Commun. Math. Phys. 337, 1–40 (2015)
Digital Object Identifier (DOI) 10.1007/s00220-014-2231-6

Communications in
**Mathematical
Physics**

Multi-Matrix Models and Noncommutative Frobenius Algebras Obtained from Symmetric Groups and Brauer Algebras

Yusuke Kimura

Okayama Institute for Quantum Physics (OIQP), Kyoyama 1-9-1, Kita-ku, Okayama 700-0015, Japan.
E-mail: londonmileend@gmail.com

Received: 16 April 2014 / Accepted: 27 June 2014
Published online: 20 November 2014 –

Abstract: It has been understood that correlation functions of multi-trace operators in $\mathcal{N} = 4$ SYM can be neatly computed using the group algebra of symmetric groups or walled Brauer algebras. On the other hand, such algebras have been known to construct 2D topological field theories (TFTs). After reviewing the construction of 2D TFTs based on symmetric groups, we construct 2D TFTs based on walled Brauer algebras. In the construction, the introduction of a dual basis manifests a similarity between the two theories. We next construct a class of 2D field theories whose physical operators have the same symmetry as multi-trace operators constructed from some matrices. Such field theories correspond to non-commutative Frobenius algebras. A matrix structure arises as a consequence of the noncommutativity. Correlation functions of the Gaussian complex multi-matrix models can be translated into correlation functions of the two-dimensional field theories.

1. Introduction

Motivated by the development of AdS/CFT correspondence [1], the study of $\mathcal{N} = 4$ super Yang–Mills theory has attracted lots of interest for the past 15 years. The anomalous dimensions are of particular interest because they correspond to energies in the dual string theory side. If we restrict our attention to the planar limit, the problem of obtaining the spectrum of anomalous dimensions is replaced with the conventional problem of diagonalising the Hamiltonian in an integrable system [2,3]. Compared to the enormous development of the planar theory, we had only limited information on non-planar corrections. The recent development of studying non-planar corrections, however, has brought us to obtain some concrete results of non-planar corrections [4–23].

Consider multi-trace gauge invariant operators constructed from a single complex matrix, relevant to the half-BPS sector in $\mathcal{N} = 4$ super Yang–Mills theory. They are classified by conjugacy classes of the symmetric group, and the two-point function can be expressed, except the trivial space-time dependence, as

$$\langle O_{[\tau]} O_{[\sigma]} \rangle = \sum_{\rho \in S_n} N^n \delta_n(\Omega_n \tau \rho \sigma \rho^{-1}). \tag{1}$$

The derivation will be given in Sect. 2. It is interesting to find that the right-hand side is written completely in terms of symmetric group data, suggesting an effective role of the symmetric group in the problem. Behind the fact that the symmetric group shows up in the evaluation of the matrix integral, there is a mathematical structure that relates the general linear group and the symmetric group known as Schur–Weyl duality. It indeed plays a central role in our idea of employing group representation theory in the recent development of non-planar physics. We will review it in Sect. 2, and some mathematical notions regarding Schur–Weyl duality are given in Appendix A. By the way, it has been known that symmetric groups give the description of coverings of two-dimensional Riemann surfaces, which played a central role in the string theoretic interpretation of the large N expansion of two-dimensional Yang–Mills [24–28]. This is an example that symmetric groups are used to describe two-dimensional field theories. In fact, if we ignore the Ω factor in the right-hand side of (1), the right-hand side is nothing but the two-point function of a two-dimensional topological field theory. In Sect. 3, we give a review of the construction of two-dimensional topological field theories associated with symmetric groups. From these facts, it is expected to be fruitful to learn four-dimensional theories from two-dimensional theories by means of symmetric groups [23].

Walled Brauer algebras [29–32] are another convenient tool to organise multi-trace gauge invariant operators constructed from some matrices [10,15]. In Sect. 4, we will explicitly construct two-dimensional topological field theories based on walled Brauer algebras. The introduction of a dual basis manifests a similarity to the construction of topological field theories by symmetric groups. We will also show that correlation functions of the Brauer topological field theory can be decomposed into correlation functions of the topological field theories obtained from symmetric groups.

The central part of this paper is, based on these connections, devoted to the study of two-dimensional field theories that are closely related to the description of multi-trace operators built from some scalar fields in $\mathcal{N} = 4$ super Yang–Mills theory. As will be reviewed in Sect. 2, when we organise gauge invariant operators that involve p kinds of matrices in terms of symmetric groups or walled Brauer algebras, the conjugation under some permutation belonging to the symmetric group $S_{n_1} \times \cdots \times S_{n_p}$ is essentially important [10–16,22]. In Sect. 5, we will construct two-dimensional field theories whose physical operators are characterised by the same conjugation. The idea of the construction is given in [23]. It has been understood that two-dimensional topological quantum field theories are in one-to-one correspondence with commutative Frobenius algebras. What we describe in Sects. 3 and 4 can be phrased mathematically, that commutative Frobenius algebras are constructed from symmetric groups and walled Brauer algebras. By contrast, algebras playing a role in the theories we consider in Sect. 5 are non-commutative Frobenius algebras. These theories have some properties owned by topological field theories, such as that the two-point function is a projector. We will see that an effective matrix-like structure shows up as a consequence of the noncommutativity, where multiplicity indices on the restricted characters [7,8,10,13,15] behave like matrix-indices. We will also discuss the connection between correlation functions of the two-dimensional field theories and correlation functions of $\mathcal{N} = 4$ super Yang–Mills theory in Sect. 6, hoping to illuminate a new geometric correspondence between the two theories. In Sect. 7, we discuss the counting problem of multi-traces from the point of view of the two-dimensional field theories. Section 8 is devoted to discussions. In some appendices we collect useful materials.

2. Correlation Functions in Multi-Matrix Models

In this section, we will review the recent approach to compute exact finite N correlation functions of multi-trace operators in $\mathcal{N} = 4$ Super Yang–Mills theory. The mathematical background behind the approach is supplemented in Appendix A.

We consider the free field theory of scalar fields, and we use the complex notation to denote them by X, Y, Z or X_a $(a = 1, 2, 3)$. The two-point function is determined by conformal symmetry to take the following form

$$\langle O_i(x)O_j(y)\rangle_{SYM} = \frac{c_{ij}}{(x-y)^{n_i+n_j}}\delta_{n_i,n_j}, \tag{2}$$

where n_i is the number of fields involved in the operator O_i. General gauge invariant operators are given by a product of an arbitrary number of single trace operators built from the fundamental fields X_a and $X_a^\dagger$. The non-trivial N-dependence of the correlator is encoded in c_{ij}, and it is obtained by solving the combinatorial problem of contractions of the free field captured by the matrix integral with the Gaussian weight [33]

$$c_{ij} = \langle O_i O_j\rangle := \int \prod_a [dX_a dX_a^\dagger] e^{-2tr(X_a X_a^\dagger)} O_i O_j, \tag{3}$$

where the measure is normalised to give $\langle (X_a)_{ij}(X_b^\dagger)_{kl}\rangle = \delta_{il}\delta_{jk}\delta_{ab}$.

The evaluation of the matrix integral will be neatly performed if we introduce an appropriate algebra as a tool to organise the multi-trace structure. Let us first consider the half BPS chiral primary operators described by a single holomorphic matrix X [4,5]. Any multi-trace operators built from n copies of X are conveniently labelled by an element of the symmetric group S_n as

$$tr_n(\sigma X^{\otimes n}) = X_{i_1}^{i_{\sigma(1)}} X_{i_2}^{i_{\sigma(2)}} \cdots X_{i_n}^{i_{\sigma(n)}} \quad (\sigma \in S_n). \tag{4}$$

Here tr_n is a trace over the tensor space $V^{\otimes n}$, where X is regarded as a linear map on V. Because two elements that are conjugate each other give the same multi-trace

$$tr_n(\sigma X^{\otimes n}) = tr_n(\rho\sigma\rho^{-1} X^{\otimes n}) \quad (\rho \in S_n), \tag{5}$$

the number of independent multi-traces is not equal to the number of elements in the symmetric group S_n. Taking the equivalence relation into account, the multi-trace operators are correctly classified by conjugacy classes. The matrix integral of the two-point function can be evaluated as

$$\begin{aligned}\langle tr_n(\tau X^{\dagger\otimes n}) tr_n(\sigma X^{\otimes n})\rangle &= \sum_{\rho\in S_n} tr_n(\tau\rho\sigma\rho^{-1}) \\ &= \sum_{\rho\in S_n}\sum_{R\vdash n} t_R \chi_R(\tau\rho\sigma\rho^{-1}) \\ &= \sum_{\rho\in S_n} N^n \delta_n(\Omega_n \tau\rho\sigma\rho^{-1}),\end{aligned} \tag{6}$$

At the first equality the Wick-contractions are expressed by elements in S_n [4]. The second and third step result from the Schur–Weyl duality, as is explained in (A.6)–(A.9). The dimension of an irreducible representation R of the symmetric group is denoted by d_R, while the dimension of an irreducible representation R of the $SU(N)$ group is denoted by t_R. The Ω_n is a specific central element in the group algebra of the symmetric group,

$$\Omega_n = \sum_{\sigma \in S_n} \sigma N^{C_\sigma - n}, \tag{7}$$

where C_σ is the number of cycles in the permutation σ. The last equality of (6) is valid for N that is larger than n. The two-point functions can be diagonalised by the basis change of (A.4) as [4]

$$\langle tr_n(p_R X^{\dagger \otimes n}) tr_n(p_S X^{\otimes n})\rangle = n! d_R t_R \delta_{RS}. \tag{8}$$

The diagonal basis is labelled by a Young diagram with n boxes.

The idea to label multi-trace operators in terms of an element in the group algebra of the symmetric group can be applied to operators described by some kinds of matrices [6,11–13]. For a multi-trace constructed from m copies of X and n copies of Y, using $\sigma \in S_{m+n}$ we have

$$tr_{m+n}(\sigma X^{\otimes m} \otimes Y^{\otimes n}) = X^{i_{\sigma(1)}}_{i_1} \cdots X^{i_{\sigma(m)}}_{i_m} Y^{i_{\sigma(m+1)}}_{i_{m+1}} \cdots Y^{i_{\sigma(m+n)}}_{i_{m+n}}. \tag{9}$$

The difference from (4) is that equivalence classes for (9) are characterised by the equivalence relation determined by the subgroup

$$tr_{m+n}(\sigma X^{\otimes m} \otimes Y^{\otimes n}) = tr_{m+n}(h\sigma h^{-1} X^{\otimes m} \otimes Y^{\otimes n}) \quad (h \in S_m \times S_n). \tag{10}$$

In this description the subgroup $H = S_m \times S_n$ plays a role. Two-point function can be evaluated as

$$\begin{aligned} \langle tr_{m+n}(\tau X^{\dagger\otimes m} \otimes Y^{\dagger\otimes n}) tr_{m+n}(\sigma X^{\otimes m} \otimes Y^{\otimes n})\rangle &= \sum_{h \in S_m \times S_n} tr_{m+n}(\tau h \sigma h^{-1}) \\ &= \sum_{h \in S_m \times S_n} N^{m+n} \delta_{m+n}(\Omega_{m+n} \tau h \sigma h^{-1}). \end{aligned} \tag{11}$$

We emphasise that the free field Wick-contractions are expressed by elements of the subgroup $S_m \times S_n$. A diagonal two-point function can be obtained by a change of basis in (A.15) [13],

$$\begin{aligned} &\langle tr_{m+n}(P^R_{A,\mu\nu} X^{\dagger\otimes m} \otimes Y^{\dagger\otimes n}) tr_{m+n}(P^S_{A',\mu'\nu'} X^{\otimes m} \otimes Y^{\otimes n})\rangle \\ &= m! n! d_A t_R \delta_{RS} \delta_{AA'} \delta_{\mu\nu'} \delta_{\nu\mu'}. \end{aligned} \tag{12}$$

The diagonal operators are labelled by a set of three Young diagrams and two multiplicity labels. Another diagonal basis is described in [11,12].

We have another way to label multi-traces constructed from some fields using walled Brauer algebras [10,15]. Walled Brauer algebras can be introduced as a Schur–Weyl dual to the $GL(N)$ groups [see (A.19)]. Mainly consider multi-trace operators constructed from m copies of X and n copies of Y, and let $B_N(m,n)$ be the walled Brauer algebra relevant for the description of such operators. The Brauer algebra contains the group algebra of $S_m \times S_n$ as a subalgebra. Gauge invariant operators are constructed by regarding $X^{\otimes m} \otimes Y^{T\otimes n}$ as operators acting on the space $V^{\otimes m} \otimes \bar{V}^{\otimes n}$, followed by the action of an element of the Brauer algebra and taking a trace:

$$tr_{m,n}(bX^{\otimes m} \otimes Y^{T\otimes n}) \quad (b \in B_N(m,n)). \tag{13}$$

The equivalence relation is very similar to (10),

$$tr_{m,n}(bX^{\otimes m} \otimes Y^{T\otimes n}) = tr_{m,n}(hbh^{-1}X^{\otimes m} \otimes Y^{T\otimes n}) \quad (h \in S_m \times S_n). \tag{14}$$

By expressing free field Wick-contractions in terms of elements in H, the two-point function is computed to give

$$\begin{aligned}\langle tr_{m,n}(bX^{\dagger\otimes m} \otimes Y^{T\dagger\otimes n})tr_{m,n}(cX^{\otimes m} \otimes Y^{T\otimes n})\rangle &= \sum_{h\in S_m\times S_n} tr_{m,n}(bhch^{-1}) \\ &= \sum_{h\in S_m\times S_n}\sum_{\gamma} t_\gamma \chi^\gamma(bhch^{-1}),\end{aligned} \tag{15}$$

where b, c are elements of the walled Brauer algebra. The dimension of an irreducible representation γ of the walled Brauer algebra is denoted by d_γ, while the dimension of an irreducible representation γ of the $GL(N)$ group is denoted by t_γ. The last equality in (15) is a consequence of the Schur–Weyl duality (A.19). By taking the linear combination in (A.28), we will obtain diagonal two-point functions [10],

$$\begin{aligned}&\langle tr_{m,n}(Q^{\gamma}_{A,\mu\nu}X^{\dagger\otimes m} \otimes Y^{T\dagger\otimes n})tr_{m,n}(Q^{\gamma'}_{A',\mu'\nu'}X^{\otimes m} \otimes Y^{T\otimes n})\rangle \\ &= m!n!d_A t_\gamma \delta_{\gamma\gamma'}\delta_{AA'}\delta_{\mu\nu'}\delta_{\nu\mu'}.\end{aligned} \tag{16}$$

Walled Brauer algebras can also be used to describe multi-trace operators constructed from more than two kinds of matrices [16,22].

Before closing this section, we will introduce some symbols to denote the equivalence classes characterising the multi-matrix structures. For a fixed element $\sigma \in S_n$ the following sum over a complete basis of S_n gives a central element

$$[\sigma] = \frac{1}{n!}\sum_{\rho\in S_n} \rho\sigma\rho^{-1} \quad (\sigma \in S_n). \tag{17}$$

In symmetric group S_{m+n} the sum over a complete basis of $S_m \times S_n$ gives an element which commutes with any elements in $S_m \times S_n$,

$$[\sigma]_H = \frac{1}{m!n!}\sum_{h\in H} h\sigma h^{-1} \quad (\sigma \in S_{m+n}), \tag{18}$$

where $H = S_m \times S_n$. Similarly an element in the Brauer algebra $B_N(m,n)$ which commutes with any elements in $S_m \times S_n$ can be constructed by

$$[b]_H = \frac{1}{m!n!}\sum_{h\in H} hbh^{-1} \quad (b \in B_N(m,n)). \tag{19}$$

The number of the equivalence classes coincides with the number of independent multi-matrix operators at large N [see also (96), (121) and (148)]. If N is small compared to the number of fields involved in the multi-traces, multi-traces are in general linearly dependent. Linearly independent multi-traces are given by the Young diagram basis [4,10–13,34].

With these new notations, the two-point functions are rewritten as

$$\begin{aligned}
&\langle tr_n([\tau]X^{\dagger\otimes n})tr_n([\sigma]X^{\otimes n})\rangle \\
&\quad = n!N^n\delta_n(\Omega_n[\tau][\sigma]) \\
&\langle tr_{m+n}([\tau]_H X^{\dagger\otimes m}\otimes Y^{\dagger\otimes n})tr_{m+n}([\sigma]_H X^{\otimes m}\otimes Y^{\otimes n})\rangle \\
&\quad = m!n!N^{m+n}\delta_{m+n}(\Omega_{m+n}[\tau]_H[\sigma]_H) \\
&\langle tr_{m,n}([b]_H X^{\dagger\otimes m}\otimes Y^{T\dagger\otimes n})tr_{m,n}([c]_H X^{\otimes m}\otimes Y^{T\otimes n})\rangle \\
&\quad = m!n!\sum_\gamma t_\gamma\chi^\gamma([b]_H[c]_H).
\end{aligned} \tag{20}$$

We will reconsider the meaning of these equations in Sect. 8.

