

5D Super Yang–Mills on $Y^{p,q}$ Sasaki–Einstein Manifolds

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Received: 20 August 2013 / Accepted: 1 August 2014

Published online: 12 October 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: On any simply connected Sasaki–Einstein five dimensional manifold one can construct a super Yang–Mills theory which preserves at least two supersymmetries. We study the special case of toric Sasaki–Einstein manifolds known as $Y^{p,q}$ manifolds. We use the localisation technique to compute the full perturbative part of the partition function. The full equivariant result is expressed in terms of a certain special function which appears to be a curious generalisation of the triple sine function. As an application of our general result we study the large N behaviour for the case of single hypermultiplet in adjoint representation and we derive the N^3 -behaviour in this case.

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1. Introduction

Recently the five dimensional Yang–Mills theory has attracted renewed attention and interesting results were derived. These theories are interesting mainly due to their relations to, first of all, the magical (2, 0) 6D theory and secondly, the 5D $N = 1$ SCFTs. Inspired by Pestun’s work [1] on localisation on S^4 , many exact results have now been derived in diverse dimensions. For the case of 5D Yang–Mills theory on S^5 , the partition function was derived and studied in a number of papers [2,3]; further extension to the squashed sphere was done in [4–7]; and last but not least the case of $S^1 \times S^4$ was studied in [8,9], and other related background, see for example [10]. These results were used in providing checks on the dualities AdS₆/CFT₅ [11, 12], AdS₇/CFT₆ [3, 7, 13] and as well as the AGT-inspired ideas [14].

In this work we go beyond the standard case of spheres and study the 5D susy gauge theory on a specific family of toric Sasaki–Einstein (SE) manifolds, the so called $Y^{p,q}$ manifolds [here (p, q) are two coprime integers]. One major motivation is the intriguing results of Lockhart and Vafa [5], which indicate how one may obtain the non-perturbative partition function from the purely perturbative part, as was inspired by the topological string consideration; thus we see it fit to extend our earlier computation on S^5 to a more intricate toric Sasaki–Einstein manifold, hoping that the non-trivial homology of $Y^{p,q}$ may provide more insight on the interpretation of the results.

Let us summarise briefly the results of the present paper. The six dimensional cone over $Y^{p,q}$ can be obtained by the standard Kähler reduction of \mathbb{C}^4 under a $U(1)$ with charge $[p + q, p - q, -p, -p]$, where we use the standard notation for $U(1)$ -actions on \mathbb{C}^4 . Although only $U(1)^3$ acts on $Y^{p,q}$, it is convenient to discuss $U(1)^4$ -actions on \mathbb{C}^4 . The full equivariant perturbative partition function for the 5D vector multiplet coupled to a hypermultiplet with mass M in representation \underline{R} has the following form

$$\begin{aligned}
 Z_{\text{pert}} = & \int_{\mathfrak{t}} dx \exp\left(-\frac{8\pi^3 r_Q}{g_{YM}^2} \text{Tr}[x^2]\right) \\
 & \times \frac{\det'_{\text{adj}} S^\Lambda(ix|\omega_1, \omega_2, \omega_3, \omega_4)}{\det_{\underline{R}} S^\Lambda\left(ix + iM + \frac{1}{2}(\omega_1 + \omega_2 + \omega_3 + \omega_4)|\omega_1, \omega_2, \omega_3, \omega_4\right)}, \tag{1}
 \end{aligned}$$

where $\varrho = \text{Vol}_{Y^{p,q}}/\text{Vol}_{S^5}$ [with $\text{Vol}_{Y^{p,q}}$ being the equivariant volume, see (41) for details] and the function S^Λ is defined as the zeta regularised infinite products

$$S^\Lambda(x|\omega_1, \omega_2, \omega_3, \omega_4) = \prod_{(i,j,k,l) \in \Lambda^+} \left(i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4 + x \right) \times \prod_{(i,j,k,l) \in \Lambda_0^+} \left(i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4 - x \right), \tag{2}$$

where the lattices are defined as follows

$$\Lambda^+ = \{i, j, k, l \in \mathbb{Z}_{\geq 0} \mid i(p + q) + j(p - q) - kp - lp = 0\}, \tag{3}$$

$$\Lambda_0^+ = \{i, j, k, l \in \mathbb{Z}_{> 0} \mid i(p + q) + j(p - q) - kp - lp = 0\}, \tag{4}$$

and $\omega_1, \omega_2, \omega_3, \omega_4$ are equivariant parameters corresponding to $U(1)^4$ -action on \mathbb{C}^4 . The lattice conditions (3) and (4) reduce the product to a three-dimensional lattice and S^Λ depends effectively only on three parameters, i.e.,

$$((p + q)\partial_{\omega_1} + (p - q)\partial_{\omega_2} - p\partial_{\omega_3} - p\partial_{\omega_4})S^\Lambda(x|\omega_1, \omega_2, \omega_3, \omega_4) = 0, \tag{5}$$

reflecting the effective $U(1)^3$ action on $Y^{p,q}$ -space. The function $S^\Lambda(x|\omega_1, \omega_2, \omega_3, \omega_4)$ resembles in many ways the triple sine functions and indicates how one may generalise the latter. The case of Sasaki–Einstein metric (or equivalently, the existence of two Killing spinors) on $Y^{p,q}$ space corresponds to the following specific choice of equivariant parameters

$$\omega_1 = 0, \quad \omega_2 = \frac{1}{(p + q)\ell}, \quad \omega_3 = \omega_4 = \frac{3}{2} - \frac{1}{2(p + q)\ell}.$$

Following the analogy with S^5 we refer to this as unsquashed $Y^{p,q}$ space, whereas the case of arbitrary equivariant parameters will be called the squashed $Y^{p,q}$ space. In this paper we study the asymptotics of S^Λ for a general set of equivariant parameters [with some minor restrictions (42) related to the moment cone and convergence]. As a concrete application of our result we study the case of $SU(N)$ gauge theory with a single hypermultiplet in the adjoint representation. For the large N -limit, in the case of large 't Hooft coupling the free energy behaves as follows

$$F = -\frac{g_{YM}^2 N^3}{96\pi r} \varrho \left(\frac{9}{4} + M^2 \right)^2, \tag{6}$$

where $\varrho = \text{Vol}_{Y^{p,q}}/\text{Vol}_{S^5}$ and this for the case of unsquashed $Y^{p,q}$ space admitting Sasaki–Einstein metric. Thus we find the result which is identical to the calculation on S^5 up to the volume factor ϱ .

The paper is organised as follows: in Sect. 2 we review the construction of supersymmetric 5D Yang–Mills theory with matter on general Sasaki–Einstein manifolds. We also review some basic properties of the Sasaki–Einstein geometry. Section 3 is a review of the definitions of $Y^{p,q}$, with some discussions on certain properties of $Y^{p,q}$ relevant for the subsequent index calculation. Section 4 contains the localisation argument and discusses the relation between supersymmetry and cohomological complexes. Section 5 contains the technical calculation of one-loop determinants and the final result is given in terms of infinite products. In Sect. 6 we study the asymptotic behaviour of these infinite products.

Using this asymptotic behaviour in Sect. 7 we provide the application of our results for the case of single hypermultiplet in adjoint representation and we derive N^3 -behaviour. Section 8 summarises the results of the paper and indicates the possible generalisation for other toric Sasaki–Einstein manifolds. Many technical details and calculations are collected in the appendices.

2. Super Yang–Mills Theory on Sasaki–Einstein 5-Folds

We start from the $N = 1$ supersymmetric gauge theory on S^5 constructed in [15], which has eight super-charges, and place the theory on some other Sasaki–Einstein (SE) manifolds. Assuming that the SE manifold is simply connected, then one is guaranteed a pair of Killing spinors of type (1, 1), and consequently a quarter supersymmetry (2 supercharges). It turns out that without turning on more background fields from the supergravity multiplet, only S^5 can have more super-charges, but on the other hand a quarter of the supersymmetry is sufficient for the purpose of localisation and we will not spend much effort in enlarging the supersymmetry, except some brief discussions in Sect. 8.

2.1. 5D Yang–Mills theory with matter on S^5 . The field content of the susy YM theory on S^5 consists of a vector-multiplet and a hyper-multiplet. The vector-multiplet contains the gauge field A_m , a scalar σ , an $SU(2)$ -triplet of scalars D_{IJ} and a symplectic Majorana gaugino λ_I , with the following off-shell supersymmetry transformation [see (84) for our notation of spinor bi-linears]

$$\begin{aligned} \delta A_m &= i\xi_I \Gamma_m \lambda^I, \\ \delta \sigma &= i\xi_I \lambda^I, \\ \delta \lambda_I &= -\frac{1}{2}(\Gamma^{mn} \xi_I) F_{mn} + (\Gamma^m \xi_I) D_m \sigma - \xi^J D_{JI} + \frac{2}{r} t_I^J \xi_J \sigma, \\ \delta D_{IJ} &= -i\xi_I \Gamma^m D_m \lambda_J + [\sigma, \xi_I \lambda_J] + \frac{i}{r} t_I^K \xi_K \lambda_J + (I \leftrightarrow J), \end{aligned} \quad (7)$$

where ξ_I is a spinor, satisfying the Killing equation

$$D_m \xi_I = \frac{1}{r} t_I^J \Gamma_m \xi_J, \quad t_I^J = \frac{i}{2} (\sigma_3)_I^J, \quad (\xi_I \xi_J) = -\frac{1}{2} \epsilon_{IJ}, \quad (8)$$

where $\sigma_3 = \text{diag}[1, -1]$. The quantity t_I^J is the vev of an $SU(2)_R$ -triplet auxiliary field in the Weyl multiplet. We remark that in checking the closure property of the susy transformation, only the Killing spinor equation and the dimensionality of the space is used.

The Lagrangian density for the vector multiplet on S^5 is

$$\begin{aligned} L_{\text{vec}} &= \frac{1}{g_{YM}^2} \text{Tr} \left[\frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D^{IJ} + \frac{2}{r} \sigma t^{IJ} D_{IJ} - \frac{10}{r^2} t^{IJ} t_{IJ} \sigma^2 \right. \\ &\quad \left. + i \lambda_I \Gamma^m D_m \lambda^I - \lambda_I [\sigma, \lambda^I] - \frac{i}{r} t^{IJ} \lambda_I \lambda_J \right], \end{aligned} \quad (9)$$

where F_{mn} is the field strength for A_m . The vector indices are raised and lowered with the metric, while the $SU(2)_R$ -indices are raised using ϵ^{IJ} (see Appendix A). The action is

susy invariant, and also in checking this, only the Killing equation (8) is used. Therefore we can take the supersymmetry transformations (7) and the action (9) from S^5 and use them on any simply connected SE manifold, but with of course only a fraction of the supersymmetry retained due to a smaller number of Killing spinor solutions.

The hyper-multiplet on S^5 consists of an $SU(2)_R$ -doublet of complex scalars q_I^A , $I = 1, 2$ and an $SU(2)_R$ -singlet fermion ψ^A , with the reality conditions ($A = 1, 2, \dots, 2N$)

$$(q_I^A)^* = \Omega_{AB} \epsilon^{IJ} q_J^B, \quad (\psi^A)^* = \Omega_{AB} C \psi^B, \tag{10}$$

where Ω_{AB} is the invariant tensor of $Sp(N)$

$$\Omega = \begin{vmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{vmatrix},$$

and C is the charge conjugation matrix.

The gauge group will be a subgroup of $Sp(N)$, in particular we consider the hyper-multiplet with the representation $\underline{N} \oplus \underline{N}$ of $SU(N)$, which is embedded in $Sp(N)$ in the standard manner

$$U \rightarrow \begin{vmatrix} U & 0 \\ 0 & U^{-T} \end{vmatrix}, \quad U \in SU(N).$$

One can rewrite the scalar field q into a more familiar form as

$$q_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_+ \\ \phi_- \end{vmatrix}, \quad q_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} -\phi_-^* \\ \phi_+^* \end{vmatrix}, \tag{11}$$

where ϕ_{\pm} transform in the \underline{N} and $\bar{\underline{N}}$ of $SU(N)$ respectively. The fermion can be written in a similar manner

$$\psi^A = \frac{1}{2} \begin{vmatrix} \psi^\alpha \\ -C \psi_\beta^* \end{vmatrix}, \tag{12}$$

where ψ^α is now an unconstrained Dirac spinor transforming in \underline{N} (here α is the index for the representation). Analogously we can discuss the adjoint representation of $SU(N)$ when two copies of the adjoint are embedded into that of $Sp(N)$.

Suppressing the gauge group index, the supersymmetry on-shell transformations are written as:

$$\begin{aligned} \delta q_I &= -2i \xi_I \psi, \\ \delta \psi &= \Gamma^m \xi_I (D_m q^I) + i \sigma \xi_I q^I - \frac{3}{r} t^{IJ} \xi_I q_J. \end{aligned} \tag{13}$$

These transformations leave invariant the action with the following Lagrangian density

$$\begin{aligned} L_{\text{hyp}} &= \epsilon^{IJ} \Omega_{AB} D_m q_I^A D^m q_J^B - \epsilon^{IJ} q_I^A \sigma_{AC} \sigma^C_B q_J^B + \frac{15}{2r^2} \epsilon^{IJ} \Omega_{AB} t^2 q_I^A q_J^B \\ &\quad - 2i \Omega_{AB} \psi^A \not{D} \psi^B - 2 \psi^A \sigma_{AB} \psi^B - 4 \Omega_{AB} \psi^A \lambda_I q^{IB} - i q_I^A D_{AB}^{IJ} q_J^B, \end{aligned} \tag{14}$$

where $t^2 = t^{IJ} t_{IJ} = 1/2$ and $\sigma_{AB} = \Omega_{AC} \sigma^C_B$. Here again, only the Killing spinor equation plus the Einstein relation $R_{mn} = 4g_{mn}$ is used for the check. A mass term can be generated through the standard trick of coupling the hypermultiplet to an auxillary vector multiplet and giving an expectation value to the scalar in the multiplet, see [15]. For the

localisation we will need to have off-shell realisation of supersymmetry transformations and for the hypermultiplet it will be resolved later on.

To summarise, with no modification, we have a superymmetric Yang–Mills theory with matter on any SE manifold, with two supersymmetries. As we saw, the existence of supersymmetry depends on the existence of Killing spinors. In the next two subsections, we quickly go over some necessary facts about SE manifolds and the explicit construction of Killing spinors. This material is quite well-known by now and thus our review is mainly to set the notations. The reader may consult the short but nice review [16].

2.2. *Sasaki–Einstein manifolds.* A contact metric structure on M^5 consists of a 1-form κ such that $\kappa(d\kappa)^2 \neq 0$ and a complex structure J on the sub-bundle $\ker \kappa$, which we call the *horizontal plane*

$$J \in \text{Aut}(\ker \kappa), \quad J^2 = -\mathbb{1},$$

and that J is compatible with $d\kappa$ in the sense that $1/2d\kappa J$ is a metric for the horizontal plane $\ker \kappa$. One can choose a unique vector field R such that

$$\iota_R \kappa = 1, \quad \iota_R d\kappa = 0,$$

and we extend J to act also on R as zero $JR = 0$, leading to

$$J^2 = -\mathbb{1} + R \otimes \kappa.$$

The metric of the tangent bundle is the direct sum of the one on $\ker \kappa$ and the one along R

$$g = \frac{1}{2}d\kappa J + \kappa \otimes \kappa.$$

As a consequence

$$\begin{aligned} g(JX, JY) &= g(X, Y) - \kappa(X)\kappa(Y), \\ d\kappa &= -2gJ, \\ R &= g^{-1}\kappa. \end{aligned} \tag{15}$$

If R is a Killing vector field, then (κ, R, J) gives a *K-contact structure*, the Killing condition is equivalent to

$$\nabla_X R = JX. \tag{16}$$

A *Sasaki-manifold* is a K-contact manifold such that its metric cone $M \times (0, \infty)$ with metric and symplectic form

$$G = r^2g + dr^2, \quad \Omega = d(r^2\kappa), \quad \mathcal{J} = 2\Omega^{-1}G.$$

is Kähler. The complex structure is written explicitly as

$$\mathcal{J} = J + r^{-1}R \otimes dr - r\partial_r \otimes \kappa,$$

it is easy to check $\mathcal{J}^2 = -\mathbb{1}_6$. The vector field

$$\epsilon = r \frac{\partial}{\partial r} \tag{17}$$

is called the *homothetic* vector field, and it is clear that

$$\mathcal{J}(\epsilon) = \kappa.$$

The Kähler condition is equivalent to the covariant constancy of \mathcal{J} with respect to the Levi-Civita connection.¹ Thus a K-contact manifold is Sasaki iff J satisfies the integrability condition

$$\langle Z, (\nabla_X J)Y \rangle = -\kappa(Z)\langle X, Y \rangle + \langle Z, X \rangle\kappa(Y), \tag{18}$$

where $\langle -, - \rangle$ is the inner product using the metric. From now on we will use the *same letter J for the complex structure as well as the 2-form gJ* .

If the cone metric is in addition Ricci-flat i.e. the cone is actually Calabi–Yau, then M is said to be Sasaki–Einstein (SE). The Ricci flatness is equivalent to

$$R_{mn} = 4g_{mn}. \tag{19}$$

Using the Reeb, one can define the *horizontal* forms

$$\omega \in \Omega_H^2(M), \quad \text{if } \iota_R \omega = 0.$$

Let us fix the volume form of M^5 as²

$$\text{vol} = \frac{1}{2}\kappa \wedge J \wedge J, \tag{20}$$

and one can define a duality for the horizontal 2-forms as

$$\omega \rightarrow *_R \omega = \iota_R * \omega, \quad \omega \in \Omega_H^2(M^5), \tag{21}$$

Next one can prove that for an SE manifold, the Weyl tensor is horizontal and anti-selfdual

$$\iota_R W_{XY} = 0, \quad *_R W_{XY} = -W_{XY}, \tag{22}$$

where

$$W_{XY} = X^m Y^n (R_{mnpq} - g_{p[m} g_{n]q}), \quad X, Y \in TM. \tag{23}$$

The proof makes use of (18) and the details can be found in chapter 5 of [18].

