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Vortex Partition Functions, Wall Crossing and Equivariant Gromov–Witten Invariants

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Abstract: In this paper we identify the problem of equivariant vortex counting in a (2, 2) supersymmetric two dimensional quiver gauged linear sigma model with that of computing the equivariant Gromov–Witten invariants of the GIT quotient target space determined by the quiver. We provide new contour integral formulae for the $\mathcal{I}$ and $\mathcal{J}$-functions encoding the equivariant quantum cohomology of the target space. Its chamber structure is shown to be encoded in the analytical properties of the integrand. This is explained both via general arguments and by checking several key cases. We show how several results in equivariant Gromov–Witten theory follow just by deforming the integration contour. In particular, we apply our formalism to compute Gromov–Witten invariants of the $\mathbb{C}^3/\mathbb{Z}_n$ orbifold, of the Uhlembeck (partial) compactification of the moduli space of instantons on $\mathbb{C}^2$, and of A_n and D_n singularities both in the orbifold and resolved phases. Moreover, we analyse dualities of quantum cohomology rings of holomorphic vector bundles over Grassmannians, which are relevant to BPS Wilson loop algebrae.

1. Introduction

One of the most exciting aspects of supersymmetric quantum field theories is the possibility to get exact non perturbative solutions via a variety of techniques. In this paper we will focus on two dimensional gauge theories with four supersymmetries. In these cases the non perturbative aspects are captured by vortex counting. This was initially developed in [1], which applied the equivariant localization of [2] to two dimensional gauge theories giving explicit vortex partition function formulas, which recently attracted attention in the context of AGT correspondence [3] and knot theory [4]. Vortex partition functions have been related to CFT degenerate conformal blocks and to topological strings in [4–9]. General contour integral formulae for vortex counting have been obtained in [10] [11] in the study of supersymmetric partition functions on S^2. These partition functions have been conjectured to compute the quantum Kähler potential of the target space of the corresponding infrared NLSM in [12]. Evidence of this conjecture was provided in [13]. Further studies along these lines have been presented in [14–18]. In this paper we will elaborate on these issues from a different viewpoint by using supersymmetric localization on S^2 to provide new contour integral formulae for the $\mathcal{I}$ and $\mathcal{J}$-functions describing the equivariant quantum cohomology of GIT quotients in terms of Givental's formalism [19] and its extension to non abelian quotients in terms of quasi-maps [20].

One of the implications of our results is thus that the equivariant vortex partition functions contain not only information about the Gromov–Witten invariants of the IR target space, but also their gravitational descendants. As will be explained more in detail in Sec.2, this is a consequence of the equivariant localization procedure with respect to a supersymmetric charge that closes on $U(1)_R$ rotations of the sphere. From the geometrical viewpoint, one thus considers S^1-equivariant maps from a sphere marked with North and South poles, where the gravitational descendants are inserted, to the target space.

We provide general rules for the calculation of supersymmetric spherical partition functions of quiver gauge theories and the corresponding $\mathcal{I}$-functions. Our formalism applies to both compact and non compact Kähler manifolds with $c_1 \geq 0$. One key result that we will obtain is the possibility of analyzing the chamber structure and wall-crossings of the GIT quotient moduli space in terms of integration contour choices. In particular, we will obtain an explicit description of the equivariant quantum cohomology and chamber structure for the resolutions of $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds and for the Uhlembeck partial compactification of the instanton moduli space.

We remark that, as observed in [21], the OPE algebra of circular BPS Wilson loops in three dimensional supersymmetric gauge theories can be reduced in some cases to the equivariant quantum K-ring of certain quasi projective varieties. In particular this led to conjecture an equivalence of the quantum cohomology rings of suitable vector bundles over complex Grassmannians using 3d dualities and circle compactification. We will use our methods to prove this conjecture in Sect. 4.

The paper is organized as follows. In Sect. 2 we provide a general discussion about the relation between the spherical partition function of a given GLSM and the quantum cohomology of the space it flows to in the IR in terms of $\mathcal{I}$ and $\mathcal{J}$-functions. In Sect. 3 and 4 we provide several examples of calculations of the quantum cohomology of abelian and non-abelian GIT quotients. We study in particular the chamber structure of the crepant resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_n$ in Sect. 3.4.2 and of the ADHM moduli space in Sect. 4.4. The duality between Grassmannians is discussed in Sect. 4.1 (with details in the Appendix) and quiver gauge theories are discussed in Sect. 4.2 and 4.3. Finally, in Sect. 5 we draw our conclusions and discuss further directions.

2. Gauge Linear Sigma Models, Stability Conditions and Wall Crossing

In this section we discuss how the exact equivariant partition functions of general $\mathcal{N} = (2, 2)$ gauged linear sigma models on the two-sphere with a $U(1)$ vector R-symmetry [10,11] encode the quantum cohomology of the target IR geometry in various stability chambers and the wall crossing among them.

The partition function for a given gauge group[1] G and matter in the representation R depends on the twisted masses which can be coupled to the system breaking its

[1] The localization applies to any classical Lie group ABCDEFG. In this paper we will focus on the $U(N)$ case.

continuous flavor symmetry group G_F to its maximal abelian subgroup T_F. The theory in general allows a gauge invariant holomorphic non singular superpotential $\mathcal{W}$.

The resulting object, in the Coulomb branch localization scheme, is given as an integral over the Cartan algebra t_G of the gauge symmetry group

$$Z^{S^2} = \frac{1}{|W(G)|} \sum_{\vec{m} \in \mathbb{Z}^{r_G}} \int_{t_G} d\vec{\tau} e^{-S_{cl}} \mu_G \mu_R \tag{2.1}$$

where $|W(G)|$ is the order of the Weyl group of G, $r_G = \dim t_G$ is the rank of the gauge group. $S_{cl} = -4\pi\vec{\xi} \cdot \vec{\tau} + i\vec{\theta} \cdot \vec{m}$ is the classical action of the GLSM depending on the FI parameters vector $\vec{\xi}$ (one for each $U(1)$ factor in G), the magnetic fluxes $\vec{m}$ and the theta-angles $\vec{\theta}$. More specific rules for quiver gauge theories will be presented in Sect. 4.

In (2.1) μ_G is the one loop determinant of the gauge multiplet

$$\prod_{r<s}^{r_G} \left(\frac{m_{rs}^2}{4} - \tau_{rs}^2 \right), \tag{2.2}$$

where $m_{rs} = m_r - m_s$ and $\tau_{rs} = \tau_r - \tau_s$, and μ_R is the one-loop determinant of the matter multiplets

$$\prod_{\rho \in R} \frac{\Gamma\left(\mathbf{q}/2 + r\rho(\tau) - \frac{\rho(m)}{2}\right)}{\Gamma\left(1 - \mathbf{q}/2 - r\rho(\tau) - \frac{\rho(m)}{2}\right)} \tag{2.3}$$

where $\mathbf{q}$ is the vector R-charge, r is the radius of S^2 and ρ is the weight of the representation the matter multiplet belongs to.

Thanks to (2.1), the computation of the partition function is reduced to residues evaluation as

$$\oint \prod_{r=1}^{r_G} \frac{d(r\lambda_r)}{2\pi i} (z\bar{z})^{-r\lambda_r} Z_{11} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{2.4}$$

where $z = e^{-2\pi\vec{\xi} + i\vec{\theta}}$ labels the different vortex sectors, $(z\bar{z})^{-r\lambda_r}$ is a contribution from the classical action, Z_{v} is the equivariant vortex partition function on the north pole patch, Z_{av} is the equivariant vortex partition function on the south pole patch and Z_{11} is the remnant one-loop measure. The contour of integration in (2.4) crucially depends on the choice of the FI-parameters and this, as we will specify better in a moment, encodes the geometric interpretation of the partition function.

One can actually read the GLSM data from a geometric perspective as in the following Table 1 [22].

Let us remark that the GLSM counterpart of the GIT stability condition is in the D-term equation which crucially depends on the FI parameters. The different stability chambers are in one-to-one correspondence with the phases of the GLSM as defined by the domains of the FI parameters. As far as the models that we study in this paper are concerned, for Abelian quotients, when the FIs are large and positive one describes a geometric phase, namely a NLSM on a Kähler target manifold [12], while for negative FIs the GLSM is in a Landau-Ginsburg phase describing an orbifold target space. In the

Table 1. GLSM vs. GIT quotient

GLSM	GW
Matter fields	Quasi-affine variety $\mathcal{A}$
Gauge group G	$G_{\mathbb{C}}$ action on $\mathcal{A}$
F/D-terms	Stable GIT quotient $\mathcal{A}//G_{\mathbb{C}}$

non-Abelian case, the possibility of having a reflection symmetry on the FI opens up leaving the orbifold phases at the fixed point of the reflection.

From the perspective of Eq. (2.4), different FI phases imply that the integral converges at different asymptotic regions of the τ-plane imposing different choices of the contour integral. As we will largely exemplify in the following, this allows us to describe the quantum cohomology of the corresponding GIT quotients in the different stability chambers. In particular, we will study the crepant resolution conjecture for both abelian and non-abelian quotients, focusing on $\mathbb{C}^3/\mathbb{Z}_n$ and on the Uhlembeck (partial) compactification of the ADHM moduli space respectively. This provides conjectural formulas for the $\mathcal{I}$ and $\mathcal{J}$-functions which are shown to reduce in the relevant particular cases to those of [23] for the $\mathbb{Z}_3$ and $\mathbb{Z}_4$ orbifolds and of [24] for the symmetric product of points in $\mathbb{C}^2$ (see later sections).

Let us now provide more details on how the quantum cohomology of the target GIT quotients is computed from the spherical partition function. It has been argued in [12] that the spherical partition function computes the vacuum amplitude of the NLSM in the infrared

$$\langle\bar{0}|0\rangle = e^{-K}, \tag{2.5}$$

where K is the quantum Kähler potential of the target space X. A general argument for the validity of this conjecture has been provided in [13], the main idea of which goes as follows. One considers the spherical partition function on the squashed two-sphere discovering that it is independent on the squashing parameter. Then the limit of extreme squashing is identified with the topological-antitopological fusion $\langle\bar{0}|0\rangle$. We remark that although [13] focused on Calabi–Yau target manifolds, its arguments apply also to Fano manifolds, for which both the A and B-twist are well defined, the latter being a Landau-Ginzburg model with cylinder as its target space. Indeed we will discuss several examples of this type including (weighted) projective spaces and (partial) flag manifolds.

Let us now draw some further steps in the analysis of the spherical partition function from a general viewpoint. Let us rewrite the above vacuum amplitude in a way which is more suitable for our purposes. Following [25,26], let us introduce the flat sections V_a of the Gauss-Manin connection spanning the vacuum bundle of the theory and satisfying

$$\left(\hbar D_a \delta_b^c + C_{ab}^c\right) V_c = 0, \tag{2.6}$$

where D_a is the covariant derivative on the vacuum line bundle and C_{ab}^c are the coefficients of the OPE in the chiral ring of observables $\phi_a\phi_b = C_{ab}^c\phi_c$. The observables $\{\phi_a\}$ provide a basis for the vector space of chiral ring operators $H^0(X) \oplus H^2(X)$ with $a = 0, 1, \ldots, b^2(X)$, ϕ_0 being the identity operator. The parameter $\hbar$ is the spectral parameter of the Gauss-Manin connection. Specifying the case $b = 0$ in (2.6), we find that $V_a = -\hbar D_a V_0$ which means that the flat sections are all generated by the fundamental solution $\mathcal{J} := V_0$ of the equation

$$\left(\hbar D_a D_b + C_{ab}^c D_c\right) \mathcal{J} = 0. \tag{2.7}$$

In order to uniquely fix the solution to (2.7) one needs to supplement some further information about the dependence on the spectral parameter. This is usually done by combining the dimensional analysis of the theory with the the $\hbar$ dependence by fixing

$$(\hbar\partial_{\hbar} + \mathcal{E})\,\mathcal{J} = 0, \tag{2.8}$$

where the covariantly constant Euler vector field $\mathcal{E} = \delta^a D_a$, δ^a being the vector of scaling dimensions of the coupling constants, scales with weight one the chiral ring structure constants as $\mathcal{E}C^c_{ab} = C^c_{ab}$ to ensure compatibility between (2.7) and (2.8).

The metric on the vacuum bundle is given by a symplectic pairing of the flat sections $g_{\bar{a}b} = \langle \bar{a}|b\rangle = V^t_{\bar{a}} E V_b$ and in particular the vacuum-vacuum amplitude, that is the spherical partition function, can be written as the symplectic pairing

$$\langle \bar{0}|0\rangle = \mathcal{J}^t E \mathcal{J} \tag{2.9}$$

for a suitable symplectic form E [25] that will be specified later.

Let us remark that in the case of non compact target, the Quantum Field Theory has to be studied in the equivariant sense to regulate its volume divergences already visible in the constant map contribution. This is accomplished by turning on the relevant twisted masses for matter fields. From the mathematical viewpoint, this amounts to work in the context of equivariant cohomology of the target space $H^\bullet_T(X)$, where T is the torus acting on X. The values of the twisted masses assign the weights of the torus action.

We point out that there is a natural correspondence of the results of supersymmetric localization on the two-sphere with the formalism developed by Givental for the computation of the flat section $\mathcal{J}$. Indeed the computation of the spherical partition function makes use of a supersymmetric charge which closes on a $U(1)$ isometry of the sphere, whose fixed points are the north and south pole. From the string viewpoint it therefore describes the embedding in the target space of a spherical world-sheet *with two marked points* where the gravitational descendants are inserted. This is precisely the setting of S^1-equivariant Gromov–Witten invariants considered by Givental [19] by studying equivariant holomorphic maps with respect to the maximal torus of the sphere automorphisms $S^1 \subset PSL(2, \mathbb{C})$. This is identified with the $U(1)$ isometry to which the supersymmetry algebra squares. As an important consequence, the equivariant parameter $\hbar$ of Givental's S^1 action gets identified with the one of the vortex partition functions arising in the localization of the spherical partition function. An excellent review of Givental's formalism can be found in [27]; here we will highlight the aspects that are strictly relevant for the subsequent discussions. The $\mathcal{J}$-function can be computed from a set of oscillatory integrals, the so called "$\mathcal{I}$-functions" which are generating functions of hypergeometric type in the variables $\hbar$ and Q_i, where $Q_i = e^{-t^i}$, t^i being the complexified Kähler parameters and $i = 1, \ldots, b_2(X)$. We observe that Givental's formalism has been developed originally for abelian quotients, more precisely for complete intersections in quasi-projective toric varieties. In this case, the $\mathcal{I}$ function is the generating function of solutions of the Picard–Fuchs equations for the mirror manifold $\check{X}$ of X, and as such can be expressed in terms of periods on $\check{X}$. From the viewpoint of the spherical partition function this has also a very nice direct interpretation by an alternative rewriting of the vacuum amplitude (2.9). Indeed, by mirror symmetry one can rewrite, in the Calabi–Yau case

$$\langle \bar{0}|0\rangle = i\int_{\check{X}} \overline{\Omega} \wedge \Omega = \Pi^t S \Pi, \tag{2.10}$$

where $\Pi = \int_{\Gamma^i} \Omega$ is the period vector and S is the symplectic pairing. The components of the $\mathcal{I}$-function can be identified with the components of the period vector Π. More generally, one can consider an elaboration of the integral form of the spherical partition function worked out in [13], where the integrand is rewritten in a mirror symmetric manifest form, by expressing the ratios of Γ-functions appearing in the Coulomb branch representation as

$$\frac{\Gamma(\Sigma)}{\Gamma(1-\bar{\Sigma})} = \int_{Im(Y)\sim Im(Y)+2\pi} \frac{d^2Y}{2\pi i} e^{\left[e^{-Y} - \Sigma Y - c.c.\right]}, \tag{2.11}$$

to obtain the right-hand-side (2.10) and then by applying the Riemann bilinear identity, one gets the left-hand side. The resulting integrals, after the integration over the Coulomb parameters and independently on the fact that the mirror representation is geometric or not, are then of the oscillatory type

$$\Pi_i = \oint_{\Gamma_i} d\vec{Y} e^{r\mathcal{W}_{eff}\left(\vec{Y}\right)}, \tag{2.12}$$

where the effective variables $\vec{Y}$ and potential $\mathcal{W}_{eff}$ are the remnants parametrizing the constraints imposed by the integration over the Coulomb parameters before getting to (2.12). Equation (2.12) is also the integral representation of Givental's $\mathcal{I}$-function for general Fano manifolds [27]. Non-abelian quotients have been studied in [20] in terms of quasi-maps theory which is the mathematical counterpart of the GLSM.

Let us now state the dictionary between Givental's formalism and the spherical partition function

$$Z^{S^2} = \oint d\lambda Z_{11} \left(z^{-r|\lambda|} Z_{\mathrm{v}}\right) \left(\bar{z}^{-r|\lambda|} Z_{\mathrm{av}}\right) \tag{2.13}$$

with $d\lambda = \prod_{\alpha=1}^{\mathrm{rank}} d\lambda_\alpha$ and $|\lambda| = \sum_\alpha \lambda_\alpha$. Our claim [28] is that Z_{v} is the $\mathcal{I}$-function of the target space X upon identifying the vortex counting parameter z with Q, λ_α with the generators of the equivariant cohomology and $r = 1/\hbar$. More precisely, the chamber structure of the GIT quotient is encoded in the choice of the FI parameters and the subsequent choice of integration contours. In particular, in the geometric phase with all the FIs large and positive, the vortex counting parameters are identified with the exponentiated complex Kähler parameters, while, in the orbifold phase they label the twisted sectors of the orbifold itself or, in other words, the basis of orbifold cohomology.

The $\mathcal{J}$-function – needed to compute the equivariant Gromov–Witten invariants of X – is then obtained from the $\mathcal{I}$-function after a suitable normalisation procedure which has been described in [28]. Actually, in some cases one can show that the $\mathcal{I}$ and the $\mathcal{J}$-functions coincide and that this normalisation procedure is not required. This is the case of Fano manifolds and ADHM moduli space for rank higher than one.

A further normalization is then required for the one-loop term in order to reproduce the classical intersection cohomology on the target manifold. In this normalization, the spherical partition function coincides with the symplectic pairing (2.9) and in particular the one-loop part reproduces in the $r \to 0$ limit the (equivariant) volume of the target space.

The above conjecture will be checked for several abelian and non abelian GIT quotients in the subsequent sections.

3. Abelian GLSMs

3.1. Projective spaces. Let us start with the basic example, that is $\mathbb{P}^{n-1}$. Its sigma model matter content consists of n chiral fields of charge 1 with respect to the $U(1)$ gauge group. In general, the Fayet-Iliopoulos parameter runs [11]; in our case

$$\xi_{\text{ren}} = \xi - \frac{n}{2\pi} \log(rM) \tag{3.1}$$

with M a SUSY-invariant ultraviolet cut-off. Notice that in the Calabi–Yau case the sum of the charges is zero, therefore[2] $\xi_{\text{ren}} = \xi$.

By defining[3] $\tau = -ir\sigma$ the $\mathbb{P}^{n-1}$ partition function reads

$$Z_{\mathbb{P}^{n-1}} = \sum_{m\in\mathbb{Z}} \int \frac{\mathrm{d}\tau}{2\pi i} e^{4\pi\xi_{\text{ren}}\tau - i\theta_{\text{ren}}m} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1-\tau-\frac{m}{2}\right)} \right)^n \tag{3.2}$$

With the change of variables

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.3}$$

we are resumming over all the poles, which are at $\lambda = 0$. Equation (3.2) then becomes

$$Z_{\mathbb{P}^{n-1}} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{11}^{\mathbb{P}^{n-1}} Z_{\text{v}}^{\mathbb{P}^{n-1}} Z_{\text{av}}^{\mathbb{P}^{n-1}} \tag{3.4}$$

where $z = e^{-2\pi\xi + i\theta}$ and

$$\begin{aligned} Z_{11}^{\mathbb{P}^{n-1}} &= (rM)^{-2nrM\lambda} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)} \right)^n \\ Z_{\text{v}}^{\mathbb{P}^{n-1}} &= z^{-rM\lambda} \sum_{l\geq 0} \frac{[(rM)^n z]^l}{(1-rM\lambda)_l^n} \\ Z_{\text{av}}^{\mathbb{P}^{n-1}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} \frac{[(-rM)^n \bar{z}]^k}{(1-rM\lambda)_k^n} \end{aligned} \tag{3.5}$$

The Pochhammer symbol $(a)_k$ is defined as

$$(a)_k = \begin{cases} \prod_{i=0}^{k-1}(a+i) & \text{for } k > 0 \\ 1 & \text{for } k = 0 \\ \prod_{i=1}^{-k} \frac{1}{a-i} & \text{for } k < 0 \end{cases} \tag{3.6}$$

[2] We will also assume that $\theta_{\text{ren}} = \theta + (s-1)\pi$, with s rank of the gauge group; this implies $\theta_{\text{ren}} = \theta$ for abelian gauge groups. This is necessary in order to reproduce the known results in the mathematical literature for Grassmannians, flag manifolds, and the Hilbert scheme of points; this shift should come from integrating out the W bosons, but we do not have a detailed explanation for it.

