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Lagrangian Reductions and Integrable Systems in Condensed Matter

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Abstract: We consider a general approach for the process of Lagrangian and Hamiltonian reduction by symmetries in chiral gauge models. This approach is used to show the complete integrability of several one dimensional texture equations arising in liquid Helium phases and neutron stars.

Contents

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1. Introduction

There is a well established relation between quantum field theory and condensed matter physics. For example, physical phenomena such as superfluidity and superconductivity are the manifestation of quantum effects at the microscopic level. On the other hand, there are classical systems, such as liquid crystals, where many phenomena, such as phase transitions between different mesophases, are described in the framework of the Landau–de Gennes theory. These systems include superfluid ^{3}He, the superfluid core of neutron stars, biaxial and uniaxial nematics.

The common feature of these different systems is that in some interval of the transition temperature, their behavior is determined by the Ginzburg–Landau equation with multidimensional order parameters. Another interesting feature of these systems is the existence of different thermodynamic phases. The description of phase transitions between different phases is a difficult and important problem in condensed matter physics. The approach, based on the identification of thermodynamic phases with orbits of the group of symmetry of the potential in the free energy, as developed in [5–7], and [1], is a useful tool for a complete classification of phases and gives a global description of these phases. From many points of view, it is possible to study these systems like a chiral field model. Viewed this way, it is known that these systems can be obtained by a general reduction procedure developed in [4], based on [2,3,8]. The goal of this paper is to unify these two approaches and techniques, to formulate a theory that contains them both, and, especially, to show its effectiveness by studying in detail the complete integrability of two systems arising in condensed matter theory: superfluid phases in ^{3}He and neutron stars.

We begin with a short review of the relevant facts of the Lagrange–Poincaré and Euler–Poincaré variational principles in Sect. 2. Its Hamiltonian counterpart, Hamilton–Poincaré and Lie–Poisson reduction, are treated in Sect. 3. We limit ourselves to the classical, as opposed to the field theoretical, description of these theories, because all examples presented in this paper necessitate only the classical theory. The field theoretical approach, which we have also developed, will be the subject of another paper. The main result of these sections is an equivalence of the two descriptions. These results are used in Sect. 4, forming the main body of the paper, to study in detail the behavior of superfluid ^{3}He and neutron star cores in different phases. The versatility of passing from one description to another, as well as between the Lagrangian and Hamiltonian formulations, is crucial in the proof of the complete integrability of the equations associated to different phases. The key to the success of our geometric method is the fact that all physical systems under study have a natural Lagrangian and Hamiltonian formulation within the Lagrange–Poincaré and Hamilton–Poincaré theories, with the Lagrangian and Hamiltonian independent on a very special group of variables. This implies that these systems have an equivalent Euler–Poincaré and Lie–Poisson description, which turns out to be considerably simpler and more appropriate to the study of the dynamics of the equations associated to the relevant phases. The possibility of using at once the four descriptions of the systems under consideration leads directly to the proof of complete integrability of the equations describing the system's behavior in different phases.

2. Lagrange–Poincaré and Euler–Poincaré Reduction on Lie Groups

In this section we shall quickly review two Lagrangian reduction processes, namely Lagrange–Poincaré and Euler–Poincaré reduction, as they apply to a Lagrangian defined on a Lie group and invariant under right translation by a closed subgroup. We shall also emphasize the case of discrete symmetry groups.

Geometric setup. Let M be smooth finite dimensional manifold, the parameter space of the theory, and let $\Phi : G \times M \to M$ be a left transitive Lie group action. Usually, M is a particular orbit of the action of G on a bigger manifold. In the context of condensed matter theory, selecting one particular orbit corresponds to choosing a particular phase of the physical system. Given a Lie group G, we shall denote by the corresponding Fraktur letter $\mathfrak{g}$ its Lie algebra.

Choose an element $m_0 \in M$ and consider the isotropy subgroup $H := G_{m_0}$. We have the diffeomorphism $G/H \ni [g] := gH \overset{\sim}{\longmapsto} gm_0 \in M$, where H acts on G by right multiplication $R_h g := gh$ for all $h \in H$ and $g \in G$. We shall always identify M with G/H via this diffeomorphism and denote by $\pi : G \to G/H$ the orbit space projection.

We suppose that the theory is described by a Lagrangian $\mathfrak{L} = \mathfrak{L}(m, \dot{m}) : TM \to \mathbb{R}$, whose associate Euler–Lagrange equations read

$$\frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{m}} - \frac{\partial \mathfrak{L}}{\partial m} = 0.$$

Recall that these equations follow from applying Hamilton's principle

$$\delta \int_{t_0}^{t_1} \mathfrak{L}(m(t), \dot{m}(t))dt = 0,$$

for arbitrary variations of the smooth curve $m(t)$ whose corresponding infinitesimal variations $\delta m(t)$ satisfies $\delta m(t_0) = \delta m(t_1) = 0$.

Since the G-action is transitive on $M = G/H$, any curve $m : [t_0, t_1] \to M$ can be written as $m(t) = \Phi_{g(t)}(m_0) =: g(t)m_0$, where $g : [t_0, t_1] \to G$. By using this relation we can rewrite the action functional in terms of the smooth curve $g(t)$ as

$$\begin{aligned}\int_{t_0}^{t_1} \mathfrak{L}(m(t), \dot{m}(t))dt &= \int_{t_0}^{t_1} \mathfrak{L}\left(g(t)m_0, \frac{d}{dt}g(t)m_0\right) dt \\ &= \int_{t_0}^{t_1} \mathfrak{L}\left(g(t)m_0, (\dot{g}(t)g(t)^{-1})_M(g(t)m_0)\right) dt,\end{aligned}$$

where, for every $\xi \in \mathfrak{g}$, $M \ni m \mapsto \xi_M(m) := \frac{d}{dt}\Big|_{t=0} \Phi_{\exp t\xi}(m) \in T_m M$ denotes the infinitesimal generator vector field of the action. This suggests the definition of the Lagrangian L_{m_0} for smooth curves $g(t)$ in the Lie group as

$$L_{m_0} : TG \to \mathbb{R}, \quad L_{m_0}(g, \dot{g}) := \mathfrak{L}\left(gm_0, (\dot{g}g^{-1})_M(gm_0)\right).$$

This Lagrangian is clearly H-invariant. Our goal is to find an explicit relation between the Euler–Lagrange equations for L_{m_0} and $\mathfrak{L}$ as well as to deduce another simpler equivalent formulation of these equations.

To do this, we shall start with a H-invariant Lagrangian $L : TG \to \mathbb{R}$ and, following the approach of [4], we shall carry out two reductions processes for L. The first one follows the Lagrange–Poincaré reduction theory developed in [2] and the second one is a generalization of the Euler–Poincaré reduction with parameters developed in [8]. These two reductions correspond to two realizations of the quotient space $(TG)/H$.

Lagrange–Poincaré approach. The Lagrange–Poincaré reduction is implemented by using the vector bundle isomorphism

$$\alpha_{\mathcal{A}} : (TG)/H \to TM \times_M \widetilde{\mathfrak{h}}$$

over $M = G/H$. Here $\mathfrak{h}$ is the Lie algebra of H and $\widetilde{\mathfrak{h}} := G \times_H \mathfrak{h} \to M$ is the adjoint bundle, where the right H-action on $G \times \mathfrak{h}$ is given by $(g, \eta) \cdot h := (gh, \mathrm{Ad}_{h^{-1}} \eta)$ for all $h \in H$, $g \in G$, and $\eta \in \mathfrak{h}$; recall that $G \times_H \mathfrak{h} := (G \times \mathfrak{h})/H$ is the orbit space of this action. The vector bundle isomorphism $\alpha_{\mathcal{A}}$ is constructed with the help of a principal connection $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$ on the principal bundle $\pi : G \to G/H = M$ and reads

$$\alpha_{\mathcal{A}}([v_g]_H) := \left(T_g\pi(v_g), \left[g, \mathcal{A}(v_g)\right]_H\right) = \left((v_g g^{-1})_M(m), \left[g, \mathcal{A}(v_g)\right]_H\right) \tag{2.1}$$

(we denote by $[x]_H$ a point in orbit space of the H-action on the manifold whose points are x).

From the given right H-invariant Lagrangian $L : TG \to \mathbb{R}$, we get the reduced Lagrangian $\mathcal{L} : TM \times_M \widetilde{\mathfrak{h}} \to \mathbb{R}$, $\mathcal{L} = \mathcal{L}(m, \dot{m}, \sigma)$, defined by

$$L(g, \dot{g}) = \mathcal{L}\left(gm_0, \dot{g}m_0, [g, \mathcal{A}(g, \dot{g})]_H\right).$$

The reduced Euler–Lagrange equations (or Lagrange–Poincaré equations) are obtained by computing the critical curve of the variational principle

$$\delta \int_{t_0}^{t_1} \mathcal{L}(m, \dot{m}, \sigma)dt, \tag{2.2}$$

for variations δm and $\delta^{\mathcal{A}}\sigma$ induced by variations $\delta g(t)$ of the curve $g(t)$, that vanish at $t = t_0, t_1$. While the variations $\delta m(t)$ are free and vanish at $t = t_0, t_1$, the variations of $\sigma(t)$ verify

$$\begin{aligned} \delta^{\mathcal{A}}\sigma(t) &:= \left.\frac{D^{\mathcal{A}}}{D\varepsilon}\right|_{\varepsilon=0} [g_\varepsilon(t), \mathcal{A}(g_\varepsilon(t), \dot{g}_\varepsilon(t))]_H \\ &= \frac{D^{\mathcal{A}}}{Dt}\eta(t) + [\eta(t), \sigma(t)] + \widetilde{\mathcal{B}}(\delta m(t), \dot{m}(t)) \in \widetilde{\mathfrak{h}}, \end{aligned}$$

where $D^{\mathcal{A}}/D\varepsilon$ denotes the covariant derivative defined by the connection one-form $\mathcal{A}$, $\widetilde{\mathcal{B}} \in \Omega^2(M, \widetilde{\mathfrak{h}})$ is the reduced curvature on the base associated to the connection $\mathcal{A}$, and $\eta(t) = [g(t), \mathcal{A}(\delta g(t))]_H \in \widetilde{\mathfrak{h}}$ is arbitrary with $\eta(t_0) = \eta(t_1) = 0$. Using these variations in (2.2) yield the Lagrange–Poincaré equations

$$\frac{D^{\mathcal{A}}}{Dt}\frac{\delta \mathcal{L}}{\delta \sigma} + \mathrm{ad}^*_\sigma \frac{\delta \mathcal{L}}{\delta \sigma} = 0, \qquad \frac{\partial \mathcal{L}}{\partial m} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{m}} = \left\langle \frac{\delta \mathcal{L}}{\delta \sigma}, \mathbf{i}_{\dot{m}}\widetilde{\mathcal{B}} \right\rangle. \tag{2.3}$$

We refer to [2] for the formulation of the general theory on principal bundles and to [4] for the detailed presentation of this special case on homogeneous spaces.

Lagrange–Poincaré equations for H discrete. Assume now that H is a closed discrete subgroup of G. Then $\mathfrak{h} = \{0\}$, $\widetilde{\mathfrak{h}}$ is the vector bundle with zero dimensional fiber and base M, $\mathcal{A} = 0$, and hence the vector bundle isomorphism $\alpha_{\mathcal{A}}$ becomes canonical, $\alpha : (TG)/H \to TM$, the source and target spaces viewed as a vector bundles over M, and it is given by

$$\alpha([v_g]_H) := T_g\pi(v_g) = (v_g g^{-1})_M(m) \in T_m M.$$

In this case, we get simply

$$\alpha\left([g(t), \dot{g}(t)]_H\right) = \left(g(t)m_0, \frac{d}{dt}g(t)m_0\right) = (m(t), \dot{m}(t)) \in TM.$$

The reduced Lagrangian $\mathcal{L} : TM \to \mathbb{R}$ yields the Lagrange–Poincaré equations (2.3) which in this case become

$$\frac{\partial \mathcal{L}}{\partial m} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{m}} = 0.$$

It is instructive to consider in more detail the isomorphism α in the case of a closed discrete subgroup. In this case, the kernel of the tangent map is zero, so that at any $g \in G$ we have the isomorphism $T_g\pi : T_gG \to T_{[g]}(G/H) = T_{gm_0}M$ which implies that

$$\alpha : (TG)/H \to T(G/H), \quad [v_g]_H \mapsto T\pi(v_g)$$

is a vector bundle isomorphism covering the identity on $M = G/H$.

Euler–Poincaré approach. The Euler–Poincaré reduction is implemented by using the vector bundle isomorphism

$$\bar{i}_{m_0} : (TG)/H \to \mathfrak{g} \times M, \quad \bar{i}_{m_0}([v_g]_H) = \left(v_g g^{-1}, \Phi_g(m_0)\right),$$

where an element $m_0 \in M$ has been fixed (see [4]). We note that a connection is not needed to write this isomorphism. If $g : [t_0, t_1] \to G$ is a given smooth curve, this formula implies

$$\bar{i}_{m_0}\left([g(t), \dot{g}(t)]_H\right) = \left(\dot{g}(t)g(t)^{-1}, \Phi_{g(t)}(m_0)\right) =: (\xi(t), m(t)). \tag{2.4}$$

Note that by composing the two vector bundle isomorphisms $\alpha_{\mathcal{A}}$ and $\bar{i}_{m_0}$ over M, we get the vector bundle isomorphism

$$\mathfrak{g} \times M \ni (\xi, m) \longmapsto \left(\xi_M(m), [g, \mathcal{A}(\xi g)]_H\right) \in TM \times \widetilde{\mathfrak{h}},$$

over M, where $g \in G$ is arbitrary such that $\pi(g) = m$.

Given $v_m \in T_m M$, $\xi_m \in \widetilde{\mathfrak{h}}_m$, the inverse of the above map is given by

$$TM \times \widetilde{\mathfrak{h}} \ni (v_m, \xi_m) \longmapsto \left(\left(\mathrm{Hor}_g(v_m)\right) g^{-1} + \mathrm{Ad}_g\,\eta, m\right) \in \mathfrak{g} \times M,$$

where $g \in G$ is such that $\pi(g) = m$, $\mathrm{Hor}_g : T_m M \to T_g G$ is the horizontal lift of the connection $\mathcal{A}$, and $\eta \in \mathfrak{h}$ is such that $\xi_m = [g, \eta]_H$. A direct verification shows that this expression does not depend on g as long as $\pi(g) = m$.

