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C_2-Cofiniteness of Cyclic-Orbifold Models

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Abstract: We prove an orbifold conjecture for conformal field theory with a solvable automorphism group. Namely, we show that if V is a C_2-cofinite simple vertex operator algebra and G is a finite solvable automorphism group of V, then the fixed point vertex operator subalgebra V^G is also C_2-cofinite, where C_2-cofiniteness is equivalent to the condition that V has only finitely many isomorphism classes of simple V-modules (including weak modules) and all finitely generated V-modules have composition series. This result offers a mathematically rigorous background to orbifold theories with solvable automorphism groups.

1. Introduction

In order to explain the moonshine phenomenon on the monster simple group and the modular functions, Borcherds [1] has introduced a concept of vertex (operator) algebra as an algebraic version of conformal field theory. It is a quadruple $(V, Y, \mathbf{1}, \omega)$ satisfying the several axioms, where V is a graded vector space $V = \oplus_{i=-K}^{\infty} V_i$, $Y(v, z) = \sum_{m\in\mathbb{Z}} v_m z^{-m-1} \in \mathrm{End}(V)[[z, z^{-1}]]$ denotes a vertex operator of $v \in V$ on V which satisfies Borcherds identity (2.1), $\mathbf{1} \in V_0$ and $\omega \in V_2$ are specified elements called the vacuum and the Virasoro element of V, respectively. We set $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

One of the main targets in the research of vertex operator algebras (shortly VOA) is a construction of VOAs of finite type, that is, all modules (including weak modules) have a composition series consisting of only finitely many isomorphism classes of simple modules. If V is a VOA and σ is a finite automorphism of order p, then a fixed point subVOA V^{σ} is called an orbifold model, (see [3,4]). So-called "orbifold conjecture" says that if V is of finite type, then so is V^{σ}. It is revealed that the above finiteness condition is equivalent to the C_2-cofiniteness by [2,8]. Here a V-module W is called to

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be C_2-cofinite when $C_2(W) = \mathrm{Span}_{\mathbb{C}}\{v_{-2}u \mid v \in V, u \in W, \mathrm{wt}(v) > 0\}$ has a finite co-dimension in W. This condition was originally introduced by [13] as a technical condition to prove the modular invariance property. It is very important and the most general theorems require this condition. For example, the author in [9] mentioned that if the orbifold model V^σ is C_2-cofinite, then we are able to get all the information of (weak) V^σ-modules from (twisted and ordinary) V-modules and every simple V^σ-module is a submodule of a (twisted or ordinary) V-module. Therefore, the C_2-cofiniteness on V^σ offers a mathematically rigorous background to all orbifold theories.

For the orbifold conjecture, there are partial answers. For example, T. Abe has proved it for a permutation automorphism of order $p = 2$. For lattice VOAs, Yamskulna [12] has shown the case $p = 2$ and the author [10] has shown the case $p = 3$, which was used to construct a new holomorphic VOA of central charge 24. In this paper, we will prove all cases for any finite order p with the powerful help of the Borcherds identity (2.1) and the skew-symmetry (4.1).

Main Theorem. *Let V be a C_2-cofinite simple VOA of CFT-type and $\sigma \in \mathrm{Aut}(V)$ of finite order p. Then a fixed point vertex operator subalgebra V^σ is also C_2-cofinite.*

As corollaries, we have:

Theorem 1. *Let V be a C_2-cofinite simple VOA of CFT-type and $G \leq \mathrm{Aut}(V)$ finite solvable. Then a fixed point vertex operator subalgebra V^G is also C_2-cofinite.*

Corollary 2. *Let V be a C_2-cofinite VOA and a subVOA U is isomorphic to a lattice VOA. Then the commutant E of U is C_2-cofinite.*

Corollary 3. *If V is C_2-cofinite and a subVOA U is isomorphic to a 2-dim. Ising model $L(\frac{1}{2}, 0)$ of central charge $\frac{1}{2}$, then the commutant E of U is C_2-cofinite.*

Here the commutant E of U is defined by $\{v \in V \mid u_m v = 0 \text{ for all } u \in U, m \geq 0\}$. We note that it is a subVOA.