[3] We are following the notation of [10], but we work with dimensionless partition functions: this means that in our integrals it appears $d(r\sigma)$ instead of $d\sigma$.

The $\mathcal{I}$-function is given by $Z_{\mathrm{v}}^{\mathbb{P}^{n-1}}$, and coincides with the one given in the mathematical literature,[4]

$$\mathcal{I}_{\mathbb{P}^{n-1}}(H, \hbar; t) = e^{\frac{tH}{\hbar}} \sum_{d \geq 0} \frac{[(\hbar)^{-n} e^t]^d}{(1 + H/\hbar)_d^n} \tag{3.7}$$

if we identify $\hbar = \frac{1}{rM}$, $H = -\lambda$, $t = \ln z$. The antivortex contribution is the conjugate $\mathcal{I}$-function, with $\hbar = -\frac{1}{rM}$, $H = \lambda$ and $\bar{t} = \ln \bar{z}$. The hyperplane class H satisfies $H^n = 0$; in some sense the integration variable λ satisfies the same relation, because the process of integration will take into account only terms up to λ^{n-1} in Z_{v} and Z_{av}.

Complete intersections in $\mathbb{P}^{n-1}$ of type $(q_0, \ldots, q_m)$, $q_j > 0$ can be obtained by adding chiral fields of charge $(-q_0, \ldots, -q_m)$. This means that the integrand in (3.2) gets multiplied by

$$\prod_{j=0}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j \tau + q_j \quad \frac{m}{2}\right)}{\Gamma\left(1 - \frac{R_j}{2} + q_j \tau + q_j \frac{m}{2}\right)} \tag{3.8}$$

The poles are still as in (3.3), but now

$$\begin{aligned} Z_{11}^{\mathbb{P}^{n-1}} &= (rM)^{-2rM(n-|q|)\lambda} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^n \prod_{j=0}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j rM\lambda\right)}{\Gamma\left(1 - \frac{R_j}{2} + q_j rM\lambda\right)} \\ Z_{\mathrm{v}}^{\mathbb{P}^{n-1}} &= z^{-rM\lambda} \sum_{l \geq 0} (-1)^{|q|l} [(rM)^{n-|q|} z]^l \frac{\prod_{j=0}^{m} (\frac{R_j}{2} - q_j rM\lambda)_{q_j l}}{(1 - rM\lambda)_l^n} \\ Z_{\mathrm{av}}^{\mathbb{P}^{n-1}} &= \bar{z}^{-rM\lambda} \sum_{k \geq 0} (-1)^{|q|k} [(-rM)^{n-|q|} \bar{z}]^k \frac{\prod_{j=0}^{m} (\frac{R_j}{2} - q_j rM\lambda)_{q_j k}}{(1 - rM\lambda)_k^n} \end{aligned} \tag{3.9}$$

where $|q| = \sum_{j=0}^{n} q_j$ and R_j is the R-charge of the j-th field. Notice that, if we want to describe a bundle over a space, we should set $R_j = 0$ and add twisted masses in the contributions coming from the fibers, since we want to separate the different cohomology generators (i.e. the different integration variables); we will do this explicitly when needed. On the other hand, complete intersections do not require and do not allow twisted masses, because the insertion of the superpotential breaks all flavour symmetry; moreover, since the superpotential must have R-charge 2, we will need some $R_j \neq 0$ (see the example of the quintic below).

3.1.1. Equivariant projective spaces. The same computation can be repeated in the more general equivariant case, with twisted masses turned on. In this case, the partition function reads (rescaling the twisted masses as $a_i \to M a_i$ in order to have dimensionless parameters)

$$Z_{\mathbb{P}^{n-1}}^{\mathrm{eq}} = \sum_{m \in \mathbb{Z}} \int \frac{\mathrm{d}\tau}{2\pi i} e^{4\pi \xi_{\mathrm{ren}} \tau - i \theta_{\mathrm{ren}} m} \prod_{i=1}^{n} \frac{\Gamma\left(\tau - \frac{m}{2} + irMa_i\right)}{\Gamma\left(1 - \tau - \frac{m}{2} - irMa_i\right)} \tag{3.10}$$

[4] This was already observed in this particular case in [4].

Choosing poles at

$$\tau = -k + \frac{m}{2} - irMa_j + rM\lambda \tag{3.11}$$

we arrive at

$$Z^{\text{eq}}_{\mathbb{P}^{n-1}} = \sum_{j=1}^{n} \oint \frac{d(rM\lambda)}{2\pi i} Z^{\mathbb{P}^{n-1}}_{\text{1l, eq}} Z^{\mathbb{P}^{n-1}}_{\text{v, eq}} Z^{\mathbb{P}^{n-1}}_{\text{av, eq}} \tag{3.12}$$

where

$$\begin{aligned} Z^{\mathbb{P}^{n-1}}_{\text{1l, eq}} &= (z\bar{z})^{irMa_j}(rM)^{-2nrM\lambda} \prod_{i=1}^{n} \frac{\Gamma(rM\lambda + irMa_{ij})}{\Gamma(1 - rM\lambda - irMa_{ij})} \\ Z^{\mathbb{P}^{n-1}}_{\text{v, eq}} &= z^{-rM\lambda} \sum_{l \geq 0} \frac{[(rM)^n z]^l}{\prod_{i=1}^{n}(1 - rM\lambda - irMa_{ij})_l} \\ Z^{\mathbb{P}^{n-1}}_{\text{av, eq}} &= \bar{z}^{-rM\lambda} \sum_{k \geq 0} \frac{[(-rM)^n \bar{z}]^k}{\prod_{i=1}^{n}(1 - rM\lambda - irMa_{ij})_k} \end{aligned} \tag{3.13}$$

and $a_{ij} = a_i - a_j$. Since there are just simple poles, the integration can be easily performed:

$$\begin{aligned} Z^{\text{eq}}_{\mathbb{P}^{n-1}} = &\sum_{j=1}^{n} (z\bar{z})^{irMa_j} \prod_{i \neq j=1}^{n} \frac{1}{irMa_{ij}} \frac{\Gamma(1 + irMa_{ij})}{\Gamma(1 - irMa_{ij})} \\ &\sum_{l \geq 0} \frac{[(rM)^n z]^l}{\prod_{i=1}^{n}(1 - irMa_{ij})_l} \sum_{k \geq 0} \frac{[(-rM)^n \bar{z}]^k}{\prod_{i=1}^{n}(1 - irMa_{ij})_k} \end{aligned} \tag{3.14}$$

In the limit $rM \to 0$ the one-loop contribution [see the first line of (3.14)] provides the equivariant volume of the target space:

$$\text{Vol}(\mathbb{P}^{n-1}_{\text{eq}}) = \sum_{j=1}^{n} (z\bar{z})^{irMa_j} \prod_{i \neq j=1}^{n} \frac{1}{irMa_{ij}} = \sum_{j=1}^{n} e^{-4\pi i \xi rMa_j} \prod_{i \neq j=1}^{n} \frac{1}{irMa_{ij}} \tag{3.15}$$

Using the fact that

$$\lim_{r \to 0} \sum_{j=1}^{n} \frac{e^{-4\pi i \xi rMa_j}}{(4\xi)^{n-1}} \prod_{i \neq j=1}^{n} \frac{1}{irMa_{ij}} = \frac{\pi^{n-1}}{(n-1)!} \tag{3.16}$$

we find the non-equivariant volume

$$\text{Vol}(\mathbb{P}^{n-1}) = \frac{(4\pi\xi)^{n-1}}{(n-1)!} \tag{3.17}$$

3.1.2. Weighted projective spaces. Another generalization consists in studying the weighted projective space $\mathbb{P}^{\mathbf{w}} = \mathbb{P}(w_0, \ldots, w_n)$, which has been studied from the mathematical point of view in [23]. This can be obtained by considering an $U(1)$ gauge theory with $n+1$ fundamentals of (positive) integer charges $w_0, \ldots, w_n$. The partition function reads

$$Z = \sum_m \int \frac{d\tau}{2\pi i} e^{4\pi\xi_{\text{ren}}\tau - i\theta_{\text{ren}}m} \prod_{i=0}^{n} \frac{\Gamma(w_i\tau - w_i\frac{m}{2})}{\Gamma(1 - w_i\tau - w_i\frac{m}{2})} \tag{3.18}$$

so one would expect $n+1$ towers of poles at

$$\tau = \frac{m}{2} - \frac{k}{w_i} + rM\lambda, \quad i = 0 \ldots n \tag{3.19}$$

with integration around $rM\lambda = 0$. Actually, in this way we might be overcounting some poles if the w_i are not relatively prime, and in any case the pole $\tau = 0$ is always counted $n+1$ times. In order to solve these problems, we will set

$$\tau = \frac{m}{2} - k + rM\lambda - F \tag{3.20}$$

where F is a set of rational numbers defined as

$$F = \left\{\frac{d}{w_i} / 0 \le d < w_i, \ \ d \in \mathbb{N}, 0 \le i \le n\right\} \tag{3.21}$$

and every number has to be counted only once. Let us explain this better with an example: if we consider just $w_0 = 2$ and $w_1 = 3$, we find the numbers (0, 1/2) and (0, 1/3, 2/3), which means $F = (0, 1/3, 1/2, 2/3)$; the multiplicity of these numbers reflects the order of the pole in the integrand, so we will have a double pole (counted by the double multiplicity of $d = 0$) and three simple poles.

The partition function then becomes

$$Z = \sum_F \oint \frac{d(rM\lambda)}{2\pi i} Z_{11}\, Z_{\text{v}}\, Z_{\text{av}} \tag{3.22}$$

with integration around $rM\lambda = 0$ and

$$\begin{aligned}
Z_{11} &= (rM)^{-2|w|rM\lambda - 2\sum_{i=0}^{n}(\omega[w_iF] - \langle w_iF\rangle)} \prod_{i=0}^{n} \frac{\Gamma(\omega[w_iF] + w_i rM\lambda - \langle w_iF\rangle)}{\Gamma(1 - \omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)} \\
Z_{\text{v}} &= z^{-rM\lambda} \sum_{l\ge 0} \frac{(rM)^{|w|l + \sum_{i=0}^{n}(\omega[w_iF] + [w_iF])} z^{l+F}}{\prod_{i=0}^{n}(1 - \omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)_{w_il + [w_iF] + \omega[w_iF]}} \\
Z_{\text{av}} &= \bar{z}^{-rM\lambda} \sum_{k\ge 0} \frac{(-rM)^{|w|k + \sum_{i=0}^{n}(\omega[w_iF] + [w_iF])} \bar{z}^{k+F}}{\prod_{i=0}^{n}(1 - \omega[w_iF] - w_i rM\lambda + \langle w_iF\rangle)_{w_ik + [w_iF] + \omega[w_iF]}}
\end{aligned} \tag{3.23}$$

In the formulae we defined $\langle w_iF\rangle$ and $[w_iF]$ as the fractional and integer part of the number w_iF, so that $w_iF = [w_iF] + \langle w_iF\rangle$, while $|w| = \sum_{i=0}^{n} w_i$. Moreover,

$$\omega[w_iF] = \begin{cases} 0 & \text{for } \langle w_iF\rangle = 0 \\ 1 & \text{for } \langle w_iF\rangle \neq 0 \end{cases} \tag{3.24}$$

This is needed in order for the $\mathcal{J}$ function to start with one in the rM expansion.

The twisted sectors in (3.21) label the base of the orbifold cohomology space.

Once more, we can also consider complete intersections in $\mathbb{P}^{\mathbf{w}}$ of type $(q_0, \ldots, q_m)$. The integrand in (3.18) has to be multiplied by

$$\prod_{j=0}^{m} \frac{\Gamma\left(\frac{R_j}{2} - q_j\tau + q_j\frac{m}{2}\right)}{\Gamma\left(1 - \frac{R_j}{2} + q_j\tau + q_j\frac{m}{2}\right)} \tag{3.25}$$

The poles do not change, and

$$\begin{aligned}
Z_{11} &= (rM)^{-2(|w|-|q|)rM\lambda - 2\sum_{i=0}^{n}(\omega[w_iF]-\langle w_iF\rangle) - 2\sum_{j=0}^{m}\langle q_jF\rangle} \\
&\quad \prod_{i=0}^{n} \frac{\Gamma(\omega[w_iF] + w_irM\lambda - \langle w_iF\rangle)}{\Gamma(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)} \prod_{j=0}^{m} \frac{\Gamma(\frac{R_j}{2} - q_jrM\lambda + \langle q_jF\rangle)}{\Gamma(1-\frac{R_j}{2} + q_jrM\lambda - \langle q_jF\rangle)} \\
Z_{\mathrm{v}} &= z^{-rM\lambda} \sum_{l\geq 0} (-1)^{|q|l + \sum_{j=0}^{m}[q_jF]} (rM)^{(|w|-|q|)l + \sum_{i=0}^{n}(\omega[w_iF]+[w_iF]) - \sum_{j=0}^{m}[q_jF]} z^{l+F} \\
&\quad \frac{\prod_{j=0}^{m}(\frac{R_j}{2} - q_jrM\lambda + \langle q_jF\rangle)_{q_jl+[q_jF]}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)_{w_il+[w_iF]+\omega[w_iF]}} \\
Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geq 0} (-1)^{|q|k + \sum_{j=0}^{m}[q_jF]} (-rM)^{(|w|-|q|)k + \sum_{i=0}^{n}(\omega[w_iF]+[w_iF]) - \sum_{j=0}^{m}[q_jF]} \bar{z}^{k+F} \\
&\quad \frac{\prod_{j=0}^{m}(\frac{R_j}{2} - q_jrM\lambda + \langle q_jF\rangle)_{q_jk+[q_jF]}}{\prod_{i=0}^{n}(1-\omega[w_iF] - w_irM\lambda + \langle w_iF\rangle)_{w_ik+[w_iF]+\omega[w_iF]}}
\end{aligned} \tag{3.26}$$

Notice that the non linear sigma model to which the GLSM flows in the IR is well defined only for $|w| \geq |q|$, which means for manifolds with $c_1 \geq 0$.

3.2. Quintic. We will now consider the most famous compact Calabi–Yau threefold, i.e. the quintic. The corresponding GLSM is a $U(1)$ gauge theory with five chiral fields Φ_a of charge +1, one chiral field P of charge -5 and a superpotential of the form $W = PG(\Phi_1, \ldots, \Phi_5)$, where G is a homogeneous polynomial of degree five. We choose the vector R-charges to be $2q$ for the Φ fields and $(2 - 5 \cdot 2q)$ for P such that the superpotential has R-charge 2. The quintic threefold is realized in the geometric phase corresponding to $\xi > 0$. For details of the construction see [22] and for the relation to the two-sphere partition function [12]. Here we want to investigate the connection to the Givental formalism. For a Calabi–Yau manifold the sum of gauge charges is zero, which implies $\xi_{\mathrm{ren}} = \xi$, and $\theta_{\mathrm{ren}} = \theta$ holds because the gauge group is abelian. The spherical partition function is

$$Z = \sum_{m\in\mathbb{Z}} \int_{i\mathbb{R}} \frac{d\tau}{2\pi i} z^{-\tau-\frac{m}{2}} \bar{z}^{-\tau+\frac{m}{2}} \left(\frac{\Gamma\left(q+\tau-\frac{m}{2}\right)}{\Gamma\left(1-q-\tau-\frac{m}{2}\right)}\right)^5 \frac{\Gamma\left(1-5q-5\tau+5\frac{m}{2}\right)}{\Gamma\left(5q+5\tau+5\frac{m}{2}\right)}. \tag{3.27}$$

Since we want to describe the phase $\xi > 0$, we have to close the contour in the left half plane. We use the freedom in q to separate the towers of poles coming from Φ's and from P. In the range $0 < q < \frac{1}{5}$ the former lie in the left half plane while the latter in the right half plane. So we pick only the poles corresponding to Φ's given by

$$\tau_k = -q - k + \frac{m}{2}, \qquad k \geq \max(0, m) \tag{3.28}$$

Then the partition function turns into a sum of residues and we express each residue by the Cauchy contour integral. Finally we arrive at

$$Z = (z\bar{z})^q \oint_{\mathcal{C}(\delta)} \frac{d(rM\lambda)}{2\pi i} Z_{11}(\lambda, rM) Z_{\text{v}}(\lambda, rM; z) Z_{\text{av}}(\lambda, rM; \bar{z}), \tag{3.29}$$

where the contour $\mathcal{C}(\delta)$ goes around $\lambda = 0$ and

$$\begin{aligned} Z_{11}(\lambda, rM) &= \frac{\Gamma(1 - 5rM\lambda)}{\Gamma(5rM\lambda)} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \right)^5 \\ Z_{\text{v}}(\lambda, rM; z) &= z^{-rM\lambda} \sum_{l \geqslant 0} (-z)^l \frac{(1 - 5rM\lambda)_{5l}}{[(1 - rM\lambda)_l]^5} \\ Z_{\text{av}}(\lambda, rM; \bar{z}) &= \bar{z}^{-rM\lambda} \sum_{k \geqslant 0} (-\bar{z})^k \frac{(1 - 5rM\lambda)_{5k}}{[(1 - rM\lambda)_k]^5} \end{aligned} \tag{3.30}$$

The vortex function $Z_{\text{v}}(\lambda, rM; z)$ reproduces the known Givental $\mathcal{I}$-function

$$\mathcal{I}(H, \hbar; t) = \sum_{d \geqslant 0} e^{(H/\hbar + d)t} \frac{(1 + 5H/\hbar)_{5d}}{[(1 + H/\hbar)_d]^5} \tag{3.31}$$

after identifying

$$H = -\lambda, \quad \hbar = \frac{1}{rM}, \quad t = \ln(-z). \tag{3.32}$$

The $\mathcal{I}$-function is valued in cohomology, where $H \in H^2(\mathbb{P}^4)$ is the hyperplane class in the cohomology ring of the embedding space. Because of dimensional reasons we have $H^5 = 0$ and hence the $\mathcal{I}$-function is a polynomial of order four in H

$$\mathcal{I} = I_0 + \frac{H}{\hbar} I_1 + \left(\frac{H}{\hbar}\right)^2 I_2 + \left(\frac{H}{\hbar}\right)^3 I_3 + \left(\frac{H}{\hbar}\right)^4 I_4. \tag{3.33}$$

This is naturally encoded in the explicit residue evaluation of (3.29), see Eq. (3.36). Now consider the Picard–Fuchs operator L. It can be easily shown that $\{I_0, I_1, I_2, I_3\} \in \text{Ker}(L)$ while $I_4 \notin \text{Ker}(L)$. L is an order four operator and so $\mathbf{I} = (I_0, I_1, I_2, I_3)^T$ form a basis of solutions. There exists another basis formed by the periods of the holomorphic $(3, 0)$ form of the mirror manifold. In homogeneous coordinates they are given as $\Pi = (X^0, X^1, \frac{\partial F}{\partial X^0}, \frac{\partial F}{\partial X^1})^T$ with F the prepotential. Thus there exists a transition matrix $\mathbf{M}$ relating these two bases

$$\mathbf{I} = \mathbf{M} \cdot \Pi \tag{3.34}$$

There are now two possible ways to proceed. One would be fixing the transition matrix using mirror construction (i.e. knowing explicitly the periods) and then showing that the pairing given by the contour integral in (3.29) after being transformed to the period basis gives the standard formula for the Kähler potential in terms of a symplectic pairing

$$e^{-K} = i\Pi^\dagger \cdot \Sigma \cdot \Pi \tag{3.35}$$

with $\Sigma = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$ being the symplectic form. The other possibility would be to use the fact that the two sphere partition function computes the Kähler potential [12] and then impose equality between (3.29) and (3.35) to fix the transition matrix. We follow this route in the following. The contour integral in (3.29) expresses the Kähler potential as a pairing in the $\mathbf{I}$ basis. It is governed by Z_{11} which has an expansion

$$Z_{11} = \frac{5}{(rM\lambda)^4} + \frac{400\,\zeta(3)}{rM\lambda} + o(1) \tag{3.36}$$

and so we get after integration (remember that $H/\hbar = -rM\lambda$)

$$\begin{aligned} Z &= -2\chi\zeta(3) I_0 \bar{I}_0 - 5(I_0\bar{I}_3 + I_1\bar{I}_2 + I_2\bar{I}_1 + I_3\bar{I}_0) \\ &= \mathbf{I}^\dagger \cdot \mathbf{A} \cdot \mathbf{I}, \end{aligned} \tag{3.37}$$

where

$$\mathbf{A} = \begin{pmatrix} -2\chi\zeta(3) & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 \\ 0 & -5 & 0 & 0 \\ -5 & 0 & 0 & 0 \end{pmatrix} \tag{3.38}$$

gives the pairing in the $\mathbf{I}$ basis and $\chi = -200$ is the Euler characteristic of the quintic threefold. From the two expressions for the Kähler potential we easily find the transition matrix as

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{5} \\ -\frac{\chi}{5}\zeta(3) & 0 & -\frac{i}{5} & 0 \end{pmatrix}. \tag{3.39}$$

Finally, we know that the mirror map is given by

$$t = \frac{I_1}{2\pi i I_0}, \quad \bar{t} = -\frac{\bar{I}_1}{2\pi i \bar{I}_0} \tag{3.40}$$

so after dividing Z by $(2\pi i)^2 I_0 \bar{I}_0$ for the change of coordinates and by a further 2π for the normalization of the $\zeta(3)$ term, we obtain the Kähler potential in terms of $t, \bar{t}$, in a form in which the symplectic product is evident.

