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Teleparallel Gravity as a Higher Gauge Theory

John C. Baez[1,2], **Derek K. Wise**[3]

[1] Department of Mathematics, University of California, Riverside, CA 92521, USA
[2] Centre for Quantum Technologies, National University of Singapore, Singapore 117543, Singapore.
E-mail: baez@math.ucr.edu
[3] Department of Mathematics, University of Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany.
E-mail: derek.wise@fau.de

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Abstract: We show that general relativity can be viewed as a higher gauge theory involving a categorical group, or 2-group, called the teleparallel 2-group. On any semi-Riemannian manifold M, we first construct a principal 2-bundle with the Poincaré 2-group as its structure 2-group. Any flat metric-preserving connection on M gives a flat 2-connection on this 2-bundle, and the key ingredient of this 2-connection is the torsion. Conversely, every flat strict 2-connection on this 2-bundle arises in this way if M is simply connected and has vanishing 2nd deRham cohomology. Extending from the Poincaré 2-group to the teleparallel 2-group, a 2-connection includes an additional piece: a coframe field. Taking advantage of the teleparallel reformulation of general relativity, which uses a coframe field, a flat connection and its torsion, this lets us rewrite general relativity as a theory with a 2-connection for the teleparallel 2-group as its only field.

1. Introduction

This paper was prompted by two puzzles in higher gauge theory. Higher gauge theory is the generalization of gauge theory where instead of a connection defining parallel transport for point particles, we have a '2-connection' defining parallel transport for particles and strings, or an 'n-connection' for higher n defining parallel transport for extended objects whose worldvolumes can have dimensions up to and including n. While the mathematics of this subject is increasingly well-developed [8,10,37,38], its potential applications to physics remain less developed.

One puzzle concerns the Poincaré 2-group. Just as ordinary gauge theory involves choosing a Lie group, higher gauge theory involves a Lie n-group. Many examples of 2-groups are known [9] and one of the simplest is the Poincaré 2-group. A 2-group can also be seen as a 'crossed module', which is a pair of groups connected by a homomorphism, say $t\colon H \to G$, together with an action of G on H obeying two equations. For the Poincaré 2-group we take G to be Lorentz group, H to be the translation group of

Minkowski spacetime, t to be trivial, and use the usual action of Lorentz transformations on the translation group. The data involved here are the same that appear in the usual construction of the Poincaré group as a semidirect product. However, the *physical meaning* of the Poincaré 2-group has until now remained obscure.

A 'spin foam model' can be seen as a way to quantize a gauge theory or higher gauge theory by discretizing spacetime and rewriting the path integral as a sum [4,20,33]. Just as we can build spin foam models starting from a group, we can try to do the same starting with a 2-group. Crane and Sheppeard proposed using the Poincaré 2-group to build such a model [17], and the mathematics needed to carry out this proposal was developed by a number of authors [5,18,43].

The resulting spin foam model [12], based on representations of the Poincaré 2-group, provides a representation-theoretic interpretation of a model developed by Baratin and Freidel [11]. These authors have conjectured a fascinating relationship between this spin foam model and Feynman diagrams in ordinary quantum field theory on Minkowski spacetime: this spin foam model could be a 'quantum model of flat spacetime'. However, the physical meaning of this spin foam model remains unclear, because the corresponding classical field theory—perhaps some sort of higher gauge theory—is not known.

This brings us to our second puzzle: the apparent shortage of interesting classical field theories involving 2-connections. The reason is fairly simple. In a gauge theory with group G, the most important field is a connection, which can be seen locally as a $\mathfrak{g}$-valued 1-form A. In a higher gauge theory based on a crossed module $t\colon H \to G$, the most important field is a 2-connection. This can be seen locally as a $\mathfrak{g}$-valued 1-form A together with an $\mathfrak{h}$-valued 2-form B. However, A and B are not independent: to define parallel transport along curves and surfaces in a well-behaved way, they must satisfy an equation, the 'fake flatness condition':

$$F = \underline{t}(B).$$

Here F is the curvature of A and we use $\underline{t}\colon \mathfrak{h} \to \mathfrak{g}$, the differential of the map $t\colon H \to G$, to convert B into a $\mathfrak{g}$-valued 2-form.

So far it seems difficult to get this condition to arise naturally in field theories, except in theories without local degrees of freedom. For example, we can take any simple Lie group G, let H be the vector space $\mathfrak{g}$ viewed as an abelian Lie group, take t to be trivial, and use the adjoint action of G on H. This gives a Lie 2-group called the 'tangent 2-group' of G [9]. In this case the fake flatness condition actually says that A is flat: $F = 0$. This equation is one of the field equations for 4d BF theory, which has this Lagrangian:

$$L = \mathrm{tr}(B \wedge F).$$

So, the solutions of 4d theory can be seen as 2-connections. However, in part because all flat connections are locally gauge equivalent, there is no physical way to distinguish between two solutions of 4d BF theory in a contractible region of spacetime. We thus say that this theory has no local degrees of freedom.

Another theory without local degrees of freedom, called 'BFCG theory', starts with a 2-connection for a fairly general Lie 2-group together with some extra fields [22,29]. The equations of motion again imply fake flatness. A proposal due to Miković and Vojinović [32] involves modifying the BFCG action for the Poincare 2-group to obtain a theory equivalent to general relativity. However, after this modification the equations of motion no longer imply fake flatness. Moreover, for solutions, the $\mathfrak{h}$-valued 2-form in the would-be 2-connection is always zero. So, the problems of finding a clear geometrical interpretation of Poincaré 2-connections, and finding a physically interesting theory involving such 2-connections, still stand.

In this paper we suggest a way to solve both these problems in one blow. The idea is to treat gravity in 4d spacetime as a higher gauge theory based on the Poincaré 2-group, or some larger 2-group. We do this using a reformulation of general relativity called 'teleparallel gravity'. This theory is locally equivalent to general relativity, at least in the presence of spinless matter. Einstein studied it intensively from 1928 to 1931 [35], and he also had a significant correspondence on the subject with Élie Cartan [19].

As in general relativity, the idea in teleparallel gravity is to start with a metric on the spacetime manifold M and then choose a metric-compatible connection ω on the tangent bundle of M. However, instead of taking ω to be the torsion-free (and typically curved) Levi-Civita connection, we take ω to be flat (and typically with nonzero torsion). There is always a way to do this, at least locally. To do it, we can pick any orthonormal coframe field e and let ω be the unique flat metric-compatible connection for which the covariant derivative of e vanishes. That is, we define ω by declaring a vector v to be parallel transported if $e(v)$ is constant along the path. This choice of ω is called the 'Weitzenböck connection' for e. And then, remarkably, one can convert all of the standard equations in general relativity into equations involving the coframe and its Weitzenböck connection, with no reference to the Levi-Civita connection.

How is this related to the Poincaré 2-group? Recall that in the Poincaré 2-group, G is the Lorentz group and H is the group of translations of Minkowski spacetime. The Weitzenböck connection ω can locally be seen as a $\mathfrak{g}$-valued 1-form, and its torsion, defined by

$$T(v, w) = \nabla_v w - \nabla_w v - [v, w]$$

can locally be seen as an $\mathfrak{h}$-valued 2-form, namely the exterior covariant derivative $d_\omega e$. So, ω and T are the right sort of objects—at least locally, but in fact globally—to form a 2-connection with the Poincaré 2-group as gauge 2-group. Even better, since the map $t\colon H \to G$ is trivial, the fake flatness condition says the Weitzenböck connection ω is flat, *which is true*.

In short, general relativity can be reformulated as a theory involving a flat metric-compatible connection ω. Even though the connection is flat, this theory has local degrees of freedom because the torsion T is nonzero and contains observable information about the local geometry of spacetime. Furthermore, the pair (ω, T) fit together to form a 2-connection, and the relevant gauge 2-group is the Poincaré 2-group.

However, teleparallel gravity involves not just the Weitzenböck connection, but also the coframe field e. So, for a higher gauge theory interpretation of teleparallel gravity, we would also like to understand the coframe field as part of a 2-connection. For this, it is helpful to recall that an ordinary Poincaré connection, say A, consists of two parts: a Lorentz connection ω and a 1-form e valued in the Lie algebra of the translation group. When e obeys a certain nondegeneracy condition, it is precisely the same as a coframe field. The curvature of A then consists of two parts, the curvature of ω:

$$R = d\omega + \omega \wedge \omega$$

and the torsion:

$$T = d_\omega e.$$

To take advantage of these well-known facts, we enlarge the Poincaré 2-group to the 'teleparallel 2-group'. This comes from the crossed module $t\colon H \to G$ where G is the

Poincaré group, H is the translation group of Minkowski spacetime, t is the inclusion, and G acts on H by conjugation, using the fact that H is a normal subgroup of G. A 2-connection with this gauge 2-group turns out to consist of three parts:

- a flat Lorentz group connection ω,
- a 1-form e valued in the Lie algebra of the translation group,
- the torsion $T = d_\omega e$.

These are precisely the data involved in teleparallel gravity.

In what follows we flesh out this story. We begin in Sect. 2 with a review of higher gauge theory, including Lie groupoids, Lie 2-groups, 2-bundles and 2-connections. To minimize subtleties that are irrelevant here, and make the paper completely self-contained, we work in the 'strict' rather than the fully general 'weak' framework. In Sect. 3, we begin by describing the relation between teleparallel geometry and Poincaré 2-connections. The highlight of this section is Theorem 21, which among other things describes Poincaré 2-connections on a principal 2-bundle called the '2-frame 2-bundle' canonically associated to any semi-Riemannian manifold.

In Sect. 4, we introduce the teleparallel 2-group. Theorem 32 does for this 2-group what Theorem 21 did for the Poincaré 2-group. Then, we describe how to express the Lagrangian for teleparallel gravity as a function of a teleparallel 2-connection. This action is not invariant under all teleparallel 2-group gauge transformations, but only under those in a sub-2-group. This may seem disappointing at first, but it mirrors what we are already familiar with in the Palatini formulation of general relativity, where the fields can be seen as forming a Poincaré connection, but the action is only invariant under gauge transformations lying in the Lorentz group. This phenomenon in Palatini gravity can be neatly understood using Cartan geometry [42]. So, it is natural to expect that the similar phenomemon in teleparallel gravity can be understood using 'Cartan 2-geometry', and we present some evidence that this is the case. We also consider what happens when we go beyond the 'strict' framework discussed here to the more general 'weak' framework.

2. Higher Gauge Theory

Here we introduce all the higher gauge theory that we will need in this paper. First we explain the Poincaré 2-group $\mathbf{Poinc}(p, q)$ and its Lie 2-algebra. Then we explain principal 2-bundles and show that any semi-Riemannian manifold of signature (p, q) has a principal 2-bundle over it whose structure 2-group is $\mathbf{Poinc}(p, q)$. We take a businesslike approach, often sacrificing generality and elegance for efficiency. For a more well-rounded introduction to higher gauge theory, see our review article [8] and the more advanced references therein.

