedness and the persistence of regularity $T(s) = T(0)$. Particularly, we prove that the H^s norms of the solutions never blow up before L^2 norms may blow up.

The widest space $L^2 \times L^2$ in which the well-posedness of the Cauchy problem (1.1) is guaranteed in Theorem 1 is far from the critical space in the sense of dilation. It is not clear whether it is optimal. We provide a counterexample for the bilinear estimate in Proposition 3 in Appendix B to show the condition $s \geq 0$ of Theorem 1 is necessary in the method based only on the Bourgain method at least.

The system (1.1) is also regarded as a semirelativistic approximation of the Schrödinger system

$$\begin{cases} i\partial_t u + \frac{\sigma_1}{2m}\Delta u = \lambda \overline{u} v, \\ i\partial_t v + \frac{\sigma_2}{2M}\Delta v = \mu u^2, \end{cases} \tag{1.3}$$

where $\sigma_j \in \{-1, 1\}$. We refer the reader to [8–11] for recent results on the Cauchy problem for (1.3). In the case of the Cauchy problem in $L^2 \times L^2$ for (1.3), the signs of σ_1, σ_2 are not essential [10]. On the other hand, in this paper the combination $(\sigma_1, \sigma_2) = (1, -1)$ or $(-1, 1)$ is essential in (1.1) in connection with the quadratic interactions on the right hand sides as far as one tries to apply the Bourgain method in $L^2 \times L^2$. For other cases, see Remark 4. We also observe that it seems difficult to close our contraction map by using only $X^{s,b}$ norms in the case $s = 0$. In Appendix B, we prove the fact that the bilinear estimate with $X^{0,b}$ norms fails. If $s > 0$, we give a simpler proof which ensures the contraction argument depending exclusively on $X^{s,b}$ norms.

Under the constraint $\lambda = c\overline{\mu}$, we have the following conservation law of charge, namely, the conservation law of the L^2 norm;

$$\|u(t)\|_{L^2}^2 + c\|v(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + c\|v_0\|_{L^2}^2 \tag{1.4}$$

for any $t \in \mathbb{R}$. Equation (1.4) can be shown by an approximation argument. Then we have the global existence of the solutions in L^2, and also H^s, $s > 0$, since $T(s) = T(0)$ in Theorem 1. Also we can show (1.4) by the argument by one of us [17], which need not take smooth approximation of solutions. In Appendix A, we give the proof of the charge conservation law with weak solutions guaranteed in $X_-^{0,1/2} \times X_+^{0,1/2}$.

We introduce some notation to be used below. For a function u of two variables (time and space), $\mathfrak{F}_x[u]$ denotes the Fourier transform in space variable x and $\widetilde{u}$ denotes the Fourier transform in space-time variables. We also write $\hat{f}$ for the Fourier transform of a one-variable function f. For $m, t \in \mathbb{R}$, $U_m(t) = \exp[-it\sqrt{m^2 - \Delta}]$ denotes the free propagator for the semirelativistic equation

$$i\partial_t v - \sqrt{m^2 - \Delta} v = 0.$$

For $s \in \mathbb{R}$, $\dot{H}^s = (-\Delta)^{-s/2} L^2(\mathbb{R})$ is the homogeneous Sobolev space of order s. For $a, b \in \mathbb{R}$, $a \vee b$ and $a \wedge b$ are the maximal and minimum, respectively. For two normed spaces X and Y, we define a norm $\|\cdot\|_{X \cap Y}$ as

$$\|a\|_{X\cap Y} = \|a\|_X \vee \|a\|_Y.$$

For $m, a, b \in \mathbb{R}$, $T_0 \in \mathbb{R}$, and $T > 0$, we define Bourgain norms

$$\|u\|_{X^{s,b}_{m,\pm}} = \left\| \langle \xi \rangle^s \langle \tau \pm \sqrt{m^2+\xi^2} \rangle^b \widetilde{u}(\tau,\xi) \right\|_{L^2_\tau L^2_\xi},$$

$$\|u\|_{X^{s,b}_{m,\pm}[T_0,T_0+T]} = \inf \left\{ \|u'\|_{X^{s,b}_{m,\pm}} ; \begin{array}{l} u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R}, \\ \operatorname{supp} u' \subset [T_0-2T, T_0+2T] \times \mathbb{R} \end{array} \right\},$$

where $\langle x \rangle = 1 + |x|$, and auxiliary norms

$$\|u\|_{Y^{s}_{m,\pm}} = \left\| \langle \xi \rangle^s \langle \tau \pm \sqrt{m^2+\xi^2} \rangle^{-1} \widetilde{u} \right\|_{L^2_\xi L^1_\tau},$$

$$\|u\|_{Y^{s}_{m,\pm}[T_0,T_0+T]} = \inf \left\{ \|u'\|_{Y^{s}_{m,\pm}} ; \begin{array}{l} u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R}, \\ \operatorname{supp} u' \subset [T_0-2T, T_0+2T] \times \mathbb{R} \end{array} \right\}.$$

We note $\|u\|_{X^{s,b}_{m,\pm}} = \|\overline{u}\|_{X^{s,b}_{m,\mp}}$ and $\|u\|_{Y^{s}_{m,\pm}} = \|\overline{u}\|_{Y^{s}_{m,\mp}}$ for any $s, b, m \in \mathbb{R}$. In addition, we abbreviate these spaces as : $X^{s,b}_\pm = X^{s,b}_{0,\pm}$, $Y^s_\pm = Y^s_{0,\pm}$. In our proof, the following space is basic for the pair of solutions (u, v);

$$\mathcal{X}^{s,b}[T_0, T_0+T] = X^{s,b}_-[T_0, T_0+T] \times X^{s,b}_+[T_0, T_0+T].$$

We use $\mathcal{X}^{s,b}[T_0, T_0+T]$ for the proof of Theorem 1.

Let ψ be a cut off function, namely, a smooth function with $0 \le \psi \le 1$, $\psi(t) = 1$ if $|t| \le 1$ and $\psi(t) = 0$ if $|t| \ge 2$. For $T > 0$, $\psi_T(t) = \psi(T^{-1}t)$.

Remark 2. For $s, b \ge 0$, $T > 0$ and $m, T_0 \in \mathbb{R}$, function space $X^{s,b}_{m,\pm}[T_0, T_0+T]$ is a quotient of a closed linear subspace of a weighted $L^2(\mathbb{R}^2)$ by another closed subspace. Since, for $s, b \ge 0$, functions whose support is restricted on a closed subset of $\mathbb{R} \times \mathbb{R}$ compose a closed linear subspace on $X^{s,b}_{m,\pm}[T_0, T_0+T]$ because $L^2(\mathbb{R}^2)$ is continuously embedded into them. Then $\|\cdot\|_{X^{s,b}_{m,\pm}[T_0,T_0+T]}$ is a quotient norm and $X^{s,b}_{m,\pm}[T_0, T_0+T]$ is a Banach space as long as for $s, b \ge 0$. However, we use the notation of $\|\cdot\|_{X^{s,b}_{m,\pm}[T_0,T_0+T]}$ even if $b < 0$. We also use the notation $\|\cdot\|_{Y^{s}_{m,\pm}[T_0,T_0+T]}$, even when $\|\cdot\|_{Y^{s}_{m,\pm}[T_0,T_0+T]}$ is only a semi-norm.

We give a brief outline of the remainder of this article. We prepare the linear and bilinear estimates in Sects. 2 and 3, respectively. We give the proof of Theorem 1 in Sect. 4. We describe the direct proof of the L^2 conservation law (1.4) in Appendix A. We give a simpler proof of the local existence in the case $s > 0$ and we show that the bilinear estimate fails with $s = 0$ in Appendix B.

2. Linear Estimates

Here we collect some basic estimates. We consider the scalar equations

$$i\partial_t u \mp \sqrt{m^2 - \Delta} u = f(u), \tag{2.1}$$

where u and f are complex-valued functions. The Cauchy problem for (2.1) with initial data $u(0,\cdot)=u_0$ is rewritten in the form of the integral equations

$$u(t)=U_m(\pm t)u_0-i\int_0^t U_m(\pm(t-t'))f(u(t'))dt'.$$

To state the proof of our theorems, the following basic estimates are necessary. From now on, the letter ψ is used exclusively for the cut-off function determined in the introduction.

Lemma 1 [7, (2.19)]. *Let $m\in\mathbb{R}$. For any $s,b\geq 0$ and $u_0\in H^s$,*

$$\|\psi(t)U_m(\pm t)u_0\|_{X^{s,b}_{m,\pm}}=\|\psi\|_{H^b}\|u_0\|_{H^s}. \tag{2.2}$$

In addition, for any $0<T<1$,

$$\|\psi_T(t)U_m(\pm t)u_0\|_{X^{s,1/2}_{m,\pm}}\lesssim\|u_0\|_{H^s}. \tag{2.3}$$

Proof. The equality (2.2) is easily seen. The estimate (2.3) follows form scaling invariance of $\dot{H}^{1/2}$. □

Proposition 1 [7, Lemma 2.1]. *Let $m\in\mathbb{R}$, $0<T\leq 1$ and let $s\geq 0$. Then*

$$\left\|\psi_T(t)\int_0^t U_m(\pm(t-t'))F(t')\,dt'\right\|_{X^{s,1/2}_{m,\pm}}\lesssim\|F\|_{X^{s,-1/2}_{m,\pm}\cap Y^s_{m,\pm}}, \tag{2.4}$$

for $F\in X^{s,-1/2}_{m,\pm}\cap Y^s_{m,\pm}$. In addition, let $\delta\geq 0$ and b satisfy $-1/2<b-1+\delta\leq 0\leq b$. Then

$$\left\|\psi_T(t)\int_0^t U_m(\pm(t-t'))F(t')\,dt'\right\|_{X^{s,b}_{m,\pm}}\lesssim T^\delta\|F\|_{X^{s,b-1+\delta}_{m,\pm}} \tag{2.5}$$

for $F\in X^{s,b-1+\delta}_{m,\pm}$.

Lemma 2 [7, Lemma 2.2]. *Let $m\in\mathbb{R}$. If $F\in Y^s_{m,\pm}$, then $\int_0^{\cdot} U_m(\cdot-t')F(t')dt'\in C(\mathbb{R};H^s)$ and it satisfies the estimate*

$$\left\|\int_0^{\cdot} U_m(\pm(\cdot-t'))F(t')dt'\right\|_{C(\mathbb{R};H^s)}\lesssim\|F\|_{Y^s_{m,\pm}}.$$

To extract a positive power of T, we use the following lemma.