3.3. Local Calabi–Yau: $\mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$. Let us now study the family of spaces $X_p = \mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$ with diagonal equivariant action on the fiber. We will find exact agreement with the $\mathcal{I}$ functions computed in [29], and we will show how the quantum corrected Kähler potential for the Kähler moduli space can be computed when equivariant parameters are turned on.

Here we will restrict only to the phase $\xi > 0$, which is the one related to X_p. The case $\xi < 0$ describes the orbifold phase of the model; this will be studied in the following sections.

3.3.1. Case $p = -1$. First of all, we have to write down the partition function; this is given by

$$Z_{-1} = \sum_{m\in\mathbb{Z}} e^{-im\theta} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1 - \tau - \frac{m}{2}\right)} \right)^2 \left(\frac{\Gamma\left(-\tau - irMa + \frac{m}{2}\right)}{\Gamma\left(1 + \tau + irMa + \frac{m}{2}\right)} \right)^2 \tag{3.41}$$

The poles are located at

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.42}$$

so we can rewrite (3.41) as

$$Z_{-1} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{11} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{3.43}$$

where

$$\begin{aligned} Z_{11} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \frac{\Gamma(-rM\lambda - irMa)}{\Gamma(1 + rM\lambda + irMa)} \right)^2 \\ Z_{\mathrm{v}} &= z^{-rM\lambda} \sum_{l\geqslant 0} z^l \frac{(-rM\lambda - irMa)_l^2}{(1 - rM\lambda)_l^2} \\ Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda} \sum_{k\geqslant 0} \bar{z}^k \frac{(-rM\lambda - irMa)_k^2}{(1 - rM\lambda)_k^2} \end{aligned} \tag{3.44}$$

Notice that our vortex partition function coincides with the Givental function given in [29]

$$\mathcal{I}_{-1}^T(q) = e^{\frac{H}{\hbar}\ln q} \sum_{d\geqslant 0} \frac{(1 - H/\hbar + \tilde{\lambda}/\hbar - d)_d^2}{(1 + H/\hbar)_d^2} q^d \tag{3.45}$$

after the usual identifications

$$H = -\lambda, \quad \hbar = \frac{1}{rM}, \quad \tilde{\lambda} = ia, \quad q = z \tag{3.46}$$

Now, expanding $\mathcal{I}_{-1}^T$ in $rM = 1/\hbar$ we find

$$\mathcal{I}_{-1}^T = 1 - rM\lambda \log z + o((rM)^2) \tag{3.47}$$

which means the mirror map is trivial and the equivariant mirror map absent, i.e. $\mathcal{I}_{-1}^T = \mathcal{J}_{-1}^T$. What remains to be specified is the normalization of the 1-loop factor. As explained in [28], this normalization is fixed by requiring the cancellation of the Euler–Mascheroni constants appearing in the Weierstrass form of the Γ-function, reproduces the classical intersection numbers and starts from 1 in the rM expansion; in our case, the factor

$$(z\bar{z})^{-irMa/2} \left(\frac{\Gamma(1 + irMa)}{\Gamma(1 - irMa)} \right)^2 \tag{3.48}$$

does the job. We can now integrate in $rM\lambda$ and expand in rM, obtaining (for $rMa = iq$)

$$Z_{-1} = \frac{2}{q^3} - \frac{1}{4q}\ln^2(z\bar{z}) + \Big[-\frac{1}{12}\ln^3(z\bar{z}) - \ln(z\bar{z})(\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})) \\ +2(\mathrm{Li}_3(z) + \mathrm{Li}_3(\bar{z})) + 4\zeta(3)\Big] + o(rM) \tag{3.49}$$

The terms inside the square brackets reproduce the Kähler potential we are interested in, once we multiply everything by $\frac{1}{2\pi(2\pi i)^2}$ and define

$$t = \frac{1}{2\pi i}\ln z, \quad \bar{t} = -\frac{1}{2\pi i}\ln \bar{z}. \tag{3.50}$$

3.3.2. Case $p = 0$. In this case case, the spherical partition function is

$$Z_0 = \sum_{m\in\mathbb{Z}} e^{-im\theta}\int \frac{d\tau}{2\pi i}e^{4\pi\xi\tau} \\ \times\left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1-\tau-\frac{m}{2}\right)}\right)^2 \frac{\Gamma(-irMa)}{\Gamma(1+irMa)}\frac{\Gamma\left(-2\tau - irMa + 2\frac{m}{2}\right)}{\Gamma\left(1+2\tau+irMa+2\frac{m}{2}\right)} \tag{3.51}$$

The poles are as in (3.42), and usual manipulations result in

$$\begin{aligned} Z_{11} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^2 \frac{\Gamma(-irMa)}{\Gamma(1+irMa)}\frac{\Gamma(-2rM\lambda - irMa)}{\Gamma(1+2rM\lambda+irMa)} \\ Z_{\mathrm{v}} &= z^{-rM\lambda}\sum_{l\geqslant 0} z^l \frac{(-2rM\lambda - irMa)_{2l}}{(1-rM\lambda)_l^2} \\ Z_{\mathrm{av}} &= \bar{z}^{-rM\lambda}\sum_{k\geqslant 0} \bar{z}^k \frac{(-2rM\lambda - irMa)_{2k}}{(1-rM\lambda)_k^2} \end{aligned} \tag{3.52}$$

Again, we recover the Givental function

$$\mathcal{I}_0^T(q) = e^{\frac{H}{\hbar}\ln q}\sum_{d\geqslant 0}\frac{(1-2H/\hbar + \tilde{\lambda}/\hbar - 2d)_{2d}}{(1+H/\hbar)_d^2}q^d \tag{3.53}$$

of [29] under the map (3.46); its expansion in rM

$$\mathcal{I}_0^T = 1 - rM\lambda\left[\log z + 2\sum_{k=1}^{\infty} z^k\frac{\Gamma(2k)}{(k!)^2}\right] - irMa\sum_{k=1}^{\infty} z^k\frac{\Gamma(2k)}{(k!)^2} + o((rM)^2) \tag{3.54}$$

implies that the mirror map is (modulo $(2\pi i)^{-1}$)

$$t = \log z + 2\sum_{k=1}^{\infty} z^k \frac{\Gamma(2k)}{(k!)^2} \tag{3.55}$$

and the equivariant mirror map is

$$\tilde{t} = \frac{1}{2}(t - \log z) = \sum_{k=1}^{\infty} z^k \frac{\Gamma(2k)}{(k!)^2} \tag{3.56}$$

The $\mathcal{J}$ function can be recovered by inverting the equivariant mirror map and changing coordinates accordingly, that is

$$\mathcal{J}_0^T(t) = e^{irMa\tilde{t}(z)} \mathcal{I}_0^T(z) = e^{irMa\tilde{t}(z)} Z_{\mathrm{v}}(z) \tag{3.57}$$

A similar job has to be done for Z_{av}. The normalization for the 1-loop factor is the same as (3.48) but in t coordinates, which means

$$(t\bar{t})^{-irMa/2} \left(\frac{\Gamma(1 + irMa)}{\Gamma(1 - irMa)} \right)^2; \tag{3.58}$$

Finally, integrating in $rM\lambda$ and expanding in rM we find

$$\begin{aligned} Z_0 = \frac{2}{q^3} - \frac{1}{4q}(t+\bar{t})^2 + \Big[-\frac{1}{12}(t+\bar{t})^3 - (t+\bar{t})(\mathrm{Li}_2(e^t) + \mathrm{Li}_2(e^{\bar{t}})) \\ +2(\mathrm{Li}_3(e^t) + \mathrm{Li}_3(e^{\bar{t}})) + 4\zeta(3) \Big] + o(rM) \end{aligned} \tag{3.59}$$

As it was shown in [29], this proves that the two Givental functions $\mathcal{J}_{-1}^T$ and $\mathcal{J}_0^T$ are the same, as well as the Kähler potentials; the $\mathcal{I}$ functions look different simply because of the choice of coordinates on the moduli space.

3.3.3. Case $p \geq 1$. In the general $p \geq 1$ case, we have

$$\begin{aligned} Z_p = \sum_{m \in \mathbb{Z}} e^{-im\theta} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau} \left(\frac{\Gamma\left(\tau - \frac{m}{2}\right)}{\Gamma\left(1 - \tau - \frac{m}{2}\right)} \right)^2 \\ \frac{\Gamma\left(-(p+2)\tau - irMa + (p+2)\frac{m}{2}\right)}{\Gamma\left(1 + (p+2)\tau + irMa + (p+2)\frac{m}{2}\right)} \frac{\Gamma\left(p\tau - irMa - p\frac{m}{2}\right)}{\Gamma\left(1 - p\tau + irMa - p\frac{m}{2}\right)} \end{aligned} \tag{3.60}$$

There are two classes of poles, given by

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.61}$$

$$\tau = -k + \frac{m}{2} + rM\lambda - F + irM\frac{a}{p} \tag{3.62}$$

where $F = \{0, \frac{1}{p}, \ldots, \frac{p-1}{p}\}$ and the integration is around $rM\lambda = 0$. This can be understood from the fact that actually the GLSM (3.60) describes the canonical bundle over the weighted projective space $\mathbb{P}_{(1,1,p)}$, which has two chambers. The regular one, associated to the poles (3.61), corresponds to the local $\mathcal{O}(p) \oplus \mathcal{O}(-2-p) \to \mathbb{P}^1$ geometry:

$$Z_p^{(0)} = \oint \frac{d(rM\lambda)}{2\pi i} Z_{11}^{(0)} Z_{\mathrm{v}}^{(0)} Z_{\mathrm{av}}^{(0)} \tag{3.63}$$

with

$$
\begin{aligned}
Z_{11}^{(0)} &= \left(\frac{\Gamma(rM\lambda)}{\Gamma(1-rM\lambda)}\right)^2 \frac{\Gamma(-(p+2)rM\lambda - irMa)}{\Gamma(1+(p+2)rM\lambda + irMa)} \frac{\Gamma(prM\lambda - irMa)}{\Gamma(1-prM\lambda + irMa)} \\
Z_{\rm v}^{(0)} &= z^{-rM\lambda} \sum_{l\geqslant 0} (-1)^{(p+2)l} z^l \frac{(-(p+2)rM\lambda - irMa)_{(p+2)l}}{(1-rM\lambda)_l^2 (1-prM\lambda + irMa)_{pl}} \\
Z_{\rm av}^{(0)} &= \bar{z}^{-rM\lambda} \sum_{k\geqslant 0} (-1)^{(p+2)k} \bar{z}^k \frac{(-(p+2)rM\lambda - irMa)_{(p+2)k}}{(1-rM\lambda)_k^2 (1-prM\lambda + irMa)_{pk}}
\end{aligned} \tag{3.64}
$$

The second chamber, associated to (3.62), is an orbifold one:

$$
Z_p^{(F)} = \sum_{\delta=0}^{p-1} \oint \frac{d(rM\lambda)}{2\pi i} Z_{11,\delta}^{(F)} Z_{{\rm v},\delta}^{(F)} Z_{{\rm av},\delta}^{(F)} \tag{3.65}
$$

where $F = \frac{\delta}{p}$. The explicit expression for $Z^{(F)}$ in the above formula can be recovered from (3.26), adding the twisted masses in the appropriate places. Notice that (3.65) can be easily integrated, since there are just simple poles.

3.4. Orbifold Gromov–Witten invariants. In this section we want to show how the analytic structure of the partition function encodes all the classical phases of the abelian GLSM. These are given by the secondary fan, which in our conventions is generated by the columns of the charge matrix Q. In terms of the partition function these phases are governed by the choice of integration contours, namely by the structure of poles we are picking up. The contour can be closed either in the left half plane (for $\xi > 0$) or in the right half plane ($\xi < 0$).[5] The transition between different phases occurs when some of the integration contours are flipped and the corresponding variable is integrated. To summarize, a single partition function contains the $\mathcal{I}$-functions of geometries corresponding to all the different phases of the GLSM. These geometries are related by minimally resolving the singularities by blow-up until the complete smoothing of the space takes place (when this is possible). Our procedure consists in considering the GLSM corresponding to the complete resolution and its partition function. Then by flipping contours and doing partial integrations one discovers all other, more singular geometries. In the following we illustrate these ideas on a couple of examples.

3.4.1. $K_{\mathbb{P}^{n-1}}$ vs. $\mathbb{C}^n/\mathbb{Z}_n$. Let us consider a $U(1)$ gauge theory with n chiral fields of charge $+1$ and one chiral field of charge $-n$. The secondary fan is generated by two vectors $\{1, -n\}$ and so has two chambers corresponding to two different phases. For $\xi > 0$ it describes a smooth geometry $K_{\mathbb{P}^{n-1}}$, that is the total space of the canonical bundle over the complex projective space $\mathbb{P}^{n-1}$, while for $\xi < 0$ the orbifold $\mathbb{C}^n/\mathbb{Z}_n$. The case $n = 3$ will reproduce the results of [30–32]. The partition function reads

$$
Z = \sum_m \int_{i\mathbb{R}} \frac{d\tau}{2\pi i} e^{4\pi\xi\tau - i\theta m} \left(\frac{\Gamma(\tau - \frac{m}{2})}{\Gamma(1-\tau-\frac{m}{2})}\right)^n \frac{\Gamma(-n\tau + n\frac{m}{2} + irMa)}{\Gamma(1+n\tau + n\frac{m}{2} - irMa)} \tag{3.66}
$$

[5] This is only true for Calabi–Yau manifolds; for $c_1 > 0$, i.e. $\sum_i Q_i > 0$, the contour is fixed.

Closing the contour in the left half plane (i.e. for $\xi > 0$) we take poles at

$$\tau = -k + \frac{m}{2} + rM\lambda \tag{3.67}$$

and obtain

$$Z = \oint \frac{d(rM\lambda)}{2\pi i} \left(\frac{\Gamma(rM\lambda)}{\Gamma(1 - rM\lambda)} \right)^n \frac{\Gamma(-nrM\lambda + irMa)}{\Gamma(1 + nrM\lambda - irMa)}$$
$$\sum_{l \geq 0} z^{-rM\lambda} (-1)^{nl} z^{nl} \frac{(-nrM\lambda + irMa)_{nl}}{(1 - rM\lambda)_l^n}$$
$$\sum_{k \geq 0} \bar{z}^{-rM\lambda} (-1)^{nk} \bar{z}^{nk} \frac{(-nrM\lambda + irMa)_{nk}}{(1 - rM\lambda)_k^n} \tag{3.68}$$

We thus find exactly the Givental function for $K_{\mathbb{P}^{n-1}}$. To switch to the singular geometry we flip the contour and do the integration. Closing in the right half plane ($\xi < 0$) we consider

$$\tau = k + \frac{\delta}{n} + \frac{m}{2} + \frac{1}{n} irMa \tag{3.69}$$

with $\delta = 0, 1, 2, \ldots, n-1$. After integrating over τ, we obtain

$$Z = \frac{1}{n} \sum_{\delta=0}^{n-1} \left(\frac{\Gamma(\frac{\delta}{n} + \frac{1}{n} irMa)}{\Gamma(1 - \frac{\delta}{n} - \frac{1}{n} irMa)} \right)^n \frac{1}{(rM)^{2\delta}}$$
$$\sum_{k \geq 0} (-1)^{nk} (\bar{z}^{-1/n})^{nk+\delta+irMa} (rM)^{\delta} \frac{(\frac{\delta}{n} + \frac{1}{n} irMa)_k^n}{(nk+\delta)!}$$
$$\sum_{l \geq 0} (-1)^{nl} (z^{-1/n})^{nl+\delta+irMa} (-rM)^{\delta} \frac{(\frac{\delta}{n} + \frac{1}{n} irMa)_l^n}{(nl+\delta)!} \tag{3.70}$$

as expected from (3.26). Notice that when the contour is closed in the right half plane, vortex and antivortex contributions are exchanged. We can compare the $n = 3$ case corresponding to $\mathbb{C}^3/\mathbb{Z}_3$ with [32], given by

$$\mathcal{I} = x^{-\lambda/z} \sum_{\substack{d \in \mathbb{N} \\ d \geq 0}} \frac{x^d}{d! z^d} \prod_{\substack{0 \leq b < \frac{d}{3} \\ \langle b \rangle = \langle \frac{d}{3} \rangle}} \left(\frac{\lambda}{3} - bz \right)^3 \mathbf{1}_{\langle \frac{d}{3} \rangle} \tag{3.71}$$

which in a more familiar notation becomes

$$\mathcal{I} = x^{-\lambda/z} \sum_{\substack{d \in \mathbb{N} \\ d \geq 0}} \frac{x^d}{d!} \frac{1}{z^{3\langle \frac{d}{3} \rangle}} (-1)^{3[\frac{d}{3}]} \left(\langle \frac{d}{3} \rangle - \frac{\lambda}{3z} \right)^3_{[\frac{d}{3}]} \mathbf{1}_{\langle \frac{d}{3} \rangle} \tag{3.72}$$

The necessary identifications are straightforward.

3.4.2. The quantum cohomology of $\mathbb{C}^3/\mathbb{Z}_{p+2}$ and its crepant resolution. We now consider the orbifold space $\mathbb{C}^3/\mathbb{Z}_{p+2}$ with weights $(1, 1, p)$ and $p > 1$. Its full crepant resolution is provided by a resolved transversal A_{p+1} singularity (namely a local Calabi–Yau threefold obtained by fibering the resolved A_{p+1} singularity over a $\mathbb{P}^1$ base space). The corresponding GLSM contains $p+2$ abelian gauge groups and $p+5$ chiral multiplets, with the following charge assignment:

$$\begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -j-1 & j & 0 & 0 & 0 & 0 & \dots & 0 & \overset{(5+j)\text{th}}{1} & 0 & \dots & 0 \\ -p-2 & p+1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.73}$$

where $1 \le j \le p$. In the following we focus on the particular chambers corresponding to the partial resolutions $K_{\mathbb{F}_p}$ and $K_{\mathbb{P}^2(1,1,p)}$. Let us start by discussing the local $\mathbb{F}_p$ chamber: this can be seen by replacing the last row in (3.73) with the linear combination

$$(\text{last row}) \quad \longrightarrow \quad (\text{last row}) - p\,(\text{second row}) - (\text{first row}) \tag{3.74}$$

which corresponds to

$$\begin{pmatrix} -p-2 & p+1 & 1 & 0 & 0 & 0 & \dots \end{pmatrix} \longrightarrow \begin{pmatrix} p-2 & 0 & 0 & 1 & 1 & -p & \dots \end{pmatrix} \tag{3.75}$$

The charge matrix (3.73) now reads ($2 \le n \le p$)

$$\begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ -n-1 & n & 0 & 0 & 0 & 0 & \dots & 0 & \overset{(5+n)\text{th}}{1} & 0 & \dots & 0 \\ p-2 & 0 & 0 & 1 & 1 & -p & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.76}$$

and, in a particular sector (i.e. for a particular choice of poles), after turning to infinity p Fayet-Iliopoulos parameters, we remain with the second and the last row:

$$Q = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ p-2 & 0 & 1 & 1 & -p \end{pmatrix} \tag{3.77}$$

which is the charge matrix of $K_{\mathbb{F}_p}$.