Given a H-invariant Lagrangian $L : TG \to \mathbb{R}$, the associated reduced Lagrangian $l : \mathfrak{g} \times M \to \mathbb{R}$ obtained through the Euler–Poincaré process is

$$L(g, \dot{g}) = l\left(\dot{g}g^{-1}, \Phi_g(m_0)\right) = l(\xi, m). \tag{2.5}$$

The Euler–Poincaré equations for l follow by applying the variational principle with constrained variations

$$\delta \int_{t_0}^{t_1} l(\xi, m)dt = 0, \quad \delta\xi = \dot{\eta} + [\eta, \xi], \quad \delta m = \eta_M(m),$$

where $\eta(t) \in \mathfrak{g}$ is arbitrary curve with $\eta(t_0) = \eta(t_1) = 0$. We thus get the equations

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} = \mathbf{J}\left(\mathbf{d}_M l(m)\right), \quad \dot{m} = \xi_M(m), \tag{2.6}$$

where $\mathbf{J} : T^*M \to \mathfrak{g}^*$ is the standard equivariant momentum map of the cotangent lifted action given by $\langle \mathbf{J}(\alpha_m), \zeta\rangle = \langle \alpha_m, \zeta_M(m)\rangle$ for all $\alpha_m \in T^*_m M$, $\zeta \in \mathfrak{g}$, and $\mathbf{d}_M$ is the exterior derivative on M.

Note that if H is closed and discrete, the Euler–Poincaré equations (2.6) for $l : \mathfrak{g} \times M \to \mathbb{R}$ do not simplify, contrary to what happens on the Lagrange–Poincaré side.

Remark 2.1. If G is Abelian, several formulas simplify due to the fact that the adjoint action is trivial. Thus, the adjoint bundle $\widetilde{\mathfrak{h}} = M \times \mathfrak{h} \to M$ is trivial and the vector bundle isomorphism (2.1) reads $\alpha_{\mathcal{A}}\left([v_g]_H\right) = \left(((v_g g^{-1})_M(m), \mathcal{A}(v_g)\right) =: (m, \dot{m}, \sigma)$. The Lagrange–Poincaré equations (2.3) simplify to

$$\frac{D^{\mathcal{A}}}{Dt}\frac{\delta \mathcal{L}}{\delta \sigma} = 0, \qquad \frac{\partial \mathcal{L}}{\partial m} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{m}} = \left\langle \frac{\delta \mathcal{L}}{\delta \sigma}, \mathbf{i}_{\dot{m}}\widetilde{\mathcal{B}}\right\rangle,$$

where the reduced curvature is a $\mathfrak{h}$-valued two-form on M. The Euler–Poincaré equations (2.6) read simply

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathbf{J}\left(\mathbf{d}_M l(m)\right), \quad \dot{m} = \xi_M(m).$$

3. Hamilton–Poincaré and Lie–Poisson Reduction on Lie Groups

In this section we summarize the necessary material that is relevant for the Hamiltonian description of condensed matter systems. This is the Hamiltonian version of the two Lagrangian reduction processes described in the preceding section.

Hamilton–Poincaré reduction. As in the preceding section, we consider a Lie group acting transitively on the left on a manifold M. Choosing $m_0 \in M$, we have the diffeomorphism $G/H \ni gH \mapsto \Phi_g(m_0) \in M$. Given a right H-invariant Hamiltonian $\mathscr{H} : T^*G \to \mathbb{R}$, we obtain, by reduction, a Hamiltonian defined on the quotient space $(T^*G)/H$. As in Sect. 2, choosing a principal connection $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$, we have a vector bundle isomorphism

$$(T^*G)/H \to T^*M \oplus \widetilde{\mathfrak{h}}^*, \quad [\alpha_g]_H \mapsto \left(\mathrm{Hor}^*_g \alpha_g, [g, \mathbf{J}(\alpha_g)]_H\right) =: (\alpha_m, \bar{\mu}),$$

where $\pi(g) = m := gm_0$, $\mathrm{Hor}_g^* : T_g^*G \to T_m^*M$ is the dual map to the horizontal lift $\mathrm{Hor}_g : T_mM \to T_gG$ associated to the connection $\mathcal{A}$, and $\mathbf{J} : T^*G \ni \alpha_g \longmapsto (T_e^*L_g\alpha_g)|_{\mathfrak{h}} \in \mathfrak{h}^*$ is a momentum map associated to the lift of right translation of H on T^*G. The reduced Hamilton equations for $\mathcal{H} : T^*M \oplus \widetilde{\mathfrak{h}}^* \to \mathbb{R}$, obtained by Poisson reduction, are called the Hamilton–Poincaré equations and read

$$\frac{Dy}{Dt} = -\frac{\partial \mathcal{H}}{\partial x} - \left\langle \bar{\mu}, \tilde{\mathcal{B}}(\dot{x}, \_) \right\rangle, \quad \dot{x} = \frac{\partial \mathcal{H}}{\partial y}, \quad \frac{D^{\mathcal{A}}\bar{\mu}}{Dt} + \mathrm{ad}^*_{\frac{\delta\mathcal{H}}{\delta\bar{\mu}}} \bar{\mu} = 0,$$

where $(x, y) \in T^*M$, $\bar{\mu} \in \widetilde{\mathfrak{h}}^*$, $\tilde{\mathcal{B}} \in \Omega^2(M, \widetilde{\mathfrak{h}})$ is the reduced curvature of $\mathcal{A}$, D/Dt in the first equation denotes the covariant derivative on T^*M associated to a given affine connection on M, and $D^{\mathcal{A}}/Dt$ in the last equation denotes the covariant derivative on $\widetilde{\mathfrak{h}}^*$ associated to the principal connection $\mathcal{A}$; for details see [3].

The symplectic leaves in $T^*M \oplus \widetilde{\mathfrak{h}}^*$ have been described in [13]; they are of the form $T^*M \times_M \widetilde{\mathcal{O}}$, where $\mathcal{O}$ is a coadjoint orbit of H, and $\widetilde{\mathcal{O}} \to M$ is the associated fiber bundle. The symplectic form is the sum of the canonical symplectic form on T^*M and a two-form on $\widetilde{\mathcal{O}}$, see [12, Theorem 2.3.12]. If the Lie group G is connected and $\mathcal{O}$ has N elements (which is happening in subsequent applications), then the fiber bundle $T^*M \times_M \widetilde{\mathcal{O}} \to M$ has N connected components, each one of them symplectomorphic to the canonical phase space T^*M.

Lie–Poisson reduction. A second realization of $(T^*G)/H$ is given by the diffeomorphism

$$(T^*G)/H \ni [\alpha_g]_H \longmapsto (\alpha_g g^{-1}, gm_0) \in \mathfrak{g}^* \times M.$$

The reduced Hamilton equations on this space read

$$\dot{\mu} + \mathrm{ad}^*_{\frac{\delta h}{\delta \mu}} \mu = \mathbf{J}\left(\mathbf{d}_M h(m)\right), \quad \dot{m} = -\left(\frac{\delta h}{\delta \mu}\right)_M (m),$$

where $\mathbf{J} : T^*M \to \mathfrak{g}^*$ is a momentum map of the cotangent lift of the left action of G on $M = G/H$ and $h : \mathfrak{g}^* \times M \to \mathbb{R}$ is the reduced Hamiltonian induced by $\mathscr{H} : T^*G \to \mathbb{R}$. These equations are Hamiltonian relative to the Poisson bracket

$$\{f, g\}(\mu, m) = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle + \left\langle \mathbf{J}\left(\mathbf{d}_M f(m)\right), \frac{\delta g}{\delta \mu}\right\rangle - \left\langle \mathbf{J}\left(\mathbf{d}_M g(m)\right), \frac{\delta f}{\delta \mu}\right\rangle. \quad (3.1)$$

See [9] for a more general situation.

We refer to [4], for further details and examples of application of these two reduction processes.

The case when M is the orbit in a G-representation space. We now suppose that V is a representation space of G and we take $M = \mathrm{Orb}(a_0) \subset V^*$. The induced Lie algebra representation $\mathfrak{g} \times V \to V$ is given by the infinitesimal operator map and is denoted by $\xi v := \xi_V(v)$, for any $\xi \in \mathfrak{g}$ and $v \in V$. We consider the semidirect product $S = G \circledS V$ and its Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$. The symplectic leaves in $\mathfrak{s}^*$ are given by the connected components of the coadjoint orbits $\mathcal{O}_{(\mu,a)}$ of S. From the formula of the coadjoint action

$$\mathrm{Ad}^*_{(g,v)^{-1}}(\mu, a) = (\mathrm{Ad}^*_{g^{-1}} \mu + v \diamond ga, ga), \quad (3.2)$$

where $(g, v) \in S$ and $(\mu, a) \in \mathfrak{s}^*$, we see that the symplectic leaves in $\mathfrak{g}^* \times M$ are $\mathcal{O}_{(\mu,a_0)}$ endowed with the minus orbit symplectic form. The diamond operation $\diamond : V \times V^* \to \mathfrak{g}^*$ in this formula is defined by $\langle v \diamond a, \xi \rangle := \langle a, \xi v \rangle$, for any $\xi \in \mathfrak{g}$, where the pairing in the left hand side is between $\mathfrak{g}^*$ and $\mathfrak{g}$, whereas in the right hand side it is between V^* and V.

These considerations provide the proof of the following theorem.

Theorem 3.1. *Given $\mu \in \mathfrak{g}^*$ and $a_0 \in V^*$, we define $\mu_{a_0} := \mu|_{\mathfrak{g}_{a_0}} \in \mathfrak{g}^*_{a_0}$, where $\mathfrak{g}_{a_0} = \{\xi \in \mathfrak{g} \mid \xi a_0 = 0\} =: \mathfrak{h}$. Let $\mathcal{O}_{\mu_{a_0}} \subset \mathfrak{g}^*_{a_0}$ be the coadjoint orbit of $H := G_{a_0}$ through μ_{a_0} and $M := G/H$. The map*

$$\mathfrak{s}^* \supset \mathcal{O}_{(\mu,a_0)} \ni (\alpha_g g^{-1}, g a_0) \longmapsto (\operatorname{Hor}^*_g \alpha_g, [g, \mathbf{J}(\alpha_g)]_H) \in T^*M \times_M \widetilde{\mathcal{O}}_{\mu_{a_0}}$$

is a symplectomorphism.

Note that the Theorem states that if $(\mu - \nu)|_{\mathfrak{g}_{a_0}} = 0$, then $\mathcal{O}_{(\mu,a_0)} = \mathcal{O}_{(\nu,a_0)}$. This can be verified directly by observing that $\operatorname{Ad}^*_{(e,v)^{-1}}(\mu, a_0) = (\mu + v \diamond a_0, a_0)$ and the map $V \ni v \mapsto v \diamond a_0 \in \mathfrak{g}^\circ_{a_0}$ (the annihilator of $\mathfrak{g}_{a_0}$ in $\mathfrak{g}^*$) is surjective (which is equivalent to $\ker(v \mapsto v \diamond a_0) = \mathfrak{g}_{a_0}$).

We now write explicitly the operator $\alpha_g \mapsto \operatorname{Hor}^*_g \alpha_g$ in the particular case when there is an Ad-invariant inner product γ on $\mathfrak{g}$. We extend γ by left invariance to a Riemannian metric on G. This Riemannian metric, also denoted γ, is right invariant. The principal connection on the right H-principal bundle $G \to G/H = M$ associated to γ has the expression $\mathcal{A}(v_g) := \mathbb{P}_{a_0}(g^{-1} v_g)$, where $\mathbb{P}_{a_0} : \mathfrak{g} \to \mathfrak{g}_{a_0}$ is the γ-orthogonal projection. The horizontal lift associated to $\mathcal{A}$ reads

$$\operatorname{Hor}_g : T_m M \to T_g G, \quad \operatorname{Hor}_g(\xi_M(m)) = g\, \mathbb{P}^\perp_{a_0}(\operatorname{Ad}_{g^{-1}} \xi),$$

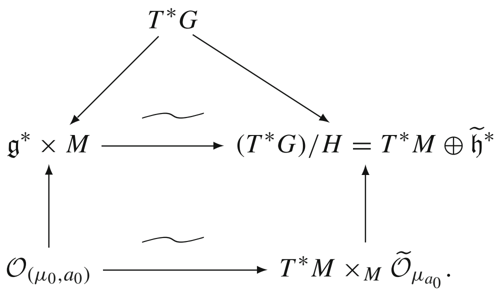
where $\mathbb{P}^\perp_{a_0} : \mathfrak{g} \to \mathfrak{g}^\perp_{a_0}$ is the γ-orthogonal projection and $m = g a_0$. We endow $M = G/H$ with the natural induced Riemannian metric, i.e.,

$$\gamma_M(\xi_M(m), \eta_M(m)) := \gamma\left(\operatorname{Hor}_g(\xi_M(m)), \operatorname{Hor}_g(\eta_M(m))\right), \tag{3.3}$$

where $g \in G$ is such that $m = g a_0$. Using the Riemannian metrics γ and γ_M, we identify TG with T^*G and TM with T^*M, respectively. With these identifications, we have

$$\operatorname{Hor}^*_g \alpha_g = (\alpha_g g^{-1})_M(m), \quad \alpha_g \in T^*_g G = T_g G. \tag{3.4}$$

We summarize the maps in this discussion in the following commutative diagram



4. Applications to Condensed Matter

4.1. Setup of the problem.

Lagrangian description. As discussed at the beginning of the previous section, for condensed matter theories the Lagrangian $\mathfrak{L}$ is defined on the tangent bundle TM of the parameter manifold M. This manifold is assumed to be a homogeneous space, relative to the transitive action of a Lie group G, with isotropy group $H = G_{m_0}$ for some preferred element $m_0 \in M$. From $\mathfrak{L}$ one can construct a Lagrangian $L_{m_0} : TG \to \mathbb{R}$, $L_{m_0}(g, \dot{g}) := \mathfrak{L}(gm_0, (\dot{g}g^{-1})_M(gm_0))$ as explained earlier. As before, we denote by $l : \mathfrak{g} \times M \to \mathbb{R}$ the reduced Lagrangian associated to L_{m_0} via Euler–Poincaré reduction.

We now state a fundamental result, to be used in the rest of the paper.

Theorem 4.1. *The following statements are equivalent.*

1. *The curve $m : [t_0, t_1] \to M$ is a solution of the Euler–Lagrange equations for $\mathfrak{L} : TM \to \mathbb{R}$, i.e.,*

$$\frac{\partial \mathfrak{L}}{\partial m} - \frac{d}{dt}\frac{\partial \mathfrak{L}}{\partial \dot{m}} = 0.$$

2. *The curve $m : [t_0, t_1] \to M$ is a solution of the Euler–Poincaré equations for $l : \mathfrak{g} \times M \to \mathbb{R}$, i.e.,*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_{\xi}\frac{\delta l}{\delta \xi} = \mathbf{J}\left(\mathbf{d}_M l(m)\right), \quad \dot{m} = \xi_M(m).$$

Proof. We will show that the Euler–Lagrange equations for L_{m_0} are equivalent to those for $\mathfrak{L}$ by implementing Lagrange–Poincaré reduction. Then we will use the equivalence with the Euler–Poincaré approach, recalled in Sect. 2 above, to write the equations in a simpler form.