Lemma 3 [7, Lemma 3.1]. *Let $s\in\mathbb{R}$, $0\leq b<b'$, $T>0$ and let $f\in X^{s,b'}_{\pm}$ satisfy* supp $f\subset[-T,T]\times\mathbb{R}$. *Then*

$$\|f\|_{X^{s,b}_{\pm}}\lesssim T^{\gamma(b',b)}\|f\|_{X^{s,b'}_{\pm}}, \tag{2.6}$$

where

$$\gamma(b',b)=\begin{cases} b'-b & \text{if } b'<1/2,\\ (1-\varepsilon)(b'-b) & \text{if } b'=1/2,\\ 1/2-b/2b' & \text{if } b'>1/2\end{cases}$$

with $\varepsilon>0$ sufficiently small.

3. Bilinear and Trilinear Estimates

In this section, we derive nonlinear estimates for $X^{s,b}_{m,\pm}$ and $Y^s_{m,\pm}$ by the method proposed in [18]. Due to the next lemma, we may put $m = 0$ without loss of generality.

Lemma 4. *For any $m, M \in \mathbb{R}$, $X^{s,b}_{m,\pm} \simeq X^{s,b}_{M,\pm}$, $Y^s_{m,\pm} \simeq Y^s_{M,\pm}$ with equivalent norms.*

Proof. The lemma follows from the following inequalities

$$\begin{aligned}\frac{\langle \tau \pm \sqrt{m^2+\xi^2}\rangle}{\langle \tau \pm \sqrt{M^2+\xi^2}\rangle} &\le 1 + \left|\frac{\langle \tau \pm \sqrt{m^2+\xi^2}\rangle - \langle \tau \pm \sqrt{M^2+\xi^2}\rangle}{\langle \tau \pm \sqrt{M^2+\xi^2}\rangle}\right| \\ &= 1 + \frac{\big||\tau \pm \sqrt{m^2+\xi^2}| - |\tau \pm \sqrt{M^2+\xi^2}|\big|}{\langle \tau \pm \sqrt{M^2+\xi^2}\rangle} \\ &\le 1 + |m - M|.\end{aligned}$$

□

In addition, we need the following bilinear estimates for Sobolev norms.

Lemma 5. *Let $\alpha, \beta, \gamma \in \mathbb{R}$. Then the inequality*

$$\|uv\|_{H^{-\alpha}} \lesssim \|u\|_{H^\beta}\|v\|_{H^\gamma}$$

holds if and only if

$$\alpha+\beta+\gamma \ge \frac{1}{2} \quad \text{and} \quad \alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha > 0,$$

or

$$\alpha+\beta+\gamma > \frac{1}{2} \quad \text{and} \quad \alpha+\beta,\ \beta+\gamma,\ \gamma+\alpha \ge 0.$$

Lemma 6. *Let $p \ge 1$ and let $\alpha,\ \beta,\ \gamma \ge 0$ satisfy $\alpha+\beta+\gamma > 1/p$. Then there exists a positive constant C such that the inequality*

$$\|\langle \tau+\delta_1\rangle^{-\alpha} f * g\|_{L^p_\tau} \le C\|\langle \tau+\delta_2\rangle^{\beta} f\|_{L^2_\tau}\|\langle \tau+\delta_3\rangle^{\gamma} g\|_{L^2_\tau}$$

holds for any real numbers $\delta_1,\ \delta_2,\ \delta_3$ and any f, g such that all the norms on the right hand side are finite.

Proof. By the Hölder and Young inequalities,

$$\begin{aligned}\|\langle \tau+\delta_1\rangle^{-\alpha} f * g(\tau)\|_{L^p_\tau} &\lesssim \|f * g\|_{L^{p(\alpha+\beta+\gamma)/(\beta+\gamma)}} \\ &\lesssim \|\langle \tau+\delta_2\rangle^{\beta} f(\tau)\|_{L^2_\tau}\|\langle \tau+\delta_3\rangle^{\gamma} g(\tau)\|_{L^2_\tau}\end{aligned}$$

from which we obtain the lemma. □

For $s \ge 0$, we define $\lambda(s)$ as

$$\lambda(s) = \begin{cases} 0 & \text{if} \quad s < 1/2, \\ s - 1/2 + \varepsilon & \text{if} \quad s \ge 1/2, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is sufficiently small. Here we state our main nonlinear estimates.

Proposition 2. *Let $s \geq 0$ and $0 \leq \rho < 1/2$. Then the inequality*

$$\|uv\|_{X_+^{s,-1/2} \cap Y_+^s} \lesssim \|u\|_{X_-^{\lambda(s),1/2}} \|v\|_{X_-^{s,1/2-\rho}} + \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}} \tag{3.2}$$

holds for any $u \in X_-^{\lambda(s),1/2}$ and $v \in X_-^{s,1/2}$.

We remark that the regularity $\lambda(s)$ in the both terms of u on the right hand side is less than the regularity s on the left hand side. Therefore, the estimate (3.2) with $s > 0$ does not follow directly from (3.2) with $s = 0$ and the Peetre's inequality $\langle\xi\rangle^{s'} \lesssim (\langle\xi - \eta\rangle^{s'} + \langle\eta\rangle^{s'})$ for $s' \geq 0$. We can exchange the smoothness with regards to the space-time variables into the smoothness with regards to the space variable by using (3.4) from the nice combination of signs $\pm$ in (3.2). This technique is found in Lemma 5 of [18].

The symmetry inequality

$$\|uv\|_{X_-^{s,-1/2} \cap Y_-^s} \lesssim \|u\|_{X_+^{\lambda(s),1/2}} \|v\|_{X_+^{s,1/2-\rho}} + \|u\|_{X_+^{\lambda(s),1/2-\rho}} \|v\|_{X_+^{s,1/2}}$$

holds by (3.2) with taking complex conjugate of u and v.

Proof. It is enough to show

$$\|uv\|_{X_+^{s,-1/2}} \lesssim \|u\|_{X_-^{\lambda(s),1/2}} \|v\|_{X_-^{s,1/2-\rho}} + \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}}$$

and

$$\|uv\|_{Y_+^s} \lesssim \|u\|_{X_-^{\lambda(s),1/2}} \|v\|_{X_-^{s,1/2-\rho}} + \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}} .$$

Let

$$M(\tau, \xi, \sigma, \eta) = \big|\tau + |\xi|\big| \vee \big|\tau - \sigma - |\xi - \eta|\big| \vee \big|\sigma - |\eta|\big|. \tag{3.3}$$

Then the triangle inequality implies

$$|\xi| + |\xi - \eta| + |\eta| \leq 3M(\tau, \xi, \sigma, \eta). \tag{3.4}$$

Also, we decompose the integral region as follows;

$$A_1 = \left\{(\tau, \xi, \sigma, \eta)\ ;\ M(\tau, \xi, \sigma, \eta) = \big|\tau + |\xi|\big|\right\},$$
$$A_2 = \left\{(\tau, \xi, \sigma, \eta)\ ;\ M(\tau, \xi, \sigma, \eta) = \big|\tau - \sigma - |\xi - \eta|\big|\right\},$$
$$A_3 = \left\{(\tau, \xi, \sigma, \eta)\ ;\ M(\tau, \xi, \sigma, \eta) = \big|\sigma - |\eta|\big|\right\}.$$

(a) X norm estimate with $s > 0$.
By the Minkowski inequality,

$$\left\| \langle\xi\rangle^s \iint \langle\tau + |\xi|\rangle^{-1/2} \chi_{A_1}\, \widetilde{u}(\tau - \sigma, \xi - \eta)\, \widetilde{v}(\sigma, \eta)\, d\sigma d\eta \right\|_{L_\tau^2 L_\xi^2}$$
$$\lesssim \left\| \int \langle\xi\rangle^{s-1/2} I_1(\xi, \eta)\, d\eta \right\|_{L_\xi^2},$$

where

$$I_1(\xi,\eta) = \left\| \int \left| \widetilde{u}(\tau-\sigma,\xi-\eta)\, \widetilde{v}(\sigma,\eta) \right| d\sigma \right\|_{L^2_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{1/2-\rho} \widetilde{u}(\tau,\xi-\eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{1/2} \widetilde{v}(\tau,\eta) \right\|_{L^2_\tau}$$

by Lemma 6. Since

$$\frac{1}{2} - s + \lambda(s) + s \geq \frac{1}{2},$$
$$\frac{1}{2} - s + \lambda(s) > 0,$$

and Lemma 5,

$$\left\| \int \langle \xi \rangle^{s-1/2} I_1(\xi,\eta)\, d\eta \right\|_{L^2_\xi} \lesssim \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}} .$$

Similarly, for $j = 2, 3$,

$$\left\| \langle \xi \rangle^s \iint \langle \tau + |\xi| \rangle^{-1/2} \chi_{A_j}\, \widetilde{u}(\tau-\sigma,\xi-\eta)\, \widetilde{v}(\sigma,\eta)\, d\sigma d\eta \right\|_{L^2_\tau L^2_\xi}$$
$$\lesssim \left\| \int \langle \xi \rangle^{s-1/2} I_j(\xi,\eta)\, d\eta \right\|_{L^2_\xi},$$

where

$$I_2(\xi,\eta) = \left\| \langle \tau + |\xi| \rangle^{-1/2} \int \langle \tau - \sigma - |\xi-\eta| \rangle^{1/2} \left| \widetilde{u}(\tau-\sigma,\xi-\eta)\, \widetilde{v}(\sigma,\eta) \right| d\sigma \right\|_{L^2_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{1/2} \widetilde{u}(\tau,\xi-\eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{1/2-\rho} \widetilde{v}(\tau,\eta) \right\|_{L^2_\tau},$$
$$I_3(\xi,\eta) = \left\| \langle \tau + |\xi| \rangle^{-1/2} \int \langle \sigma - |\eta| \rangle^{1/2} \left| \widetilde{u}(\tau-\sigma,\xi-\eta)\, \widetilde{v}(\sigma,\eta) \right| d\sigma \right\|_{L^2_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{1/2-\rho} \widetilde{u}(\tau,\xi-\eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{1/2} \widetilde{v}(\tau,\eta) \right\|_{L^2_\tau},$$

by Lemma 6. Then, we obtain by Lemma 5

$$\left\| \int \langle \xi \rangle^{s-1/2} I_j(\xi,\eta) d\eta \right\|_{L^2_\xi} \lesssim \|u\|_{X_-^{\lambda(s),1/2}} \|v\|_{X_-^{s,1/2-\rho}} + \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}} .$$

(b) X norm estimate with $s = 0$.

$$\|uv\|_{X_+^{0,-1/2}} \leq \sum_{j=1}^{3} \left\| \iint \langle M(\tau,\xi,\sigma,\eta) \rangle^{-1/2} \chi_{A_j} \langle \tau + |\xi| \rangle^{-1/2} \langle M(\tau,\xi,\sigma,\eta) \rangle^{1/2} \right.$$
$$\left. \cdot \widetilde{u}(\tau-\sigma,\xi-\eta) \widetilde{v}(\sigma,\eta) d\sigma d\eta \right\|_{L^2_\tau L^2_\xi}$$

$$\lesssim \sum_{j=1}^{3} \left\| \int \langle \xi \rangle^{-1/4} \langle \eta \rangle^{-1/4} I_j(\xi, \eta)\, d\eta \right\|_{L^2_\xi}$$
$$\lesssim \|u\|_{X_-^{0,1/2}} \|v\|_{X_-^{0,1/2-\rho}} + \|u\|_{X_-^{0,1/2-\rho}} \|v\|_{X_-^{0,1/2}} ,$$

where M is defined in (3.3) and the last estimate follows from Lemmas 5 and 6.