¹ It can be shown that the closedness of the wouldbe Kähler form $G\mathcal{J}$ plus the vanishing of the Nijenhuis tensor is equivalent to the covariant constancy of \mathcal{J} , see lemma 4.15 in [17].

² Due to a historical accident, the choice of volume form in [2] is opposite to the current one, see also footnote 4. The reader should bear this in mind when comparing results between the two papers, especially some anti-self-dualities there will become self-dualities here.

2.3. *Sasaki–Einstein 5-folds and killing spinors.* Using the horizontal complex structure one can define the so called canonical spin^c-structure. Let

$$W_{\text{can}} = \bigoplus \Omega_H^{0,\bullet}(M), \tag{24}$$

where $\Omega_H^{0,\bullet}$ consists of horizontal forms anti-holomorphic with respect to J . One then has a representation of the Clifford algebra: let ψ be any section of W_{can} and χ a 1-form, define the Clifford action

$$\chi \cdot \psi = \begin{cases} \sqrt{2}\chi \wedge \psi & \chi \in \Omega_H^{0,1}(M) \\ \sqrt{2}i_{g^{-1}\chi}\psi & \chi \in \Omega_H^{1,0}(M) \\ (-1)^{\text{deg}+1}\psi & \chi = \kappa \end{cases} \tag{25}$$

In this way, one has a spin^c-structure whose characteristic line bundle (see chapter 5 in [19]) is the anti-canonical line bundle associated with the complex structure J . If M is actually SE, then condition (19), (15) together with the condition $H_1(M, \mathbb{Z})_{\text{tor}} = 0$ would imply that M is spin (theorem 7.5.27 in [20]).

For a 5D SE spin manifold one can show that there exists a pair of killing spinors satisfying

$$D_m \bar{\xi}^1 = -\frac{i}{2}\Gamma_m \xi^1, \quad D_m \xi^2 = +\frac{i}{2}\Gamma_m \xi^2,$$

we will review the construction from [21]. Consider the following dimension 1 sub-bundle W_0 within the spin bundle W

$$\psi \in W_0 \subset W, \quad R\psi = -\psi, \quad \frac{1}{2}(1 + i\mu J)X^\perp \cdot \psi = 0, \quad \forall X \in \Gamma(TM), \tag{26}$$

where $\mu = \pm 1$ and to keep the formulae neat, we have omitted Γ whenever the Clifford multiplication is obvious. In view of the construction (25), the condition above says that ψ is in $\Omega_H^{0,0}(M)$ or $\Omega_H^{0,2}(M)$ depending on μ . One can rewrite the second condition in (26) into

$$A(X)\psi = \left(\mu JX - \frac{i}{2}RX - \frac{i}{2}X\right)\psi = 0.$$

One then defines a connection for the subbundle W_0

$$\tilde{D}_X = D_X + \frac{i\mu}{2}X,$$

and checks that this is indeed a connection, i.e.

$$(1) [\tilde{D}_X, R]\psi = 0, \quad (2) [\tilde{D}_X, A(Y)]\psi = 0, \quad \psi \in W_0.$$

The crucial step now is to show that the curvature of \tilde{D} is zero when restricted to W_0 . First an explicit calculation shows that the curvature is given by the Weyl-tensor

$$\begin{aligned} [\tilde{D}_X, \tilde{D}_Y] - \tilde{D}_{[X,Y]} &= \frac{1}{4}R_{XYpq}\Gamma^{pq} - \frac{1}{4}[X, Y] = \frac{1}{4}X^m Y^n (R_{mnpq} - g_{p[m}g_{n]q})\Gamma^{pq} \\ &= \frac{1}{4}W_{XYpq}\Gamma^{pq}. \end{aligned}$$

which was shown to be horizontal and anti-self-dual. In fact, for any such 2-form Z_{pq} one has $\mathcal{Z}\psi = 0$ for $\psi \in W_0$

$$\begin{aligned} \mathcal{Z}\psi &= Z_{pq}\Gamma^{pq}\psi = -\frac{\sqrt{g}}{2}Z_{rs}\epsilon^{rs}_{pqt}R^t\Gamma^{pq}\psi \\ &= Z^{rs}\Gamma_{rst}R^t\psi = Z^{rs}(\Gamma_{rs}\Gamma_t - 2\Gamma_r g_{st})R^t\psi = -\mathcal{Z}\psi, \end{aligned} \tag{27}$$

where we used $\iota_R Z = 0$ and (86). To summarise the foregoing arguments, we have reached the conclusion:

If a Sasaki–Einstein manifold M^5 is simply connected, then the solution to the Killing equation

$$D_X\psi = -\frac{i\mu}{2}X\psi, \quad \mu = \pm 1$$

exists and is unique up to a constant scale factor. The simply connectedness is needed to ensure we have no non-trivial flat bundle. Moreover, the solution satisfy

$$R\psi = -\psi, \quad (\mu JX - \frac{i}{2}(1 + R)X)\psi = 0, \tag{28}$$

$$\mathcal{J}\psi = -4i\mu\psi, \tag{29}$$

where the second line is a simple consequence of the first.

3. Geometry of $Y^{p,q}$ Manifolds

In this section we briefly review some relevant facts about the family of toric SE manifolds known as $Y^{p,q}$ manifold. For further details the reader may consult [22–24].

3.1. $Y^{p,q}$ from reduction. Take \mathbb{C}^4 with coordinates $[z_1, z_2, z_3, z_4]$ and the standard complex structure. Let us introduce the vector fields e_i and one forms η_i

$$e_i = i(z^i\partial_{z_i} - c.c), \quad \eta_i = \frac{i}{2}(z^i d\bar{z}^i - c.c), \quad \eta = \sum_i \eta_i, \tag{30}$$

where e_i generates the phase rotation of z_i and η_i is its dual 1-form. It is clear that $d\eta$ gives the standard Kähler form on \mathbb{C}^4 .

Consider now a $U(1)_T$ acting on \mathbb{C}^4 with charge $[p + q, p - q, -p, -p]$, the vector field generated by the action is³

$$T = (p + q)e_1 + (p - q)e_2 - pe_3 - pe_4. \tag{31}$$

We perform a Kähler reduction with respect to T . The moment map for T is

$$\mu_T = (p + q)|z_1|^2 + (p - q)|z_2|^2 - p|z_3|^2 - p|z_4|^2,$$

and the reduction

$$C(Y^{p,q}) = \mu_T^{-1}(0)/U(1)_T$$

³ Here we follow the convention from [22], but the explicit metric given in [25] corresponds to the charge vector $[p, p, -(p - q), -(p + q)]$.

is again Kähler and has a cone structure due to the special form of μ_T . This will be the Kähler cone over a base which is by definition a Sasaki-manifold. One can get the base of the cone by imposing $\sum_i a_i |z_i|^2 = 1$, with $a_i \geq 0$. One possible choice is that of a squashed S^7

$$(p + q)|z_1|^2 + (p - q)|z_2|^2 + p|z_3|^2 + p|z_4|^2 = 1, \tag{32}$$

which together with $\mu_T = 0$ leads to $S^3 \times S^3$

$$\begin{aligned} \mu_T^{-1}(0)|_{S^7} &= \{(z_1, z_2, z_3, z_4) \mid (p + q)|z_1|^2 + (p - q)|z_2|^2 = 1/2, \\ p|z_3|^2 + p|z_4|^2 &= 1/2\} \sim S^3 \times S^3. \end{aligned}$$

Of course, there is nothing special about the choice (32), if one chooses instead $\sum_i |z_i|^2 = 1$, one still gets $S^3 \times S^3$. We remark that the T action is free on $S^3 \times S^3$ if $\gcd(p, q) = 1$, since $z_{1,2}$ cannot be zero together nor can $z_{3,4}$. So $Y^{p,q}$ can also be presented as a quotient of $S^3 \times S^3$ by T .

With the choice (32), one has the Reeb vector field

$$r_1 = (p + q)e_1 + (p - q)e_2 + pe_3 + pe_4, \tag{33}$$

it is easy to check that on $Y^{p,q}$, we have $\iota_{r_1}\eta = 1$, and $\iota_{r_1}d\eta = 0$. Moreover η will descend to $Y^{p,q}$ and be the contact 1-form κ there. This Reeb will not admit an SE metric, but it does help us to get a better handle on the geometry of our SE manifold.

- The orbifold Base of $Y^{p,q}$.

In contrast to T , the Reeb vector field (33) does not in general induce a free action, even though it is nowhere zero. Indeed, if one takes the quotient of $Y^{p,q}$ with respect to r_1 , one gets the weighted projective space

$$\mathbb{C}P(p + q, p - q) \times \mathbb{C}P(1, 1).$$

In fact, this is easier to see if one takes the double quotient of $S^3 \times S^3$ with respect to both T and r_1 . The spaces $\mathbb{C}P(r, s)$ are orbifolds except when $r = s$. This special case happens when $p = 1$ and $q = 0$, which leads to the SE manifold called $T^{1,1}$, whose cone is the well-known conifold

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 0. \tag{34}$$

In fact $T^{1,1}$ is a regular SE manifold, since the Reeb orbit is closed and thus $T^{1,1}$ is a $U(1)$ -fibration over $S^2 \times S^2$ of degree 1 and 1.

For an orbifold, we can still apply some version of the Riemann–Roch theorem, and our computation of the super-determinant in Sect. 5.3 is similar to this in spirit. In the next section, we will instead look at the possibility of presenting $Y^{p,q}$ as a $U(1)$ -fibration. Our result will concur with the geometric interpretation of the explicit metric found in [25].

3.2. *Looking for free $U(1)$ in $Y^{p,q}$.* Next we try to find free $U(1)$ -actions in this geometry. Let α be the $U(1)$ with charge $[a, c, b, d]$

$$\begin{array}{c|cccc} & z_1 & z_2 & z_3 & z_4 \\ \hline T & p+q & p-q & -p & -p \\ \alpha & a & c & b & d \\ \hline R1 & p+q & p-q & p & p \end{array}.$$

The result is that if $\gcd(p, q) = 1$, then a free $U(1)$ has charge vector

$$[a, c, b, d] = [a, -a - 2b, b, b], \quad \text{where } a, b, c, d \in \mathbb{Z}, \quad (a + b)p + bq = 1. \tag{35}$$

For the proof, note first that for α to be free, it can be nowhere parallel with T , otherwise it will have a zero on the quotient. Even when this is satisfied, at special points there might still be discrete stability groups. For example, when $z_2 = z_4 = 0$, if (where $k, l \in \mathbb{Z}$)

$$\begin{aligned} \frac{\theta}{2\pi}a &= \frac{\phi}{2\pi}(p+q) + k, \\ \frac{\theta}{2\pi}b &= \frac{\phi}{2\pi}(-p) + l, \end{aligned}$$

then a rotation $e^{i\theta\alpha}$ can be undone by $e^{-i\phi T}$. So if $\theta/(2\pi)$ and $\phi/(2\pi)$ have rational (but non-integer) solutions, then there is a non-trivial stability group. The solution to these equations is given by

$$\begin{aligned} \frac{\theta}{2\pi} &= \frac{pk + (p+q)l}{ap + b(p+q)}, \\ \frac{\phi}{2\pi} &= \frac{-bk + al}{ap + b(p+q)}. \end{aligned}$$

To exclude any non-integer solution, we need to have $\theta/(2\pi)$ to be an integer for *any* k and l , but as p, q are coprime, one can find k, l s.t. $kp + l(p+q) = 1$, thus the denominator $ap + b(p+q)$ must be ± 1 . And similar reasonings lead to three more equations

$$\begin{cases} (a+b)p + bq = \pm 1 \\ (a+d)p + dq = \pm 1, \\ (c+b)p - bq = \pm 1 \\ (c+d)p - dq = \pm 1. \end{cases}$$

Let us first assume that the rhs of the first pair is $+1$, which implies $b = d$, and that the rhs of the second pair must also have the same sign. Take this first to be $+1$, then we have

$$(a+b)p + bq = 1, \quad (c+b)p - bq = 1,$$

which would have a solution only when $p = 1, 2$, so we do not consider this possibility. Now take the -1 option

$$(a+b)p + bq = 1, \quad (c+b)p - bq = -1,$$

which admits a family of solutions

$$[a, c, b, d] = [a, -a - 2b, b, b], \quad (a + b)p + bq = 1.$$

One possibility left is when the right hand side of the first pair have +1, -1

$$\begin{cases} (a + b)p + bq = +1 \\ (a + d)p + dq = -1 \end{cases}$$

then subtracting the two equations one get $(b - d)(p + q) = 2$, which again admits no solutions since $p > q > 0$. Thus we have only the set of solutions (35) that works for generic p, q .

3.3. *The base of the U(1) fibration.* We investigate the base of the $U(1)$ fibration by a taking further quotient of $Y^{p,q}$ by α . To this end, one may consider modding out $S^3 \times S^3$ by any $SL(2, \mathbb{Z})$ combinations of the two $U(1)$'s T and α . We choose

$$\begin{aligned} -aT + (p + q)\alpha &= [0, -2, 1, 1], \\ -bT - p\alpha &= [-1, 1, 0, 0], \quad \det \begin{pmatrix} -a & p + q \\ -b & -p \end{pmatrix} = 1. \end{aligned} \tag{36}$$

This shows that the quotient consists of a twisted product $S^2 \rtimes S^2$. Indeed, from the first equation, one sees that the first S^2 is fibred over the second one with degree -2 , as such we will call the two S^2 's the *fibre* and *base* S^2 respectively. From the second equation, the complex structure of the fibre S^2 is quite unconventional, if we choose it to agrees with the standard one at $z_2 = 0$ then at $z_1 = 0$, it will be opposite, see section 5.3 in [23] for more discussion on this fact.

We can cover the base of the $U(1)$ -bundle with four patches, let U_{00} denote the patch with $z_2 \neq 0, z_4 \neq 0$, and U_{10} be the patch $z_1 \neq 0, z_4 \neq 0$, etc. For later computation of the index, we pick on patch U_{00} two vector fields $e_3 = [0, 0, 1, 0]$ and $e_1 = [1, 0, 0, 0]$, and we tabulate the expression of these vector fields in other patches (change of coordinates)

| | e_3 | e_1 |
|----------|---|-------------------------------|
| U_{00} | [0, 0, 1, 0] | [1, 0, 0, 0] |
| U_{01} | $[2, 0, 0, -1] + (q - p)\alpha - (a + 2b)T$ | [1, 0, 0, 0] |
| U_{10} | [0, 0, 1, 0] | $[0, 1, 0, 0] + p\alpha + bT$ |
| U_{11} | $[0, 2, 0, -1] - aT + (p + q)\alpha$ | $[0, 1, 0, 0] + bT + p\alpha$ |

(37)

where T of course descends to zero on $Y^{p,q}$. In the appendix, we will see that this change of coordinates can be precisely reproduced by analyzing the explicit SE metric.

3.4. *The Reeb that admits a Sasaki–Einstein metric.* The above Reeb vector field will lead only to a Sasaki structure on the manifold, but not an Einstein metric. In a beautiful paper [26], it is shown how to find the Reeb vector that admits an SE metric, which for the computation of partition function is sufficient.

The SE manifold $Y^{p,q}$ correspond to what is known as being of the *Reeb type*, meaning that it is a torus fibration over a base and that the Reeb vector field is generated

by the torus action. Let us again start from \mathbb{C}^4 with the standard Kähler structure given by $d\eta$. Performing the symplectic reduction as before

$$\mathbb{C}^4 // T = \mu_T^{-1}(0)/T, \quad \mu_T = (p + q)|z_1|^2 + (p - q)|z_2|^2 - p(|z_3|^2 + |z_4|^2),$$

one obtains a cone over a base $Y^{p,q}$. This cone inherits a Kähler structure from \mathbb{C}^4 , which will be *held fixed* in the deformation to come later. One can choose the following effectively acting 3-tori

$$T_1 : [0, 0, 1, 0], \quad T_2 = [-1, 0, 1, 0], \quad T_3 = \alpha, \tag{38}$$

there is quite some freedom in the choice of the tori, here we are following [23]. It is easy to write down the moment map

$$\mu_1 = |z_3|^2, \quad \mu_2 = |z_3|^3 - |z_1|^2, \quad p\mu_3 = |z_1|^2 - |z_2|^2,$$

where for the third relation, we used the second equation of (36). From $|z_i|^2 \geq 0$, the range of μ_i is a polytope cone, described as

$$\mathcal{C} = \{[\mu_1, \mu_2, \mu_3] \mid \boldsymbol{\mu} \cdot \mathbf{v}_a \geq 0\}, \tag{39}$$

where \mathbf{v}_a , $a = 1 \dots 4$ are the inward pointing normal of the facets of the cone

$$\mathbf{v}_1 = [1, 0, 0], \quad \mathbf{v}_2 = [1, -2, -p + q], \quad \mathbf{v}_3 = [1, -1, -p], \quad \mathbf{v}_4 = [1, -1, 0].$$

Then all possible Reeb vectors must live in the interior of the cone \mathcal{C}^* generated by \mathbf{v}_a , i.e.

$$\mathbf{r} = b_1 T_1 + b_2 T_2 + b_3 T_3, \quad \mathbf{b} = \sum \lambda_a \mathbf{v}_a, \quad \lambda_a > 0.$$

This is easy to see, let μ_R be the moment map corresponding to \mathbf{r} , then $\mu_R = \text{const}$ will intersect the cone \mathcal{C} at a polygon precisely when $\mathbf{b} \in (\mathcal{C}^*)^\circ$. This polygon will serve as the base of the torus fibration mentioned earlier.