Since L_{m_0} is H-invariant, by fixing a connection $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$, we get the Lagrange–Poincaré Lagrangian $\mathcal{L}$, that we now compute. We have

$$\mathcal{L}\left(m, \dot{m}, [g, \mathcal{A}(g, \dot{g})]_H\right) = L_{m_0}(g, \dot{g}) = \mathfrak{L}(gm_0, (\dot{g}g^{-1})_M(gm_0)) = \mathfrak{L}(m, \dot{m}).$$

This means that $\mathcal{L} : TM \times_M \widetilde{\mathfrak{h}} \to \mathbb{R}$ does not depend on the second variable, so $\frac{\delta \mathcal{L}}{\delta \sigma} = 0$, and $\mathcal{L} = \mathfrak{L}$. Thus, in the general system (2.3), the first equation disappears and the right hand side of the second vanishes. Therefore, the Lagrange–Poincaré equations in (2.3) reduce to the Euler–Lagrange equations for $\mathfrak{L}$ on TM.

We now compute the Euler–Poincaré reduced Lagrangian. We have

$$\begin{aligned} l(\xi, m) &= l\left(\dot{g}g^{-1}, \Phi_g(m_0)\right) = L_{m_0}(g, \dot{g}) \\ &= \mathfrak{L}(gm_0, (\dot{g}g^{-1})_M(gm_0)) = \mathfrak{L}(m, \xi_M(m)). \end{aligned}$$

From the results recalled in Sect. 2, we know that the Euler–Lagrange equations for L_{m_0} are equivalent to the Euler–Poincaré equations

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} + \mathrm{ad}^*_{\xi}\frac{\delta l}{\delta \xi} = \mathbf{J}\left(\mathbf{d}_M l(m)\right), \quad \dot{m} = \xi_M(m).$$

This proves the statement of the Theorem. □

Ginzburg–Landau theory of phase transitions. We briefly review the major steps in Landau's theory of phase transitions.

Phenomenological Ginzburg–Landau theory, initially formulated to describe the behavior of superconductivity and superfluidity of ^{4}He near points of phase transition, turned out to be also very convenient in the determination of of phase transitions of superfluid ^{3}He. We recall here briefly the main statements of Landau's second order theory of phase transitions. For a detailed presentation, see [10, §83, §141–153, §162].

(1) At a phase transition point, the symmetry of the system spontaneously changes.
(2) The system is characterized by some macroscopic quantity, an *order parameter*. For example, in the case of liquid crystals, this quantity is taken to be the director field, a symmetric traceless 2-tensor, or the wryness tensor, depending on the chosen model.
(3) Near the transition point, due to the smallness of the parameters $\alpha_i(T-T_c)$, the free energy (i.e., the thermodynamic potential) F admits an expression of the following type

$$F(p,T) = F_0(p) + \alpha_1(T-T_c)V(Q)^2 + \alpha_2(T-T_c)V(Q)^2V(Q)^4 + \|\operatorname{grad} Q\|^2,$$

where V is an invariant function under the symmetry group of a system (e.g., the trace), T is the temperature, T_c is the critical temperature at which the phase transition occurs, p is the pressure, Q is an order parameter of the given system which is chosen by the concrete physical situation under study (usually Q is a matrix), and the functions $\alpha_1(T-T_c)$, $\alpha_2(T-T_c)$ depending on the temperature are phenomenological parameters.
(4) The change of symmetry in the transition is determined only by the order parameter.
(5) To small fluctuations of the order parameter, it is possible to ignore terms of higher order in the Ginzburg–Landau expansion. The so called Levanuyk–Ginzburg criterion ensures the validity of the expression of $F(p,T)$ given above, if the mean square fluctuation of the order parameter φ, averaged over the correlation volume, is small compared with the characteristic value of $\langle\varphi\rangle$ (see [10]).

The advantage of the Ginzburg–Landau approach is based on the fact that, with relatively few basic assumptions, it is possible to reduce the investigation (in many important cases) of an infinite dimensional quantum particle system to the study of a finite dimensional mechanical problem. Superfluid ^{3}He provides such an example. Of course, this is a more complicated system than superfluid ^{4}He since there are more thermodynamic phases.

We describe now the concrete method implementing this Ginzburg–Landau phase transition theory. One is given a free energy F, the sum of a potential $U(A)$ depending on some order parameter A, but not on its spatial derivatives, and a gradient term $F_{grad}(A,\nabla A)$ that depends on both the order parameter A and its derivatives ∇A.

(A) Expand the potential $U(A)$ up to fourth order and replace it with this expression.
(B) Find the largest possible Lie group that leaves this fourth degree polynomial $U(A)$ invariant and determine the Lie group action (very often a representation) on the space of all order parameters A.
(C) Find the formal minima of the potential $U(A)$, i.e., $\frac{\delta U}{\delta A}=0$ and $\frac{\delta^2 U}{\delta A^2}\geq 0$ (positive Hessian).
(D) Take the Lie group orbit through each minimum and consider it as a configuration space of a Lagrangian system given by the free energy. Note that it is not necessary

to add the potential $U(A)$ to the gradient term $F_{grad}(A, \nabla A)$, since it is constant on each such orbit, by construction. The goal is the study of each Lagrangian system on such a Lie group orbit, because the Ginzburg–Landau equations turn out to be the Euler–Lagrange equations for F_{grad}. The function F_{grad} determines a metric on the orbit.

Sometimes, steps (B) and (C) are hard to carry out. In practice one starts with a Lie group that is, on physical grounds, a symmetry of the system and then determines, using invariant theory, the most general polynomial of fourth degree, invariant under this group. This polynomial is then taken as the potential $U(A)$. Then one classifies all orbits of this Lie group on the space of all order parameters A (or, at least, determines enough orbits) and finds, in this way, orbits of physical interest that describe different thermodynamic phases of the system. This problem is solved using techniques developed in [1,5,6,15], where thermodynamic phases are identified with orbits containing a minimum of the potential $U(A)$ of the free energy.

It turns out that different types of textures for the system are given as solutions to the Ginzburg–Landau equations for a given phase and that, on each Lie group orbit, the Ginzburg–Landau equations are the Euler–Lagrange equations for F_{grad}.

We shall apply this method to the study of different phases in superfluid ^{3}He [1] and rotating neutron stars ([16]). Using the same techniques one can also study one-dimensional textures in liquid crystals and superfluids as well as phase transitions between biaxial and uniaxial nematics; we leave these latter topics for a future publication.

4.2. One-dimensional textures in the A- and B-phases of liquid Helium ^{3}He. The order parameter of superfluid ^{3}He is given by complex 3×3 matrices $A \in \mathfrak{gl}(3, \mathbb{C})$.[1]

The free energy is given by

$$\mathcal{F}(A, \nabla A) = F_{grad}(A, \nabla A) + U(A),$$

where

$$F_{grad}(A, \nabla A) = \gamma_1 \sum_{i,p,k} \left(\partial_k \bar{A}_{pi}\right)\left(\partial_k A_{pi}\right) + \gamma_2 \sum_{i,p,k} \left(\partial_k \bar{A}_{pi}\right)\left(\partial_i A_{pk}\right) + \gamma_3 \sum_{i,p,k} \left(\partial_k \bar{A}_{pk}\right)\left(\partial_i A_{pi}\right),$$

$\gamma_1, \gamma_2, \gamma_3 > 0$ are constants, and $U(A)$ is in the Ginzburg–Landau form, namely,

$$U(A) = \alpha \operatorname{Tr}(AA^*) + \beta_1 |\operatorname{Tr}(AA^{\mathsf{T}})|^2 + \beta_2 \left[\operatorname{Tr}(AA^*)\right]^2 + \beta_3 \operatorname{Tr}\left[(A^*A)\overline{(A^*A)}\right] + \beta_4 \operatorname{Tr}\left[(AA^*)^2\right] + \beta_5 \operatorname{Tr}\left[(AA^*)\overline{(AA^*)}\right],$$

for $\alpha, \beta_1, \ldots, \beta_5 \in \mathbb{R}$. Note that these expressions are real valued.

In one dimension, we compute

$$\begin{aligned} F_{grad}(A, \partial_z A) &= \gamma_1 \partial_z \bar{A}_{pi} \partial_z A_{pi} + \gamma_2 \partial_z \bar{A}_{p3} \partial_z A_{p3} + \gamma_3 \partial_z \bar{A}_{p3} \partial_z A_{p3} \\ &= \operatorname{Re} \operatorname{Tr}(\Gamma \partial_z A^* \partial_z A) = \langle\langle \partial_z A, \partial_z A \rangle\rangle\,, \end{aligned} \tag{4.1}$$

[1] The matrices A are related to $\Delta'_{\sigma\sigma'}$, the "energetic gap" of the triplet pairing of interacting quasiparticles of ^{3}He, and so this gap can be expressed in terms of A. Thus A can be regarded as the order parameter of superfluid ^{3}He; see [15, Sect. 5.2.1].

where $\Gamma := \operatorname{diag}(\gamma_1, \gamma_1, \gamma_1 + \gamma_2 + \gamma_3)$ and we defined the inner product on $\mathfrak{gl}(3, \mathbb{C})$ by

$$\langle\langle A, B\rangle\rangle := \operatorname{Re}\operatorname{Tr}(\Gamma A^* B). \tag{4.2}$$

The following identities are useful in the computations:

$$\langle\langle A, B\rangle\rangle = \langle\langle B, A\rangle\rangle\,, \qquad \langle\langle uA, B\rangle\rangle = \langle\langle A, \overline{u}B\rangle\rangle\,,$$

for any $A, B \in \mathfrak{gl}(3, \mathbb{C})$ and $u \in \mathbb{C}$. In addition, $\langle\langle\, ,\, \rangle\rangle$ is $\mathbb{R}$-bilinear.

Group representation, orbits, and thermodynamic phases. The potential function $U(A)$ is invariant under the left representation of the compact Lie group $G = U(1) \times SO(3)_L \times SO(3)_R$ on $\mathfrak{gl}(3, \mathbb{C})$ given by

$$\left(e^{i\varphi}, R_1, R_2\right) \cdot A := e^{i\varphi} R_1 A R_2^{-1}, \tag{4.3}$$

where $A \in \mathfrak{gl}(3, \mathbb{C})$ and $\left(e^{i\varphi}, R_1, R_2\right) \in G$. As the formula above shows, the indices L and R on the two groups $SO(3)$ indicate the side of the multiplication on the matrix A.

Note that the term F_{grad} is not G-invariant. However, to determine the thermodynamic phases, it suffices to study $U(A)$. The phases correspond to different orbits. A partial classification of the orbits is given in [1]. Below we shall consider only some of these orbits that are physically relevant for the phases of superfluid ^{3}He (see [15, Sect. 5.2]).

The A-phase of superfluid ^{3}He has two regimes depending on whether $L \ll L_{dip}$ or $L \gg L_{dip}$, where L and L_{dip} are the characteristic and dipole length, respectively. The first regime corresponds to minimal degeneracy and the dipole interaction can be neglected. The order parameter matrix $A \in \mathfrak{gl}(3, \mathbb{C})$ is representable as an element of the $U(1) \times SO(3)_L \times SO(3)_R$-orbit through the point A_0 given in (4.4), i.e., $A = e^{i\varphi} R_1 A_0 R_2^{-1}$. In the second regime, the energy of the dipole interaction should be taken into account. As a consequence, the order parameter matrix $A \in \mathfrak{gl}(3, \mathbb{C})$ is representable as an element of the $SO(3)$-orbit through the same matrix A_0 under the different action $A = RA_0R^{-1}$. For details, see [15, Sect. 5.2.3].

4.2.1. The A-phase—first regime. We consider the orbit M of $U(1) \times SO(3)_L \times SO(3)_R$ through the point

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & i & 0 \end{pmatrix} \in \mathfrak{gl}(3, \mathbb{C}). \tag{4.4}$$

We note that $e^{i\varphi} A_0 = \rho(\varphi) A_0 \rho(-\varphi)$, where $\rho(\varphi) := \exp(\varphi\widehat{\mathbf{e}}_3)$.

Proposition 4.2. (i) *The isotropy subgroup of* A_0 *is*

$$\begin{aligned} H &= \{(e^{i\varphi}, \rho(\alpha)J_+, \rho(\varphi)\tilde{J}_+), (e^{i\varphi}, \rho(\alpha)J_-, \rho(\varphi)\tilde{J}_-)\} \subset G \\ &= U(1) \times SO(3)_L \times SO(3)_R, \end{aligned}$$

where $\rho(\alpha) = \exp(\alpha\widehat{\mathbf{e}}_3)$ *and*

$$J_\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \quad \tilde{J}_\pm = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.5}$$

(ii) *Let* $\mathbb{Z}_2 = \{\pm 1\}$ *act on* $(A, \mathbf{x}) \in SO(3) \times S^2$ *as* $(-1) \cdot (A, \mathbf{x}) = (A\tilde{J}_-, -\mathbf{x})$. *Then the map*

$$M = G/H \ni [e^{i\varphi}, R_1, R_2]_H \longmapsto [R_2\rho(-\varphi), R_1\mathbf{e}_3]_{\mathbb{Z}_2} \in (SO(3) \times S^2)/\mathbb{Z}_2, \quad (4.6)$$

is a diffeomorphism.

(iii) *The map*

$$(SO(3) \times S^2)/\mathbb{Z}_2 \ni [A, \mathbf{x}]_{\mathbb{Z}_2} \longmapsto \mathbf{x} \otimes (A_1 + iA_2) \in \mathrm{Orb}(A_0), \quad (4.7)$$

where A_i *denotes the* i*th column of the matrix* A, *is a diffeomorphism.*

Proof. (i) Writing $A_0 = \Re(A_0) + i\Im(A_0)$, where $\Re(A_0)$ and $\Im(A_0)$ are real and imaginary parts of A_0, the equality $e^{i\varphi} R_1 A_0 R_2^{-1} = A_0$ is equivalent to the two equations

$$(\cos\varphi) R_1 \Re(A_0) R_2^{-1} - (\sin\theta) R_1 \Im(A_0) R_2^{-1} = \Re(A_0)$$
$$(\sin\varphi) R_1 \Re(A_0) R_2^{-1} + (\cos\theta) R_1 \Im(A_0) R_2^{-1} = \Im(A_0).$$

The proof then follows from a direct computation which is done by writing the matrices R_i in terms of their rows.