(c) Y norm estimate with $s > 0$.
By the Minkowski inequality,

$$\left\| \langle \xi \rangle^s \iint \langle \tau + |\xi| \rangle^{-1} \chi_{A_1}\, \widetilde{u}(\tau - \sigma, \xi - \eta)\, \widetilde{v}(\sigma, \eta)\, d\sigma d\eta \right\|_{L^2_\xi L^1_\tau}$$
$$\lesssim \left\| \int \langle \xi \rangle^{s-1/2} J_1(\xi, \eta)\, d\eta \right\|_{L^2_\xi},$$

where

$$J_1(\xi, \eta) = \left\| \langle \tau + |\xi| \rangle^{-1/2} \int \left| \widetilde{u}(\tau - \sigma, \xi - \eta)\, \widetilde{v}(\sigma, \eta) \right| d\sigma \right\|_{L^1_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi - \eta| \rangle^{1/2-\rho} \widetilde{u}(\tau, \xi - \eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{1/2} \widetilde{v}(\tau, \eta) \right\|_{L^2_\tau}$$

by Lemma 6. Then we obtain

$$\left\| \int \langle \xi \rangle^{s-1/2} J_1(\xi, \eta)\, d\eta \right\|_{L^2_\xi} \lesssim \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}}$$

by Lemma 5. Similarly, for $j = 2, 3$,

$$\left\| \iint \langle \tau + |\xi| \rangle^{-1} \chi_{A_j}\, \widetilde{u}(\tau - \sigma, \xi - \eta)\, \widetilde{v}(\sigma, \eta)\, d\sigma d\eta \right\|_{L^2_\xi L^1_\tau}$$
$$\lesssim \left\| \int \langle \xi \rangle^{s-1/2} J_j(\xi, \eta)\, d\eta \right\|_{L^2_\xi},$$

where

$$J_2(\xi, \eta) = \left\| \langle \tau + |\xi| \rangle^{-1} \int \langle \tau - \sigma - |\xi - \eta| \rangle^{1/2} \left| \widetilde{u}(\tau - \sigma, \xi - \eta)\, \widetilde{v}(\sigma, \eta) \right| d\sigma \right\|_{L^1_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi - \eta| \rangle^{1/2} \widetilde{u}(\tau, \xi - \eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{1/2-\rho} \widetilde{v}(\tau, \eta) \right\|_{L^2_\tau},$$
$$J_3(\xi, \eta) = \left\| \langle \tau + |\xi| \rangle^{-1} \int \langle \sigma - |\eta| \rangle^{1/2} \left| \widetilde{u}(\tau - \sigma, \xi - \eta) \widetilde{v}(\sigma, \eta) \right| d\sigma \right\|_{L^1_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi - \eta| \rangle^{1/2-\rho} \widetilde{u}(\tau, \xi - \eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{1/2} \widetilde{v}(\tau, \eta) \right\|_{L^2_\tau}.$$

Then we obtain from Lemma 5

$$\left\| \int \langle \xi \rangle^{s-1/2} J_j(\xi, \eta)\, d\eta \right\|_{L^2_\xi} \lesssim \|u\|_{X_-^{\lambda(s),1/2}} \|v\|_{X_-^{s,1/2-\rho}} + \|u\|_{X_-^{\lambda(s),1/2-\rho}} \|v\|_{X_-^{s,1/2}} .$$

(d) Y norm estimate with $s = 0$.
By Lemmas 5 and 6,

$$\|uv\|_{X_+^{0,-1/2}} \leq \sum_{j=1}^{3} \left\| \iint \langle M(\tau,\xi,\sigma,\eta)\rangle^{-1/2} \chi_{A_j} \langle \tau + |\xi|\rangle^{-1} \langle M(\tau,\xi,\sigma,\eta)\rangle^{1/2} \cdot \widetilde{u}(\tau-\sigma,\xi-\eta)\widetilde{v}(\sigma,\eta) d\sigma d\eta \right\|_{L_\xi^2 L_\tau^1}$$
$$\lesssim \sum_{j=1}^{3} \left\| \int \langle \xi\rangle^{-1/4} \langle \eta\rangle^{-1/4} J_j(\xi,\eta)\, d\eta \right\|_{L_\xi^2}$$
$$\lesssim \|u\|_{X_-^{0,1/2}} \|v\|_{X_-^{0,1/2-\rho}} + \|u\|_{X_-^{0,1/2-\rho}} \|v\|_{X_-^{0,1/2}}.$$

□

Corollary 1. *Let* $s \geq 0$, $0 \leq \rho' < 1/2$ *and let* $T > 0$. *Then*

$$\|uv\|_{X_\pm^{s,-1/2} \cap Y_\pm^s} \lesssim T^{\rho'} \|u\|_{X_\mp^{\lambda(s),1/2}} \|v\|_{X_\mp^{s,1/2}} \tag{3.5}$$

for any $u \in X_\mp^{\lambda(s),1/2}$ *and* $v \in X_\mp^{s,1/2}$ *such that* $\operatorname{supp} u,\ \operatorname{supp} v \subset [-T,T] \times \mathbb{R}$.

Proof. By Proposition 2 with $\rho > \rho'$ and Lemma 3 with $\varepsilon > 0$ such that $(1-\varepsilon)\rho = \rho'$, we obtain (3.5). □

Remark 3. Here we observe that Proposition 2 is optimal in some sense. We show the estimates in $X^{s,b}$ fail for $s < 0$ in Proposition 3 below. This is the reason why we require $s \geq 0$ in our $X^{s,b}$ argument for the proof of Theorem 1, even though the critical exponent in the sense of dilation is $-1/2$. >From the simple consideration, the $X^{s,b}$ argument requires the following bilinear estimate

$$\|uv\|_{X_+^{s,b-1}} \lesssim \|u\|_{X_-^{s,b}} \|v\|_{X_-^{s,b}}$$

with some $b \in \mathbb{R}$ since the gain of regularity with respect to time-space is at most 1 from the point of view of Proposition 1. We shall observe this inequality in Proposition 3, and see also Proposition 6.

Proposition 3. *For any* $b \geq 0$ *and* $s < 0$, *there exists a pair* $u, v \in X_-^{s,b}$ *such that*

$$\|uv\|_{X_+^{s,b-1}} = \infty. \tag{3.6}$$

Proof. Suppose $b \geq 1/2$. Let $0 < \varepsilon < -s/2$ and let

$$\widetilde{u_1}(\tau,\xi) = \widetilde{v_1}(\tau,\xi) = \langle \xi\rangle^{-s-1/2-\varepsilon} \langle \tau - |\xi|\rangle^{-b-1/2-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$\begin{aligned}
&\langle \xi \rangle^s \langle \tau + |\xi| \rangle^{b-1} \iint \langle \eta \rangle^{-s-1/2-\varepsilon} \langle \xi - \eta \rangle^{-s-1/2-\varepsilon} \\
&\quad \cdot \langle \sigma - |\eta| \rangle^{-b-1/2-\varepsilon} \langle \tau - \sigma - |\xi - \eta| \rangle^{-b-1/2-\varepsilon} d\sigma d\eta \\
&\quad \gtrsim \langle \xi \rangle^s \langle \tau \rangle^{b-1} \int_0^{\xi} \langle \eta \rangle^{-s-1/2-\varepsilon} \langle \xi - \eta \rangle^{-s-1/2-\varepsilon} \langle \tau - |\xi - \eta| - |\eta| \rangle^{-2b-2\varepsilon} d\eta \\
&\quad \gtrsim \langle \xi \rangle^{-s-1-2\varepsilon} \langle \tau \rangle^{b-1} \int_{\xi/3}^{2\xi/3} d\eta \\
&\quad \gtrsim \langle \tau \rangle^{-1/2}.
\end{aligned}$$

This implies $u_1 v_1 \notin X_+^{s,b-1}$. Moreover, suppose $0 \le b < 1/2$. Let b and δ satisfy $0 < 2\varepsilon \le 1/2 - b$ and let

$$\begin{aligned}
\widetilde{u_2}(\tau, \xi) &= \langle \xi \rangle^{-s-1/2-\varepsilon} \langle \tau - |\xi| \rangle^{-b-1/2-\varepsilon}, \\
\widetilde{v_2}(\tau, \xi) &= \langle \xi \rangle^{-s-1/2-\varepsilon} \langle \tau - |\xi| \rangle^{-b} \langle \tau + |\xi| \rangle^{-1/2-\varepsilon}.
\end{aligned}$$

Since for any real number a and b, $\langle a + b \rangle \le \langle a \rangle \langle b \rangle$, for $\xi > 0$,

$$\begin{aligned}
&\langle \xi \rangle^s \langle \tau + |\xi| \rangle^{b-1} \iint \langle \eta \rangle^{-s-1/2-\varepsilon} \langle \xi - \eta \rangle^{-s-1/2-\varepsilon} \langle \sigma - |\eta| \rangle^{-b-1/2-\varepsilon} \\
&\quad \cdot \langle \tau - \sigma - |\xi - \eta| \rangle^{-b} \langle \tau - \sigma + |\xi - \eta| \rangle^{-1/2-\varepsilon} d\sigma d\eta \\
&\quad \gtrsim \langle \tau + |\xi| \rangle^{b-1} \iint \langle \eta \rangle^{-1/2-\varepsilon} \langle \xi - \eta \rangle^{-b-1/2-\varepsilon} \langle \sigma - |\eta| \rangle^{-b-1/2-\varepsilon} \\
&\quad \cdot \langle \tau - \sigma + |\xi - \eta| \rangle^{-b-1/2-\varepsilon} d\sigma d\eta \\
&\quad \gtrsim \langle \tau + \xi \rangle^{b-1} \int_{-\infty}^{0} \langle \eta \rangle^{-1/2-\varepsilon} \langle \xi - \eta \rangle^{-b-1/2-\varepsilon} \langle \tau + \xi \rangle^{-2b-2\varepsilon} d\eta \\
&\quad \gtrsim \langle \tau + \xi \rangle^{-b-1-2\varepsilon} \langle \xi \rangle^{-1/2} \notin L^2_{\xi > 0}(L^2_\tau).
\end{aligned}$$