3. 2D Topological Field Theories and Semisimple Algebras

In the previous section we have reviewed that employing the group algebra of the symmetric group and the walled Brauer algebra is very effective for labelling multi-trace gauge invariant operators in the multi-matrix models. On the other hand it has been known that two-dimensional topological quantum field theories can be defined from such algebras. This fact would bring up a new aspect on the role of the algebras, suggesting a new connection with two-dimensional field theories. In this section we will review the construction of two-dimensional topological quantum field theories. For our purpose it is convenient to define them in terms of lattice [35–37], where it was shown that semisimple algebras have a one-to-one correspondence with two-dimensional topological quantum field theories.

Consider the triangulation of a two-dimensional compact orientable surface. The structure of the surface is encoded in the way of gluing the triangles. To each edge we assign a colour index i, and to each triangle with edges labelled by i, j, k, we assign a complex number C_{ijk}. We assume that C_{ijk} is invariant under cyclic permutations of the colour indices

$$C_{ijk} = C_{jki} = C_{kij}, \tag{21}$$

but no relation is imposed between C_{ijk} and the orientation-reversed object C_{ikj}. Two adjacent edges are identified by introducing a gluing operator g^{ij}, which is assumed to be symmetric $g^{ij} = g^{ji}$ and to have the inverse g_{ij}. In the dual diagrams g^{ij} and C_{ikj} correspond to the propagator and the three-point vertex respectively. The partition function of a triangulated surface is given by the product of complex numbers C_{ijk} assigned to each triangle with each edge glued by the gluing operator g^{ij} and the summation over all possible triangulations.

Topological quantum field theories are characterised by the invariance under local deformations of the background. In the lattice construction, topological models are constructed by imposing the invariance of partition functions under any local change of the triangulations. It is known that two basic moves are sufficient to generate all topologically equivalent triangulations. We will use so called bubble move and 2-2 move (see Fig. 1). From the invariance of partition functions under the bubble moves, we obtain

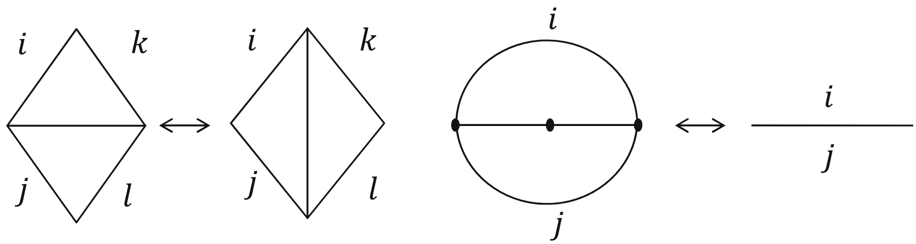

Fig. 1. *Left* the 2-2 move, *right* the bubble move

$$C_{ik}{}^{l} C_{jl}{}^{k} = g_{ij}, \tag{22}$$

where

$$C_{ij}{}^{k} = C_{ijl} g^{lk}. \tag{23}$$

We use the usual summation convention that repeated indices are summed. From the invariance of partition functions under the 2-2 moves, we have

$$C_{ij}{}^{p} C_{pk}{}^{l} = C_{ip}{}^{l} C_{jk}{}^{p}. \tag{24}$$

These two conditions are solved by introducing a semisimple associative algebra [35,36]. Let a basis and the structure constant of the algebra be ϕ_i and $C_{ij}{}^{k}$, that is, $\phi_i \phi_j = C_{ij}{}^{k} \phi_k$. It is easy to find that the condition (24) is derived from the associativity of the algebra; $(\phi_i \phi_j)\phi_k = \phi_i(\phi_j \phi_k)$. The condition (22) indicates that we can define a nondegenerate metric g_{ij} given in the equation. With these conditions, partition functions only depend on the genus of the manifold.

In order to make the construction more concrete, we will consider the group algebra of symmetric group S_n as an example of semisimple algebras [35]. Introducing the regular representation, the structure constant is given by

$$C_{ij}{}^{k} = \frac{1}{n!} tr^{(r)}(\sigma_k^{-1} \sigma_i \sigma_j), \tag{25}$$

where $tr^{(r)}$ is the trace of the regular representation. Defining the delta function over the group algebra of S_n by $\delta_n(\sigma) = 1$ if $\sigma = 1$ and 0 otherwise, the trace of the regular representation can be expressed by

$$tr^{(r)}(\sigma) = n! \delta_n(\sigma). \tag{26}$$

It is also convenient to consider the expansion in terms of the character of the symmetric group as

$$tr^{(r)}(\sigma) = \sum_{R \vdash n} d_R \chi_R(\sigma), \tag{27}$$

where $R \vdash n$ means that R is a Young diagram with n boxes. From this formula we can verify $tr^{(r)}(1) = \sum_{R \vdash n} (d_R)^2$, where d_R is the dimension of an irreducible representation R of S_n.

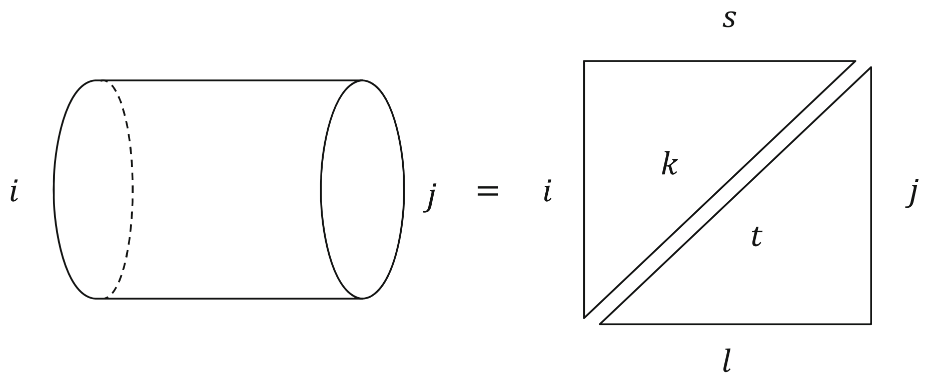

Fig. 2. A triangulation of the cylinder with two boundaries i, j: the edges k and t and the edges s and l being glued by the operator g^{ij} as in (33)

From (22) and (23) we determine

$$\begin{aligned} g_{ij} &= tr^{(r)}(\sigma_i\sigma_j) = n!\delta_n(\sigma_i\sigma_j) \\ C_{ijk} &= tr^{(r)}(\sigma_i\sigma_j\sigma_k) = n!\delta_n(\sigma_i\sigma_j\sigma_k), \end{aligned} \tag{28}$$

and

$$g^{ij} = \frac{1}{(n!)^2} tr^{(r)}(\sigma_i^{-1}\sigma_j^{-1}) = \frac{1}{n!}\delta(\sigma_i^{-1}\sigma_j^{-1}). \tag{29}$$

Let us now introduce a dual basis defined by

$$\sigma^i := g^{ij}\sigma_j = \frac{1}{n!}\sigma_i^{-1}, \tag{30}$$

which satisfies

$$\sum_i \sigma^i \sigma_i = 1. \tag{31}$$

If we use the dual basis, we do not have an extra factor arising from raising or lowering indices

$$\begin{aligned} g^{ij} &= tr^{(r)}(\sigma^i\sigma^j) = n!\delta(\sigma^i\sigma^j), \\ C_{ij}{}^k &= tr^{(r)}(\sigma_i\sigma_j\sigma^k) = n!\delta(\sigma_i\sigma_j\sigma^k). \end{aligned} \tag{32}$$

We next construct the two-point function on a sphere. A simple triangulation shown in Fig. 2 leads to

$$\eta_{ij} = C_{iks}C_{ljt}g^{sl}g^{kt} = C_{ik}{}^l C_{lj}{}^k. \tag{33}$$

We can show that

$$\begin{aligned} \eta_{ij} &= \sum_{R\vdash n} \chi^R(\sigma_i)\chi^R(\sigma_j) \\ \eta^{ij} &= \sum_{R\vdash n} \chi^R(\sigma^i)\chi^R(\sigma^j) \\ \eta_i{}^j &= \sum_{R\vdash n} \chi^R(\sigma_i)\chi^R(\sigma^j), \end{aligned} \tag{34}$$

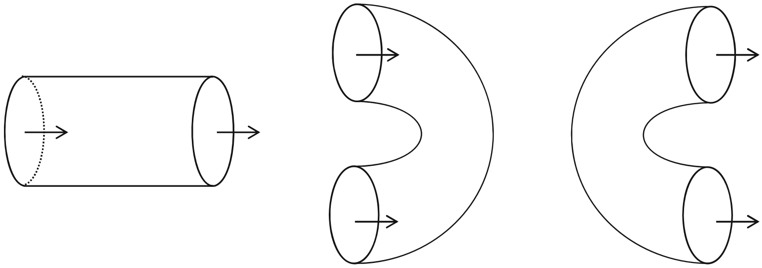

Fig. 3. Three oriented cylinders, being drawn so that all *arrows* are turning to the *right*

or we have

$$\begin{aligned}\eta_{ij} &= \sum_k tr^{(r)}(\sigma_i\sigma_k\sigma_j\sigma^k) = tr^{(r)}([\sigma_i][\sigma_j])\\ \eta^{ij} &= \sum_k tr^{(r)}(\sigma^i\sigma_k\sigma^j\sigma^k) = tr^{(r)}([\sigma^i][\sigma^j])\\ \eta_i{}^j &= \sum_k tr^{(r)}(\sigma_i\sigma_k\sigma^j\sigma^k) = tr^{(r)}([\sigma_i][\sigma^j]).\end{aligned} \tag{35}$$

One important property is that this is a projection operator, $\eta^2 = \eta$. More explicitly, we have

$$\eta_{ij}\eta^{jk} = \eta_i{}^k, \quad \eta_i{}^j\eta_j{}^k = \eta_i{}^k, \quad \eta_{ij}\eta^j{}_k = \eta_{ik}, \tag{36}$$

and so on. The fact that a cylinder is a projection operator is a general property of topological field theories [38]. Lower indices and upper indices correspond to different orientations. The difference comes from the difference between the basis and the dual basis. In order to distinguish them it is convenient to to call the two kinds of boundaries *in-boundaries* and *out-boundaries*. We can draw three kinds of cylinders as in Fig. 3, corresponding to $\eta_i{}^j$, η_{ij} and η^{ij}. The readers can find the diagrammatic meaning of the relation (36) using the oriented cylinders in Fig. 3. In this theory, the difference is just the factor of $1/n!$,

$$\chi^R(\sigma^i) = \frac{1}{n!}\chi^R(\sigma_i^{-1}) = \frac{1}{n!}\chi^R(\sigma_i), \tag{37}$$

because σ^{-1} is conjugate to σ. The difference between different orientations will be more important in theories we will consider in what follows.

The projection operator determines the space of physical operators [35]. It can be shown that the η is the projection operator from the algebra onto the centre of the algebra,

$$\eta_i{}^j\sigma_j = \sum_{R\vdash n}\frac{1}{d_R}\chi^R(\sigma_i)p_R = [\sigma_i], \tag{38}$$

where p_R is a central element given in (A.4), and $[\sigma]$ is defined in (17). In other words, physical states are invariant under time translation,

$$\eta_i{}^j[\sigma_j] = [\sigma_i]. \tag{39}$$

We thus have a one-to-one correspondence between central elements and physical states. The number of the physical states is the number of conjugacy classes.

Because the cylinder is a projection operator onto the physical Hilbert space, the torus, which is obtained by gluing two boundaries of the cylinder, gives the dimension of the vector space [38,39],

$$Z_{G=1} = \eta_i{}^i = \sum_{R \vdash n} 1. \tag{40}$$

This counts the number of Young diagrams built from n boxes. It is equivalent to the number of multi-matrices built from n copies of a matrix at large N.

The three-point function N_{ijk} is obtained from the structure constant of the operator product of physical states,

$$[\sigma_i][\sigma_j] = N_{ij}{}^k[\sigma_k]. \tag{41}$$

From this we find

$$N_{ijk} = tr^{(r)}([\sigma_i][\sigma_j][\sigma_k]) = \sum_{R \vdash n} \frac{1}{d_R} \chi^R(\sigma_i)\chi^R(\sigma_j)\chi^R(\sigma_k) \tag{42}$$

and

$$N_{ij}{}^k = tr^{(r)}([\sigma_i][\sigma_j][\sigma^k]) = \sum_{R \vdash n} \frac{1}{d_R} \chi^R(\sigma_i)\chi^R(\sigma_j)\chi^R(\sigma^k), \tag{43}$$

where $N_{ijk} = N_{ik}{}^l \eta_{lk}$. Because this operator algebra is commutative

$$[\sigma_i][\sigma_j] = [\sigma_j][\sigma_i], \tag{44}$$

we have $N_{ij}{}^k = N_{ji}{}^k$. The associativity $([\sigma_i][\sigma_j])[\sigma_k] = [\sigma_i]([\sigma_j][\sigma_k])$ gives

$$N_{ij}{}^k N_{kl}{}^n = N_{ik}{}^n N_{jl}{}^k. \tag{45}$$

From $[\sigma_i] = \eta_i{}^p \sigma_p$, we find that the cylinder and the three-holed sphere are also obtained by acting with the projection operator on g_{ij} and C_{ijk} as

$$\begin{aligned} \eta_k{}^i \eta_l{}^j g_{ij} &= \eta_{kl} \\ \eta_l{}^i \eta_m{}^j \eta_n{}^k C_{ijk} &= N_{lmn}. \end{aligned} \tag{46}$$

These are also obtained by a simple triangulation [35].

An associative algebra with a non-degenerate inner product is called Frobenius algebra.[1] It has been understood that Frobenius algebras play a role in the algebraic and axiomatic formulation of topological quantum field theories, and commutative Frobenius algebras are in one-to-one correspondence with two-dimensional topological quantum field theories [40,41]. In the example, the algebra of $[\sigma_i]$ with the metric η_{ij} is a

[1] Important remarks are found in p. 98 of [41]. A Frobenius algebra is defined by providing with its Frobenius structure (Frobenius pairing or Frobenius form). If we choose two different Frobenius structures in an algebra, we will obtain two different Frobenius algebras. Being a Frobenius algebra is not a property of an algebra, but it is a structure of the algebra.

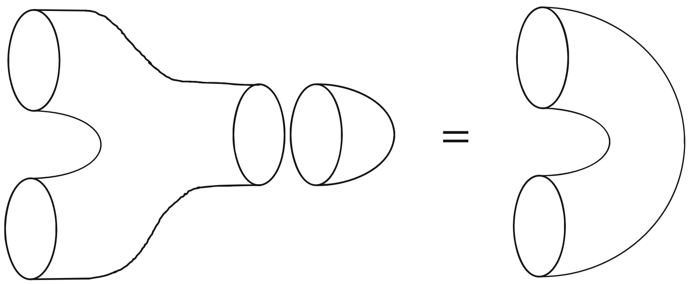

Fig. 4. Drawing of $N_{ij}{}^k \eta_{k0} = \eta_{ij}$

commutative Frobenius algebra. The η_{ij} is called Frobenius pairing. The pairing also determines a linear map from the vector space to the ground field,

$$\eta_{0i} = \sum_{R \vdash n} d_R \chi_R(\sigma_i) = n!\delta(\sigma_i), \tag{47}$$

where $\sigma_0 = 1$. This is called Frobenius form. Formally it is the sphere with a hole. The three objects, η_{ij}, N_{ijk} and η_{i0} can be basic building blocks to build up all surfaces. There is a relation among them, $N_{ij}{}^k \eta_{k0} = \eta_{ij}$ (see Fig. 4).

All partition functions are built up from the building blocks η_{ij} and N_{ijk}. The partition function of a Riemann surface of genus G is

$$Z_G = \sum_{R \vdash n} \frac{1}{(d_R)^{2G-2}}. \tag{48}$$

The correlation function on the manifold of genus G with B_1 in-boundaries and B_2 out-boundaries is given by

$$Z_{G,B_1,B_2} = \sum_{R \vdash n} \frac{1}{(d_R)^{2G+B_1+B_2-2}} \chi^R(\sigma_{i_1}) \cdots \chi^R(\sigma_{i_{B_1}}) \chi^R(\sigma^{j_1}) \cdots \chi^R(\sigma^{j_{B_2}}). \tag{49}$$

The repeated use of (B.3) and (B.4) helps us to find an equivalent form

$$Z_{G,B_1,B_2} = \sum_{k,l} tr^{(r)} \left((\rho_{k_1} \tau_{l_1} \rho^{k_1} \tau^{l_1}) \cdots (\rho_{k_G} \tau_{l_G} \rho^{k_G} \tau^{l_G}) [\sigma_{i_1}] \cdots [\sigma_{i_{B_1}}][\sigma^{j_1}] \cdots [\sigma^{j_{B_2}}] \right). \tag{50}$$

We note that $\sum_{k,l} \rho_k \tau_l \rho^k \tau^l$ and $[\sigma]$ are central elements of the group algebra of the symmetric group, so their positions inside the trace are irrelevant. The form shows that the correlation functions are also obtained by representing the surface as a polygone with edges properly identified [42].