(ii) Let us first show that the map is well-defined. Given $(e^{i\varphi}, R_1, R_2) \in G$, any element in the equivalence class $[(e^{i\varphi}, R_1, R_2)]_H$ has the form

$$(e^{i\varphi}, R_1, R_2)(e^{i\psi}, \rho(\alpha)J_\pm, \rho(\psi)\tilde{J}_\pm) = (e^{i(\varphi+\psi)}, R_1\rho(\alpha)J_\pm, R_2\rho(\psi)\tilde{J}_\pm),$$

where $(e^{i\psi}, \rho(\alpha)J_\pm, \rho(\psi)\tilde{J}_\pm) \in H$. Applying formula (4.6) to the expression (4.8), yields

$$[R_2\rho(\psi)\tilde{J}_\pm\rho(-\varphi)\rho(-\psi), R_1\rho(\alpha)J_\pm\mathbf{e}_3]_{\mathbb{Z}_2} = [R_2\rho(-\varphi)\tilde{J}_\pm, \pm R_1\mathbf{e}_3]_{\mathbb{Z}_2}$$
$$= [R_2\rho(-\varphi), R_1\mathbf{e}_3]_{\mathbb{Z}_2},$$

where we used the properties

$$\rho(\alpha)\tilde{J}_\pm = \tilde{J}_\pm\rho(\alpha) \quad \text{and} \quad \rho(\alpha)J_\pm = J_\pm\rho(\pm\alpha). \quad (4.8)$$

The map is clearly surjective. To show the injectivity, we take $(e^{i\varphi}, R_1, R_2), (e^{i\varphi'}, R_1', R_2') \in G$ such that $[R_2\rho(-\varphi), R_1\mathbf{e}_3]_{\mathbb{Z}_2} = [R_2'\rho(-\varphi'), R_1'\mathbf{e}_3]_{\mathbb{Z}_2}$. We thus have the equalities $R_2\rho(-\varphi) = R_2'\rho(-\varphi')\tilde{J}_\pm$ and $R_1\mathbf{e}_3 = \pm R_1'\mathbf{e}_3$. From the first equality, there exists $\rho(\psi) \in U(1)$ such that $R_2 = R_2'\rho(\psi)\tilde{J}_\pm$, by using (4.8). Rewriting the second equality as $R_1\mathbf{e}_3 = R_1' J_\pm\mathbf{e}_3$, we obtain the existence of $\rho(\alpha) \in U(1)$ such that $R_1 = R_1'\rho(\alpha)J_\pm$. This proves that $[(e^{i\varphi}, R_1, R_2)]_H = [(e^{i\varphi'}, R_1', R_2')]_H$.

(iii) Given $[A, \mathbf{x}]_{\mathbb{Z}_2} \in (SO(3) \times S^2)/\mathbb{Z}_2$, let $(e^{i\varphi}, R_1, R_2) \in G$ be such that $[R_2\rho(-\varphi), R_1\mathbf{e}_3]_{\mathbb{Z}_2} = [A, \mathbf{x}]_{\mathbb{Z}_2}$. A possible choice is $(e^{i\varphi}, R_1, R_2) = (1, R_1, A)$, where $R_1 \in SO(3)$ is such that $R_1\mathbf{e}_3 = \mathbf{x}$. With this choice, an easy computation shows that

$$(1, R_1, A) \cdot A_0 = \mathbf{x} \otimes (A_1 + iA_2).$$

□

Remark 4.3. We observe that the subgroup $\tilde{G} := SO(3)_L \times SO(3)_R \subset G$ acts transitively on the orbit $\mathrm{Orb}(A_0)$, see (4.6), (4.7). The isotropy subgroup of A_0 is $\tilde{G}_{A_0} = H \cap \tilde{G} = \{1, \rho(\alpha)J, \tilde{J}\}$, which is isomorphic to $O(2)$. Therefore the orbit can be equally well described as the homogeneous space $(SO(3)_L \times SO(3)_R)/\tilde{G}_{A_0}$.

Lagrangian formulation. We now apply Theorem 4.1 with this description of the orbit, so the Lie algebra is $\mathfrak{so}(3)_L \times \mathfrak{so}(3)_R$. On this orbit M, we consider the Lagrangian density given by the gradient part only, i.e.,

$$\mathcal{L}(A, \partial_z A) = F_{grad}(A, \partial_z A) = \langle (\partial_z A)\Gamma, \partial_z A \rangle = \langle\!\langle \partial_z A, \partial_z A \rangle\!\rangle\,, \tag{4.9}$$

where

$$\langle A, B \rangle := \mathrm{Re}\,\mathrm{Tr}(A^* B). \tag{4.10}$$

Note that

$$\frac{\delta \mathcal{L}}{\delta \partial_z A} = 2\partial_z A \Gamma. \tag{4.11}$$

The texture equations are given by the Euler–Lagrange equations for $\mathcal{L}$ on the orbit M.

The reduced velocity $\xi = \partial_z g g^{-1}$ of the general theory (see (2.4)) is given here by $\xi = (\mathbf{v}, \mathbf{w}) : \mathbb{R} \to \mathfrak{so}(3)_L \times \mathfrak{so}(3)_R$, where $\mathbf{v}$ and $\mathbf{w}$ are the chiral velocities $\mathbf{v} = (\partial_z R_1) R_1^{-1}$ and $\mathbf{w} = R_2^{-1}(\partial_z R_2)$, see [15, formula (5.133)]. The second formula in (2.6) is given here by

$$\partial_z A = \widehat{\mathbf{v}} A + A \widehat{\mathbf{w}}.$$

Using this expression and formula (4.1), the Euler–Poincaré Lagrangian

$$l = l(\xi, m) : \mathfrak{so}(3)_L \times \mathfrak{so}(3)_R \times M \to \mathbb{R}$$

of the general theory given in (2.5), is computed in this case to be

$$l(\mathbf{v}, \mathbf{w}, A) = \mathrm{Re}\,\mathrm{Tr}(\Gamma \partial_z A^* \partial_z A) = \langle\!\langle \widehat{\mathbf{v}} A + A \widehat{\mathbf{w}}, \widehat{\mathbf{v}} A + A \widehat{\mathbf{w}} \rangle\!\rangle\,,$$

(see (4.2)). Defining

$$I_{ab}(A) = \langle\!\langle A \widehat{\mathbf{e}}_a, A \widehat{\mathbf{e}}_b \rangle\!\rangle\,, \quad \chi_{ab}(A) = \langle\!\langle \widehat{\mathbf{e}}_a A, \widehat{\mathbf{e}}_b A \rangle\!\rangle\,, \quad \text{and} \quad \Sigma_{ab}(A) = \langle\!\langle \widehat{\mathbf{e}}_a A, A \widehat{\mathbf{e}}_b \rangle\!\rangle\,,$$

the formula for the Lagrangian above becomes

$$\begin{aligned} l(\mathbf{v}, \mathbf{w}, A) &= \sum_{a,b=1}^{3} \left(I_{ab}(A) w_a w_b + \chi_{ab}(A) v_a v_b + 2\Sigma_{ab}(A) v_a w_b \right) \\ &= \mathbf{w}^{\mathsf{T}} \mathbf{I}(A) \mathbf{w} + \mathbf{v}^{\mathsf{T}} \boldsymbol{\chi}(A) \mathbf{v} + 2\mathbf{v}^{\mathsf{T}} \boldsymbol{\Sigma}(A) \mathbf{w}. \end{aligned}$$

Thus, the Euler–Poincaré equations (2.6) read

$$\partial_z \frac{\delta l}{\delta \mathbf{v}} + \mathrm{ad}^*_{\mathbf{v}} \frac{\delta l}{\delta \mathbf{v}} = \mathbf{J}_1 \left(\frac{\delta l}{\delta A} \right), \quad \partial_z \frac{\delta l}{\delta \mathbf{w}} - \mathrm{ad}^*_{\mathbf{w}} \frac{\delta l}{\delta \mathbf{w}} = \mathbf{J}_2 \left(\frac{\delta l}{\delta A} \right), \quad \partial_z A = \widehat{\mathbf{v}} A + A \widehat{\mathbf{w}},$$

where $\mathbf{J}_1 : T^*M \to \mathfrak{so}^*(3)$ is a momentum map of the left action and $\mathbf{J}_2 : T^*M \to \mathfrak{so}^*(3)$ is a momentum map of the right action of $SO(3)$ on the orbit M, respectively.

Using the duality pairing $\langle A, B\rangle = \operatorname{Re}\operatorname{Tr}(A^*B)$ on $\mathfrak{gl}(3,\mathbb{C})$, we get the Euler–Poincaré equations

$$\frac{d}{dz}\frac{\delta l}{\delta \mathbf{v}} + \frac{\delta l}{\delta \mathbf{v}} \times \mathbf{v} = \overrightarrow{2\operatorname{Re}\left(\frac{\delta l}{\delta A}A^*\right)}, \qquad \frac{d}{dz}\frac{\delta l}{\delta \mathbf{w}} - \frac{\delta l}{\delta \mathbf{w}} \times \mathbf{w} = \overrightarrow{2\operatorname{Re}\left(A^*\frac{\delta l}{\delta A}\right)},$$

where $\overrightarrow{A} \in \mathbb{R}^3$ is defined by $\widehat{\overrightarrow{A}} := A^{skew} := \frac{1}{2}(A - A^\mathsf{T})$, and where we have

$$\begin{aligned}\frac{\delta l}{\delta \mathbf{v}} &= 2\boldsymbol{\chi}\mathbf{v} + 2\boldsymbol{\Sigma}\mathbf{w}, \quad \frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{I}\mathbf{w} + 2\boldsymbol{\Sigma}^\mathsf{T}\mathbf{w},\\ \frac{\delta l}{\delta A} &= -2\left(A\widehat{\mathbf{w}}\Gamma\widehat{\mathbf{w}} + \widehat{\mathbf{v}}\widehat{\mathbf{v}}A\Gamma + \widehat{\mathbf{v}}A\widehat{\mathbf{w}}\Gamma + \widehat{\mathbf{v}}A\Gamma\widehat{\mathbf{w}}\right).\end{aligned}$$

Hamiltonian formulation. As expected from the general theory, the Euler–Poincaré Lagrangian $l(\mathbf{v},\mathbf{w},A)$ is degenerate, since for all $A \in M$, the quadratic form $(\mathbf{v},\mathbf{w}) \mapsto \langle\langle \widehat{\mathbf{v}}A + A\widehat{\mathbf{w}}, \widehat{\mathbf{v}}A + A\widehat{\mathbf{w}}\rangle\rangle$ has a one dimensional kernel given by the isotropy Lie algebra $\mathfrak{g}_A = \{(\mathbf{v},\mathbf{w}) \in \mathfrak{g} \mid \widehat{\mathbf{v}}A + A\widehat{\mathbf{w}} = 0\}$.

Since the Lagrangian (4.9) is nondegenerate, we consider the associated Hamiltonian on T^*M, given by

$$\mathcal{H}(\alpha_A) = \frac{1}{4}\operatorname{Re}\operatorname{Tr}(\Gamma^{-1}\alpha_A^*\alpha_A) = \frac{1}{4}\left\langle \alpha_A\Gamma^{-1}, \alpha_A\right\rangle. \tag{4.12}$$

Now, we apply Theorem 3.1 in this particular case. The element a_0 is given by A_0 in (4.4). The groups are $G = SO(3)_L \times SO(3)_R$, $H = \widetilde{G}_{A_0} = \{\rho(\alpha)J, \tilde{J}\}$. Given $\mu = (\mathbf{m},\mathbf{n}) \in \mathfrak{so}(3)^* \times \mathfrak{so}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$, since $\mathfrak{g}_{A_0} = \{(\lambda\mathbf{e}_3, \mathbf{0}) \mid \lambda \in \mathbb{R}\}$, we have $\mu_{a_0} = (m_3\mathbf{e}_3, \mathbf{0})$. We now compute the G_{A_0}-coadjoint orbit $\mathcal{O}_{\mu_{a_0}}$. We have the formulas

$$\begin{aligned}\operatorname{Ad}_{(\rho(\alpha)J,\tilde{J})}(\widehat{\mathbf{v}},\widehat{\mathbf{w}}) &= (\rho(\alpha)J\widehat{\mathbf{v}}J\rho(-\alpha), \tilde{J}\widehat{\mathbf{w}}\tilde{J})\\ \operatorname{Ad}^*_{(\rho(\alpha)J,\tilde{J})}(m_3\mathbf{e}_3,\mathbf{0}) &= ((-1)^{|J|}m_3\mathbf{e}_3, \mathbf{0}),\end{aligned}$$

where $|J| = 0$ if $J = I_3$ and $|J| = 1$ otherwise. Thus, $\mathcal{O}_{(m_3\mathbf{e}_3,\mathbf{0})} = \{(\pm m_3\mathbf{e}_3, \mathbf{0})\}$ and hence the fibers of the associated fiber bundle $\widetilde{\mathcal{O}}_{(m_3\mathbf{e}_3,\mathbf{0})} \to M$ are two points sets. In this special situation, the symplectic structure on $T^*M \times_M \widetilde{\mathcal{O}}_{(m_3\mathbf{e}_3,\mathbf{0})}$ is given by the canonical symplectic form on T^*M since the Lie algebra $\mathfrak{g}_{A_0}$ is one-dimensional and the fiber is discrete, see [12, Theorem 2.3.12]. We conclude that the coadjoint orbit $\mathcal{O}_{(\mathbf{m},\mathbf{n},A_0)}$ has two connected components each one symplectically diffeomorphic to T^*M for any $\mathbf{m},\mathbf{n} \in \mathbb{R}^3$. In particular, the dimension of the coadjoint orbit $\mathcal{O}_{(\mathbf{m},\mathbf{n},A_0)}$ is ten.

Now, we extend the Hamiltonian (4.12) to the symplectic manifold $T^*M \times_M \widetilde{\mathcal{O}}_{(m_3\mathbf{e}_3,\mathbf{0})}$. Hamilton's equations are unchanged. Using the symplectomorphism of Theorem 3.1, we get a Hamiltonian function on the coadjoint orbit $\mathcal{O}_{(\mathbf{m},\mathbf{n},A_0)}$ of the semidirect product $(SO(3)_L \times SO(3)_R)Ⓢ\mathfrak{gl}(3,\mathbb{C})$. It is a symplectic leaf of the Lie-Poisson manifold $[(\mathfrak{so}(3)_L \times \mathfrak{so}(3)_R)Ⓢ\mathfrak{gl}(3,\mathbb{C})]^*$ and hence of its Poisson submanifold $(\mathfrak{so}(3)_L \times \mathfrak{so}(3)_R)^* \times M$, endowed with the Lie-Poisson bracket

$$\begin{aligned}\{f,h\}(\mathbf{m},\mathbf{n},A) &= \mathbf{m}\cdot\frac{\delta f}{\delta \mathbf{m}} \times \frac{\delta h}{\delta \mathbf{m}} - \mathbf{n}\cdot\frac{\delta f}{\delta \mathbf{n}} \times \frac{\delta h}{\delta \mathbf{n}}\\ &\quad + \left\langle \frac{\delta f}{\delta A}, \widehat{\frac{\delta h}{\delta \mathbf{m}}}A + A\widehat{\frac{\delta h}{\delta \mathbf{n}}}\right\rangle - \left\langle \frac{\delta h}{\delta A}, \widehat{\frac{\delta f}{\delta \mathbf{m}}}A + A\widehat{\frac{\delta f}{\delta \mathbf{n}}}\right\rangle.\end{aligned} \tag{4.13}$$

A direct computation shows that the kernel of the Poisson tensor is one dimensional at all points $(\mathbf{m}, \mathbf{n}, A_0)$. This means that the dimension of the symplectic leaves through $(\mathbf{m}, \mathbf{n}, A_0)$ is ten. We have recovered the previous result stating that the dimension of the coadjoint orbit $\mathcal{O}_{(\mathbf{m},\mathbf{n},A_0)}$ is ten. We note that the function $C(\mathbf{m}, \mathbf{n}, A) = \frac{1}{2}\operatorname{Re}\operatorname{Tr}(A^*A)$ is a Casimir function of this bracket. Indeed, since $\frac{\delta C}{\delta A} = A$, a direct computation that involves only the third term in the expression above shows that $\{C, f\} = 0$ for all functions f.