In deforming the Reeb vector, one holds the Kähler form fixed and deforms the complex structure and hence the Kähler metric of the cone. One can write down the Einstein–Hilbert action of the metric

$$S[\mathbf{r}] = \int_{Y^{p,q}} d\mu (R - 12),$$

where R is the Ricci scalar and the stationary point of the action gives the Einstein metric $R_{mn} = 4g_{mn}$. The insight from [26] is that the action only depends on the Reeb vector and the action has a unique stationary point within the interior of the dual cone \mathcal{C}^* . Quite remarkably, the value of the Hilbert–Einstein action can be written as some simple elementary geometrical quantities of the moment cone \mathcal{C} , which can in turn be related to the volume of the $Y^{p,q}$ space

$$S[\mathbf{r}] = 8(b_1 - 2)\text{Vol } Y^{p,q}[\mathbf{r}], \tag{40}$$

where $\text{Vol } Y^{p,q}[\mathbf{r}]$ is a function of the Reeb. With our choice of basis of $U(1)$'s in (38), the parameters b_i are related to the general equivariant parameters as

$$b_1 = \omega_1 + \omega_2 + \omega_3 + \omega_4, \quad b_2 = -\omega_1 - \omega_2 - 2\omega_4, \quad b_3 = -p\omega_2 + (q - p)\omega_4,$$

and the volume is computed to be

$$\begin{aligned} \varrho[R] &= \frac{\text{Vol } Y^{p,q}[R]}{\text{Vol } S^5} \\ &= \frac{(p^2\omega_1 + p^2\omega_2 + p^2\omega_3 + p^2\omega_4 - pq\omega_1 + pq\omega_2 - q^2\omega_3 - q^2\omega_4) p}{(p\omega_1 + p\omega_4 + q\omega_4)(p\omega_2 + p\omega_4 - q\omega_4)(p\omega_2 + p\omega_3 - q\omega_3)(p\omega_1 + p\omega_3 + q\omega_3)}. \end{aligned} \tag{41}$$

The condition that \mathbf{b} be within the dual cone translates to some conditions on ω_i

$$\begin{aligned} p(\omega_2 + \omega_4) &> q\omega_4, \quad p(\omega_2 + \omega_3) > q\omega_3, \quad p(\omega_3 + \omega_1) + q\omega_3 > 0, \\ p(\omega_4 + \omega_1) + q\omega_4 &> 0, \end{aligned} \tag{42}$$

as a by product, one sees that any free $U(1)$ found in Sect. 3.2 will necessarily be disqualified as a Reeb.

To continue, one can then find the unique Reeb vector field admitting the SE metric by finding the stationary point of (40), the result is

$$R = \left(\frac{3}{2} - \frac{1}{2(p+q)\ell}\right)(e_3 + e_4) + \frac{1}{(p+q)\ell}e_2, \tag{43}$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}} = \frac{2p^2 - 3q^2 + p\sqrt{4p^2 - 3q^2}}{9q(p^2 - q^2)}. \tag{44}$$

This is exactly the one found using the explicit metric from [25]. One can plug back the value of R and get the volume of $Y^{p,q}$ with the SE metric

$$\begin{aligned} \varrho &= \frac{\text{Vol}_{Y^{p,q}}}{\text{Vol}_{S^5}} = \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]} \\ &= \frac{1}{27p^2(q^2 - p^2)}(-8p^3 + 9pq^2 - (4p^2 - 3q^2)^{3/2}). \end{aligned} \tag{45}$$

As for $Y^{p,q}$ with a general R , we will call it the *squashed* $Y^{p,q}$, whose volume is given by (41).

Notice that ℓ here appears in the explicit metric as the period of the fibre coordinate of the $U(1)$ -fibre over the base $S^2 \times S^2$, see Appendix C. Moreover, since ℓ is generically irrational, the resulting Reeb vector field is of an *irregular* type, i.e. its orbit is not closed. But for the $T^{1,1}$ case ℓ^{-1} goes to zero when one sets $p = 1$ and $q = 0$, and the Reeb vector $R = 3/2[0, 0, 1, 1]$ is freely acting and has closed orbit.

4. The Cohomological Complex and Localisation

The key step of the localisation procedure is to make a change of variable in the fields, so that the fields would behave as coordinates and their conjugate momenta (both even and odd) on some space. In this process, a combination of the susy transformation will behave like the equivariant differential, and thus one has the standard localisation in equivariant cohomology.

The process of change of variable is given in [2] and here we will just sketch the steps.

4.1. *Vector-multiplet.* We first define some geometrical quantities using the Killing spinor. First, define the vector field

$$R^P = -\xi_I \Gamma^P \xi^I.$$

The Killing spinor equation implies that R is a Killing vector field, in fact, it is the Reeb vector field of our SE geometry,⁴ from the way our Killing spinor is constructed in Sect. 2.3. Furthermore, the spinor bi-linear

$$t^{IJ} \xi_I \Gamma_{mn} \xi_J = -\frac{1}{2} J_{mn}$$

is the horizontal Kähler 2-form.⁵ It satisfies

$$\iota_R J = 0, \quad *_R J = J,$$

where $*_R$ is defined in (21).

The gaugino λ_I in (7) can be converted into a 1-form and a 2-form:

$$\Psi_m = \xi_I \Gamma_m \lambda^I, \quad \chi_{mn} = \xi_I \Gamma_{mn} \lambda^I + R_{[m} \xi_I \Gamma_{n]} \lambda^I, \tag{46}$$

where $R_m = g_{mn} R^n$ is the contact 1-form κ . The 1-form Ψ is an unrestricted 1-form, the 2-form χ satisfies the same conditions as J :

$$\iota_R \chi = 0, \quad \iota_R * \chi = \chi. \tag{47}$$

The formula (46) can be inverted to write λ_I as

$$\lambda_I = -\frac{1}{2} \xi^J (\xi_J \Gamma^{mn} \xi_I) \chi_{mn} + (\Gamma^m \xi_I) \Psi_m, \tag{48}$$

The fermion λ_I has eight real components which is the same as the five components from Ψ plus 3 more from χ .

In the new variables the susy transformations (7) can be rewritten as

$$\begin{aligned} \delta A_m &= i \Psi_m, & \delta \Psi_m &= -R^n F_{nm} + D_m \sigma, \\ \delta \chi_{mn} &= H_{mn}, & \delta H_{mn} &= -i \mathcal{L}_R^A \chi_{mn} - [\sigma, \chi_{mn}], \\ \delta \sigma &= -i R^m \Psi_m, \end{aligned} \tag{49}$$

where $\mathcal{L}_R^A = L_R + i[\ , \iota_R A]$. Here the 2-form H is a redefinition of D

$$H_{mn} = 2(F_H^+)_{mn} + (\xi^I \Gamma_{mn} \xi^J) (D_{IJ} + \frac{2}{r} t_{IJ} \sigma), \tag{50}$$

where

$$F_H^+ = \frac{1}{2} (1 + *_R) F - \frac{1}{2} (\kappa \wedge \iota_R F)$$

is the horizontal self-dual component of F . The field H satisfies exactly the same conditions (47) as χ .

⁴ The vector field— R takes the place of v used in [2].

⁵ Recall that we use the same letter for the horizontal complex structure and the Kähler 2-form.

The square of the transformations (49) reads

$$\delta^2 = -iL_R + G_{i(\sigma + \iota_R A)}, \tag{51}$$

where $G_{i(\sigma + \iota_R A)}$ is a gauge transformation with parameter $i(\sigma + \iota_R A)$. With our conventions G_ϵ acts as

$$G_\epsilon A = d\epsilon - i[A, \epsilon], \quad G_\epsilon \phi = -i[\phi, \epsilon],$$

where ϕ is any field in the adjoint.

As promised, (51) shows that the susy transformation written in the new variables can be regarded as an equivariant differential, with the fields A_m, χ_{mn} being the coordinates of some space and Ψ_m, H_{mn} their conjugate momenta. The field σ is a bit special, since the combination $\Phi = \sigma - \iota_\nu A$ is annihilated by δ , but this field will also be incorporated as a momentum once the ghost sector is included [1].

4.2. Hyper-multiplet. For the hyper-multiplet, we would do what is opposite to the previous section, and combine the scalar field q_I in (13) with the Killing spinors and formulate the cohomological complex in terms of spinors (the goal is always to work with a complex that is $SU(2)_R$ -singlet). We define a new bosonic spinor field q

$$q = \xi_I q^I, \quad q_I = -2\xi_I q.$$

From the reality condition satisfied by ξ_I and q_I one can see that the spinor field q now satisfies the same reality condition as ψ .

The susy transformation (13) expressed in terms of q and ψ will only close on-shell, to mend this we need introduce a bosonic spinor field \mathcal{F} of opposite $\gamma_5 = -R \cdot \Gamma$ eigenvalue. One can obtain an off-shell susy enlarging (13)

$$\begin{aligned} \delta q^A &= iP_+ \psi^A, \\ \delta \psi^A &= -\frac{1}{4r} J_{pq} (\Gamma^{pq} q^A) + (\not{D} + i\sigma) q^A + \mathcal{F}^A, \\ \delta \mathcal{F}^A &= -iP_- \not{D} \psi^A - \sigma P_- \psi^A - \Psi^m (\Gamma_m + R_m) q^A, \end{aligned} \tag{52}$$

where we use the projector $P_\pm = 1/2(1 \pm \gamma_5)$ and $P_+ q = q, P_- \mathcal{F} = \mathcal{F}$. Notice that σq^A should be understood as $\sigma^A_B q^B$ and similarly for the term involving Ψ_m . The transformations above square to the following:

$$\delta^2 \Phi = (-iL_R^s - \sigma - \iota_R A) \Phi, \quad \Phi = \{q, \psi, \mathcal{F}\}, \tag{53}$$

where L_R^s is the spinor Lie derivative, see Appendix A. After a further linear shift of \mathcal{F} , one can break the middle line of (52) according to its eigenvalue under γ_5 , and get a nice complex that parallels (49)

$$\begin{aligned} \delta q^A &= i\psi_+^A, & \delta \psi_+^A &= (-L_R^s + i(\sigma + \iota_R A)) q^A, \\ \delta \psi_-^A &= \tilde{\mathcal{F}}^A, & \delta \tilde{\mathcal{F}}^A &= (-iL_R^s - (\sigma + \iota_R A)) \psi_-^A. \end{aligned} \tag{54}$$

The above complex is written in terms of the fields q, ψ and $\tilde{\mathcal{F}}$ which satisfy the reality conditions. We can solve these reality conditions in terms of the unconstrained fields as we did in (12). Let

$$q^A = [q^\alpha, -Cq_\beta^*]^T, \quad \psi^A = [\psi^\alpha, -C\psi_\beta^*]^T, \quad \tilde{\mathcal{F}}^A = [\tilde{\mathcal{F}}^\alpha, -C\tilde{\mathcal{F}}_\beta^*]^T. \tag{55}$$

Now we can rewrite the complex (54) in terms of the new fields and it looks exactly the same, except for the change of indices $A \rightarrow \alpha$

$$\delta q^\alpha = i\psi_+^\alpha, \quad \delta\psi_+^\alpha = \dots$$

One property that we will need for the transformations is that it acts holomorphically, in that it does not mix $q^\alpha, \psi^\alpha, \tilde{\mathcal{F}}^\alpha$ with their conjugates. This point will be important later when we decide over what spaces do we compute the determinant of the operator δ^2 , see next section.

4.3. *Localisation.* The localisation argument has now become fairly standard, so we will not give all the details. Take a finite dimensional example

$$\int d^n x d^n \psi \omega(x, \psi),$$

where x is regarded as the coordinate (even or odd) of certain space and ψ its momentum of the opposite statistics. Assume that $\omega(x, \psi)$ is invariant under an odd symmetry

$$\delta x = \psi, \quad \delta\psi = L_v x.$$

Pick a function V odd, such that $\delta^2 V = 0$, one can insert into the integral a factor

$$\int d^n x d^n \psi \omega(x, \psi) e^{-t\delta V}, \quad \delta^2 V = 0,$$

without changing the value of the integral. The last statement can be seen by differentiating with respect to t , and using $\int \delta(\dots) = 0$. If one then sends $t \rightarrow \infty$, the integral will be concentrated at the critical points of the bosonic part of δV

$$\int d^n x d^n \psi \omega(x, \psi) e^{-t\delta V} = \sum_{\text{cr pt}} \omega \text{sdet}^{-1/2}((\delta V)''),$$

where of course we assume that the critical points are non-degenerate.

Furthermore, at each critical point, the equality $\delta^2 V = 0$ leads to certain relations among the coefficients of Hessian $(\delta V)''$, and consequently the simplification

$$\text{sdet}^{-1/2}((\delta V)') = \text{sdet}_x^{-1/2}(\delta^2),$$

where sdet_x means to take the super-determinant only on the coordinates.

In the case one has complex coordinates x, \bar{x} , complex momenta $\psi, \bar{\psi}$ and δ acts holomorphically, then a similar argument as above gives, up to a constant phase, the determinant

$$\text{sdet}^{-1/2}((\delta V)') = \text{sdet}_{\text{hol } x}^{-1}(\delta^2),$$

where the subscript $\text{hol } x$ means taking the super-determinant only on the holomorphic coordinates. Furthermore, one has $\text{sdet}_{\text{hol } x}(\delta^2) = \text{sdet}_{\text{ahol } x}(\delta^2)$, again up to a constant phase. Note that this phase can be computed for a finite dimensional case, but for an infinite dimensional problem, we have yet no means to handle it systematically, this problem is left open both in [2] and in the current work.

To summarise, the final result of the integral is just the sum of contribution from each critical point of the above form. The first case above applies to the vector multiplet while the second to the hyper-multiplet.

4.4. *Localisation locus.* To put this knowledge to practice, we look at the vector multiplet first. One adds to the action an exact term

$$S_{\text{vec}} \rightarrow S_{\text{vec}} + t \delta \int V_{\text{vec}},$$

$$V_{\text{vec}} = \text{Tr} \left[\frac{1}{2} \Psi \wedge *(-\iota_R F - D\sigma) - \chi \wedge *H + 2\chi \wedge *F \right].$$

The bosonic part of δV_{vec} is

$$\delta V_{\text{vec}}|_{\text{bos}} = \text{Tr} \left[\frac{1}{2} (-\iota_R F + D\sigma) \wedge *(-\iota_R F - D\sigma) - H \wedge *H + 2H \wedge *F \right]$$

$$= \text{Tr} \left[\frac{1}{2} (\iota_R F)^2 - \frac{1}{2} (D\sigma)^2 + F_H^+ \wedge *F_H^+ - (H - F_H^+) \wedge *(H - F_H^+) \right].$$

One can integrate out H , while the rest is a perfect square,⁶ so the localisation locus is

$$F_H^+ = 0, \quad \iota_R F = 0, \quad D\sigma = 0. \tag{56}$$

The first two equations describe the so called ‘contact instanton’ while the last says σ is a covariant constant. To keep the discussion lucid, we have not mentioned the gauge fixing ghost sector, but the details can be found in [27].

For the geometry S^5 , the Reeb vector is generated by a free $U(1)$ action and the quotient is $\mathbb{C}P^2$, one calls these SE manifolds *regular*. From the analysis of [2], the contact instantons on S^5 correspond to usual instantons on $\mathbb{C}P^2$, and only the irreducible ones contribute. The argument leading to this relies on a specific choice of gauge. To reach this gauge one needs to use the regularity condition. The SE manifolds $Y^{p,q}$ are generically *irregular* as the Reeb vector field does not have closed orbit. In the best case, for certain values of p, q , when the Reeb generates a locally free action, then the manifolds are a $U(1)$ fibration over a base orbifold. Due to these concerns, it is not clear to us whether the contact instanton on $Y^{p,q}$ can be thought of as usual instantons on the base orbifold in the quasi-regular case, let alone the irregular case. But at any rate, it is likely to be more advantageous to reverse the game and study 4D instantons by lifting them to 5D rather than pushing 5D instantons to 4D. We leave this subject for a future enterprise and focus on the zero-instanton sector for the rest of the paper. We do want to point out that the vanishing argument proved in Sect. 4.5 is valid for all contact instanton backgrounds.

- For the hyper-multiplet, one can add to the action a δ -exact term

$$S_{\text{hyp}} \rightarrow S_{\text{hyp}} + t \delta \int V_{\text{hyp}}, \quad \text{where } V_{\text{hyp}} = \frac{1}{2} (\delta\psi^A)^\dagger \psi^A.$$

The bosonic part of δV_{hyp} is⁷

$$\delta V_{\text{hyp}}|_{\text{bos}} = \left| \left(\frac{1}{4} \not{J} + \not{D} \right) q \right|^2 + |\sigma q|^2 + |\mathcal{F}|^2,$$

⁶ Keep in mind that σ has been Wick rotated and hence is valued in $i\mathfrak{g}$.

⁷ In writing δV_{hyp} we used the original complex (52), in particular we used the field \mathcal{F} instead of the shifted variable $\tilde{\mathcal{F}}$. This is different from what we did in [2].

where we have already used (55) to write all fields in terms of their unconstrained components. In the manipulation above σ and $\tilde{\mathcal{F}}$ are Wick rotated, which is crucial for having a complete square as above.

Since δV is positive definite the localization locus is given by the following equations

$$\left(\frac{1}{4}\mathcal{J} + \mathcal{D}\right)q^\alpha = 0, \quad \sigma^\alpha_\beta q^\beta = 0, \quad \mathcal{F}^\alpha = 0. \tag{57}$$

We first point out that by applying the chiral projector to the first equation, one gets two conditions

$$P_- \mathcal{D}q = 0, \quad \left(-R \cdot D - \frac{1}{4}\mathcal{J}\right)q = \left(-L^s_R + i\iota_R A\right)q = 0, \tag{58}$$

in fact if one had used the complex (54), then $P_- \mathcal{D}q$ is absorbed into $\tilde{\mathcal{F}}$.

It was shown in [2] that this set of conditions implies $q = \mathcal{F} = 0$ at the $A = 0$ configuration. Now we make a digression and prove a vanishing theorem that strengthens this result, then we will resume with the partition function.