It is possible to choose a diagonal basis so that the operator product is given by $O_\alpha O_\beta = \delta_{\alpha\beta} O_\alpha$. This is realised by

$$p_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma^{-1})[\sigma] = d_R \sum_i \chi_R(\sigma^i)[\sigma_i]. \tag{51}$$

The role of p_R is explained around (A.4). In this basis, $\eta_R{}^S = \delta_R{}^S$.

4. 2D Topological Field Theories Obtained from Walled Brauer Algebras

4.1. Construction of Brauer topological field theory. In this section we present the topological lattice field theory obtained from the walled Brauer algebra $B_N(m, n)$, following the construction in the previous section. Walled Brauer algebras have a parameter N, which is identified with the matrix size N when we use them to organise the multi-trace structure of $N \times N$ matrices. We will assume that N is large enough so that $m + n < N$ is satisfied. This large N condition secures the semisimplicity of the algebra, which we need to construct a non-degenerate metric. In this construction we will use the idea presented in [43] that general semisimple algebras can be analogously introduced to finite groups.

Let a basis of the algebra be b_i. We first define a dual basis b_i^* with respect to the bilinear form given by the trace $tr_{m,n}$ over the mixed tensor space $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ in (A.19),

$$tr_{m,n}(b_i b_j^*) = \delta_i{}^j. \tag{52}$$

This dual basis was exploited in [10].

The trace of the regular representation is introduced by

$$tr^{(r)}(b) = \sum_i tr_{m,n}(b_i^* b b_i) = \sum_\gamma d^\gamma \chi^\gamma(b), \tag{53}$$

where (A.20) and (B.10) are helpful to confirm the second equality, and γ are irreducible representations of the walled Brauer algebra. An irreducible representation γ is labelled by a bi-partition [see below Eq. (A.19)]. Here d^γ is the dimension of an irreducible representation γ of the walled Brauer algebra, while t^γ will be that of the $GL(N)$ group. The appearance of the $GL(N)$ group is a consequence of the Schur–Weyl duality. The multiplication of the algebra, $b_i b_j = C_{ij}{}^k b_k$, is determined by

$$C_{ij}{}^k = tr_{m,n}(b_k^* b_i b_j). \tag{54}$$

Using orthogonality relations in Appendix B we can derive

$$g_{ij} = tr^{(r)}(b_i b_j), \tag{55}$$

$$C_{ijk} = tr^{(r)}(b_i b_j b_k), \tag{56}$$

and

$$g^{ij} = \sum_\gamma \frac{(t^\gamma)^2}{d^\gamma} \chi^\gamma(b_i^* b_j^*). \tag{57}$$

Let us now introduce another dual basis by

$$b^i = g^{ij} b_j, \tag{58}$$

and we can confirm the following equation

$$\sum_i b^i b_i = 1. \tag{59}$$

The following equation is helpful to relate the first dual basis b_i^* and the second dual basis b^i,

$$D^\gamma(b^i) = g^{ij} D^\gamma(b_j) = \frac{t^\gamma}{d^\gamma} D^\gamma(b_i^*), \tag{60}$$

where $D^\gamma(b)$ is the representation matrix of b in γ. From this, we find that b^i is a dual basis with respect to the bilinear form determined by the trace of the regular representation,

$$tr^{(r)}(b_i b^j) = \delta_i{}^j. \tag{61}$$

Defining a dual basis allows us to obtain the representation theory of general semisimple algebras in a very similar way to finite groups [43]. In fact, we will find that almost all formulae are very similar to the formulae used in the symmetric group. In terms of the dual basis, we will obtain a handy expression for quantities with upper indices,

$$\begin{aligned} g^{ij} &= tr^{(r)}(b^i b^j) \\ C_{ij}{}^k &= tr^{(r)}(b_i b_j b^k). \end{aligned} \tag{62}$$

In what follows we will exploit the second dual basis b^i.

The computation of the two-point function is the same as (33). We obtain

$$\begin{aligned} \eta_{ij} &= \sum_\gamma \chi^\gamma(b_i)\chi^\gamma(b_j) \\ \eta^{ij} &= \sum_\gamma \chi^\gamma(b^i)\chi^\gamma(b^j) \\ \eta_i{}^j &= \sum_\gamma \chi^\gamma(b_i)\chi^\gamma(b^j). \end{aligned} \tag{63}$$

If we use (B.21), we find another expression of the two-point function,

$$\begin{aligned} \eta_{ij} &= \sum_k tr^{(r)}(b_i b_k b_j b^k) \\ \eta^{ij} &= \sum_k tr^{(r)}(b^i b_k b^j b^k) \\ \eta_i{}^j &= \sum_k tr^{(r)}(b_i b_k b^j b^k). \end{aligned} \tag{64}$$

Because $\chi^\gamma(b_i)$ is not proportional to $\chi^\gamma(b^i)$, the difference between in-boundaries (corresponding to lower indices) and out-boundaries (corresponding to upper indices) is more important.

$\sum_\gamma$ denotes the sum over all irreducible representations of the walled Brauer algebra. An irreducible representation of $B_N(m, n)$ is labelled by a bi-partition (γ_+, γ_-), where $\gamma_+ \vdash (m-k)$ and $\gamma_- \vdash (n-k)$, and k is an integer in the range $0 \le k \le min(m, n)$. The sum will be performed by

$$\sum_\gamma = \sum_{k=0}^{min(m,n)} \sum_{\gamma_+ \vdash (m-k)} \sum_{\gamma_- \vdash (n-k)}. \tag{65}$$

The Hilbert space of physical states is determined by the projection operator η. It is the projection onto the centre of the algebra,

$$\eta_i{}^j b_j = \sum_\gamma \chi^\gamma(b_i)\chi^\gamma(b^j)b_j = \sum_\gamma \frac{1}{d_\gamma}\chi^\gamma(b_i)P^\gamma. \tag{66}$$

where P^γ, which is given in (A.23), is a central element in the Brauer algebra. If we define

$$[[b]] = \sum_i b_i b b^i, \tag{67}$$

then we can show that

$$\eta_i{}^j b_j = [[b_i]]. \tag{68}$$

Equivalently we have

$$\eta_i{}^j [[b_j]] = [[b_i]]. \tag{69}$$

Physical operators are invariant under time evolution.

The three-point function is determined by the operator product of physical states. From

$$[[b_i]][[b_j]] = N_{ij}{}^k [[b_k]], \tag{70}$$

we obtain

$$N_{ijk} = N_{ij}{}^l \theta_{lk} = tr^{(r)}([[b_i]][[b_j]][[b_k]]) = \sum_\gamma \frac{1}{d^\gamma}\chi^\gamma(b_i)\chi^\gamma(b_j)\chi^\gamma(b_k). \tag{71}$$

We can also use (46) to derive this. The associativity $([[b_i]][[b_j]])[[b_k]] = [[b_i]]([[b_j]][[b_k]])$ is expressed by $N_{ij}{}^k N_{kl}{}^n = N_{ik}{}^n N_{jl}{}^k$. The diagonal operator product is realised by switching to the representation basis P^γ, that is, $P^\gamma P^{\gamma'} = \delta^{\gamma\gamma'}P^\gamma$. The algebra of $[[b_i]]$ with the bilinear form η is a commutative Frobenius algebra.

The partition function of a Riemann surface of genus G is

$$Z_G = \sum_\gamma \frac{1}{(d^\gamma)^{2G-2}}, \tag{72}$$

and this is equivalent to

$$Z_G = \sum_{i,j} tr^{(r)}\left((c_{i_1}d_{j_1}c^{i_1}d^{j_1})\cdots(c_{i_G}d_{j_G}c^{i_G}d^{j_G})\right), \tag{73}$$

where c, d are elements of the Brauer algebra. The partition function of $G = 1$ gives the dimension of the vector space

$$Z_{G=1} = \sum_i \eta_i{}^i = \sum_\gamma 1. \tag{74}$$

This counts the number of all irreducible representations. Because a γ is given by the bi-partition as in (65), we have

$$Z_{G=1} = \sum_{k}^{min(m,n)} \sum_{\gamma_+ \vdash m-k} 1 \sum_{\gamma_- \vdash n-k} 1 = \sum_{k=0}^{min(m,n)} p(m-k)p(n-k),$$

where $p(n)$ is a partition of a number n.

Likewise for a Riemann manifold of genus G with B_1 in-boundaries and B_2 out-boundaries the correlation function is

$$Z_{G,B_1,B_2} = \sum_{\gamma} \frac{1}{(d^\gamma)^{2G+B_1+B_2-2}} \chi^\gamma(b_{i_1}) \cdots \chi^\gamma(b_{i_{B_1}}) \chi^\gamma(b^{j_1}) \cdots \chi^\gamma(b^{j_{B_2}}). \quad (75)$$

An equivalent form is

$$Z_{G,B_1,B_2} = \sum_{k,l} tr^{(r)} \left((c_{k_1} d_{l_1} c^{k_1} d^{l_1}) \cdots (c_{k_G} d_{l_G} c^{k_G} d^{l_G}) [[b_{i_1}]] \cdots [[b_{i_{B_1}}]][[b^{j_1}]] \cdots [[b^{j_{B_2}}]] \right). \quad (76)$$

Both $\sum_{k,l}(c_k d_l c^k d^l)$ and $[[b]]$ are central elements of the Brauer algebra. The indices are raised or lowered by η. For example we can use η^{ij} to convert an in-boundary into an out-boundary, finding from

$$\eta^{ij} \chi^\gamma(b_j) = \chi^\gamma(b^i). \quad (77)$$

There is a formal correspondence between the symmetric group S_{m+n} and the walled Brauer algebra $B_N(m,n)$, clarified thanks to the introduction of the dual basis. The correspondence is as follows

$$\begin{aligned} (R, A) &\leftrightarrow (\gamma, A), \\ t_R &\leftrightarrow t_\gamma, \\ d_R &\leftrightarrow d_\gamma, \\ tr^{(r)}(\sigma) &\leftrightarrow tr^{(r)}(b), \\ \sigma_i &\leftrightarrow b_i, \\ \sigma^i &\leftrightarrow b^i. \end{aligned} \quad (78)$$

The only difference between them is that in the symmetric group $\sigma_i \sigma^i$ is proportional to the unit element, while in the Brauer algebra $b_i b^i$ is not proportional to the unit, where the repeated index i is not summed in each case.

4.2. Brauer topological field theory as a collection of symmetric group topological field theories. The partition functions obtained from the symmetric group in Sect. 3 can be regarded as counting of n-fold coverings of a general Riemann manifold, and this idea plays a crucial role in the string theoretic description of large N two-dimensional Yang–Mills theory [24–28]. On the other hand, an interpretation in terms of coverings is less clear for walled Brauer algebras, but it is naturally expected to give two kinds of coverings because the walled Brauer algebra contains the group algebra of $S_m \times S_n$ as a subalgebra. In fact, in [24–26] the coupled representations, which are irreducible representations of the Brauer algebra corresponding to $k = 0$, were considered to describe non-holomorphic maps from worldsheets to the target space. In [44] the full expansion of two-dimensional Yang–Mills is described in terms of only holomorphic maps from a new formula for the couple representation of $GL(N)$ derived with the help of the walled Brauer algebra.

In this subsection, we will show that the correlation functions of the Brauer topological field theory can be expressed in terms of correlation functions obtained from the symmetric group $S_{m-k} \times S_{n-k}$, where k takes all integers in $1 \leq k \leq min(m,n)$.

The partition functions of a surface of genus G are easily expressed in terms of the symmetric group data, if we use the formula (A.22), as

$$\begin{aligned} Z_G^{B_N(m,n)} &= \sum_{\gamma} \frac{1}{(d^{\gamma})^{2G-2}} \\ &= \sum_{k=0}^{min(m,n)} \sum_{\gamma_+ \vdash (m-k)} \sum_{\gamma_- \vdash (n-k)} \left(\frac{(m-k)!(n-k)!k!}{m!n!} \right)^{2-2G} \frac{1}{(d^{\gamma_+})^{2G-2}} \frac{1}{(d^{\gamma_-})^{2G-2}} \\ &= \sum_{k=0}^{min(m,n)} \left(\frac{(m-k)!(n-k)!k!}{m!n!} \right)^{2-2G} Z_G^{S_{m-k}} Z_G^{S_{n-k}}, \end{aligned} \tag{79}$$

where $Z_G^{S_{m-k}}$ is the partition function (48) corresponding to S_{m-k},

$$Z_G^{S_{m-k}} = \sum_{R \vdash (m-k)} \frac{1}{(d_R)^{2G-2}}. \tag{80}$$

Because the right-hand side in (79) has a clear interpretation in terms of coverings of the Riemann surface of genus G, the partition function of the Brauer topological field theory can have a meaning in terms of $(m-k)$-coverings and $(n-k)$-coverings of the Riemann surface. It is interesting that a smaller number of sheets than m, n also come in the game.

We next consider correlation functions. For the purpose we will use a character formula of the walled Brauer algebra. The character of a general element b in the walled Brauer algebra can be expressed in terms of the character of an element of the form $C^{\otimes h} \otimes b_+ \otimes b_-$ as

$$\chi^{\gamma}(b) = N^{z(b)-h} \chi^{\gamma}(C^{\otimes h} \otimes b_+ \otimes b_-), \tag{81}$$

where b_+ and b_- are elements in S_{m-h} and S_{n-h} respectively. $C^{\otimes h}$ denotes h factors of the contraction, $C \otimes \cdots \otimes C$. The $z(b)$ is a quantity that is read from the element b. This formula is more explained in (C.1). Then the two-point function can be

$$\begin{aligned} \eta^{B_N(m,n)}(b, \tilde{b}) &= \sum_{\gamma} \chi^{\gamma}(b) \chi^{\gamma}(\tilde{b}) \\ &= N^{z(b)+z(\tilde{b})-h-\tilde{h}} \eta^{B_N(m,n)}(C^{\otimes h} \otimes b_+ \otimes b_-, C^{\otimes \tilde{h}} \otimes \tilde{b}_+ \otimes \tilde{b}_-). \end{aligned} \tag{82}$$

We will next use

$$\begin{aligned} &\chi^{\gamma}(C^{\otimes h} \otimes b_+ \otimes b_-) \\ &= \frac{N^h}{(m-k)!(n-k)!} \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(b_+, b_-; \sigma_2, \tau_2) \chi_{\gamma_+}(\sigma_2^{-1}) \chi_{\gamma_-}(\tau_2^{-1}), \end{aligned} \tag{83}$$

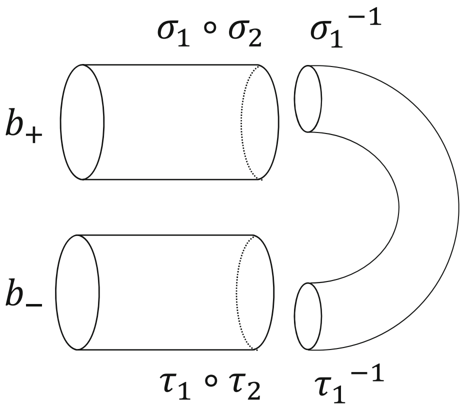

Fig. 5. Pictorial drawing of (84)

where

$$\Delta(b_+, b_-; \sigma_2, \tau_2) = \frac{1}{(k-h)!^2} \sum_{\sigma_1, \tau_1 \in S_{k-h}} \eta^{S_{m-h}}(b_+, \sigma_1 \circ \sigma_2)\eta^{S_{n-h}}(b_-, \tau_1 \circ \tau_2)\eta^{S_{k-h}}(\sigma_1^{-1}, \tau_1^{-1}). \quad (84)$$

This will be derived in (C.6). The Δ is composed of three two-point functions defined in the topological field theory of the symmetric group (see Fig. 5).