Let us see how this happens in detail; since it is easier for our purposes, we will consider the charge matrix (3.76). For generic p, the partition function with the addition of a twisted mass for the field corresponding to the first column of (3.76) is given by

$$\begin{aligned} Z = \sum_{m_0,\dots,m_{p+1}} \oint &\left[\prod_{i=0}^{p+1} \frac{d\tau_i}{2\pi i} z_i^{-\tau_i - \frac{m_i}{2}} \bar{z}_i^{-\tau_i + \frac{m_i}{2}} \right] \left[\prod_{j=0}^{p} \frac{\Gamma(\tau_j - \frac{m_j}{2})}{\Gamma(1 - \tau_j - \frac{m_j}{2})} \right] \\ &\frac{\Gamma(\tau_1 - p\tau_{p+1} - \frac{m_1}{2} + p\frac{m_{p+1}}{2})}{\Gamma(1 - \tau_1 + p\tau_{p+1} - \frac{m_1}{2} + p\frac{m_{p+1}}{2})} \left(\frac{\Gamma(-\tau_0 + \tau_{p+1} + \frac{m_0}{2} - \frac{m_{p+1}}{2})}{\Gamma(1 + \tau_0 - \tau_{p+1} + \frac{m_0}{2} - \frac{m_{p+1}}{2})} \right)^2 \\ &\frac{\Gamma(\tau_0 + \sum_{j=1}^{p} j\tau_j - \frac{m_0}{2} - \sum_{j=1}^{p} j\frac{m_j}{2})}{\Gamma(1 - \tau_0 - \sum_{j=1}^{p} j\tau_j - \frac{m_0}{2} - \sum_{j=1}^{p} j\frac{m_j}{2})} \\ &\frac{\Gamma(-\sum_{j=1}^{p}(j+1)\tau_j + (p-2)\tau_{p+1} + \sum_{j=1}^{p}(j+1)\frac{m_j}{2} - (p-2)\frac{m_{p+1}}{2} + irMa)}{\Gamma(1 + \sum_{j=1}^{p}(j+1)\tau_j - (p-2)\tau_{p+1} + \sum_{j=1}^{p}(j+1)\frac{m_j}{2} - (p-2)\frac{m_{p+1}}{2} - irMa)} \end{aligned} \tag{3.78}$$

Now, choosing the sector

$$\begin{aligned}\tau_0 &= -k_0 + \frac{m_0}{2} \\ \tau_n &= -k_n + \frac{m_n}{2}, \quad 2 \le n \le p\end{aligned} \tag{3.79}$$

and integrating over these variables we arrive at

$$\begin{aligned}Z = &\sum_{k_0,k_n\ge 0}\sum_{l_0,l_n\ge 0}\frac{z_0^{l_0}}{l_0!}\frac{(-1)^{k_0}\bar{z}_i^{k_0}}{k_0!}\prod_{n=2}^{p}\frac{z_i^{l_i}}{l_i!}\frac{(-1)^{k_i}\bar{z}_i^{k_i}}{k_i!}\sum_{m_1,m_{p+1}} \\ &\oint\frac{d\tau_1}{2\pi i}\frac{d\tau_{p+1}}{2\pi i}e^{4\pi\xi_1\tau_1-i\theta_1 m_1}e^{4\pi\xi_{p+1}\tau_{p+1}-i\theta_{p+1}m_{p+1}} \\ &\frac{\Gamma(\tau_1-p\tau_{p+1}-\frac{m_1}{2}+p\frac{m_{p+1}}{2})}{\Gamma(1-\tau_1-\frac{m_1}{2}+p\tau_{p+1}+p\frac{m_{p+1}}{2})} \\ &\left(\frac{\Gamma(k_0+\tau_{p+1}-\frac{m_{p+1}}{2})}{\Gamma(1-l_0-\tau_{p+1}-\frac{m_{p+1}}{2})}\right)^2\frac{\Gamma(-k_0+\tau_1-\sum_{n=2}^{p}nk_n-\frac{m_1}{2})}{\Gamma(1+l_0-\tau_1+\sum_{n=2}^{p}nl_n-\frac{m_1}{2})} \\ &\frac{\Gamma(-2\tau_1+\sum_{n=2}^{p}(n+1)k_n+(p-2)\tau_{p+1}+2\frac{m_1}{2}-(p-2)\frac{m_{p+1}}{2}+irMa)}{\Gamma(1+2\tau_2-\sum_{n=2}^{p}(n+1)l_n-(p-2)\tau_{p+1}+2\frac{m_2}{2}-(p-2)\frac{m_{p+1}}{2}-irMa)}\end{aligned} \tag{3.80}$$

which defines a linear sigma model with charges (3.77) for $k_0 = k_n = 0$, $l_0 = l_n = 0$ (i.e. when $\xi_0 = \xi_n = \infty$).

The secondary fan of this model has four chambers, but here we concentrate only on three of them, describing $K_{\mathbb{F}_p}$, $K_{\mathbb{P}^2(1,1,p)}$ and $\mathbb{C}^3/\mathbb{Z}_{p+2}$ respectively. Its partition function is given by

$$\begin{aligned}Z = &\sum_{m_1,m_{p+1}}\int\frac{d\tau_1}{2\pi i}\frac{d\tau_{p+1}}{2\pi i}e^{4\pi\xi_1\tau_1-i\theta_1 m_1}e^{4\pi\xi_{p+1}\tau_{p+1}-i\theta_{p+1}m_{p+1}}\left(\frac{\Gamma(\tau_{p+1}-\frac{m_{p+1}}{2})}{\Gamma(1-\tau_{p+1}-\frac{m_{p+1}}{2})}\right)^2 \\ &\frac{\Gamma(\tau_1-\frac{m_1}{2})}{\Gamma(1-\tau_1-\frac{m_1}{2})}\frac{\Gamma(-p\tau_{p+1}+\tau_1+p\frac{m_{p+1}}{2}-\frac{m_1}{2})}{\Gamma(1+p\tau_{p+1}-\tau_1+p\frac{m_{p+1}}{2}-\frac{m_1}{2})} \\ &\frac{\Gamma((p-2)\tau_{p+1}-2\tau_1-(p-2)\frac{m_{p+1}}{2}+2\frac{m_1}{2}+irMa)}{\Gamma(1-(p-2)\tau_{p+1}+2\tau_1-(p-2)\frac{m_{p+1}}{2}+2\frac{m_1}{2}-irMa)}\end{aligned} \tag{3.81}$$

If we consider the set of poles

$$\begin{aligned}\tau_{p+1} &= -k_{p+1} + \frac{m_{p+1}}{2} + rM\lambda_{p+1} \\ \tau_1 &= -k_1 + \frac{m_1}{2} + rM\lambda_1\end{aligned} \tag{3.82}$$

we are describing the canonical bundle over $\mathbb{F}_p$:

$$Z_{K_{\mathbb{F}_p}} = \oint \frac{d(rM\lambda_1)}{2\pi i} \frac{d(rM\lambda_{p+1})}{2\pi i} \left(\frac{\Gamma(rM\lambda_{p+1})}{\Gamma(1-rM\lambda_{p+1})} \right)^2 \frac{\Gamma(rM\lambda_1)}{\Gamma(1-rM\lambda_1)}$$
$$\frac{\Gamma(-prM\lambda_{p+1}+rM\lambda_1)}{\Gamma(1+prM\lambda_{p+1}-rM\lambda_1)} \frac{\Gamma((p-2)rM\lambda_{p+1}-2rM\lambda_1+irMa)}{\Gamma(1-(p-2)rM\lambda_{p+1}+2rM\lambda_1-irMa)}$$
$$\sum_{l_1,l_{p+1}} (-1)^{(p-2)l_{p+1}} z_{p+1}^{l_{p+1}-rM\lambda_{p+1}} z_1^{l_1-rM\lambda_1}$$
$$\frac{((p-2)rM\lambda_{p+1}-2rM\lambda_1+irMa)_{2l_1-(p-2)l_{p+1}}}{(1-rM\lambda_{p+1})^2_{l_{p+1}}(1-rM\lambda_1)_{l_1}(1+prM\lambda_{p+1}-rM\lambda_1)_{l_1-pl_{p+1}}}$$
$$\sum_{k_1,k_{p+1}} (-1)^{(p-2)k_{p+1}} \bar{z}_{p+1}^{k_{p+1}-rM\lambda_{p+1}} \bar{z}_1^{k_1-rM\lambda_1}$$
$$\frac{((p-2)rM\lambda_{p+1}-2rM\lambda_1+irMa)_{2k_1-(p-2)k_{p+1}}}{(1-rM\lambda_{p+1})^2_{k_{p+1}}(1-rM\lambda_1)_{k_1}(1+prM\lambda_{p+1}-rM\lambda_1)_{k_1-pk_{p+1}}} \tag{3.83}$$

On the other hand, taking poles for

$$\tau_1 = p\tau_{p+1} - p\frac{m_{p+1}}{2} + \frac{m_1}{2} - k_1 \tag{3.84}$$

and integrating over τ_1 we obtain the canonical bundle over $\mathbb{P}^2_{(1,1,p)}$:

$$Z_{K_{\mathbb{P}^2_{(1,1,p)}}} = \sum_{k_1,l_1\geq 0} \frac{z_1^{l_1}}{l_1!} \frac{(-1)^{k_1}\bar{z}_1^{k_1}}{k_1!}$$
$$\sum_{m_{p+1}} \int \frac{d\tau_{p+1}}{2\pi i} e^{4\pi(\xi_{p+1}+p\xi_1)\tau_{p+1}-i(\theta_{p+1}+p\theta_1)m_{p+1}} \left(\frac{\Gamma(\tau_{p+1}-\frac{m_{p+1}}{2})}{\Gamma(1-\tau_{p+1}-\frac{m_{p+1}}{2})} \right)^2$$
$$\frac{\Gamma(p\tau_{p+1}-p\frac{m_{p+1}}{2}-k_1)}{\Gamma(1-p\tau_{p+1}-p\frac{m_{p+1}}{2}+l_1)} \frac{\Gamma(-(p+2)\tau_{p+1}+(p+2)\frac{m_{p+1}}{2}+irMa+2k_1)}{\Gamma(1+(p+2)\tau_{p+1}+(p+2)\frac{m_{p+1}}{2}-irMa-2l_1)} \tag{3.85}$$

with $l_1 = k_1 - m_1 + pm_{p+1}$ and $z_1 = e^{-2\pi\xi_1+i\theta_1}$. In fact, in the limit $\xi_1 \to \infty$ with $\xi_{p+1}+p\xi_1$ finite, only the $k_1 = l_1 = 0$ sector contributes, leaving the linear sigma model of $K_{\mathbb{CP}^2_{(1,1,p)}}$ for $\xi_{p+1}+p\xi_1 > 0$.

From the point of view of the charge matrix, the choice (3.84) corresponds to take linear combinations of the rows, in particular

$$\begin{pmatrix} p-2 & 0 & 1 & 1 & -p \end{pmatrix} \longrightarrow \begin{pmatrix} p-2 & 0 & 1 & 1 & -p \end{pmatrix} + p\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{3.86}$$

which implies $\xi_{p+1} \to \xi_{p+1} + p\xi_1$, $\theta_{p+1} \to \theta_{p+1} + p\theta_1$ and

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ p-2 & 0 & 1 & 1 & -p \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ -p-2 & p & 1 & 1 & 0 \end{pmatrix} \tag{3.87}$$

while the process of integrating in τ_1 is equivalent to eliminate the second row (notice that we have a simple pole, in this case, i.e. the column $(1 \;\; 0)^T$ appears with multiplicity 1).

The case $p = 2$ appears in [32,33] and corresponds to a full crepant resolution. So, by one blow down we arrived at $K_{\mathbb{P}^2(1,1,p)}$ whose charge matrix is given by

$$Q = \begin{pmatrix} 1 & 1 & p & -p-2 \end{pmatrix} \tag{3.88}$$

The associated two sphere partition function is correspondingly

$$Z = \sum_{m\in\mathbb{Z}} \int \frac{d\tau}{2\pi i} e^{4\pi\xi\tau - i\theta m} \left(\frac{\Gamma(\tau - \frac{m}{2})}{\Gamma(1-\tau-\frac{m}{2})} \right)^2 \frac{\Gamma(p\tau - p\frac{m}{2})}{\Gamma(1-p\tau-p\frac{m}{2})} \\ \frac{\Gamma(-(p+2)\tau + (p+2)\frac{m}{2} + irMa)}{\Gamma(1+(p+2)\tau + (p+2)\frac{m}{2} - irMa)} \tag{3.89}$$

It has two phases, $K_{\mathbb{P}^2(1,1,p)}$ and a more singular $\mathbb{C}^3/\mathbb{Z}_{p+2}$. The first phase corresponds to close the integration contour in the left half plane of this effective model; since the result is rather ugly, we will simply state that it can be obtained from (3.26), with the necessary modifications (i.e. twisted masses). For $p = 2$ it matches the formula presented in [32].

The second phase describing $\mathbb{C}^3/\mathbb{Z}_{p+2}$ can be obtained by flipping the contour to the right half plane and doing the integration in the single variable. Finally, we arrive at

$$Z = \frac{1}{p+2} \sum_{\delta=0}^{p+1} \left(\frac{\Gamma(\frac{\delta}{p+2} + \frac{1}{p+2} irMa)}{\Gamma(1 - \frac{\delta}{p+2} - \frac{1}{p+2} irMa)} \right)^2 \\ \frac{\Gamma(\langle \frac{p\delta}{p+2} \rangle + \frac{p}{p+2} irMa)}{\Gamma(1 - \langle \frac{p\delta}{p+2} \rangle - \frac{p}{p+2} irMa)} \frac{1}{(rM)^{2\left(\delta - \left[\frac{p\delta}{p+2}\right]\right)}} \\ \sum_{k\geq 0} (-1)^{(p+2)k} (\bar{z}^{-\frac{1}{p+2}})^{(p+2)k+\delta+irMa} (rM)^{\delta - \left[\frac{p\delta}{p+2}\right]} \\ \frac{(\frac{\delta}{p+2} + \frac{1}{p+2} irMa)_k^2 (\langle \frac{p\delta}{p+2} \rangle + \frac{p}{p+2} irMa)_{pk + \left[\frac{p\delta}{p+2}\right]}}{((p+2)k+\delta)!} \\ \sum_{l\geq 0} (-1)^{(p+2)l} (z^{-\frac{1}{p+2}})^{(p+2)l+\delta+irMa} (-rM)^{\delta - \left[\frac{p\delta}{p+2}\right]} \\ \frac{(\frac{\delta}{p+2} + \frac{1}{p+2} irMa)_l^2 (\langle \frac{p\delta}{p+2} \rangle + \frac{p}{p+2} irMa)_{pl + \left[\frac{p\delta}{p+2}\right]}}{((p+2)l+\delta)!} \tag{3.90}$$

The $\mathcal{I}$-function of the orbifold case in the δ-sector of the orbifold cohomology is then obtained from the second line of the above formula and for $p = 2$ it matches with [32].

4. Non-abelian GLSM

In this section we apply our methods to non-abelian gauged linear sigma models and give new results for some non-abelian GIT quotients. These are also tested against results in the mathematical literature when available.

The first case that we analyse are complex Grassmannians. On the way we also give an alternative proof for the conjecture of Hori and Vafa which can be rephrased stating that the $\mathcal{I}$-function of the Grassmannian can be obtained from that corresponding to a product of projective spaces after acting with an appropriate differential operator.

One can also study a more general theory corresponding to holomorphic vector bundles over Grassmannians. These spaces arise in the context of the study of BPS Wilson loop algebra in three dimensional supersymmetric gauge theories. In particular we will discuss the mathematical counterpart of a duality proposed in [21] which extends the standard Grassmannian duality to holomorphic vector bundles over them.

We also study flag manifolds and more general non-abelian quiver gauge theories for which we provide the rules to compute the spherical partition function and the $\mathcal{I}$-function.

4.1. Grassmannians. The sigma model for the complex Grassmannian $Gr(s,n)$ contains n chirals in the fundamental representation of the $U(s)$ gauge group. Its partition function is given by

$$Z_{Gr(s,n)} = \frac{1}{s!}\sum_{m_1,\dots,m_s}\int\prod_{i=1}^{s}\frac{\mathrm{d}\tau_i}{2\pi i}e^{4\pi\xi_{\text{ren}}\tau_i - i\theta_{\text{ren}}m_i}\prod_{i<j}^{s}\left(\frac{m_{ij}^2}{4}-\tau_{ij}^2\right)\prod_{i=1}^{s}\left(\frac{\Gamma\left(\tau_i-\frac{m_i}{2}\right)}{\Gamma\left(1-\tau_i-\frac{m_i}{2}\right)}\right)^n \tag{4.1}$$

As usual, we can write it as

$$\frac{1}{s!}\oint\prod_{i=1}^{s}\frac{d(rM\lambda_i)}{2\pi i}Z_{11}Z_{\text{v}}Z_{\text{av}} \tag{4.2}$$

where

$$\begin{aligned}
Z_{11} &= \prod_{i=1}^{s}(rM)^{-2nrM\lambda_i}\left(\frac{\Gamma(rM\lambda_i)}{\Gamma(1-rM\lambda_i)}\right)^n\prod_{i<j}^{s}(rM\lambda_i - rM\lambda_j)(-rM\lambda_i + rM\lambda_j)\\
Z_{\text{v}} &= z^{-rM|\lambda|}\sum_{l_1,\dots,l_s}\frac{[(rM)^n(-1)^{s-1}z]^{l_1+\dots+l_s}}{(1-rM\lambda_1)_{l_1}^n\dots(1-rM\lambda_s)_{l_s}^n}\prod_{i<j}^{s}\frac{l_i - l_j - rM\lambda_i + rM\lambda_j}{-rM\lambda_i + rM\lambda_j}\\
Z_{\text{av}} &= \bar{z}^{-rM|\lambda|}\sum_{k_1,\dots,k_s}\frac{[(-rM)^n(-1)^{s-1}\bar{z}]^{k_1+\dots+k_s}}{(1-rM\lambda_1)_{k_1}^n\dots(1-rM\lambda_s)_{k_s}^n}\prod_{i<j}^{s}\frac{k_i - k_j - rM\lambda_i + rM\lambda_j}{-rM\lambda_i + rM\lambda_j}.
\end{aligned} \tag{4.3}$$

We normalized the vortex and antivortex terms in order to have them starting from one in the rM series expansion and we defined $|\lambda| = \lambda_1 + \dots + \lambda_s$. The resulting $\mathcal{I}$-function Z_{v} coincides with the one given in [34]

$$\mathcal{I}_{Gr(s,n)} = e^{\frac{t\sigma_1}{\hbar}}\sum_{(d_1,\dots,d_s)}\frac{\hbar^{-n(d_1+\dots+d_s)}[(-1)^{s-1}e^t]^{d_1+\dots+d_s}}{\prod_{i=1}^{s}(1+x_i/\hbar)_{d_i}^n}\prod_{i<j}^{s}\frac{d_i - d_j + x_i/\hbar - x_j/\hbar}{x_i/\hbar - x_j/\hbar} \tag{4.4}$$

if we match the parameters as we did in the previous cases. Here the λ's are interpreted as Chern roots of the tautological bundle.

4.1.1. The Hori–Vafa conjecture. Hori and Vafa conjectured [35] that $\mathcal{I}_{Gr(s,n)}$ can be obtained by $\mathcal{I}_{\mathbb{P}}$, where $\mathbb{P} = \prod_{i=1}^{s} \mathbb{P}_{(i)}^{n-1}$, by acting with a differential operator. This has been proved in [34]; here we remark that in our formalism this is a simple consequence of the fact that the partition function of non-abelian vortices can be obtained from copies of the abelian ones upon acting with a suitable differential operator [5]. In fact we note that $Z_{Gr(s,n)}$ can be obtained from $Z_{\mathbb{P}}$ simply by dividing by $s!$ and identifying

$$
\begin{aligned}
Z_{11}^{Gr} &= \prod_{i<j}^{s} (rM\lambda_i - rM\lambda_j)(-rM\lambda_i + rM\lambda_j) Z_{11}^{\mathbb{P}} \\
Z_{\mathrm{v}}^{Gr}(z) &= \prod_{i<j}^{s} \frac{\partial_{z_i} - \partial_{z_j}}{-rM\lambda_i + rM\lambda_j} Z_{\mathrm{v}}^{\mathbb{P}}(z_1, \ldots, z_s)\Big|_{z_i=(-1)^{s-1}z} \\
Z_{\mathrm{av}}^{Gr}(\bar{z}) &= \prod_{i<j}^{s} \frac{\partial_{\bar{z}_i} - \partial_{\bar{z}_j}}{-rM\lambda_i + rM\lambda_j} Z_{\mathrm{av}}^{\mathbb{P}}(\bar{z}_1, \ldots, \bar{z}_s)\Big|_{\bar{z}_i=(-1)^{s-1}\bar{z}}.
\end{aligned}
\tag{4.5}
$$

4.2. Holomorphic vector bundles over Grassmannians. The $U(N)$ gauge theory with N_f fundamentals and N_a antifundamentals flows in the infra-red to a non-linear sigma model with target space given by a holomorphic vector bundle of rank N_a over the Grassmannian $Gr\left(N, N_f\right)$. We adopt the notation $Gr\left(N, N_f|N_a\right)$ for this space.