4.5. The vanishing argument for hyper-multiplet. We start from the equation $(\mathcal{D} + \mathcal{J}/4)q = 0$ and show that $q = 0$ at an instanton background. Consider the intergral

$$\begin{aligned} 0 &= \int_M \left(\frac{1}{4}\mathcal{J} + \mathcal{D}\right)q^\dagger \cdot \left(\frac{1}{4}\mathcal{J} + \mathcal{D}\right)q = \int_M q^\dagger \left(-\frac{1}{4}\mathcal{J} + \overleftarrow{\mathcal{D}}\right) \left(\frac{1}{4}\mathcal{J} + \mathcal{D}\right)q \\ &= \int_M q^\dagger \left(-\mathcal{D}^2 - \frac{1}{4}\mathcal{J}\mathcal{D} + \frac{1}{4}\overleftarrow{\mathcal{D}}\mathcal{J} - \frac{1}{16}\mathcal{J}^2\right)q = \int_M q^\dagger \left(-\mathcal{D}^2 + \frac{1}{8}\mathcal{J}^2 - \frac{1}{16}\mathcal{J}^2\right)q \\ &= \int_M q^\dagger \left(\frac{1}{16}\mathcal{J}^2 - \mathcal{D}^2\right)q. \end{aligned}$$

The two terms in the integral are

$$\mathcal{D}^2 = D^2 - 5 - \frac{i}{2}\mathcal{F}, \quad \mathcal{J}^2 = -16P_+.$$

We also put the gauge field in an instanton configuration $\iota_v F = 0 = F_H^+$. Then we have

$$q^\dagger \mathcal{F} q = q^\dagger (F_H^+)_{mn} \Gamma^{mn} q = 0.$$

Assembling everything altogether

$$0 = \int_M q^\dagger \left(\frac{1}{16}\mathcal{J}^2 - D^2 + 5\right)q = \int_M q^\dagger \left(-D^2 + 4\right)q = \int_M (D_m q)^\dagger (D^m q) + 4 \int_M q^\dagger q.$$

So we must have $q = 0$.

Now we can write down schematically the perturbative part of the partition function. From the discussion of Sect. 4.3, we first need to evaluate the classical action at the localisation locus. The hyper-multiplet action completely vanishes, while from the vector multiplet action (9), we get only

$$S_{cr} = \int d^5x \sqrt{g} \frac{1}{g_{YM}^2} \text{Tr} \left[-\frac{1}{2} D_{IJ} D^{IJ} + \frac{2}{r} \sigma t^{IJ} D_{IJ} - \frac{10}{r^2} t^{IJ} t_{IJ} \sigma^2 \right] \Big|_{D_{IJ} = -2t_{IJ} \sigma/r, \sigma = \text{const}},$$

where $D_{IJ} = -2t_{IJ}\sigma/r$ is deduced from the change of variable (50), and that at the critical point $H = F_H^+ = 0$. Thus we get

$$S_{cr} = \frac{\text{Vol}_{Y^{p,q}}}{g_{YM}^2} \text{Tr} \left[-\frac{16}{r^2} t^{IJ} t_{IJ} \sigma^2 \right] = -\frac{8\text{Vol}_{Y^{p,q}}}{g_{YM}^2 r^2} \text{Tr}[\sigma^2].$$

The volume of $Y^{p,q}$ is given in Eq. 45. So we trade $\text{Vol}_{Y^{p,q}}$ for $q\text{Vol}_{S^5} = \pi^3 q r^5$

$$S_{cr} = -\frac{8\pi^3 r q}{g_{YM}^2} \text{Tr}[r^2 \sigma^2]. \tag{59}$$

Putting together also the determinant factor at the localisation locus, we get (keep in mind that σ is purely imaginary)

$$Z_{\text{pert}} = \int_{\text{ig}} d\sigma \exp \left(\frac{8\pi^3 q r}{g_{YM}^2} \text{Tr}[r^2 \sigma^2] \right) \frac{\text{sdet}'_{\text{vec}}(-iL_R - \sigma)^{1/2}}{\text{sdet}'_{\text{hyp}}(-iL_R^s - \sigma)},$$

where the super-determinant for the vector multiplet is taken over the sections of the complex

$$E : 0 \rightarrow \Omega^0(M) \rightarrow \Omega^1_H(M) \rightarrow \Omega^{2+}_H(M) \rightarrow 0, \tag{60}$$

and sdet' denotes the exclusion of constant modes.⁸ In contrast, the super-determinant for the hyper-multiplet is taken over the sections of the spin bundle

$$W_{\text{can}} : 0 \rightarrow \Omega^{0,0}(M) \rightarrow \Omega^{0,1}_H(M) \rightarrow \Omega^{0,2}_H(M) \rightarrow 0.$$

In fact one can directly see that $E \simeq W_{\text{can}} \oplus W_{\text{can}}^*$, which shows that if one considers the hyper-multiplet in the adjoint, there will be extra cancellation between the vector and hyper-multiplet.

We can do some re-writings of the result, first it is convenient to use the dimensionless combination $x = r\sigma$, and factor out (an infinite power of) r , and consider the determinant of $(-irL_R^s - x)$ and $(-irL_R - x)$ instead. Secondly, since the function to be integrated is ad-invariant, one can write the integral over \mathfrak{g} as an integral over \mathfrak{h} with a Jacobian factor

$$Z_{\text{pert}} = \frac{1}{|W|} \frac{\text{Vol}(G)}{\text{Vol}(T)} \int_{\text{it}} dx \left(\prod_{\beta > 0} \langle \beta, x \rangle^2 \right) \exp \left(\frac{8\pi^3 r q}{g_{YM}^2} \text{Tr}[x^2] \right) \frac{\text{sdet}'_{\text{vec}}(-irL_R - x)^{1/2}}{\text{sdet}'_{\text{hyp}}(-irL_R^s - x)}, \tag{61}$$

where β runs over positive roots.

The remainder of the paper is about computing these two determinants, we write down here the result, *discarding all the irrelevant multiplicative constants*. Let $R =$

⁸ See Appendices C.2 and C.3 in [27]. We only consider the trivial background $A = 0$, so the gauge sector of [27] derived for S^5 is still applicable.

$[\omega_1, \omega_2, \omega_3, \omega_4]$, assuming of course that R is in the interior of the cone dual to the moment map cone (see Sect. 3.4)

$$\begin{aligned}
 Z_{\text{pert}} &= \int_{it} dx \prod_{\beta > 0} \langle \beta, x \rangle^2 \cdot \exp\left(\frac{8\pi^3 r_Q[R]}{g_{YM}^2} \text{Tr}[x^2]\right) \frac{P_{\text{vec}}}{P_{\text{hyp}}}, \\
 P_{\text{hyp}} &= \det_{\underline{R}} \prod_{i,j,k,l \in \Lambda^+} \left((i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4 + \frac{1}{2}(\omega_1 + \omega_2 + \omega_3 + \omega_4))^2 - x^2 \right), \\
 P_{\text{vec}} &= \det_{\text{adj}} \prod_{i,j,k,l \in \Lambda_0^+} \left((i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4)^2 - x^2 \right) \\
 &\quad \cdot \prod_{i,j,k,l \in \Lambda_1^+} \left((i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4)^2 - x^2 \right)^{1/2}, \tag{62}
 \end{aligned}$$

where the lattices for the products are defined

$$\begin{aligned}
 \Lambda^+ &= \{i, j, k, l \in \mathbb{Z}_{\geq 0} \mid i(p+q) + j(p-q) = kp + lp\}, \\
 \Lambda_0^+ &= \{i, j, k, l \in \mathbb{Z}_{> 0} \mid i(p+q) + j(p-q) = kp + lp\}, \\
 \Lambda_1^+ &= \Lambda^+ \setminus (\Lambda_0^+ \cup \{0, 0, 0, 0\}).
 \end{aligned}$$

We have also used the volume associated with a general Reeb correspondingly⁹ given in (41). If one sets

$$\omega_1 = 0, \quad \omega_2 = \frac{1}{(p+q)\ell}, \quad \omega_3 = \omega_4 = \frac{3}{2} - \frac{1}{2(p+q)\ell},$$

one obtains the partition function of the supersymmetric theory. However one may leave the parameters $(\omega_1, \omega_2, \omega_3, \omega_4)$ unfixed and study how the partition function responds to the $U(1)$ isometries of the geometry.

We can do some more rewriting and relate these infinite products to certain generalisation of the Barnes’s function, or triple sine function. For P_{vec} factor, we can write $\det_{\text{adj}} f(x) = f(0)^{\text{rk}G} \prod_{\beta \in \text{roots}} f(\langle \beta, x \rangle)$, but $f(0)^{\text{rk}G}$ is a (non-zero) multiplicative constant that can be discarded. We will also combine the factor $\prod_{\beta > 0} \langle \beta, x \rangle^2$ with P_{vec} to get

$$P_{\text{vec}} \prod_{\beta > 0} \langle \beta, x \rangle^2 = \det'_{\text{adj}} S^\Lambda(x | \omega_1, \omega_2, \omega_3, \omega_4),$$

where $\det'_{\text{adj}} f(x)$ is short for $\prod_{\beta \in \text{roots}} f(\langle \beta, x \rangle)$, i.e. the determinant taken in the adjoint representation with the zero weight subspace excluded. The function S^Λ is defined as

$$\begin{aligned}
 S^\Lambda(x | \omega_1, \omega_2, \omega_3, \omega_4) &= \prod_{(i,j,k,l) \in \Lambda^+} \left(i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4 + x \right) \\
 &\quad \times \prod_{(i,j,k,l) \in \Lambda_0^+} \left(i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4 - x \right). \tag{63}
 \end{aligned}$$

⁹ Surely, for a general Reeb, one does not have an SE metric and hence no susy a priori, but one may take the cohomological complexes (49) and (54) as the starting point and compute the partition function.

With this new function, the factor P_{hyp} is just

$$P_{\text{hyp}} = \det_R S^\Lambda(x + \frac{1}{2}(\omega_1 + \omega_2 + \omega_3 + \omega_4) | \omega_1, \omega_2, \omega_3, \omega_4).$$

Thus finally we get the answer in the form (1) presented in introduction. Since mass M for hypermultiplet can be generated through the standard trick of coupling the hypermultiplet to an auxiliary vector multiplet we end up with the general expression (1) for the massive hypermultiplet. We will leave the investigation of the function $S^\Lambda(x | \omega_1, \omega_2, \omega_3, \omega_4)$ to the future work. Here we will be content with establishing that we do get a sensible matrix model, by computing in Sect. 6 the asymptotic behaviour of the products, the result is

$$Z_{\text{pert}} = \int_{\text{it}} dx \exp\left(\frac{8\pi^3 r \varrho}{g_{YM}^2} \text{Tr}[x^2]\right) \exp(\pi V(x)), \tag{64}$$

$$V(x) \sim \text{Tr}_{\text{adj}}\left(|x|\left(\frac{q}{2p}\ell + \frac{3}{4}\varrho\right) - \frac{|x|^3}{6}\varrho\right) - \text{Tr}_R\left(|x|\left(\frac{q}{2p}\ell - \frac{3}{8}\varrho\right) - \frac{|x|^3}{6}\varrho\right),$$

which is similar to the case of S^5 . There is a limit of how big the hyper-multiplet representation can be, beyond this limit, the potential flips over and the matrix is ill-defined. If the hyper multiplet is in the adjoint, then the cubic term cancels, and the linear potential has coefficient $9/4\varrho$, we have exact matching with the S^5 result up to volume factor. If we will look at the large volume limit (analogously to the analysis in [13]) then we reproduce the known flat case results.

Let us also set $p = 1$ and $q = 0$ to get the result for $T^{1,1}$. In particular, one has $\varrho = 16/27$ and $q\ell = 4/9$, and the potential tends to

$$V(x) \sim \text{Tr}_{\text{adj}}\left(\frac{2}{3}|x| - \frac{8}{81}|x|^3\right) + \text{Tr}_R\left(\frac{8}{81}|x|^3\right).$$

5. Computation of the Partition Function

5.1. *Spectrum of L_R^s .* The computation of the super-determinant requires one to find the spectrum of the operators L_v and L_v^s . The former is quite straightforward, while the latter will be shown to be equivalent to the former up to a constant but important shift.

We pick the Killing spinor satisfying $D_X \psi_0 = -i/2X\psi_0$ [in fact ψ_0 would correspond to a zero form in the canonical spin representation (25)], which satisfies

$$\frac{1}{2}(1 + iJ)X\psi_0 = 0, \quad \forall X,$$

so ψ_0 serves as the ‘vacuum’ and any other form can be constructed by applying the ‘raising operators’

$$\alpha^1 \wedge \dots \wedge \alpha^p \psi_0 \in W_{\text{can}}, \quad \text{where } \alpha^i \in \Gamma(T^*M), \quad \frac{1}{2}(1 + iJ)\alpha^i = 0,$$

to the vacuum.

By using the fact that L_X^s preserves J and $[L_X^s, \Gamma \cdot \alpha] = \Gamma \cdot (L_X \alpha)$, we get

$$L_X^s(\alpha^1 \wedge \dots \wedge \alpha^p \psi_0) = L_X(\alpha^1 \wedge \dots \wedge \alpha^p) \psi_0 + \alpha^1 \wedge \dots \wedge \alpha^p L_X^s \psi_0,$$

i.e. up to its action on the vacuum, the operator L_X^s is just the usual Lie-derivative on forms.

The next few steps are a bit technical and is presented in the appendix. It turns out that $L_X^s \psi_0 = i/r f_X \psi_0$ with f_X being a real constant, and so L_X^s can be identified as L_X with a numerical shift

$$L_X^s = L_X + \frac{i}{r} f_X.$$

The value of this shift is crucial in many ways, and we have our rule of thumb: assume that X has a zero (or is decomposable into a sum of commuting Killing vectors $X = \sum u_i$, $[u_i, v] = 0$, each with a zero). At one of its zeros, one can assume that X induces rotations of disjoint 2-planes, for each such rotation of degree k on a 2-plane with the standard complex structure, one gets a shift of $k/2$, and the sum of all the shifts gives f_X .

As an example, for the case of S^5 embedded in \mathbb{C}^3 , the Reeb vector field $R = e_1 + e_2 + e_3$, where e_i is the rotation of the i th factor in \mathbb{C}^3 , thus we get 3 shifts of $1/2$ and $f_R = 3/2$ (this shift was obtained in [2] through more or less brute force). While for squashed S^5 , the relevant R is decomposed as $v = \sum a_i e_i$, where a_i are the squashing parameters and the shift becomes $f_R = 1/2 \sum a_i$, and this shift plays an important role in [5, 7]. For our situation we have the decomposition (43), and the shift is also $f_R = 3/2$.

5.2. Transversally elliptic operators. In this section, we will temporarily suppress the dimensionful parameter r , and also treat x as a c -number. Both ingredients can be reinstated easily later. The type of super-determinants we need to compute for the hyper-multiplet is

$$\text{sdet}(-iL_X^s + x) = \frac{\det(-iL_X + f_X + x)|_{\psi_-}}{\det(-iL_X + f_X + x)|_{q_+}}, \tag{65}$$

where we have replaced the vector field v with a general isometry X , which descends from a vector field acting on \mathbb{C}^4

$$X = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 + \omega_4 e_4.$$

Clearly X commutes with R and $L_X J = 0$. We do so to put extra knobs onto the partition function, and to see how the latter responds to the symmetry of the geometry.

For the hyper-multiplet, the statistics of the fields q_+ and ψ_- is determined by their eigen-value under $\gamma_5 = -R \cdot \Gamma$, thus one can use the operator

$$D = P_-(\Gamma^i D_i)P_+, \quad P_{\pm} = \frac{1}{2}(1 \pm \gamma_5) = \frac{1}{2}(1 \mp R \cdot \Gamma) \tag{66}$$

to keep track of the cancellation between bosons and fermions. That is to say, the operator D sends every non-zero mode of q to a non-zero mode of ψ , with equal L_X^s eigenvalue, and hence the non-zero modes do not contribute anything to (65). The mismatch of eigenmodes of L_X^s between q and ψ is then in the kernel and cokernel of D and is captured by the index theorem.

To summarise, we need to obtain the kernel and cokernel of D , moreover, we need to decompose the (co)kernel into the simultaneous eigen-modes of e_i . Put more formally, we need to compute the equivariant index

$$\text{ind}_G(D) = \text{char}_G(\ker D) - \text{char}_G(\text{coker } D), \tag{67}$$

where G is the group generated by the three $U(1)$'s and char_G denotes the character. In the case D is an elliptic operator, the right hand side of (67) is also known as the *Lefschetz number*, which, upon evaluating the character at $1 \in G$, gives the usual index. But now the key difference is that D is not elliptic, its symbol $\sigma(D)$ (obtained by replacing ∂_m with p_m in D , where p_m is coordinate for T^*M) has a kernel along the Reeb direction.

What we here is a *transversally elliptic operator*. To be more precise, let G be a Lie group acting on M by isometry and $\tau : \mathfrak{g} \rightarrow TM$ be the infinitesimal action. Let E, F be two G -equivariant vector bundles on M and $D : E \rightarrow F$ be a differential operator that commutes with the G -action. Define

$$T_G^*M = \{\alpha \in T^*M \mid \langle \alpha, \tau(x) \rangle = 0, \forall x \in \mathfrak{g}\},$$

then D is transversally elliptic if its symbol is invertible on T_G^*M except at the zero section. In our case, since κ is a linear combination of e_i , so the possible kernel of D is precluded by the above condition for T_G^*M .

It is important to remember that the equivariant index defined in (67) will not be a function (ad-invariant if G is not abelian) on G , but rather a *distribution*. For example, over S^1 with the standard $U(1)$ -action, the 0 operator is transversally elliptic, and has delta function $\delta(1 - t)$ as index, where $t \in U(1)$. Thus in general, the index will have torsions and it will be illegal to evaluate the index at a given point.

In the lectures notes [28] of Sir Michael Atiyah, the index homomorphism was in principle completely worked out. In the following, we will use two different formulae given in [28]. The first is quite simple and is applicable only because of the simplicity of the $Y^{p,q}$ geometry, especially since it is a quotient of $S^3 \times S^3$.

5.3. Computing the index on $Y^{p,q}$ -method I. The easiest derivation of index theorem is through the equivariant K-theory. To explain the index calculation, we find ourself in need of making a big digression to review a small portion of the book by Atiyah. The symbol $\sigma(D)$ induces a bundle map $\pi^*E \rightarrow \pi^*F$ (where $\pi : T_G^*M \rightarrow M$ is the projection), and since $\sigma(D)$ is an isomorphism away from the zero section, it gives a complex

$$0 \rightarrow \pi^*E \xrightarrow{\sigma(D)} \pi^*F \rightarrow 0$$

exact except at the zero section, thus the complex is an element of

$$K_G^0(T_G^*M)$$

which by definition consists of stable isomorphism classes of pairs of bundles $[E_1, E_2]$ such that $E_1 \simeq E_2$ outside of a compact subset of T_G^*M . It is also convenient to choose a G -invariant metric and thereby identify T_G^*M with $T_G M$, where the latter consists of tangent vectors perpendicular to the G -action.