Remark 1. In this paper, we assume that V is of CFT-type. This is because of simplifying the proof. From our proof, it is not difficult to see that we have the same conclusion without the assumption of CFT-type.

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2. Truncation Property

From the axiom of VOAs, for $v \in V_r$ and $u \in V_n$, we have $v_m u \in V_{r-m-1+n}$. Hence there is an integer N such that $v_n u = 0$ for any $n > N$. This is called a truncation property. To simplify the notation, we will say that v is truncated on u.

Set $V^* = \mathrm{Hom}(V, \mathbb{C})$ and define a pairing $\langle \cdot, \cdot \rangle$ by $\langle v, \xi \rangle = \xi(v)$ for $\xi \in V^*$ and $v \in V$. For $v \in V$ and $m \in \mathbb{Z}$, actions v_m on V^* are defined by

$$\langle w, Y^*(v, z)\xi \rangle = \langle Y(e^{L(1)z}(-z^{-2})^{L(0)}v, z^{-1})w, \xi \rangle$$

for $w \in V, \xi \in \mathrm{Hom}(V,\mathbb{C})$, where $Y^*(v,z)=\sum_{m\in\mathbb{Z}} v_m z^{-m-1}$ is called an adjoint operator of v. An important fact is that $(\oplus_{m=0}^{\infty}\mathrm{Hom}(V_m,\mathbb{C}), Y^*)$ becomes a V-module, see [6] for the proof. This module is called a restricted dual of V which is denoted by V'. In particular, $Y^*(\cdot,z)$ satisfy the Borcherds identity:

$$\sum_{i=0}^{\infty}\binom{m}{i}(u_{r+i}v)_{m+n-i}\xi=\sum_{j=0}^{\infty}(-1)^j\binom{r}{j}(u_{r+m-j}v_{n+j}\xi-(-1)^r v_{r+n-j}u_{m+j}\xi) \tag{2.1}$$

for any $m,n,r\in\mathbb{Z}$, $v,u\in V$, $\xi\in V'$. Since $V^*=\prod_n \mathrm{Hom}(V_n,\mathbb{C})$, we can express $\xi\in V^*$ by $\prod_n \xi_{(n)}$ with $\xi_{(n)}\in\mathrm{Hom}(V_n,\mathbb{C})$. We call $\xi\in V^*$ $L(0)$-free if $\dim\mathbb{C}[L(0)]\xi=\infty$, that is, $\xi_{(n)}\neq 0$ for infinitely many n. We note that if V is C_2-cofinite, then any (weak) module does not contain $L(0)$-free elements.

The weight of the terms in (2.1) for $\xi\in\mathrm{Hom}(V_t,\mathbb{C})$ and that for $\xi\in\mathrm{Hom}(V_s,\mathbb{C})$ are different when $t\neq s$. We also have that the both sides of (2.1) are well-defined for each $\xi\in\mathrm{Hom}(V_t,\mathbb{C})$. Therefore the Borcherds' identity is also well-defined on V^*, as Haisheng Li has pointed out in [7]. However, V^* is not a V-module. The problem is a failure of truncation properties.