Lemma 4.4. *The Riemannian metric on M induced by the* Ad*-invariant inner product $\gamma((\mathbf{a}, \mathbf{b}), (\mathbf{v}, \mathbf{w})) = \mathbf{a}\cdot\mathbf{v} + \mathbf{b}\cdot\mathbf{w}$ on $\mathfrak{so}(3)_L \times \mathfrak{so}(3)_R$ (M is viewed here as the orbit $(SO(3)_L \times SO(3)_R)/\tilde{G}_{A_0}$ as in Remark 4.3) coincides with the metric induced by the inner product* (4.10) (*here, $M \subset \mathfrak{gl}(3,\mathbb{C})$*)*, that is,*

$$\gamma_M(\widehat{\mathbf{a}}A + A\widehat{\mathbf{b}}, \widehat{\mathbf{v}}A + A\widehat{\mathbf{w}}) = \operatorname{Re}\operatorname{Tr}\left((\widehat{\mathbf{a}}A + A\widehat{\mathbf{b}})^*(\widehat{\mathbf{v}}A + A\widehat{\mathbf{w}})\right).$$

Proof. We need to verify identity (3.3). It is readily checked that at A_0, we have $\operatorname{Re}\operatorname{Tr}\left((\widehat{\mathbf{a}}A_0 + A_0\widehat{\mathbf{b}})^*(\widehat{\mathbf{v}}A_0 + A_0\widehat{\mathbf{w}})\right) = a_1v_1 + a_2v_2 + \mathbf{b}\cdot\mathbf{w} = \mathbb{P}^\perp_{A_0}(\mathbf{a}, \mathbf{b}) \cdot \mathbb{P}^\perp_{A_0}(\mathbf{v}, \mathbf{w})$, where $\mathbb{P}^\perp_{A_0}(\mathbf{a}, \mathbf{b}) = ((a_1, a_2, 0), \mathbf{b})$. Since $(\mathbf{a}, \mathbf{b})_M(A) = \widehat{\mathbf{a}}A + A\widehat{\mathbf{b}}$, inserting the expression $A = R_1A_0R_2^{-1}$, we get

$$\begin{aligned}\operatorname{Re}\operatorname{Tr}\left((\widehat{\mathbf{a}}A + A\widehat{\mathbf{b}})^*(\widehat{\mathbf{v}}A + A\widehat{\mathbf{w}})\right) &= (R_1^{-1}\mathbf{a})_1(R_1^{-1}\mathbf{v})_1 + (R_1^{-1}\mathbf{a})_2(R_1^{-1}\mathbf{a})_2 + R_2^{-1}\mathbf{b}\cdot R_2^{-1}\mathbf{w}\\ &= \mathbb{P}^\perp_{A_0}(R_1^{-1}\mathbf{a}, R_2^{-1}\mathbf{b}) \cdot \mathbb{P}^\perp_{A_0}(R_1^{-1}\mathbf{b}, R_2^{-1}\mathbf{w}),\end{aligned}$$

which proves the formula. □

It follows that formula (3.4) can be applied. Therefore, we get

$$\operatorname{Hor}^*_{(R_1,R_2)}(\widehat{\mathbf{m}}R_1, R_2\widehat{\mathbf{n}}) = \widehat{\mathbf{m}}A + A\widehat{\mathbf{n}} \in T^*_AM.$$

Fixing $\mu = (\mathbf{m}_0, \mathbf{n}_0)$ and applying Theorem 3.1 we get the Hamiltonian function on the coadjoint orbit $\mathcal{O}_{(\mathbf{m}_0,\mathbf{n}_0,A_0)}$ by pulling back the Hamiltonian $\mathcal{H}$ in (4.12). We obtain

$$h(\mathbf{m}, \mathbf{n}, A) = \mathcal{H}(\widehat{\mathbf{m}}A + A\widehat{\mathbf{n}}) = \frac{1}{4}\left\langle(\widehat{\mathbf{m}}A + A\widehat{\mathbf{n}})\Gamma^{-1}, \widehat{\mathbf{m}}A + A\widehat{\mathbf{n}}\right\rangle, \tag{4.14}$$

where $(\mathbf{m}, \mathbf{n}, A) \in \mathcal{O}_{(\mathbf{m}_0,\mathbf{n}_0,A_0)}$.

The general formula for the coadjoint action on a semidirect product (3.2) (see, e.g., [14]) yields in this case

$$\operatorname{Ad}^*_{(R_1,R_2,V)^{-1}}(\mathbf{m}, \mathbf{n}, A) = \left(R_1\mathbf{m} + 2\overrightarrow{\operatorname{Re}(R_1AR_2^{-1}V)}, R_2\mathbf{n} + 2\overrightarrow{\operatorname{Re}(VR_1AR_2^{-1})}, R_1AR_2^{-1}\right),$$

where $(R_1, R_2, V) \in (SO(3)_L \times SO(3)_R) \circledS \mathfrak{gl}(3,\mathbb{C})$.

Integrability. We will now consider subgroup actions of the coadjoint action that are symmetries of the Hamiltonian (4.14) and compute the associated momentum maps.

The first one is given by the $U(1)$-action $\operatorname{Ad}^*_{(I_3,\rho(\varphi),0)}(\mathbf{m}, \mathbf{n}, A) = (\mathbf{m}, \rho(\varphi)\mathbf{n}, A\rho(-\varphi))$. This action is automatically Poisson and leaves the Hamiltonian (4.14) invariant because $\rho(\varphi)\Gamma^{-1} = \Gamma^{-1}\rho(\varphi)$. The infinitesimal generator of this action is $(\mathbf{m}, \mathbf{n}, A) \mapsto (\mathbf{0}, \mathbf{e}_3 \times \mathbf{n}, -A\widehat{\mathbf{e}}_3)$ and a momentum map is found to be $\mathbf{J}^{\text{orb}}_3(\mathbf{m}, \mathbf{n}, A) = -\mathbf{e}_3 \cdot \mathbf{n}$. Therefore, $\{\mathbf{J}^{\text{orb}}_3, h\} = 0$.

The second symmetry is given by the $SO(3)$-action $\mathrm{Ad}^*_{(R,I_3,0)}(\mathbf{m},\mathbf{n},A) = (R\mathbf{m},\mathbf{n}, RA)$ whose infinitesimal generator associated to $\widehat{\mathbf{v}} \in \mathfrak{so}(3)$ is $(\mathbf{m},\mathbf{n},A) \mapsto (\mathbf{v}\times\mathbf{m},\mathbf{0},\widehat{\mathbf{v}}A)$. This action leaves the Hamiltonian (4.14) invariant. The momentum map is $\mathbf{J}^{\mathrm{spin}}(\mathbf{m},\mathbf{n},A) = \mathbf{m}$.

To find the next conserved quantity is considerably more involved. We start with the Euler–Lagrange equations for the Lagrangian $\mathcal{L}(A,\partial_z A)$ in (4.9) on TM. This Lagrangian is $U(1)$-invariant under the tangent lift of the action $A \mapsto e^{i\varphi}A$. The infinitesimal generator associated to $\theta \in \mathbb{R}$ is $\theta_M(A) = \mathrm{i}\theta A$. Using (4.11), we compute the associated momentum map as follows

$$j_m(A,\partial_z A) = \left\langle \frac{\delta\mathcal{L}}{\delta(\partial_z A)}, \mathrm{i}A \right\rangle = 2\left\langle \partial_z A\,\Gamma, \mathrm{i}A \right\rangle . \tag{4.15}$$

Taking into account that $\partial_z A = \widehat{\mathbf{v}}A + A\widehat{\mathbf{w}}$, for some $\mathbf{v},\mathbf{w} \in \mathbb{R}^3$, this formula becomes

$$j_m(A,\widehat{\mathbf{v}}A + A\widehat{\mathbf{w}}) = -2\,\mathrm{Re}\,\mathrm{Tr}(A\Gamma A^*\widehat{\mathbf{v}}\mathrm{i}) - 2\,\mathrm{Re}\,\mathrm{Tr}(\Gamma\widehat{\mathbf{w}}A^*A\mathrm{i}) = 2\,\langle\!\langle \widehat{\mathbf{v}}A + A\widehat{\mathbf{w}}, \mathrm{i}A\rangle\!\rangle .$$

Theorem 4.5. *The five functions h, j_m, $\mathbf{J}^{\mathrm{orb}}_3$, $\mathbf{J}^{\mathrm{spin}}_3$, $\|\mathbf{J}^{\mathrm{spin}}\|^2$ form a completely integrable system on the ten dimensional coadjoint orbit $\mathcal{O}_{(\mathbf{m}_0,\mathbf{n}_0,A_0)}$.*

Proof. We need to check that the five functions pairwise commute and their differentials are almost everywhere linearly independent. First we check that they pairwise commute. It is clear that $\{\mathbf{J}^{\mathrm{spin}}_{\mathbf{v}}, h\} = 0$, for all $\mathbf{v} \in \mathbb{R}^3$. In particular, $\{\mathbf{J}^{\mathrm{spin}}_3, h\} = 0$ and $\{\|\mathbf{J}^{\mathrm{spin}}\|^2, h\} = 0$. In addition, formula (4.13) implies that $\{\mathbf{J}^{\mathrm{spin}}_3, \mathbf{J}^{\mathrm{orb}}_3\} = 0$ and $\{\|\mathbf{J}^{\mathrm{spin}}\|^2, \mathbf{J}^{\mathrm{orb}}_3\} = 0$.

Since j_m is conserved on the solutions of the Euler–Lagrange equations associated to $\mathcal{L}$, its pull-back to $\mathcal{O}_{(\mathbf{m}_0,\mathbf{n}_0,A_0)}$ commutes with the Hamiltonian h.

In order to see that j_m commutes with $\mathbf{J}^{\mathrm{orb}}_3$ and $\mathbf{J}^{\mathrm{spin}}$, we will consider the induced $U(1)$ and $SO(3)$-actions on TM and T^*M and observe that they are the tangent and cotangent lift of commuting actions. Therefore, viewed as momentum maps on T^*M and TM, via the change of variables $\mathcal{O}_{(\mathbf{m}_0,\mathbf{n}_0,A_0)} \to T^*M \to TM$ (see Theorem 3.1 and 4.11), these momentum maps commute. Concerning $\mathbf{J}^{\mathrm{orb}}_3$, the $U(1)$-action induced on TM is the tangent lift of the action $A \mapsto A\rho(-\varphi)$. For $\mathbf{J}^{\mathrm{spin}}$, the $SO(3)$-action induced on TM is the tangent lift of the action $A \mapsto RA$, $R \in SO(3)$. They evidently commute with the action $A \mapsto e^{i\varphi}A$ yielding j_m.

One can also check directly that the expressions of the momentum maps $j^{\mathrm{orb}}_3(A,\partial_z A)$ and $j^{\mathrm{spin}}(A,\partial_z A)$ on TM associated to the tangent lifted actions of $A \mapsto A\rho(-\varphi)$ and $A \mapsto RA$ are consistent with those of $\mathbf{J}^{\mathrm{orb}}_3(\mathbf{m},\mathbf{n},A)$ and $\mathbf{J}^{\mathrm{spin}}(\mathbf{m},\mathbf{n},A)$, respectively.

The five functions commute in view of the discussion above. We need to show that their differentials are linearly independent except on a set of measure zero in $\mathcal{O}_{(\mathbf{m}_0,\mathbf{n}_0,A_0)}$. It turns out that showing their independence on M, rather than on $\mathcal{O}_{(\mathbf{m}_0,\mathbf{n}_0,A_0)}$, is considerably simpler computationally. The functional derivatives on TM are

$$\frac{\delta j_m}{\delta A} = -2\mathrm{i}\partial_z A\,\Gamma, \quad \frac{\delta j_m}{\delta \partial_z A} = 2\mathrm{i}A\Gamma, \quad \frac{\delta j_3^{\mathrm{orb}}}{\delta A} = 2\partial_z A\Gamma\widehat{\mathbf{e}}_3, \quad \frac{\delta j_3^{\mathrm{orb}}}{\delta \partial_z A} = -2A\widehat{\mathbf{e}}_3\Gamma,$$
$$\frac{\delta j_k^{\mathrm{spin}}}{\delta A} = -2\widehat{\mathbf{e}}_k\partial_z A\,\Gamma, \quad \frac{\delta j_k^{\mathrm{spin}}}{\delta \partial_z A} = 2\widehat{\mathbf{e}}_k A\Gamma, \quad \frac{\delta \mathcal{L}}{\delta A} = 0, \quad \frac{\delta \mathcal{L}}{\delta \partial_z A} = 2\partial_z A\,\Gamma,$$
$$\frac{\delta \|\mathbf{J}^{\mathrm{spin}}\|^2}{\delta A} = -4j_1^{\mathrm{spin}}\widehat{\mathbf{e}}_1\partial_z A\,\Gamma - 4j_2^{\mathrm{spin}}\widehat{\mathbf{e}}_2\partial_z A\,\Gamma,$$
$$\frac{\delta \|\mathbf{J}^{\mathrm{spin}}\|^2}{\delta \partial_z A} = 4j_1^{\mathrm{spin}}\widehat{\mathbf{e}}_1 A\Gamma + 4j_2^{\mathrm{spin}}\widehat{\mathbf{e}}_2 A\Gamma.$$

In order to show the independence, we have to show that the equations

$$\alpha_1\frac{\delta j_m}{\delta A} + \alpha_2\frac{\delta j_3^{\mathrm{orb}}}{\delta A} + \alpha_3\frac{\delta j_3^{\mathrm{spin}}}{\delta A} + \alpha_4\frac{\delta \|\mathbf{J}^{\mathrm{spin}}\|^2}{\delta A} + \alpha_5\frac{\delta \mathcal{L}}{\delta A} = 0 \tag{4.16}$$
$$\alpha_1\frac{\delta j_m}{\delta \partial_z A} + \alpha_2\frac{\delta j_3^{\mathrm{orb}}}{\delta \partial_z A} + \alpha_3\frac{\delta j_3^{\mathrm{spin}}}{\delta \partial_z A} + \alpha_4\frac{\delta \|\mathbf{J}^{\mathrm{spin}}\|^2}{\delta \partial_z A} + \alpha_5\frac{\delta \mathcal{L}}{\delta \partial_z A} = 0 \tag{4.17}$$

imply $\alpha_i = 0$, for all $i = 1, \ldots, 5$ and for all $A \in M$ except on a set of measure zero in M.