Therefore, $u_2 v_2 \notin X_+^{0,b-1}$. We complete the proof of (3.6). □

Remark 4. The trick of exchanging smoothness above is not applicable to the bilinear estimate $X_+^{s,b} X_\pm^{s,b} \hookrightarrow X_+^{s,b-1}$ nor $X_-^{s,b} X_\pm^{s,b} \hookrightarrow X_-^{s,b-1}$. For example, the estimate $X_-^{s,b} X_-^{s,b} \hookrightarrow X_-^{s,b-1}$ corresponds to the investigation for the Cauchy problem

$$\begin{cases} i\partial_t u + \sqrt{m^2 - \Delta}\, u = \lambda \bar{u} v, \\ i\partial_t v + \sqrt{M^2 - \Delta}\, v = \mu u^2, \\ (u(0), v(0)) = (u_0, v_0). \end{cases}$$

Indeed, for any $s \le 1/2$ and $b \in \mathbb{R}$, let $\widetilde{u}_\pm = \langle \tau \pm \xi \rangle^{-b-1} \langle \xi \rangle^{-s-1/2} \log \langle \xi \rangle^{-3/4}$. Then $u_\pm \in X_\pm^{s,b}$ and

$$\| u_+ u_\pm \|_{X_+^{s,b-1}} = \| u_- u_\pm \|_{X_-^{s,b-1}} = \infty.$$

These estimates are calculated as follows;

$$
\begin{aligned}
\|u_+u_+\|_{X_+^{s,b-1}} &= \Big\| \langle\xi\rangle^s \langle\tau+|\xi|\rangle^{b-1} \iint \langle\tau-\sigma+|\xi-\eta|\rangle^{-b-1}\langle\sigma+|\eta|\rangle^{-b-1} \\
&\qquad \cdot \langle\xi-\eta\rangle^{-s-1/2}\log\langle\xi-\eta\rangle^{-3/4}\langle\eta\rangle^{-s-1/2}\log\langle\eta\rangle^{-3/4} d\sigma d\eta \Big\|_{L^2L^2} \\
&\geq \Big\| \langle\xi\rangle^s \langle\tau+\xi\rangle^{b-1} \int_0^{\xi}\int_{-\eta-1}^{-\eta+1} \langle\tau-\sigma+\xi-\eta\rangle^{-b-1}\langle\sigma+\eta\rangle^{-b-1} \\
&\qquad \cdot \langle\xi-\eta\rangle^{-s-1/2}\log\langle\xi-\eta\rangle^{-3/4}\langle\eta\rangle^{-s-1/2}\log\langle\eta\rangle^{-3/4} d\sigma d\eta \Big\|_{L^2_{\xi\geq 2}L^2_{-\xi-1\leq\tau\leq-\xi+1}} \\
&\gtrsim \Big\| \langle\xi\rangle^{-1/2}\log\langle\xi\rangle^{-3/4} \int_0^{\xi} \langle\eta\rangle^{-1}\log\langle\eta\rangle^{-3/4} d\eta \Big\|_{L^2_{\xi\geq 2}} \\
&\gtrsim \Big\| \langle\xi\rangle^{-1/2}\log\langle\xi\rangle^{-1/2} \Big\|_{L^2_{\xi\geq 2}}, \\
\|u_+u_-\|_{X_+^{s,b-1}} &= \Big\| \langle\xi\rangle^s \langle\tau+|\xi|\rangle^{b-1} \iint \langle\tau-\sigma+|\xi-\eta|\rangle^{-b-1}\langle\sigma-|\eta|\rangle^{-b-1} \\
&\qquad \cdot \langle\xi-\eta\rangle^{-s-1/2}\log\langle\xi-\eta\rangle^{-3/4}\langle\eta\rangle^{-s-1/2}\log\langle\eta\rangle^{-3/4} d\sigma d\eta \Big\|_{L^2L^2} \\
&\geq \Big\| \langle\xi\rangle^s \langle\tau+\xi\rangle^{b-1} \int_{-\xi}^{0}\int_{-\eta-1}^{-\eta+1} \langle\tau-\sigma+\xi-\eta\rangle^{-1}\langle\sigma+\eta\rangle^{-1} \\
&\qquad \cdot \langle\xi-\eta\rangle^{-s-1/2}\log\langle\xi-\eta\rangle^{-3/4}\langle\eta\rangle^{-s-1/2}\log\langle\eta\rangle^{-3/4} d\sigma d\eta \Big\|_{L^2_{\xi\geq 2}L^2_{-\xi-1\leq\tau\leq-\xi+1}} \\
&\gtrsim \Big\| \langle\xi\rangle^{-1/2}\log\langle\xi\rangle^{-3/4} \int_0^{\xi} \langle\eta\rangle^{-1}\log\langle\eta\rangle^{-3/4} d\eta \Big\|_{L^2_{\xi\geq 2}} = \infty,
\end{aligned}
$$

and the remainders are estimated similarly.

4. Proof of Theorem 1

We separate the proof for the existence and for the persistence of regularity.

4.1. Proof of existence. Let $s \geq 0$, $(u_0, v_0) \in H^s \times H^s$ and let $0 < T \leq 1$. We define $\Phi : (u, v) \mapsto (\Phi_1(u, v),\ \Phi_2(u, v))$ as

$$
\begin{cases}
(\Phi_1(u,v))(t) = U_m(-t)u_0 - i\lambda \int_0^t U_m(t'-t)\,\overline{u(t')}\,v(t')dt', \\
(\Phi_2(u,v))(t) = U_M(t)v_0 - i\mu \int_0^t U_M(t-t')\,u(t')^2 dt'.
\end{cases}
\tag{4.1}
$$

We also define a metric space

$$
B^s(R, [0,T]) = \Big\{ (u,v) \in \mathcal{X}^{s,1/2}[0,T];\ \|(u,v)\|_{\mathcal{X}^{s,1/2}[0,T]} \leq R \Big\}
$$

with metric

$$
\begin{aligned}
d^s(\,(u_1,v_1),\ (u_2,v_2)\,) &= \|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{X}^{s,1/2}[0,T]} \\
&= \|u_1-u_2\|_{X_-^{s,1/2}[0,T]} + \|v_1-v_2\|_{X_+^{s,1/2}[0,T]}.
\end{aligned}
$$

We see $(B^s(R,[0,T]), d^s)$ is a complete metric space for any $s \geq 0$. We prove that Φ is a contraction map on $B^s(R,[0,T])$ for sufficiently large R and sufficiently small T.

Let $(u,v) \in B^s(R,[0,T])$ and let $(u',v') \in X_-^{s,1/2} \times X_+^{s,1/2}$ satisfy

$$\begin{aligned} u' &= u \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} u' \subset [-2T,2T]\times\mathbb{R},\\ v' &= v \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} v' \subset [-2T,2T]\times\mathbb{R}. \end{aligned}$$

Then $\Phi_1(u,v)$ and $\Phi_2(u,v)$ are defined on $[0,T]\times\mathbb{R}$. Moreover,

$$\begin{aligned} \psi_T(t)\int_0^t U_m(t'-t)\,\overline{u'(t')}\,v'(t')\,dt' &= \int_0^t U_m(t'-t)\,\overline{u(t')}\,v(t')dt',\\ \psi_T(t)\int_0^t U_M(t-t')\,u'(t')^2\,dt' &= \int_0^t U_M(t-t')\,u(t')^2\,dt' \end{aligned}$$

on $[0,T]\times\mathbb{R}$ and their supports are contained in $[-2T,2T]\times\mathbb{R}$. Then,

$$\begin{aligned} &\|\Phi_1(u,v)\|_{X_-^{s,1/2}[0,T]}\\ &\leq \|U_m(-t)\,u_0\|_{X_-^{s,1/2}[0,T]} + \left\|\lambda\int_0^t U_m(t'-t)\,\overline{u(t')}\,v(t')dt'\right\|_{X_-^{s,1/2}[0,T]}. \end{aligned}$$

By Proposition 1,

$$\|U_m(-t)\,u_0\|_{X_-^{s,1/2}[0,T]} \leq \|\psi_T(t)U_m(-t)\,u_0\|_{X_-^{s,1/2}} \lesssim \|u_0\|_{H^s}.$$

By Proposition 1 and Corollary 1, for $0<\rho<1/2$,

$$\begin{aligned} &\left\|\int_0^t U_m(t'-t)\,\overline{u(t')}\,v(t')\,dt'\right\|_{X_-^{s,1/2}[0,T]}\\ &\leq \inf_{u',v'}\left\|\psi_T(t)\int_0^t U_m(t'-t)\,\overline{u'(t')}\,v'(t')\,dt'\right\|_{X_-^{s,1/2}}\\ &\lesssim \inf_{u',v'}\left\|\overline{u'}\,v'\right\|_{X_-^{s,-1/2}\cap Y_-^s}\\ &\lesssim \inf_{u',v'} T^\rho\,\|u'\|_{X_-^{s,1/2}}\,\|v'\|_{X_+^{s,1/2}}\\ &\lesssim T^\rho\,\|u\|_{X_-^{s,1/2}[0,T]}\,\|v\|_{X_+^{s,1/2}[0,T]} \leq T^\rho R^2 \end{aligned}$$

for $0<\rho<1/2$. Similarly,

$$\|\Phi_2(u,v)\|_{X_+^{s,1/2}[0,T]} \lesssim \|v_0\|_{H^s} + T^\rho R^2.$$

This implies that Φ is a map from $B^s(R,[0,T])$ into itself for some R and T. Moreover, let $(u_j,v_j) \in B^s(R,[0,T])$ for $j=1,2$ and let $(u_j',v_j') \in X_-^s \times X_+^s$ satisfy

$$\begin{aligned} u_j' &= u_j \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} u_j' \subset [-2T,2T]\times\mathbb{R},\\ v_j' &= v_j \ \text{ on } \ [0,T]\times\mathbb{R}, \quad \operatorname{supp} v_j' \subset [-2T,2T]\times\mathbb{R}. \end{aligned}$$

We have

$$
\begin{aligned}
&\|\Phi_1(u_1, v_1) - \Phi_1(u_2, v_2)\|_{X_-^{s,1/2}[0,T]} \\
&\lesssim \inf_{u_1',u_2',v_1',v_2'} \left\{ \|\overline{(u_1' - u_2')}\, v_1'\|_{X_-^{s,-1/2} \cap Y_-^s} \left\| \overline{u_2'}\, (v_1' - v_2') \right\|_{X_-^{s,-1/2} \cap Y_-^s} \right\} \\
&\le T^\rho \inf_{u_1'-u_2',v_1'} \left\{ \|v_1'\|_{X_+^{s,1/2}} \|u_1' - u_2'\|_{X_-^{s,1/2}} \right\} \\
&\quad + T^\rho \inf_{u_2',v_1'-v_2'} \left\{ \|u_2'\|_{X_-^{s,1/2}} \|v_1' - v_2'\|_{X_+^{s,1/2}} \right\} \\
&\lesssim T^\rho R \, \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{X}^{s,1/2}[0,T]} .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|\Phi_2(u_1, v_1) - \Phi_2(u_2, v_2)\|_{X_+^{s,1/2}} \\
&\lesssim T^\rho \inf_{u_1',u_2'} \|u_1' + u_2'\|_{X_-^{s,1/2}} \|u_1' - u_2'\|_{X_-^{s,1/2}} \\
&\lesssim T^\rho \|u_1 + u_2\|_{X_-^{s,1/2}[0,T]} \|u_1 - u_2\|_{X_-^{s,1/2}[0,T]} \\
&\lesssim T^\rho R \, \|(u_1, u_2) - (v_1, v_2)\|_{\mathcal{X}^{s,1/2}[0,T]} .
\end{aligned}
$$

Therefore Φ is a contraction map on $B^s(R, [0, T])$ with sufficiently small T.