Lemma 4. *If u and v are truncated on ξ, then $v_m u$ is also truncated on ξ for any m. In particular, if V is generated by $\Omega\subseteq V$ as a vertex algebra and all elements in Ω are truncated on ξ, then all elements in V are truncated on ξ.*

Proof. We may assume $u_n\xi=v_n\xi=u_nv=0$ for $n\geq N$. We assert that for $s\in\mathbb{N}$ and $n\geq 2N+s$, we have $(u_{N-s}v)_n\xi=0$. Suppose false and let s be a minimal counterexample. Substituting $r=N-s$, $n=N+s+p$, $m=N+q$ in (2.1) with $p,q\geq 0$, the left side equals

$$\begin{aligned}\mathrm{LH}&=\sum_{i=0}^{\infty}\binom{N+q}{i}(u_{N-s+i}v)_{2N+q+s+p-i}\xi=\sum_{i=0}^{s}\binom{N+q}{i}(u_{N-(s-i)}v)_{2N+s-i+p+q}\xi\\&=(u_{N-s}v)_{2N+s+p+q}\xi\end{aligned}$$

by the minimality of s. On the other hand, the right side is

$$\mathrm{RH}=\sum_{i=0}^{\infty}(-1)^i\binom{N-s}{i}\left(u_{2N-s+q-i}v_{N+s+p+i}\xi-(-1)^{N-s}v_{2N-s+p-i}u_{N+q+i}\xi\right)=0,$$

which contradicts the choice of s. □

Since $v_nu_m\xi=u_mv_n\xi+\sum_{i=0}^{\infty}\binom{n}{i}(v_iu)_{n+m-i}\xi$, Lemma 2 (see also Li [7]) implies:

Lemma 5. *If $v,u\in V$ are truncated on $\xi\in V^*$, then v is truncated on $u_m\xi$ for any m. In particular, if all elements of V are truncated on ξ, then $\mathrm{Span}_{\mathbb{C}}\{u^1_{m_1}\cdots u^k_{m_k}\xi \mid u^i\in V, m_i\in\mathbb{Z}\}$ is a V-module.*

3. General Setting

Let $(V, Y, \mathbf{1}, \omega)$ be a C_2-cofinite VOA and σ an automorphism of V of order p. Viewing V as a $<\sigma>$-module, we decompose

$$V = V^{(0)} \oplus V^{(1)} \oplus \cdots \oplus V^{(p-1)}$$

where $V^{(m)} = \{v \in V \mid v^{\sigma} = e^{2\pi\sqrt{-1}m/p}v\}$. If $V^{(1)}$ and $V^{(p-1)}$ are C_2-cofinite, then so is $V^{(0)}$ by the main theorem in [11]. Therefore we assume that $V^{(1)}$ is not C_2-cofinite.

For $A, B \subseteq V$ and $m \in \mathbb{Z}$, $A_{(m)}B$ denotes a subspace $\mathrm{Span}_{\mathbb{C}}\{a_m b \mid a \in A, b \in B\}$.

Lemma 6. *$(V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)}$ has a finite co-dimension in $V^{(1)}$.*

Proof. Suppose false, i.e. $V^{(1)}_m/((V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)})_m \neq 0$ for infinitely many m. Then there is a $L(0)$-free element $\xi \in (V^{(1)})^*$ such that $\langle (V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)}, \xi\rangle = 0$. In other words,

$$0 = \langle v_{-2-\mathbb{N}}u, \xi\rangle = \langle u_{-2-\mathbb{N}}v, \xi\rangle$$

for any $v \in V^{(1)}$ and $u \in V^{(0)}$. Since $V^{(1)}$ is a direct summand of V, we may view $(V^{(1)})^* \subseteq V^*$. By taking adjoint operators, we have:

$$\langle u, v_{2\mathrm{wt}(v)+\mathbb{N}}\xi\rangle = \langle (-1)^{\mathrm{wt}(v)}\sum_{s=0}^{\infty}\frac{1}{s!}(L(1)^s v)_{-2-s-\mathbb{N}}u, \xi\rangle = 0$$
$$\langle v, u_{2\mathrm{wt}(u)+\mathbb{N}}\xi\rangle = \langle (-1)^{\mathrm{wt}(u)}\sum_{s=0}^{\infty}\frac{1}{s!}(L(1)^s u)_{-2-s-\mathbb{N}}v, \xi\rangle = 0,$$

which imply that $v \in V^{(1)}$ and $u \in V^{(0)}$ truncate on ξ. However, since V is simple, $V^{(1)} + V^{(0)}$ generates a C_2-cofinite VOA V by normal products, which contradicts that ξ is $L(0)$-free. □

So, there is a finite dimensional subspace P of $V^{(1)}$ such that $V^{(1)} = (V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)} + P$. We may assume that P is a direct sum of homogeneous spaces.