One can prove the equality of the partition functions for $Gr\left(N, N_f|N_a\right)$ and $Gr\left(N_f - N, N_f|N_a\right)$ after a precise duality map in a certain range of parameters. All this will be specified in the Appendix. At the level of $\mathcal{I}$-functions this proves the isomorphism among the relevant quantum cohomology rings conjectured in [21]. In analysing this duality we follow the approach of [10], where also the main steps of the proof were outlined. However we will detail their calculations and note some differences in the explicit duality map, which we refine in order to get a precise equality of the partition functions.

The partition function of the $Gr\left(N, N_f|N_a\right)$ GLSM is

$$
\begin{aligned}
Z = \frac{1}{N!} \sum_{\{m_s \in \mathbb{Z}\}_{s=1}^{N}} \int_{(i\mathbb{R})^N} \prod_{s=1}^{N} \frac{d\tau_s}{2\pi i} z_{\mathrm{ren}}^{-\tau_s - \frac{m_s}{2}} \bar{z}_{\mathrm{ren}}^{-\tau_s + \frac{m_s}{2}} \prod_{s<t}^{N} \left(\frac{m_{st}^2}{4} - \tau_{st}^2 \right) \\
\prod_{s=1}^{N} \prod_{i=1}^{N_f} \frac{\Gamma\left(\tau_s - i\frac{a_i}{\hbar} - \frac{m_s}{2}\right)}{\Gamma\left(1 - \tau_s + i\frac{a_i}{\hbar} - \frac{m_s}{2}\right)} \prod_{s=1}^{N} \prod_{j=1}^{N_a} \frac{\Gamma\left(-\tau_s + i\frac{\widetilde{a}_j}{\hbar} + \frac{m_s}{2}\right)}{\Gamma\left(1 + \tau_s - i\frac{\widetilde{a}_j}{\hbar} + \frac{m_s}{2}\right)},
\end{aligned}
\tag{4.6}
$$

while the one of $Gr\left(N_f - N, N_f|N_a\right)$ reads

$$
Z = \frac{1}{N^D!} \sum_{\{m_s \in \mathbb{Z}\}_{s=1}^{N^D}} \int_{(i\mathbb{R})^{N^D}} \prod_{s=1}^{N^D} \frac{d\tau_s}{2\pi i} (z_{ren}^D)^{-\tau_s - \frac{m_s}{2}} (\bar{z}_{ren}^D)^{-\tau_s + \frac{m_s}{2}} \prod_{s<t}^{N^D} \left(\frac{m_{st}^2}{4} - \tau_{st}^2 \right)
$$

$$\prod_{s=1}^{N^D}\prod_{i=1}^{N_f}\frac{\Gamma\left(\tau_s+i\frac{a_i^D}{\hbar}-\frac{m_s}{2}\right)}{\Gamma\left(1-\tau_s-i\frac{a_i^D}{\hbar}-\frac{m_s}{2}\right)}\prod_{s=1}^{N^D}\prod_{j=1}^{N_a}\frac{\Gamma\left(-\tau_s-i\frac{\widetilde{a}_j^D}{\hbar}+\frac{m_s}{2}\right)}{\Gamma\left(1+\tau_s+i\frac{\widetilde{a}_j^D}{\hbar}+\frac{m_s}{2}\right)}$$
$$\prod_{i=1}^{N_f}\prod_{j=1}^{N_a}\frac{\Gamma\left(-i\frac{a_i-\widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1+i\frac{a_i-\widetilde{a}_j}{\hbar}\right)}, \tag{4.7}$$

The proof of the equality of the two is shown in detail in the Appendix to hold under the duality map

$$z^D=(-1)^{N_a}z \tag{4.8}$$
$$\frac{a_j^D}{\hbar}=-\frac{a_j}{\hbar}+C \tag{4.9}$$
$$\frac{\widetilde{a}_j^D}{\hbar}=-\frac{\widetilde{a}_j}{\hbar}-(C+i) \tag{4.10}$$

where

$$C=\frac{1}{N_f-N}\sum_{i=1}^{N_f}\frac{a_i}{\hbar}. \tag{4.11}$$

4.3. Flag manifolds. Let us consider now a linear sigma model with gauge group $U(s_1)\times\cdots\times U(s_l)$ and with matter in the $(s_1,\bar{s}_2)\oplus\cdots\oplus(s_{l-1},\bar{s}_l)\oplus(s_l,n)$ representations, where $s_1<\cdots<s_l<n$. This flows in the infrared to a non-linear sigma model whose target space is the flag manifold $Fl(s_1,\ldots,s_l,n)$. The partition function is given by

$$Z_{Fl}=\frac{1}{s_1!\ldots s_l!}\sum_{\substack{\vec{m}^{(a)}\\ a=1\ldots l}}\int\prod_{a=1}^{l}\prod_{i=1}^{s_a}\frac{d\tau_i^{(a)}}{2\pi i}e^{4\pi\xi_{\mathrm{ren}}^{(a)}\tau_i^{(a)}-i\theta_{\mathrm{ren}}^{(a)}m_i^{(a)}}Z_{\mathrm{vector}}Z_{\mathrm{bifund}}Z_{\mathrm{fund}}$$
$$Z_{\mathrm{vector}}=\prod_{a=1}^{l}\prod_{i<j}^{s_a}\left(\frac{(m_{ij}^{(a)})^2}{4}-(\tau_{ij}^{(a)})^2\right)$$
$$Z_{\mathrm{bifund}}=\prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}}\frac{\Gamma\left(\tau_i^{(a)}-\tau_j^{(a+1)}-\frac{m_i^{(a)}}{2}+\frac{m_j^{(a+1)}}{2}\right)}{\Gamma\left(1-\tau_i^{(a)}+\tau_j^{(a+1)}-\frac{m_i^{(a)}}{2}+\frac{m_j^{(a+1)}}{2}\right)}$$
$$Z_{\mathrm{fund}}=\prod_{i=1}^{s_l}\left(\frac{\Gamma\left(\tau_i^{(l)}-\frac{m_i^{(l)}}{2}\right)}{\Gamma\left(1-\tau_i^{(l)}-\frac{m_i^{(l)}}{2}\right)}\right)^n \tag{4.12}$$

This is computed by taking poles at

$$\tau_i^{(a)} = \frac{m_i^{(a)}}{2} - k_i^{(a)} + rM\lambda_i^{(a)} \tag{4.13}$$

which gives

$$Z_{Fl} = \frac{1}{s_1! \dots s_l!} \oint \prod_{a=1}^{l} \prod_{i=1}^{s_a} \frac{d(rM\lambda_i^{(a)})}{2\pi i} Z_{\text{1-loop}} Z_{\text{v}} Z_{\text{av}} \tag{4.14}$$

where

$$\begin{aligned}
Z_{\text{1-loop}} &= (rM)^{-2rM\left[\sum_{a=1}^{l-1}(|\lambda^{(a)}|s_{a+1} - |\lambda^{(a+1)}|s_a) + n|\lambda^{(l)}|\right]} \\
&\quad \prod_{a=1}^{l}\prod_{i<j}^{s_a}(rM\lambda_i^{(a)} - rM\lambda_j^{(a)})(rM\lambda_j^{(a)} - rM\lambda_i^{(a)}) \\
&\quad \prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}} \frac{\Gamma\left(rM\lambda_i^{(a)} - rM\lambda_j^{(a+1)}\right)}{\Gamma\left(1 - rM\lambda_i^{(a)} + rM\lambda_j^{(a+1)}\right)} \prod_{i=1}^{s_l}\left(\frac{\Gamma\left(rM\lambda_i^{(l)}\right)}{\Gamma\left(1 - rM\lambda_i^{(l)}\right)}\right)^n \\
Z_{\text{v}} &= \sum_{\vec{l}^{(a)}} (rM)^{\sum_{a=1}^{l-1}(|l^{(a)}|s_{a+1} - |l^{(a+1)}|s_a) + n|l^{(l)}|} \prod_{a=1}^{l} (-1)^{(s_a-1)|l^{(a)}|} z_a^{|l^{(a)}| - rM|\lambda^{(a)}|} \\
&\quad \prod_{a=1}^{l}\prod_{i<j}^{s_a} \frac{l_i^{(a)} - l_j^{(a)} - rM\lambda_i^{(a)} + rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)} + rM\lambda_j^{(a)}} \\
&\quad \prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}} \frac{1}{(1 - rM\lambda_i^{(a)} + rM\lambda_j^{(a+1)})_{l_i^{(a)} - l_j^{(a+1)}}} \prod_{i=1}^{s_l} \frac{1}{\left[(1 - rM\lambda_i^{(l)})_{l_i^{(l)}}\right]^n} \\
Z_{\text{av}} &= \sum_{\vec{k}^{(a)}} (-rM)^{\sum_{a=1}^{l-1}(|k^{(a)}|s_{a+1} - |k^{(a+1)}|s_a) + n|k^{(l)}|} \prod_{a=1}^{l} (-1)^{(s_a-1)|k^{(a)}|} \bar{z}_a^{|k^{(a)}| - rM|\lambda^{(a)}|} \\
&\quad \prod_{a=1}^{l}\prod_{i<j}^{s_a} \frac{k_i^{(a)} - k_j^{(a)} - rM\lambda_i^{(a)} + rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)} + rM\lambda_j^{(a)}} \\
&\quad \prod_{a=1}^{l-1}\prod_{i=1}^{s_a}\prod_{j=1}^{s_{a+1}} \frac{1}{(1 - rM\lambda_i^{(a)} + rM\lambda_j^{(a+1)})_{k_i^{(a)} - k_j^{(a+1)}}} \prod_{i=1}^{s_l} \frac{1}{\left[(1 - rM\lambda_i^{(l)})_{k_i^{(l)}}\right]^n}
\end{aligned} \tag{4.15}$$

k's and l's are non-negative integers.

This result can be compared with the one in [36]. Indeed our fractions with Pochhammers at the denominator are equivalent to the products appearing there and we find perfect agreement with the Givental $\mathcal{I}$-functions under the by now familiar identification $\hbar = \frac{1}{rM}$, $\lambda = -H$ in Z_{v} and $\hbar = -\frac{1}{rM}$, $\lambda = H$ in Z_{av}.

4.4. Quivers. The techniques we used in the flag manifold case can be easily generalized to more general quivers; let us write down the rules to compute their partition functions. Every node of the quiver, i.e. every gauge group $U(s_a)$, contributes with:

- Integral:

$$\frac{1}{s_a!}\oint\prod_{i=1}^{s_a}\frac{d(rM\lambda_i^{(a)})}{2\pi i} \tag{4.16}$$

- One-loop factor:

$$(rM)^{-2rM|\lambda^{(a)}|\sum_i Q_i^{(a)}}\prod_{i<j}^{s_a}(rM\lambda_i^{(a)}-rM\lambda_j^{(a)})(rM\lambda_j^{(a)}-rM\lambda_i^{(a)}) \tag{4.17}$$

- Vortex factor:

$$\sum_{\vec{l}^{(a)}}(rM)^{|l^{(a)}|\sum_i Q_i^{(a)}}(-1)^{(s_a-1)|l^{(a)}|}z_a^{|l^{(a)}|-rM|\lambda^{(a)}|} \times\prod_{i<j}^{s_a}\frac{l_i^{(a)}-l_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \tag{4.18}$$

- Anti-vortex factor:

$$\sum_{\vec{k}^{(a)}}(-rM)^{|k^{(a)}|\sum_i Q_i^{(a)}}(-1)^{(s_a-1)|k^{(a)}|}\bar{z}_a^{|k^{(a)}|-rM|\lambda^{(a)}|} \times\prod_{i<j}^{s_a}\frac{k_i^{(a)}-k_j^{(a)}-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}}{-rM\lambda_i^{(a)}+rM\lambda_j^{(a)}} \tag{4.19}$$

Here $Q_i^{(a)}$ is the charge of the i-th chiral matter field with respect to the abelian subgroup $U(1)_a\subset U(s_a)$ corresponding to $\xi^{(a)}$ and $\theta^{(a)}$.
Every matter field in a representation of $U(s_a)\times U(s_b)$ and R-charge R contributes with:

- One-loop factor:

$$\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\frac{\Gamma\left(\frac{R}{2}+q_a rM\lambda_i^{(a)}+q_b rM\lambda_j^{(b)}\right)}{\Gamma\left(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)}\right)} \tag{4.20}$$

- Vortex factor:

$$\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\frac{1}{(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)})_{q_a l_i^{(a)}+q_b l_j^{(b)}}} \tag{4.21}$$

- Anti-vortex factor:

$$(-1)^{q_a s_b|k^{(a)}|+q_b s_a|k^{(b)}|}\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\frac{1}{(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)})_{q_a k_i^{(a)}+q_b k_j^{(b)}}} \tag{4.22}$$

In particular, the bifundamental $(s_a, \bar{s}_b)$ is given by $q_a = 1$, $q_b = -1$. A field in the fundamental can be recovered by setting $q_a = 1$, $q_b = 0$; for an antifundamental, $q_a = -1$ and $q_b = 0$. We can recover the usual formulae if we use (3.6). Multifundamental representations can be obtained by a straightforward generalization: for example, a trifundamental representation gives

$$\prod_{i=1}^{s_a}\prod_{j=1}^{s_b}\prod_{k=1}^{s_c}\frac{1}{(1-\frac{R}{2}-q_a rM\lambda_i^{(a)}-q_b rM\lambda_j^{(b)}-q_c rM\lambda_k^{(c)})_{q_a l_i^{(a)}+q_b l_j^{(b)}+q_c l_k^{(c)}}} \tag{4.23}$$

for the vortex factor.

In principle, these formulae are also valid for adjoint fields, if we set $s_a = s_b$, $q_a = 1$, $q_b = -1$; in practice, the diagonal contribution will give a $\Gamma(0)^{s_a}$ divergence, so the only way we can make sense of adjoint fields is by giving them a twisted mass.

4.5. Orbifold cohomology of the ADHM moduli space. The formalism described so far has been applied in [28] to the study of the equivariant quantum cohomology of the ADHM moduli space. This is encoded in the following $\mathcal{I}$-function

$$\begin{aligned}\mathcal{I}_{k,N} = &\sum_{d_1,\ldots,d_k\geq 0}((-1)^N z)^{d_1+\cdots+d_k}\prod_{r=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_r - ira_j + ir\epsilon)_{d_r}}{(1-r\lambda_r - ira_j)_{d_r}}\\ &\times\prod_{r<s}^{k}\frac{d_s-d_r-r\lambda_s+r\lambda_r}{-r\lambda_s+r\lambda_r}\frac{(1+r\lambda_r-r\lambda_s-ir\epsilon)_{d_s-d_r}}{(r\lambda_r-r\lambda_s+ir\epsilon)_{d_s-d_r}}\frac{(r\lambda_r-r\lambda_s+ir\epsilon_1)_{d_s-d_r}}{(1+r\lambda_r-r\lambda_s-ir\epsilon_1)_{d_s-d_r}}\\ &\times\frac{(r\lambda_r-r\lambda_s+ir\epsilon_2)_{d_s-d_r}}{(1+r\lambda_r-r\lambda_s-ir\epsilon_2)_{d_s-d_r}}\end{aligned} \tag{4.24}$$

The purpose of this section is to use the wallcrossing approach developed here to analyze the equivariant quantum cohomology of the Uhlembeck (partial) compactification of the moduli space of instantons by tuning the FI parameter ξ of the GLSM to zero. Indeed, as we will shortly discuss, in this case there is a reflection symmetry $\xi \to -\xi$ showing that the sign of the FI is not relevant to fix the phase of the GLSM. Actually, fixing $\xi = 0$ allows pointlike instantons. This produces a conjectural formula for the $\mathcal{I}$-function of the ADHM space in the orbifold chamber. In particular for rank one instantons, namely Hilbert schemes of points, our results are in agreement with those in [24].

Let us recall some elementary aspects on the moduli space $\mathcal{M}_{k,N}$ of k $SU(N)$ instantons on $\mathbb{C}^2$. This space is non compact both because the manifold $\mathbb{C}^2$ is non compact and because of point-like instantons. The first source of non compactness is cured by the introduction of the so-called Ω-background which, mathematically speaking, corresponds to work in the equivariant cohomology with respect to the maximal torus of rotations on $\mathbb{C}^2$. The second one can be approached in different ways. A compactification scheme is provided by the Uhlembeck one

$$\mathcal{M}_{k,N}^U = \bigsqcup_{l=0}^{k}\mathcal{M}_{k-l,N}\times S^l\left(\mathbb{C}^2\right) \tag{4.25}$$

Due to the presence of the symmetric product factors this space contains orbifold singularities. A desingularization is provided by the moduli space of torsion free sheaves on $\mathbb{P}^2$ with a framing on the line at infinity. This is described in terms of the ADHM complex linear maps $(B_1, B_2) : \mathbb{C}^k \to \mathbb{C}^k$ and $(I, J^\dagger) : \mathbb{C}^k \to \mathbb{C}^N$ which satisfy the F-term equation

$$[B_1, B_2] + IJ = 0$$

and the D-term equation

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \xi \mathbb{I}$$

where ξ is a parameter that gets identified with the FI parameter of the GLSM and that ensures the stability condition of the sheaf.

Notice that the ADHM equations are symmetric under the reflection $\xi \to -\xi$ and

$$(B_i, I, J) \to (B_i^\dagger, -J^\dagger, I^\dagger)$$

The Uhlembeck compactification is recovered in the $\xi \to 0$ limit. This amounts to set the vortex expansion parameter as

$$(-1)^N z = e^{i\theta} \tag{4.26}$$

giving therefore the orbifold $\mathcal{I}$-function

$$\begin{aligned} \mathcal{I}^U_{k,N} = & \sum_{d_1,\dots,d_k \geq 0} (e^{i\theta})^{d_1+\dots+d_k} \prod_{r=1}^{k} \prod_{j=1}^{N} \frac{(-r\lambda_r - ira_j + ir\epsilon)_{d_r}}{(1 - r\lambda_r - ira_j)_{d_r}} \\ & \times \prod_{r<s}^{k} \frac{d_s - d_r - r\lambda_s + r\lambda_r}{-r\lambda_s + r\lambda_r} \frac{(1 + r\lambda_r - r\lambda_s - ir\epsilon)_{d_s-d_r}}{(r\lambda_r - r\lambda_s + ir\epsilon)_{d_s-d_r}} \\ & \times \frac{(r\lambda_r - r\lambda_s + ir\epsilon_1)_{d_s-d_r}}{(1 + r\lambda_r - r\lambda_s - ir\epsilon_1)_{d_s-d_r}} \frac{(r\lambda_r - r\lambda_s + ir\epsilon_2)_{d_s-d_r}}{(1 + r\lambda_r - r\lambda_s - ir\epsilon_2)_{d_s-d_r}} \end{aligned} \tag{4.27}$$

In the abelian case, namely for $N = 1$, the above $\mathcal{I}$-function reproduces the results of [24] for the equivariant quantum cohomology of the symmetric product of k points in $\mathbb{C}^2$. Indeed, by using the map to the Fock space formalism for the equivariant quantum cohomology developed in [28], it is easy to see that both approaches produce the same small equivariant quantum cohomology. Notice that the map (4.26) reproduces in the $N = 1$ case the one of [24].

5. A_p and D_p Singularities

The k-instanton moduli space for $U(N)$ gauge theories on ALE spaces $\mathbb{C}^2/\Gamma$ has been described by [37] in terms of quiver representation theory. We can therefore apply the same procedure we used in the previous section and in [28] and compute the partition function on S^2 for the relevant quiver. This will give us information about the quantum cohomology of these ALE spaces. Similar results were discussed in [38]. We will focus on A_p and D_p singularities and consider the Hilbert scheme of points on their resolutions as well as the orbifold phase given by the symmetric product of points.