4.2. Proof of persistence regularity. Let $s \ge 0$ and let $(u_0, v_0) \in H^s \times H^s$. By the previous subsection, we have the maximal existence time $T(s') > 0$ for $0 \le s' \le s$ such that there is a unique pair of local solutions $(u, v) \in C([0, T(s')), H^{s'} \times H^{s'})$. Since $s \ge \lambda(s)$, we have $T(s) \le T(\lambda(s))$, where $\lambda(s)$ is as in (3.1). We show that if $T(s) < T(\lambda(s))$, then

$$
\sup_{t \in [0, T(s))} \|(u, v)(t)\|_{H^s \times H^s} < \infty, \tag{4.2}
$$

namely, $T(s) = T(\lambda(s))$. Let $T_1 = 1 \wedge \frac{T(\lambda(s)) - T(s)}{2}$. For sufficiently large c, we define $R_1 > 0$ as follows

$$
R_1 = 2c \left(1 + \sup_{t \in [0, T(s) + T_1]} \|(u, v)(t)\|_{H^{\lambda(s)} \times H^{\lambda(s)}} \right) < \infty.
$$

We have $0 < T_2 < T_1$ such that for any $0 < T_0 < T(s)$ and any $0 < T < T_2$, Φ is a contraction map on $B^{\lambda(s)}(R_1, [T_0, T_0 + T])$. Let $0 < \rho < 1/2$, and let $(u_j, v_j) \in B^{\lambda(s)}(R_1, [T_0, T_0 + T])$. Let $u_j' \in X_-^{s,1/2}$, $u_j'' \in X_-^{\lambda(s),1/2}$, $v_j' \in X_+^{s,1/2}$, $v_j'' \in X_+^{\lambda(s),1/2}$ satisfy

$$
\begin{aligned}
u_j' &= u_j \ \text{ on } \ [T_0, T_0 + T] \times \mathbb{R}, && \operatorname{supp} u_j' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R}, \\
u_j'' &= u_j \ \text{ on } \ [T_0, T_0 + T] \times \mathbb{R}, && \operatorname{supp} u_j'' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R}, \\
v_j' &= v_j \ \text{ on } \ [T_0, T_0 + T] \times \mathbb{R}, && \operatorname{supp} v_j' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R}, \\
v_j'' &= v_j \ \text{ on } \ [T_0, T_0 + T] \times \mathbb{R}, && \operatorname{supp} v_j'' \subset [T_0 - 2T, T_0 + 2T] \times \mathbb{R}
\end{aligned}
$$

for $j = 1, 2$. Then by Proposition 2, for $0 < \rho < 1/2$,

$$\begin{aligned}\|\Phi_1(u_1, v_1)\|_{X_-^{s,1/2}[T_0,T_0+T]} &\le \|U_m(-t)u(T_0)\|_{X_-^{s,1/2}[T_0,T_0+T]} \\ &\quad + \left\|\lambda \int_{T_0}^{t} U_m(t'-t)\overline{u_1(t')}\, v_1(t')\, dt'\right\|_{X_-^{s,1/2}[T_0,T_0+T]} \\ &\le c\|u(T_0)\|_{H^s} + cT^\rho \inf_{u_1'', v_1'} \|u_1''\|_{X_-^{\lambda(s),1/2}} \|v_1'\|_{X_+^{s,1/2}} \\ &\le c\|u(T_0)\|_{H^s} + cT^\rho R_1 \|v_1\|_{X_+^{s,1/2}[T_0,\ T_0+T]}\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\Phi_2(u_1, v_1)\|_{X_+^{s,1/2}[T_0,T_0+T]} &\le \|U_M(t)v(T_0)\|_{X_+^{s,1/2}[T_0,T_0+T]} \\ &\quad + \left\|\mu \int_{T_0}^{t} U_M(t-t')u_1(t')^2\, dt'\right\|_{X_+^{s,1/2}[T_0,T_0+T]} \\ &\le c\|v(T_0)\|_{H^s} + cT^\rho \inf_{u_1'', u_1'} \|u_1''\|_{X_-^{\lambda(s),1/2}} \|u_1'\|_{X_-^{s,1/2}} \\ &\le c\|v(T_0)\|_{H^s} + cT^\rho R_1 \|u_1\|_{X_-^{s,1/2}[T_0,T_0+T]}.\end{aligned}$$

Let

$$R_2(T_0) = 2c\{1 + \|u(T_0)\|_{H^s} + \|v(T_0)\|_{H^s}\}$$

and let

$$T_3 = T_2 \wedge (8cR_1)^{-1/\rho} \wedge (T(s) - T_0).$$

Then for $0 < T < T_3$, Φ is a map on

$$B^{\lambda(s)}(R_1, [T_0, T_0 + T]) \cap B^s(R_2(T_0), [T_0, T_0 + T]).$$

In addition,

$$\begin{aligned}&\|\Phi_1(u_1, v_1) - \Phi_1(u_2, v_2)\|_{X_-^{s,1/2}[T_0,T_0+T]} \\ &\le \left\|\lambda \int_{T_0}^{t} U_m(t'-t)\overline{u_1(t')}\left\{v_1(t') - v_2(t')\right\} dt'\right\|_{X_-^{s,1/2}[T_0,T_0+T]} \\ &\quad + \left\|\lambda \int_{T_0}^{t} U_m(t'-t)v_2(t')\,\overline{\left\{u_1(t') - u_2(t')\right\}}\, dt'\right\|_{X_-^{s,1/2}[T_0,T_0+T]} \\ &\le cT^\rho \inf_{u_1'', v_1'-v_2'} \|u_1''\|_{X_-^{\lambda(s),1/2}} \|v_1' - v_2'\|_{X_+^{s,1/2}} \\ &\quad + cT^\rho \inf_{v_2'', u_1'-u_2'} \|v_2''\|_{X_+^{\lambda(s),1/2}} \|u_1' - u_2'\|_{X_-^{s,1/2}} \\ &\le \frac{1}{4}\|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{X}^{s,1/2}[T_0,\ T_0+T]}\,.\end{aligned}$$

Similarly,

$$\begin{aligned}&\|\Phi_2(u_1, v_1) - \Phi_2(u_2, v_2)\|_{X_+^{s,1/2}[T_0,\ T_0+T]}\\&\le cT^\rho \inf_{u_1',u_1'',u_2',u_2''} \|u_1'' + u_2''\|_{X_-^{\lambda(s),1/2}} \left\|u_1' - u_2'\right\|_{X_-^{s,1/2}}\\&\le \frac{1}{4}\|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{X}^{s,1/2}[T_0,T_0+T]}.\end{aligned}$$

Therefore Φ is a contraction map and the pair of solutions (u, v) is guaranteed in both $\mathcal{X}^{\lambda(s),1/2}[T_0, T_0 + T]$ and $\mathcal{X}^{s,1/2}[T_0, T_0 + T]$. If $(T(s) - T_0) < T_2 \wedge (8cR_1)^{-1/\rho}$, then $T_3 = T(s) - T_0$ and

$$\sup_{T\in[0,T(s)-T_0)} \|(u, v)\|_{\mathcal{X}^{s,1/2}[T_0,\ T_0+T]} \le R_2(T_0),$$

which together with Proposition 2 implies

$$\sup_{T\in[0,T(s)-T_0)} \left\{\|\overline{u}v\|_{Y_-^s[T_0,\ T_0+T]} + \|u^2\|_{Y_+^s[T_0,\ T_0+T]}\right\} \le cR_2(T_0)^2.$$

Then by Lemma 2,

$$\sup_{t\in[T_0,T(s))} \|(u, v)(t)\|_{H^s\times H^s} \le c^2R_2(T_0)^2.$$

Thus, we obtain (4.2) and $T(s) = T(\lambda(s)) = T(0)$.

Acknowledgement. The authors are grateful to the referee for important remarks and suggestions.

Appendix A: Proof of the L^2 Conservation and Theorem 2

In this appendix, we prove the L^2 conservation for Theorem 2.

Although we can justify a formal proof of the L^2 conservation by the approximation argument by smooth solutions, we give a different approach here. We derive the conservation laws without approximation. We derive it in the framework of the Bourgain method, as we studied in the previous sections. For the Schrödinger equation, there is a proof of the conservation laws in the framework of the Strichartz estimate [17]. To our knowledge, the direct proof of conservation law without smooth approximation had not been studied unless the Strichartz estimate holds. We have to justify each step of the calculation. Especially we show the integrability of terms in the argument. We use the following Lemma and Proposition for it.

Lemma 7. *Let p and α satisfy $p \ge 1$ and $0 \le \alpha \le 1/p$. Let β, γ, κ satisfy $0 \le \beta, \gamma, \kappa \le 1/2$ and $\alpha+\beta+\gamma+\kappa = 1/p+1/2+\varepsilon$ with $\varepsilon > 0$. Then there exists a positive constant C such that the inequality*

$$\|\langle\tau + \delta_1\rangle^{-\alpha} f * g * h\|_{L_\tau^p} \le C\|\langle\tau + \delta_2\rangle^\beta f\|_{L_\tau^2}\|\langle\tau + \delta_3\rangle^\gamma g\|_{L_\tau^2}\|\langle\tau + \delta_4\rangle^\kappa h\|_{L_\tau^2}$$

holds for any real numbers $\delta_1, \delta_2, \delta_3, \delta_4$ and any f, g, h such that all the norms on the right hand side are finite.