Proposition 7. *$V^{(1)} = (V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)]P$.*

Proof. Suppose false and we choose $0 \neq w \in V^{(1)} - ((V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)]P)$ with minimal weight. Since $V^{(1)} = (V^{(1)})_{(-2)}V^{(0)} + (V^{(0)})_{(-2)}V^{(1)} + P$, we may assume $w \in (V^{(1)})_{(-2)}V^{(0)}$. We may also assume $w = a_{-2}u$ with $a \in V^{(1)}$ and $u \in V^{(0)}$. Then by the skew-symmetry (4.1), we have

$$w = -u_{-2}a - \sum_{j=1}^{\infty}\frac{(-1)^j}{j!}L(-1)^j u_{-2+j}a.$$

Since $\mathrm{wt}(u_{-2+j}a) < \mathrm{wt}(u_{-2}a) = \mathrm{wt}(w)$ for $j \geq 1$, we have

$$\begin{aligned} w &\in (V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)](V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)]P \\ &\subseteq (V^{(0)})_{(-2)}V^{(1)} + \mathbb{C}[L(-1)]P \end{aligned}$$

by the minimality of $\mathrm{wt}(w)$, which contradicts the choice of w. □

4. The Coefficient Functions

Since $L(-1)C_2(V^{(1)}) \subseteq C_2(V^{(1)})$, $V^{(1)}/C_2(V^{(1)})$ is a finitely generated $\mathbb{C}[L(-1)]$-module by Proposition 7 and so the $L(-1)$-torsion part has a finite dimension. Let $\widetilde{T}$ be the inverse image in $V^{(1)}$ of the $L(-1)$-torsion submodule of $V^{(1)}/C_2(V^{(1)})$. Since $\dim V^{(1)}/C_2(V^{(1)}) = \infty$, there is a set of free generators $\{\alpha^i : i = 1, \ldots, t\}$ such that

$$V^{(1)} = \left(\oplus_{i=1}^{t} \mathbb{C}[L(-1)]\alpha^i\right) \oplus \widetilde{T}.$$

Set $\widehat{T} = \left(\oplus_{i=2}^{t} \mathbb{C}[L(-1)]\alpha^i\right) \oplus \widetilde{T}$ and $\alpha = \alpha^1$, then $\widehat{T}$ is $\mathbb{C}[L(-1)]$-invariant and

$$V^{(1)} = \mathbb{C}[L(-1)]\alpha \oplus \widehat{T}.$$

We may assume that α is a homogeneous element.

We also introduce an equivalent relation $\equiv$ on $V^{(1)}$ by modulo $\widehat{T}$. Under this setting, for any $n \in \mathbb{N}$ and any homogeneous elements $a \in V^{(k)}$ and $b \in V^{(p-k+1)}$, there are complex numbers $f^{a,b}(n)$ such that

$$a_{-n-\mathrm{wt}(\alpha)+\mathrm{wt}(a)+\mathrm{wt}(b)-1}b \equiv f^{a,b}(n)\alpha_{-n-1}\mathbf{1} \pmod{\widehat{T}}.$$

We note $\frac{L(-1)^n}{n!}\alpha = \alpha_{-n-1}\mathbf{1}$ for $n \in \mathbb{N}$. From now on, for $a, b \in V$, we always use M to denote $\mathrm{wt}(a)+\mathrm{wt}(b)-\mathrm{wt}(\alpha)$ for simplifying the notation. We view $f^{a,b}$ as a map from $\mathbb{N}$ to $\mathbb{C}$. We note that since $\mathrm{wt}(a_{-n+M-1}b) < \mathrm{wt}(\alpha)$ for $n \in \mathbb{Z}_{<0}$, we have $a_{-n+M-1}b \in T$ by the choice of α and so we may also consider $f^{a,b}(n) = 0$ for $n \in \mathbb{Z}_{<0}$.