2.1. Lie groupoids. The first step towards higher gauge theory is to generalize the concept of 'manifold' to a kind of space that has, besides points, also arrows between points. There are many ways to do this, all closely related but differing in technical details. Here we use the concept of a 'Lie groupoid'. A **groupoid** is a category where every morphism has an inverse; we can visualize the objects of a groupoid as points and the morphisms as arrows. A Lie groupoid is basically a groupoid where the set of points forms a smooth manifold, and so does the set of arrows. More precisely:

Definition 1. A **Lie groupoid X** is a groupoid where:

- the collection of objects is a manifold, say $\mathbf{X}_0$;
- the collection of morphisms is a manifold, say $\mathbf{X}_1$;
- the maps $s, t\colon \mathbf{X}_1 \to \mathbf{X}_0$ sending each morphism to its source and target are smooth;
- the map sending each object to its identity morphism is smooth;
- the map sending each morphism to its inverse is smooth;
- the set of composable pairs of morphisms is a submanifold of $\mathbf{X}_1 \times \mathbf{X}_1$, and composition is a smooth map from this submanifold to $\mathbf{X}_1$.

To ensure that the set of composable pairs of morphisms is a submanifold, it suffices to assume that the map s, or equivalently t, is a submersion. This assumption is commonly taken as part of the definition of a Lie groupoid, and the reader is welcome to include it, since it holds in all our examples, but we will not actually need it.

We will use the obvious naive notion of map between Lie groupoids:

Definition 2. Given Lie groupoids $\mathbf{X}$ and $\mathbf{Y}$, a **map** $f\colon \mathbf{X} \to \mathbf{Y}$ is a functor for which:

- the map sending objects to objects, say $f_0\colon \mathbf{X}_0 \to \mathbf{Y}_0$, is smooth;
- the map sending morphisms to morphisms, say $f_1\colon \mathbf{X}_1 \to \mathbf{Y}_1$, is smooth.

There is another more general notion of map between Lie groupoids, and in this more general context Lie groupoids may be identified with 'differentiable stacks' [14,23]. However, in this paper we only need maps of the above type.

Many interesting examples of Lie groupoids can be found in Mackenzie's book [27]. Here are two rather trivial examples we will need:

Example 3. Any manifold can be seen as a Lie groupoid with only identity morphisms. In what follows, we freely treat manifolds as Lie groupoids in this way. Note that a map between Lie groupoids of this type is the same as a smooth map between manifolds.

Example 4. Given Lie groupoids $\mathbf{X}$ and $\mathbf{Y}$, there is a **product** Lie groupoid $\mathbf{X} \times \mathbf{Y}$ with $(\mathbf{X} \times \mathbf{Y})_0 = \mathbf{X}_0 \times \mathbf{Y}_0$ and $(\mathbf{X} \times \mathbf{Y})_1 = \mathbf{X}_1 \times \mathbf{Y}_1$, where the source and target maps, identity-assigning map and composition of morphisms are defined componentwise. Note that this product comes with **projection** maps from $\mathbf{X} \times \mathbf{Y}$ to $\mathbf{X}$ and to $\mathbf{Y}$.

2.2. 2-Groups. In general, a Lie 2-group is a Lie groupoid equipped with a multiplication that obeys the group axioms *up to isomorphism* [9,24,36]. But the 2-groups needed in this paper, including the Poincaré 2-group, are all 'strict': the group laws hold on the nose, as equations. So, we need only the definition of strict Lie 2-groups:

Definition 5. A **(strict) Lie 2-group** $\mathbf{G}$ is a Lie groupoid equipped with an **identity** object $1 \in \mathbf{G}$, a **multiplication** map $m\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, and an **inverse** map $\mathrm{inv}\colon \mathbf{G} \to \mathbf{G}$ such that the usual group axioms hold.

Note that given a Lie 2-group $\mathbf{G}$, the manifold of objects $\mathbf{G}_0$ forms a Lie group with multiplication m_0, and the manifold of morphisms $\mathbf{G}_1$ forms a Lie group with multiplication m_1. So, a 2-group consists of two groups, but with further structure given by the source and target maps $s, t\colon \mathbf{G}_1 \to \mathbf{G}_0$, the identity-assigning map $i\colon \mathbf{G}_0 \to \mathbf{G}_1$, and composition of morphisms.

We are mainly interested in physics on spacetimes with several space dimensions and one time dimension, but the construction of the Poincaré 2-group works for any signature. So, let us fix natural numbers $p, q \geq 0$ and define $\mathbb{R}^{p,q}$ to be the vector space $\mathbb{R}^{p+q}$ equipped with the metric

$$ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2.$$

We often think of $\mathbb{R}^{p,q}$ as a group, and call it the **translation group**. We write $\mathrm{O}(p, q)$ to mean the group of linear isometries of $\mathbb{R}^{p,q}$. With some abuse of language, let us call this group the **Lorentz group**. Similarly, we define the **Poincaré group** to be the semidirect product

$$\mathrm{IO}(p, q) = \mathrm{O}(p, q) \ltimes \mathbb{R}^{p,q}$$

where $\mathrm{O}(p, q)$ acts on $\mathbb{R}^{p,q}$ as linear transformations in the obvious way. Multiplication in the Poincaré group is given by

$$(g, h)(g', h') = (gg', h + gh'),$$

where an element (g, h) consists of an element $g \in \mathrm{O}(p, q)$ and an element $h \in \mathbb{R}^{p,q}$. We denote the Poincaré group as $\mathrm{IO}(p, q)$ because it is also called the **inhomogeneous orthogonal group**. We will not need the smaller inhomogenous special orthogonal group $\mathrm{ISO}(p, q)$.

The Poincaré 2-group is very similar to the Poincaré group. The ingredients used to build it are just the same: the Lorentz group and its action on the translation group. The difference is that now the translations enter at a higher categorical level than the Lorentz transformations: namely, as morphisms rather than objects.

Definition 6. The **Poincaré 2-group Poinc**(p, q) is the Lie 2-group where:

- The Lie group of objects is the Lorentz group $\mathrm{O}(p, q)$.
- The Lie group of morphisms is the Poincaré group $\mathrm{IO}(p, q)$.
- The source and target of a morphism $(g, h) \in \mathrm{IO}(p, q)$ are both equal to g.
- The composite morphism $(g, h') \circ (g, h)$ is $(g, h' + h)$, where addition is done as usual in $\mathbb{R}^{p,q}$.

In what follows, we will also need to see the Lorentz group as a 2-group of a degenerate sort. This works as follows:

Example 7. In Example 3 we saw that any manifold may be seen as a Lie groupoid with only identity morphisms. As a corollary, any Lie group G may be seen as a 2-group whose morphisms are all identity morphisms. This 2-group has G as its group of objects and also G as its group of morphisms; by abuse of language we call this 2-group simply G.

2.3. Actions of Lie 2-groups. Just as Lie groups can act on manifolds, Lie 2-groups can act on Lie groupoids. Here we only consider 'strict' actions:

Definition 8. Given a Lie 2-group $\mathbf{G}$, a **(strict) left G 2-space** is a Lie groupoid $\mathbf{X}$ equipped with a map $\alpha\colon \mathbf{G} \times \mathbf{X} \to \mathbf{X}$ obeying the usual axioms for a left group action. Similarly, a **(strict) right G 2-space** is a Lie groupoid $\mathbf{X}$ equipped with a map $\alpha\colon \mathbf{X} \times \mathbf{G} \to \mathbf{X}$ obeying the usual axioms for a right group action.

Definition 9. Given a Lie 2-group $\mathbf{G}$ and left $\mathbf{G}$ 2-spaces $\mathbf{X}$ and $\mathbf{Y}$, we define a **(strict) map of G 2-spaces** to be a map of Lie groupoids $f: \mathbf{X} \to \mathbf{Y}$ such that acting by $\mathbf{G}$ and then mapping by f is the same as mapping and then acting. In other words, this diagram commutes:

$$\begin{array}{ccc} \mathbf{G} \times \mathbf{X} & \xrightarrow{\alpha_{\mathbf{X}}} & \mathbf{X} \\ {\scriptstyle 1\times f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbf{G} \times \mathbf{Y} & \xrightarrow[\alpha_Y]{} & \mathbf{Y} \end{array}$$

where $\alpha_{\mathbf{X}}$ is the action of $\mathbf{G}$ on $\mathbf{X}$, and $\alpha_{\mathbf{Y}}$ is the action of $\mathbf{G}$ on $\mathbf{Y}$.

The examples we need are these:

Example 10. Using the idea in Examples 3 and 7, any manifold can be seen as a Lie groupoid, and any Lie group can be seen as a Lie 2-group. Continuing this line of thought, if the manifold X is a left G space, we can think of it as a left 2-space for the 2-group. Similarly, any right G space can be seen as a right G 2-space for this 2-group.

Example 11. Given a Lie 2-group $\mathbf{G}$, the multiplication $m: \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ makes $\mathbf{G}$ into both a left $\mathbf{G}$ 2-space and a right $\mathbf{G}$ 2-space.

Example 12. If we treat the Lorentz group $\mathrm{O}(p, q)$ as a Lie 2-group following the ideas in Example 7, the result is a 'sub-2-group' of $\mathbf{Poinc}(p, q)$. This Lie 2-group has a left action on $\mathbf{Poinc}(p, q)$, coming from left multiplication. Explicitly, the action

$$\alpha: \mathrm{O}(p, q) \times \mathbf{Poinc}(p, q) \to \mathbf{Poinc}(p, q)$$

has

$$\alpha_0: \mathrm{O}(p, q) \times \mathrm{O}(p, q) \to \mathrm{O}(p, q)$$

given by multiplication in the Lorentz group, and

$$\alpha_1: \mathrm{O}(p, q) \times \mathrm{IO}(p, q) \to \mathrm{IO}(p, q)$$

given by restricting multiplication in the Poincaré group.

2.4. Crossed modules. A 'Lie crossed module' is an alternative way of describing a Lie 2-group. The reader can find the full definition of a Lie crossed module elsewhere [8,9]; we do not need it here. All we need to know is that a Lie crossed module is a quadruple (G, H, t, α) that we can extract from a Lie 2-group $\mathbf{G}$ as follows:

- G is the Lie group $\mathbf{G}_0$ of objects of $\mathbf{G}$;
- H is the normal subgroup of $\mathbf{G}_1$ consisting of morphisms with source equal to $1 \in G$;
- $t: H \to G$ is the homomorphism sending each morphism in H to its target;
- α is the action of G as automorphisms of H defined using conjugation in $\mathbf{G}_1$ as follows: $\alpha(g)h = 1_g h {1_g}^{-1}$, where $1_g \in \mathbf{G}_1$ is the identity morphism of $g \in G$.