Proof. By the Hölder and the Young inequalities,

$$\begin{aligned}\|\langle\tau+\delta_1\rangle^{-\alpha} f*g*h\|_{L^p_\tau} &\lesssim \|f*g*h\|_{L^{p_1}}\\ &\lesssim \|f\|_{L^{p_2}}\,\|g*h\|_{L^{p_3}}\\ &\lesssim \|f\|_{L^{p_2}}\,\|g\|_{L^{p_4}}\,\|h\|_{L^{p_5}}\\ &\lesssim \|\langle\tau+\delta_2\rangle^{\beta} f\|_{L^2_\tau}\|\langle\tau+\delta_3\rangle^{\gamma} g\|_{L^2_\tau}\|\langle\tau+\delta_4\rangle^{\kappa} h\|_{L^2_\tau},\end{aligned}$$

where

$$\frac{1}{p_1}=\frac{1}{p}-\alpha+\frac{\alpha\varepsilon}{\alpha+\beta+\gamma+\kappa},\qquad \frac{1}{p_2}=\frac{1}{2}+\beta-\frac{\beta\varepsilon}{\alpha+\beta+\gamma+\kappa},$$

$$\frac{1}{p_3}=\frac{1}{p_1}+1-\frac{1}{p_2},\qquad \frac{1}{p_4}=\frac{1}{2}+\gamma-\frac{\gamma\varepsilon}{\alpha+\beta+\gamma+\kappa},$$

$$\frac{1}{p_5}=\frac{1}{2}+\kappa-\frac{\kappa\varepsilon}{\alpha+\beta+\gamma+\kappa},$$

from which we obtain the lemma. □

Proposition 4.

$$\begin{aligned}&\left\|\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\right\|_{L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta}\\ &\lesssim \|u\|_{X^{0,1/2}_\pm}\,\|v\|_{X^{0,1/2}_\pm}\,\|w\|_{X^{0,1/2}_\pm}\end{aligned}$$

for any $u,v,w\in X^{0,1/2}_\pm$.

Proof. Let

$$N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\tau\big|\vee\big|\sigma-\rho\pm|\xi-\eta|\big|\vee\big|\rho\pm|\eta|\big|\vee\big|\tau-\sigma\pm|\xi|\big|,$$

Then we have $|\xi|+|\xi-\eta|+|\eta|\le 4N$. We also separate the integral region as follows

$$\begin{aligned}B_1&=\big\{(\tau,\sigma,\xi,\rho,\eta)\;;\;N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\tau\big|\big\},\\ B_2&=\big\{(\tau,\sigma,\xi,\rho,\eta)\;;\;N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\sigma-\rho\pm|\xi-\eta|\big|\big\},\\ B_3&=\big\{(\tau,\sigma,\xi,\rho,\eta)\;;\;N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\rho\pm|\eta|\big|\big\},\\ B_4&=\big\{(\tau,\sigma,\xi,\rho,\eta)\;;\;N(\tau,\xi,\sigma,\rho,\varepsilon)=\big|\tau-\sigma\pm|\xi|\big|\big\}.\end{aligned}$$

By Lemmas 5, 7 and the Hölder inequality,

$$\begin{aligned}&\left\|\chi_{B_1}\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\right\|_{L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta}\\ &\lesssim \|\langle\tau\rangle^{-1/2}\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\\ &\qquad\cdot\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta}\\ &\lesssim \left\|\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\|\langle\tau\pm|\xi-\eta|\rangle^{1/2}\widetilde{u}(\tau,\xi-\eta)\|_{L^2_\tau}\|\langle\tau\pm|\eta|\rangle^{1/2}\widetilde{v}(\tau,\eta)\|_{L^2_\tau}\right\|_{L^2_\xi L^1_\eta}\\ &\qquad\|\langle\tau\pm|\xi|\rangle^{1/2}\widetilde{w}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\\ &\lesssim \|u\|_{X^{0,1/2}_\pm}\,\|v\|_{X^{0,1/2}_\pm}\,\|w\|_{X^{0,1/2}_\pm}.\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|\chi_{B_2}\langle\tau\rangle^{-1}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta}\\
&\quad\lesssim \|\langle\tau\rangle^{-1}\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\langle\sigma-\rho\pm|\xi-\eta|\rangle^{1/2}\\
&\qquad\cdot\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{v}(\rho,\eta)\widetilde{w}(\tau-\sigma,\xi)\|_{L^1_\tau L^1_\xi L^1_\sigma L^1_\rho L^1_\eta}\\
&\quad\lesssim \Big\|\langle\xi\rangle^{-1/4}\langle\eta\rangle^{-1/4}\|\langle\tau\pm|\xi-\eta|\rangle^{1/2}\widetilde{u}(\tau,\xi-\eta)\|_{L^2_\tau}\|\langle\tau\pm|\eta|\rangle^{1/2}\widetilde{v}(\tau,\eta)\|_{L^2_\tau}\Big\|_{L^2_\xi L^1_\eta}\\
&\qquad\|\langle\tau\pm|\xi|\rangle^{1/2}\widetilde{w}(\tau,\xi)\|_{L^2_\tau L^2_\xi}\\
&\quad\lesssim \|u\|_{X^{0,1/2}_\pm}\|v\|_{X^{0,1/2}_\pm}\|w\|_{X^{0,1/2}_\pm}.
\end{aligned}$$

The other integrations are estimated similarly. □

Then we show the charge conservation with Proposition 4.

Let $(u_0, v_0) \in L^2 \times L^2$ and let $T > 0$ sufficiently small. Then we have a pair of extensions $(u, v) \in X^{0,1/2}_- \times X^{0,1/2}_+$ of the solutions for the Cauchy problem (1.1) such that for any $t \in [0, T]$,

$$u(t) = U_m(-t)u_0 - i\lambda\int_0^t U_m(t'-t)\overline{u(t')}v(t')dt',$$
$$v(t) = U_M(t)v_0 - ic^{-1}\overline{\lambda}\int_0^t U_M(t-t')u(t')^2dt'.$$

Then

$$\begin{aligned}
\|u(t)\|^2_{L^2} &= \|U_m(t)u\|^2_{L^2}\\
&= \left\|u_0 - i\lambda\int_0^t U_m(t')\overline{u(t')}v(t')dt'\right\|^2_{L^2}\\
&= \|u_0\|^2_{L^2} - 2\mathrm{Im}\left(\hat{u}_0, \lambda\int_0^t \mathfrak{F}_x\left[U_m(t')\overline{u(t')}v(t')\right]dt'\right)\\
&\quad+\left\|\lambda\int_0^t \mathfrak{F}_x[U_m(t')\overline{u}(t')v(t')]dt'\right\|^2_{L^2},
\end{aligned}$$

where $(\cdot,\cdot)$ is the $L^2(\mathbb{R})$ inner product. We have

$$\int_0^t f(t')dt' = \int\frac{\exp[it\tau]-1}{i\tau}\hat{f}(\tau)d\tau$$

for any $f \in L^1$ such that $\hat{f} \in \langle\tau\rangle L^1_\tau$. Moreover, the inequalities

$$\|\mathfrak{F}_x\,[\overline{u}v]\|_{L^\infty_\xi L^1_t} \le \|u\|_{L^2L^2}\,\|v\|_{L^2L^2} \le \|u\|_{X^{0,1/2}_-}\,\|v\|_{X^{0.1/2}_+}$$

hold by the Hölder inequality and

$$\iiiint\frac{\exp[it\cdot]-1}{i\cdot}\overline{\widetilde{u}(\rho-\sigma,\eta-\xi)}\widetilde{v}(\rho,\eta)\overline{\widetilde{u}(\sigma-\cdot,\xi)}d\xi d\sigma d\eta d\rho \in L^1$$

by Proposition 4. Then

$$\begin{aligned}
&\left\|\lambda \int_0^t \mathfrak{F}_x \left[U_m(t')\overline{u}(t')v(t')\right] dt'\right\|_{L^2}^2 \\
&= 2\mathrm{Re} \int \int_0^t \lambda \mathfrak{F}_x[\overline{u}v](t') \overline{\lambda \mathfrak{F}_x \left[\int_0^{t'} U_m(t''-t')\overline{u}(t'')v(t'')dt''\right]} dt'd\xi \\
&= -2\mathrm{Im} \int \int_0^t \lambda \mathfrak{F}_x[\overline{u(t')}v(t')] \overline{\mathfrak{F}_x[U_m(-t')]\hat{u}_0 - \mathfrak{F}_x\left[u(t')\right]} dt'd\xi \\
&= 2\mathrm{Im}\left(\hat{u}_0,\ \lambda \int_0^t \mathfrak{F}_x[U_m(t')\overline{u(t')}v(t')]dt'\right) \\
&\quad + 2\mathrm{Im}\,\lambda \iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau} \overline{\widetilde{u}(\rho-\sigma,\eta-\xi)}\widetilde{v}(\rho,\eta)\overline{\widetilde{u}(\sigma-\tau,\xi)} d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 - \|u_0\|_{L^2}^2 \\
&\quad = 2\mathrm{Im}\,\lambda \iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau} \overline{\widetilde{u}(\rho-\sigma,\eta-\xi)\widetilde{u}(\sigma-\tau,\xi)}\widetilde{v}(\rho,\eta) d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|v(t)\|_{L^2}^2 = \|U_M(-t)v\|_{L^2}^2 = \|v_0\|_{L^2} - 2\mathrm{Im}\left(\hat{v}_0,\ c^{-1}\overline{\lambda} \int_0^t \mathfrak{F}_x\left[U_M(-t')u(t')^2\right] dt'\right) \\
+ \left\|c^{-1}\overline{\lambda} \int_0^t \mathfrak{F}_x\left[U_M(-t')u(t')^2 dt'\right]\right\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
&\left\|c^{-1}\overline{\lambda} \int_0^t \mathfrak{F}_x\left[U_M(-t')u(t')^2\right] dt'\right\|_{L^2}^2 \\
&= -2\mathrm{Im} \int \int_0^t c^{-1}\overline{\lambda}\mathfrak{F}_x[u(t')^2] \overline{\mathfrak{F}_x \left[ic^{-1}\overline{\lambda} \int_0^{t'} U_M(t'-t'')u(t'')^2 dt''\right]} dt'd\xi \\
&= 2\mathrm{Im}\left(\hat{v}_0,\ c^{-1}\overline{\lambda} \int_0^t \mathfrak{F}_x\left[U_M(-t')u(t')^2\right] dt'\right) \\
&\quad + \frac{2}{c}\mathrm{Im}\,\overline{\lambda} \iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau} \widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)} d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}$$

Then

$$\begin{aligned}
&\|v(t)\|_{L^2}^2 - \|v_0\|_{L^2}^2 \\
&\quad = 2c^{-1}\mathrm{Im}\,\overline{\lambda} \iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau} \widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)} d\tau d\xi d\sigma d\eta d\rho.
\end{aligned}$$

In addition,

$$
\begin{aligned}
&-\operatorname{Im}\bar{\lambda}\iiiint\!\!\int \frac{\exp[it\tau]-1}{i\tau}\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)\overline{\widetilde{v}(\sigma-\tau,\xi)}d\tau d\xi d\sigma d\eta d\rho\\
&\quad=\operatorname{Im}\lambda\iiiint\!\!\int \frac{\exp[-it\tau]-1}{i(-\tau)}\overline{\widetilde{u}(\sigma-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)}\,\widetilde{v}(\sigma-\tau,\xi)d\xi d\sigma d\eta d\rho d\tau\\
&\quad=\operatorname{Im}\lambda\iiiint\!\!\int \frac{\exp[-it\tau]-1}{i(-\tau)}\overline{\widetilde{u}(\tau+\rho'-\rho,\xi-\eta)\widetilde{u}(\rho,\eta)}\,\widetilde{v}(\rho',\xi)d\xi d\sigma d\eta d\rho' d\tau\\
&\quad=\operatorname{Im}\lambda\iiiint\!\!\int \frac{\exp[it\tau']-1}{i\tau'}\overline{\widetilde{u}(\rho'-\sigma',\xi-\eta)\widetilde{u}(\sigma'-\tau',\eta)}\,\widetilde{v}(\rho',\xi)d\xi d\sigma' d\eta d\rho' d\tau'\\
&\quad=\operatorname{Im}\lambda\iiiint\!\!\int \frac{\exp[it\tau']-1}{i\tau'}\\
&\qquad\times\overline{\widetilde{u}(\rho'-\sigma',\eta'-\xi')\widetilde{u}(\sigma'-\tau',\xi')}\,\widetilde{v}(\rho',\eta')\,d\xi' d\sigma' d\eta' d\rho' d\tau',
\end{aligned}
$$

where $\rho'=\sigma-\tau$, $\sigma'=\rho-\tau$, $\tau'=-\tau$, $\xi'=\eta$, and $\eta'=\xi$. Finally we have

$$\|u(t)\|_{L^2}^2+c\|v(t)\|_{L^2}^2=\|u_0\|_{L^2}^2+c\|v_0\|_{L^2}^2$$

for $t\in[0,T]$.