For $k = 0, \ldots, p-1$, we set

$$\mathcal{F}_k = \mathrm{Span}_{\mathbb{C}}\left\{f^{a,b} \mid a \in V^{(k)}, b \in V^{(p-k+1)}\right\}.$$

For a map $f : \mathbb{N} \to \mathbb{C}$, we define a map xf by $(xf)(n) = nf(n)$.

Lemma 8. *$\mathcal{F}_k$ are all $\mathbb{C}[x]$-invariant.*

Proof. Clearly, $\mathcal{F}_k$ is a vector space. Since

$$(L(-1)a)_{-n+M}b = (n-M)a_{-n+M-1}b \equiv (n-M)f^{a,b}(n)\alpha_{-n-1}\mathbf{1} \pmod{\widehat{T}},$$

we have $xf^{a,b} = f^{L(-1)a,b} + Mf^{a,b} \in \mathcal{F}_k$. □

Lemma 9. *For $f \in \mathcal{F}_0$, $Q_f = \{n \in \mathbb{Z} \mid f(n) \neq 0\}$ is a finite set. We will call such a function finite type.*

Proof. For $a \in V^{(0)}$, $b \in V^{(1)}$, we have $a_{-n-1+M}b \equiv f^{a,b}(n)\alpha_{-n-1}\mathbf{1} \pmod{\widehat{T}}$. Therefore, $a_{-n+M-1}b \in C_2(V^{(1)})$ and $f^{a,b}(n) = 0$ for $n \geq M+1$. Since all elements in $\mathcal{F}_0$ are linear combinations of such elements, we have the desired result. □

For a map $f : \mathbb{N} \to \mathbb{C}$, we introduce two operators S and T as follows:

$$Sf(n) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k f(k) \quad \text{for } n \in \mathbb{N}$$
$$Tf(n) = (-1)^n f(n) \quad \text{for } n \in \mathbb{N}.$$

Clearly, $S^2 = T^2 = \mathrm{id}$. We also have the following by induction.

Lemma 10.

$$(ST)^k f(n) = \sum_{j=0}^{n} \binom{n}{j} k^j f(n-j) \quad \textit{for } k = 1, \ldots$$

The following is a key result.

Lemma 11. *For $k = 1, 2, \ldots, p-1$, we have:*

$$\mathcal{F}_{1+p-k} = S(\mathcal{F}_k) \quad \textit{and} \quad \mathcal{F}_{p-k} = T(\mathcal{F}_k).$$

Proof. The operator S comes from the skew-symmetry. In fact, for $a \in V^{(k)}, b \in V^{(p-k+1)}$ and $a_{-n+M-1}b \equiv f^{a,b}(n)\alpha_{-n-1}\mathbf{1}$ for $x \in \mathbb{N}$, then

$$\begin{aligned} b_{-n+M-1}a &\equiv (-1)^{n+M} \sum_{k=0}^{\infty} \frac{L(-1)^k}{k!} (-1)^k a_{-(n-k)+M-1} b \\ &\equiv (-1)^{n+M} \sum_{k=0}^{n} \frac{L(-1)^k}{k!} (-1)^k f^{a,b}(n-k)\alpha_{-n+k-1}\mathbf{1} \\ &\qquad \text{since } L(-1)^k a_{-n+k+M-1} b \in \widehat{T} \text{ for } k > n, \\ &\equiv (-1)^{n+M} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f^{a,b}(n-k)\alpha_{-n-1}\mathbf{1} \\ &\equiv (-1)^{M} \sum_{k=0}^{n} \binom{n}{k} (-1)^k f^{a,b}(n)\alpha_{-n-1}\mathbf{1} \equiv (-1)^M S f^{a,b}(n)\alpha_{-n-1}\mathbf{1}. \end{aligned} \tag{4.1}$$

Therefore, $f^{b,a}(n) = (-1)^M S f^{a,b}(n)$ and $\mathcal{F}_{1+p-k} = S(\mathcal{F}_k)$.