The formula enables us to compute

$$\begin{aligned}
&\eta^{B_N(m,n)}(b, \tilde{b}) \\
&= N^{z(b)+z(\tilde{b})-h-\tilde{h}} \eta^{B_N(m,n)}(C^{\otimes h} \otimes b_+ \otimes b_-, C^{\otimes \tilde{h}} \otimes \tilde{b}_+ \otimes \tilde{b}_-) \\
&= N^{z(b)+z(\tilde{b})-h-\tilde{h}} \sum_{\gamma} \chi^{\gamma}(C^{\otimes h} \otimes b_+ \otimes b_-)\chi^{\gamma}(C^{\otimes \tilde{h}} \otimes \tilde{b}_+ \otimes \tilde{b}_-) \\
&= N^{z(b)+z(\tilde{b})} \sum_{k=max(h,\tilde{h})}^{min(m,n)} \sum_{\gamma_+ \vdash (m-k)} \sum_{\gamma_- \vdash (n-k)} \\
&\quad \times \frac{1}{(m-k)!(n-k)!} \sum_{\sigma_2 \in S_{m-k}} \sum_{\tau_2 \in S_{n-k}} \Delta(b_+, b_-; \sigma_2, \tau_2)\chi_{\gamma_+}(\sigma_2^{-1})\chi_{\gamma_-}(\tau_2^{-1}) \\
&\quad \times \frac{1}{(m-k)!(n-k)!} \sum_{\tilde{\sigma}_2 \in S_{m-k}} \sum_{\tilde{\tau}_2 \in S_{n-k}} \Delta(\tilde{b}_+, \tilde{b}_-; \tilde{\sigma}_2, \tilde{\tau}_2)\chi_{\gamma_+}(\tilde{\sigma}_2^{-1})\chi_{\gamma_-}(\tilde{\tau}_2^{-1}) \\
&= N^{z(b)+z(\tilde{b})} \sum_{k=max(h,\tilde{h})}^{min(m,n)} \sum_{\sigma_2, \tilde{\sigma}_2 \in S_{m-k}} \sum_{\tau_2, \tilde{\tau}_2 \in S_{n-k}} \Delta(b_+, b_-; \sigma_2, \tau_2)\Delta(\tilde{b}_+, \tilde{b}_-; \tilde{\sigma}_2, \tilde{\tau}_2) \\
&\quad \times \eta^{S_{m-k} \times S_{n-k}}(\sigma_2^{-1} \otimes \tau_2^{-1}, \tilde{\sigma}_2^{-1} \otimes \tilde{\tau}_2^{-1}), \qquad (85)
\end{aligned}$$

where η is the two-point function (34) corresponding to $S_{m-k} \times S_{n-k}$,

$$\eta^{S_{m-k} \times S_{n-k}}(\sigma_2^{-1} \otimes \tau_2^{-1}, \tilde{\sigma}_2^{-1} \otimes \tilde{\tau}_2^{-1}) = \eta^{S_{m-k}}(\sigma_2^{-1}, \tilde{\sigma}_2^{-1})\eta^{S_{n-k}}(\tau_2^{-1}, \tilde{\tau}_2^{-1}), \quad (86)$$

and

$$\eta^{S_{m-k}}(\sigma_2^{-1}, \tilde{\sigma}_2^{-1}) = \frac{1}{((m-k)!)^2} \sum_{\gamma_+ \vdash (m-k)} \chi_{\gamma_+}(\sigma_2^{-1}) \chi_{\gamma_+}(\tilde{\sigma}_2^{-1}). \tag{87}$$

We have shown that the two-point function of the Brauer topological field theory can be expressed in terms of the two-point function of the topological field theory obtained from the group algebra of $S_{m-k} \times S_{n-k}$, where k takes all integers in $max(h, \tilde{h}) \leq k \leq min(m, n)$.

It is straightforward to extend this to any correlation functions. For example the three-holed sphere (71) is as follows:

$$\begin{aligned}
&Z^{B_N(m,n)}_{G=0,B=3}(b^1, b^2, b^3) \\
&\quad = N^{z(b^1)+z(b^2)+z(b^3)-h_1-h_2-h_3} Z^{B_N(m,n)}_{G=0,B=3} \\
&\qquad \times (C^{\otimes h_1} \otimes b^1_+ \otimes b^1_-, C^{\otimes h_2} \otimes b^2_+ \otimes b^2_-, C^{\otimes h_3} \otimes b^3_+ \otimes b^3_-,) \\
&\quad = N^{z(b^1)+z(b^2)+z(b^3)} \sum_{k=max(h_1,h_2,h_3)}^{min(m,n)} \sum_{\sigma_1,\sigma_2,\sigma_3 \in S_{m-k}} \sum_{\tau_1,\tau_2,\tau_3 \in S_{n-k}} \\
&\qquad \times \Delta(b^1_+, b^1_-; \sigma_1, \tau_1) \Delta(b^2_+, b^2_-; \sigma_2, \tau_2) \Delta(b^3_+, b^3_-; \sigma_3, \tau_3) \\
&\qquad \times \frac{k!(m-k)!(n-k)!}{m!n!} Z^{S_{m-k} \times S_{n-k}}_{G=0,B=3}(\sigma_1^{-1} \otimes \tau_1^{-1}, \sigma_2^{-1} \otimes \tau_2^{-1}, \sigma_3^{-1} \otimes \tau_3^{-1}),
\end{aligned} \tag{88}$$

where $Z^{S_{m-k} \times S_{n-k}}_{G=0,B=3}(\sigma_1^{-1} \otimes \tau_1^{-1}, \sigma_2^{-1} \otimes \tau_2^{-1}, \sigma_3^{-1} \otimes \tau_3^{-1})$ is the three-holed sphere given in (42) associated with $S_{m-k} \times S_{n-k}$. Thus we can express any correlation functions obtained from the Brauer algebra as a set of correlation functions obtained from the symmetric group $S_{m-k} \times S_{n-k}$.

5. 2D Field Theories with the Restricted Structure

As we have reviewed in Sect. 2, gauge invariant operators in $\mathcal{N} = 4$ super Yang–Mills built from p complex matrices can be labelled by an element of the form $[\sigma_i]_H$ or $[b_i]_H$, where $H = S_{n_1} \times \cdots \times S_{n_p}$. With the motivation to associate the description of the matrix models with two-dimensional field theories, we will construct a class of two-dimensional field theories whose physical states are given by $[\sigma_i]_H$ or $[b_i]_H$. The idea of such theories is given in [23]. For simplicity in what follows we will consider the case $H = S_m \times S_n$, but the generalisation to $S_{n_1} \times \cdots \times S_{n_p}$ is straightforward.

We will first consider the symmetric group S_{m+n}. As we have seen in the last two sections, physical operators are determined by the two-point function, because it is a projection operator onto the Hilbert space of physical operators. In order to obtain physical operators labelled by $[\sigma_i]_H$, we will consider the two-point function drawn in Fig. 6 in stead of the cylinder (33). On the double line in the figure we restrict the sum to the subgroup H [23], and triangulations should be done consistently with the restriction. A simple triangulation of such a cylinder is shown in the RHS of Fig. 6:

$$\theta_{ij} = \sum_k \sum_a C_{ikb} C_{ajl} h^{ab} g^{kl}, \tag{89}$$

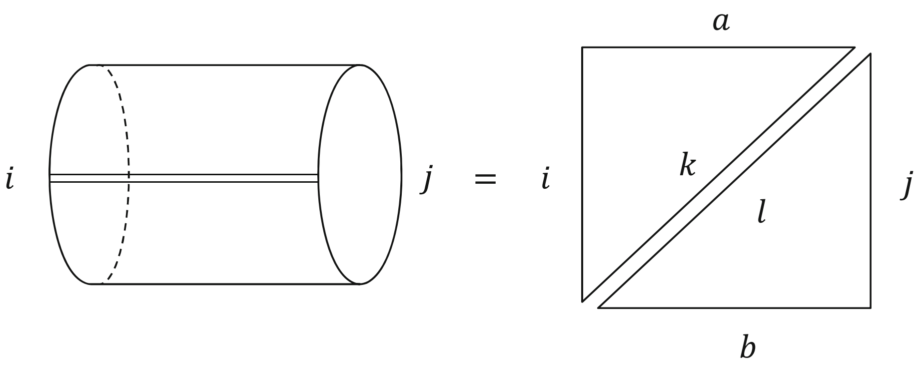

Fig. 6. *LHS* a cylinder with the restriction on the double line, *RHS* a simple triangulation

where a runs over a complete set of the subgroup H. The g^{ij} and C_{ijk} are defined in (28) and (29). The h is the metric defined in H as

$$\begin{aligned} h_{ab} &= m!n!\delta_{m,n}(\sigma_a\sigma_b) \\ h^{ab} &= m!n!\delta_{m,n}(\sigma^a\sigma^b), \end{aligned} \tag{90}$$

where σ^a is the dual basis of σ_a, which is defined by $\sigma^a = \frac{1}{m!n!}\sigma_a^{-1}$. The delta function defiend over the group algebra of H is denoted by $\delta_{m,n}(\sigma)$.

We can show that

$$\begin{aligned} \theta_{ij} &= \sum_a tr^{(r)}(h_a\sigma_i h^a\sigma_j) = tr^{(r)}([\sigma_i]_H[\sigma_j]_H) \\ \theta^{ij} &= \sum_a tr^{(r)}(h_a\sigma^i h^a\sigma^j) = tr^{(r)}([\sigma^i]_H[\sigma^j]_H) \\ \theta_i{}^j &= \sum_a tr^{(r)}(h_a\sigma_i h^a\sigma^j) = tr^{(r)}([\sigma_i]_H[\sigma^j]_H), \end{aligned} \tag{91}$$

and the projector relation $\theta^2 = \theta$ can be checked explicitly. The use of (B.9) leads to another expression:

$$\begin{aligned} \theta_{ij} &= \sum_{R,A,\mu,\nu} \frac{d^R}{d_A}\chi^R_{A,\mu\nu}(\sigma_i)\chi^R_{A,\nu\mu}(\sigma_j) \\ \theta^{ij} &= \sum_{R,A,\mu,\nu} \frac{d^R}{d_A}\chi^R_{A,\mu\nu}(\sigma^i)\chi^R_{A,\nu\mu}(\sigma^j) \\ \theta_i{}^j &= \sum_{R,A,\mu,\nu} \frac{d^R}{d_A}\chi^R_{A,\mu\nu}(\sigma_i)\chi^R_{A,\nu\mu}(\sigma^j), \end{aligned} \tag{92}$$

where d_R is the dimension of an irreducible representation R of the symmetric group S_{m+n}, while d_A is the dimension of an irreducible representation A of the symmetric group $S_m \times S_n$. The $\chi^R_{A,\mu\nu}(\sigma)$ is the restricted character [see around (A.15)]. The indices μ, ν run over $1, \ldots, M^R_A$, where M^R_A counts the number of times the representation A appears in the R. This projector (93) was obtained in [23] from the invariance (10), which leads to $\theta_i{}^j tr_{m+n}(\sigma_j X^{\otimes m} \otimes Y^{\otimes n}) = tr_{m+n}(\sigma_i X^{\otimes m} \otimes Y^{\otimes n})$. We have another expression [23] which is closely related to [11,12], instead of (93).

From the cyclic property of the restricted character

$$\begin{aligned} \chi^R_{A,\mu\nu}(h\sigma) &= \chi^R_{A,\mu\nu}(\sigma h) \quad (h \in S_m \times S_n) \\ \chi^R_{A,\mu\nu}(\rho\sigma) &\neq \chi^R_{A,\mu\nu}(\sigma\rho) \quad (\rho \in S_{m+n}/S_m \times S_n), \end{aligned} \tag{93}$$

we find that boundary operators are more sensitive to the orientations corresponding to upper and lower indices because σ_i is not conjugate by some permutation in the subgroup $S_m \times S_n$ to the dual element σ^i.

Physical operators are determined by the projection operator

$$\theta_i{}^j \sigma_j = \sum_{R,A,\mu,\nu} \frac{1}{d_R} \chi^R_{A,\mu\nu}(\sigma_i) P^R_{A,\mu\nu} = [\sigma_i]_H, \tag{94}$$

where $[\sigma_i]_H$ is defined in (18), and $P^R_{A,\mu\nu}$ is a basis of elements commuting with any elements in $H = S_m \times S_n$ [13]. See (A.15). The Eq. (94) also means that physical states are invariant under time evolution,

$$\theta_i{}^j [\sigma_j]_H = [\sigma_i]_H. \tag{95}$$

Because $\theta_i{}^j$ is a projector, we expect to obtain the the dimension of the vector space by taking a trace in the same way as the topological field theories in Sects. 3 and 4,

$$Z_{G=1} = \sum_i \theta_i{}^i = \sum_{R,A} (M^R_A)^2, \tag{96}$$

which comes from (B.6). This is the partition function of a torus. The dimension of the vector space gives the expected result of the counting of the number of multi-traces constructed from m copies of X and n copies of Y via the method of using the basis (9), as shown in [13,45]. The counting of multi-traces is associated with $Z_{G=1}$ by considering the counting of orbits of the group actions via Burnside's lemma in [23,46].

The three-point function (three-holed sphere) Ξ_{ijk} can be obtained as the structure constant of the algebra spanned by $[\sigma_i]_H$. From

$$[\sigma_i]_H [\sigma_j]_H = \Xi_{ij}{}^k [\sigma_k]_H, \tag{97}$$

we find that

$$\begin{aligned} \Xi_{ij}{}^k &= tr^{(r)}([\sigma_i]_H [\sigma_j]_H [\sigma^k]_H), \\ \Xi_{ijk} &= tr^{(r)}([\sigma_i]_H [\sigma_j]_H [\sigma_k]_H) = \Xi_{ij}{}^l \theta_{lk}. \end{aligned} \tag{98}$$

Note that the two-point function and the three-point function are also obtained by acting with the projection operator on g_{ij} and C_{ijk}

$$\begin{aligned} \theta_i{}^k \theta_j{}^l g_{kl} &= \theta_{ij} \\ \theta_i{}^p \theta_j{}^q \theta_k{}^r C_{pqr} &= \Xi_{ijk}, \end{aligned} \tag{99}$$

which is analogous to (46). It is also convenient to rewrite the expression in terms of the restricted characters:

$$\Xi_{ij}{}^k = \sum_{R,A,\mu,\nu,\lambda} \frac{d^R}{(d_A)^2} \chi^R_{A,\mu\nu}(\sigma_i) \chi^R_{A,\nu\lambda}(\sigma_j) \chi^R_{A,\lambda\mu}(\sigma^k). \tag{100}$$

The associativity of the algebra, $([\sigma_i]_H[\sigma_j]_H)[\sigma_k]_H = [\sigma_i]_H([\sigma_j]_H[\sigma_k]_H)$, is translated into

$$\Xi_{ij}{}^{k}\Xi_{kl}{}^{n} = \Xi_{ik}{}^{n}\Xi_{jl}{}^{k}. \tag{101}$$

The vector space whose elements are $[\sigma]_H$, equipped with the bilinear form θ_{ij}, is a Frobenius algebra. The Frobenius form is θ_{i0}. One can check

$$\Xi_{ij}{}^{k}\theta_{k0} = \theta_{ij}. \tag{102}$$

A big difference from the Frobenius algebras in Sects. 3 and 4 is that the Frobenius algebra in this section is *noncommutative*, that is,

$$\Xi_{ij}{}^{k} \neq \Xi_{ji}{}^{k} \tag{103}$$

as a consequence of the non-commutative operator algebra

$$[\sigma_i]_H[\sigma_j]_H \neq [\sigma_j]_H[\sigma_i]_H. \tag{104}$$

By a change of basis, we will obtain the following (almost diagonal) operator product

$$P^{R}_{A,\mu\nu}P^{R'}_{A',\mu'\nu'} = \delta^{RR'}\delta_{AA'}\delta_{\nu\mu'}P^{R}_{A,\mu\nu'}. \tag{105}$$

The existence of the μ,ν indices reflects the noncommutativity, and we also find the μ,ν indices indeed behave like matrix-indices. (See also orthogonality relations for restricted characters in Appendix B.) Due to the non-commutativity, this field theory is not topological.

The partition functions can be computed from the building blocks θ_{ij}, Ξ_{ijk}. The partition function of a Riemann surface of genus G can be computed as

$$\sum_{R,A}\frac{1}{(d^R d_A)^{G-1}}(M^R_A)^{G+1}. \tag{106}$$

Equations in Appendix B will be helpful to derive this. The partition function depends only on G. If we set $M^R_A = 1$ and $d_A = d_R$, we reproduce the result (48) as expected. The partition function has another form,

$$Z_G = \sum_{a,j} tr^{(r)}\left((h_{a_1}\sigma_{j_1}h^{a_1}\sigma^{j_1})\cdots(h_{a_G}\sigma_{j_G}h^{a_G}\sigma^{j_G})\right), \tag{107}$$

where h_a is a basis of the subgroup $S_m \times S_n$ and h^a is the dual basis. The indices $a_1,\ldots,a_G$ run over a complete set of $S_m \times S_n$.