The *index homomorphism* associates an element of $\sigma(D) \in K_G^0(T_G M)$ with the index of D . But the crucial point is that it is possible to give a *topological characterisation*

of this homomorphism, which then tremendously simplifies the index calculation since manipulating isomorphism classes of vector bundles is much simpler than manipulating the differential operators.

We start from a simple example that could actually carry us a long way in our computation for $Y^{p,q}$. Take a single sphere S^{2n-1} embedded in \mathbb{C}^n and a group $G = U(1)$ acts on \mathbb{C}^n with charge vector $[m_1, \dots, m_n]$, with $m_i > 0$. Let H also be a $U(1)$ acting with charge $[1, \dots, 1]$, we remark that through demanding all $m_i > 0$ (of course all of them < 0 is equally good), the two K-groups $K_G^0(T_G S^{2n-1})$ and $K_G^0(T_H S^{2n-1})$ are isomorphic, because the bundle $T_G S^{2n-1}$ is isomorphic to $T_H S^{2n-1}$. The latter isomorphism is constructed by simply projecting $T_H S^{2n-1}$ to $T_G S^{2n-1}$; since on the sphere the vector field given by the charge vector $[m_1, \dots, m_n]$ is nowhere orthogonal to the one with charge vector $[1, \dots, 1]$, and hence the projection has no kernel. On $T_H S^{2n-1}$ there is a $\bar{\partial}$ symbol which is formed by pulling back the $\bar{\partial}$ operator defined on the quotient $S^{2n-1}/H \simeq \mathbb{C}P^{n-1}$. We will call this symbol the horizontal $\bar{\partial}$ -symbol, denoted as $[\bar{\partial}_H]$, as it is defined on the plane perpendicular to the Reeb vector H of S^{2n-1} . This symbol, when regarded as an element of $K_G^0(T_G S^{2n-1})$, has index (proposition 5.4 in [28])

$$\text{ind}_G[\bar{\partial}_H] = \left[\prod \frac{1}{1-t^{-m_j}} \right]^- - \left[\prod \frac{1}{1-t^{-m_j}} \right]^+,$$

where $[\]^\pm$ means to expand the content in the brace in positive/negative powers of t . For example, let $n = 2$ and $m_i = 1$

$$\text{ind}_G[\bar{\partial}_H] = \sum_{k \in \mathbb{Z}} (k+1)t^{-k},$$

the way to read the result is that the coefficient of t^{-k} is the index

$$\dim H^0(S^2, \mathcal{O}(k)) - \dim H^1(S^2, \mathcal{O}(k)).$$

It is well-known that $\dim H^0(S^2, \mathcal{O}(k)) = k + 1$, $k \geq 0$ and 0 if $k < 0$; while $\dim H^1(S^2, \mathcal{O}(k)) = 0$, $k \geq 0$ and $-1 - k$ if $k < 0$. In general, if some $m_i \neq 1$, the index becomes

$$\text{ind}_G[\bar{\partial}] = \sum_{k,l \geq 0} t^{-km_1 - lm_2} - \sum_{k,l < 0} t^{-km_1 - lm_2}, \tag{68}$$

the coefficient of t^{-m} then can be construed as the Riemann–Roch theorem of the weighted projective space (proposition 10.1 in [28])

$$\dim H^0(\mathbb{C}P(\bar{m}_1, \bar{m}_2), \mathcal{O}(m)) - \dim H^1(\mathbb{C}P(\bar{m}_1, \bar{m}_2), \mathcal{O}(m)),$$

where $\bar{m}_1/\bar{m}_2 = m_1/m_2$ and $\text{gcd}(\bar{m}_1, \bar{m}_2) = 1$. The weighted projective space is an orbifold unless $\bar{m}_1 = \bar{m}_2 = 1$.

Back to operator D defined in (66). The group in question will be denoted $H = U(1)^3$ (recall that $Y^{p,q}$ is toric). We need to manipulate D into something manageable. First notice that only the homotopy type of the symbol of D is important for the index computation; second, if one uses the spinor representation by means of the horizontal anti-holomorphic forms, one can unfold the complex $0 \rightarrow \Omega_H^{0,\text{even}} \xrightarrow{D} \Omega_H^{0,\text{odd}} \rightarrow 0$ into

$$0 \rightarrow \Omega_H^{0,0} \rightarrow \Omega_H^{0,1} \rightarrow \Omega_H^{0,2} \rightarrow 0,$$

and the symbol $[D]$ turns into the symbol $[\bar{\partial}_R]$ (the operator $\bar{\partial}$ does not exist, but its symbol does), the subscript R is there to remind us that the symbol depends on the horizontal plane and hence R . We want to deform $[\bar{\partial}_R]$ into $[\bar{\partial}_{R_0}]$ that is defined with a more convenient Reeb. Notice that by looking at the charge vector of R_0 and R , one easily checks that R and R_0 are never anti-parallel (in the dense open subset of $Y^{p,q}$, this is obvious, one needs only pay attention to those points where certain torus action degenerates). One then has an isomorphism between the two complexes $\Omega_H^{0,\cdot} \simeq \Omega_{H_0}^{0,\cdot}$, defined for R and R_0 respectively, by simply projecting one to the other. That R and R_0 are never anti-parallel guarantees that the projection has no kernel, and hence an isomorphism. Under this isomorphism, the symbol $[\bar{\partial}_R]$ induces a symbol homotopic to $[\bar{\partial}_{R_0}]$, and we can thus compute the index using the new symbol $[\bar{\partial}_{R_0}]$. To compute the index, we lift the symbol $[\bar{\partial}_{R_0}]$ from $Y^{p,q}$ to $S^3 \times S^3$, and enlarge the group H to $G = H \times U(1)$, where the extra $U(1)$ is the freely acting $U(1)$ called $U(1)_T$ before. We will lift the symbol to $S^3 \times S^3$ through the projection $S^3 \times S^3 \rightarrow Y^{p,q}$. Because this $U(1)_T$ is free, once we have computed the index on $S^3 \times S^3$, we can pick out the terms that correspond to the trivial representations of $U(1)_T$ (see theorem 3.1 in [28]), so as to go back down to the quotient space $Y^{p,q}$. In lifting from $Y^{p,q}$ to $S^3 \times S^3$, the horizontal complex $[\bar{\partial}_{R_0}]$ is lifted to the $\bar{\partial}$ -complex that is horizontal w.r.t both T and R_0 . To see this, it is useful to keep in mind the following picture. One can obtain the horizontal $\bar{\partial}$ -complex on $Y^{p,q}$ in two steps. One starts from the standard complex structure on \mathbb{C}^4 , restricting oneself to the constant moment map level $\mu_T^{-1}(0)$ (where μ_T is the moment map for $U(1)_T$), one has a transverse complex structure transverse to T . This complex structure is clearly invariant under $U(1)_T$ and will go down the symplectic quotient to be the complex structure \mathcal{J} on $\mathbb{C}^4//U(1)_T$, and as we recall, the latter is the Kähler cone $C(Y^{p,q})$. In the second step, we restrict \mathcal{J} to the plane transverse to both the homothetic vector field $r\partial_r$ and the Reeb vector $\mathcal{J}(r\partial_r)$, and we obtain the desired transverse complex structure on $Y^{p,q}$. Since $S^3 \times S^3$ is obtained from \mathbb{C}^4 by imposing $\mu_T = 0$ and $\sum_i |z_i|^2 = 1$, we need only restrict ourselves to the directions transverse to T and Reeb to get the last mentioned transverse complex structure.

Having established this, the $\bar{\partial}_{R_0}$ -complex splits into the horizontal $\bar{\partial}$ -complex of the two S^3 's individually. What remains is to compute the index of $[\bar{\partial}]$ on the two S^3 , take the product, and pick out the terms of trivial representation of $U(1)_T$. Let now G_1 (resp. G_2) be $U(1)^2$ acting on the first (resp. second) $S^3 \subset \mathbb{C}^2$ in the standard manner. Denote by $s_{1,2}$ and $t_{1,2}$ the coordinates of the two $U(1)$'s of G_1 (resp. G_2). Then as a simple generalisation from the above example, the index is

$$\begin{aligned} \text{ind}_{G_1}[\bar{\partial}] &= \sum_{i,j \geq 0} s_1^{-i} s_2^{-j} - \sum_{i,j < 0} s_1^{-i} s_2^{-j}, \\ \text{ind}_{G_2}[\bar{\partial}] &= \sum_{k,l \geq 0} t_1^{-k} t_2^{-l} - \sum_{k,l < 0} t_1^{-k} t_2^{-l}, \end{aligned}$$

and the index of the product $S^3 \times S^3$ is the product of the two indices above. Since T has charge vector $[p + q, p - q, -p, -p]$, the terms in the product of the two indices above that correspond to the trivial representation under T must satisfy

$$i(p + q) + j(p - q) = (k + l)p. \tag{69}$$

This is possible only between the following combination

$$\sum_{i,j \geq 0} s_1^{-i} s_2^{-j} \cdot \sum_{k,l \geq 0} t_1^{-k} t_2^{-l} + \sum_{i,j < 0} s_1^{-i} s_2^{-j} \cdot \sum_{k,l < 0} t_1^{-k} t_2^{-l} \Big|_{i(p+q)+j(p-q)=(k+l)p}. \tag{70}$$

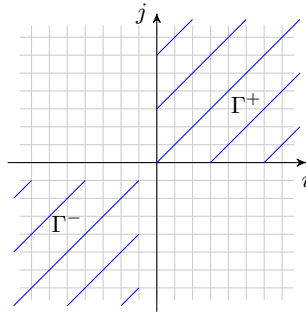


Fig. 1. The lattice Γ for summation for $p = 3$ and $q = 1$

We define a lattice

$$\begin{aligned} \Lambda^+ &= \{i, j, k, l \geq 0, i(p + q) + j(p - q) = p(k + l)\}, \\ \Lambda^- &= \{i, j, k, l < 0, i(p + q) + j(p - q) = p(k + l)\}, \end{aligned}$$

and our summation of the indices will take place on this lattice.

Let us pause for a minute and understand what we got. It was shown in Sect. 3 that $Y^{p,q}$ is a $U(1)$ -fibration over an orbifold base if one uses R_1 instead of the irregular R . The Reeb R_1 is only locally free, and for given i, j, k, l its mode is $n = (p+q)i + (p-q)j + p(k+l)$. One can try to reorganise the lattice by fixing the level n , then the intersection of constant n plane with the lattice Λ^\pm has only finite number of lattice points (this amounts to looking at Riemann–Roch theorem on an orbifold). But the intersection is quite jagged in the sense that there is no general formula for the number of lattice points at each given n , this reflects the fact that the Reeb acts non-freely and we have an orbifold base. However, in the case of $T^{1,1}$, the intersection is nice, since $T^{1,1}$ is regular SE and the base of the Reeb fibration is a (Kähler–Einstein) manifold. To see this, let us set $p = 1, q = 0$, then for fixed n , the intersection has exactly $(n + 1)^2$ lattice points, which shows again that $T^{1,1}$ is a $U(1)$ bundle over $S^2 \times S^2$ of degree 1 and 1.

To investigate the lattice Λ further, we notice that the condition (69) is satisfied iff

$$\begin{cases} i + j - (k + l) = mq \\ i - j = -mp \end{cases}, \tag{71}$$

showing that i, j must be on the lattice Γ in Fig. 1, and $k + l$ is determined by i and j . For later use, we point out that the mode m appearing here is (negative) of the mode along the free $U(1)$ called α in Sect. 3.2, to see this, we plug (71) into the charge vector of the free $U(1)$:

$$\begin{aligned} [a, -a - 2b, b, b] &\rightarrow ai - (a + 2b)(i + mp) + b(2i + m(p - q)) \\ &= -m(ap + b(p + q)) = -m. \end{aligned} \tag{72}$$

5.4. The super-determinant of L_X^s . Now we can finish the calculation of the super-determinant (65). From the index (70), a summand such as

$$s_1^{-i} s_2^{-j} t_1^{-k} t_2^{-l},$$

corresponds to a mode with charge i, j, k, l under e_1, e_2, e_3, e_4 , thus this mode has $-iL_X$ eigen-value

$$\omega_1 i + \omega_2 j + \omega_3 k + \omega_4 l$$

One then easily writes down the super-determinant as the product

$$\begin{aligned} \text{sdet}_{\text{hyp}}(-iL_X^s + x) &= \prod_{\Lambda^+} (h(i, j, k, l, x)) \cdot \prod_{\Lambda^-} (h(i, j, k, l, x)), \\ h(i, j, k, l, x) &= \omega_1 i + \omega_2 j + \omega_3 k + \omega_4 l + (\omega_1 + \omega_2 + \omega_3 + \omega_4)/2 + x. \end{aligned}$$

where we have the extra shift worked out earlier. We remark here that whenever we write an infinite product, we mean implicitly the *zeta regulated product* [29], i.e.

$$\prod_k \lambda_k = \exp \left(- \frac{\partial}{\partial s} \frac{1}{\Gamma(s)} \int_0^\infty \sum_k e^{-\lambda_k t} t^{s-1} dt \Big|_{s=0} \right) \tag{73}$$

and analytically continued.

Notice that the product can be put in a form symmetric in $x \rightarrow -x$. For the lattice Λ^- , one redefine $i = -i - 1, j = -j - 1, k = -k - 1$ and $l = -l - 1$, then the product can be written as

$$\prod_{\Lambda^-} h(i, j, k, l, x) = \prod_{\Lambda^+} h(-i, -j, -k, -l, x),$$

from the rhs one can pull out (an infinite number of) minus signs and rewrite

$$rhs = (-1)^\infty \prod_{\Lambda^+} h(i, j, k, l, -x).$$

If one so wishes, one can use the zeta function to regulate also $(-1)^\infty$, but as it is a constant, we will just discard it. Finally the super-determinant is

$$\text{sdet}_{\text{hyp}}(-iL_X^s + x) = \prod_{\Lambda^+} h(i, j, k, l, x) \cdot h(i, j, k, l, -x). \tag{74}$$

We remark that the symmetry of $x \rightarrow -x$ in the result is no coincidence, since the matter content of the hyper-multiplet transforms in the representation $\underline{R} \oplus \bar{\underline{R}}$ of the gauge group, and $x \rightarrow -x$ corresponds to taking $\underline{R} \rightarrow \bar{\underline{R}}$, so here we have a nice confirmation of this symmetry.

For the sake of variation as well as double check, we present in the appendix a different calculation also based on formulae given in [28]. This method resembles in appearance the Lefschetz fixed point formulae (see [30]) but differs drastically in its interpretation. The computation can be performed on $Y^{p,q}$ without lifting the geometry to $S^3 \times S^3$.

We need to put the product into a form more suitable for the asymptotic analysis. Note that (71) has two subcases

$$\begin{aligned} m \geq 0, i \geq 0 : \quad & j = i + mp, \quad k + l = 2i + m(p - q), \\ m > 0, j \geq 0 : \quad & i = j + mp, \quad k + l = 2j + m(p + q), \end{aligned}$$

the first applies to the part of Γ^+ above (and including) the diagonal while the second below the diagonal. Plugging these into the product

$$\begin{aligned} \text{sdet}_{\text{hyp}}(-iL_X^s + x) &= \prod_{m, i \geq 0} \prod_{k=0}^{2i+m(p-q)} h(i, i + mp, k, 2i + m(p - q) - k, x) \cdot h(\dots, -x) \\ &\prod_{m > 0, j \geq 0} \prod_{k=0}^{2j+m(p+q)} h(j + mp, j, k, 2j + m(p + q) - k, x) \cdot h(\dots, -x). \end{aligned} \tag{75}$$

For the vector multiplet, one can recycle most of the results above by mapping the complex in (60) to $E \simeq W_{\text{can}} \oplus W_{\text{can}}^*$. As such, the super determinant $\text{sdet}'_{\text{vec}}(-iL_X + x)$ will be the norm squared of that of the hyper-multiplet, but without the shift and without the constant modes. In total we get

$$\begin{aligned} \text{sdet}'_{\text{vec}}(-iL_X + x) &= \prod_{\Lambda^+ \setminus \{0\}} g(i, j, k, l, x) \cdot \prod_{\Lambda^-} g(i, j, k, l, x), \\ g(i, j, k, l, x) &= ((\omega_1 i + \omega_2 j + \omega_3 k + \omega_4 l)^2 - x^2). \end{aligned} \tag{76}$$

One can equally flip the Λ^- part above to Λ^+ . It is convenient to divide $\Lambda^+ \setminus \{0\}$ into one part $\Lambda_0^+ = \Lambda^+ \cap \{i, j, k, l > 0\}$ and another part Λ_1^+ where at least one but not all of i, j, k, l are zero. Finally the product becomes

$$\text{sdet}'_{\text{vec}}(-iL_X + x) = \prod_{\Lambda_0^+} g(i, j, k, l, x)^2 \cdot \prod_{\Lambda_1^+} g(i, j, k, l, x).$$

6. Asymptotic Analysis of the Partition Function

6.1. *General formulae.* We can work out a slightly more general formula. Let x be a complex variable with $\text{Re } x = 0$ and $|\text{Im } x| \gg 0$ and let $g(t)$ be a function such that the two integrals below converge.

$$\int_1^\infty |g(t)e^{-xt}| dt < \infty, \quad \int_1^\infty |g(t)'e^{-xt}| dt < \infty.$$

From the absolute convergence above, we know that

$$I = \int_1^\infty g(t)e^{-xt} dt \sim \mathcal{O}(x^{-1}). \tag{77}$$

The technique that leads to this estimate is rather standard in asymptotic analysis. Consider

$$I = \int_1^\infty g(t)e^{-xt} dt \stackrel{\text{ibp}}{=} \frac{1}{x} e^{-x} g(1) + \frac{1}{x} \int_1^\infty e^{-xt} g(t)' dt,$$

it is clear that both terms are of order $\mathcal{O}(x^{-1})$.