For any $m \in \mathbb{Z}$ and $h \in \mathbb{N}$, by substituting $n = -m-2-h$ and $r = -x+N+h$ in the Borcherds' identity (2.1), we have

$$\begin{aligned} 0 &\equiv \sum_{i=0}^{\infty} \binom{m}{i} (u_{-n+N+h+i}v)_{-2-h-i}\xi \\ &= \sum_{j=0}^{\infty} \binom{-n+N+h}{j} (-1)^j u_{-n+N+h+m-j} v_{-m-2-h+j}\xi \\ &\quad - (-1)^{-n} \sum_{j=0}^{-m+\mathrm{wt}(u)+\mathrm{wt}(\xi)} \binom{-n+N+h}{j} (-1)^{j+N+h} v_{-n+N-m-2-j} u_{m+j}\xi \end{aligned}$$

for $u \in V^{(k)}, v \in V^{(p-k)}, \xi \in V^{(1)}$, where $N = \mathrm{wt}(\xi) + \mathrm{wt}(u) + \mathrm{wt}(v)$. We note $u_{m+j}\xi = 0$ for $j \geq Q = \mathrm{wt}(u) + \mathrm{wt}(\xi) - m$. Since we will treat only $v_{-n+N-m-2}u_m\xi$ later, we may assume $u_m\xi \neq 0$ and so $Q \geq 1$. Let us consider a $Q \times Q$-matrix

$$A := ((-1)^{h-j+N} \binom{-x+h+N}{j})_{h,j=0,\ldots,Q-1}$$

consisting of coefficients of $(-1)^x v_{-x-2+N-j-m}(u_{m+j}\xi)$, where we view n as a variable x. It is easy to see $\det A = \pm 1$ since $\binom{s+1}{j} - \binom{s}{j} = \binom{s}{j-1}$. Therefore, there are polynomials $\lambda_h^m(x) \in \mathbb{C}[x]$ for $0 \le h < Q$ such that

$$(-1)^x v_{-x-2+N-m} u_m \xi$$
$$\equiv \sum_{h=0}^{Q-1} \lambda_h^m(x) \left(\sum_{j=0}^{N+h+m+2} \binom{-x+h+N}{j} (-1)^j u_{-x+h+m+N-j}(v_{-2-m-h+j}\xi) \right).$$

Since the coefficients of the right side at $\alpha_{-x-1}\mathbf{1}$ are all in $\mathcal{F}_k$ by Lemma 8, the above equation implies that a function defined by $v_{-x-2+N-m}u_m\xi$ is in $T(\mathcal{F}_k)$ for any $v \in V^{(p-k)}$, $u \in V^{(k)}$, $\xi \in V^{(1)}$ and $m \in \mathbb{Z}$. Since $V^{(1+k)}$ is a simple $V^{(0)}$-module, $V^{(1+k)}$ is spanned by elements with the form $u_m\xi$ with $u \in V^{(k)}$, $\xi \in V^{(1)}$ and $m \in \mathbb{Z}$ and so we have $T(\mathcal{F}_{p-k}) \subseteq \mathcal{F}_k$ for any k. Since $T^2 = 1$, we have the equality $T(\mathcal{F}_k) = \mathcal{F}_{p-k}$. □