Appendix B: Proof of Local Well-Posedness Independent of *Y* Norm

In this Appendix, we clarify why the auxiliary space Y is important in our argument. We give an alternative proof of the existence of solutions for $s>0$, without using the auxiliary norm Y. On the other hand, we shall explain why we need the norm Y at least in our argument in the case $s=0$. It is important that $\delta(s)$ in this proof below is strictly positive. We exchange it into the positive power of T. Then the contraction argument is completed when T is sufficiently small.

For the alternative proof, we use the following proposition.

Proposition 5. *Let $\varepsilon>0$, $\rho\ge 0$, $b,\delta\in\mathbb{R}$ satisfy*

$$
\begin{aligned}
&1+b-\delta>\frac{1}{2}+\varepsilon+\rho,\\
&b+\delta+\varepsilon,\ \rho+\delta+\varepsilon\le 1,\\
&b-\varepsilon,\ b-\rho\ge 0,\\
&s+\varepsilon\ge 1/2.
\end{aligned}
$$

Then

$$\|uv\|_{X_{\pm}^{s,b-1+\delta}}\lesssim\|u\|_{X_{\mp}^{s,b}}\ \|v\|_{X_{\mp}^{s,b-\rho}}+\|u\|_{X_{\mp}^{s,b-\rho}}\ \|v\|_{X_{\mp}^{s,b}}\tag{B.1}$$

for any $u,v\in X_{\mp}^{s,b}$.

Remark 5. $b=1/2$, $\delta=0$, $\varepsilon=1/2$ are the only numbers that ensures (B.1) for $s=0$. For detail, see Proposition 6.

Proof. We use the same notation as in the proof of Proposition 2. Since $|\xi|, |\xi-\eta|, |\eta| \leq 3M(\tau, \xi, \sigma, \eta)$ and Lemma 6,

$$\left\| \langle \xi \rangle^s \iint \langle \tau + |\xi| \rangle^{b-1+\delta} \chi_{A_1}\, \widetilde{u}(\tau-\sigma, \xi-\eta)\, \widetilde{v}(\sigma, \eta)\, d\sigma d\eta \right\|_{L^2_\tau L^2_\xi}$$
$$\lesssim \left\| \int \langle \xi \rangle^{s-\varepsilon/3} \langle \xi - \eta \rangle^{-\varepsilon/3} \langle \eta \rangle^{-\varepsilon/3} K_1\, d\eta \right\|_{L^2_\xi},$$

where

$$K_1 = \left\| \langle \tau + |\xi| \rangle^{b-1+\delta+\varepsilon} \int |\widetilde{u}(\tau-\sigma, \xi-\eta)\, \widetilde{v}(\sigma, \eta)|\, d\sigma \right\|_{L^2_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{b-\rho} \widetilde{u}(\tau, \xi-\eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{b} \widetilde{v}(\tau, \eta) \right\|_{L^2_\tau}.$$

Similarly, for $j = 2, 3$,

$$\left\| \langle \xi \rangle^s \iint \langle \tau + |\xi| \rangle^{b-1+\delta} \chi_{A_j}\, \widetilde{u}(\tau-\sigma, \xi-\eta)\, \widetilde{v}(\sigma, \eta)\, d\sigma d\eta \right\|_{L^2_\tau L^2_\xi}$$
$$\lesssim \left\| \int \langle \xi \rangle^{s-\varepsilon/3} \langle \xi - \eta \rangle^{-\varepsilon/3} \langle \eta \rangle^{-\varepsilon/3} K_j\, d\eta \right\|_{L^2_\xi},$$

where

$$K_2 = \left\| \langle \tau + |\xi| \rangle^{b-1+\delta} \int \langle \tau - |\xi-\eta| \rangle^{\varepsilon} |\widetilde{u}(\tau-\sigma, \xi-\eta)\, \widetilde{v}(\sigma, \eta)|\, d\sigma \right\|_{L^2_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{b} \widetilde{u}(\tau, \xi-\eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{b-\rho} \widetilde{v}(\tau, \eta) \right\|_{L^2_\tau},$$
$$K_3 = \left\| \langle \tau + |\xi| \rangle^{b-1+\delta} \int \left| \widetilde{u}(\tau-\sigma, \xi-\eta)\, \langle \tau - |\eta| \rangle^{\varepsilon} \widetilde{v}(\sigma, \eta) \right| d\sigma \right\|_{L^2_\tau}$$
$$\lesssim \left\| \langle \tau - |\xi-\eta| \rangle^{b-\rho} \widetilde{u}(\tau, \xi-\eta) \right\|_{L^2_\tau} \left\| \langle \tau - |\eta| \rangle^{b} \widetilde{v}(\tau, \eta) \right\|_{L^2_\tau}.$$

We obtain (B.1) by Lemma 5. □

Proof [The alternative proof of Theorem 1 for $s > 0$]. Let $s > 0$, $(u_0, v_0) \in H^s \times H^s$ and let $0 < T \leq 1$. We take $b(s) = 3/4 \wedge (1+s)/2 > 1/2$ and $\delta(s) = 1/4 \wedge s/2 > 0$ for Proposition 5.

Let

$$\|u\|_{X'^{s,b}_{m,\pm}[T_0, T_0+T]} = \inf \left\{ \|u'\|_{X^{s,b}_{m,\pm}} \ ; u'(t,x) = u(t,x) \text{ on } [T_0, T_0+T] \times \mathbb{R} \right\}.$$

and

$$\mathcal{X}'^{s,b}[T_0, T_0+T] = X'^{s,b}_{-}[T_0, T_0+T] \times X'^{s,b}_{+}[T_0, T_0+T].$$

We define a metric space

$$B'^{s}(R, T) = \left\{ (u, v) \in \mathcal{X}'^{s,b(s)}[0, T] \ ; \ \|u\|_{X'^{s,b(s)}_{-}[0,T]} + \|v\|_{X'^{s,b(s)}_{+}[0,T]} \leq R \right\}$$

with metric

$$d'(\,(u_1, v_1),\ (u_2, v_2)\,) = \|(u_1, v_1) - (u_2, v_2)\|_{\mathcal{X}'^{s,b(s)}[0,T]}\,.$$

We see $(B'^s(R,T), d')$ is a complete metric space. We prove that Φ defined as (4.1) is a contraction map on $B'^s(R,T)$ for sufficiently large R and sufficiently small T.

Let $(u,v) \in B'^s(R,T)$ and let $(u',v') \in X_-^{s,b(s)} \times X_+^{s,b}$ satisfy

$$u' = u \ \text{ on } \ [0,T]\times\mathbb{R}, \quad v' = v \ \text{ on } \ [0,T]\times\mathbb{R}.$$

We have

$$\begin{aligned}
&\|\Phi_1(u,v)\|_{X_-'^{s,b(s)}[0,T]} \\
&\le \|U_m(-t)\,u_0\|_{X_-'^{s,b(s)}[0,T]} + \left\|\lambda\int_0^t U_m(t'-t)\,\overline{u(t')}\,v(t')dt'\right\|_{X_-'^{s,b(s)}[0,T]}.
\end{aligned}$$

By Lemma 1,

$$\|U_m(-t)\,u_0\|_{X_-'^{s,b}[0,T]} \le \|\psi(t)U_m(-t)\,u_0\|_{X_-'^{s,b}} \lesssim \|u_0\|_{H^s}.$$

By Propositions 1 and 5, we obtain

$$\begin{aligned}
&\left\|\int_0^t U_m(t'-t)\,\overline{u(t')}\,v(t')\,dt'\right\|_{X_-'^{s,b(s)}[0,T]} \\
&\le \inf_{u',v'}\left\|\psi_T\int_0^t U_m(t'-t)\,\overline{u'(t')}\,v'(t')\,dt'\right\|_{X_-^{s,b(s)}} \\
&\lesssim \inf_{u',v'}\ T^{\delta(s)}\left\|\overline{u'}\,v'\right\|_{X_-^{s,b(s)-1+\delta(s)}} \\
&\lesssim \inf_{u',v'}\ T^{\delta(s)}\left\|u'\right\|_{X_-^{s,b(s)}}\ \left\|v'\right\|_{X_+^{s,b(s)}} \\
&\lesssim T^{\delta(s)}\,\|u\|_{X_-'^{s,b(s)}[0,T]}\ \|v\|_{X_+'^{s,b(s)}[0,T]} \le T^{\delta(s)}R^2.
\end{aligned}$$

Similarly,

$$\|\Phi_2(u,v)\|_{X_+'^{s,b(s)}[0,T]} \lesssim \|v_0\|_{H^s} + T^{\delta(s)}R^2.$$

Thus, Φ is a map from $B'^s(R,T)$ to $B'^s(R,T)$ for some R and T. Moreover, let $(u_j, v_j) \in B'^s(R,T)$ for $j=1,2$ and let $(u_j', v_j') \in X_-^{s,b(s)} \times X_+^{s,b(s)}$ satisfy

$$u_j' = u_j \ \text{ on } \ [0,T]\times\mathbb{R}, \qquad v_j' = v_j \ \text{ on } \ [0,T]\times\mathbb{R}.$$

Then we have

$$\begin{aligned}
&\|\Phi_1(u_1,v_1) - \Phi_1(u_2,v_2)\|_{X_-'^{s,b(s)}[0,T]} \\
&\lesssim \inf_{u_1',u_2',v_1',v_2'} T^{\delta(s)}\left\{\left\|\overline{(u_1'-u_2')}\,v_1'\right\|_{X_-^{s,b(s)-1+\delta(s)}} + \left\|\overline{u_2'}\,(v_1'-v_2')\right\|_{X_-^{s,b(s)-1+\delta(s)}}\right\} \\
&\lesssim T^{\delta(s)} \inf_{u_1',u_2',v_1',v_2'}\left\{\|v_1'\|_{X_+^{s,b(s)}}\|u_1'-u_2'\|_{X_-^{s,b(s)}} + \|u_2'\|_{X_-^{s,b(s)}}\|v_1'-v_2'\|_{X_+^{s,b(s)}}\right\} \\
&\lesssim T^{\delta(s)}R\,\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{X}'^{s,b(s)}[0,T]}\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\Phi_2(u_1, v_1) - \Phi_2(u_2, v_2)\|_{X_+^{\prime s,b(s)}} \\
&\lesssim T^{\delta(s)} \inf_{u_1', u_2'} \left\|u_1' + u_2'\right\|_{X_-^{s,b(s)}} \left\|u_1' - u_2'\right\|_{X_-^{s,b(s)}} \\
&\lesssim T^{\delta(s)} \|u_1 + u_2\|_{X_-^{\prime s,b(s)}[0,T]} \|u_1 - u_2\|_{X_-^{\prime s,b(s)}[0,T]} \\
&\lesssim T^{\delta(s)} R \,\|(u_1, u_2) - (v_1, v_2)\|_{\mathcal{X}^{\prime s,b(s)}[0,T]} .
\end{aligned}$$

Thus, Φ is a contraction map on $B'^s(R, T)$ for sufficiently small T. □

The following proposition implies that we can not take $\delta > 0$ when $s = 0$ in the above proof.