Conversely, any Lie crossed module gives a Lie 2-group.

If we take all the data in a Lie crossed module and differentiate it, we get:

- the Lie algebra $\mathfrak{g}$ of G,

- the Lie algebra $\mathfrak{h}$ of H,
- the Lie algebra homomorphism $\underline{t}\colon \mathfrak{h} \to \mathfrak{g}$ obtained by differentiating $t\colon H \to G$, and
- the Lie algebra homomorphism $\underline{\alpha}\colon \mathfrak{g} \to \mathfrak{aut}(H)$ obtained by differentiating $\alpha\colon G \to \mathrm{Aut}(H)$.

Here we write $\underline{t}$ and $\underline{\alpha}$ instead of dt and $d\alpha$ to reduce confusion in later formulas involving these maps and also differential forms, where d stands for the exterior derivative. The quadruple $(\mathfrak{g}, \mathfrak{h}, \underline{t}, \underline{\alpha})$ is called an **infinitesimal crossed module**. Just as a Lie crossed module is an alternative way to describe a Lie 2-group, an infinitesimal crossed module is a way to describe a Lie 2-algebra [6,10]. We will use this approach in our description of 2-connections.

The crossed module (G, H, t, α) coming from the Poincaré 2-group works as follows:

- G is the Lorentz group $\mathrm{O}(p, q)$;
- H is the translation group $\mathbb{R}^{p,q}$ viewed as an abelian Lie group;
- t is trivial.
- α is the obvious representation of $\mathrm{O}(p, q)$ on $\mathbb{R}^{p,q}$.

Differentiating all this, we obtain an infinitesimal crossed module where:

- $\mathfrak{g} = \mathfrak{o}(p, q)$;
- $\mathfrak{h} = \mathbb{R}^{p,q}$ viewed as an abelian Lie algebra;
- $\underline{t}$ is trivial;
- $\underline{\alpha}$ is the obvious representation of $\mathfrak{o}(p, q)$ on $\mathbb{R}^{p,q}$.

2.5. Principal 2-bundles. In what follows we take a lowbrow, pragmatic approach to 2-bundles. In particular, we only define 'strict' 2-bundles, since those are all we need here. The interested reader can find more sophisticated treatments elsewhere [13,15,16,38].

Just as a bundle involves a 'projection' $p\colon E \to M$ that is a map between manifolds, a strict 2-bundle involves a map $p\colon \mathbf{E} \to M$. Here $\mathbf{E}$ is a Lie groupoid, but for our applications M is a mere manifold, regarded as a Lie groupoid as in Example 3.

The key property we require of a bundle is 'local triviality'. For this, note that given any open subset $U \subseteq M$, there is a Lie groupoid $\mathbf{E}|_U$ whose objects and morphisms are precisely those of $\mathbf{E}$ that map under p to objects and morphisms lying in U. Then we can restrict p to $\mathbf{E}|_U$ and obtain a map we call $p|_U : \mathbf{E}|_U \to U$. The local triviality assumption says that every point $x \in M$ has a neighborhood U such that $p|_U : \mathbf{E}|_U \to U$ is a 'trivial 2-bundle'. More precisely:

Definition 13. A **(strict) 2-bundle** consists of:

- a Lie groupoid $\mathbf{E}$ (the **total 2-space**),
- a manifold M (the **base space**),
- a map $p\colon \mathbf{E} \to M$ (the **projection**), and
- a Lie groupoid $\mathbf{F}$ (the **fiber**),

such that for every point $p \in M$ there is an open neighborhood U containing p and an isomorphism

$$t\colon E|_U \to U \times \mathbf{F},$$

called a **local trivialization**, such that this diagram commutes:

$$\begin{array}{ccc} E|_U & \xrightarrow{t} & U \times \mathbf{F} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_U} & \\ & U & \end{array}$$

Definition 14. Given a Lie 2-group $\mathbf{G}$, a **(strict) principal G 2-bundle** is a 2-bundle $p\colon \mathbf{P} \to M$ where:

- the fiber $\mathbf{F}$ is $\mathbf{G}$,
- the total 2-space $\mathbf{P}$ is a right $\mathbf{G}$ 2-space, and
- for each point $p \in M$ there is a neighborhood U containing p and a local trivialization $t\colon E|_U \to U \times \mathbf{F}$ that is an isomorphism of right $\mathbf{G}$ 2-spaces.

It is worth noting that just as a 2-group consists of two groups and some maps relating them, a principal 2-bundle consists of two principal bundles and some maps relating them. Suppose $p\colon \mathbf{P} \to M$ is a principal $\mathbf{G}$ 2-bundle. Then the bundle of objects, $p_0\colon \mathbf{P}_0 \to M$, is a principal $\mathbf{G}_0$ bundle, and the bundle of morphisms, $p_1\colon \mathbf{P}_1 \to M$, is a principal $\mathbf{G}_1$ bundle. These are related by the source and target maps $s, t\colon \mathbf{P}_1 \to \mathbf{P}_0$, the identity-assigning map $i\colon \mathbf{P}_0 \to \mathbf{P}_1$, and the map describing composition of morphisms. All these are maps between bundles over M.

We can build principal 2-bundles using transition functions. First recall the situation in ordinary gauge theory: suppose G is a Lie group and M a manifold. In this case we can build a G bundle over M using transition functions. To do this, we write M as the union of open sets or **patches** $U_i \subseteq M$:

$$M = \bigcup_i U_i .$$

Then, choose a smooth transition function on each double intersection of patches:

$$g_{ij}\colon U_i \cap U_j \to G.$$

These transition functions give gauge transformations. We can build a principal G bundle over all of M by gluing together trivial bundles over the patches with the help of these gauge transformations. However, this procedure will only succeed if the transition functions satisfy a consistency condition on each triple intersection:

$$g_{ij}(x) g_{jk}(x) = g_{ik}(x)$$

for all $x \in U_i \cap U_j \cap U_k$. This equation is called the **cocycle condition**.

A similar recipe works for higher gauge theory. Now let $\mathbf{G}$ be a Lie 2-group with G as its Lie group of objects. To build a principal $\mathbf{G}$ 2-bundle, it suffices to choose transition functions on double intersections of patches:

$$g_{ij}\colon U_i \cap U_j \to G$$

such that the cocycle condition holds. Conversely, given a strict principal $\mathbf{G}$ 2-bundle, a choice of open neighborhoods containing each point and local trivializations gives rise to transition functions obeying the cocycle condition. In a more general 'weak' principal

2-bundle, the cocycle condition would need to hold only up to isomorphism [13]. While this generalization is very important, none of our work here requires it.

For now we only give a rather degenerate example of a principal 2-bundle. The next section will introduce a more interesting example, one that really matters to us: the '2-frame 2-bundle'.

Example 15. Suppose $p\colon P \to M$ is a principal G bundle. Then we can regard G as a Lie 2-group as in Example 7, and P as a right 2-space of this Lie 2-group as in Example 10. Then $p\colon P \to M$ becomes a principal 2-bundle with this Lie 2-group as its structure 2-group.

2.6. Associated 2-bundles. Just as principal bundles have associated bundles, principal 2-bundles have associated 2-bundles. Given a principal $\mathbf{G}$ 2-bundle $p\colon \mathbf{P} \to M$ and a left $\mathbf{G}$ 2-space $\mathbf{F}$, we can construct an **associated 2-bundle**

$$q\colon \mathbf{P} \times_G \mathbf{F} \to M$$

with fiber $\mathbf{F}$. To do this, we first construct a Lie groupoid $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$. This Lie groupoid has a manifold of objects $(\mathbf{P} \times_{\mathbf{G}} \mathbf{F})_0$ and a manifold of morphisms $(\mathbf{P} \times_{\mathbf{G}} \mathbf{F})_1$. Both these are given as quotient spaces:

$$(\mathbf{P} \times_{\mathbf{G}} \mathbf{F})_i = \frac{\mathbf{P}_i \times \mathbf{F}_i}{(xg, f) \sim (x, gf)}$$

where $xg \in \mathbf{P}_i$ is the result of letting $g \in \mathbf{G}_i$ act on the right on $x \in \mathbf{P}_i$, and $gf \in \mathbf{F}_i$ is the result of letting $g \in \mathbf{G}_i$ act on the left on $f \in \mathbf{F}_i$. One can check that these quotient spaces are indeed manifolds. One can also check that the usual composition of morphisms in $\mathbf{P} \times \mathbf{F}$ descends to $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$, and that $\mathbf{P} \times_{\mathbf{G}} \mathbf{F}$ is a Lie groupoid. Then, we can define a map

$$q\colon \mathbf{P} \times_G \mathbf{F} \to M$$

sending the equivalence class of any object (x, f) to the object $p(x)$, and doing the only possible thing on morphisms. Finally, we can check that $q: \mathbf{P} \times_G \mathbf{F} \to M$ is a 2-bundle with fiber $\mathbf{F}$.

To connect teleparallel gravity to higher gauge theory, we take advantage of the fact that any manifold with a metric comes equipped with principal 2-bundle whose structure 2-group is the Poincaré 2-group. We call this the '2-frame 2-bundle'. To build it, we start with a manifold M equipped with a semi-Riemannian metric g of signature (p, q). Let $p\colon FM \to M$ be the bundle of orthonormal frames, a principal $\mathrm{O}(p, q)$ bundle. As in Example 15 we can think of $\mathrm{O}(p, q)$ as a Lie 2-group and $p\colon FM \to M$ as a principal $\mathrm{O}(p, q)$ 2-bundle.

Next, recall from Example 12 that the Poincaré 2-group $\mathbf{Poinc}(p, q)$ is a left 2-space for the 2-group $\mathrm{O}(p, q)$. This lets us form the associated 2-bundle $q\colon FM \times_{\mathrm{O}(p,q)} \mathbf{Poinc}(p, q) \to M$.