Let $f(t)$ be a function that has Laurent expansion at $t = 0$ of the form

$$f(t) = f_n t^{-n} + \dots + f_1 t^{-1} + f_0 + \mathcal{O}(t) = \bar{f}(t) + \mathcal{O}(t), \tag{78}$$

and that $f(t)$ tends to constant at infinity. We investigate the asymptotic behaviour of the following

$$J_0^\infty(f) = \partial_s \left(\Gamma(s)^{-1} \int_0^\infty f(t) e^{-xt} t^{s-1} dt \right) \Big|_{s=0}.$$

From the estimate (77), as well as the fact that $\Gamma(0)^{-1} = 0$ we will be free to alter the limits of the integral as it suits us

$$J_0^\infty(f) = J_0^1(f) + \mathcal{O}(x^{-1}) = J_0^1(\bar{f}) + \mathcal{O}(x^{-1}) = J_0^\infty(\bar{f}) + \mathcal{O}(x^{-1}),$$

where \bar{f} is defined in (78). It is easy to see

$$J_0^\infty(t^{-k}) = \frac{\partial}{\partial s} \frac{\Gamma(s-k)}{\Gamma(s)} x^{-s+k} \Big|_{s=0} = \frac{\partial}{\partial s} \frac{1}{(s-k) \cdots (s-1)} x^{-s+k} \Big|_{s=0}$$

for some small $k \geq 0$ we have¹⁰

$$J_0^\infty(t^{-k}) : \begin{aligned} k = 0, & \quad -\ln x; \quad k = 1, \quad x \ln x - x; \\ k = 2, & \quad -\frac{1}{2}x^2 \ln x + \frac{3}{4}x^2; \quad k = 3, \quad \frac{1}{6}x^3 \ln x - \frac{11}{36}x^3. \end{aligned} \quad (79)$$

From these and the Laurent expansion (78) one can completely determine the asymptotic behaviour of $J_0^\infty(f(t))$.

6.2. Application to $Y^{p,q}$. For the hyper-multiplet, we need to work out the asymptotic behaviour of a product of the form [the first line of (75)]

$$\star = \prod_{m,i \geq 0} \prod_{k=0}^{2i+m(p-q)} h(i, i+mp, k, 2i+m(p-q)-k, \pm x).$$

From the definition of zeta-regulated product, the relevant integral to consider is

$$\begin{aligned} -\ln \star &= \partial_s \frac{1}{\Gamma(s)} \sum_{m,i \geq 0} \sum_{k=0}^{2i+m(p-q)} \int_0^\infty e^{-h(i,j,k,l,\pm x)t} t^{s-1} dt \Big|_{s=0} \\ &= \partial_s \frac{1}{\Gamma(s)} \int_0^\infty e^{-(\frac{1}{2}(\omega_1+\omega_2+\omega_3+\omega_4)\pm x)t} t^{s-1} dt \cdot \frac{1}{1-e^{-(\omega_3-\omega_4)t}} \\ &\quad \left(\frac{1}{1-e^{-(\omega_1+\omega_2+2\omega_4)t}} \frac{1}{1-e^{-(\omega_2 p+\omega_4(p-q))t}} \right. \\ &\quad \left. - \frac{1}{1-e^{-(\omega_1+\omega_2+2\omega_3)t}} \frac{e^{-(\omega_3-\omega_4)t}}{1-e^{-(\omega_2 p+\omega_3(p-q))t}} \right) \Big|_{s=0}. \end{aligned}$$

In performing the summation over i and m , one needs $\omega_1 + \omega_2 + 2\omega_{3,4} > 0$ and $\omega_2 p + \omega_{3,4}(p - q) > 0$, but these are implied by the dual cone condition (42).

The second line of (75) can be done similarly. It also follows from (42) that $\sum \omega_i > 0$, and then the function after the \cdot satisfies the criteria laid out before and will serve as our $f(t)$. We can expand $f(t)$ into a Laurent series, and follow the procedure given

¹⁰ Where $\ln x$ denotes $\log x$ in its principle branch i.e. $-\pi < \text{Im} \ln x \leq \pi$.

previously to get the asymptotic behaviour. This involves a bit of meticulous calculation, so we only give the results

$$\begin{aligned}
 -\ln \text{sdet}_{\text{hyp}} &= i\pi \operatorname{sgn}(\operatorname{Im} x) \left(\frac{1}{6} \varrho[R] \cdot x^3 + B_h \cdot x \right), \\
 B_h &= -\frac{1}{24} \left((\omega_3 - \omega_4)^2 + (\omega_1 - \omega_2)^2 \right) \varrho[R] \\
 &\quad - \frac{p^2 q (\omega_1 - \omega_2) (\omega_3 \omega_4 - \omega_1 \omega_2) - pq^2 (\omega_1 \omega_2 (\omega_3 + \omega_4) + (\omega_1 + \omega_2) \omega_3 \omega_4)}{6((\omega_1 + \omega_3)p + q\omega_3)((\omega_1 + \omega_4)p + \omega_4 q)((\omega_2 + \omega_3)p - q\omega_3)((\omega_2 + \omega_4)p - \omega_4 q)}.
 \end{aligned} \tag{80}$$

That the coefficient of x^3 term is proportional to the general squashed volume is perhaps not surprising. The coefficient B_h satisfies the condition (5) as expected, but only when taking the two terms together. The geometrical interpretation of B_h is beyond us for now, but it may be related to some correction coming from the non-trivial topology, such as the volume of the generator of $H^2(Y^{p,q}) \sim \mathbb{Z}$. It will be extremely interesting if one can fix this. If we plug in the actual value of ω_i for the Reeb vector (43), the result is

$$i\pi \operatorname{sgn}(\operatorname{Im} x) \left(x \left(\frac{q}{2p} \ell - \frac{3}{8} \varrho \right) + \frac{1}{6} \varrho x^3 \right).$$

One can also allow x to have a real part, but it must be in the range

$$\operatorname{Re} x \in \left[0, \frac{1}{2} (\omega_1 + \omega_2 + \omega_3 + \omega_4) \right),$$

to make sure both $(\omega_1 + \omega_2 + \omega_3 + \omega_4)/2 \pm x$ have a positive real part so as to converge the integral. We remark that, in the same computation on S^5 , the product gives the Barne’s function and can be explicitly written in terms of poly-logarithms, which do exhibit branching behaviour when $\operatorname{Re} x$ exceeds certain range. Here our function is a generalisation of the Barne’s function, and it shows similar branching behaviours.

The calculation for the vector-multiplet is far less pleasant as it lacks the elegant symmetry possessed by the hyper-multiplet, hence we will be just giving the result. But we do point out that the exclusion of the constant modes in the vector part is crucial for the convergence of the integral.

$$\begin{aligned}
 -\frac{1}{2} \ln \text{sdet}'_{\text{vec}} &= \frac{1}{2} \ln(-x^2) + i\pi \operatorname{sgn}(\operatorname{Im} x) \left(\frac{1}{6} \varrho[R] \cdot x^3 + B_v \cdot x \right), \\
 B_v &= \frac{1}{12} (\omega_3^2 + \omega_4^2 + 4\omega_3 \omega_4 + 3(\omega_1 + \omega_2)(\omega_3 + \omega_4) + \omega_1^2 + 4\omega_1 \omega_2 + \omega_2^2) \varrho[R] \\
 &\quad - \frac{p^2 q (\omega_1 - \omega_2) (\omega_3 \omega_4 - \omega_1 \omega_2) - pq^2 (\omega_1 \omega_2 (\omega_3 + \omega_4) + (\omega_1 + \omega_2) \omega_3 \omega_4)}{6((\omega_1 + \omega_3)p + q\omega_3)((\omega_1 + \omega_4)p + \omega_4 q)((\omega_2 + \omega_3)p - q\omega_3)((\omega_2 + \omega_4)p - \omega_4 q)}.
 \end{aligned} \tag{81}$$

If we plug in the value of ω_i for the Reeb again, we get

$$\frac{1}{2} \ln(-x^2) + i\pi \operatorname{sgn}(\operatorname{Im} x) \left(x \left(\frac{q}{2p} \ell + \frac{3}{4} \varrho \right) + \frac{1}{6} \varrho x^3 \right).$$

It is an interesting feature that if one considers a single hyper-multiplet in the adjoint representation, then the leading x^3 term cancel just as in the S^5 case and the asymptotic behaviour of the potential for the matrix model will be $|x|$. In the next section we will discuss the implication of this fact.

7. N^3 -Behavior from Matrix Model

In this section we present one of the possible applications of our general result (1). Let us consider the gauge group $SU(N)$ and the matter content consisting of a single hypermultiplet in adjoint representation with mass M . We are interested in the large N behaviour of the free energy for this model. Our analysis is completely analogous to the treatment of the model on S^5 from [13,31]. Thus we present the final result and for the details we refer the readers to the references just given.

Let us introduce 't Hooft coupling constant $\lambda = g_{YM}^2 N/r$ and rewrite the matrix model (1) in terms of eigenvalues ϕ_i . In the limit $\lambda \gg 1$ we can assume the large separation of eigenvalues $|\phi_i - \phi_j| \gg 1$ and moreover we also assume $|M| \ll \lambda$. Using the asymptotic expansions (80) and (81) the matrix model is drastically simplified the large 't Hooft coupling

$$Z = \int \prod_i d\phi_i \exp\left(-\frac{8\pi^3 N}{\lambda} \varrho \sum_i \phi_i^2 + \frac{\pi}{2} \left[\frac{1}{4} \left(\sum_{i=1}^4 \omega_i\right)^2 + M^2 \right] \varrho \sum_{i,j} |\phi_i - \phi_j|\right),$$

where the matrix model is written in terms of ϕ eigenvalues. Following the same logic as in [31] we can evaluate the free energy on the saddle point and obtain the following expression

$$F = -\log Z = -\frac{g_{YM}^2 N^3}{96\pi r} \varrho \left(\frac{1}{4} \left(\sum_{i=1}^4 \omega_i\right)^2 + M^2 \right)^2. \tag{82}$$

for the general squashed $Y^{p,q}$ space. If we consider the case of unsquashed $Y^{p,q}$ space and set $\sum \omega_i = 3$, in fact, it was shown in [26] that $\sum \omega_i = 3$ is a necessary condition for the Reeb vector to admit an SE metric, then we arrive to

$$F = -\log Z = -\frac{g_{YM}^2 N^3}{96\pi r} \varrho \left(\frac{9}{4} + M^2 \right)^2. \tag{83}$$

Surprisingly the result is identical to that of the theory on S^5 up to a volume factor ϱ .

Using the results presented in this paper, it is straightforward exercise to generalise to squashed $Y^{p,q}$ space the treatment of matrix models for 5D SCFTs presented in [11].

8. Discussion

In this paper we considered the 5D Yang–Mills theory with matter on $Y^{p,q}$ manifolds. This theory preserves two supersymmetries and this is sufficient for us to localise the model. The partition function is localised on contact instantons, however, a general treatment of the instanton sector presents some challenges due to complications in geometry and is left for the future. We perform explicit calculation only for the zero instanton sector and obtained the full perturbative result in terms of certain special functions and we studied their asymptotic properties.

Let us briefly discuss some related topics:

- **Isometry and enlargement of supersymmetry**

We can wonder whether we can enlarge the number of susy for a given $Y^{p,q}$ geometry. The answer seems to be a disappointing ‘no’, at least when one does not turn on extra background fields.

Let us investigate the extreme case of S^5 embedded in \mathbb{R}^6 , which has an $SO(6)$ isometry. Choose three $SO(2)$ -rotations along the 1–2, 3–4 and 5–6 direction as the Cartan of $\mathfrak{so}(6)$. The Reeb vector is the sum $R = \sum e_i$, let X be an isometry that does not commute with R , and let L_X^s act on the Killing spinors. Since L_X^s preserves the Killing spinors, then $L_X^s \psi$ must also be a Killing spinor linearly independent of ψ , this way we have enlarged the number of super charges (this method of analyzing the susy algebra is used in [32,33]). Indeed for S^5 with the round metric, we get eight super-charges transforming in $\underline{4} \oplus \overline{\underline{4}}$ of $\mathfrak{so}(6)$. For S^5 with various squashed metric, one has reduced isometry and hence only a fraction of susy is preserved [4].

For the $Y^{p,q}$ manifold, the infinitesimal isometry is shown to be $\mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ [25], but unfortunately all these isometries commute with the Reeb and hence generate no new susy. Actually this follows from a deeper reason. Bär’s cone construction [34] shows that the Killing spinors on M correspond to parallel spinors on the metric cone $C(M)$, thereby converting the classification problem of Killing spinors to the classification of holonomy on $C(M)$. The latter problem is by now well-understood and the situation is summarised in theorem 5.15 from [35], which shows that S^5 is the only 5D manifold with Killing spinors of type (4, 4) (this means 4 Killing spinors with $\mu = 1$ and 4 with $\mu = -1$); the other SE manifolds admit only Killing spinors of type (1, 1).

• **Generalisation to $L^{a,b,c}$ spaces**

The spaces $L^{a,b,c}$ contain $Y^{p,q}$ as a subclass (see [36], whereas the SE metric is presented in [37]). The construction of such spaces is similar, one takes the quotient of \mathbb{C}^4 with respect to a $U(1)$ of charge $[a, b, -c, -a - b + c]$, with appropriate coprimeness conditions on the three positive integers a, b, c , to form the Kähler cone over the desired SE space. Hence, we expect that most of our calculation can be generalised to $L^{a,b,c}$ spaces rather easily, and one obtains the partition function written in terms of the function S^Λ , but with the lattice determined by the new charge vector $[a, b, -c, -a - b + c]$. The result may reveal more how the partition function depends on the non-trivial homology of the space.

Acknowledgements. We thank Johan Källén, Joseph Minahan, Anton Nedelin and Konstantin Zarembo for discussion of this and related topics. The research of J.Q. is supported by the Luxembourg FNR grant PDR 2011-2, and by the UL grant GeoAlgPhys 2011–2013. The research of M.Z. is supported in part by Vetenskapsrådet under grant # 2011-5079.

A. Notations and Conventions

We follow the convention for spinors from [15]. The gamma matrices satisfy the Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\delta^{ab},$$

and the charge conjugation matrix satisfies

$$C^{-1}(\Gamma^a)^T C = \Gamma^a, \quad C^T = -C, \quad C^* = C.$$

Note that the type-writer fonts are reserved for *flat indices*.

The spinor bi-linears are formed using C ,

$$\psi^T C \chi \xrightarrow{\text{abbreviate}} \psi \chi, \tag{84}$$

though throughout the paper, these bi-linears are abbreviated as $(\psi \chi)$, following the notation of [15]. Due to the symmetry property of C , one has

$$(\psi \chi) = -(\chi \psi), \quad (\psi \Gamma^a \chi) = -(\chi \Gamma^a \psi), \quad (\psi \Gamma^{ab} \chi) = (\chi \Gamma^{ab} \psi),$$

where $\Gamma^{a_1 \dots a_n} = (1/n!) \Gamma^{[a_1} \dots \Gamma^{a_n]}$ and all spinors appearing above are *bosonic* (even). The product of three or more gamma matrices can be reduced

$$\Gamma^{abc} e^{abcde} = -6 \Gamma^{de}, \tag{85}$$

where $e^{a \dots b}$ is the Levi-Civita symbol $e^{12345} = 1$.

On a curved manifold, one defines the gamma matrices by means of the veilbeins, i.e. a set of mutually orthogonal (local) sections of the tangent bundle

$$E^a \in \Gamma(TM), \quad \langle E^a, E^b \rangle = \delta^{ab},$$

where $\langle -, - \rangle$ is the pairing using the metric g . The gamma matrices are defined as

$$\Gamma^m = E^{ma} \Gamma^a, \quad \Gamma_m = g_{mn} \Gamma^n,$$

and the duality (85) turns into

$$\frac{1}{3!} g^{1/2} \Gamma_{mnp} \epsilon_{qr}^{mnp} = -\Gamma_{qr}. \tag{86}$$

One may also consult the appendix of [2] for the Fierz identities.

We also use Dirac’s slash notation

$$\not{M} = M_{i_1 \dots i_p} \Gamma^{i_1 \dots i_p}, \quad M \in \Omega^p(M),$$

and to keep the formulae neat, *we will even dispense with the slash whenever confusion is unlikely.*

The $SU(2)$ R-symmetry index are raised and lowered from the left

$$\xi^I = \epsilon^{IJ} \xi_J, \quad \xi_I = \epsilon_{IJ} \xi^J, \quad \epsilon^{IK} \epsilon_{KJ} = \delta^I_J, \quad \epsilon^{12} = -\epsilon_{12} = 1.$$

• **The spinor Lie derivative.** This notion is quite old [38], but it is explained more transparently in the physics context by Figueroa-O’Farrill [33]. For a Killing vector, one defines

$$L_X^s = D_X - \frac{1}{8} \nabla_{[m} X_n] \Gamma^{mn} = D_X - \frac{1}{8} dX,$$

which satisfies (f is a function and Y is any vector field)

$$\begin{aligned} [L_X^s, \not{Y}] &= [X, Y] \cdot \Gamma, & [L_X^s, f] &= L_X f, \\ [L_X^s, L_Y^s] &= L_{[X, Y]}^s, & [L_X^s, D_Y] &= D_{[X, Y]}. \end{aligned} \tag{87}$$

In this paper almost all covariant derivatives will be denoted as D , be it on the spin bundle or the gauge bundle; but ∇ will be reserved for the Levi-Civita covariant derivative. The vector indices are raised and lowered with the metric tacitly, e.g. the dX above means identifying X with a 1-form using the metric.

B. The Shift

The simple task to accomplish here is to compute $L_X^s \psi$ where X is a Killing vector commuting with R and ψ is a Killing spinor. We just outline the procedure here.

First it is easy to see that $L_X^s \psi$ will remain a Killing spinor even if X does not commute with R , by using the last property of (87). But we need to show something stronger, i.e. that L_X^s preserves the subbundle defined in (28). The first condition of (28) is easy, since

$$[L_X^s, R]\psi = (L_X R)\psi = 0,$$

using $L_X R = 0$. For the second condition, one needs some computation. The following relation will also be useful

$$\begin{aligned} 0 &= L_X J_q^p = \nabla_X J_q^p - J_q^r \nabla_r X^p + J_r^p \nabla_q X^r \\ &= -R^p X_q + X^p R_q - \frac{1}{2}(dX)_r^p J_q^r + \frac{1}{2}J_r^p (dX)_q^r, \end{aligned} \tag{88}$$

which places a restraint on the $(2, 0) + (0, 2)$ part of dX .