The next interest will be a Riemann surface with boundaries. This should be paid more attention because the Frobenius algebra is noncommutative. Correlation functions depend on the way of gluing the building blocks. As a simple example consider the case of $G = 1$ and $B = 2$. We have two possible ways, $\Xi_{ijk}\Xi^{jkl}$ and $\Xi_{ijk}\Xi^{kjl}$, giving different answers. The first one is computed to give

$$\Xi_{ijk}\Xi^{jkl} = \sum_{R,A,\mu,\nu}\frac{1}{(d_A)^2}\chi^R_{A,\mu\mu}(\sigma_i)\chi^R_{A,\nu\nu}(\sigma^l), \tag{108}$$

while the second one is

$$\Xi_{ijk}\Xi^{kjl} = \sum_{R,A,\mu,\nu} \frac{1}{(d_A)^2} M_A^R \chi^R_{A,\mu\nu}(\sigma_i)\chi^R_{A,\nu\mu}(\sigma^l). \tag{109}$$

The second one is the same as $\Xi_i{}^l{}_k\Xi_p{}^{pk}$. If we set $\sigma_i = 1$ and $\sigma^l = 1$, both reduce to the case $G = 1$ in (106). The difference between the two correlation functions is how multiplicity indices μ, ν are contracted. Introducing $M_A^R \times M_A^R$ matrices M_i, the structure of $tr(M_1)tr(M_2)$ arises effectively from the first one, while $tr1tr(M_1M_2)$ from the second one. The first one can also be written as

$$\Xi_{ijk}\Xi^{jkl} = \sum_{a,p} tr^{(r)}(h_a\sigma_p[\sigma_i]_H h^a\sigma^p[\sigma^l]_H), \tag{110}$$

while the second can be

$$\Xi_{ijk}\Xi^{kjl} = \sum_{a,p} tr^{(r)}(h_a\sigma_p h^a\sigma^p[\sigma_i]_H[\sigma^l]_H). \tag{111}$$

Note that

$$[\sigma_i]_H h = h[\sigma_i]_H \quad (h \in S_m \times S_n), \tag{112}$$

but

$$[\sigma_i]_H \tau \neq \tau[\sigma_i]_H \quad (\tau \in S_{m+n}/S_m \times S_n). \tag{113}$$

As we have seen in the example, the correlation functions are not uniquely determined when we specify G and B, and they are sensitive to the positions of boundaries. Reflecting the matrix structure, they have cyclic symmetries acting on the boundary positions. For example the following correlation function, which is one of correlation functions of a surface of genus G with B boundaries,

$$\sum_{R,A} \frac{1}{(d^R)^{G-1}(d_A)^{G+B-1}} (M_A^R)^G \sum_{\mu,\nu,\tau} \chi^R_{A,\mu\nu}(\sigma_{i_1})\chi^R_{A,\nu\lambda}(\sigma_{i_2})\cdots\chi^R_{A,\tau\mu}(\sigma_{i_B}) \tag{114}$$

is invariant under cyclic permutations acting on the indices $i_1, \ldots, i_B$. One may read the single trace structure $tr(M_1\cdots M_B)$ from this. The restricted characters were first introduced to describe open strings on giant gravitons [6–8]. It is interesting to study if the origin of of the noncommutativity can be explained in terms of D-branes in the context of two-dimensional theories.

So far we have considered a new kind of two-dimensional field theories based on the symmetric group, and the same construction can be applied to the walled Brauer algebra. The formal replacement (78) with

$$\begin{aligned} {}_H &\leftrightarrow [b_i]_H, \\ M_A^R &\leftrightarrow M_A^\gamma, \end{aligned} \tag{115}$$

where M_A^γ is the number of times A appears in γ, will work to obtain the new two-dimensional field theory based on the walled Brauer algebra. We will show some of them explicitly for convenience. The two-point function is given by

$$\theta_{ij} = tr^{(r)}([b_i]_H[b_j]_H) = \sum_{\gamma,A,\mu,\nu} \frac{d^\gamma}{d_A} \chi^\gamma_{A,\mu\nu}(b_i)\chi^\gamma_{A,\nu\mu}(b_j)$$
$$\theta^{ij} = tr^{(r)}([b^i]_H[b^j]_H) = \sum_{\gamma,A,\mu,\nu} \frac{d^\gamma}{d_A} \chi^\gamma_{A,\mu\nu}(b^i)\chi^\gamma_{A,\nu\mu}(b^j) \tag{116}$$
$$\theta_i{}^j = tr^{(r)}([b_i]_H[b^j]_H) = \sum_{\gamma,A,\mu,\nu} \frac{d^\gamma}{d_A} \chi^\gamma_{A,\mu\nu}(b_i)\chi^\gamma_{A,\nu\mu}(b^j),$$

and this determines the Hilbert space of physical operators,

$$\theta_i{}^j b_j = \sum_{\gamma,A,\mu,\nu} \frac{1}{d_A} \chi^\gamma_{A,\mu\nu}(b_i) Q^\gamma_{A,\mu\nu} = [b_i]_H. \tag{117}$$

Changing to the basis (A.28), the almost diagonal operator product is obtained [10,15]

$$Q^\gamma_{A,\mu\nu} Q^{\gamma'}_{A',\mu'\nu'} = \delta^{\gamma\gamma'}\delta_{AA'}\delta_{\nu\mu'} Q^\gamma_{A,\mu\nu'}, \tag{118}$$

where μ, ν are multiplicity labels running over $1, \dots, M^\gamma_A$. The three-point function is the structure constant (i.e. $[b_i]_H[b_j]_H = \Xi_{ij}{}^k[b_k]_H$),

$$\Xi_{ijk} = tr^{(r)}([b_i]_H[b_j]_H[b_k]_H) = \sum_{\gamma,A,\mu,\nu,\lambda} \frac{d^\gamma}{(d_A)^2} \chi^\gamma_{A,\mu\nu}(b_i)\chi^\gamma_{A,\nu\lambda}(b_j)\chi^\gamma_{A,\lambda\mu}(b_k). \tag{119}$$

The partition function of a Riemann surface of genus G is given by

$$Z_G = \sum_{a,j} tr^{(r)} \left((h_{a_1} b_{j_1} h^{a_1} b^{j_1}) \cdots (h_{a_G} b_{j_G} h^{a_G} b^{j_G}) \right)$$
$$= \sum_{\gamma,A} \frac{1}{(d^\gamma d_A)^{G-1}} (M^\gamma_A)^{G+1}. \tag{120}$$

Furnished with the bilinear form (117), the algebra formed by $[b_i]_H$ is a noncommutative Frobenius algebra. The partition function of $G = 1$ gives the dimension of the vector space

$$Z_{G=1} = \sum_i \theta_i{}^i = \sum_{\gamma,A} (M^\gamma_A)^2. \tag{121}$$

This is equivalent to the counting of the number of multi-traces built from m copies of X and n copies of Y via the method of using the basis (13), as shown in [10,47].

6. 2D Theoretic Interpretation of the Multi-Matrix Models

In this section we will study a relation between correlators of the Gaussian matrix models and correlation functions of the two-dimensional field theories.

First consider the method of using symmetric group elements to describe multi-trace operators. Recall (11),

$$\begin{aligned} S_{\sigma_i,\sigma_j} &= \langle tr_{m+n}(\sigma_i X^{\dagger\otimes m} \otimes Y^{\dagger\otimes n}) tr_{m+n}(\sigma_j X^{\otimes m} \otimes Y^{\otimes n}) \rangle \\ &= \sum_{h\in H} tr_{m+n}(h\sigma_i h^{-1}\sigma_j) \\ &= \frac{N^{m+n}}{(m+n)!} \sum_{h\in H} tr^{(r)}(\Omega_{m+n} h\sigma_i h^{-1}\sigma_j), \end{aligned} \tag{122}$$

where $\sigma_i, \sigma_j \in S_{m+n}$ and $H = S_m \times S_n$. With the notation (20), one may write it as

$$S_{[\sigma_i]_H,[\sigma_j]_H} = N^{m+n} \frac{m!n!}{(m+n)!} tr^{(r)}(\Omega_{m+n}[\sigma_i]_H[\sigma_j]_H). \tag{123}$$

Comparing to the three-point function (98), we find that the S_{σ_i,σ_j} can be interpreted to be the three-point function with the Omega factor put at one of the boundaries (see also [23]). More explicitly, we have

$$S_{\sigma_i,\sigma_j} = N^{m+n} \frac{m!n!}{(m+n)!} \sum_k \Xi_{ijk} N^{C_{\sigma_k}-(m+n)}. \tag{124}$$

Because Ω_{m+n} is a central element in the group algebra of S_{m+n}, the ordering of the three objects is not important.

Next consider the way of labelling multi-traces in terms of Brauer elements. Recall the two-point function (15):

$$\begin{aligned} B_{b_i,b_j} &= \langle tr_{m,n}(b_i X^{\otimes m} \otimes Y^{T\otimes n}) tr_{m,n}(b_j X^{\dagger\otimes m} \otimes Y^{*\otimes n}) \rangle \\ &= \sum_{h\in S_m\times S_n} tr_{m,n}(h b_i h^{-1} b_j) \\ &= \sum_{\gamma} \sum_{h\in S_m\times S_n} t^{\gamma} \chi^{\gamma}(h b_i h^{-1} b_j). \end{aligned} \tag{125}$$

We can also write it as

$$B_{[b_i]_H,[b_j]_H} = \frac{1}{m!n!} \sum_{\gamma} t^{\gamma} \chi^{\gamma}([b_i]_H[b_j]_H). \tag{126}$$

The sum is over all Young diagrams $\gamma = (\gamma_+, \gamma_-)$ by means of (65), where γ_+ and γ_- are partitions of $m-k$ and $n-k$ respectively. We now use the fact that the dimension of an irreducible representation γ of the $GL(N)$ group can be expressed in terms of elements in $S_{m-k} \times S_{n-k}$ as

$$t_{\gamma} = \frac{N^{m+n-2k}}{(m-k)!(n-k)!} \chi_{(\gamma_+,\gamma_-)}(\Omega_{m-k,n-k}). \tag{127}$$

The character of $S_{m-k} \times S_{n-k}$ is denoted by $\chi_{(\gamma_+,\gamma_-)}$, and $\Omega_{m-k,n-k}$ is a central element in the group algebra of $S_{m-k} \times S_{n-k}$ which is called coupled Omega factor [24–26]. We do not use an explicit form of $\Omega_{m-k,n-k}$, but we should keep in mind that the N-dependence of t_{γ} is encoded in it, and it is not just the product $\Omega_{m-k} \times \Omega_{n-k}$. In order to combine t^γ and $\chi^\gamma(hb_ih^{-1}b_j)$, we will express $\chi^\gamma(hb_ih^{-1}b_j)$ in terms of characters in $S_{m-k} \times S_{n-k}$ by using the formulae (81) and (83).

Let us first consider the simplest case that both b_i and b_j are elements in the group algebra of $S_m \times S_n$ as an exercise before going to the general case. Suppose $b_i = b_i^+ \otimes b_i^-$, $b_j = b_j^+ \otimes b_j^-$, where b_i^+ and b_j^+ are elements in S_m and b_i^- and b_j^- are elements in S_n. For $h = \sigma \otimes \tau \in S_m \times S_n$,

$$hb_ih^{-1}b_j = (\sigma b_i^+ \sigma^{-1} b_j^+) \otimes (\tau b_i^- \tau^{-1} b_j^-) \in S_m \times S_n. \tag{128}$$

Applying the formula (83) to $\chi^\gamma(hb_ih^{-1}b_j)$, we obtain

$$\chi^\gamma(hb_ih^{-1}b_j) = \frac{1}{(m-k)!(n-k)!} \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(\sigma b_i^+ \sigma^{-1} b_j^+, \tau b_i^- \tau^{-1} b_j^-; \sigma_2, \tau_2) \times \chi_{(\gamma_+,\gamma_-)}(\sigma_2^{-1} \otimes \tau_2^{-1}), \tag{129}$$

where

$$\chi_{(\gamma_+,\gamma_-)}(\sigma_2^{-1} \otimes \tau_2^{-1}) = \chi_{\gamma_+}(\sigma_2^{-1})\chi_{\gamma_-}(\tau_2^{-1}). \tag{130}$$

Then the two-point function can be written as

$$\begin{aligned} &B_{b_i,b_j} \\ &= \sum_\gamma \sum_{h \in S_m \times S_n} t^\gamma \chi^\gamma(hb_ih^{-1}b_j) \\ &= \sum_{k=0}^{min(m,n)} \sum_{\gamma_+ \vdash (m-k), \gamma_- \vdash (n-k)} \sum_{\sigma \in S_m, \tau \in S_n} \frac{N^{m+n-2k}}{(m-k)!(n-k)!} \chi_{(\gamma_+,\gamma_-)}(\Omega_{m-k,n-k}) \\ &\quad \times \frac{1}{(m-k)!(n-k)!} \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(\sigma b_i^+ \sigma^{-1} b_j^+, \tau b_i^- \tau^{-1} b_j^-; \sigma_2, \tau_2) \\ &\quad \times \chi_{(\gamma_+,\gamma_-)}(\sigma_2^{-1} \otimes \tau_2^{-1}) \\ &= \sum_{k=0}^{min(m,n)} \sum_{\sigma \in S_m, \tau \in S_n} N^{m+n-2k} \frac{1}{(m-k)!(n-k)!} \\ &\quad \times \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(\sigma b_i^+ \sigma^{-1} b_j^+, \tau b_i^- \tau^{-1} b_j^-; \sigma_2, \tau_2) \\ &\quad \times \delta_{m-k,n-k}(\Omega_{m-k,n-k}\sigma_2^{-1} \otimes \tau_2^{-1}), \end{aligned} \tag{131}$$

where $\delta_{m-k,n-k}(\sigma \otimes \tau)$ is the delta function defined over the group algebra of $S_{m-k} \times S_{n-k}$.

By the way in this case we know that the second line in (125) is factorised to give

$$\begin{aligned} &\sum_{h \in S_m \times S_n} tr_{m,n}(hb_ih^{-1}b_j) \\ &= \sum_{\sigma \in S_m, \tau \in S_n} tr_m(\sigma b_i^+ \sigma^{-1} b_j^+) tr_n(\sigma b_i^- \sigma^{-1} b_j^-) \\ &= \sum_{\sigma \in S_m, \tau \in S_n} N^{m+n} \delta_m(\Omega_m \sigma b_i^+ \sigma^{-1} b_j^+) \delta_n(\Omega_n \sigma b_i^- \sigma^{-1} b_j^-). \end{aligned} \tag{132}$$

Comparing (131) with (132) gives an identity

$$\delta_{m,n}(\Omega_m\sigma \otimes \Omega_n\tau) = \sum_k N^{-2k}\frac{1}{(m-k)!(n-k)!} \times \sum_{\sigma_2\in S_{m-k},\tau_2\in S_{n-k}} \Delta(\sigma,\tau;\sigma_2,\tau_2)\delta_{m-k,n-k}(\Omega_{m-k,n-k}\sigma_2^{-1}\otimes\tau_2^{-1}), \quad (133)$$

where σ and τ are elements in S_m and S_n respectively.

Let us next study a general situation, where b_i and b_j are general elements in the Brauer algebra. We define $b_{ij}^{\alpha} := \alpha b_i \alpha^{-1} b_j$. (We will use α for elements in $S_m \times S_n$ in stead of h not to get an extra confusion with the number of contractions h.) Using the formula (81),

$$\sum_{\alpha\in S_m\times S_n} \chi^{\gamma}(\alpha b_i\alpha^{-1}b_j) = \sum_{\alpha\in S_m\times S_n} N^{z(b_{ij}^{\alpha})-h_\alpha}\chi^{\gamma}(C^{\otimes h_\alpha}\otimes b_{ij}^{\alpha+}\otimes b_{ij}^{\alpha+}), \quad (134)$$

where $b_{ij}^{\alpha+}\otimes b_{ij}^{\alpha+}$ is an element in the group algebra of $S_{m-h_\alpha}\times S_{n-h_\alpha}$, and $z(b_{ij}^{\alpha})$ is the number of zero-cycles in b_{ij}^{α}, and h_α, $b_{ij}^{\alpha+}$, $b_{ij}^{\alpha-}$ are defined by this equation. We have added the subscript α to mean that they depend on α.

We will get an expression for B_{b_i,b_j},

$$\begin{aligned} B_{b_i,b_j} &= \sum_{\gamma}\sum_{\alpha\in S_m\times S_n} t^{\gamma}\chi^{\gamma}(\alpha b_i\alpha^{-1}b_j) \\ &= \sum_{\gamma}\sum_{\alpha\in S_m\times S_n} t^{\gamma}N^{z(b_{ij}^{\alpha})-h_\alpha}\chi^{\gamma}(C^{\otimes h_\alpha}\otimes b_{ij}^{\alpha+}\otimes b_{ij}^{\alpha-}) \\ &= \sum_{\alpha\in S_m\times S_n}\sum_{k=h_\alpha}^{min(m,n)} N^{m+n-2k}N^{z(b_{ij}^{\alpha})}\frac{1}{(m-k)!(n-k)!} \\ &\quad\times \sum_{\sigma_2\in S_{m-k},\tau_2\in S_{n-k}} \Delta(b_{ij}^{\alpha+}, b_{ij}^{\alpha-};\sigma_2,\tau_2)\delta_{m-k,n-k}(\Omega_{m-k,n-k}\sigma_2^{-1}\otimes\tau_2^{-1}). \end{aligned} \quad (135)$$

In this case we do not have a factorisation like (132). We can also write it as

$$\begin{aligned} B_{b_i,b_j} = &\sum_{\alpha\in S_m\times S_n}\sum_{k=h_\alpha}^{min(m,n)} N^{m+n-2k}N^{z(b_{ij}^{\alpha})} \\ &\times\frac{1}{((k-h_\alpha)!(m-k)!(n-k)!)^2}\sum_{\sigma_2\in S_{m-k},\tau_2\in S_{n-k}} \\ &\times\sum_{\sigma_1,\tau_1\in S_{k-h_\alpha}} \eta^{S_{m-h_\alpha}}(b_{ij}^{\alpha+},\sigma_1\circ\sigma_2)\eta^{S_{n-h_\alpha}}(b_{ij}^{\alpha-},\tau_1\circ\tau_2)\eta^{S_{k-h_\alpha}}(\sigma_1^{-1},\tau_1^{-1}) \\ &\times\eta^{S_{m-k}\times S_{n-k}}(\Omega_{m-k,n-k},\sigma_2^{-1}\otimes\tau_2^{-1}). \end{aligned} \quad (136)$$

A pictorial drawing is presented in Fig. 7. Each piece in the above equation has an interpretation as a correlation function of the topological field theory based on the symmetric group. What we have done is almost the same as what we have done in Sect. 4.2. Correlation functions expressed in terms of elements in the Brauer algebra have an interpretation as an assemblage of correlation functions in terms of symmetric group elements.