Definition 16. Given a manifold M equipped with a metric of signature (p, q), we set

$$\mathbf{2F}M = FM \times_{\mathrm{O}(p,q)} \mathbf{Poinc}(p, q)$$

and define the **2-frame 2-bundle** of M to be

$$q\colon \mathbf{2F}M \to M.$$

Note that since $\mathbf{Poinc}(p, q)$ is a right 2-space for itself, $\mathbf{2F}M$ is also a right $\mathbf{Poinc}(p, q)$ 2-space. Using this fact, it is easy to check the following:

Proposition 17. *Given a manifold M equipped with a metric of signature (p, q), the 2-frame 2-bundle $\mathbf{2F}M$ is a principal 2-bundle with structure 2-group $\mathbf{Poinc}(p, q)$.*

2.7. 2-Connections. Just as a connection on a bundle over a manifold M allows us to describe parallel transport along curves in M, a '2-connection' on a 2-bundle over M allows us to describe parallel transport along *curves and surfaces*. Just as a connection on a G bundle can be described locally as 1-form taking values in the Lie algebra $\mathfrak{g}$, a connection on a $\mathbf{G}$ 2-bundle can be described locally as a $\mathfrak{g}$-valued 1-form A together with an $\mathfrak{h}$-valued 2-form B. In order to consistently define parallel transport along curves and surfaces, A and B need to be related by an equation called the 'fake flatness condition'. The concept of 'fake curvature' was first introduced by Breen and Messing [16], but only later did it become clear that a consistent theory of parallel transport in higher gauge theory requires the fake curvature to vanish [10,38].

Suppose $\mathbf{G}$ is a 2-group whose infinitesimal crossed module is $(\mathfrak{g}, \mathfrak{h}, \underline{t}, \underline{\alpha})$. Then a **2-connection** on the trivial principal $\mathbf{G}$ 2-bundle over a smooth manifold M consists of

- a $\mathfrak{g}$-valued 1-form A on M and
- an $\mathfrak{h}$-valued 2-form B on M

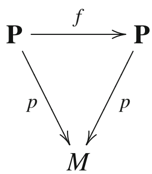
such that the **fake flatness** condition holds:

$$dA + A \wedge A = \underline{t}(B).$$

Here we use $\underline{t}\colon \mathfrak{h} \to \mathfrak{g}$, to convert B into a $\mathfrak{g}$-valued 2-form. As usual, $A \wedge A$ really stands for $\frac{1}{2}[A, A]$, defined using the wedge product of 1-forms together with the Lie bracket in $\mathfrak{g}$.

To describe 2-connections on a general strict $\mathbf{G}$ 2-bundle, we need to know how 2-connections on a trivial 2-bundle transform under gauge transformations. Here we only consider 'strict' gauge transformations:

Definition 18. Given a principal $\mathbf{G}$ 2-bundle $p\colon \mathbf{P} \to M$, a **(strict) gauge transformation** is a map of right $\mathbf{G}$ spaces $f\colon \mathbf{P} \to \mathbf{P}$ such that this diagram commutes:



It is easy to check that strict gauge transformations on the trivial principal $\mathbf{G}$ 2-bundle over M are in one-to-one correspondence with smooth functions $g\colon M \to G$, where G is the group of objects of $\mathbf{G}$. The proof is exactly like the proof for ordinary principal bundles. So, henceforth we identify gauge transformations on the trivial $\mathbf{G}$ 2-bundle over M with functions of this sort.

Now suppose we have a 2-connection (A, B) on the trivial principal $\mathbf{G}$ 2-bundle over M. By definition, a gauge transformation $g\colon M \to G$ acts on this 2-connection to give a new 2-connection (A', B') as follows:

$$\begin{aligned} A' &= gAg^{-1} \; + \; g\,dg^{-1} \\ B' &= \alpha'(g)(B) \end{aligned}$$

The second formula deserves a bit of explanation. Here we are composing $\alpha'\colon G \to \mathrm{Aut}(\mathfrak{h})$ with $g\colon M \to G$ and obtaining an $\mathrm{Aut}(\mathfrak{h})$-valued function $\alpha'(g)$, which then acts on the $\mathfrak{h}$-valued 2-form B to give a new $\mathfrak{h}$-valued 2-form $\alpha'(g)(B)$.

Now, suppose we have a principal $\mathbf{G}$ 2-bundle $p\colon P \to M$ built using transition functions

$$g_{ij}\colon U_i \cap U_i \to G$$

as described in Sect. 2.5. Note that each function g_{ij} can be seen as a gauge transformation. To equip P with a **strict 2-connection**, we first put a strict 2-connection on the trivial 2-bundle over each patch U_i. So, on each patch we choose a $\mathfrak{g}$-valued 1-form A_i and an $\mathfrak{h}$-valued 2-form B_i obeying

$$dA_i + A_i \wedge A_i = \underline{t}(B_i)$$

Then, we require that on each intersection of patches $U_i \cap U_j$, the 2-connection (A_i, B_i) is the result of applying the gauge transformation g_{ij} to (A_j, B_j):

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij}\, dg_{ij}^{-1} \tag{1}$$

$$B_i = \alpha'(g_{ij})(B_j) \tag{2}$$

For more details, including the more general case of weak 2-connections on weak 2-bundles, see [8] and the references therein.

We can also define strict 2-connections in terms of *global* data:

Proposition 19. *Suppose* $p\colon \mathbf{P} \to B$ *is a principal* $\mathbf{G}$ *2-bundle. Then a strict 2-connection* (A, B) *on* $\mathbf{P}$ *is the same as a connection* A *on the principal* $\mathbf{G}_0$ *bundle* $p_0\colon \mathbf{P}_0 \to B$ *together with a* $(\mathbf{P}_0 \times_{\mathbf{G}_0} \mathfrak{h})$*-valued 2-form* B *obeying the fake flatness condition.*

Proof. The equivalence follows simply by inspection of the local data used to build a strict 2-bundle with strict 2-connection. Choosing a suitable cover U_i of B, the principal $\mathbf{G}$ 2-bundle $\mathbf{P}$ and the principal $\mathbf{G}_0$ bundle $\mathbf{P}_0$ are constructed via the same transition functions $g_{ij}\colon U_i \cap U_j \to \mathbf{G}_0$. Then (1) is precisely the condition that assembles a family of Lie algebra-valued 1-forms into a connection on $\mathbf{P}_0$. Similarly, (2) is just the equation relating the local descriptions of a 2-form with values in the associated bundle $\mathbf{P}_0 \times_{\mathbf{G}_0} \mathfrak{h}$. The only additional condition for the 2-connection (A, B) is fake flatness, so imposing this, we are done. □

This theorem gives us a way to understand the fake flatness condition in a global way. The curvature of a connection A on a principal G bundle P is a $(P \times_G \mathfrak{g})$-valued 2-form F, where the vector bundle $P \times_G \mathfrak{g}$ is built using the adjoint representation of G. The map $\underline{t}\colon \mathfrak{h} \to \mathfrak{g}$ in the infinitesimal crossed module is an intertwiner of representations of G:

$$\mathrm{Ad}(g)\underline{t}(X) = \underline{t}(\alpha'(g)X) \qquad \forall g \in G,\ X \in \mathfrak{h}$$

where α' denotes the representation of G on $\mathfrak{h}$ given by differentiating α. Thus $\underline{t}$ induces a vector bundle map $\underline{t}\colon P \times_G \mathfrak{h} \to P \times_G \mathfrak{g}$. Using this, the fake flatness condition

$$F = \underline{t}(B)$$

makes sense globally as an equation between $(P \times_G \mathfrak{g})$-valued forms.

2.8. Curvature. There are three kinds of curvature for 2-connections. Suppose (A, B) is a strict 2-connection on a **G** 2-bundle $P \to M$. By working locally, we may choose a trivialization for P and treat A as a $\mathfrak{g}$-valued 1-form and B as an $\mathfrak{h}$-valued 2-form. This simplifies the discussion a bit.

First, just as in ordinary gauge theory, we may define the **curvature** to be the $\mathfrak{g}$-valued 2-form given by:

$$F = dA + A \wedge A.$$

Second, we define the **fake curvature** to be the $\mathfrak{g}$-valued 2-form $F - \underline{t}(B)$. However, this must equal zero. Third, we define the **2-curvature** to be the $\mathfrak{h}$-valued 3-form given by:

$$G = dB + \underline{\alpha}(A) \wedge B.$$

Beware: the symbol G here has nothing to do with the group G. In the second term on the right-hand side, we compose $\underline{\alpha}\colon \mathfrak{g} \to \mathfrak{aut}(H)$ with the $\mathfrak{g}$-valued 1-form A and obtain an $\mathfrak{aut}(H)$-valued function $\underline{\alpha}(g)$. Then we wedge this with B, letting $\mathfrak{aut}(H)$ act on $\mathfrak{h}$ as part of this process, and obtain an $\mathfrak{h}$-valued 2-form.

The intuitive idea of 2-curvature is this: just as the curvature describes the holonomy of a connection around an infinitesimal loop, the 2-curvature describes the holonomy of a 2-connection over an infinitesimal 2-sphere. This can be made precise using formulas for holonomies over surfaces [30,31,38], which we will not need here.

If the 2-curvature of a 2-connection vanishes, the holonomy over a surface will not change if we apply a smooth homotopy to that surface while keeping its edges fixed. A 2-connection whose curvature and 2-curvature both vanish truly deserves to be called **flat**.

3. Teleparallel Geometry and the Poincaré 2-Group

With tools of higher gauge theory in hand, we turn now to our main geometric and physical applications—teleparallel geometry and teleparallel gravity.

The simplest version of teleparallel gravity may be viewed as a rewriting of Einstein gravity in which torsion, as opposed to curvature, plays the lead role. This results in a theory that is conceptually quite different from general relativity: if gravity is interpreted within the teleparallel framework, then Einstein's original vision appears wrong on several counts. Unlike in general relativity, the spacetime of teleparallel gravity is *flat*. As a consequence, it is possible to compare vectors at distant points to decide, for example, whether the velocity vectors of two distant observers are parallel (hence the term 'teleparallelism' or 'distant parallelism'). Flat spacetime clearly flies in the face of Einstein's geometric picture of gravity as spacetime curvature: in fact, in teleparallel theories, gravity is a *force*.

With these features, teleparallel gravity sounds like such a throwback to the Newtonian understanding of gravity that it would be easy to dismiss, except for one fact: teleparallel gravity is *locally equivalent* to general relativity.

Our main interest in teleparallel gravity here is that it involves a flat connection ω and its torsion $d_\omega e$. As we shall see, these are precisely the data needed for a Poincaré 2-connection on the 2-frame 2-bundle of spacetime—or more precisely, a 2-bundle isomorphic to this. In what follows we briefly sketch some of the main ideas of teleparallel gravity, and discuss how it may be viewed as a higher gauge theory. For more on teleparallel gravity, we refer the reader to the work of Pereira and others [1–3,25].

3.1. General relativity in teleparallel language. The 'coframe field' or 'vielbein' is important in many approaches to gravity, and especially in teleparallel gravity. Locally, a coframe field is an $\mathbb{R}^n$-valued 1-form $e\colon TM \to \mathbb{R}^n$, where M is a $(p+q)$-dimensional manifold. When e is nondegenerate—that is, an isomorphism when restricted to each tangent space—it gives a metric on M via pullback.