One then writes

$$[L_X^s, A(Y)] = [\tilde{D}_X - \frac{i\mu}{2}X + \frac{1}{8}dX, A(Y)],$$

we already know that $[\tilde{D}_X, A(Y)]$ restricts to zero. The two latter terms are computed to be

$$\begin{aligned} -\frac{i\mu}{2}[X, A(Y)] &= -i\mu A(Y) + i(XJY) + \frac{\mu}{2}((X \cdot Y) - XY_R + 2X(R \cdot Y) \\ &\quad - (X \cdot R)Y + R(X \cdot Y)), \\ [dX, A(Y)] &= -8\mu l_Y(R \wedge X) - 4A(l_Y dX) - 4i\mu Y A(X) + 2\mu Y(1 + R)X \\ &\quad + 8i(YJX). \end{aligned}$$

Combine the two pieces, ignoring anything containing A and using $R \cdot \psi = -\psi$, one gets $[L_X^s, A(Y)] = 0$. So the second condition of (28) is preserved by L_X^s .

Since we know that $L_X^s \psi$ remains a Killing spinor, furthermore the subbundle (28) is of rank one, one then sees that $L_X^s \psi$ is a constant multiple of ψ . To find this proportionality coefficient, we look at $L_X^s \psi$ more closely

$$L_X^s \psi = \left(-\frac{i\mu}{2}X^{\parallel} + \frac{1}{8}(dX^\perp)_{pq} \Gamma^{pq} \right) \psi,$$

where $X^{\parallel} = (R \cdot X)_R$ and dX^\perp is the horizontal part of dX . Now consider dX^\perp , it can be decomposed into self and anti-self dual part, the latter vanishes when acting on ψ , as was shown in (27). As for the self-dual part, it can be either $(1, 1)$ or $(2, 0) + (0, 2)$ w.r.t J , the latter again vanishes due to (88). What remains is the self-dual $(1, 1)$ part of dX^\perp , but note that such a 2-form is necessarily proportional to J . To see this assume M is $(1, 1)$ self-dual and choose the volume form to be $1/2J \wedge J$

$$\begin{aligned} M &= *_4 M = \frac{1}{2}\langle M, J \rangle J + J M J = \frac{1}{2}\langle M, J \rangle J + J M J = \frac{1}{2}\langle M, J \rangle J - M, \\ \Rightarrow M &= \frac{1}{4}\langle M, J \rangle J, \end{aligned}$$

where $\langle M, J \rangle = M_{pq} J^{pq}$.

Thus L_X^s acting on ψ simplifies to a multiplicative factor

$$L_X^s \psi = \left(\frac{i\mu}{2} \langle X, R \rangle + \frac{1}{32} \langle dX, J \rangle J \right) \psi = i\mu \left(\frac{1}{2} \langle X, R \rangle - \frac{1}{8} \langle dX, J \rangle \right) \psi = i\mu f_X \psi, \tag{89}$$

where we used (29).

As a check that f_X is a constant, let us look at $X = R$ and compute f_R from (89)

$$f_R = \frac{1}{2} \langle R, R \rangle - \frac{1}{8} (-2) \langle J, J \rangle = \frac{1}{2} + \frac{1}{4} (+4) = \frac{3}{2},$$

which is the correct result from our S^5 computation [2]. It is nonetheless worth the effort to figure out f_X for a more general X , which then allows us to get the fully equivariant partition function.

Now that we know f_X is a constant, we can evaluate it at a convenient point, say, a point where $X = 0$, and (89) simplifies to

$$f_X = -\frac{1}{4} (\partial_p X^q) J_q^p,$$

where the covariant derivative has been swapped for ordinary derivative since $X = 0$. For our particular case, we know $Y^{p,q}$ comes from reduction data, and the complex structure is inherited from the canonical one on \mathbb{C}^4 , we can compute the above expression in \mathbb{C}^4 . For each e_i , go to one of its fixed point, one has

$$(\partial_p e_i^q) J_q^p = -2,$$

the computation is valid passing down to the quotients, since $\iota_T \eta = \mu_T = 0$. As a final check, let us plug in the Reeb vector decomposed into $U(1)$'s, namely

$$R = \left(\frac{3}{2} - \frac{1}{2(p+q)\ell} \right) (e_3 + e_4) + \frac{1}{(p+q)\ell} e_2,$$

since each e_i contributes $1/2$, we get once again $3/2$.

C. The Explicit Metric

The metric is given by

$$\begin{aligned}
 ds^2 &= \frac{1-cy}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} (d\psi - \cos \theta d\phi)^2 \\
 &\quad + w(y) \left[d\alpha + f(y) (d\psi - \cos \theta d\phi) \right]^2, \tag{90} \\
 w(y) &= \frac{2(a-y^2)}{1-cy}, \quad q(y) = \frac{a-3y^2+2cy^3}{a-y^2}, \quad f(y) = \frac{ac-2y+y^2c}{6(a-y^2)}.
 \end{aligned}$$

The parameter c will be set to be 1 when it is not zero. The angle variable α has period $2\pi\ell$, where ℓ is defined in (44). The parameter a is given by

$$a = \frac{1}{2} + \frac{1}{4} (-1 + 3\lambda^{-2}) \sqrt{4 - 3\lambda^{-2}}, \quad \lambda = p/q,$$

and y is restricted to the range $y_1 \leq y \leq y_2$, where $y_{1,2}$ is the negative and the smaller positive root of the cubic equation $a - 3y^2 + 2y^3 = 0$. In particular

$$y_1 = \frac{3}{4}(1 - \lambda^{-1}) - \frac{y_3}{2}, \quad y_2 = \frac{3}{4}(1 + \lambda^{-1}) - \frac{y_3}{2}, \quad y_3 = \frac{3}{2}(1 - \lambda^{-2}) + \frac{1}{2p\lambda\ell}.$$

The following is easy to check, and will be important later

$$f(y_1) = \frac{\ell}{2}(p + q), \quad f(y_2) = \frac{\ell}{2}(q - p). \tag{91}$$

With these choice of parameters, the manifold is homeomorphic to a $U(1)$ bundle over $S^2 \times S^2$ with degree p and q .

Next, we will try to read off the change of coordinates from the metric, in particular, we want to reproduce table (37). The first term is the metric of the round sphere, which is what we call the base sphere. The second and third term describes the fibre sphere, note that the term $-\cos\theta d\phi$ in the combination $d\psi - \cos\theta d\phi$ is the connection for the fibration. In this original way of presenting the metric, it may seem that $d\phi$ is ill-defined at $\theta = 0, \pi$, since at the two poles the azimuth angle is undetermined. The reason for this is that the fibration associated to the degree $-2 U(1)$ bundle is trivialised everywhere except at the two poles, where each pole hosts a -1 point charge (note that $dd\phi$ is not zero but a delta function). In this paper, we prefer to present the metric in the traditional way, namely, we will remove the delta function charge but re-introduce the transition function at the equator. To do so, in the patch $U_{00} : \theta < \pi, y > y_1$, redefine $\psi_{00} = \psi - \phi$ and $\phi_{00} = \phi$, but at the patch $U_{10} : \theta < \pi, y < y_2$, $\psi_{10} = -\psi + \phi$ and $\phi_{10} = \phi$. This way, in U_{10} , the combination

$$d\psi - \cos\theta d\phi = d\psi_{00} + (1 - \cos\theta)d\phi_{00},$$

and similarly for U_{10} :

$$d\psi - \cos\theta d\phi = -d\psi_{10} + (1 - \cos\theta)d\phi_{10}.$$

But on the intersection $U_{00} \cap U_{10}$, we have $\psi_{10} = -\psi_{00}$.

One has to do the same for the $U(1)$ fibre parameterised by α . In the patches U_{00} and U_{10} , define $\alpha_{00} = \alpha + f_2\psi_{00}$, and $\alpha_{10} = \alpha - f_1\psi_{10}$, then

$$\begin{aligned} d\alpha + f(y)(d\psi - \cos\theta d\phi) &= d\alpha_{00} - f_2d\psi_{00} + f(y)(d\psi_{00} + (1 - \cos\theta)d\phi_{00}) \\ &= d\alpha_{10} + f_1d\psi_{10} + f(y)(-d\psi_{10} + (1 - \cos\theta)d\phi_{10}), \end{aligned}$$

then the singularity at $y = y_{1,2}$ is removed. On the intersection $U_{10} \cap U_{00}$,

$$\phi_{10} = \phi_{00}, \quad \psi_{10} = -\psi_{00}, \quad \alpha_{10} = \alpha_{00} + (f_1 - f_2)\psi_{00}.$$

Now we get go to the patches covering $\theta = \pi$,

$$\begin{aligned} U_{01} : \{\theta > 0, y > y_1\}, \quad \phi_{01} &= -\phi_{00}, \quad \psi_{01} = \psi + \phi = \psi_{00} + 2\phi_{00}, \\ \alpha_{01} &= \alpha + f_2(\psi + \phi) = \alpha_{00} + 2f_2\phi_{00}, \end{aligned}$$

$$\begin{aligned} U_{11} : \{\theta > 0, y < y_2\}, \quad \phi_{11} &= -\phi_{00}, \quad \psi_{11} = -\phi - \psi = -2\phi_{00} - \psi_{00}, \\ \alpha_{11} &= \alpha + f_1(\phi + \psi) = \alpha_{00} + (f_1 - f_2)\psi_{00} + 2f_1\phi_{00}. \end{aligned}$$

From these change of coordinates we get the table below

| | U_{10} | U_{01} | U_{11} |
|------------------------|---|--|--|
| $\partial_{\phi_{00}}$ | $\partial_{\phi_{10}}$ | $-\partial_{\phi_{01}} + 2\partial_{\psi_{01}} + 2f_2\partial_{\alpha_{01}}$ | $-\partial_{\phi_{11}} - 2\partial_{\psi_{11}} + 2f_1\partial_{\alpha_{11}}$ |
| $\partial_{\psi_{00}}$ | $-\partial_{\psi_{10}} + (f_1 - f_2)\partial_{\alpha_{10}}$ | $\partial_{\psi_{01}}$ | $-\partial_{\psi_{11}} + (f_1 - f_2)\partial_{\alpha_{11}}$ |

Identifying $\partial_{\phi_{00}} = \partial_{\phi_{10}} = [0, 0, 1, 0]$, $\partial_{\phi_{01}} = \partial_{\phi_{11}} = [0, 0, 0, 1]$, $\partial_{\psi_{00}} = \partial_{\psi_{01}} = [1, 0, 0, 0]$, $\partial_{\psi_{10}} = \partial_{\psi_{11}} = [0, -1, 0, 0]$, and taking into account (91) and some normalisation, this table is identical to table (37).

Finally, the Reeb vector in the original coordinates is

$$R = \frac{\partial}{\partial \psi} - \frac{1}{6} \frac{\partial}{\partial \alpha}.$$

Rewriting this in the new coordinates gives for example in patches U_{00}, U_{10}

$$\begin{aligned} R &= \frac{\partial}{\partial \psi_{00}} + \left(f_2 - \frac{1}{6}\right) \frac{\partial}{\partial \alpha_{00}} \\ &= -\frac{\partial}{\partial \psi_{10}} + \left(f_1 - \frac{1}{6}\right) \frac{\partial}{\partial \alpha_{10}}. \end{aligned}$$

Close to $y = y_{1,2}$, the first terms both vanish, and the second terms $f_{2,1} - 1/6 = -y_{2,1}/6$ change sign. This fact plays a crucial role in method II of the index calculation.

D. Calculation of the Index Method II

This approach is less demanding on one’s knowledge of the global geometry on the spcae in question but is certainly more long winded. The material we present here is in lecture 8 of [28]. As was explained earlier, a transversally elliptic operator D induces an element $[\sigma(D)] \in K_G(T_G^*X)$, and the ellipticity ensures that the symbol $\sigma(D)$ is an isomorphism except along the zero section of T_G^*X . It is convenient to pick a G -invariant metric and identify T_G^*X with $T_G X$, consisting of the tangent vectors orthogonal to the group action. One can deform the symbol of the operator slightly along a vector field, and make the symbol into an isomorphism even on the zero section of $T_G X$, provided one stays away from the zero of the vector field in question. It will be natural to use a vector field generated by the group action for this job.

Now let us take $G = U(1)^n$, and define a filtration of X

$$X = X_0 \supset X_1 \supset \dots \supset X_{n+1} = \emptyset,$$

where $X_i = \{x \in X : \dim G_x \geq i\}$ with G_x being the stability group at x . The key to the localisation of the index calculation is the following exact sequence

$$0 \rightarrow K_G(T_G(X - X_i)) \rightarrow K_G(T_G(X - X_{i+1})) \xrightarrow{\theta_i} K_G(T_G X|_{X_i - X_{i+1}}) \rightarrow 0,$$

The sequence above is split exact, and we will describe the splitting map θ_i shortly. From the split exactness, the middle term is just the direct sum of the two terms on the end

$$K_G(T_G(X - X_{i+1})) \simeq K_G(T_G(X - X_i)) \oplus \theta_i K_G(T_G X|_{X_i - X_{i+1}}).$$

One can continue with the same game with $K_G(T_G(X - X_i))$ and bootstrap oneself all the way from $X - X_1$ to $X - X_{n+1} = X$.

One can further determine $K_G(T_G X|_{X_i - X_{i+1}})$ by relating it to $K_G(T_G(X_i - X_{i+1}))$ through the Thom isomorphism, since $T_G X|_{X_i - X_{i+1}}$ is a complex vector bundle over $T_G(X_i - X_{i+1})$. To summarise, we have that $K_G(T_G X)$ is a direct sum

$$K_G(T_G X) = \bigoplus_i \phi_i K_G(T_G(X_i - X_{i+1})), \tag{92}$$

where ϕ_i is the composition of the Thom isomorphism and θ_i . Hence given any symbol $[D]$ for which we want to compute the index, we can decompose $[D] \in K_G(T_G X)$ as the sum of classes $[D] = \phi_i K_G(T_G(X_i - X_{i+1}))$, in the hope that the index homomorphism of the summands can be determined by other means. In our situation, we will only need the cases when $X_i - X_{i+1}$ are either circles or points, and their K -group is

$$K_G(pt) = R(G), \quad K_{U(1)}(T_{U(1)}S^1) = K_{U(1)}(S^1) = R(U(1))/(\lambda), \quad \lambda = 1 - t^{-1}, \tag{93}$$

where we use a Laurent polynomial $f(t)$ to denote both a function on $U(1)$ and direct sum of representations, for example $f(t) = t^{-1} + 2t^2$ means a direct sum of 3 representations of $U(1)$, one with charge -1 and two with charge 2. The index homomorphism is given by assigning a representation to its character, in particular for the zero symbol $[0] \in K_{U(1)}(T_{U(1)}S^1)$, its index is

$$\text{ind}[0] = \delta(1 - t), \tag{94}$$

which is annihilated by $(1 - t^{-1})$, as it should from (93).

Amongst the two parts making up ϕ , the Thom isomorphism is well-understood, what is tricky is θ_i . For clarity, we take a simpler situation, which is actually sufficient for all our calculations later

$$0 \rightarrow K_G(T_G(\mathbb{C} - \{0\})) \rightarrow K_G(T_G\mathbb{C}) \xrightarrow{\theta_{\pm}} K_G(\mathbb{C}) \rightarrow 0,$$

where $G = U(1)$ acting on \mathbb{C} in the standard way.¹¹ Take for example $1 \in K_G(\{0\})$ and the Thom isomorphism multiplies to it the class $\sigma = [\bar{\partial}] \in K_G(\mathbb{C})$. One would like to ‘insert’ σ into $K_G(T_G\mathbb{C})$, in such a way that when restricted to \mathbb{C} one gets σ back. One can certainly extend σ to $K_G(T_GU)$ where U is a small neighbourhood of $\{0\}$ (using the retraction $U \rightarrow \{0\}$) for example). But this alone will not do, since in order to insert σ into $T_G\mathbb{C}$, one needs σ to have support only in a compact subset of T_GU so as to make the insertion ‘local’, but σ is never an isomorphism on the zero section of T_GU , then as U is open, the support of σ is not compact. This is where one needs to use the vector field generated by G to deform σ so that σ is an isomorphism outside a compact subset of T_GU . Depending on in which direction one deforms σ , one gets two classes $[\bar{\partial}^{\pm}]$, and if one applies the index homomorphism to these two classes, one gets

$$\text{ind}_{U(1)}([\bar{\partial}^{\pm}]) = \left[\frac{1}{1 - s^{-1}} \right]^{\pm}, \tag{95}$$

where s is the coordinate of $G = U(1)$. The superscript \pm corresponds to two regularisations that send the rational function $1/(1 - s^{-1})$ into distributions such that

$$(1 - s^{-1}) \left[\frac{1}{1 - s^{-1}} \right]^{\pm} = 1.$$

¹¹ In this case $X = \mathbb{C}$, $X_1 = \{0\}$, and $K_G(\{0\}) \rightarrow K_G(\mathbb{C}) = K_G(T_G\mathbb{C}|_{\{0\}})$ is the Thom isomorphism.

The explicit expression for $\text{ind}([\bar{\partial}^\pm])$ is

$$\begin{aligned}
 P(s) &= \left[\frac{1}{1-s^{-1}} \right]^+ = -s - s^2 - \dots, \\
 N(s) &= \left[\frac{1}{1-s^{-1}} \right]^- = 1 + s^{-1} + s^{-2} + \dots, \\
 \delta(1-s) &= N(s) - P(s),
 \end{aligned}$$

it is easy to verify $(1-s^{-1})N(s) = (1-s^{-1})P(s) = 1$ and hence $(1-s^{-1})\delta(1-s) = 0$ as delta function does.