Writing $A = \mathbf{x}(\mathbf{A}_1 + \mathrm{i}\mathbf{A}_2)^{\mathsf{T}}$, where $\|\mathbf{x}\| = 1$, $\|\mathbf{A}_i\| = 1$, $\mathbf{A}_1 \cdot \mathbf{A}_2 = 0$, and using the formula $\partial_z A = (\partial_z\mathbf{x})(\mathbf{A}_1 + \mathrm{i}\mathbf{A}_2)^{\mathsf{T}} + \mathbf{x}(\partial_z\mathbf{A}_1 + \mathrm{i}\partial_z\mathbf{A}_2)^{\mathsf{T}}$, where $\partial_z\mathbf{x}\cdot\mathbf{x} = 0$, $\partial_z\mathbf{A}_i \cdot \mathbf{A}_i = 0$, $\partial_z\mathbf{A}_1 \cdot \mathbf{A}_2 + \mathbf{A}_1 \cdot \partial_z\mathbf{A}_2 = 0$, and evaluating Eq. (4.16) on the vector $(\mathbf{A}_1 + \mathrm{i}\mathbf{A}_2) \times (\partial_z\mathbf{A}_1 + \mathrm{i}\partial_z\mathbf{A}_2)$, we get $\alpha_2(\partial_z\mathbf{A}_1 + \mathrm{i}\partial_z\mathbf{A}_2)^{\mathsf{T}}(\partial_z\mathbf{A}_1 + \mathrm{i}\partial_z\mathbf{A}_2)(\mathbf{e}_3 \cdot (\mathbf{A}_1 + \mathrm{i}\mathbf{A}_2)) = 0$. This implies $\alpha_2 = 0$ except on a set of measure zero in M.

Using this and evaluating Eq. (4.17) on $\mathbf{A}_1 \times \mathbf{A}_2$, we get $\alpha_5(\partial_z\mathbf{A}_1 + \mathrm{i}\partial_z\mathbf{A}_2)^{\mathsf{T}}(\mathbf{A}_1 \times \mathbf{A}_2) = 0$ which again implies $\alpha_5 = 0$ except on a set of measure zero in M. Then multiplying (4.17) on the right by $A^*\mathbf{x}$, taking the dot product with $\mathbf{x}$, and using the formula $AA^*\mathbf{x} = 2\mathbf{x}$, we get $\alpha_1 = 0$. Multiplying the remaining equation on the left by $\mathbf{e}_3^{\mathsf{T}}$, we get $\alpha_4(j_1^{\mathrm{spin}}(A, \partial_z A)\mathbf{e}_2^{\mathsf{T}} - j_2^{\mathrm{spin}}(A, \partial_z A)\mathbf{e}_1^{\mathsf{T}})\partial_z A = 0$, which again, except on a set of measure zero in M, implies $\alpha_4 = 0$. From this it follows that $\alpha_3 = 0$ except on a set of measure zero in M. □

4.2.2. The A-phase—second regime. In this situation, we consider the orbit $M = \{RA_0R^{-1} \mid R \in SO(3)\}$ of $SO(3)$ through A_0 given by (4.4). A direct verification proves the following result.

Proposition 4.6. (i) *The isotropy subgroup $SO(3)_{A_0}$ equals*

$$SO(3)_{A_0} = \left\{ I_3, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

the group isomorphism being given by

$$I_3 \longleftrightarrow (1,1), \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \longleftrightarrow (1,-1),$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow (-1,1), \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \longleftrightarrow (-1,-1).$$

(ii) *The map*

$$SO(3)/SO(3)_{A_0} \ni [R]_{SO(3)_{A_0}} \longmapsto RA_0R^{-1} \in \mathrm{Orb}(A_0).$$

is a diffeomorphism.

Lagrangian formulation. We apply Theorem 4.1 with this description of the orbit. The reduced velocity (see (2.4)) is given here by $\mathbf{w} = (\partial_z R)R^{-1}$. The second formula in (2.6) becomes

$$\partial_z A = \widehat{\mathbf{w}}A - A\widehat{\mathbf{w}} = [\widehat{\mathbf{w}}, A].$$

Using this expression and formula (4.1), the Euler–Poincaré Lagrangian (2.5) reads

$$l(\mathbf{w}, A) = \mathrm{Re}\,\mathrm{Tr}(\Gamma \partial_z A^* \partial_z A) = \langle\!\langle [A, \widehat{\mathbf{w}}], [A, \widehat{\mathbf{w}}] \rangle\!\rangle\,,$$

(see (4.2)). Defining

$$J_{ab}(A) = \langle\!\langle [A, \widehat{\mathbf{e}}_a], [A, \widehat{\mathbf{e}}_b] \rangle\!\rangle\,, \tag{4.18}$$

the Lagrangian (4.9) reads

$$l(\mathbf{w}, A) = \sum_{a,b=1}^{3} J_{ab}(A) w_a w_b = \mathbf{w}^{\mathsf{T}} \mathbf{J}(A)\mathbf{w}.$$

The Euler–Poincaré equations (2.6) are

$$\partial_z \frac{\delta l}{\delta \mathbf{w}} + \mathrm{ad}^*_{\mathbf{w}} \frac{\delta l}{\delta \mathbf{w}} = -\mathbf{J}\left(\frac{\delta l}{\delta A}\right),$$

where $\mathbf{J} : T^*\,\mathrm{Orb}(A_0) \to \mathfrak{so}(3)^*$ is a momentum map of the right action of $SO(3)$ on $T^*\,\mathrm{Orb}(A_0)$. Using the duality pairing $\langle A, B\rangle = \mathrm{Re}\,\mathrm{Tr}(A^*B)$ on $\mathfrak{gl}(3,\mathbb{C})$, we get

$$\frac{d}{dz}\frac{\delta l}{\delta \mathbf{w}} + \frac{\delta l}{\delta \mathbf{w}} \times \mathbf{w} = 2\,\mathrm{Re}\,\overrightarrow{\left[\frac{\delta l}{\delta A}^*, A\right]}, \tag{4.19}$$

where we have

$$\frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{J}(A)\mathbf{w}, \qquad \frac{\delta l}{\delta A} = 2[[\widehat{\mathbf{w}}, A]\Gamma, \widehat{\mathbf{w}}].$$

Hamiltonian formulation. Using the general formula (3.2) for the coadjoint action of the semidirect product $SO(3) \circledS \mathfrak{gl}(3,\mathbb{C})$, it is it easily seen that the coadjoint orbit through $(0, A_0)$ is $\mathfrak{so}(3)^* \times \mathrm{Orb}(A_0)$. This is consistent with the general theory in Theorem 3.1: since $\mathfrak{g}_{a_0} = 0$, the dimension of the orbit is six.

Define $\mathbf{m} := \frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{J}(A)\mathbf{w}$. Thus the Hamiltonian associated to the Lagrangian l has the expression

$$h(\mathbf{m}, A) = \mathbf{m}^{\mathsf{T}}\mathbf{w} - l(\mathbf{w}, A) = \frac{1}{4}\mathbf{m}^{\mathsf{T}}\mathbf{J}(A)^{-1}\mathbf{m}. \tag{4.20}$$

The non-degenerate Lie–Poisson bracket on the coadjoint orbit $\mathfrak{so}(3)^* \times \mathrm{Orb}(A_0)$ is

$$\{f, h\}(\mathbf{m}, A) = \mathbf{m} \cdot \left(\frac{\delta f}{\delta \mathbf{m}} \times \frac{\delta h}{\delta \mathbf{m}}\right) + \left\langle \frac{\delta h}{\delta A}, \left[A, \widehat{\frac{\delta f}{\delta \mathbf{m}}}\right]\right\rangle - \left\langle \frac{\delta f}{\delta A}, \left[A, \widehat{\frac{\delta h}{\delta \mathbf{m}}}\right]\right\rangle \tag{4.21}$$

and hence the equations $\partial_z f = \{f, h\}$, for any f, are

$$\partial_z \mathbf{m} + \mathbf{m} \times \frac{\delta h}{\delta \mathbf{m}} = -2\,\mathrm{Re}\,\overrightarrow{\left[\frac{\delta h}{\delta A}^*, A\right]} \qquad \partial_z A = \left[\widehat{\frac{\delta h}{\delta \mathbf{m}}}, A\right]. \tag{4.22}$$

Integrability. In the following Theorem, we prove that the Hamiltonian system given by (4.20) relative to the Poisson bracket (4.21) on the six dimensional coadjoint obit $\mathfrak{so}(3)^* \times \mathrm{Orb}(A_0)$ is completely integrable.

Theorem 4.7. *The three functions h, j_m, $\mathbf{J}_3$ form a completely integrable system on the six dimensional coadjoint orbit* $\mathfrak{so}(3)^* \times \mathrm{Orb}(A_0)$.

Proof. The three integrals of motion are the Hamiltonian (4.20), the momentum map j_m given in (4.15), i.e., $j_m(\mathbf{m}, A) = \left\langle\!\left\langle \left[\frac{1}{2}\widehat{\mathbf{J}(A)^{-1}\mathbf{m}}, A\right], \mathrm{i}A \right\rangle\!\right\rangle$ after transforming to the variables $(\mathbf{m}, A)$, and $\mathbf{J}_3(\mathbf{m}, A) := \mathbf{e}_3 \cdot \mathbf{m}$. As in the discussion of the A-phase, the previous regime, we note that j_m is a momentum map associated to the circle action on configuration space given by $A \mapsto e^{\mathrm{i}\varphi}A$. Puling back j_m to the Hamiltonian side, i.e., expressing it in the variables $(\mathbf{m}, A)$, it follows that $\{h, j_m\} = 0$. It is important to note that this $U(1)$-action with momentum map j_m is expressed in the variables $(\mathbf{m}, A)$ as: $(\mathbf{m}, A) \mapsto \left(\mathbf{m}, e^{\mathrm{i}\varphi}A\right)$.

Now, consider a second circle action on $\mathfrak{so}(3)^* \times \mathrm{Orb}(A_0)$ given by

$$(\mathbf{m}, A) \mapsto (\rho(\varphi)\mathbf{m}, \rho(\varphi)A\rho(\varphi)^{-1}), \quad \text{where} \quad \rho(\varphi) := \exp(\varphi\widehat{\mathbf{e}}_3).$$

This is the coadjoint action of a subgroup of $SO(3)\,\circledS\,\mathfrak{gl}(3, \mathbb{C})$ and hence it is Poisson. It admits a momentum map which is $\mathbf{J}_3$. The Hamiltonian h given by (4.20) is invariant under this action and so we conclude that $\{h, \mathbf{J}_3\} = 0$. The action $(\mathbf{m}, A) \mapsto (\rho(\varphi)\mathbf{m}, \rho(\varphi)A\rho(\varphi)^{-1})$ is induced via the cotangent bundle version of Theorem 4.1 by the cotangent lift of the action $A \mapsto \rho(\varphi)A\rho(\varphi)^{-1}$. This action on configuration space commutes with the previously considered circle action $A \mapsto e^{\mathrm{i}\varphi}A$. Therefore, the associated momentum maps commute, i.e., $\{j_m, \mathbf{J}_3\} = 0$. Concluding, we have $\{h, j_m\} = 0$, $\{h, \mathbf{J}_3\} = 0$, $\{j_m, \mathbf{J}_3\} = 0$.

Finally, we prove the functional independence of the three integrals h, j_m, $\mathbf{J}_3$. Instead of showing that their differentials are linearly independent away from a subset of measure zero in $\mathfrak{so}(3)^* \times \mathrm{Orb}(A_0)$, we will show that the Hamiltonian vector fields generated by these integrals are linearly independent on such a set. Since j_m and $\mathbf{J}_3$ are momentum maps, their Hamiltonian vector fields relative to the Lie–Poisson bracket (4.21) coincide with the infinitesimal generator vector fields of the corresponding $U(1)$-actions. These vector fields are hence $(\mathbf{m}, A) \mapsto (\mathbf{m}, A; \mathbf{0}, \mathrm{i}A)$ and $(\mathbf{m}, A) \mapsto (\mathbf{m}, A; \mathbf{e}_3 \times \mathbf{m}, [\widehat{\mathbf{e}}_3, A])$.

Now we compute the Hamiltonian vector field for h given by (4.20). We have $\delta h/\delta \mathbf{m} = \frac{1}{2}\mathbf{J}(A)^{-1}\mathbf{m}$. A direct computation shows that

$$\frac{\delta h}{\delta A} = 2\,[\widehat{\mathbf{w}}, [\widehat{\mathbf{w}}, A]\,\Gamma]\,, \quad \text{where} \quad \mathbf{w} := \frac{1}{2}\mathbf{J}(A)^{-1}\mathbf{m}.$$

Therefore, from (4.22), we obtain the expression of the Hamiltonian vector field defined by h, namely,

$$(\mathbf{m}, A) \longmapsto X_h(\mathbf{m}, A) = \left(\mathbf{m}, A; \mathbf{w} \times \mathbf{m} - 4\,\mathrm{Re}\,\overrightarrow{\left[\,[\Gamma\,[A, \widehat{\mathbf{w}}]\,, \widehat{\mathbf{w}}]\,, A\right]}, [\widehat{\mathbf{w}}, A]\right).$$

We need to show that

$$\begin{cases} \alpha_1 \left(\mathbf{w} \times \mathbf{m} - 4\,\mathrm{Re}\,\overrightarrow{X}\right) + \alpha_3 \mathbf{e}_3 \times \mathbf{m} = 0, \\ \alpha_1\,[\widehat{\mathbf{w}}, A] + \alpha_2 \mathrm{i} A + \alpha_3\,[\widehat{\mathbf{e}}_3, A] = 0, \end{cases}$$

where $X := \left[\,[\Gamma\,[A, \widehat{\mathbf{w}}]\,, \widehat{\mathbf{w}}]\,, A\right]$, implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ for generic $(\mathbf{m}, A)$. Taking the dot product of the first equation with $\mathbf{m}$ yields $\alpha_1\,\mathrm{Re}\,\overrightarrow{X} \cdot \mathbf{m} = 0$. It is easy to find points $(\mathbf{m}, A)$ for which $\mathrm{Re}\,\overrightarrow{X} \cdot \mathbf{m} \neq 0$. Since this expression is polynomial in $\mathbf{w}$ and A and since it does not vanish identically, its set of zeros is of measure zero in $\mathfrak{so}(3)^* \times \mathrm{Orb}(A_0)$. This shows that for a set of measure zero on this phase space, $\alpha_1 = 0$. Choosing $\mathbf{m}$ not collinear with $\mathbf{e}_3$, implies that $\alpha_3 = 0$. Now, for any $A \in \mathrm{Orb}(A_0) \neq 0$, we get $\alpha_2 = 0$. This proves the Theorem. □

4.2.3. The B-phase. We work, as before, with the Lie group action (4.3) and consider the element

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathfrak{gl}(3, \mathbb{C}).$$

Proposition 4.8. (i) *The isotropy subgroup of* I_3 *is*

$$H = \{(1, R, R) \mid R \in SO(3)\} \subset G = U(1) \times SO(3)_L \times SO(3)_R.$$

(ii) *We have the diffeomorphism*

$$M = G/H \ni [e^{\mathrm{i}\varphi}, R_1, R_2]_H \longmapsto (e^{\mathrm{i}\varphi}, R_1 R_2^{-1}) \in U(1) \times SO(3). \tag{4.23}$$

(iii) *We have the diffeomorphism*

$$U(1) \times SO(3) \ni (e^{\mathrm{i}\varphi}, R) \longmapsto e^{\mathrm{i}\varphi} R \in \mathrm{Orb}(I_3). \tag{4.24}$$

We observe that the subgroup $\widetilde{G} := U(1) \times SO(3)_L \subset G$ acts transitively on the orbit $\mathrm{Orb}(I_3)$, see (4.23), (4.24). The isotropy subgroup of I_3 is $\widetilde{G}_{I_3} = H \cap \widetilde{G} = \{1, I_3, I_3\}$. We recover the fact that the orbit $\mathrm{Orb}(I_3) \subset \mathfrak{gl}(3, \mathbb{C})$ is diffeomorphic to $U(1) \times SO(3)$.