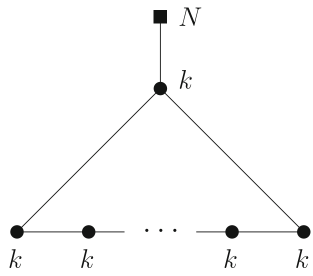

Fig. 1. The $A^{(1)}_{p-1}$ quiver

Let us start by considering the A_p case. Define $\vec{k} = (k, \ldots, k)$ vector of p components; the instanton number is given by k. The Nakajima quiver (Fig. 1) describing instantons on $\mathbb{C}^2/\mathbb{Z}_p$ consists of a gauge group $U(k)^p$ with matter I, J in fundamental, antifundamental representation of the first $U(k)$ and matter $B_{b,b\pm1}$ in bifundamental representations of all the $U(k)$ groups, together with adjoint fields χ_b and a superpotential $W = \mathrm{Tr}_1[\chi_1(B_{1,2}B_{2,1} - B_{1,p}B_{p,1} + IJ)] + \sum_{b=2}^{p} \mathrm{Tr}_b[\chi_b(B_{b,b+1}B_{b+1,b} - B_{b,b-1}B_{b-1,b})]$.[6]

The spherical partition function for this model is given by[7]

$$Z_{\vec{k},N} = \frac{1}{(k!)^p} \oint \prod_{b=1}^{p} \prod_{s=1}^{k} \frac{d(r\lambda_s^{(b)})}{2\pi i} Z_{11} Z_{\mathrm{v}} Z_{\mathrm{av}} \tag{5.1}$$

$$\begin{aligned} Z_{11} &= \left(\frac{\Gamma(1-ir\epsilon)}{\Gamma(ir\epsilon)}\right)^{pk} \prod_{b=1}^{p}\prod_{s=1}^{k} (z_b\bar{z}_b)^{-r\lambda_s^{(b)}} \prod_{b=1}^{p}\prod_{s=1}^{k}\prod_{t\neq s}^{k} (r\lambda_s^{(b)} - r\lambda_t^{(b)}) \\ &\quad \times \frac{\Gamma(1+r\lambda_s^{(b)} - r\lambda_t^{(b)} - ir\epsilon)}{\Gamma(-r\lambda_s^{(b)} + r\lambda_t^{(b)} + ir\epsilon)} \\ &\quad \prod_{b=1}^{p}\prod_{s=1}^{k}\prod_{t=1}^{k} \frac{\Gamma(r\lambda_s^{(b)} - r\lambda_t^{(b-1)} + ir\epsilon_1)}{\Gamma(1 - r\lambda_s^{(b)} + r\lambda_t^{(b-1)} - ir\epsilon_1)} \frac{\Gamma(-r\lambda_s^{(b)} + r\lambda_t^{(b-1)} + ir\epsilon_2)}{\Gamma(1 + r\lambda_s^{(b)} - r\lambda_t^{(b-1)} - ir\epsilon_2)} \\ &\quad \prod_{s=1}^{k}\prod_{j=1}^{N} \frac{\Gamma(r\lambda_s^{(1)} + ira_j)}{\Gamma(1 - r\lambda_s^{(1)} - ira_j)} \frac{\Gamma(-r\lambda_s^{(1)} - ira_j + ir\epsilon)}{\Gamma(1 + r\lambda_s^{(1)} + ira_j - ir\epsilon)} \end{aligned} \tag{5.2}$$

$$\begin{aligned} Z_{\mathrm{v}} &= \sum_{\{\vec{l}\}} \prod_{s=1}^{k} (-1)^{Nl_s^{(1)}} \prod_{b=1}^{p} z_b^{l_s^{(b)}} \prod_{b=1}^{p}\prod_{s<t}^{k} \frac{l_t^{(b)} - l_s^{(b)} - r\lambda_t^{(b)} + r\lambda_s^{(b)}}{-r\lambda_t^{(b)} + r\lambda_s^{(b)}} \\ &\quad \frac{(1 + r\lambda_s^{(b)} - r\lambda_t^{(b)} - ir\epsilon)_{l_t^{(b)} - l_s^{(b)}}}{(r\lambda_s^{(b)} - r\lambda_t^{(b)} + ir\epsilon)_{l_t^{(b)} - l_s^{(b)}}} \end{aligned}$$

[6] In order to keep a light notation, here $b = p + 1$ has to be intended as $b = 1$.

[7] Similarly, here $b = 0$ has to be intended as $b = p$.

$$\prod_{b=1}^{p}\prod_{s=1}^{k}\prod_{t=1}^{k}\frac{1}{(1-r\lambda_s^{(b)}+r\lambda_t^{(b-1)}-ir\epsilon_1)_{l_s^{(b)}-l_t^{(b-1)}}}$$
$$\frac{1}{(1+r\lambda_s^{(b)}-r\lambda_t^{(b-1)}-ir\epsilon_2)_{l_t^{(b-1)}-l_s^{(b)}}}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s^{(1)}-ira_j+ir\epsilon)_{l_s^{(1)}}}{(1-r\lambda_s^{(1)}-ira_j)_{l_s^{(1)}}} \tag{5.3}$$

$$Z_{\text{av}} = \sum_{\{\vec{k}\}}\prod_{s=1}^{k}(-1)^{Nk_s^{(1)}}\prod_{b=1}^{p}\bar{z}_b^{k_s^{(b)}}$$
$$\prod_{b=1}^{p}\prod_{s<t}^{k}\frac{k_t^{(b)}-k_s^{(b)}-r\lambda_t^{(b)}+r\lambda_s^{(b)}}{-r\lambda_t^{(b)}+r\lambda_s^{(b)}}\frac{(1+r\lambda_s^{(b)}-r\lambda_t^{(b)}-ir\epsilon)_{k_t^{(b)}-k_s^{(b)}}}{(r\lambda_s^{(b)}-r\lambda_t^{(b)}+ir\epsilon)_{k_t^{(b)}-k_s^{(b)}}}$$
$$\prod_{b=1}^{p}\prod_{s=1}^{k}\prod_{t=1}^{k}\frac{1}{(1-r\lambda_s^{(b)}+r\lambda_t^{(b-1)}-ir\epsilon_1)_{k_s^{(b)}-k_t^{(b-1)}}}$$
$$\frac{1}{(1+r\lambda_s^{(b)}-r\lambda_t^{(b-1)}-ir\epsilon_2)_{k_t^{(b-1)}-k_s^{(b)}}}\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s^{(1)}-ira_j+ir\epsilon)_{k_s^{(1)}}}{(1-r\lambda_s^{(1)}-ira_j)_{k_s^{(1)}}} \tag{5.4}$$

From Z_{11} we can recover in the limit $r \to 0$ an integral formula for the A_{p-1} ALE Nekrasov partition function:

$$Z_{\text{ALE}} = \frac{1}{r^{2Npk}}\frac{(i\epsilon)^{pk}}{(k!)^p}\oint\prod_{b=1}^{p}\prod_{s=1}^{k}\frac{d\lambda_s^{(b)}}{2\pi i}\prod_{b=1}^{p}\prod_{s=1}^{k}\prod_{t\neq s}^{k}(\lambda_s^{(b)}-\lambda_t^{(b)})(-\lambda_s^{(b)}+\lambda_t^{(b)}+i\epsilon)$$
$$\prod_{b=1}^{p}\prod_{s=1}^{k}\prod_{t=1}^{k}\frac{1}{(\lambda_s^{(b)}-\lambda_t^{(b-1)}+i\epsilon_1)(-\lambda_s^{(b)}+\lambda_t^{(b-1)}+i\epsilon_2)}$$
$$\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{1}{(\lambda_s^{(1)}+ia_j)(-\lambda_s^{(1)}-ia_j+i\epsilon)} \tag{5.5}$$

We can now study a few examples. In particular, we will be interested in the computation of the equivariant mirror map: this will be non-trivial only in the case $N = 1$, by the same argument proposed in [28]. Even if we are not able to provide a general combinatorial proof, a few examples can convince us that the equivariant mirror map is given by $(1+\prod_{b=1}^{p} z_b)^{ikr\epsilon}$, as known from the mathematical literature on the subject [39]: this has been checked in the cases $k = 1, 2$ for $p = 2$ and in the case $k = 1$ for $p = 3, 4$.

We now consider the quiver associated to a D_{p+1} singularity (Fig. 2). In this case, the gauge group will be $U(k)^4 \times U(2k)^{p-2}$, with matter I, J in the fundamental, antifundamental representation of the first $U(k)$, matter $B_{b,b\pm1}$ in bifundamental representations, and matter χ_b in the adjoint representation, with superpotential

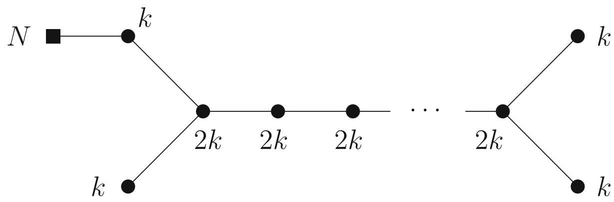

Fig. 2. The $D^{(1)}_{p+1}$ quiver

$$\begin{aligned} W = {} & \mathrm{Tr}_1[\chi_1(B_{1,3}B_{3,1}+IJ)]+\mathrm{Tr}_2[\chi_2(B_{2,3}B_{3,2})] \\ & +\mathrm{Tr}_3[\chi_3(B_{3,4}B_{4,3}-B_{3,1}B_{1,3}-B_{3,2}B_{2,3})] \\ & +\mathrm{Tr}_p[\chi_p(-B_{p,p-1}B_{p-1,p}+B_{p,p+1}B_{p+1,p}+B_{p,p+2}B_{p+2,p})] \\ & +\sum_{b=4}^{p-1}\mathrm{Tr}_b[\chi_b(B_{b,b+1}B_{b+1,b}-B_{b,b-1}B_{b-1,b})] \\ & +\mathrm{Tr}_{p+1}[\chi_{p+1}(-B_{p+1,p}B_{p,p+1})]+\mathrm{Tr}_{p+2}[\chi_{p+2}(-B_{p+2,p}B_{p,p+2})]. \end{aligned} \tag{5.6}$$

Defining the $(p+2)$-components vector $\vec{k}=(k,k,2k,\ldots,2k,k,k)$, the spherical partition function for this model will be

$$Z_{\vec{k},N} = \frac{1}{(k!)^4(2k!)^{p-2}}\oint\prod_{b=1}^{p+2}\prod_{s=1}^{k_b}\frac{d(r\lambda_s^{(b)})}{2\pi i}Z_{11}Z_{\mathrm{v}}Z_{\mathrm{av}} \tag{5.7}$$

$$\begin{aligned} Z_{11} = {} & \left(\frac{\Gamma(1-ir\epsilon)}{\Gamma(ir\epsilon)}\right)^{2pk}\prod_{b=1}^{p+2}\prod_{s=1}^{k_b}(z_b\bar{z}_b)^{-r\lambda_s^{(b)}}\prod_{b=1}^{p+2}\prod_{s=1}^{k_b}\prod_{t\neq s}(r\lambda_s^{(b)}-r\lambda_t^{(b)}) \\ & \times\frac{\Gamma(1+r\lambda_s^{(b)}-r\lambda_t^{(b)}-ir\epsilon)}{\Gamma(-r\lambda_s^{(b)}+r\lambda_t^{(b)}+ir\epsilon)} \\ & \prod_{b=3}^{p}\prod_{s=1}^{2k}\prod_{t=1}^{2k}\frac{\Gamma(r\lambda_s^{(b+1)}-r\lambda_t^{(b)}+ir\epsilon_1)}{\Gamma(1-r\lambda_s^{(b+1)}+r\lambda_t^{(b)}-ir\epsilon_1)}\frac{\Gamma(-r\lambda_s^{(b+1)}+r\lambda_t^{(b)}+ir\epsilon_2)}{\Gamma(1+r\lambda_s^{(b+1)}-r\lambda_t^{(b)}-ir\epsilon_2)} \\ & \prod_{b=1}^{2}\prod_{s=1}^{2k}\prod_{t=1}^{k}\frac{\Gamma(r\lambda_s^{(3)}-r\lambda_t^{(b)}+ir\epsilon_1)}{\Gamma(1-r\lambda_s^{(3)}+r\lambda_t^{(b)}-ir\epsilon_1)}\frac{\Gamma(-r\lambda_s^{(3)}+r\lambda_t^{(b)}+ir\epsilon_2)}{\Gamma(1+r\lambda_s^{(3)}-r\lambda_t^{(b)}-ir\epsilon_2)} \\ & \prod_{b=p+1}^{p+2}\prod_{s=1}^{k}\prod_{t=1}^{2k}\frac{\Gamma(r\lambda_s^{(b)}-r\lambda_t^{(p)}+ir\epsilon_1)}{\Gamma(1-r\lambda_s^{(b)}+r\lambda_t^{(p)}-ir\epsilon_1)}\frac{\Gamma(-r\lambda_s^{(b)}+r\lambda_t^{(p)}+ir\epsilon_2)}{\Gamma(1+r\lambda_s^{(b)}-r\lambda_t^{(p)}-ir\epsilon_2)} \\ & \prod_{s=1}^{k}\prod_{j=1}^{N}\frac{\Gamma(r\lambda_s^{(1)}+ira_j)}{\Gamma(1-r\lambda_s^{(1)}-ira_j)}\frac{\Gamma(-r\lambda_s^{(1)}-ira_j+ir\epsilon)}{\Gamma(1+r\lambda_s^{(1)}+ira_j-ir\epsilon)} \end{aligned} \tag{5.8}$$

$$Z_{\mathrm{v}} = \sum_{\{\vec{l}\}}\prod_{s=1}^{k}(-1)^{Nl_s^{(1)}}\prod_{b=1}^{p+2}z_b^{l_s^{(b)}}\prod_{b=1}^{p+2}\prod_{s<t}^{k_b}\frac{l_t^{(b)}-l_s^{(b)}-r\lambda_t^{(b)}+r\lambda_s^{(b)}}{-r\lambda_t^{(b)}+r\lambda_s^{(b)}}$$

$$
\begin{aligned}
&\times \frac{(1+r\lambda_s^{(b)}-r\lambda_t^{(b)}-ir\epsilon)_{l_t^{(b)}-l_s^{(b)}}}{(r\lambda_s^{(b)}-r\lambda_t^{(b)}+ir\epsilon)_{l_t^{(b)}-l_s^{(b)}}}\\
&\prod_{b=3}^{p}\prod_{s=1}^{2k}\prod_{t=1}^{2k}\frac{1}{(1-r\lambda_s^{(b+1)}+r\lambda_t^{(b)}-ir\epsilon_1)_{l_s^{(b+1)}-l_t^{(b)}}}\\
&\times\frac{1}{(1+r\lambda_s^{(b+1)}-r\lambda_t^{(b)}-ir\epsilon_2)_{l_t^{(b)}-l_s^{(b+1)}}}\\
&\prod_{b=1}^{2}\prod_{s=1}^{2k}\prod_{t=1}^{k}\frac{1}{(1-r\lambda_s^{(3)}+r\lambda_t^{(b)}-ir\epsilon_1)_{l_s^{(3)}-l_t^{(b)}}}\frac{1}{(1+r\lambda_s^{(3)}-r\lambda_t^{(b)}-ir\epsilon_2)_{l_t^{(b)}-l_s^{(3)}}}\\
&\prod_{b=p+1}^{p+2}\prod_{s=1}^{k}\prod_{t=1}^{2k}\frac{1}{(1-r\lambda_s^{(b)}+r\lambda_t^{(p)}-ir\epsilon_1)_{l_s^{(b)}-l_t^{(p)}}}\frac{1}{(1+r\lambda_s^{(b)}-r\lambda_t^{(p)}-ir\epsilon_2)_{l_t^{(p)}-l_s^{(b)}}}\\
&\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s^{(1)}-ira_j+ir\epsilon)_{l_s^{(1)}}}{(1-r\lambda_s^{(1)}-ira_j)_{l_s^{(1)}}}
\end{aligned}
\tag{5.9}
$$

$$
\begin{aligned}
Z_{\text{av}} = &\sum_{\{\vec{k}\}}\prod_{s=1}^{k}(-1)^{Nk_s^{(1)}}\prod_{b=1}^{p+2}\bar{z}_b^{k_s^{(b)}}\prod_{b=1}^{p+2}\prod_{s<t}^{k_b}\frac{k_t^{(b)}-k_s^{(b)}-r\lambda_t^{(b)}+r\lambda_s^{(b)}}{-r\lambda_t^{(b)}+r\lambda_s^{(b)}}\\
&\times\frac{(1+r\lambda_s^{(b)}-r\lambda_t^{(b)}-ir\epsilon)_{k_t^{(b)}-k_s^{(b)}}}{(r\lambda_s^{(b)}-r\lambda_t^{(b)}+ir\epsilon)_{k_t^{(b)}-k_s^{(b)}}}\\
&\prod_{b=3}^{p}\prod_{s=1}^{2k}\prod_{t=1}^{2k}\frac{1}{(1-r\lambda_s^{(b+1)}+r\lambda_t^{(b)}-ir\epsilon_1)_{k_s^{(b+1)}-k_t^{(b)}}}\\
&\times\frac{1}{(1+r\lambda_s^{(b+1)}-r\lambda_t^{(b)}-ir\epsilon_2)_{k_t^{(b)}-k_s^{(b+1)}}}\\
&\prod_{b=1}^{2}\prod_{s=1}^{2k}\prod_{t=1}^{k}\frac{1}{(1-r\lambda_s^{(3)}+r\lambda_t^{(b)}-ir\epsilon_1)_{k_s^{(3)}-k_t^{(b)}}}\frac{1}{(1+r\lambda_s^{(3)}-r\lambda_t^{(b)}-ir\epsilon_2)_{k_t^{(b)}-k_s^{(3)}}}\\
&\prod_{b=p+1}^{p+2}\prod_{s=1}^{k}\prod_{t=1}^{2k}\frac{1}{(1-r\lambda_s^{(b)}+r\lambda_t^{(p)}-ir\epsilon_1)_{k_s^{(b)}-k_t^{(p)}}}\frac{1}{(1+r\lambda_s^{(b)}-r\lambda_t^{(p)}-ir\epsilon_2)_{k_t^{(p)}-k_s^{(b)}}}\\
&\prod_{s=1}^{k}\prod_{j=1}^{N}\frac{(-r\lambda_s^{(1)}-ira_j+ir\epsilon)_{k_s^{(1)}}}{(1-r\lambda_s^{(1)}-ira_j)_{k_s^{(1)}}}
\end{aligned}
\tag{5.10}
$$

From Z_{11} we can recover an integral expression for the D_{p+1} ALE Nekrasov partition function by taking the limit $r \to 0$, as we did for the previous case. The structure of the poles for this model is quite involved, and we leave its study to future work. Nevertheless, an analysis of the simplest cases gives $(1+z_1z_2\prod_{b=3}^{p} z_b^2 z_{p+1}z_{p+2})^{irk\epsilon}$ as the equivariant mirror map, again in agreement with [39]. In line with these computation, we expect also the equivariant mirror map for the E-type ALE spaces to depend only on the dual Dynkin label of the affine Dynkin diagram for the corresponding algebra.

As far as the orbifold phase is concerned, the discussion goes along the same lines as in previous Sect. 4: by reversing the sign of all Fayet-Iliopoulos parameters one obtains the same phase due to the symmetry of ADHM constraints. The orbifold phase is then reached by analytic continuation on the product of circles $|z_b| = 1$. This provides conjectural formulae for the equivariant $\mathcal{I}$ and $\mathcal{J}$ functions of the symmetric product of points of A_p and D_p singularities that it would be interesting to check against rigorous mathematical results.

6. Conclusions

In this paper we exploited some properties of the spherical partition function for supersymmetric $(2, 2)$ GLSMs to provide contour integral formulae for the $\mathcal{I}$ and the $\mathcal{J}$-functions encoding the equivariant quantum cohomology of general GIT quotients. We have given a toolbox to compute the S^2 partition function for gauge theory quivers.

We have developed two particular applications of our formulas. The first concerns the analysis of the contour integral applied to the wall crossing phenomenon among the various chambers of a given GIT quotient. We used this method to provide conjectural formulae for the quantum cohomology of the $\mathbb{C}^3/\mathbb{Z}_n$ orbifold and of the Uhlembeck (partial) compactification of the instanton moduli space on $\mathbb{C}^2$. The second has to do with the use of the Cauchy theorem to prove gauge theory/quantum cohomology dualities. This allowed us to prove a conjectural equivalence of quantum cohomology of vector bundles over Grassmannians proposed in the context of the study of Wilson loop algebrae in three dimensional supersymmetric gauge theories [21].

There are several directions worth further investigation. Concerning orbifold quantum cohomology, we underline that our approach can be applied to any classical gauge group and thus could be exploited for example to compute the Gromov–Witten invariants of D and E type finite groups quotients.