To define coframe fields globally, even when M is not parallelizable, we start by fixing a vector bundle $\mathcal{T} \to M$ isomorphic to the tangent bundle, and equipped with a metric η of signature (p, q). We call $\mathcal{T}$ a **fake tangent bundle** for M. We then define a **coframe field** to be a vector bundle isomorphism

$$\begin{array}{ccc} TM & \xrightarrow{\;e\;} & \mathcal{T} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

The coframe field lets us pull back the inner product on $\mathcal{T}$ and get an inner product on the tangent bundle, making M into a semi-Riemannian manifold with a metric g of signature (p, q).

As mentioned, these ideas are useful in a number of approaches to gravity, most notably the Palatini approach, where the fiber $\mathcal{T}_x$ is sometimes called the 'internal space'. But in teleparallel gravity we must go further and assume that $\mathcal{T}$ is also equipped with a flat metric-preserving connection, say D. We can then use the coframe field to pull this back to a connection on TM, the **Weitzenböck connection**. We write this as ∇. By construction, the Weitzenböck connection is flat and metric-compatible. However, its torsion:

$$T(v, w) = \nabla_v w - \nabla_w v - [v, w]$$

is typically nonzero.

The idea behind teleparallel gravity is to take familiar notions from general relativity and write them in terms of the coframe field e and its Weitzenböck connection ∇ instead of the metric g and its Levi-Civita connection $\tilde{\nabla}$. To do this it is useful to consider the **contorsion**, which is the difference of the Weitzenböck and Levi-Civita connections:

$$K = \nabla - \tilde{\nabla}$$

Since the Levi-Civita connection can be computed from the metric, which in turn can be computed from the coframe field, we can write K explicitly in terms of the coframe field e and its Weitzenböck connection ∇. We do not need the explicit formula here: we merely want to note how the contorsion exhibits the physical meaning of teleparallel gravity.

For example, consider the motion of a particle in free fall. In general relativity, the particle's worldline $\gamma(s)$ obeys the geodesic equation for the Levi-Civita connection, stating that its covariant acceleration vanishes:

$$\tilde{\nabla}_{v(s)} v(s) = 0$$

where $v(s) = \gamma'(s)$ is the particle's velocity. In teleparallel gravity, on the other hand, the equation governing a particle's motion becomes

$$\nabla_{v(s)} v(s) = K_{v(s)} v(s)$$

From this perspective, the particle accelerates away from the geodesic determined by the connection ∇, and the contorsion is interpreted as the *gravitational force* responsible for the acceleration.

This may seem like a shell game designed to hide the 'true meaning' of general relativity. On the other hand, it is interesting that gravity admits such a qualitatively different alternative interpretation. Moreover, while we are only considering the teleparallel equivalent of general relativity here, more general versions of teleparallel gravity make physical predictions different from those of general relativity [3,25].

In any case, using the same strategy to rewrite the Einstein–Hilbert action of general relativity, one obtains, up to a boundary term, the teleparallel gravity action [3,28], which is proportional to:

$$S[e] = \int d^n x \ \det(e) \left(\frac{1}{4} T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu \right) \tag{3}$$

Here the components of the torsion in a coordinate frame x^μ are given by

$$T(\partial_\mu, \partial_\nu) = T^\rho{}_{\mu\nu} \partial_\rho,$$

and indices are moved using the metric pulled back along e. We have written $d^n x \det(e)$ for the volume form, where $n = p + q$ is the dimension of spacetime. Of course, one usually considers the case of 3+1 dimensions, but the action can be written in arbitrary dimension and signature. Note also that while the action involves the torsion T, this is a function of the Weitzenböck connection, which is a function of the coframe field, so the action is a function only of e.

To obtain the field equations, one can vary the action $S[e]$ with respect to the coframe field e. Of course, we can also 'cheat', using the known answer from general relativity. We simply convert $S[e]$ back into the usual Einstein–Hilbert action, and then recast the resulting Einstein equations in terms of the coframe and Weitzenböck connection. We shall not do this here, since we make no use of these equations; we refer the reader to references already cited in this section for more details.

3.2. Torsion and the coframe field. Since torsion is less widely studied than curvature—many geometry and physics books set torsion to zero from the outset—we recall some ideas about it here. We take an approach suited to teleparallel gravity, but also to our interpretation of it in terms of the Poincaré 2-group.

We start with a manifold M equipped with a fake tangent bundle $\mathcal{T} \to M$. From this we can build a principal $\mathrm{O}(p, q)$ bundle $\mathcal{F} \to M$, called the **fake frame bundle**. The idea is to mimic the usual construction of the frame bundle of a semi-Riemannian manifold. Thus, we let the fiber $\mathcal{F}_x$ at a point $x \in M$ be the space of linear isometries $\mathbb{R}^{p,q} \to \mathcal{T}_x$. The group $\mathrm{O}(p, q)$ acts on each fiber in an obvious way, and we can check that these fibers fit together to form the total space $\mathcal{F}$ of a principal $\mathrm{O}(p, q)$ bundle over M.

Next, suppose we have a coframe field

$$e \colon TM \xrightarrow{\sim} \mathcal{T}.$$

This allows us to pull back the inner product on $\mathcal{T}$ and get an inner product on the tangent bundle, making M into a semi-Riemannian manifold with a metric g given by

$$g(v, w) = \eta(e(v), e(w)).$$

Next, suppose we have a flat connection ω on $\mathcal{F}$. This determines a flat metric-preserving connection on $\mathcal{T}$, say D. We can then pull this back using the coframe field to obtain the Weitzenböck connection ∇ on TM. Explicitly, the **Weitzenböck connection** is determined by requiring that

$$e(\nabla_v w) = D_v(e(w))$$

for all vector fields v, w on M. We note that this condition is invariant under gauge transformations of $\mathcal{F}$. Hence the Weitzenböck connection itself is gauge invariant.

The torsion of the Weitzenböck connection is the TM-valued 2-form given by

$$T(v, w) = \nabla_v w - \nabla_w v - [v, w]$$

On the other hand, the coframe field e can be seen as a $\mathcal{T}$-valued 1-form, so its covariant exterior derivative is a $\mathcal{T}$-valued 2-form $d_\omega e$. Crucially, this is just the torsion in disguise. More precisely, we can translate between $d_\omega e$ and T using the coframe field:

$$e(T(v, w)) = (d_\omega e)(v, w)$$

for all vector fields v, w on M. To see this recall that the covariant exterior derivative d_ω is defined just like the ordinary differential d, but where all directional derivatives are replaced by covariant derivatives. Hence:

$$\begin{aligned} (d_\omega e)(v, w) &:= D_v(e(w)) - D_w(e(v)) - e([v, w]) \\ &= e\left(\nabla_v w - \nabla_w v - [v, w]\right) \\ &= e(T(v, w)). \end{aligned} \tag{4}$$

There is also a simple relationship between the notions of torsion and contorsion. If $\tilde{\omega}$ is the torsion-free connection for the coframe field e, then we find

$$\begin{aligned} d_\omega e &= d_\omega e - d_{\tilde{\omega}} e \\ &= (de + \omega \wedge e) - (de + \tilde{\omega} \wedge e) \\ &= K \wedge e \end{aligned}$$

where $K = \omega - \tilde{\omega}$ is the contorsion.

3.3. Poincaré 2-connections. Now we reach our first main result, a relationship between flat connections with torsion and Poincaré 2-connections. As we saw in Proposition 17, any semi-Riemannian manifold comes equipped with a principal $\mathbf{Poinc}(p, q)$ 2-bundle, its '2-frame 2-bundle'. There is also a 'fake' version of this construction when our manifold M is equipped with a fake tangent bundle $\mathcal{T} \to M$:

Definition 20. If $\mathcal{T} \to M$ is a fake tangent bundle with a metric of signature (p, q), and $\mathcal{F} \to M$ is its fake frame bundle, then we define the **fake 2-frame 2-bundle** to be the principal $\mathbf{Poinc}(p, q)$ 2-bundle $\mathbf{2}\mathcal{F} \to M$ where

$$\mathbf{2}\mathcal{F} = \mathcal{F} \times_{\mathrm{O}(p,q)} \mathbf{Poinc}(p, q).$$

As we have seen, the raw ingredients of teleparallel gravity are a flat connection ω on the fake frame bundle, together with a $\mathcal{T}$-valued 1-form e. The torsion, which plays a crucial role in teleparallel gravity, can then be reinterpreted as the $\mathcal{T}$-valued 2-form $d_\omega e$. Our result is that the pair $(\omega, d_\omega e)$ is then a flat 2-connection on the fake 2-frame 2-bundle. Conversely, if some topological conditions hold, every such flat 2-connection arises this way:

Theorem 21. *Suppose $\mathcal{T} \to M$ is a fake tangent bundle with a metric of signature (p, q). If ω is a flat connection on the fake frame bundle of M and e is a $\mathcal{T}$-valued 1-form, then $(\omega, d_\omega e)$ is a flat 2-connection on the fake 2-frame 2-bundle $\mathbf{2}\mathcal{F}$. Conversely, if M is simply connected and has vanishing 2nd de Rham cohomology, every flat 2-connection on $\mathbf{2}\mathcal{F}$ arises in this way.*

Proof. By construction, $\mathbf{2}\mathcal{F}$ has $\mathcal{F}$ as its manifold of objects. Thus, by Proposition 19, a 2-connection on $\mathbf{2}\mathcal{F}$ amounts to a connection ω on $\mathcal{F}$ together with a 2-form on M with values in $\mathcal{T} = (\mathcal{F} \times_{\mathrm{O}(p,q)} \mathbb{R}^{p,q})$, satisfying fake flatness. For the Poincaré 2-group, $\underline{t} = 0$, so fake flatness simply means the connection ω is flat. We thus get a 2-connection $(\omega, d_\omega e)$ on $\mathbf{2}\mathcal{F}$ from any flat connection ω on $\mathcal{F}$ and $\mathcal{T}$-valued 1-form e.

To see that the 2-connection $(\omega, d_\omega e)$ is flat, we must also check that its 2-curvature vanishes. This 2-curvature is just the exterior covariant derivative $d_\omega(d_\omega e)$. However, by the Bianchi identity

$$d_\omega(d_\omega e) = R \wedge e$$

where R is the curvature of ω, and $R = 0$ because ω is flat. It follows that the 2-connection $(\omega, d_\omega e)$ is flat.