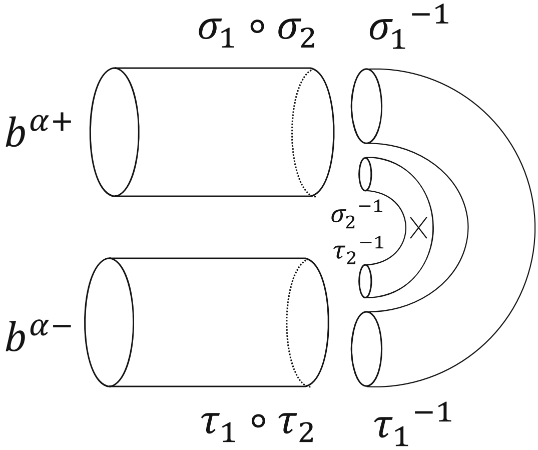

Fig. 7. Drawing of (136): the *cross* represents the insertion of $\Omega_{m-k,n-k}$

7. Some Relations Among the 2D Quantum Field Theories

In this section we will explore some relations among the field theories we have shown, by exploiting a map Σ between elements of the Brauer algebra and elements of the symmetric group,[2] which was exploited in [10,13,44].

Under the map, elements of the Brauer algebra $B_N(m,n)$ are related to elements of the symmetric group S_{m+n}. Suppose that a basis in $B_N(m,n)$ is related to a basis in S_{m+n} by

$$\sigma_i = \Sigma(b_i), \tag{137}$$

and the inverse map is denoted by

$$b_i = \Sigma^{-1}(\sigma_i). \tag{138}$$

When two bases are related by the map in the above way, we can show

$$tr^{(r)}(b_i b^j) = tr^{(r)}(\sigma_i \sigma^j). \tag{139}$$

Note that b^j is the dual basis obtained from a basis b_i and σ^j is the dual basis obtained from a basis σ_i, and the two dual bases are not related by the map, i.e. $\sigma^i \neq \Sigma(b^i)$. In fact we find that the both sides are equal to $\delta_i{}^j$, which come from the definition of the dual bases. Here we will give a proof of the equality (139) using the property of Σ,

[2] Elements of the symmetric group S_{m+n} are expressed diagrammatically by $m+n$ vertical edges between two horizontal lines where each horizontal line has $m+n$ points. Similarly elements of the walled Brauer algebra $B_N(m,n)$ are expressed by $m+n$ lines between the two horizontal lines with a vertical barrier separating the m points from the n points. Vertical edges do not cross the wall, and horizontal edges start and end on opposite sides of the wall. The map Σ reflects the upper right segment and the lower right segment into each other. For more details see section 3.3 in [10].

$$
\begin{aligned}
tr^{(r)}(b_i b^j) &= tr_{m,n}(b_i b_j^*) \\
&= tr_{m+n}(\Sigma(b_i)\Sigma(b_j^*)) \\
&= tr_{m+n}(\Sigma(b_i)(\Sigma(b_j))^{-1}\Sigma(1^*)) \\
&= tr_{m+n}(\Sigma(b_i)(\Sigma(b_j))^{-1}\Omega_{m+n}^{-1})\frac{1}{N^{m+n}} \\
&= \delta_{m+n}(\Sigma(b_i)(\Sigma(b_j))^{-1}) \\
&= tr^{(r)}(\sigma_i \sigma^j),
\end{aligned} \tag{140}
$$

where we have used the following equations found in [10],

$$
\begin{aligned}
\Sigma(b^*) &= \Sigma(1^*)(\Sigma(b))^{-1} \\
\Sigma(1^*) &= \frac{1}{N^{m+n}}\Omega_{m+n}^{-1}.
\end{aligned} \tag{141}
$$

Note that neither $tr^{(r)}(b_i b_j) = tr^{(r)}(\sigma_i \sigma_j)$ nor $tr^{(r)}(b^i b^j) = tr^{(r)}(\sigma^i \sigma^j)$ is satisfied. The relation (139) can be generalised to

$$
tr^{(r)}(\Sigma^{-1}(\tau)b^j) = tr^{(r)}(\tau\sigma^j) \tag{142}
$$

for any element τ in the group algebra of S_{m+n}.

We will consider the two-point function of the topological field theories obtained from the symmetric group S_{m+n} and the Brauer algebra $B_N(m,n)$. The two correlation functions are related if boundary elements belong to the group algebra of $S_m \times S_n$, as we will show below. For $h_1 = h_1^+ \otimes h_1^-, h_2 = h_2^+ \otimes h_2^- \in S_m \times S_n$, using (142),

$$
\begin{aligned}
\eta^{S_{m+n}}(h_1, h_2) &= \sum_i tr^{(r)}(h_1\sigma_i h_2 \sigma^i) \\
&= \sum_i tr^{(r)}(\Sigma^{-1}(h_1\sigma_i h_2)b^i) \\
&= \sum_i tr^{(r)}((h_1^+ \circ (h_2^-)^{-1})\Sigma^{-1}(\sigma_i)(h_2^+ \circ (h_1^-)^{-1})b^i) \\
&= \sum_i tr^{(r)}((h_1^+ \circ (h_2^-)^{-1})b_i(h_2^+ \circ (h_1^-)^{-1})b^i) \\
&= \eta^{B_N(m,n)}(h_1^+ \circ (h_2^-)^{-1}, h_2^+ \circ (h_1^-)^{-1}),
\end{aligned} \tag{143}
$$

where we have used the following equation

$$
\Sigma^{-1}(h_1\sigma h_2) = (h_1^+ \circ (h_2^-)^{-1})\Sigma^{-1}(\sigma)(h_2^+ \circ (h_1^-)^{-1}), \tag{144}
$$

which come from the definition of Σ. Note that we have used the same symbol for the regular representation of S_{m+n} and that of $B_N(m,n)$.

Gluing two boundaries of the cylinders, we have

$$
\begin{aligned}
\sum_{h \in S_m \times S_n} \eta^{S_{m+n}}(h, h^{-1}) &= \sum_{h^+ \in S_m, h^- \in S_n} \eta^{B_N(m,n)}(h^+ \circ h^-, (h^+)^{-1} \circ (h^-)^{-1}) \\
&= \sum_{h \in S_m \times S_n} \eta^{B_N(m,n)}(h, h^{-1}).
\end{aligned} \tag{145}
$$

One find that gluing two ends of the cylinder with boundary elements restricted to $S_m \times S_n$ gives the torus given in Sect. 5,

$$\sum_{h \in S_m \times S_n} \eta^{S_{m+n}}(h, h^{-1}) = \theta^{S_{m+n}}{}_i{}^i$$
$$\sum_{h \in S_m \times S_n} \eta^{B_N(m,n)}(h, h^{-1}) = \theta^{B_N(m,n)}{}_i{}^i. \tag{146}$$

Therefore (145) means

$$\theta^{S_{m+n}}{}_i{}^i = \theta^{B_N(m,n)}{}_i{}^i. \tag{147}$$

Taking into account that the partition function of a torus gives the dimension of the vector space, this equation states that the vector space of $[\sigma_i]_H$ and the vector space of $[b_i]_H$ have the same dimension.

In fact the result itself has been known from the counting of gauge invariant operators at large N. Let the number of gauge invariant operators built from m copies of X and n copies of Y be $N(m, n)$. There are several ways to obtain this. If we use the restricted Schur basis (12), we obtain $N(m, n) = \sum_{R,A} (M_A^R)^2$ [13,45], which is equal to $\theta^{S_{m+n}}{}_i{}^i$. If we use the Brauer basis (15), we have $N(m, n) = \sum_{\gamma,A} (M_A^\gamma)^2$ [10,47], which is equal to $\theta^{B_N(m,n)}{}_i{}^i$. Thus (147) indeed is an expected equation from the counting of multi-trace operators at large N. It is interesting that

$$\sum_{R,A} (M_A^R)^2 = \sum_{\gamma,A} (M_A^\gamma)^2 \tag{148}$$

has been derived as a consequence of the relation (143) between two topological field theories.

8. Discussions

Having studied two-dimensional (almost topological) field theories related to the multi-matrix models, it is now good to get back to the equations in (20). Let us first see the left-hand side. It is given in the $U(N)$ gauge theory language, and gauge invariant quantities are integrated. Multi-trace operators are characterised by the invariance under the gauge transformation,

$$X \to gXg^{-1}, \quad Y \to gYg^{-1}. \tag{149}$$

(Enhanced symmetries at the free theory of $\mathcal{N} = 4$ SYM were discussed in [15].) On the other hand, the equations on the right-hand side are completely expressed in terms of the symmetric group or the walled Brauer algebra. The quantities inside the trace, $[\sigma]$, $[\sigma]_H$ and $[b]_H$, are characterised by the invariance under the following gauge transformation,

$$\sigma \to h\sigma h^{-1}, \quad b \to hbh^{-1} \quad (h \in S_m \times S_n). \tag{150}$$

One might regard the equations in (20) as an analogue of the GKPW relation [48,49]. Mathematical manipulations behind the equality can be understood from the Schur–Weyl duality. Frobenius algebras are algebras of these gauge invariant quantities, and the bilinear forms (the Frobenius forms) are projection operators onto gauge invariant quantities. These Frobenius algebras are noncommutative except for the case of the one-complex matrix model. The noncommutativity is related to the existence of the multiplicity indices μ, ν on the restricted characters, which are originally introduced to describe open strings on giant gravitons [6–8]. Noncommutative Frobenius algebras would play a role in the description of two-dimensional field theories with a certain structure relevant for the noncommutativity. It would be interesting to use axiomatic notions of noncommutative Frobenius algebras to understand what kind of field theories are described reflecting the noncommutative nature. For example see [50–53] for references of the direction. Even apart from the connection to $\mathcal{N} = 4$ SYM, noncommutative Frobenius algebras themselves seem to be an interesting subject to learn.

In Sects. 4.2 and 6, we have decomposed correlation functions of two-dimensional quantum field theories obtained from walled Brauer algebras into correlation functions of two-dimensional quantum field theories obtained from symmetric groups, with exploiting the property that the character of the walled Brauer algebra can be expressed in terms of the character of the symmetric group $S_{m-k} \times S_{n-k}$. From the link between permutations and coverings, walled Brauer algebras can also be interesting mathematical tools incorporate two kinds of maps, i.e., a holomorphic map from m worldsheets to the target and an anti-holomorphic map from n worldsheets. The complete large N expansion of two-dimensional Yang–Mills is a well-known example [24–28]. A non-holomorphic extension of [54,55] exploiting walled Brauer algebras might be a possible future direction.

Throughout this paper, we have assumed that N is large compared to the number of fields involved in multi-traces (i.e. $n < N$ for the one-matrix model, and $m+n < N$ for the two-matrix model—we call this large N). If the number of fields exceed the bound N (i.e. $n > N$ or $m+n > N$—we call this small N), it is good to use the diagonal operator basis, $\langle O_R O_S \rangle = N^n tr^{(r)}(\Omega_n p_R p_S)$ instead of (20). In the AdS/CFT correspondence, gauge invariant operators are considered to be dual to D-branes (called giant gravitons) or geometries if the number of fields are comparable to N or N^2, and considering representation bases clarifies the correspondence between gauge theory operators and string states. It would be interesting to study a two-dimensional interpretation of representation bases, focusing on the role of Frobenius algebras.

Acknowledgements. I would like to thank Sanjaye Ramgoolam for valuable discussions. E-mail conversations with him motivated me to initiate this work.

A. Symmetric Groups, Brauer Algebras and Schur–Weyl Duality

In this section we will make a brief introduction of symmetric groups and walled Brauer algebras, focusing on the role of Schur–Weyl duality in the description of multi-traces.

Let V be an N-dimensional vector space over $\mathbb{C}$ on which an $N \times N$ matrix X is supposed to act.

The tensor product $X^{\otimes n} = X \otimes \cdots \otimes X$ can be viewed as an operator acting on the tensor space $V^{\otimes n}$. We define an action of the symmetric group S_n as permuting n vector spaces,

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}. \tag{A.1}$$

In addition to this the $GL(N)$ group acts in the standard way on it by the simultaneous matrix multiplication,

$$g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n. \tag{A.2}$$

These two actions can be shown to commute, and the Schur–Weyl duality says that the tensor space is decomposed into the direct sum of irreducible representations for these groups as

$$V^{\otimes n} = \bigoplus_{R \vdash n} V_R^{GL(N)} \otimes V_R^{S_n}. \tag{A.3}$$

The sum is taken for all Young diagrams with n boxes satisfying $c_1(R) \leq N$, where $c_1(R)$ is the number of rows in R. The projection operator p_R of an irreducible representation R can be introduced as an element in the group algebra of S_n,

$$p_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma^{-1})\sigma, \tag{A.4}$$

which acts on the tensor space as

$$p_R V^{\otimes n} = V_R^{GL(N)} \otimes V_R^{S_n}. \tag{A.5}$$

We have denoted by d_R the dimension of an irreducible representation R of the symmetric group, and let t_R be the dimension of an irreducible representation R of the $GL(N)$ group.

Let tr_n be a trace over the tensor space $V^{\otimes n}$. From the Schur–Weyl duality (A.3), we have

$$tr_n(\sigma) = \sum_R t_R \chi_R(\sigma) \tag{A.6}$$

for an element σ in the symmetric group. If we introduce C_σ, the number of cycles in the permutation σ, we have $tr_n(\sigma) = N^{C_\sigma}$. The orthogonality relation of the characters leads to

$$t_R = \frac{N^n}{n!} \chi_R(\Omega_n), \tag{A.7}$$

where Ω_n is a central element in the group algebra of S_n called Omega factor

$$\Omega_n = \sum_{\sigma \in S_n} \sigma N^{C_\sigma - n}. \tag{A.8}$$

By substituting (A.7) back to (A.6), the trace of an element τ in the group algebra of S_n can be written as

$$\begin{aligned} tr_n(\tau) &= \frac{N^n}{n!} \sum_{R \vdash n} \chi_R(\Omega_n)\chi_R(\tau) \\ &= \frac{N^n}{n!} \sum_{R \vdash n} d_R \chi_R(\Omega_n \tau) \\ &= N^n \delta_n(\Omega_n \tau), \end{aligned} \tag{A.9}$$

where we have introduced the delta function defined over the group algebra of S_n by $\delta_n(\sigma) = 1$ if $\sigma = 1$ and 0 otherwise,

$$\delta_n(\sigma) = \frac{1}{n!} \sum_{R \vdash n} d_R \chi_R(\sigma). \tag{A.10}$$

In order to derive (A.9), we have assumed $N > n$. When this is not satisfied, only irreducible representations satisfying the constraint $c_1(R) \leq N$ are summed in the first line and the second line of (A.9). On the other hand all Young diagrams with n boxes are summed in (A.10).

We next consider the vector space that are relevant for the description of multi-trace operators made from two matrices X and Y. When m copies of X and n copies of Y are considered, the tensor space we will consider is $V^{\otimes(m+n)}$, and the Schur–Weyl duality claims

$$V^{\otimes(m+n)} = \bigoplus_{R\vdash(m+n)} V_R^{GL(N)} \otimes V_R^{S_{m+n}}. \tag{A.11}$$

We now consider the decomposition of the irreducible representation R of S_{m+n} into irreducible representations of the subgroup $S_m \times S_n$,

$$V_R^{S_{m+n}} = \bigoplus_A M_A^R\ V_A^{S_m\times S_n}. \tag{A.12}$$

On restricting to the subgroup, some copies of A appear. The number of times A appears in R is denoted by M_A^R, which is given by the Littlewood-Richardson coefficient

$$M_A^R = g(\alpha, \beta; R) \tag{A.13}$$

where we have expressed A with two Young diagrams $\alpha \vdash m$ and $\beta \vdash n$. We now define an operator $P^R_{A,\mu\nu}$ playing a role under the decomposition. Here μ, ν run over $1, \ldots, M_A^R$, labelling which copy of A we are using. If the multiplicity is trivial, P_A^R is the projection operator onto the irreducible representation A inside the R. If the multiplicity is non-trivial, $P^R_{A,\mu\nu}$ is an intertwiner mapping the ν-th copy of the representations A to the μ-th copy of the representations A. It satisfies

$$P^R_{A,\mu\nu} P^{R'}_{A',\mu'\nu'} = \delta^{RR'} \delta_{AA'} \delta_{\nu\mu'} P^R_{A,\mu\nu'}. \tag{A.14}$$

In terms of elements in the symmetric group S_{m+n}, the operator $P^R_{A,\mu\nu}$ can be explicitly written as

$$P^R_{A,\mu\nu} = \frac{d_R}{(m+n)!} \sum_{\sigma\in S_{m+n}} \chi^R_{A,\nu\mu}(\sigma)\sigma^{-1}, \tag{A.15}$$

where $\chi^R_{A,\nu\mu}(\sigma)$ is a quantity called restricted character [6–8, 13]. This can be computed by the trace of the matrix σ in the representation R, but the trace is only over the subspace A appearing in the (μ, ν) component of the $M_A^R \times M_A^R$ matrix.