Another interesting issue is the extension of the approach developed in this paper to the computation of open Gromov–Witten invariants by implementing suitable boundary conditions via Brini's remodelling technique [40].

Vortex partition functions have been shown to satisfy differential equations of Hypergeometric type and this has a clear counterpart in the context of AGT correspondence being the null state equations for degenerate conformal blocks [4–7]. Differential equations of similar type are obeyed by $\mathcal{I}$ and $\mathcal{J}$-functions associated to general GIT quotients whose explicit form would be useful to spell out in detail in order to study the mirror geometries and the link to classical integrable systems.

These equations are naturally promoted to finite difference equations in K-theoretic vortex counting [6,7,41]. The AGT-like dual of these have been recently studied in [42] where their interpretation in terms of the q-deformed Virasoro algebra null state equation is proposed. We plan to study the relation between K-theoretic vortex counting, refined topological strings, quantum K-theory and quantum integrable systems in a forthcoming work.

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A. Duality $Gr\left(N, N_f|N_a\right) \simeq Gr\left(N_f - N, N_f|N_a\right)$

The Grassmannian $Gr\left(N, N_f|N_a\right)$ is defined as a $U(N)$ gauge theory with N_f fundamentals and N_a antifundamentals, so we can write the partition function in the form

$$Z = \frac{1}{N!}\sum_{\{m_s\in\mathbb{Z}\}_{s=1}^N}\int_{(i\mathbb{R})^N}\prod_{s=1}^N\frac{d\tau_s}{2\pi i}z_{\text{ren}}^{-\tau_s-\frac{m_s}{2}}\bar{z}_{\text{ren}}^{-\tau_s+\frac{m_s}{2}}\prod_{s<t}^N\left(\frac{m_{st}^2}{4}-\tau_{st}^2\right)$$
$$\prod_{s=1}^N\prod_{i=1}^{N_f}\frac{\Gamma\left(\tau_s - i\frac{a_i}{\hbar}-\frac{m_s}{2}\right)}{\Gamma\left(1-\tau_s+i\frac{a_i}{\hbar}-\frac{m_s}{2}\right)}\prod_{s=1}^N\prod_{j=1}^{N_a}\frac{\Gamma\left(-\tau_s+i\frac{\widetilde{a}_j}{\hbar}+\frac{m_s}{2}\right)}{\Gamma\left(1+\tau_s-i\frac{\widetilde{a}_j}{\hbar}+\frac{m_s}{2}\right)}, \tag{A.1}$$

where $\hbar$ relates to the radius of the sphere and the renormalization scale M as $\hbar = \frac{1}{rM}$ and $a_j, \widetilde{a}_j$ are the dimensionless (rescaled by M^{-1}) equivariant weights for fundamentals and antifundamentals respectively. The renormalized Kahler coordinate z_{ren} is defined as

$$z_{\text{ren}} = e^{-2\pi\xi_{\text{ren}}+i\theta_{\text{ren}}} = \hbar^{N_a-N_f}(-1)^{N-1}z. \tag{A.2}$$

since we have

$$\xi_{\text{ren}} = \xi - \frac{1}{2\pi}(N_f - N_a)\log(rM), \theta_{\text{ren}} = \theta + (N-1)\pi \tag{A.3}$$

From now on we are setting $M = 1$. We close the contours in the left half planes, so that we pick only poles coming from the fundamentals. We need to build an N-pole to saturate the integration measure. Hence the partition function becomes a sum over all possible choices of N-poles, i.e. over all combinations how to pick N objects out of N_f. Now the proposal is that duality holds separately for a fixed choice of an N-pole and its corresponding dual. For simplicity of notation let us prove the duality for a particular choice of an N-pole and its $(N_f - N)$-dual

$$(\underbrace{\square,\ldots,\square}_{N},\underbrace{\bullet,\ldots,\bullet}_{N_f-N})\overset{\text{dual}}{\longleftrightarrow}(\underbrace{\bullet,\ldots,\bullet}_{N},\underbrace{\square,\ldots,\square}_{N_f-N}), \tag{A.4}$$

where boxes denote the choice of poles forming the N-pole.

A.1. $Gr\left(N, N_f|N_a\right)$. The poles are at positions

$$\tau_s = -k_s + \frac{m_s}{2} + \frac{\lambda_s}{\hbar} \tag{A.5}$$

and it still remains to be integrated over λ's around $\lambda_s = ia_s$, where s runs from 1 to N. This fully specifies from which fundamental we took the pole. Plugging this into (A.1), the integral reduces to the following form

$$Z = \oint_{\mathcal{M}}\left\{\prod_{s=1}^N\frac{d\lambda_s}{2\pi i\hbar}\right\}Z_{11}\left(\frac{\lambda_s}{\hbar},\frac{a_i}{\hbar},\frac{\widetilde{a}_j}{\hbar}\right)z^{-\sum_{s=1}^N\frac{\lambda_s}{\hbar}}\widetilde{I}\left((-1)^{N_a}\kappa z,\frac{\lambda_s}{\hbar},\frac{a_i}{\hbar},\frac{\widetilde{a}_j}{\hbar}\right)$$
$$\times\,\bar{z}^{-\sum_{s=1}^N\frac{\lambda_s}{\hbar}}\widetilde{I}\left((-1)^{N_a}\bar{\kappa}\bar{z},\frac{\lambda_s}{\hbar},\frac{a_i}{\hbar},\frac{\widetilde{a}_j}{\hbar}\right), \tag{A.6}$$

where we defined $\kappa = \hbar^{N_a - N_f}(-1)^{N-1}$, $\bar{\kappa} = (-\hbar)^{N_a - N_f}(-1)^{N-1}$. Here we are integrating over a product of circles $\mathcal{M} = \bigotimes_{r=1}^{k} S^1(ia_r, \delta)$ with δ small enough such that only the pole at the center of the circle is included. From this form we can read of the I function for $Gr\left(N, N_f|N_a\right)$ as

$$I = z^{-\sum_{s=1}^{N}\frac{\lambda_s}{\hbar}} \sum_{\{l_s\geq 0\}_{s=1}^{N}} \left((-1)^{N_a}\kappa z\right)^{\sum_{s=1}^{N} l_s} \prod_{s<t}^{N} \frac{\lambda_{st} - \hbar l_{st}}{\lambda_{st}} \prod_{s=1}^{N} \frac{\prod_{j=1}^{N_a}\left(\frac{-\lambda_s + i\widetilde{a}_j}{\hbar}\right)_{l_s}}{\prod_{i=1}^{N_f}\left(1 + \frac{-\lambda_s + ia_i}{\hbar}\right)_{l_s}}, \tag{A.7}$$

where $x_{st} := x_s - x_t$. Now we integrate over λ's in (A.6), which is straightforward since Z_{1l} contains only simple poles and the rest is holomorphic in λ's. Finally, we get

$$Z^{(\square,\ldots,\square,\bullet,\ldots,\bullet)} = Z_{\text{class}} Z_{11} Z_{\text{v}} Z_{\text{av}}, \tag{A.8}$$

where the individual pieces are given as follows

$$Z_{\text{class}} = \prod_{s=1}^{N}\left(\hbar^{2(N_a - N_f)} z\bar{z}\right)^{-\frac{ia_s}{\hbar}} \tag{A.9}$$

$$Z_{11} = \prod_{s=1}^{N}\prod_{i=N+1}^{N_f} \frac{\Gamma\left(\frac{ia_{si}}{\hbar}\right)}{\Gamma\left(1 - \frac{ia_{si}}{\hbar}\right)} \prod_{s=1}^{N}\prod_{j=1}^{N_a} \frac{\Gamma\left(-\frac{i(a_s - \widetilde{a}_j)}{\hbar}\right)}{\Gamma\left(1 + \frac{i(a_s - \widetilde{a}_j)}{\hbar}\right)} \tag{A.10}$$

$$Z_{\text{v}} = \sum_{\{l_s\geq 0\}_{s=1}^{N}} \left((-1)^{N_a}\kappa z\right)^{\sum_{s=1}^{N} l_s} \prod_{s<t}^{N}\left(1 - \frac{\hbar l_{st}}{ia_{st}}\right) \prod_{s=1}^{N} \frac{\prod_{j=1}^{N_a}\left(-i\frac{a_s - \widetilde{a}_j}{\hbar}\right)_{l_s}}{\prod_{i=1}^{N_f}\left(1 - i\frac{a_{si}}{\hbar}\right)_{l_s}} \tag{A.11}$$

$$Z_{\text{av}} = Z_{\text{v}}\left[\kappa z \to \bar{\kappa}\bar{z}\right] \tag{A.12}$$

To prove the duality it is actually better to manipulate Z_{v} to a more convenient form (combining the contributions of the vectors and fundamentals by using identities between the Pochhammers)

$$Z_{\text{v}} = \sum_{l=0}^{\infty}\left[(-1)^{N_a + N - N_f}\kappa z\right]^{l} Z_l \tag{A.13}$$

with Z_l given by

$$Z_l = \sum_{\{l_s\geq 0|\sum_{s=1}^{N} l_s = l\}} \prod_{s=1}^{N} \frac{\prod_{j=1}^{N_a}\left(-i\frac{a_s - \widetilde{a}_j}{\hbar}\right)_{l_s}}{l_s!\prod_{i\neq s}^{N}\left(i\frac{a_{si}}{\hbar} - l_s\right)_{l_i}\prod_{i=N+1}^{N_f}\left(i\frac{a_{si}}{\hbar} - l_s\right)_{l_s}}. \tag{A.14}$$

A.2. The dual theory $Gr\left(N_f - N, N_f|N_a\right)$. Going to the dual theory not only the rank of the gauge group changes to $N_f - N$, but there is a new feature arising. New matter fields $M^i_{\bar{j}}$ appear, they are singlets under the gauge group and couple to the fundamentals and antifundamentals via a superpotential $W^D = \widetilde{\phi}^{\mu\bar{j}} M^i_{\bar{j}} \phi_{\mu i}$. So the partition function gets a new contribution from the mesons M (we set $N^D = N_f - N$)

$$Z = \frac{1}{N^D!} \sum_{\{m_s \in \mathbb{Z}\}_{s=1}^{N^D}} \int_{(i\mathbb{R})^{N^D}} \prod_{s=1}^{N^D} \frac{d\tau_s}{2\pi i} (z^D_{ren})^{-\tau_s - \frac{m_s}{2}} (\bar{z}^D_{ren})^{-\tau_s + \frac{m_s}{2}} \prod_{s<t}^{N^D} \left(\frac{m_{st}^2}{4} - \tau_{st}^2 \right)$$
$$\prod_{s=1}^{N^D} \prod_{i=1}^{N_f} \frac{\Gamma\left(\tau_s + i\frac{a^D_i}{\hbar} - \frac{m_s}{2}\right)}{\Gamma\left(1 - \tau_s - i\frac{a^D_i}{\hbar} - \frac{m_s}{2}\right)} \prod_{s=1}^{N^D} \prod_{j=1}^{N_a} \frac{\Gamma\left(-\tau_s - i\frac{\widetilde{a}^D_j}{\hbar} + \frac{m_s}{2}\right)}{\Gamma\left(1 + \tau_s + i\frac{\widetilde{a}^D_j}{\hbar} + \frac{m_s}{2}\right)}$$
$$\times \prod_{i=1}^{N_f} \prod_{j=1}^{N_a} \frac{\Gamma\left(-i\frac{a_i - \widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1 + i\frac{a_i - \widetilde{a}_j}{\hbar}\right)}, \tag{A.15}$$

where the last factor is the new contribution of the mesons (note that it depends on the original equivariant weights, not on the dual ones). All the computations are analogue to the previous case, so we give the result right after integration

$$Z^{(\bullet,\ldots,\bullet,\square,\ldots,\square)} = Z^D_{\text{class}} Z^D_{11} Z^D_{\text{v}} Z^D_{\text{av}}, \tag{A.16}$$

where the building blocks are

$$Z^D_{\text{class}} = \prod_{s=N+1}^{N_f} \left(\hbar^{2(N_a - N_f)} z^D \bar{z}^D\right)^{-\frac{i a^D_s}{\hbar}} \tag{A.17}$$

$$Z^D_{11} = \prod_{s=N+1}^{N_f} \prod_{i=N+1}^{N_f} \frac{\Gamma\left(\frac{i a^D_{si}}{\hbar}\right)}{\Gamma\left(1 - \frac{i a^D_{si}}{\hbar}\right)} \prod_{j=1}^{N_a} \frac{\Gamma\left(-\frac{i(a^D_s - \widetilde{a}^D_j)}{\hbar}\right)}{\Gamma\left(1 + \frac{i(a^D_s - \widetilde{a}^D_j)}{\hbar}\right)} \prod_{i=1}^{N_f} \prod_{j=1}^{N_a} \frac{\Gamma\left(-i\frac{a_i - \widetilde{a}_j}{\hbar}\right)}{\Gamma\left(1 + i\frac{a_i - \widetilde{a}_j}{\hbar}\right)} \tag{A.18}$$

$$Z^D_{\text{v}} = \sum_{l=0}^{\infty} \left[(-1)^{N_a - N} (\kappa z)^D\right]^l Z^D_l \tag{A.19}$$

$$Z^D_{\text{av}} = \sum_{k=0}^{\infty} \left[(-1)^{N_a - N} (\bar{\kappa}\bar{z})^D\right]^k Z^D_k \tag{A.20}$$

with Z^D_l given by

$$Z^D_l = \sum_{\{l_s \geq 0 | \sum_{s=N+1}^{N_f} l_s = l\}} \prod_{s=N+1}^{N_f} \frac{\prod_{j=1}^{N_a} \left(-i\frac{a^D_s - \widetilde{a}^D_j}{\hbar}\right)_{l_s}}{l_s! \prod_{\substack{i=N+1 \\ i \neq s}}^{N_f} \left(i\frac{a^D_{si}}{\hbar} - l_s\right)_{l_i} \prod_{i=1}^{N} \left(i\frac{a^D_{si}}{\hbar} - l_s\right)_{l_s}}. \tag{A.21}$$

A.3. Duality map. We are now ready to discuss the duality between the two theories. The statement is the following. For $N_f \geq N_a + 2$, there exists a duality map $z^D = z^D(z)$ and $a_j^D = a_j^D(a_j)$, $\widetilde{a}_j^D = \widetilde{a}_j^D(\widetilde{a}_j)$ under which the partition functions for $Gr\left(N, N_f|N_a\right)$ and $Gr\left(N_f - N, N_f|N_a\right)$ are equal.[8] In the first step we will construct the duality map and then we will show that (A.9–A.14) indeed match with (A.17–A.21). The partition function is a double power series in z and $\bar{z}$ multiplied by Z_{class}. In order to achieve equality of the partition functions, Z_{class} have to be equal after duality map and then the power series have to match term by term. Moreover we can look only at the holomorphic piece Z_{v}, for the antiholomorphic everything goes in a similar way. The constant term is Z_{11}, which is a product of gamma functions with arguments linear in the equivariant weights. This implies that the duality map for the equivariant weights is linear. But then the map between the Kahler coordinates can be only a rescaling since a constant term would destroy the matching of Z_{11}. So we arrive at the most general ansatz for the duality map

$$z^D = sz \tag{A.22}$$

$$\frac{a_i^D}{\hbar} = -E\frac{a_i}{\hbar} + C \tag{A.23}$$

$$\frac{\widetilde{a}_j^D}{\hbar} = -F\frac{\widetilde{a}_j}{\hbar} + D \tag{A.24}$$

Matching the constant terms Z_{11} gives the constraints

$$E = F = 1, \;\; D = -(C + i). \tag{A.25}$$

Imposing further the equivalence of Z_{class} fixes C to be

$$C = \frac{1}{N_f - N}\sum_{i=1}^{N_f}\frac{a_i}{\hbar}. \tag{A.26}$$

We are now at a position where Z_{class} and Z_{11} match, while the only remaining free parameter in the duality map is s. We fix it by looking at the linear terms in Z_{v} and Z_{v}^D. Of course this does not assure that all higher order terms do match, but we will show that this is the case for $N_f \geq N_a + 2$.[9] So taking only $k = 1$ contributions in Z_{v} and Z_{v}^D we get for s

$$s = (-1)^{N-1}\frac{\mathcal{N}}{\mathcal{D}}, \tag{A.27}$$

where

$$\mathcal{N} = \sum_{s=1}^{N}\frac{\prod_{j=1}^{N_a}\left(-i\,\frac{a_s - \widetilde{a}_j}{\hbar}\right)}{\prod_{i\neq s}^{N}\left(-i\,\frac{a_{si}}{\hbar}\right)\prod_{i=N+1}^{N_f}\left(1 - i\,\frac{a_{si}}{\hbar}\right)} \tag{A.28}$$

[8] We will see the reason for this range later.

[9] A direct computation for a handful of examples suggests that higher order terms do not match for s obtained as just outlined if $N_f < N_a + 2$.

$$\mathcal{D} = \sum_{s=N+1}^{N_f} \frac{\prod_{j=1}^{N_a} \left(1 + i\frac{a_s - \widetilde{a}_j}{\hbar}\right)}{\prod_{i=1}^{N} \left(1 + i\frac{a_{si}}{\hbar}\right) \prod_{\substack{i=N+1 \\ j \neq s}}^{N_f} \left(-i\frac{a_{si}}{\hbar}\right)}. \tag{A.29}$$

The proposal is that for $N_f \geq N_a + 2$

$$s = (-1)^{N_a}. \tag{A.30}$$

Out of this range s is a complicated rational function in the equivariant parameters. This completes the duality map for $N_f \geq N_a + 2$ and suggests that there is no duality map for $N_f < N_a + 2$.

A.4. Proof of equivalence of the partition functions. By construction of the mirror map we know that Z_{class}, Z_{11} and moreover also the linear terms in Z_{v} match. Now we will prove (d.m. is the shortcut for duality map)

$$Z_{\text{v}} = Z_{\text{v}}^D|_{d.m.} \tag{A.31}$$

for $N_f \geq N_a + 2$. Looking at (A.13) and (A.19) we see that this boils down to

$$Z_l = (-1)^{N_a l} Z_l^D|_{d.m.}. \tag{A.32}$$

The key to prove the above relation is to write Z_l as a contour integral

$$Z_l = \int_{\mathcal{C}_u} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right)\Big|_{\epsilon=1}, \tag{A.33}$$

where $\mathcal{C}_u$ is a product of contours having the real axes as base and then are closed in the upper half plane by a semicircle. The integrand has the form

$$f = \frac{1}{\epsilon^l l!} \prod_{\alpha<\beta}^{l} \frac{(\phi_\alpha - \phi_\beta)^2}{(\phi_\alpha - \phi_\beta)^2 - \epsilon^2} \prod_{\alpha=1}^{l} \frac{\prod_{j=1}^{N_a} \left(i\frac{\widetilde{a}_j}{\hbar} + \phi_\alpha\right)}{\prod_{i=1}^{N} \left(\phi_\alpha + i\frac{a_i}{\hbar}\right) \prod_{i=N+1}^{N_f} \left(-i\frac{a_i}{\hbar} - \epsilon - \phi_\alpha\right)}. \tag{A.34}$$

It is necessary to add small imaginary parts to ϵ and a_i, $\epsilon \to \epsilon + i\delta$, $-ia_i \to -ia_i + i\hbar\delta'$ with $\delta > \delta'$. The proof of (A.33) goes by direct evaluation. First we have to classify the poles. Due to the imaginary parts assignments, they are at [10]

$$\phi_\alpha = -i\frac{a_i}{\hbar}, \qquad \alpha = 1, \ldots, l, \quad i = 1, \ldots, N \tag{A.35}$$

$$\phi_\beta = \phi_\alpha + \epsilon, \qquad \beta \geq \alpha \tag{A.36}$$

We have to build an l-pole, which means that the poles are classified by partitions of l into N parts, $l = \sum_{I=1}^{N} l_I$. The I-th Young tableau $YT(l_I)$ with l_I boxes can be only

[10] One has to assume a_i to be imaginary at this point. The general result is obtained by analytic continuation after integration.