As a consequence, we apply Theorem 4.1 with this description of the orbit and hence the Lie algebra one has to consider is $\mathbb{R} \times \mathfrak{so}(3)_L$. On this orbit M we consider the Lagrangian density given by the gradient part only, i.e., $\mathcal{L}(A, \nabla A) = F_{grad}(A, \nabla A)$.

The reduced velocity $\xi(z) = \partial_z g g^{-1}$ of the general theory (see (2.4)) is given here by $\xi = (v, \mathbf{w}) : X \to \mathbb{R} \times \mathfrak{so}(3)_L$, where $v = \partial_z \varphi$, and $\mathbf{w} = R^{-1}(\partial_z R)$. The Euler–Poincaré Lagrangian

$$l = l(\xi, m) : \mathbb{R} \times \mathfrak{so}(3)_L \times M \to \mathbb{R}$$

of the general theory given in (2.5), is computed in this case to be

$$\begin{aligned} l(v, \mathbf{w}, A) &= F_{grad}(A, \partial_z A) = \langle\langle \mathrm{i}vA + A\widehat{\mathbf{w}}, \mathrm{i}vA + A\widehat{\mathbf{w}} \rangle\rangle \\ &= v^2 \langle\langle A, A \rangle\rangle + \mathbf{w}^{\mathsf{T}} I(A)\mathbf{w} + 2v \langle\langle A\mathrm{i}, A\widehat{\mathbf{w}} \rangle\rangle \\ &= 2\gamma_1 \|\mathbf{w}\|^2 + (\gamma_2 + \gamma_3)(w_1^2 + w_2^2) + (3\gamma_1 + \gamma_2 + \gamma_3)v^2, \end{aligned}$$

where $I_{ab}(A) = \langle\langle A\widehat{\mathbf{e}}_a, A\widehat{\mathbf{e}}_b \rangle\rangle$ and we used $A = e^{\mathrm{i}\varphi} R$ so that $A^* A = I_3$.

The Euler–Poincaré equations (2.6) are

$$\partial_z \frac{\delta l}{\delta v} = \mathbf{J}_1 \left(\frac{\delta l}{\delta A} \right), \qquad \partial_z \frac{\delta l}{\delta \mathbf{w}} - \mathrm{ad}^*_{\mathbf{w}} \frac{\delta l}{\delta \mathbf{w}} = \mathbf{J}_2 \left(\frac{\delta l}{\delta A} \right), \tag{4.25}$$

where $\mathbf{J}_1 : T^*M \to \mathbb{R}$ is a momentum map of the $U(1)$ action $A \mapsto e^{\mathrm{i}\psi} A$, and $\mathbf{J}_2 : T^*M \to \mathfrak{so}(3)^*_L$ is a momentum map of the $SO(3)$ action $A \mapsto QA$. They have the expressions

$$\mathbf{J}_1(\alpha_A) = \mathrm{Re}\,\mathrm{Tr}(\alpha_A^* \mathrm{i} A) \quad \text{and} \quad \mathbf{J}_2(\alpha_A) = \overrightarrow{2\mathrm{Re}\left(\alpha_A^* A\right)}. \tag{4.26}$$

The second equation in (2.6) is given, in this case, by

$$\partial_z A = \mathrm{i}vA + A\widehat{\mathbf{w}}.$$

Since

$$\frac{\delta l}{\delta v} = 2(3\gamma_1 + \gamma_2 + \gamma_3)v, \quad \frac{\delta l}{\delta \mathbf{w}} = 4\gamma_1 \mathbf{w} + 2(\gamma_2 + \gamma_3)(w_1, w_2, 0)^{\mathsf{T}} = \mathbf{J}\mathbf{w}, \quad \frac{\delta l}{\delta A} = 0,$$

where $\mathbf{J} = \mathrm{diag}(4\gamma_1 + 2\gamma_2 + 2\gamma_3, 4\gamma_1 + 2\gamma_2 + 2\gamma_3, 4\gamma_1)$, we get

$$\partial_z v = 0, \qquad \partial_z \mathbf{J}\mathbf{w} + \mathbf{w} \times \mathbf{J}\mathbf{w} = 0, \qquad \partial_z A = ivA + A\widehat{\mathbf{w}}. \tag{4.27}$$

The equation for $\mathbf{w}$ is the Euler equation for a free symmetric rigid body with moment of inertia $\mathbf{J}$ whose solutions are well-known (see e.g., [11, Sect. 5.7]). These are then substituted in the third equation of (4.27) which is a linear equation in A; the system can be solved explicitly.

Remark 4.9. An interesting question is the transition from one phase to another. In the case of liquid crystals, this means the transition from biaxial nematics to uniaxial nematics which leads to the deformation of the orbit $SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ to $SO(3)/\mathbb{Z}_2$. We do not consider this problem in the present paper.

4.3. Neutron stars. The classification of the phases in neutron stars parallels the procedure used in the classification of the phases for superfluid liquid Helium ^{3}He, taking into account the differences associated to the order parameter of these two problems. For details, see [16]. We shall use the notations of this paper below and consider several examples of orbits.

Define $S^2(\mathbb{C}^3)_0 := \left\{A \in \mathfrak{gl}(3, \mathbb{C}) \mid A^\mathsf{T} = A,\ \operatorname{Tr}(A) = 0\right\}$. The direct product Lie group $U(1) \times SO(3)$ acts on $S^2(\mathbb{C}^3)_0$ by

$$A \longmapsto e^{\mathrm{i}\varphi} RAR^{-1}, \tag{4.28}$$

where $e^{\mathrm{i}\varphi} \in U(1)$ and $R \in SO(3)$. The possible non-trivial orbits of this group action are given in [16] and denoted by Ω_i, where $i = 1, \ldots, 10$. We shall consider below only the most interesting examples $i = 1, 4, 6, 8$.

4.3.1. Ω_1-phase. Consider the orbit of the group $U(1) \times SO(3)$ through

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \in S^2(\mathbb{C}^3)_0.$$

A direct verification shows that the isotropy subgroup of A_0 for the action (4.28) is the one-dimensional subgroup $\{(1, \rho(\varphi)J_\pm) \mid \varphi \in \mathbb{R}\} \subset U(1) \times SO(3)$, where $\rho(\varphi) = \exp(\varphi\widehat{\mathbf{e}}_3)$, $J_\pm$ is defined in (4.5), and $\rho(\varphi)J_\pm = J_\pm\rho(\varphi)$. So, the $(U(1) \times SO(3))$-orbit though A_0 is diffeomorphic to $U(1) \times [SO(3)/(\mathbb{Z}_2 \times SO(2))]$ which is the Ω_1-orbit in the classification [16]. Explicitly, this $U(1) \times SO(3)$-orbit is

$$M = \operatorname{Orb}(A_0) = \left\{e^{\mathrm{i}\varphi}\left(\mathbf{u}\mathbf{u}^\mathsf{T} + \mathbf{v}\mathbf{v}^\mathsf{T} - 2\mathbf{w}\mathbf{w}^\mathsf{T}\right)\Big|\,\mathbf{u}, \mathbf{v} \in S^2, \mathbf{u}\cdot\mathbf{v} = 0, \mathbf{w} = \mathbf{u}\times\mathbf{v}\right\}.$$

Note that, in accordance with the isotropy subgroup described above, the identities characterizing $M = \operatorname{Orb}(A_0)$ are unchanged if we replace

- $\mathbf{u}$ by $\mathbf{u}\cos\theta + \mathbf{v}\sin\theta$ and $\mathbf{v}$ by $-\mathbf{u}\sin\theta + \mathbf{v}\cos\theta$ (this is the $SO(2)$-action)
- $\mathbf{v}$ by $-\mathbf{v}$ and hence $\mathbf{w}$ by $-\mathbf{w}$ (this is the J_--action).

Note that since the isotropy subgroup is Abelian, its coadjoint orbits are points. Therefore, by Theorem 3.1, the canonical cotangent bundle T^*M is symplectically diffeomorphic to the coadjoint orbit $\mathcal{O}_{(p,\mathbf{m},A_0)}$ endowed, as usual, with the minus orbit symplectic structure, $(p, \mathbf{m}) \in (\mathbb{R} \times \mathfrak{so}(3))^*$.

To find the Lagrangian associated to this phase, we note that the reduced velocity $\xi = \partial_z g g^{-1}$ of the general theory (see (2.4)) is given here by $\xi = (v, \mathbf{w}) : \mathbb{R} \to \mathbb{R} \times \mathfrak{so}(3)$, where $v = \partial_z\varphi$ and $\mathbf{w} = (\partial_z R)R^{-1}$. The infinitesimal generator of the group action, i.e., the second formula in (2.6), has in this case the expression

$$\partial_z A = \mathrm{i}vA + [\widehat{\mathbf{w}}, A]\,.$$

This formula and (4.1) show that the Euler–Poincaré Lagrangian (2.5) of the general theory becomes in this case the function $l = l(\xi, m) : \mathbb{R} \times \mathfrak{so}(3) \times M \to \mathbb{R}$ given by

$$\begin{aligned} l(v, \mathbf{w}, A) &= \langle\!\langle \mathrm{i}vA + [\widehat{\mathbf{w}}, A]\,, \mathrm{i}vA + [\widehat{\mathbf{w}}, A]\rangle\!\rangle \\ &= v^2\,\langle\!\langle A, A\rangle\!\rangle + 2v\,\langle\!\langle \mathrm{i}A, [\widehat{\mathbf{w}}, A]\rangle\!\rangle + \mathbf{w}^\mathsf{T}\mathbf{J}(A)\mathbf{w} \end{aligned}$$

(see (4.2) and (4.18)). Thus, the Euler–Poincaré equations (2.6) read

$$\partial_z \frac{\delta l}{\delta v} = \operatorname{Re}\operatorname{Tr}\left(\frac{\delta l}{\delta A}^{*} A\mathrm{i}\right), \qquad \partial_z \frac{\delta l}{\delta \mathbf{w}} + \frac{\delta l}{\delta \mathbf{w}} \times \mathbf{w} = 2\overrightarrow{\operatorname{Re}\left[\frac{\delta l}{\delta A}^{*}, A\right]}, \tag{4.29}$$
$$\partial_z A = \mathrm{i} v A + [\widehat{\mathbf{w}}, A],$$

and where we have

$$\frac{\delta l}{\delta v} = 2v \left\langle\!\left\langle A, A \right\rangle\!\right\rangle + 2 \left\langle\!\left\langle \mathrm{i}A, [\widehat{\mathbf{w}}, A] \right\rangle\!\right\rangle, \qquad \frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{J}(A)\mathbf{w} - 4v\overrightarrow{\operatorname{Re}\left[\mathrm{i}\Gamma A^{*}, A\right]},$$
$$\frac{\delta l}{\delta A} = [[\widehat{\mathbf{w}}, A]\,\Gamma, \widehat{\mathbf{w}}] + [[\widehat{\mathbf{w}}, A]\,\Gamma, \widehat{\mathbf{w}}]^{\mathsf{T}} + X,$$

where X is the traceless symmetric part of $v^2 A\Gamma - 2v\mathrm{i}\,[\widehat{\mathbf{w}}, A]\,\Gamma - 2v\mathrm{i}\,[\widehat{\mathbf{w}}, A\Gamma]$.

Theorem 4.10. *The three functions h, j_m, $\mathbf{J}_3$ form a completely integrable system on all six dimensional coadjoint orbits $\mathcal{O}_{(p,\mathbf{m},A_0)}$.*

Proof. As in Sect. 4.2.2, the two commuting circle actions

$$(p, \mathbf{m}, A) \mapsto \left(p, \mathbf{m}, e^{\mathrm{i}\varphi} A\right), \qquad (p, \mathbf{m}, A) \mapsto (p, \rho(\varphi)\mathbf{m}, \rho(\varphi) A \rho(\varphi)^{-1}),$$

where $\rho(\varphi) := \exp(\varphi\widehat{\mathbf{e}}_3)$, generate the momentum maps j_m and $\mathbf{J}_3$, respectively. Therefore, h, j_m, and $\mathbf{J}_3$ (see the text after (4.22) for the expressions of j_m and $\mathbf{J}_3$) are in involution.