For the converse, suppose we have any flat 2-connection (A, B) on $\mathbf{2}\mathcal{F}$; we will bring in the additional topological assumptions as we need them. As just discussed, the fake flatness condition implies that $A = \omega$ is a *flat* connection on $\mathcal{F}$. By Proposition 19, B is a $\mathcal{T}$-valued 2-form, and since the 2-curvature vanishes, B is covariantly closed:

$$d_\omega B = 0.$$

Now for any $\mathcal{T}$-valued form X we have $(d_\omega)^2 X = R \wedge X = 0$, since ω is flat. We thus get a cochain complex of $\mathcal{T}$-valued forms:

$$\Omega^0(M, \mathcal{T}) \xrightarrow{d_\omega} \Omega^1(M, \mathcal{T}) \xrightarrow{d_\omega} \Omega^2(M, \mathcal{T}) \xrightarrow{d_\omega} \cdots$$

Let us denote the cohomology of this complex by $H^\bullet_\omega(M, \mathcal{T})$. If we had $H^2_\omega(M, \mathcal{T}) = 0$, then $d_\omega B = 0$ would imply $B = d_\omega e$ for some $e \in \omega^1(M, \mathcal{T})$. This would complete the proof.

We now begin imposing topological conditions to guarantee that $H^2_\omega(M, \mathcal{T}) = 0$. First, suppose that (each component of) M is simply connected. In this case, note that FM has a flat connection if and only if M is parallelizable, meaning that the frame bundle FM is trivializable. For, any trivialization of FM determines a unique connection such that parallel transport preserves this trivialization. And conversely, given a flat connection, $\pi_1(M) = 1$ implies parallel transport is completely path independent, so we can trivialize FM by parallel translation, starting from a frame at one point in each component of M.

Since $\mathcal{F}$ is isomorphic to FM, the same is true for it: if M is simply connected, $\mathcal{F}$ admits a flat connection if and only if M is parallelizable.

Thus we may assume M is parallelizable. By trivializing the fake frame bundle of M, we may think of the 2-form B as taking values not in $\mathcal{T}$, but simply in $\mathbb{R}^{p,q}$. We are thus reduced to a cochain complex of the form

$$\Omega^0(M, \mathbb{R}^{p,q}) \xrightarrow{d_\omega} \Omega^1(M, \mathbb{R}^{p,q}) \xrightarrow{d_\omega} \Omega^2(M, \mathbb{R}^{p,q}) \xrightarrow{d_\omega} \cdots$$

and B determines a class in $H^2_\omega(M, \mathbb{R}^{p,q})$.

Gauge transformations act on the cohomology $H^\bullet_\omega$ in a natural way. Having trivialized $\mathcal{F}$, a gauge transformation may be written $\omega' = g\omega g^{-1} + gdg^{-1}$. Using this formula, it is easy to show that

$$d_{\omega'}(gX) = g(d_\omega X).$$

Thus, if Y and Y' are cohomologous for ω, then $Y - Y' = d_\omega X$, hence $gY - gY' = d_{\omega'}(gX)$, so that gY and gY' are cohomologous for ω'. Since M is simply connected, all flat connections on $\mathcal{F}$ are gauge equivalent. In particular, they are all gauge equivalent to the 'zero connection' in our chosen trivialization of $\mathcal{F}$.

In this gauge, the covariant differential d_ω becomes the ordinary differential d, and the question is simply whether $dB = 0$ implies $B = de$ for some e. This is essentially a question of de Rham cohomology: thinking of $B \in \Omega^2(M, \mathbb{R}^{p,q})$ as a $(p+q)$-tuple of 2-forms $B_1, \ldots, B_{p+q}$, it is clear that B is exact precisely when each B_i is. So, demanding now that the second de Rham cohomology of M vanishes, we get $B = de$ for some coframe field e, in the chosen gauge. Switching back to an arbitrary gauge, this shows that our 2-connection is really of the form $(\omega, d_\omega e)$. □

In particular, we can take the fake tangent bundle of M to be the actual tangent bundle:

Corollary 22. *Suppose M is a manifold equipped with a metric g of signature (p, q). If ω is a flat metric-compatible connection on the frame bundle of M, then the pair (ω, T), where T is the torsion of ω, is a flat strict 2-connection on the 2-frame 2-bundle of M.*

Proof. The tangent bundle TM, equipped with the metric g, is a particular case of a fake tangent bundle on M, and the corresponding fake frame bundle is just the usual (orthonormal) frame bundle. Taking the coframe field $e: TM \to TM$ to be the identity map, we have $\nabla = D$, so the calculation (4) simplifies to

$$(d_\omega e)(v, w) = \nabla_v w - \nabla_w v - [v, w] = T(v, w).$$

Thus $d_\omega e = T$, and by the previous theorem, (ω, T) is a flat 2-connection on the fake 2-frame 2-bundle, which here is just the original 2-frame 2-bundle of Definition 16. □

While the converse in Theorem 21 assumes the manifold is simply connected, this condition is not really necessary. We only introduced it to make it easy to check that the cohomology group $H^2_\omega(M, \mathcal{T})$ vanishes. Our proof gives more:

Corollary 23. *Suppose $\mathcal{T} \to M$ is a fake tangent bundle with a metric of signature (p, q). Suppose (A, B) is a flat 2-connection on* **2FM**. *Then $A = \omega$ is a flat connection on $\mathcal{T}$, and if $H^2_\omega(M, \mathcal{T}) = 0$, then $B = d_\omega e$ for some $\mathcal{T}$-valued 1-form e on M.*

Proof. This was established in the proof of Theorem 21. □

4. Teleparallel Gravity and the Teleparallel 2-Group

In the previous section, we showed that the Poincaré 2-group is related to 'teleparallel geometry'—the geometry of flat connections with torsion. However, the relevance to teleparallel *gravity* is not yet clear. The main field in the action for teleparallel gravity is the *coframe field*, while the Poincaré 2-connection $(\omega, d_\omega e)$ seems to play more of a supporting role. It would be much more satisfying if we could view the coframe field in terms of a 2-connection as well. In fact we can do just this if we extend the Poincaré 2-group to a larger 2-group: the 'teleparallel 2-group'.

4.1. The teleparallel 2-group. Let us start by defining this 2-group directly:

Definition 24. The **teleparallel 2-group** is the Lie 2-group $\mathbf{Tel}(p, q)$ for which:

- $\mathrm{IO}(p, q)$ is the Lie group of objects,
- $\mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q}$ is the Lie group of morphisms, where the semidirect product is defined using the following action:

$$\begin{aligned} \alpha\colon \mathrm{IO}(p,q) \times \mathbb{R}^{p,q} &\to \mathbb{R}^{p,q} \\ ((g,v),w) &\mapsto gw, \end{aligned}$$

  where $(g, v) \in \mathrm{IO}(p, q) = \mathrm{O}(p, q) \ltimes \mathbb{R}^{p,q}$ and $w \in \mathbb{R}^{p,q}$.
- the source of the morphism $((g, v), w)$ is (g, v),
- the target of the morphism $((g, v), w)$ is $(g, v + w)$
- the composite $((g, v), w) \circ ((g', v'), w')$, when defined, is $((g', v'), w + w')$

Explicitly, the product in the group of morphisms is given by

$$((g,v),w)((g',v'),w') = ((gg', v + gv'), w + gw').$$

The Lie crossed module corresponding to $\mathbf{Tel}(p, q)$ has:

- $G = \mathrm{IO}(p, q)$
- $H = \mathbb{R}^{p,q}$
- $t\colon \mathbb{R}^{p,q} \to \mathrm{IO}(p, q)$ the inclusion homomorphism $v \mapsto (1, v)$
- $\alpha(g, v)w = gw$.

The corresponding Lie 2-algebra, the **teleparallel Lie 2-algebra**, has an infinitesimal crossed module with

- $\mathfrak{g} = \mathfrak{io}(p, q)$
- $\mathfrak{h} = \mathbb{R}^{p,q}$
- $\underline{t}\colon \mathbb{R}^{p,q} \to \mathfrak{io}(p, q)$ the inclusion homomorphism $v \mapsto (0, v)$
- $\underline{\alpha}(\xi, v)w = \xi w$.

Another way to think about the teleparallel 2-group starts with the Poincaré group. Given any Lie group G there is a Lie 2-group $\mathbf{E}(G)$ with G as its group of objects and a unique morphism between any two objects. This corresponds to the crossed module $1\colon G \to G$, where G acts on itself by conjugation. This Lie 2-group plays an important role in topology, as first noted by Segal [7,39]: it is closely related to the contractible G space EG that is the total space of the universal principal G-bundle $EG \to BG$. However, what matters to us here is that teleparallel 2-group is a sub-2-group of $\mathbf{E}(\mathrm{IO}(p, q))$.

Definition 25. Given a Lie group G, define $\mathbf{E(G)}$ to be the Lie 2-group for which:

- G is the Lie group of objects,
- $G \ltimes G$ is the Lie group of morphisms, where G acts on itself by conjugation,
- the source of the morphism $(g, h) \in G \ltimes G$ is g,
- the target of the morphism $(g, h) \in G \ltimes G$ is hg
- the composite $(g, h) \circ (g', h')$, when defined, is (g', hh')

It is worth noting that $G \ltimes G$ is isomorphic to $G \times G$. However, the description of $\mathbf{E}(G)$ using a semidirect project makes it easier to understand the inclusion

$$i\colon \mathbf{Tel}(p, q) \to \mathbf{E}(\mathrm{IO}(p, q)).$$

Namely, the inclusions at the object and morphism levels, which we denote as i_0 and i_1, are the obvious ones:

$$\begin{aligned} i_0 &\colon & \mathrm{IO}(p, q) &\to \mathrm{IO}(p, q) \\ i_1 &\colon & \mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q} &\to \mathrm{IO}(p, q) \ltimes \mathrm{IO}(p, q). \end{aligned}$$

The 2-group $\mathbf{E}(G)$ can also be obtained from the crossed module (G, H, t, α) for which G and H are the same group G, $t\colon G \to G$ is the identity homomorphism, and α is the action of G on itself by conjugation.

There is also an inclusion of Poincaré 2-group in the teleparallel 2-group:

$$j\colon \mathbf{Poinc}(p, q) \to \mathbf{Tel}(p, q)$$

where the inclusions at the object and morphism levels:

$$\begin{aligned} j_0 &\colon & \mathrm{O}(p, q) &\to \mathrm{IO}(p, q) \\ j_1 &\colon & \mathrm{O}(p, q) \ltimes \mathbb{R}^{p,q} &\to \mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q} \end{aligned}$$

are the obvious ones. Here we have written the group of morphisms of $\mathbf{Poinc}(p, q)$ as $\mathrm{O}(p, q) \ltimes \mathbb{R}^{p,q}$ rather than $\mathrm{IO}(p, q)$ to emphasize that j_1 maps the $\mathbb{R}^{p,q}$ into the second factor of $\mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q}$, not into the copy of $\mathbb{R}^{p,q}$ hiding in the first factor.