As an example (and a rather pedestrian one, but it will help us establish some conventions), let us compute the index of $[\bar{\partial}] \in K_G(T_G^* \mathcal{O}(p))$ where $\mathcal{O}(p)$ is the total space of the $U(1)$ bundle over S^2 of degree $p \geq 0$, which can be presented as

$$[z_1, z_2, e^{i\theta}] / \sim, \quad [z_1, z_2, e^{i\theta}] \sim [z_1 e^{i\alpha}, e^{i\alpha} z_2, e^{ip\alpha+i\theta}].$$

The group G is $U(1)^2$ with the action

$$s : [sz_1, z_2, e^{i\theta}], \quad u : [z_1, z_2, ue^{i\theta}]. \tag{96}$$

Note the second $U(1)$ is free. The above is valid close to the north pole $z_1 = 0$, while at the south pole, the action of the two $U(1)$'s becomes

$$s : [z_1, s^{-1}z_2, s^{-p}e^{i\theta}], \quad u : [z_1, z_2, ue^{i\theta}]. \tag{97}$$

It is not hard to decompose $[\bar{\partial}]$ into the direct sum (92). At $z_1 = 0$ or $z_2 = 0$ one of the $U(1)$ degenerates, by applying the restriction map to, say, $z_1 = 0$, then $K_G(T_G \mathcal{O}(p)) \rightarrow K_G(T_G \mathcal{O}(p)|_{z_1=0}) = K_G(\mathbb{C} \times S^1)$. In this process $[\bar{\partial}]$ restricts to the 0 symbol on the S^1 factor, so we just get the contribution $[\bar{\partial}^\pm] \cdot \delta(1-u)$ from the fixed point $z_1 = 0$, depending on our choice of the splitting map θ^\pm . One gets a similar contribution $\theta^\pm K_G(\mathbb{C} \times S^1)$ from $z_2 = 0$, where the $U(1)^2$ actions on the two factors should be read off from (97).

How do we decide consistently which splitting map θ^\pm to use? We will choose a global vector field generated by a $U(1)$ -action, e.g. the first $U(1)$ in (96), and trivialise the symbol $[\bar{\partial}]$ globally except at the zeros of the vector field, then by scrutinising the symbol $[\bar{\partial}]$ close to the zeros, we can find out whether it should be $[\bar{\partial}^+]$ or $[\bar{\partial}^-]$.

The $U(1)$ we have chosen is just the standard rotation of a sphere. Denote the inhomogeneous coordinate of S^2 as $z = z_1/z_2$ and the vector field of this rotation is $v = i(z\bar{\partial} - \bar{z}\partial)$. Recall that the symbol $\sigma(\bar{\partial})$ sends a tangent vector X to a bundle morphism $\sigma(\bar{\partial})(X) : \Omega^{0,0}(S^2, \mathcal{O}(p)) \rightarrow \Omega^{0,1}(S^2, \mathcal{O}(p))$, one can deform the symbol into

$$\sigma^\pm(\bar{\partial})(X) = \sigma(\bar{\partial})(X \pm v),$$

which will remain a bundle isomorphism even when $X = 0$, thus trivialising the symbol $\sigma(\bar{\partial})$ except at the two poles. This deformation correspond to the symbol $[\bar{\partial}^\pm]$ introduced earlier. We choose the + option (choosing - makes no difference).

Now we can assemble the contributions from the two poles [using also (95)]

$$\begin{aligned}
 \text{north pole} : & \left[\frac{1}{1-s^{-1}} \right]^+ \delta(1-u) = -(s + s^2 + \dots) \delta(1-u), \\
 \text{south pole} : & \left[\frac{1}{1-s} \right]^+ \delta(1-us^{-p}) = (1 + s + s^2 + \dots) \delta(1-us^{-p}).
 \end{aligned}$$

Combining the the two terms, and we collect the coefficient proportional to u^n

$$\text{ind}([\bar{\partial}]) (u^n) = -(s + s^2 + \dots) + (1 + s + s^2 + \dots) s^{-pn} = \begin{cases} n \geq 0 & 1 + \dots + s^{-pn} \\ n < 0 & -s - \dots - s^{-pn-1}, \end{cases}$$

in particular, evaluating at $s = 1$, the coefficient of u^n is the index

$$\dim H^0(S^2, \mathcal{O}(pn)) - \dim H^1(S^2, \mathcal{O}(pn)).$$

Notice that in general one does not have the luxury of evaluating an index at certain value, since the indices will be a distribution not a function.

Now we are ready to take on our problem of computing the index on $Y^{p,q}$. We take $G = U(1)^3$ generated by e_1, e_3 of table (37) and α which is free, and denote by s, t and u the coordinates of the three $U(1)$'s.

The operator D we want to compute is the $\bar{\partial}$ -operator whose complex structure is the one determined by J , which as we saw in Sect. 3.3 agrees with one on the base S^2 of the fibration $S^2 \times S^2$. While over the fibre S^2 , it is homotopic to $\bar{\partial}$ close to the north pole, but to $-\partial$ close to the south pole. We will again choose e_1 and e_3 to trivialise the symbol $\sigma(D)$ everywhere except the four poles, and from the previous discussion of the complex structure, the trivialisation procedure is identical to the previous example when restricted to the base S^2 . On the fibre sphere, the situation is similar to the case in lemma 6.4 in [28], and one can trivialise D by trivialising $\bar{\partial}$ with the positive regularisation close to the north pole and $-\partial$ with the negative regularisation close to the south pole. This choice of regularisation will then mesh together at the equator.

To summarise, we have the following contribution (where the notation of the four patches U_{ij} were defined in Sect. 3.3)

$$\begin{aligned} \text{pole in } U_{00} &: \left[\frac{1}{1-s^{-1}} \right]^+ \left[\frac{1}{1-t^{-1}} \right]^+ \delta(1-u), \\ \text{pole in } U_{01} &: \left[\frac{1}{1-v^{-1}} \right]^+ \Big|_{v=t^2s} \left[\frac{1}{1-t} \right]^+ \delta(1-ut^{q-p}), \\ \text{pole in } U_{10} &: \left[\frac{1}{1-s^{-1}} \right]^- \left[\frac{1}{1-t^{-1}} \right]^+ \delta(1-us^p), \\ \text{pole in } U_{11} &: \left[\frac{1}{1-v^{-1}} \right]^- \Big|_{v=t^2s} \left[\frac{1}{1-t} \right]^+ \delta(1-us^p t^{p+q}), \end{aligned}$$

where the change of variable can be read off from the table (37).

Now combine the U_{00} and U_{10} contribution, and single out the term of power u^m , then the sum over s looks like

$$m < 0 : \left(- \sum_1^\infty s^k + \sum_{-\infty}^{pm} s^k \right) \left[\frac{1}{1-t^{-1}} \right]^+, \quad m \geq 0 : \left(- \sum_{pm+1}^\infty s^k + \sum_{-\infty}^0 s^k \right) \left[\frac{1}{1-t^{-1}} \right]^+.$$

The combination of U_{10} and U_{11} contribution can be obtained by replacing $s \rightarrow t^2s$, $t \rightarrow t^{-1}$ as well as including an overall factor $t^{m(q-p)}$

$$\begin{aligned} m < 0 & \quad t^{m(q-p)} \left(- \sum_1^\infty (t^2s)^k + \sum_{-\infty}^{pm} (t^2s)^k \right) \left[\frac{1}{1-t} \right]^+ \\ m \geq 0 & \quad t^{m(q-p)} \left(- \sum_{pm+1}^\infty (t^2s)^k + \sum_{-\infty}^0 (t^2s)^k \right) \left[\frac{1}{1-t} \right]^+. \end{aligned}$$

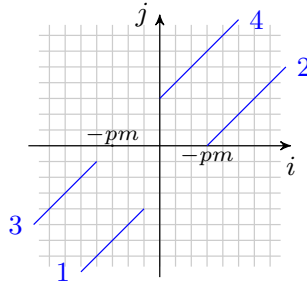


Fig. 2. The two lower lines have $m < 0$ and the two upper $m \geq 0$. And each lattice point on the lines in the first quadrant has multiplicity $1 + 2i + m(p - q)$, while those in the third quadrant $-2i + m(q - p) - 1$

Once all four contributions are combined, one observes that one can send $t = 1$ safely, this is so because D restricts to an elliptic operator on the base sphere. We will keep t and expand out $[1/(1 - t^{\pm 1})]^+$, there will be some partial cancellations, which correspond to the missing segments between line 1, 2 and 3, 4 in Fig. 2.

$$\begin{aligned}
 m < 0 & \left(\sum_1^{\infty} s^k \sum_{j=1}^{2k+m(q-p)-1} t^j + \sum_{-\infty}^{pm} s^k \sum_{2k+m(q-p)}^0 t^j \right) \\
 m \geq 0 & \left(\sum_{pm+1}^{\infty} s^k \sum_{j=1}^{2k+m(q-p)-1} t^j + \sum_{-\infty}^0 s^k \sum_{2k+m(q-p)}^0 t^j \right).
 \end{aligned}$$

To facilitate comparison with the calculation in Sect. 5.3, we rename $k = -i, j = -k$ and our index becomes

$$\begin{aligned}
 m < 0 & \left(\sum_{i=-\infty}^{-1} s^{-i} \sum_{k=2i-m(q-p)+1}^{-1} t^{-k} + \sum_{i=-pm}^{\infty} s^{-i} \sum_{k=0}^{2i-m(q-p)} t^{-k} \right) \\
 m \geq 0 & \left(\sum_{i=-\infty}^{-pm-1} s^{-i} \sum_{k=2i-m(q-p)+1}^{-1} t^{-k} + \sum_{i=0}^{\infty} s^{-i} \sum_{k=0}^{2i-m(q-p)} t^{-k} \right).
 \end{aligned}$$

The four terms correspond each to the four lines in the Fig. 2, and the agreement with (70) is crystal clear. As we have already remarked in (72) that the mode m in (71) is the mode of the free $U(1)$ denoted as α , and here by doing the calculation differently, we are merely changing the order of summation (but in a manner that is allowed for an infinite sum).

References

1. Pestun, V.: Localization of gauge theory on a four-sphere and supersymmetric Wilson loops. Commun. Math. Phys. **313**, 71–129 (2012). [arXiv:0712.2824](#) [hep-th]
2. Källén, J., Qiu, J., Zabzine, M.: The perturbative partition function of supersymmetric 5D Yang–Mills theory with matter on the five-sphere. JHEP **1208**, 157 (2012). [arXiv:1206.6008](#) [hep-th]
3. Kim, H.-C., Kim, S.: M5-branes from gauge theories on the 5-sphere. JHEP **1305**, 144 (2013). [arXiv:1206.6339](#) [hep-th]
4. Imamura, Y.: Supersymmetric theories on squashed five-sphere. Prog. Theor. Exp. Phys. 013B04 (2013). [arXiv:1209.0561](#) [hep-th]

5. Lockhart, G., Vafa, C.: Superconformal partition functions and non-perturbative topological strings. [arXiv:1210.5909](https://arxiv.org/abs/1210.5909) [hep-th]
6. Imamura, Y.: Perturbative partition function for squashed S^5 . *Prog. Theor. Exp. Phys.* 073B01 (2013). [arXiv:1210.6308](https://arxiv.org/abs/1210.6308) [hep-th]
7. Kim, H.-C., Kim, J., Kim, S.: Instantons on the 5-sphere and M5-branes. [arXiv:1211.0144](https://arxiv.org/abs/1211.0144) [hep-th]
8. Kim, H.-C., Kim, S.-S., Lee, K.: 5-Dim superconformal index with enhanced global symmetry. *JHEP* **1210**, 142 (2012). [arXiv:1206.6781](https://arxiv.org/abs/1206.6781) [hep-th]
9. Terashima, S.: On supersymmetric gauge theories on $S^4 \times S^1$. *Phys. Rev. D* **89**, 125001 (2014). [arXiv:1207.2163](https://arxiv.org/abs/1207.2163) [hep-th]
10. Fujitsuka, M., Honda, M., Yoshida, Y.: Maximal super Yang–Mills theories on curved background with off-shell supercharges. *JHEP* **1301**, 162 (2013). [arXiv:1209.4320](https://arxiv.org/abs/1209.4320) [hep-th]
11. Jafferis, D.L., Pufu, S.S.: Exact results for five-dimensional superconformal field theories with gravity duals. *J. High Energy Phys.* **2014**, 32 (2014). [arXiv:1207.4359](https://arxiv.org/abs/1207.4359) [hep-th]
12. Assel, B., Estes, J., Yamazaki, M.: Wilson loops in 5d $N = 1$ SCFTs and AdS/CFT. *Ann. Henri Poincaré* **15**, 589 (2014). [arXiv:1212.1202](https://arxiv.org/abs/1212.1202) [hep-th]
13. Minahan, J.A., Nedelin, A., Zabzine, M.: 5D super Yang–Mills theory and the correspondence to AdS₇/CFT₆. *J. Phys. A Math.* **46**, 355401 (2013). [arXiv:1304.1016](https://arxiv.org/abs/1304.1016) [hep-th]
14. Nieri, F., Pasquetti, S., Passerini, F.: 3d & 5d gauge theory partition functions as q-deformed CFT correlators. [arXiv:1303.2626](https://arxiv.org/abs/1303.2626) [hep-th]
15. Hosomichi, K., Seong, R.-K., Terashima, S.: Supersymmetric gauge theories on the five-sphere. [arXiv:1203.0371](https://arxiv.org/abs/1203.0371) [hep-th]
16. Boyer, C.P., Galicki, K.: Sasakian geometry, holonomy, and supersymmetry. *Handbook of pseudo-Riemannian geometry and supersymmetry*, IRMA Lect. Math. Theor. Phys. Evr. Math. Soc. Zürich **16**, 39–83 (2010). [arXiv:math/0703231](https://arxiv.org/abs/math/0703231)
17. McDuff, D., Salamon, D.: Introduction to symplectic topology. In: *Oxford Mathematical Monographs*, 2 edn. Oxford University Press, New York (1999)
18. Blair, D.E.: Contact manifolds in Riemannian geometry. *Lecture notes in mathematics*, vol. 509. Springer, New York (1976)
19. Salamon, D.: Spin geometry and Seiberg–Witten invariants (1996)
20. Boyer, C.P., Galicki, K.: *Sasakian Geometry*. Oxford University Press, USA (2008)
21. Friedrich T., Kath I.: Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator. *J. Differ. Geom.* **29**, 263–279 (1989)
22. Boyer, C.P.: Completely integrable contact Hamiltonian systems and toric contact structures on $S^2 \times S^3$. *SIGMA* **7**, 58 (2011). [arXiv:1101.5587](https://arxiv.org/abs/1101.5587) [math.SG]
23. Martelli, D., Sparks, J.: Toric geometry, Sasaki–Einstein manifolds and a new infinite class of AdS/CFT duals. *Commun. Math. Phys.* **262**, 51–89 (2006). [arXiv:hep-th/0411238](https://arxiv.org/abs/hep-th/0411238) [hep-th]
24. Boyer, C.P., Pati, J.: On the equivalence problem for toric contact structures on S^3 -bundles over S^2 . *Pac. J. Math.* **267**(2), 277–324 (2014). [arXiv:1204.2209](https://arxiv.org/abs/1204.2209) [math.SG]
25. Gauntlett, J.P., Martelli, D., Sparks, J., Waldram, D.: Sasaki–Einstein metrics on $S^2 \times S^3$. *Adv. Theor. Math. Phys.* **8**, 711–734 (2004). [arXiv:hep-th/0403002](https://arxiv.org/abs/hep-th/0403002) [hep-th]
26. Martelli, D., Sparks, J., Yau, S.-T.: The geometric dual of a-maximisation for toric Sasaki–Einstein manifolds. *Commun. Math. Phys.* **268**, 39–65 (2006). [arXiv:hep-th/0503183](https://arxiv.org/abs/hep-th/0503183) [hep-th]
27. Källén, J., Zabzine, M.: Twisted supersymmetric 5D Yang–Mills theory and contact geometry. *JHEP* **1205**, 125 (2012). [arXiv:1202.1956](https://arxiv.org/abs/1202.1956) [hep-th]
28. Atiyah, M.F.: *Elliptic Operators and Compact Groups*, vol. 401. Springer, Berlin (1974)
29. Quine, J.R., Heydari, S.H., Song, R.Y.: Zeta regularized products. *Trans. Am. Math. Soc.* **338**(1), 213–231 (1993). <http://www.jstor.org/stable/2154453>
30. Atiyah, M.F., Bott, R.: A Lefschetz fixed point formula for elliptic complexes: II. *Appl. Ann. Math.* **88**(3), 451–491 (1968). <http://www.jstor.org/stable/1970721>
31. Källén, J., Minahan, J., Nedelin, A., Zabzine, M.: N^3 -behavior from 5D Yang–Mills theory. *JHEP* **1210**, 184. [arXiv:1207.3763](https://arxiv.org/abs/1207.3763) [hep-th]
32. Moroianu, A.: On the infinitesimal isometries of manifolds with killing spinors. *J. Geom. Phys.* **35**(1), 63–74 (2000). <http://www.sciencedirect.com/science/article/pii/S0393044099000790>
33. Figueroa-O’Farrill, J.M.: On the supersymmetries of Anti-de Sitter vacua. *Class. Quantum Gravity* **16**, 2043–2055 (1999). [arXiv:hep-th/9902066](https://arxiv.org/abs/hep-th/9902066) [hep-th]
34. Bär C.: Real Killing spinors and holonomy. *Commun. Math. Phys.* **154**, 509–521 (1993)
35. Boyer, C.P., Galicki, K.: 3-Sasakian manifolds. *Surv. Differ. Geom.* **7**, 123–184 (1999). [arXiv:hep-th/9810250](https://arxiv.org/abs/hep-th/9810250) [hep-th]
36. Martelli, D., Sparks, J.: Toric sasakieinstein metrics on. *Phys. Lett. B* **621**(12), 208–212 (2005). <http://www.sciencedirect.com/science/article/pii/S0370269305008609>

37. Cvetič, M., Lu, H., Page, D.N., Pope, C.: New Einstein–Sasaki spaces in five and higher dimensions. *Phys. Rev. Lett.* **95**, 071101 (2005). [arXiv:hep-th/0504225](https://arxiv.org/abs/hep-th/0504225) [hep-th]
38. Kosmann, Y.: Dérivées de Lie des spineurs. *Ann. di Mat. Pura Appl. (IV)* **91**, 317–395 (1972)

Communicated by N. A. Nekrasov