1-dimensional (we choose a row) since we have only one ϵ to play with. To illustrate what we have in mind, we show an example of a possible partition

$$(\underbrace{\square\square\square}_{l_1}, \bullet, \square\square, \square, \ldots, \square\square, \underbrace{\bullet}_{l_N}). \tag{A.37}$$

Residue theorem then turns the integral into a sum over all such partitions and the poles corresponding to a given partition are given as

$$\phi^I_{n_I} = -i\frac{a_I}{\hbar} + (n_I - 1)\epsilon + \lambda^I_{n_I}, \tag{A.38}$$

where $I = 1, \ldots, N$ labels the position of the Young tableau in the N-vector and $n_I = 1, \ldots, l_I$ labels the boxes in $YT(l_I)$. Substituting this in (A.33) we get (the $l!$ gets cancelled by the permutation symmetry of the boxes)

$$\begin{aligned} Z_l = \frac{1}{\epsilon^l} \sum_{\{l_I \geq 0 | \sum_{I=1}^N l_I = l\}} \oint_{\mathcal{M}} \prod_{\substack{I=1\\ l_I\neq 0}}^N \prod_{n_I=1}^{l_I} \frac{d\lambda^I_{n_I}}{2\pi i} \prod_{\substack{I\neq J\\ l_I\neq 0, l_J\neq 0}}^N \prod_{n_I=1}^{l_I} \prod_{n_J=1}^{l_J} \\ \times \frac{\left(-i\frac{a_{IJ}}{\hbar} + n_{IJ}\epsilon + \lambda^{I,J}_{n_I,n_J}\right)}{\left(-i\frac{a_{IJ}}{\hbar} + (n_{IJ}-1)\epsilon + \lambda^{I,J}_{n_I,n_J}\right)} \prod_{\substack{I=1\\ l_I\neq 0}}^N \prod_{n_I\neq n_J}^{l_I} \frac{\left(n_{IJ}\epsilon + \lambda^{I,I}_{n_I,n_J}\right)}{\left((n_{IJ}-1)\epsilon + \lambda^{I,I}_{n_I,n_J}\right)} \\ \times \prod_{\substack{I=1\\ l_I\neq 0}}^N \prod_{n_I=1}^{l_I} \frac{\prod_{j=1}^{N_a}\left(i\frac{\tilde{a}_j}{\hbar} - i\frac{a_I}{\hbar} + (n_I-1)\epsilon + \lambda^I_{n_I}\right)}{\prod_{r=1}^N \left(-i\frac{a_{Ir}}{\hbar} + (n_I-1)\epsilon + \lambda^I_{n_I}\right) \prod_{r=N+1}^{N_f}\left(-i\frac{a_{Ir}}{\hbar} - n_I\epsilon - \lambda^I_{n_I}\right)}, \end{aligned} \tag{A.39}$$

where we integrate over $\mathcal{M} = \bigotimes_{r=1}^l S^1(0,\delta)$. The computation continues as follows. We separate the poles in λ's (there are only simple poles), the rest is a holomorphic function, so we can effectively set the λ's to zero there. Eventually, we obtain

$$\begin{aligned} Z_l = \frac{1}{\epsilon^l} \sum_{\{l_I \geq 0 | \sum_{I=1}^N l_I = l\}} \left[\oint_{\mathcal{M}} \prod_{\substack{I=1\\ l_I\neq 0}}^N \left\{ \left(\prod_{n_I=1}^{l_I} \frac{d\lambda^I_{n_I}}{2\pi i} \right) \left(\frac{1}{\lambda^I_1} \prod_{n_I=1}^{l_I-1} \frac{1}{\lambda^{I,I}_{n_I+1,n_I}} \right) \right\} \right] \\ \times \prod_{I\neq J}^N \frac{\left(1 + i\frac{a_{IJ}}{\hbar\epsilon} - l_I\right)_{l_J}}{\left(1 + i\frac{a_{IJ}}{\hbar\epsilon}\right)_{l_J}} \prod_{\substack{I=1\\ l_I\neq 0}}^N \frac{\epsilon^{l_I-1}}{l_I} \\ \times \frac{\prod_{I=1}^N \prod_{j=1}^{N_a} \epsilon^{l_I} \left(\frac{i\frac{\tilde{a}_j}{\hbar} + a_I}{\epsilon}\right)}{\prod_{I=1}^N \prod_{r\neq I}^N \epsilon^{l_I}\left(-i\frac{a_{Ir}}{\hbar\epsilon}\right) \prod_{\substack{I=1\\ l_I\neq 0}}^N \epsilon^{l_I-1}(l_I-1)! \prod_{I=1}^N \prod_{r=N+1}^{N_f} \epsilon^{l_I}\left(-i\frac{a_{rI}}{\hbar\epsilon}\right)}, \end{aligned} \tag{A.40}$$

where the integration gives $[\ldots] = 1$. We are left with products of ratios including the equivariant parameters, which we express as Pochhammer symbols and after heavy Pochhammer algebra we finally arrive at (A.14), which proves (A.33).

Now, if the integrand f does not have poles at infinity, which happens exactly for $N_f \geq N_a + 2$, we can write

$$\int_{\mathcal{C}_u} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) = (-1)^l \int_{\mathcal{C}_d} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right) \tag{A.41}$$

with $\mathcal{C}_d$ having the same base as $\mathcal{C}_u$ but is closed in the lower half plane by a semicircle. Both contours are oriented counterclockwise. The lovely fact is that the r.h.s. of the above equation gives the desired result

$$(-1)^l \int_{\mathcal{C}_d} \prod_{\alpha=1}^{l} \frac{d\phi_\alpha}{2\pi i} f\left(\phi, \epsilon, \frac{a}{\hbar}, \frac{\widetilde{a}}{\hbar}\right)\Big|_{\epsilon=1} = (-1)^{N_a l} Z_l^D|_{d.m.} \tag{A.42}$$

after direct evaluation of the integral, completely analogue to that of (A.33).

A.5. Example: the $Gr(1,3) \simeq Gr(2,3)$ case. Let us show this isomorphism explicitly in a simple case: we will consider $Gr(1,3)$ and $Gr(2,3)$ in a completely equivariant setting.
Let us first compute the equivariant partition function for $Gr(1,3)$:

$$\begin{aligned} Z_{Gr(1,3)} &= \sum_m \int \frac{d\tau}{2\pi i} e^{4\pi \xi_{\text{ren}} \tau - i\theta_{\text{ren}} m} \prod_{j=1}^{3} \frac{\Gamma(\tau + irMa_j - \frac{m}{2})}{\Gamma(1 - \tau - irMa_j - \frac{m}{2})} \\ &= \sum_{i=1}^{3} ((rM)^6 z\bar{z})^{irMa_i} \prod_{\substack{j=1 \\ j\neq i}}^{3} \frac{\Gamma(-irMa_{ij})}{\Gamma(1 + irMa_{ij})} \sum_{l\geq 0} \frac{[(rM)^3 z]^l}{\prod_{j=1}^{3}(1 + irMa_{ij})_l} \\ &\quad \times \sum_{k\geq 0} \frac{[(-rM)^3 \bar{z}]^k}{\prod_{j=1}^{3}(1 + irMa_{ij})_k} \end{aligned} \tag{A.43}$$

Here we defined $a_{ij} = a_i - a_j$, and the twisted masses have been rescaled according to $a_i \to Ma_i$, so they are now dimensionless. For $Gr(2,3)$ we have (with $\tilde{\theta}_{\text{ren}} = \tilde{\theta} + \pi = \tilde{\theta} + 3\pi$, being $\tilde{\theta} \longrightarrow \tilde{\theta} + 2\pi$ a symmetry of the theory)

$$\begin{aligned} Z_{Gr(2,3)} &= \frac{1}{2} \sum_{m_1, m_2} \int \frac{d\tau_1}{2\pi i} \frac{d\tau_2}{2\pi i} e^{4\pi \tilde{\xi}_{\text{ren}}(\tau_1+\tau_2) - i\tilde{\theta}_{\text{ren}}(m_1+m_2)} \\ &\quad \left(-\tau_{12}^2 + \frac{m_{12}^2}{4}\right) \prod_{r=1}^{2} \prod_{j=1}^{3} \frac{\Gamma(\tau_r + irM\tilde{a}_j - \frac{m_r}{2})}{\Gamma(1 - \tau_r - irM\tilde{a}_j - \frac{m_r}{2})} \\ &= \sum_{i<j}^{3} ((rM)^6 \tilde{z}\bar{\tilde{z}})^{irM(\tilde{a}_i + \tilde{a}_j)} \prod_{\substack{k=1 \\ k\neq i,j}}^{3} \frac{\Gamma(-irM\tilde{a}_{ik})}{\Gamma(1 + irM\tilde{a}_{ik})} \frac{\Gamma(-irM\tilde{a}_{jk})}{\Gamma(1 + irM\tilde{a}_{jk})} \\ &\quad \sum_{l_1, l_2 \geq 0} \frac{[(-rM)^3 \tilde{z}]^{l_1+l_2}}{\prod_{k=1}^{3}(1 + irM\tilde{a}_{ik})_{l_1} \prod_{k=1}^{3}(1 + irM\tilde{a}_{jk})_{l_2}} \frac{l_1 - l_2 + irM\tilde{a}_i - irM\tilde{a}_j}{irM\tilde{a}_i - irM\tilde{a}_j} \end{aligned}$$

$$\sum_{k_1,k_2\geq 0} \frac{[(rM)^3\tilde{\bar{z}}]^{k_1+k_2}}{\prod_{k=1}^{3}(1+irM\tilde{a}_{ik})_{k_1}\prod_{k=1}^{3}(1+irM\tilde{a}_{jk})_{k_2}} \frac{k_1-k_2+irM\tilde{a}_i-irM\tilde{a}_j}{irM\tilde{a}_i-irM\tilde{a}_j} \tag{A.44}$$

In both situations, we are assuming $a_1 + a_2 + a_3 = 0$ and $\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 = 0$. Consider now the partition $(\bullet, \bullet, \Box)$ for $Gr(1, 3)$ and the dual partition $(\Box, \Box, \bullet)$ for $Gr(2, 3)$; we have respectively

$$\begin{aligned}
Z_{Gr(1,3)}^{(\bullet,\bullet,\Box)} &= ((rM)^6 z\bar{z})^{irMa_3} \frac{\Gamma(-irMa_{31})}{\Gamma(1+irMa_{31})} \frac{\Gamma(-irMa_{32})}{\Gamma(1+irMa_{32})} \\
&\quad \sum_{l\geq 0} \frac{[(rM)^3 z]^l}{l!(1+irMa_{31})_l(1+irMa_{32})_l} \\
&\quad \sum_{k\geq 0} \frac{[(-rM)^3 \bar{z}]^k}{k!(1+irMa_{31})_k(1+irMa_{32})_k} \\
Z_{Gr(2,3)}^{(\Box,\Box,\bullet)} &= ((rM)^6 \tilde{z}\tilde{\bar{z}})^{irM(\tilde{a}_1+\tilde{a}_2)} \frac{\Gamma(-irM\tilde{a}_{13})}{\Gamma(1+irM\tilde{a}_{13})} \frac{\Gamma(-irM\tilde{a}_{23})}{\Gamma(1+irM\tilde{a}_{23})} \\
&\quad \sum_{l_1,l_2\geq 0} \frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{i=1}^{2} l_i! \prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{l_i}} \frac{l_1-l_2+irM\tilde{a}_1-irM\tilde{a}_2}{irM\tilde{a}_1-irM\tilde{a}_2} \\
&\quad \sum_{k_1,k_2\geq 0} \frac{[(rM)^3\tilde{\bar{z}}]^{k_1+k_2}}{\prod_{i=1}^{2} k_i! \prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{k_i}} \frac{k_1-k_2+irM\tilde{a}_1-irM\tilde{a}_2}{irM\tilde{a}_1-irM\tilde{a}_2}
\end{aligned} \tag{A.45}$$

Since

$$\begin{aligned}
&\sum_{l_1,l_2\geq 0} \frac{[(-rM)^3\tilde{z}]^{l_1+l_2}}{\prod_{i=1}^{2} l_i! \prod_{j\neq i}^{3}(1+irM\tilde{a}_{ij})_{l_i}} \frac{l_1-l_2+irM\tilde{a}_1-irM\tilde{a}_2}{irM\tilde{a}_1-irM\tilde{a}_2} \\
&\quad = \sum_{l\geq 0} \frac{[(-rM)^3\tilde{z}]^l}{l!(1+irM\tilde{a}_{13})_l(1+irM\tilde{a}_{23})_l} c_l
\end{aligned} \tag{A.46}$$

and

$$c_l = \sum_{l_1=0}^{l} \frac{l!}{l_1!(l-l_1)!} \frac{(1+irM\tilde{a}_{23}+l-l_1)_{l_1}(1+irM\tilde{a}_{13}+l_1)_{l-l_1}}{(irM\tilde{a}_{12}-l+l_1)_{l_1}(-irM\tilde{a}_{12}-l_1)_{l-l_1}} = (-1)^l = (-1)^{3l}$$

we can conclude that $Z_{Gr(1,3)}^{(\bullet,\bullet,\Box)} = Z_{Gr(2,3)}^{(\Box,\Box,\bullet)}$ if we identify $a_i = -\tilde{a}_i$ and $\xi = \tilde{\xi}$, $\theta = \tilde{\theta}$ (i.e., $z = \tilde{z}$). It is then easy to prove that $Z_{Gr(1,3)} = Z_{Gr(2,3)}$.

References

1. Shadchin, S.: On F-term contribution to effective action. J. High Energy Phys. **8**, 52 (2007). arXiv:hep-th/0611278
2. Nekrasov, N.A.: Seiberg–Witten Prepotential from instanton counting. Adv. Theor. Math. Phys. **7**, 831–864 (2004). arXiv:hep-th/0206161

3. Alday, L.F., Gaiotto, D., Tachikawa, Y.: Liouville correlation functions from four-dimensional gauge theories. Lett. Math. Phys. **91**, 167–197 (2010). arXiv:0906.3219
4. Dimofte, T., Gukov, S., Hollands, L.: Vortex counting and Lagrangian 3-manifolds. Lett. Math. Phys. **98**, 225–287 (2011). arXiv:1006.0977
5. Bonelli, G., Tanzini, A., Zhao, J.: Vertices, vortices and interacting surface operators. JHEP **1206**, 178 (2012). arXiv:1102.0184
6. Bonelli, G., Tanzini, A., Zhao, J.: The Liouville side of the vortex. JHEP **1109**, 096 (2011). arXiv:1107.2787
7. Kozcaz, C., Pasquetti, S., Passerini, F., Wyllard, N.: Affine sl(N) conformal blocks from N=2 SU(N) gauge theories. JHEP **1101**, 045 (2011). arXiv:1008.1412
8. Kanno, H., Tachikawa, Y.: Instanton counting with a surface operator and the chain-saw quiver. JHEP **1106**, 119 (2011). arXiv:1105.0357
9. Bulycheva, K., Chen, H.-Y., Gorsky, A., Koroteev, P.: BPS states in omega background and integrability. JHEP **1210**, 116 (2012). arXiv:1207.0460
10. Benini, F., Cremonesi, S.: Partition functions of N=(2,2) gauge theories on S^2 and vortices. Commun. Math. Phys. (to appear, 2014). arXiv:1206.2356
11. Doroud, N., Gomis, J., Le Floch, B., Lee, S.: Exact results in D = 2 supersymmetric gauge theories. JHEP **1305**, 093 (2013). arXiv:1206.2606
12. Jockers, H., Kumar, V., Lapan, J.M., Morrison, D.R., Romo, M.: Two-sphere partition functions and Gromov–Witten invariants. Commun. Math. Phys. **325**(3), 1139–1170 (2014). arXiv:1208.6244
13. Gomis, J., Lee, S.: Exact Kahler potential from gauge theory and mirror symmetry. JHEP **1304**, 019 (2013). arXiv:1210.6022
14. Park, D.S., Song, J.: The Seiberg–Witten Kahler potential as a two-sphere partition function. JHEP **1301**, 142 (2013). arXiv:1211.0019
15. Sharpe, E.: Predictions for Gromov–Witten invariants of noncommutative resolutions. J. Geom. Phys. **74**, 256–265 (2013). arXiv:1212.5322
16. Honma, Y., Manabe, M.: Exact Kähler potential for Calabi–Yau fourfolds. J. High Energy Phys. **5**, 102 (2013). arXiv:1302.3760
17. Halverson, J., Kumar, V., Morrison, D.R.: New methods for characterizing phases of 2D supersymmetric gauge theories. J. High Energy. Phys **2013**, 143 (2013). arXiv:1305.3278
18. Sharpe, E.: A few Ricci-flat stacks as phases of exotic GLSM's. Phys. Lett. B. **726**(1–3), 390–395 (2013). arXiv:1306.5440
19. Givental, A.B.: Equivariant Gromov–Witten Invariants. Int. Math. Res. Notices **1996**, 613–663 (1996). arXiv:alg-geom/9603021
20. Ciocan-Fontanine, I., Kim, B., Maulik, D.: Stable quasimaps to GIT quotients. J. Geom. Phys. **75**, 17–47 (2014). arXiv:1106.3724
21. Kapustin, A., Willett, B.: Wilson loops in supersymmetric Chern-Simons-matter theories and duality. ArXiv e-prints (2013). arXiv:1302.2164
22. Witten, E.: Phases of N=2 theories in two-dimensions. Nucl. Phys. **B403**, 159–222 (1993). arXiv:hep-th/9301042
23. Coates, T., Corti, A., Lee, Y.-P., Tseng, H.-H.: The quantum orbifold cohomology of weighted projective spaces. Acta. Math. **202**, 139–193 (2009). arXiv:math/0608481
24. Bryan, J., Graber, T.: The crepant resolution conjecture. Proc. Symp. Pure Math. **80.1**, 1–20 (2009). arXiv:math/0610129
25. Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Holomorphic anomalies in topological field theories. Nucl.Phys. **B405**, 279–304 (1993). arXiv:hep-th/9302103
26. Dubrovin, B.: Geometry of 2-D topological field theories. In: Integrable Systems and Quantum Groups (Montecatini Terme, 1993). Lecture Notes in Mathematics, vol. 1620, pp. 120–348. Springer, New York (1996). arXiv:hep-th/9407018
27. Coates, T., Givental, A.: Quantum Riemann—Roch, Lefschetz and Serre. Ann. Math. **165**, 15–53 (2007). arXiv:math/0110142
28. Bonelli, G., Sciarappa, A., Tanzini, A., Vasko, P.: The stringy instanton partition function. J. High Energy. Phys. **2014**, 38 (2014). arXiv:1306.0432
29. Forbes, B., Jinzenji, M.: J functions, non-nef toric varieties and equivariant local mirror symmetry of curves. Int. J. Mod. Phys. **A22**, 2327–2360 (2007). arXiv:math/0603728
30. Aganagic, M., Bouchard, V., Klemm, A.: Topological strings and (almost) modular forms. Commun. Math. Phys. **277**, 771–819 (2008). arXiv:hep-th/0607100
31. Coates, T., Corti, A., Iritani, H., Tseng, H.-H.: Computing genus-zero twisted Gromov–Witten Invariants. Duke Math J. **147**(3), 377–438 (2009). arXiv:math/0702234
32. Coates, T.: Wall-crossings in toric Gromov–Witten Theory II: local examples. ArXiv e-prints (2008). arXiv:0804.2592

33. Brini, A., Tanzini, A.: Exact results for topological strings on resolved Y**p,q singularities. Commun. Math. Phys. **289**, 205–252 (2009). arXiv:0804.2598
34. Bertram, A., Ciocan-Fontanine, I., Kim, B.: Two proofs of a conjecture of Hori and Vafa. Duke Math. J. **126**(1), 101–136 (2005). arXiv:math/0304403
35. Hori, K., Vafa, C.: Mirror symmetry. arXiv:hep-th/0002222
36. Bertram, A., Ciocan-Fontanine, I., Kim, B.: Gromov–Witten invariants for abelian and nonabelian quotients. J. Algebraic Geom. **17**, 275–294 (2008). arXiv:math/0407254
37. Kronheimer, P.B., Nakajima, H.: Yang–Mills instantons on ALE gravitational instantons. Math. Ann. **288**, 263–307 (1990)
38. Ciocan-Fontanine, I., Diaconescu, D.-E., Kim, B.: From I to J in two dimensional (4, 4) quiver gauge theories (preprint)
39. Maulik, D., Oblomkov, A.: Quantum cohomology of the Hilbert scheme of points on A_n-resolutions. J. Am. Math. Soc. **22**, 1055–1091 (2009). arXiv:0802.2737 [math.AG]
40. Brini, A.: The Local Gromov–Witten theory of $\{\{C\}\{P\}^\wedge 1\}$ and integrable hierarchies. Commun. Math. Phys. **313**, 571–605 (2012). arXiv:1002.0582
41. Givental, A., Lee, Y.-P.: Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups. Invent. Math. **151**, 193–219 (2003). arXiv:math/0108105
42. Nieri, F., Pasquetti, S., Passerini, F.: 3d and 5d gauge theory partition functions as q-deformed CFT correlators. arXiv:1303.2626

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