Remember that we can regard the group $\mathrm{O}(p, q)$ as a Lie 2-group whose morphisms are all identities. As pointed out by Urs Schreiber, this 2-group is equivalent in the 2-category of Lie 2-groups, though not isomorphic, to $\mathbf{Tel}(p, q)$. The obvious inclusion $\mathrm{O}(p, q) \to \mathbf{Tel}(p, q)$ is an equivalence. In 'weak' higher gauge theory, equivalent 2-groups are often interchangeable. Nonetheless, $\mathrm{O}(p, q)$ and $\mathbf{Tel}(p, q)$ play different roles in our work. This remains somewhat mysterious and deserves further exploration.

The 2-groups that play a role in this paper can be summarized as follows. Each is included in the next:

$\mathbf{G}$	$\mathrm{O}(p, q)$	$\mathbf{Poinc}(p, q)$	$\mathbf{Tel}(p, q)$	$\mathbf{E}(\mathrm{IO}(p, q))$
$\mathbf{G}_0$	$\mathrm{O}(p, q)$	$\mathrm{O}(p, q)$	$\mathrm{IO}(p, q)$	$\mathrm{IO}(p, q)$
$\mathbf{G}_1$	$\mathrm{O}(p, q)$	$\mathrm{O}(p, q) \ltimes \mathbb{R}^{p,q}$	$\mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q}$	$\mathrm{IO}(p, q) \ltimes \mathrm{IO}(p, q)$

4.2. Poincaré connections. Our next goal is to describe 2-connections on certain 2-bundles with the teleparallel 2-group as gauge 2-group. Such a 2-bundle has a bundle of objects that is a principal Poincaré group bundle. Thus, a 2-connection on a principal **Tel**(p, q) 2-bundle consists partly of a Poincaré connection. It will therefore be helpful to have a few facts about Poincaré connections at hand.

First, if we pull the adjoint representation of the Poincaré group $\mathrm{IO}(p, q)$ back to the Lorentz group $\mathrm{O}(p, q)$, it splits into a direct sum

$$\mathfrak{io}(p, q) = \mathfrak{o}(p, q) \oplus \mathbb{R}^{p,q}$$

of irreducible $\mathrm{O}(p, q)$ representations. Since a connection A on a principal $\mathrm{IO}(p, q)$ bundle can locally be described by an $\mathfrak{io}(p, q)$-valued 1-form, we can split it as:

$$A = \omega + e,$$

where ω is $\mathfrak{o}(p, q)$-valued and e is $\mathbb{R}^{p,q}$-valued. Note that ω can be seen as a Lorentz connection, while e can be seen as a coframe field, at least when it restricts to an isomorphism on each tangent space. It is also easy to check that the same direct sum of representations splits the curvature of A, $F = dA + A \wedge A$, into two parts as follows:

$$F = R + d_\omega e.$$

Here the $\mathfrak{o}(p, q)$-valued part

$$R = d\omega + \omega \wedge \omega$$

is the curvature of ω, while the $\mathbb{R}^{p,q}$-valued part is the torsion $d_\omega e$.

To see how this all works globally, it helps to note that the coframe field, defined in Sect. 3.1 as an isomorphism $TM \to \mathcal{T}$, can equivalently be viewed as a certain kind of 1-form on $\mathcal{F}$:

Lemma 26. *Let $\mathcal{T}$ be a fake tangent bundle on M, and $p\colon \mathcal{F} \to M$ the corresponding fake frame bundle. Then there is a canonical one-to-one correspondence between:*

- *vector bundle morphisms $e\colon TM \to \mathcal{T}$, and*
- *$\mathbb{R}^{p,q}$-valued 1-forms ε on $\mathcal{F}$ that are:*
  - *horizontal: ε vanishes on $\ker(dp)$*
  - *$\mathrm{O}(p, q)$-equivariant: $R_h^*\varepsilon = h^{-1} \circ \varepsilon$ for all $h \in \mathrm{O}(p, q)$.*

Moreover, the first of these is an isomorphism precisely when the second is nondegenerate, meaning that each restriction $\varepsilon\colon T_f\mathcal{F} \to \mathbb{R}^{p,q}$ has rank $p + q$.

Proof. Suppose $e\colon TM \to \mathcal{T}$ is a vector bundle morphism. Let π be the projection map for the tangent bundle of $\mathcal{F}$:

$$\pi : T\mathcal{F} \to \mathcal{F}.$$

Then we can define $\tilde{e}\colon T\mathcal{F} \to \mathbb{R}^{p,q}$, an $\mathbb{R}^{p,q}$-valued 1-form on $\mathcal{F}$, as:

$$\tilde{e}(v) = \pi(v)^{-1}\big(e \circ dp(v)\big) \qquad \forall v \in T\mathcal{F}.$$

where $dp\colon T\mathcal{F} \to TM$ is the differential of the fake frame bundle $p\colon \mathcal{F} \to M$. In this formula we are using the fact that $\pi(v) \in \mathcal{F}$ is itself an isomorphism $\mathbb{R}^{p,q} \to \mathcal{T}_{p(\pi(v))}$, by the definition of the fake frame bundle in Sect. 3.2, and taking the inverse of this. This $\tilde{e}$ is horizontal, since it obviously vanishes on the kernel of dp. All that remains to check is equivariance. This follows simply from the equivariance of $\pi\colon T\mathcal{F} \to \mathcal{F}$ and $dp\colon T\mathcal{F} \to TM$, where the action $\mathrm{O}(p,q)$ on TM is trivial, induced from the trivial action on M:

$$\begin{aligned} R_h^*\tilde{e}(v) = \tilde{e}(R_{h*}v) &= \pi(R_{h*}v)^{-1}\big(e \circ dp(R_{h*}v)\big) \\ &= h^{-1} \circ \pi(v)^{-1}(e \circ dp(v)) \\ &= h^{-1} \circ \tilde{e}(v). \end{aligned}$$

Conversely, suppose we are given a horizontal and equivariant 1-form $\varepsilon\colon T\mathcal{F} \to \mathbb{R}^{p,q}$, and let us construct a vector bundle morphism $\bar{\varepsilon}\colon TM \to \mathcal{T}$. Given $w \in T_xM$, pick any $f \in \mathcal{F}_x$ and $v \in T_f\mathcal{F}$. Since ε is horizontal, $\varepsilon(v)$ does not depend on which $v \in T_f\mathcal{F}$ we pick; it does depend on the point f in the fiber, but this dependence is $\mathrm{O}(p,q)$-equivariant. Using this, it is easy to check that demanding

$$\bar{\varepsilon}(w) = \pi(v)\big(\varepsilon(v)\big) \quad \forall w \in TM, v \in T\mathcal{F} \text{ with } dp(v) = w$$

uniquely determines a vector bundle morphism $\bar{\varepsilon}\colon TM \to \mathcal{T}$.

It is also easy to see that these processes are inverse, namely that $\bar{\tilde{e}} = e$ and $\tilde{\bar{\varepsilon}} = \varepsilon$. Finally, it is easy to check that the vector bundle morphism $e\colon TM \to \mathcal{T}$ is an isomorphism precisely when the corresponding horizontal equivariant 1-form $\varepsilon\colon T\mathcal{F} \to \mathbb{R}^{p,q}$ is nondegenerate. □

Every manifold equipped with a fake tangent bundle $\mathcal{T} \to M$ has a principal Poincaré group bundle over it. To build this, we start with the fake frame bundle $\mathcal{F} \to M$ as described in Sect. 3.2. This is a principal Lorentz group bundle. We then extend this to a principal Poincaré group bundle over M, the **extended fake frame bundle** $\boldsymbol{I\mathcal{F}}$:

$$I\mathcal{F} := \mathcal{F} \times_{\mathrm{O}(p,q)} \mathrm{IO}(p,q).$$

A connection on the extended fake frame bundle is an $\mathfrak{io}(p,q)$-valued 1-form A on the total space $I\mathcal{F}$ satisfying the usual equations. But such connections have a more intuitive description:

Proposition 27. *There is a canonical* $\mathrm{O}(p,q)$*-equivariant correspondence between the following kinds of data:*

- *a connection A on the extended fake frame bundle $I\mathcal{F}$, with nondegenerate $\mathbb{R}^{p,q}$ part;*
- *a connection ω on the fake frame bundle $\mathcal{F}$ together with a coframe field e.*

Proof. A connection A on $I\mathcal{F} \to M$ is an $\mathfrak{io}(p,q)$-valued 1-form on the total space satisfying the usual equivariance properties under $\mathrm{IO}(p,q)$. This connection pulls back to a 1-form on $\mathcal{F}$, and the direct sum of $\mathrm{O}(p,q)$ representations $\mathfrak{io}(p,q) = \mathfrak{o}(p,q) \oplus \mathbb{R}^{p,q}$ splits this into 1-forms ω and e with values in $\mathfrak{o}(p,q)$ and $\mathbb{R}^{p,q}$. One can check that ω is a connection on $\mathcal{F}$ and that when e nondegenerate, it is equivalent to a coframe field, via Lemma 26.

Conversely, a connection ω on $\mathcal{F}$ and a coframe field, viewed as an equivariant 1-form $e : T\mathcal{F} \to \mathbb{R}^{p,q}$, assemble into an $\mathfrak{io}(p,q)$-valued 1-form on $\mathcal{F}$, thanks to our direct sum of $\mathrm{O}(p,q)$ representations. This 1-form has a unique equivariant extension to the associated bundle $I\mathcal{F} = \mathcal{F} \times_{\mathrm{O}(p,q)} \mathrm{IO}(p,q)$, and this is a connection A. The $\mathbb{R}^{p,q}$ part of A is nondegenerate because e is.

This correspondence is clearly equivariant under gauge transformations of the principal $\mathrm{O}(p,q)$ bundle $\mathcal{F}$. □

It follows that the local description of the curvature of an $\mathrm{IO}(p,q)$ connection, given at the beginning of this section, holds globally as well:

Proposition 28. *Let A be a connection on the extended fake frame bundle $\mathbf{I}\mathcal{F}$ described by a connection ω on $\mathcal{F}$ and a coframe field e. Then the curvature F of A consists of the curvature $R = d\omega + \omega \wedge \omega$ of ω and the torsion $d_\omega e$.*

4.3. Teleparallel 2-connections. We now have all the ingredients we need to build $\mathbf{Tel}(p,q)$ 2-connections. The first step is to note that any manifold M equipped with a fake tangent bundle has a principal $\mathbf{Tel}(p,q)$ 2-bundle over it. We can build this by forming the fake frame bundle and then extending its gauge 2-group from the Lorentz group to the teleparallel 2-group:

Definition 29. If $\mathcal{T} \to M$ is a fake tangent bundle with a metric of signature (p,q), and $\mathcal{F} \to M$ is its fake frame bundle, then we define the **teleparallel 2-bundle** to be the principal $\mathbf{Tel}(p,q)$ 2-bundle $\mathbf{Tel}(\mathcal{F}) \to M$ where

$$\mathbf{Tel}(\mathcal{F}) = \mathcal{F} \times_{\mathrm{O}(p,q)} \mathbf{Tel}(p,q).$$

Alternatively, we can start with the fake 2-frame 2-bundle and extend its gauge 2-group from the Poincaré 2-group to the teleparallel 2-group. This gives a 2-bundle

$$\mathbf{2}\mathcal{F} \times_{\mathbf{Poinc}(p,q)} \mathbf{Tel}(p,q)$$

which is canonically isomorphic to $\mathbf{Tel}(\mathcal{F})$. So, we are free to also think of this as the teleparallel 2-bundle.