Now we are able to complete the proof of Main Theorem. As we have shown, every elements in $\mathcal{F}_0$ is of finite type. In particular, since $\mathbf{1}_{-x-1}\alpha = \delta_{0,x}\alpha$, we have $\delta_{0,x} \in \mathcal{F}_0$. On the other hand, by Lemma 11, we have

$$\begin{array}{cccccccccccc} \mathcal{F}_0 & & \mathcal{F}_1 & & \mathcal{F}_2 & & \cdots & & \mathcal{F}_{p-1} & & \mathcal{F}_0 \\ \downarrow T & \nearrow S & \downarrow T & \nearrow S & \downarrow T & \nearrow S & \cdots & \nearrow S & \downarrow T & \nearrow S & \\ \mathcal{F}_0 & & \mathcal{F}_{p-1} & & \mathcal{F}_{p-2} & & \cdots & & \mathcal{F}_1 & & \end{array}$$

where p is the order of σ. In particular, we have $(ST)^p(\mathcal{F}_0) = \mathcal{F}_0$. However, since $(ST)^p(\delta_{0,x})(n) = p^n$ is not of finite type. We have a contradiction.

This completes the proof of Main Theorem. □

As last, we will prove corollaries of Main Theorem.

Proof of Theorem 1. Let V be a C_2-cofinite simple VOA and G a finite solvable subgroup of $\mathrm{Aut}(V)$. We will prove Theorem 1 by the induction on $|G|$. Since G is solvable, G has a normal abelian subgroup $A \neq 1$. We first assume that $G = A$ and let $1 \neq \sigma \in G$ be an element of prime order. Then V^σ is C_2-cofinite by Main Theorem. Furthermore, V^σ is simple by [5]. Therefore, $V^G = (V^\sigma)^{G/<\sigma>}$ is also C_2-cofinite by the induction, which proves the assertion of Theorem 1. So, we have $A < G$. By the minimality of $|G|$, V^A is C_2-cofinite and it is also simple by [5]. Therefore, by the minimality of $|G|$, $V^G = (V^A)^{G/A}$ is also C_2-cofinite.

Proof of Corollary 1. Assume $U \cong V_L$ for some lattice L and set $L^* = \{a \in \mathbb{Q}L \mid \langle a, L\rangle \subseteq \mathbb{Z}\}$. We view V as a V_L-module. Since V_L is rational and the category of V_L-modules have a L^*/L-module structure, the actions of $G = \mathrm{Hom}(L^*/L, \mathbb{C}^\times)$ on V are induced from this structure. Then $V^G \cong U \otimes E$, where E is a commutant of U in V. By Theorem 1, $U \otimes E$ is C_2-cofinite. If E is not C_2-cofinite, then E has a weak module B containing $L(0)$-free element by [8] and so $U \otimes E$ has a weak module $U \otimes B$ containing $L(0)$-free elements, which contradicts the C_2-cofiniteness on $U \otimes E$.

Proof of Corollary 2. Let $U \cong L(\frac{1}{2}, 0)$ and we view V as a U-module. Since $L(\frac{1}{2}, 0)$ is rational, V is a direct sum of simple U-modules. Then τ defined by Id on $L(\frac{1}{2}, 0)$ and $L(\frac{1}{2}, \frac{1}{2})$ and $-$Id on $L(\frac{1}{2}, \frac{1}{16})$ as U-modules, respectively, becomes an automorphism of V by the fusion rules of $L(\frac{1}{2}, 0)$-modules. Then V^τ is C_2-cofinite by Main Theorem and simple by [5]. We then view it as a U-module, whose compositions are isomorphic to $L(\frac{1}{2}, 0)$ or $L(\frac{1}{2}, \frac{1}{2})$ as U-modules. Then σ defined by Id on $L(\frac{1}{2}, 0)$ and $-$Id on $L(\frac{1}{2}, \frac{1}{2})$

as U-modules, respectively, is again an automorphism of V^{τ}. Then $(V^{\tau})^{\sigma} = U \otimes E$, where E is a commutant of U in V. By Main Theorem, $U \otimes E$ is C_2-cofinite and so is E by the same argument as above.

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