The sum of the diagonal copies of all possible irreducible representations in $S_m \times S_n$ inside the representation R gives rise to the usual character $\chi^R(\sigma) = \sum_{A,\mu} \chi^R_{A,\mu\mu}(\sigma)$, and the projector of an irreducible representation R is given by

$$P^R = \sum_{A,\mu} P^R_{A,\mu\mu}. \tag{A.16}$$

The intertwiner operator has the symmetry,

$$hP^R_{A,\mu\nu} = P^R_{A,\mu\nu}h \quad (h \in S_m \times S_n). \tag{A.17}$$

But elements in $S_{m+n}/S_m \times S_n$ do not commute with it. Due to this, the cyclicity of the restricted character works for elements in the subalgebra,

$$\chi^R_{A,\mu\nu}(h\sigma) = \chi^R_{A,\mu\nu}(\sigma h) \quad (h \in S_m \times S_n). \tag{A.18}$$

Let us next consider the mixed tensor space $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ by including the complex conjugate space $\bar{V}$. This is relevant for the description of multi-trace operators made out of X and $X^\dagger$, or X and Y^T. Similar to the previous cases, we can consider two commuting actions on this space, resulting in the following Schur–Weyl duality,

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \bigoplus_\gamma V_\gamma^{GL(N)} \otimes V_\gamma^{B_N(m,n)}. \tag{A.19}$$

Here $B_N(m,n)$ is the walled Brauer algebra [29–32]. (This algebra is sensitive to N, which is a big difference from the group algebra of the symmetric group.) The γ is a set of two Young diagrams (γ_+, γ_-), where γ_+ is a Young diagram with $m-k$ boxes and γ_- is a Young diagram with $n-k$ boxes. The k is an integer in $0 \le k \le min(m,n)$. The sum over γ in (A.19) is constrained by $c_1(\gamma_+) + c_1(\gamma_-) \le N$.

From the Schur–Weyl duality, we have

$$tr_{m,n}(b) = \sum_\gamma t_\gamma \chi^\gamma(b) \quad (b \in B_N(m,n)) \tag{A.20}$$

where t_γ is the dimension of an irreducible representation γ of the $GL(N)$ group, and $\chi^\gamma(b)$ is the character of an irreducible representation γ of the Brauer algebra. The $tr_{m,n}$ denotes a trace over the mixed tensor space. Let d_γ be the dimension of γ in the Brauer algebra. We have a formula of t_γ using elements in $S_m \times S_n$,

$$t_\gamma = \frac{N^{m+n-2k}}{(m-k)!(n-k)!}\chi_{(\gamma_+,\gamma_-)}(\Omega_{m-k,n-k}), \tag{A.21}$$

where $\chi_{(\gamma_+,\gamma_-)}(\sigma \otimes \tau)$ is the character of $S_{m-k} \times S_{n-k}$, and $\Omega_{m-k,n-k}$ is a central element in the group algebra of $S_{m-k} \times S_{n-k}$ called coupled Omega factor [24–26]. A formula to express $\Omega^{-1}_{m,n}$ in terms of Ω^{-1}_{m+n} is given in [44]. We have the following formula for d^γ,

$$d^\gamma = \frac{m!n!}{(m-k)!(n-k)!k!} d_{\gamma_+} d_{\gamma_-}, \tag{A.22}$$

where d_{γ_+} and d_{γ_-} are the dimensions of S_{m-k} and S_{n-k} respectively.

We can construct the projection operator P^γ as[3]

$$P^\gamma = d^\gamma \sum_i \chi^\gamma(b^i) b_i, \tag{A.23}$$

where b^i is the dual basis in (30). The Schur–Weyl duality asserts

$$P^\gamma V^{\otimes m} \otimes \bar{V}^{\otimes n} = V_\gamma^{GL(N)} \otimes V_\gamma^{B_N(m,n)}. \tag{A.24}$$

[3] The expression is valid for $m + n \le N$.

Because the walled Brauer algebra $B_N(m,n)$ contains the group algebra of $S_m \times S_n$, which we denote by $\mathbb{C}(S_m \times S_n)$, we have a decomposition similar to (A.12)

$$V_\gamma^{B_N(m,n)} = \bigoplus_A M_A^\gamma \ V_A^{\mathbb{C}(S_m\times S_n)}, \tag{A.25}$$

Here the multiplicity associated with the decomposition is given by

$$M_A^\gamma = \sum_{\delta\vdash k} g(\gamma_+,\delta;\alpha)g(\gamma_-,\delta;\beta), \quad A=(\alpha,\beta). \tag{A.26}$$

We can introduce an operator $Q^\gamma_{A,\mu\nu}$ as an element in the Brauer algebra [10,15] that satisfies

$$Q^\gamma_{A,\mu\nu}Q^{\gamma'}_{A',\mu'\nu'} = \delta^{\gamma\gamma'}\delta_{AA'}\delta_{\nu\mu'}Q^\gamma_{A,\mu\nu'}. \tag{A.27}$$

The role of the operator $Q^\gamma_{A,\mu\nu}$ is completely the same as $P^R_{A,\mu\nu}$. Introducing the restricted character of the walled Brauer algebra, we have

$$Q^\gamma_{A,\mu\nu} = d^\gamma \sum_i \chi^\gamma_{A,\nu\mu}(b^i)b_i. \tag{A.28}$$

The relation between P^γ and $Q^\gamma_{A,\mu\nu}$ is

$$P^\gamma = \sum_{A,\mu} Q^\gamma_{A,\mu\mu}. \tag{A.29}$$

The intertwiner and the restricted character have the symmetry

$$\begin{aligned} hQ^\gamma_{A,\mu\nu}h^{-1} &= Q^\gamma_{A,\mu\nu}, \\ \chi^\gamma_{A,\mu\nu}(hbh^{-1}) &= \chi^\gamma_{A,\mu\nu}(b) \quad (h\in S_m\times S_n). \end{aligned} \tag{A.30}$$

There is another way of using the Schur–Weyl duality in the description of multi-traces [11,12,16].

B. Orthogonality Relations

In this section we summarise orthogonality relations of representations and characters of the symmetric group and the walled Brauer algebra.

Orthogonality relations of the symmetric group S_n are

$$\frac{1}{n!}\sum_{\sigma\in S_n} D^R(\sigma)_{ij}D^S(\sigma^{-1})_{kl} = \frac{1}{d_R}\delta_{il}\delta_{jk}\delta_{RS}, \tag{B.1}$$

$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\chi_S(\sigma^{-1}) = \delta_{RS}, \tag{B.2}$$

$$\frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\chi_S(\sigma^{-1}\tau) = \frac{1}{d_R}\chi_R(\tau)\delta_{RS}, \tag{B.3}$$

$$\frac{1}{d_R}\chi_R(\sigma)\chi_R(\tau) = \frac{1}{n!}\sum_{\rho\in S_n}\chi_R(\rho\sigma\rho^{-1}\tau), \tag{B.4}$$

$$\sum_{R\vdash n}\chi_R(\sigma)\chi_R(\tau) = \sum_{\rho\in S_n}\delta_n(\rho\sigma\rho^{-1}\tau). \tag{B.5}$$

We have similar formulae for the restricted characters. For the symmetric group S_{m+n} with the restriction to $S_m \times S_n$ considered, we have

$$\frac{1}{(m+n)!}\sum_{\sigma\in S_{m+n}} \chi^R_{A,\mu\nu}(\sigma)\chi^{R'}_{A',\mu'\nu'}(\sigma^{-1}) = \frac{d_A}{d_R}\delta_{RR'}\delta_{AA'}\delta_{\mu\nu'}\delta_{\nu\mu'}, \tag{B.6}$$

This formula is consistent with (B.2) due to $d_R = \sum_A d_A M^R_A$. We also have

$$\frac{1}{(m+n)!}\sum_{\sigma\in S_{m+n}} \chi^R_{A,\mu\nu}(\sigma)\chi^{R'}_{A',\mu'\nu'}(\sigma^{-1}\tau) = \frac{1}{d_R}\chi^R_{A\mu\nu'}(\tau)\delta_{RR'}\delta_{AA'}\delta_{\mu'\nu}, \tag{B.7}$$

$$\frac{1}{(m+n)!}\sum_{\sigma\in S_{m+n}} \chi^R_{A,\mu\nu}(\sigma)\chi^{R'}_{A',\mu'\nu'}(\tau\sigma^{-1}) = \frac{1}{d_R}\chi^R_{A\mu'\nu}(\tau)\delta_{RR'}\delta_{AA'}\delta_{\mu\nu'}, \tag{B.8}$$

$$\sum_{A,\mu,\nu}\frac{1}{d_A}\chi^R_{A,\mu\nu}(\sigma_1)\chi^R_{A,\nu\mu}(\sigma_2) = \frac{1}{m!n!}\sum_{h\in S_m\times S_n}\chi^R(h\sigma_1 h^{-1}\sigma_2). \tag{B.9}$$

If we introduce the dual basis $\sigma^i = \frac{1}{n!}\sigma_i^{-1}$, factorials disappear from these formulae.

For the walled Brauer algebra $B_N(m,n)$, we have

$$\sum_{b\in B_N(m,n)} D^\gamma(b)_{ij}D^{\gamma'}(b^*)_{kl} = \frac{1}{t^\gamma}\delta_{il}\delta_{jk}\delta^{\gamma\gamma'}, \tag{B.10}$$

$$\sum_{b\in B_N(m,n)} \chi^\gamma(b)\chi^{\gamma'}(b^*) = \frac{d^\gamma}{t^\gamma}\delta^{\gamma\gamma'}, \tag{B.11}$$

$$\sum_{b\in B_N(m,n)} \chi^\gamma(b)\chi^{\gamma'}(b^*c) = \frac{1}{t^\gamma}\delta^{\gamma\gamma'}\chi^\gamma(c), \tag{B.12}$$

$$\sum_{\gamma}\chi^\gamma(b)\chi^\gamma(c) = \sum_{d\in B_N(m,n)} t^\gamma\chi^\gamma(dbd^*c), \tag{B.13}$$

$$\sum_{b\in B_N(m,n)} \chi^\gamma_{A,\mu\nu}(b)\chi^{\gamma'}_{A',\mu'\nu'}(b^*) = \frac{d_A}{t^\gamma}\delta_{\gamma\gamma'}\delta_{AA'}\delta_{\mu\nu'}\delta_{\mu'\nu}, \tag{B.14}$$

$$\sum_{b\in B_N(m,n)} \chi^\gamma_{A,\mu\nu}(b)\chi^{\gamma'}_{A',\mu'\nu'}(b^*c) = \frac{1}{t^\gamma}\delta_{\gamma\gamma'}\delta_{AA'}\chi^\gamma_{A,\mu\nu'}(c)\delta_{\mu'\nu}, \tag{B.15}$$

$$\sum_{b\in B_N(m,n)} \chi^\gamma_{A,\mu\nu}(b)\chi^{\gamma'}_{A',\mu'\nu'}(cb^*) = \frac{1}{t^\gamma}\delta_{\gamma\gamma'}\delta_{AA'}\chi^\gamma_{A,\mu'\nu}(c)\delta_{\mu\nu'}, \tag{B.16}$$

$$\sum_{A,\mu,\nu}\frac{1}{d_A}\chi^\gamma_{A,\mu\nu}(b_1)\chi^\gamma_{A,\nu\mu}(b_2) = \frac{1}{m!n!}\sum_{h\in S_m\times S_n}\chi^\gamma(hb_1h^{-1}b_2). \tag{B.17}$$

The orthogonality relations in terms of the dual basis $b^i = g^{ij}b_j$ are as follows:

$$\sum_i D^\gamma(b_i)_{ij}D^{\gamma'}(b^i)_{kl} = \frac{1}{d^\gamma}\delta_{il}\delta_{jk}\delta^{\gamma\gamma'}, \tag{B.18}$$

$$\sum_i \chi^\gamma(b_i)\chi^{\gamma'}(b^i) = \delta^{\gamma\gamma'}, \tag{B.19}$$

$$\sum_i \chi^\gamma(b_i)\chi^{\gamma'}(b^i c) = \frac{1}{d^\gamma}\delta^{\gamma\gamma'}\chi^\gamma(c), \tag{B.20}$$

$$\sum_\gamma \chi^\gamma(b)\chi^\gamma(c) = \sum_i d^\gamma \chi^\gamma(b_i b b^i c), \tag{B.21}$$

$$\sum_i \chi^\gamma_{A,\mu\nu}(b_i)\chi^{\gamma'}_{A',\mu'\nu'}(b^i) = \frac{d_A}{d^\gamma}\delta_{\gamma\gamma'}\delta_{AA'}\delta_{\mu\nu'}\delta_{\mu'\nu}, \tag{B.22}$$

$$\sum_i \chi^\gamma_{A,\mu\nu}(b_i)\chi^{\gamma'}_{A',\mu'\nu'}(b^i c) = \frac{1}{d^\gamma}\delta_{\gamma\gamma'}\delta_{AA'}\chi^\gamma_{A,\mu\nu'}(c)\delta_{\mu'\nu}, \tag{B.23}$$

$$\sum_i \chi^\gamma_{A,\mu\nu}(b_i)\chi^{\gamma'}_{A',\mu'\nu'}(c b^i) = \frac{1}{d^\gamma}\delta_{\gamma\gamma'}\delta_{AA'}\chi^\gamma_{A,\mu'\nu}(c)\delta_{\mu\nu'}. \tag{B.24}$$

The rewriting of orthogonality relations of the symmetric group using the dual basis $\sigma^i = \frac{1}{n!}\sigma_i^{-1}$ allows us to find the perfect similarity between the symmetric group and the Brauer algebra (see also [43]).

C. A Character Formula of the Walled Brauer Algebra

In this section we will derive (83).

The character of an element b in the walled Brauer algebra $B_N(m,n)$ is related to the character of an element of the form $C^{\otimes h}\otimes b_+\otimes b_-$ as

$$\chi^\gamma(b) = N^{z(b)-h}\chi^\gamma(C^{\otimes h}\otimes b_+\otimes b_-), \tag{C.1}$$

where b_+ is an element in S_{m-h} and b_- is an element in S_{n-h}, and $C^{\otimes h} = C\otimes\cdots\otimes C$ is h disjoint contractions. (See section 3 of [10] for a brief explanation of the contraction.) The $z(b)$ denotes the number of zero cycles[4] involved in b. This formula is proved in theorem 3.1 of [56] and in theorem 5.13 of [57]. Note that h is the number of contractions in $C^{\otimes h}\otimes b_+\otimes b_-$, not the number of contractions in b.

The formula can be derived using some basic properties of the algebra. Here instead of giving a proof, we will derive the formula for some cases in $B_N(3,3)$. Elements $C_{1\bar{1}}C_{2\bar{2}}$ and $C_{2\bar{2}}(13)(\bar{1}\bar{3})$ are already of the form $C^h\otimes b_+\otimes b_-$, where $C_{i\bar{j}}$ is the contraction acting on the i-th vector space of $V^{\otimes m}$ and the j-th vector space of $\bar{V}^{\otimes n}$. On the other hand, $C_{1\bar{1}}C_{2\bar{2}}(23)$ and $C_{2\bar{2}}(123)(\bar{1}\bar{3})$ are not of the form, which have $z = 1, 0$ respectively. Using $C^2 = NC$ and the cyclicity of the character, we can show

$$\begin{aligned}\chi^\gamma(C_{1\bar{1}}C_{2\bar{2}}(23)) &= \frac{1}{N^2}\chi^\gamma(C_{1\bar{1}}C_{2\bar{2}}(23)C_{2\bar{2}}C_{1\bar{1}}) = \frac{1}{N}\chi^\gamma(C_{1\bar{1}}C_{2\bar{2}}),\\ \chi^\gamma(C_{2\bar{2}}(123)(\bar{1}\bar{3})) &= \frac{1}{N}\chi^\gamma(C_{2\bar{2}}(123)(\bar{1}\bar{3})C_{2\bar{2}}) = \frac{1}{N}\chi^\gamma(C_{2\bar{2}}(13)(\bar{1}\bar{3})),\end{aligned} \tag{C.2}$$

which have reproduced the formula (C.1). The diagrammatic computation explained in section 3.2 in [10] is convenient to derive the last equalities.

---

4 When we write the conventional diagram of a cycle of an element b in the walled Brauer algebra (for example see [56,57]), if the number of vertical edges on the left side of the wall minus the number of vertical edges on the right side of the wall is zero, we say that the cycle type of the cycle is zero, or it is a zero-cycle.