To show their functional independence on a dense open subset of phase space, we shall work on the Hamiltonian side. The infinitesimal generator vector fields of the circle actions given above coincide with the Hamiltonian vector fields of j_m and $\mathbf{J}_3$, respectively, i.e., they are

$$X_{j_m}(p, \mathbf{m}, A) = (0, \mathbf{0}, \mathrm{i}A), \qquad X_{\mathbf{J}_3}(p, \mathbf{m}, A) = (0, \mathbf{e}_3 \times \mathbf{m}, [\widehat{\mathbf{e}}_3, A]).$$

Finally the Hamiltonian vector field for h is obtained by using (4.29) and the relations $\frac{\delta h}{\delta A} = -\frac{\delta l}{\delta A}$, $\mathbf{w} = \frac{\delta h}{\delta \mathbf{m}}$, $v = \frac{\delta h}{\delta v}$. A direct computation yields

$$X_h(p, \mathbf{m}, A) = \left(-\operatorname{ReTr}\left(\frac{\delta h}{\delta A}^{*} \mathrm{i}A\right), \frac{\delta h}{\delta \mathbf{m}} \times \mathbf{m} - 2\overrightarrow{\operatorname{Re}\left[\frac{\delta h}{\delta A}^{*}, A\right]}, \mathrm{i}\frac{\delta h}{\delta p} A + \left[\widehat{\frac{\delta h}{\delta \mathbf{m}}}, A\right] \right).$$

Writing $\alpha_1 X_h(p, \mathbf{m}, A) + \alpha_2 X_{\mathbf{J}_3}(p, \mathbf{m}, A) + \alpha_3 X_{j_m}(p, \mathbf{m}, A) = 0$, implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ on an open dense set of points $(p, \mathbf{m}, A)$ in the coadjoint orbit. □

4.3.2. Ω_4-phase. The Ω_4-phase corresponds to

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \in S^2(\mathbb{C}^3)_0,$$

where $\omega^3 = 1$. The isotropy subgroup is

$$\big(U(1)\times SO(3)\big)_{A_0} = \left\{ \left(1, \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}\right), \left(\omega, \begin{pmatrix} 0 & 0 & \epsilon_1 \\ \epsilon_2 & 0 & 0 \\ 0 & \epsilon_3 & 0 \end{pmatrix}\right), \left(\omega^2, \begin{pmatrix} 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \\ \epsilon_3 & 0 & 0 \end{pmatrix}\right) \,\middle|\, \epsilon_i = \pm 1,\ i = 1,2,3,\ \epsilon_1\epsilon_2\epsilon_3 = 1 \right\}$$

and it is isomorphic to the tetrahedral group (the 12 elements alternating group $\mathfrak{A}_4$ on 4 letters). The $(U(1) \times SO(3))$-orbit $\mathrm{Orb}(A_0) = M$ is hence diffeomorphic to $[U(1) \times SO(3)]/\mathfrak{A}_4$, which is the Ω_4-coadjoint orbit in the classification [16].

To compute the texture equations associated to this phase, we note that the reduced velocity $\xi = \partial_z g g^{-1}$ of the general theory (see (2.4)) is given here by $\xi = (v, \mathbf{w}) : \mathbb{R} \to \mathbb{R} \times \mathfrak{so}(3)$, where $v = \partial_z \varphi$ and $\mathbf{w} = (\partial_z R)R^{-1}$. The second formula in (2.6) (the infinitesimal generator of the action) is given here by

$$\partial_z A = \mathrm{i}vA + [\widehat{\mathbf{w}}, A]\,.$$

Using this expression and formula (4.1), the Euler–Poincaré Lagrangian

$$l = l(\xi, m) : \mathbb{R} \times \mathfrak{so}(3) \times M \to \mathbb{R}$$

of the general theory given in (2.5), is computed in this case to be

$$\begin{aligned} l(v, \mathbf{w}, A) &= \operatorname{Re}\operatorname{Tr}(\Gamma \partial_z A^* \partial_z A) \\ &= \mathbf{w}^{\mathsf{T}}\mathbf{J}(A)\mathbf{w} + 2v\,\langle\!\langle \mathrm{i}A, [\widehat{\mathbf{w}}, A]\rangle\!\rangle + (3\gamma_1 + \gamma_2 + \gamma_3)v^2, \end{aligned}$$

(see (4.2) and (4.18)). Thus, the Euler–Poincaré equations (2.6) read

$$\partial_z \frac{\delta l}{\delta v} = \operatorname{Re}\operatorname{Tr}\left(\frac{\delta l}{\delta A}^* A\mathrm{i}\right), \quad \partial_z \frac{\delta l}{\delta \mathbf{w}} + \frac{\delta l}{\delta \mathbf{w}} \times \mathbf{w} = \overrightarrow{2\operatorname{Re}\left[\frac{\delta l}{\delta A}^*, A\right]}, \quad \partial_z A = \mathrm{i}vA + [\widehat{\mathbf{w}}, A]\,,$$

where we have

$$\frac{\delta l}{\delta v} = 2(3\gamma_1 + \gamma_2 + \gamma_3)v + \langle\!\langle \mathrm{i}A, [\widehat{\mathbf{w}}, A]\rangle\!\rangle\,, \qquad \frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{J}(A)\mathbf{w} - 4v\overrightarrow{\operatorname{Re}\left[\mathrm{i}\Gamma A^*, A\right]},$$
$$\frac{\delta l}{\delta A} = [[\widehat{\mathbf{w}}, A]\,\Gamma, \widehat{\mathbf{w}}] + [[\widehat{\mathbf{w}}, A]\,\Gamma, \widehat{\mathbf{w}}]^{\mathsf{T}} + X,$$

where X is the traceless symmetric part of $v^2 A\Gamma - 2v\mathrm{i}\,[\widehat{\mathbf{w}}, A]\,\Gamma - 2v\mathrm{i}\,[\widehat{\mathbf{w}}, A\Gamma]$.

Passing to the Hamiltonian formulation, a direct computation using (3.1) gives the Lie–Poisson bracket

$$\begin{aligned} \{f, h\}(p, \mathbf{m}, A) = \mathbf{m} \cdot \left(\frac{\delta f}{\delta \mathbf{m}} \times \frac{\delta h}{\delta \mathbf{m}}\right) &+ \left\langle \frac{\delta h}{\delta A}, \left[A, \widehat{\frac{\delta f}{\delta \mathbf{m}}}\right] - \mathrm{i}\frac{\delta f}{\delta p}A \right\rangle \\ &- \left\langle \frac{\delta f}{\delta A}, \left[A, \widehat{\frac{\delta h}{\delta \mathbf{m}}}\right] - \mathrm{i}\frac{\delta h}{\delta p}A \right\rangle, \end{aligned}$$

where $p := \frac{\delta l}{\delta v} = 2(3\gamma_1 + \gamma_2 + \gamma_3)v$ and $\mathbf{m} := \frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{J}(A)\mathbf{w}$. Thus, the equations $\partial_z f = \{f, h\}$ for any f are

$$\partial_z p = -\operatorname{ReTr}\left(\frac{\delta h}{\delta A}^* A\mathrm{i}\right), \qquad \partial_z \mathbf{m} + \mathbf{m} \times \frac{\delta h}{\delta \mathbf{m}} = -2\operatorname{Re}\overrightarrow{\left[\frac{\delta h}{\delta A}^*, A\right]}$$
$$\partial_z A = \mathrm{i}\frac{\delta h}{\delta p} A + \left[\widehat{\frac{\delta h}{\delta \mathbf{m}}}, A\right].$$

The Hamiltonian system generated by h has three integrals of motion in involution: $h, \mathbf{J}_3, j_m = p$. The integrals $\mathbf{J}_3$ and j_m commute with h because on the Lagrangian side they are momentum maps and hence constant on the solutions of the Euler–Lagrange equations. The maps $\mathbf{J}_3$ and j_m commute because they are momentum maps of two commuting circle actions (this is the same argument as given in Sect. 4.2.2). The fact that $p = j_m$ is a direct computation replacing $\partial_z A = \mathrm{i}vA + [\widehat{\mathbf{w}}, A]$ in the defining formula (4.15) for j_m and using $A = e^{\mathrm{i}\varphi} R A_0 R^{-1}$, $A_0 A_0^* = I_3$. For the complete integrability of this system, one more integral is needed.

4.3.3. Ω_6-phase. Consider the orbit of this group through

$$A_0 = \begin{pmatrix} 1 & \mathrm{i} & 0 \\ \mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in S^2(\mathbb{C}^3)_0.$$

We note that

$$e^{\mathrm{i}\varphi} A_0 = \rho\left(-\frac{\varphi}{2}\right) A_0 \rho\left(\frac{\varphi}{2}\right), \tag{4.30}$$

where $\rho(\varphi) = \exp(\varphi\widehat{\mathbf{e}}_3)$. A direct verification shows that the isotropy subgroup of $SO(3)$ for the action (4.28) is $\{\tilde{J}_\pm\} \cong \mathbb{Z}_2$ (see (4.5)). As a consequence, using (4.30), it follows that the isotropy subgroup of the action (4.28) equals

$$(U(1) \times SO(3))_{A_0} = \left\{\left(e^{\mathrm{i}\varphi}, \tilde{J}_\pm \rho\left(\frac{\varphi}{2}\right)\right) \,\middle|\, e^{\mathrm{i}\varphi} \in U(1)\right\} \cong U(1) \times \mathbb{Z}_2.$$

The group $\{\tilde{J}_\pm\} \cong \mathbb{Z}_2 = \{\pm 1\}$ acts on $SO(3)$ by $R \mapsto R\tilde{J}_\pm$ and $(U(1) \times SO(3))_{A_0}$ on $U(1) \times SO(3)$ by $\left(e^{\mathrm{i}\psi}, R\right) \mapsto \left(e^{\mathrm{i}\psi}, R\right)\left(e^{\mathrm{i}\varphi}, \tilde{J}_\pm \rho\left(\frac{\varphi}{2}\right)\right)$. Thus

$$(U(1) \times SO(3)) \,/\, (U(1) \times SO(3))_{A_0} \ni \left[e^{\mathrm{i}\psi}, R\right] \longmapsto \left[R\rho\left(-\frac{\varphi}{2}\right)\right] \in SO(3)/\mathbb{Z}_2$$

is a diffeomorphism which shows that the orbit $(U(1) \times SO(3)) \cdot A_0$ is diffeomorphic to $SO(3)/\mathbb{Z}_2$. We have found the orbit of type Ω_6 in the classification given in [16].

It is easy to see that

$$R^{\mathsf{T}} A_0 R = (R_1 + \mathrm{i}R_2)(R_1 + \mathrm{i}R_2)^{\mathsf{T}},$$

where R_j is the jth column of $R \in SO(3)$, $j = 1, 2$. In view of the previous considerations, this shows that

$$\operatorname{Orb}(A_0) = \left\{(\mathbf{x} + \mathrm{i}\mathbf{y})(\mathbf{x} + \mathrm{i}\mathbf{y})^{\mathsf{T}} \mid \mathbf{x}, \mathbf{y} \in S^2,\ \mathbf{x} \cdot \mathbf{y} = 0\right\} \cong SO(3)/\mathbb{Z}_2. \tag{4.31}$$

Lagrangian formulation. Note that the setting is very similar to the setup for superfluid liquid Helium ^{3}He in the second regime of phase A, since the action turns out the be the same, namely, $A \mapsto RAR^{-1}$. The orbit is, however, different because the matrices A_0 in these two cases do not lie on the same $SO(3)$-orbit.

The Lagrangian used in this case (see [16]) is given by (4.1). Therefore, the Euler–Poincaré Lagrangian takes the same form as in the second regime of the A-phase for superfluid liquid Helium ^{3}He, namely,

$$l(\mathbf{w}, A) = \sum_{a,b=1}^{3} J_{ab}(A) w_a w_b = \mathbf{w}^\mathsf{T} \mathbf{J}(A) \mathbf{w},$$

where $\mathbf{J}(A)$ is given in (4.18). The Euler–Poincaré equations are thus (4.19). The functional derivatives are in this case

$$\frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{J}(A)\mathbf{w}, \quad \frac{\delta l}{\delta A} = [[\widehat{\mathbf{w}}, A]\Gamma, \widehat{\mathbf{w}}] + [[\widehat{\mathbf{w}}, A]\Gamma, \widehat{\mathbf{w}}]^\mathsf{T} \in S^2(\mathbb{C}^3)_0.$$

Hamiltonian formulation. All formulas, with the changes noted above, are identical to the ones in the second regime of phase A for superfluid liquid Helium ^{3}He, i.e., (4.20) and (4.21) hold. The same considerations about the complete integrability of the equations given at the end of Sect. 4.2.2 hold because the action given by multiplication with $e^{\mathrm{i}\varphi}$ preserves the orbit $\mathrm{Orb}(A_0)$ given by (4.31) in view of (4.30). We get the following result.

Theorem 4.11. *The three functions h, j_m, $\mathbf{J}_3$ form a completely integrable system on the six dimensional coadjoint orbit Ω_6.*

4.3.4. Ω_8-phase. The Ω_8-phase corresponds to

$$A_0^{\pm} = \begin{pmatrix} 0 & 1 & \pm\mathrm{i} \\ 1 & 0 & 0 \\ \pm\mathrm{i} & 0 & 0 \end{pmatrix} \in S^2(\mathbb{C}^3)_0.$$

Note that $e^{\mp\mathrm{i}\varphi} A_0^{\pm} = \rho(\varphi) A_0^{\pm} \rho(-\varphi)$, where $\rho(\varphi) := \exp(\varphi \widehat{\mathbf{e}}_1)$. Hence $SO(3)$ acts transitively on the orbit through A_0, i.e., $\mathrm{Orb}(A_0) = \{RA_0R^{-1} \mid R \in SO(3)\}$. A direct computation shows that the isotropy group $SO(3)_{A_0} = \{I_3\}$ and hence $\mathrm{Orb}(A_0)$ is diffeomorphic to $SO(3)$.

It is easy to verify that the isotropy subgroup of the original action equals

$$(U(1) \times SO(3))_{A_0^{\pm}} = \left\{ \left(e^{\mathrm{i}\varphi}, \rho(\pm\varphi) \right) \mid e^{\mathrm{i}\varphi} \in U(1) \right\} \cong U(1).$$

Following the same method as for the other phases, we conclude that

$$l(\mathbf{w}, A) = \langle\langle [\widehat{\mathbf{w}}, A], [\widehat{\mathbf{w}}, A] \rangle\rangle = \mathbf{w}^\mathsf{T} \mathbf{J}(A) \mathbf{w},$$

where $\widehat{\mathbf{w}} = (\partial_z R) R^{-1}$ and $\mathbf{J}(A)$ is given in (4.18). Thus, the Euler–Poincaré equations (2.6) become in this case

$$\partial_z \frac{\delta l}{\delta \mathbf{w}} + \frac{\delta l}{\delta \mathbf{w}} \times \mathbf{w} = 2\overrightarrow{\mathrm{Re}\left[\frac{\delta l}{\delta A}^{*}, A \right]}, \quad \partial_z A = [\widehat{\mathbf{w}}, A],$$

and we have

$$\frac{\delta l}{\delta \mathbf{w}} = 2\mathbf{J}(A)\mathbf{w}, \quad \frac{\delta l}{\delta A} = [[\widehat{\mathbf{w}}, A]\,\Gamma, \widehat{\mathbf{w}}] + [[\widehat{\mathbf{w}}, A]\,\Gamma, \widehat{\mathbf{w}}]^{\mathsf{T}}.$$

As in the case of the Ω_6-phase we obtain the following result.

Theorem 4.12. *The three functions h, j_m, $\mathbf{J}_3$ form a completely integrable system on the six dimensional coadjoint orbit Ω_8.*

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