Proposition 6. *For any* $b \in [0, 1/2) \cup (1/2, 1]$*, there exists a pair* $(u, v) \in X_-^{0,b} \times X_-^{0,b}$ *such that*

$$\|uv\|_{X_+^{0,b-1}} = \infty. \tag{B.2}$$

Also for any $\delta > 0$*, there exists a pair* $(u, v) \in X_-^{0,1/2} \times X_-^{0,1/2}$ *such that*

$$\|uv\|_{X_+^{0,-1/2+\delta}} = \infty. \tag{B.3}$$

Remark 6. This is the reason why we use not only the norm $X_\pm^{s,b}$ but also the norm $Y_\pm^s$ and support restricted functions to obtain solutions of the Cauchy problem (1.1).

Proof. Suppose $1/2 < b \le 1$. Let $0 < 2\varepsilon \le b - 1/2$ and let

$$\widetilde{u_1}(\tau, \xi) = \widetilde{v_1}(\tau, \xi) = \langle\xi\rangle^{-\frac{1}{2}-\varepsilon}\langle\tau - |\xi|\rangle^{-b-1/2-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$\begin{aligned}
&\langle\tau + |\xi|\rangle^{b-1} \\
&\quad\cdot \iint \langle\eta\rangle^{-1/2-\varepsilon}\langle\xi - \eta\rangle^{-1/2-\varepsilon}\langle\sigma - |\eta|\rangle^{-b-1/2-\varepsilon}\langle\tau - \sigma - |\xi - \eta|\rangle^{-b-1/2-\varepsilon} d\sigma d\eta \\
&\gtrsim \langle\tau\rangle^{b-1} \int_{\xi/3}^{2\xi/3} \langle\eta\rangle^{-1/2-\varepsilon}\langle\xi - \eta\rangle^{-1/2-\varepsilon}\langle\tau - |\xi - \eta| - |\eta|\rangle^{-2b-2\varepsilon} d\eta \\
&\gtrsim \langle\tau\rangle^{b-1}\langle\xi\rangle^{-1-2\varepsilon} \int_{\xi/3}^{2\xi/3} d\eta \\
&\gtrsim \langle\tau\rangle^{-1/2}.
\end{aligned}$$

This implies $u_1 v_1 \notin X_+^{0,b-1}$.

Moreover, suppose $0 \le b < 1/2$. Let b and δ satisfy $0 < 2\varepsilon \le 1/2 - b$ and let

$$\begin{aligned}
\widetilde{u_2}(\tau, \xi) &= \langle\xi\rangle^{-1/2-\varepsilon}\langle\tau - |\xi|\rangle^{-b-1/2-\varepsilon}, \\
\widetilde{v_2}(\tau, \xi) &= \langle\xi\rangle^{-1/2-\varepsilon}\langle\tau - |\xi|\rangle^{-b}\langle\tau + |\xi|\rangle^{-1/2-\varepsilon}.
\end{aligned}$$

Since for any real number a and b, $\langle a+b\rangle \le \langle a\rangle\langle b\rangle$, for $\xi > 0$,

$$\begin{aligned}
&\langle \tau + |\xi|\rangle^{b-1} \iint \langle\eta\rangle^{-1/2-\varepsilon}\langle\xi-\eta\rangle^{-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon} \\
&\quad \cdot \langle\tau-\sigma-|\xi-\eta|\rangle^{-b}\langle\tau-\sigma+|\xi-\eta|\rangle^{-1/2-\varepsilon} d\sigma d\eta \\
&\gtrsim \langle \tau + |\xi|\rangle^{b-1} \iint \langle\eta\rangle^{-1/2-\varepsilon}\langle\xi-\eta\rangle^{-b-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-b-1/2-\varepsilon} \\
&\quad \cdot \langle\tau-\sigma+|\xi-\eta|\rangle^{-b-1/2-\varepsilon} d\sigma d\eta \\
&\gtrsim \langle\tau+\xi\rangle^{b-1} \int_{-\infty}^{0} \langle\eta\rangle^{-1/2-\varepsilon}\langle\xi-\eta\rangle^{-b-1/2-\varepsilon}\langle\tau+\xi\rangle^{-2b-2\varepsilon} d\eta \\
&\gtrsim \langle\tau+\xi\rangle^{-b-1-2\varepsilon}\langle\xi\rangle^{-1/2} \notin L^2_{\xi>0}(L^2_\tau).
\end{aligned}$$

Therefore $u_2v_2 \notin X_+^{0,b-1}$. We complete the proof of (B.2).

Suppose $\delta > 0$ and $b = 1/2$. Let ε satisfy $0 < 2\varepsilon \le \delta$ and let

$$\widetilde{u_3}(\tau,\xi) = \widetilde{v_3}(\tau,\xi) = \langle\xi\rangle^{-1/2-\varepsilon}\langle\tau-|\xi|\rangle^{-1-\varepsilon}.$$

If $\tau > 2$, $\tau - 1 < \xi < \tau + 1$, then

$$\begin{aligned}
&\langle\tau+|\xi|\rangle^{-1/2+\delta} \\
&\quad \cdot \iint \langle\eta\rangle^{-1/2-\varepsilon}\langle\xi-\eta\rangle^{-1/2-\varepsilon}\langle\sigma-|\eta|\rangle^{-1-\varepsilon}\langle\tau-\sigma-|\xi-\eta|\rangle^{-1-\varepsilon} d\sigma d\eta \\
&\gtrsim \langle\tau\rangle^{-1/2+\delta} \int_{\xi/3}^{2\xi/3} \langle\eta\rangle^{-1/2-\varepsilon}\langle\xi-\eta\rangle^{-1/2-\varepsilon}\langle\tau-|\xi-\eta|-|\eta|\rangle^{-1-2\varepsilon} d\eta \\
&\gtrsim \langle\tau\rangle^{-1/2+\delta}\langle\xi\rangle^{-1-2\varepsilon} \int_{\xi/3}^{2\xi/3} d\eta \\
&\gtrsim \langle\tau\rangle^{-1/2}.
\end{aligned}$$

This yields $u_3v_3 \notin X_+^{0,b-1}$ and we obtain (B.3). □

References

1. Borgna, J.P., Rial, D.F.: Existence of ground states for a one-dimensional relativistic Schrödinger equation. J. Math. Phys. **53**, 062301 (2012)
2. Bournaveas, N.: Local well-posedness for a nonlinear Dirac equation in spaces of almost critical dimension. Discret. Contin. Dyn. Syst. **20**, 605–616 (2008)
3. Bournaveas, N., Candy, T., Machihara, S.: Local and global well posedness for the Chern–Simons–Dirac system in one dimension. Differ. Integral Equ. **25**, 699–718 (2012)
4. Cho, Y., Ozawa, T.: On the semirelativistic Hartree-type equation. SIAM J. Math. Anal. **38**, 1060–1074 (2006)
5. D'Ancona, P., Foschi, D., Selberg, S.: Null structure and almost optimal local regularity for the Dirac–Klein–Gordon system. J. Eur. Math. Soc. **9**, 877–899 (2007)
6. Fröhlich, J., Lenzmann, E.: Blowup for nonlinear wave equations describing boson stars. Commun. Pure Appl. Math. **60**, 1691–1705 (2007)
7. Ginibre, J., Tsutsumi, Y., Velo, G.: On the Cauchy problem for the Zakharov system. J. Funct. Anal. **151**, 384–436 (1997)
8. Hayashi, N., Li, C., Naumkin, P.I.: On a system of nonlinear Schrödinger equations in 2d. Differ. Integral Equ. **24**, 417–434 (2011)

9. Hayashi, N., Li, C., Ozawa, T.: Small data scattering for a system of nonlinear Schrödinger equations. Differ. Equ. Appl. **3**, 415–426 (2011)
10. Hayashi, N., Ozawa, T., Tanaka, K.: On a system of nonlinear Schrödinger equations with quadratic interaction. Ann. I. H. Poincaré **30**, 661–690 (2013)
11. Hoshino, G., Ozawa, T.: Analytic smoothing effect for a system of nonlinear Schrödinger equations. Differ. Equ. Appl. **5**, 395–408 (2013)
12. Krieger, J., Lenzmann, E., Raphaël, P.: Nondispersive solutions to the L^2-critical half-wave equation. Arch. Ration. Mech. Anal. **209**, 61–129 (2013)
13. Lenzmann, E.: Well-posedness for semi-relaivistic Hartree equations of critical type. Math. Phys. Anal. Geom. **10**, 43–64 (2007)
14. Machihara, S., Nakanishi, K., Tsugawa, K.: Well-posedness for nonlinear Dirac equations in one dimension. Kyoto J. Math. **50**, 403–451 (2010)
15. Ogawa, T.: A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations. Nonlinear Anal. **14**, 765–769 (1990)
16. Ogawa, T., Ozawa, T.: Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem. J. Math. Anal. Appl. **155**, 531–540 (1991)
17. Ozawa, T.: Remarks on proofs of conservation laws for nonlinear Schrödinger equations. Calc. Var. Partial Differ. Equ. **25**, 403–408 (2006)
18. Selberg, S., Tesfahun, A.: Low regularity well-posedness of the Dirac–Klein–Gordon equations in one space dimension. Commun. Contemp. Math. **10**, 181–194 (2008)
19. Vladimirov, M.V.: On the solvability of a mixed problem for a nonlinear equation of Schrödinger type. Dokl. Akad. Nauk SSSR **275**, 780–783 (1984)

Communicated by W. Schlag