For an element which is conjugate by some permutation in $S_m \times S_n$ to the form $C^{\otimes h} \otimes b_+ \otimes b_-$, the character of the element can be expanded in terms of the character in $S_{m-h} \times S_{n-h}$ as

$$\chi^{\gamma}(C^{\otimes h} \otimes b_+ \otimes b_-) = N^h \sum_{\lambda \vdash (m-h)} \sum_{\pi \vdash (n-h)} \sum_{\delta \vdash (k-h)} g(\delta, \gamma_+; \lambda) g(\delta, \gamma_-; \pi) \chi_{\lambda}(b_+) \chi_{\pi}(b_-), \tag{C.3}$$

where $\gamma = (\gamma_+, \gamma_-)$, $\gamma_+ \vdash (m-k)$, $\gamma_- \vdash (n-k)$. Note that the character vanishes if $k - h < 0$. This formula is found in theorem 7.20 of [57]. If we set $h = 0$, it becomes a familiar formula

$$\begin{aligned}\chi^{\gamma}(\sigma \otimes \tau) &= \sum_{R \vdash m} \sum_{S \vdash n} \sum_{\delta \vdash k} g(\delta, \gamma_+; \lambda) g(\delta, \gamma_-; \pi) \chi_R(\sigma) \chi_S(\tau) \\ &= \sum_{\lambda \vdash m} \sum_{\pi \vdash n} M^{\gamma}_{(R,S)} \chi_R(\sigma) \chi_S(\tau),\end{aligned} \tag{C.4}$$

where $M^{\gamma}_{(R,S)}$ is the multiplicity of the representation (R, S) inside the representation γ.

In order to rewrite (C.3) further, we will use a formula of Littlewood-Richardson coefficients,

$$g(R, S; T) = \frac{1}{n_1! n_2!} \sum_{\sigma_1 \in S_{n_1}} \sum_{\sigma_2 \in S_{n_2}} \chi_R(\sigma_1^{-1}) \chi_S(\sigma_2^{-1}) \chi_T(\sigma_1 \circ \sigma_2), \tag{C.5}$$

where $R \vdash n_1$, $S \vdash n_2$ and $T \vdash (n_1 + n_2)$. Substituting this into (C.3), we obtain

$$\begin{aligned}
&\chi^{\gamma}(C^{\otimes h} \otimes b_+ \otimes b_-) \\
&= N^h \sum_{\lambda \vdash (m-h), \pi \vdash (n-h)} \sum_{\delta \vdash (k-h)} \\
&\quad \times \frac{1}{(k-h)!(m-k)!} \sum_{\sigma_1 \in S_{k-h}, \sigma_2 \in S_{m-k}} \chi_{\delta}(\sigma_1^{-1}) \chi_{\gamma_+}(\sigma_2^{-1}) \chi_{\lambda}(\sigma_1 \circ \sigma_2) \\
&\quad \times \frac{1}{(k-h)!(n-k)!} \sum_{\tau_1 \in S_{k-h}, \tau_2 \in S_{n-k}} \chi_{\delta}(\tau_1^{-1}) \chi_{\gamma_-}(\tau_2^{-1}) \chi_{\pi}(\tau_1 \circ \tau_2) \\
&\quad \times \chi_{\lambda}(b_+) \chi_{\pi}(b_-) \\
&= N^h \frac{(m-h)!(n-h)!}{(k-h)!(m-k)!(n-k)!} \sum_{\sigma_i, \tau_i} \delta_{m-h}(b_+[\sigma_1 \circ \sigma_2]) \\
&\quad \times \delta_{n-h}(b_-[\tau_1 \circ \tau_2]) \delta_{k-h}([\sigma_1^{-1}] \tau_1^{-1}) \\
&\quad \times \chi_{\gamma_+}(\sigma_2^{-1}) \chi_{\gamma_-}(\tau_2^{-1}) \\
&= N^h \frac{1}{(m-k)!(n-k)!} \sum_{\sigma_2 \in S_{m-k}, \tau_2 \in S_{n-k}} \Delta(b_+, b_-; \sigma_2, \tau_2) \chi_{\gamma_+}(\sigma_2^{-1}) \chi_{\gamma_-}(\tau_2^{-1}),
\end{aligned} \tag{C.6}$$

where we have defined

$$\begin{aligned}
&\Delta(b_+, b_-; \sigma_2, \tau_2) \\
&= \frac{(m-h)!(n-h)!}{(k-h)!} \sum_{\sigma_1,\tau_1\in S_{k-h}} \delta_{m-h}(b_+[\sigma_1\circ\sigma_2])\delta_{n-h}(b_-[\tau_1\circ\tau_2])\delta_{k-h}([\sigma_1^{-1}]\tau_1^{-1}) \\
&= \frac{1}{((k-h)!)^2} \sum_{\sigma_1,\tau_1\in S_{k-h}} tr^{(r)}_{m-h}(b_+[\sigma_1\circ\sigma_2])tr^{(r)}_{n-h}(b_-[\tau_1\circ\tau_2])tr^{(r)}_{k-h}([\sigma_1^{-1}]\tau_1^{-1}).
\end{aligned} \tag{C.7}$$

Note again that $b_+ \in S_{m-h}$ and $b_- \in S_{n-h}$, while $\sigma_2 \in S_{m-k}$ and $\tau_2 \in S_{n-k}$. $\delta_l(\sigma)$ is the delta function defined over the group algebra of S_l. To obtain the second equality in (C.6) we have used the formula (B.5), and $[\sigma]$ is defined in (17). We have introduced the trace of the regular representation of S_n as

$$tr^{(r)}_n(\sigma) = n!\delta_n(\sigma) \quad (\sigma \in S_n). \tag{C.8}$$

If we use the two-point function of the topological field theory given in Sect. 3, we have

$$\begin{aligned}
&\Delta(b_+, b_-; \sigma_2, \tau_2) \\
&= \frac{1}{(k-h)!^2} \sum_{\sigma_1,\tau_1\in S_{k-h}} \eta^{S_{m-h}}(b_+, \sigma_1\circ\sigma_2)\eta^{S_{n-h}}(b_-, \tau_1\circ\tau_2)\eta^{S_{k-h}}(\sigma_1^{-1}, \tau_1^{-1}).
\end{aligned} \tag{C.9}$$

It is convenient to keep in our mind that $\sigma^{-1} \in S_n$ always appears with $1/n!$. Using the dual basis $\sigma^i = \frac{1}{n!}\sigma_i^{-1}$, we get tidy expressions without factorials;

$$\chi^\gamma(C^{\otimes h} \otimes b_+ \otimes b_-) = N^h \sum_{i\in S_{m-k}, j\in S_{n-k}} \Delta(b_+, b_-; \sigma_i, \tau_j)\chi_{\gamma_+}(\sigma^i)\chi_{\gamma_-}(\tau^j), \tag{C.10}$$

and

$$\Delta(b_+, b_-; \sigma_i, \tau_j) = \sum_{k,l\in S_{k-h}} \eta^{S_{m-h}}(b_+, \sigma_k\circ\sigma_i)\eta^{S_{n-h}}(b_-, \tau_l\circ\tau_j)\eta^{S_{k-h}}(\sigma^k, \tau^l). \tag{C.11}$$

These look like pleasant but having too many indices, which might confuse the readers, so we do not use (C.10) and (C.11) in the main text.

References

1. Maldacena, J.M.: The Large N limit of superconformal field theories and supergravity. Adv. Theor. Math. Phys. **2**, 231 (1998). hep-th/9711200
2. Minahan, J.A., Zarembo, K.: The Bethe ansatz for N=4 super Yang–Mills. JHEP **0303**, 013 (2003). hep-th/0212208
3. Beisert, N., et al.: Review of AdS/CFT Integrability: an Overview. Lett. Math. Phys. **99**, 3 (2012). arXiv:1012.3982 [hep-th]
4. Corley, S., Jevicki, A., Ramgoolam, S.: Exact correlators of giant gravitons from dual N=4 SYM theory. Adv. Theor. Math. Phys. **5**, 809 (2002). hep-th/0111222

5. Corley, S., Ramgoolam, S.: Finite factorization equations and sum rules for BPS correlators in N = 4 SYM theory. Nucl. Phys. B **641**, 131 (2002). hep-th/0205221
6. Balasubramanian, V., Berenstein, D., Feng, B., Huang, M.-X.: D-Branes in Yang–Mills theory and emergent gauge symmetry. JHEP **0503**, 006 (2005). hep-th/0411205
7. de Mello Koch, R., Smolic, J., Smolic, M.: Giant gravitons—with strings attached (I). JHEP **0706**, 074 (2007). hep-th/0701066
8. de Mello Koch, R., Smolic, J., Smolic, M.: Giant gravitons—with strings attached (II). JHEP **0709**, 049 (2007). hep-th/0701067
9. Bekker, D., de Mello Koch, R., Stephanou, M.: Giant gravitons—with strings attached (III). JHEP **0802**, 029 (2008). arXiv:0710.5372 [hep-th]
10. Kimura, Y., Ramgoolam, S.: Branes, anti-branes and brauer algebras in gauge-gravity duality. JHEP **0711**, 078 (2007). arXiv:0709.2158 [hep-th]
11. Brown, T.W., Heslop, P.J., Ramgoolam, S.: Diagonal multi-matrix correlators and BPS operators in N = 4 SYM. JHEP **0802**, 030 (2008). arXiv:0711.0176 [hep-th]
12. Brown, T.W., Heslop, P.J., Ramgoolam, S.: Diagonal free field matrix correlators, global symmetries and giant gravitons. JHEP **0904**, 089 (2009). arXiv:0806.1911 [hep-th]
13. Bhattacharyya, R., Collins, S., Koch, R.d.M.: Exact multi-matrix correlators. JHEP **0803**, 044 (2008). arXiv:0801.2061 [hep-th]
14. Bhattacharyya, R., de Mello Koch, R., Stephanou, M.: Exact multi-restricted Schur polynomial correlators. JHEP **0806**, 101 (2008). arXiv:0805.3025 [hep-th]
15. Kimura, Y., Ramgoolam, S.: Enhanced symmetries of gauge theory and resolving the spectrum of local operators. Phys. Rev. D **78**, 126003 (2008). arXiv:0807.3696 [hep-th]
16. Kimura, Y.: Non-holomorphic multi-matrix gauge invariant operators based on Brauer algebra. JHEP **0912**, 044 (2009). arXiv:0910.2170 [hep-th]
17. Kimura, Y.: Quarter BPS classified by Brauer algebra. JHEP **1005**, 103 (2010). arXiv:1002.2424 [hep-th]
18. Pasukonis, J., Ramgoolam, S.: From counting to construction of BPS states in N = 4 SYM. JHEP **1102**, 078 (2011). arXiv:1010.1683 [hep-th]
19. Carlson, W., Koch, R.d.M., Lin, H.: Nonplanar integrability. JHEP **1103**, 105 (2011). arXiv:1101.5404 [hep-th]
20. Koch, R.D.M., Dessein, M., Giataganas, D., Mathwin, C.: Giant graviton oscillators. JHEP **1110**, 009 (2011). arXiv:1108.2761 [hep-th]
21. Koch, R.D.M., Ramgoolam, S.: A double coset ansatz for integrability in AdS/CFT. JHEP **1206**, 083 (2012). arXiv:1204.2153 [hep-th]
22. Kimura, Y.: Correlation functions and representation bases in free N = 4 super Yang–Mills. Nucl. Phys. B **865**, 568 (2012). arXiv:1206.4844 [hep-th]
23. Pasukonis, J., Ramgoolam, S.: Quivers as Calculators: Counting, correlators and Riemann surfaces. JHEP **1304**, 094 (2013). arXiv:1301.1980 [hep-th]
24. Gross, D.J.: Two-dimensional QCD as a string theory. Nucl. Phys. B **400**, 161 (1993) [hep-th/9212149]
25. Gross, D.J., Taylor, W.: Two-dimensional QCD is a string theory. Nucl. Phys. B **400**, 181 (1993). hep-th/9301068
26. Gross, D.J., Taylor, W.: Twists and Wilson loops in the string theory of two-dimensional QCD. Nucl. Phys. B **403**, 395A (1993). hep-th/9303046
27. Cordes, S., Moore, G.W., Ramgoolam, S.: Large N 2-D Yang–Mills theory and topological string theory. Commun. Math. Phys. **185**, 543 (1997). hep-th/9402107
28. Cordes, S., Moore, G.W., Ramgoolam, S.: Lectures on 2-d Yang–Mills theory, equivariant cohomology and topological field theories. Nucl. Phys. Proc. Suppl. **41**, 184 (1995). hep-th/9411210
29. Stembridge, J.R.: Rational tableaux and the tensor algebra of gl(n). J. Comb. Theory A **46**, 79 (1987)
30. Koike, K.: On the decomposition of tensor products of the representations of the classical groups : by means of the universal characters. Adv. Math. **74**, 57–86 (1989)
31. Turaev, V.G.: Operator invariants of tangles, and R-matrices. Izv. Akad. Nauk SSSR Ser. Mat. **53**(5), 1073–1107, 1135 (1989)
32. Benkart, M., Chakrabarti, M., Halverson, T., Lee, C., Leduc, R., Stroomer, J.: Tensor product representations of general linear groups and their connections with Brauer algebras. J. Algebra **166**, 529 (1994)
33. Kristjansen, C., Plefka, J., Semenoff, G.W., Staudacher, M.: A new double scaling limit of N = 4 super Yang–Mills theory and PP wave strings. Nucl. Phys. B **643**, 3 (2002). hep-th/0205033
34. Balasubramanian, V., Berkooz, M., Naqvi, A., Strassler, M.J.: Giant gravitons in conformal field theory. JHEP **0204**, 034 (2002). hep-th/0107119
35. Fukuma, M., Hosono, S., Kawai, H.: Lattice topological field theory in two-dimensions. Commun. Math. Phys. **161**, 157 (1994). hep-th/9212154
36. Bachas, C., Petropoulos, P.M.S.: Topological models on the lattice and a remark on string theory cloning. Commun. Math. Phys. **152**, 191 (1993). hep-th/9205031

37. Chung, S.-W., Fukuma, M., Shapere, A.D.: Structure of topological lattice field theories in three-dimensions. Int. J. Mod. Phys. A **9**, 1305 (1994). hep-th/9305080
38. Atiyah, M.: Topological quantum field theories. Inst. Hautes Etudes Sci. Publ. Math. **68**, 175 (1989)
39. Dijkgraaf, R., Witten, E.: Topological gauge theories and group cohomology. Commun. Math. Phys. **129**, 393 (1990)
40. Abrams, L.: Two dimensional topological quantum field theories and Frobenius algebras. J. Knot Theory Ramif. **5**, 569–587 (1996)
41. Kock, J.: Frobenius Algebras and 2-D Topological Quantum Field Theories. Cambridge University Press, Cambridge (2004)
42. Witten, E.: On quantum gauge theories in two-dimensions. Commun. Math. Phys. **141**, 153 (1991)
43. Ram, A.: Thesis, Univ. Cal. San Diego, Chapter 1 (1991)
44. Kimura, Y., Ramgoolam, S.: Holomorphic maps and the complete 1/N expansion of 2D SU(N) Yang–Mills. JHEP **0806**, 015 (2008). arXiv:0802.3662 [hep-th]
45. Collins, S.: Restricted Schur Polynomials and Finite N Counting. Phys. Rev. D **79**, 026002 (2009). arXiv:0810.4217 [hep-th]
46. de Mello Koch, R., Ramgoolam, S.: Strings from Feynman graph counting: without large N. Phys. Rev. D **85**, 026007 (2012). arXiv:1110.4858 [hep-th]
47. Kimura, Y., Ramgoolam, S., Turton, D.: Free particles from Brauer algebras in complex matrix models. JHEP **1005**, 052 (2010). arXiv:0911.4408 [hep-th]
48. Gubser, S.S., Klebanov, I.R., Polyakov, A.M.: Gauge theory correlators from noncritical string theory. Phys. Lett. B **428**, 105 (1998). hep-th/9802109
49. Witten, E.: Anti-de Sitter space and holography. Adv. Theor. Math. Phys. **2**, 253 (1998). hep-th/9802150
50. Lazaroiu, C.I.: On the structure of open - closed topological field theory in two-dimensions. Nucl. Phys. B **603**, 497 (2001). hep-th/0010269
51. Alexeevski, A., Natanzon, S.: Non-commutative extensions of two-dimensional topological field theories and Hurwitz numbers for real algebraic curves (2002). arXiv:math/0202164
52. Moore, G.W., Segal, G.: D-branes and K-theory in 2D topological field theory (2006). hep-th/0609042
53. Lauda, A.D.: Frobenius algebras and planar open string topological field theories (2005). arXiv:math/0508349
54. Koch, R.d.M., Ramgoolam, S.: From Matrix Models and Quantum Fields to Hurwitz Space and the absolute Galois Group (2010). arXiv:1002.1634 [hep-th]
55. Gopakumar, R.: What is the simplest gauge-string duality? (2011). arXiv:1104.2386 [hep-th]
56. Ram, A.: Characters of Brauer's centralizer algebras. Pac. J. Math. **169**(1), 1–200 (1995)
57. Halverson, T.: Characters of the centralizer algebras of mixed tensor representations of Gl(r,C) and the quantum group Uq(gl(r,C)). Pac. J. Math. **174**(2), 295–587 (1996)

Communicated by H. Ooguri