To describe 2-connections on the teleparallel 2-bundle, it is perhaps easiest to go one step further and extend the gauge 2-group all the way to $\mathbf{E}(\mathrm{IO}(p,q))$. More generally, one can describe $\mathbf{E}(G)$ 2-connections for an arbitrary Lie group G:

Proposition 30. *Let P be a principal G-bundle, $\mathbf{E}(P) = P \times_G \mathbf{E}(G)$ the associated $\mathbf{E}(G)$ 2-bundle. For each connection A on P there is a unique strict 2-connection on $\mathbf{E}(P)$ whose 1-form part is A; the 2-form part is the curvature $F = dA + A \wedge A$.*

Proof. The bundle of objects of $\mathbf{E}(P)$ is P, so the 1-form part of a 2-connection (A, B) is a connection on P. The fake flatness condition in this case simply says $B = F$, since '$\underline{t}$' in the infinitesimal crossed module is the identity map. □

As a corollary, an $\mathbf{E}(\mathrm{IO}(p,q))$ 2-connection consists of four parts: connection, coframe, curvature, and torsion. However, the first two parts determine the others:

Proposition 31. *Let $\mathcal{F}$ be a fake frame bundle. A 2-connection on $\mathcal{F} \times_{\mathrm{O}(p,q)} \mathbf{E}(\mathrm{IO}(p,q))$ is specified uniquely by a connection ω on $\mathcal{F}$ and a $\mathcal{T}$-valued 1-form e. The 2-form part of this 2-connection consists of the curvature $R = d\omega + \omega \wedge \omega$ together with the torsion $d_\omega e$.*

Proof. $\mathcal{F} \times_{\mathrm{O}(p,q)} \mathbf{E}(\mathrm{IO}(p,q))$ is canonically isomorphic to $\mathbf{E}(\mathbf{I}\mathcal{F})$, where $\mathbf{I}\mathcal{F}$ is the extended fake frame bundle. In light of Propositions 27 and 28, the result then follows immediately from Proposition 30. □

Now consider $\mathbf{Tel}(p,q) \subseteq \mathbf{E}(\mathrm{IO}(p,q))$. A $\mathbf{Tel}(p,q)$ 2-connection can be thought of as a special case of an $\mathbf{E}(\mathrm{IO}(p,q))$ 2-connection. We can still interpret it as an $\mathrm{O}(p,q)$ connection ω and coframe e, together with the torsion $d_\omega e$. However, fake flatness now implies that $R = d\omega + \omega \wedge \omega = 0$, so ω must be a *flat* connection. Summarizing:

Theorem 32. *Let M be a manifold equipped with a fake tangent bundle. A 2-connection on the* $\mathbf{Tel}(p,q)$ *2-bundle*

$$\mathcal{F} \times_{\mathrm{O}(p,q)} \mathbf{Tel}(p,q)$$

consists of:

- *a* flat *connection ω on $\mathcal{F}$*
- *a $\mathcal{T}$-valued 1-form e, and*
- *the $\mathcal{T}$-valued 2-form $d_\omega e$.*

In the case where e is an isomorphism, it can be viewed as a coframe field, and we have just the fields needed for describing teleparallel geometry.

From this perspective, $\mathbf{Tel}(p,q)$ results from $\mathbf{E}(\mathrm{IO}(p,q))$ by truncating the part where the curvature lives, leaving only torsion at the morphism level. Crucially, this truncation does not simply 'forget' the curvature part: the fake flatness condition forces the curvature to vanish.

The reader may well wonder if we could do an analogous truncation that would allow us to use 2-connections to describe geometries that are not flat but rather torsion-free, removing the $\mathrm{O}(p,q)$ part of the group of morphisms in the 2-group, rather than the $\mathbb{R}^{p,q}$ part. In the language of crossed modules, this would mean taking the group H to be $\mathrm{O}(p,q)$. But unlike $\mathbb{R}^{p,q}$, $\mathrm{O}(p,q)$ is not a normal subgroup of $\mathrm{IO}(p,q)$, so the action of $\mathrm{IO}(p,q)$ does not restrict to an action on this subgroup. Thus, this strategy fails to give a 2-group.

Something interesting happens when we calculate the curvature of a $\mathbf{Tel}(p,q)$ 2-connection:

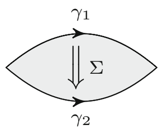
Proposition 33. *Every strict 2-connection on a strict* $\mathbf{Tel}(p,q)$ *2-bundle has vanishing 2-curvature.*

Proof. Let $(A,B) = ((\omega,e), d_\omega e)$ be a 2-connection on a trivial principal $\mathbf{Tel}(p,q)$ 2-bundle. A direct calculation of the 2-curvature 3-form G gives

$$\begin{aligned} G &= d_A(d_\omega e) \\ &= R \wedge d_\omega e + e \wedge d_\omega e. \end{aligned}$$

The first term here is zero, since $R = 0$. The second term is also zero: the wedge product really involves the bracket of Lie algebra valued forms, where e and $d_\omega e$ both live in the abelian subalgebra $\mathbb{R}^{p,q} \subseteq \mathfrak{io}(p,q)$. The condition $G = 0$ is a local condition, so this proves any 2-connection has vanishing 2-curvature, on nontrivial 2-bundles as well. □

This phenomenon can also be seen by noting that the holonomy of a teleparallel 2-connection does not change as we deform the surface. Suppose we have a pair of paths γ_1, γ_2, bounding a surface Σ:



The general recipe for computing surface holonomies simplifies when we have a $\mathbf{Tel}(p,q)$ 2-connection, because ω is flat and the group $\mathbb{R}^{p,q}$ is abelian. Since ω is flat, we can pick a local trivialization of the fake tangent bundle for which $\omega = 0$. Using this trivialization, the 2-form $d_\omega e$ can be interpreted as an $\mathbb{R}^{p,q}$-valued 2-form. To obtain the surface holonomy, we simply integrate this 2-form over Σ. Stokes' theorem implies that this surface holonomy can be rewritten as

$$\int_\Sigma de = \int_{\gamma_2} e - \int_{\gamma_1} e.$$

Geometrically, $\int_\gamma e$ is just the 'translational holonomy' of the Poincaré connection along γ. So, the surface holonomy of a teleparallel 2-connection simply measures the difference between the translational holonomies along the two bounding edges. In particular, the surface holonomy does not change as we apply a smooth homotopy to Σ while keeping its edges γ_1 and γ_2 fixed, so its 2-curvature G must vanish.

For a flat $\mathbf{Poinc}(p,q)$ 2-connection, the surface holonomy can likewise be seen as measuring the difference between translational holonomies along its two bounding edges. An immediate corollary of Theorem 21 is that, under the conditions of that theorem (or, in any case, locally), a flat $\mathbf{Poinc}(p,q)$ 2-connection extends to $\mathbf{Tel}(p,q)$ 2-connection, simply by adjoining a coframe field. This coframe is unique up to a covariantly closed 1-form, and adding a covariantly closed 1-form clearly does not change the 2-holonomy.

4.4. Teleparallel gravity as a $\mathbf{Tel}(p,q)$ *higher gauge theory.* We can now view teleparallel gravity as a theory whose only field is a $\mathbf{Tel}(p,q)$ 2-connection. Summarizing our geometric framework, we start with a manifold M equipped with a fake tangent bundle $\mathcal{T}$ and its corresponding fake frame bundle $\mathcal{F}$. From this, we build the principal $\mathbf{Tel}(p,q)$ 2-bundle

$$\mathbf{Tel}(\mathcal{F}) = \mathcal{F} \times_{\mathrm{O}(p,q)} \mathbf{Tel}(p,q).$$

A 2-connection on this, by Theorem 32, is equivalent to a flat connection ω on $\mathcal{F}$ together with a $\mathcal{T}$-valued 1-form e. When $e\colon TM \to \mathcal{T}$ is an isomorphism, pulling back along e gives a metric on TM as well the Weitzenböck connection ∇, with torsion

$$T(v,w) = e^{-1}(d_\omega e(v,w)).$$

In brief, the 2-connection provides us with everything we need for teleparallel gravity: e is the coframe field, while ω is the 'internal' counterpart of the Weitzenböck connection.

Using this, we can compute the action for a $\mathbf{Tel}(p,q)$ 2-connection $(\omega, e, d_\omega e)$ using the same formula (3) already given:

$$S[\omega, e, d_\omega e] = \int d^n x \ \det(e) \left(\frac{1}{4} T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu \right)$$

This requires a bit of explanation. Previously, we presented this action as a function of just the coframe field e, whereas we now think of it as a function of a $\mathbf{Tel}(p,q)$ 2-connection, which includes both e and the flat connection ω. This does not change the critical points of the action, since flat connections ω are locally gauge equivalent, and the action, considered as a function of both e and ω, is invariant under gauge transformations of $\mathcal{F}$.

In other words, while it is common practice in teleparallel gravity to fix ω once and for all—usually to the standard flat connection on the trival $\mathrm{O}(p,q)$ bundle $\mathcal{F} =$

$M \times \mathrm{O}(p,q)$—here we allow ω to vary. However, it can only vary in a rather innocuous way: it must be *flat*, since it is part of a $\mathbf{Tel}(p,q)$ 2-connection. One simple way to impose this restriction is to write $\omega = h\omega_0 h^{-1} + h dh^{-1}$, where ω_0 is a fixed flat connection, and vary ω indirectly by varying h. As we shall see in Sect. 4.6, Theorem 37, we can in fact parametrize $\mathbf{Tel}(p,q)$ 2-connections by 'weak' $\mathbf{Poinc}(p,q)$ gauge transformations.

An alternative would be to first allow ω to be an arbitrary connection on $\mathcal{F}$ and then arrange for the equations of motion to imply ω is flat, for example by introducing a Lagrange multiplier. This is a bit closer to the approach behind other actions for higher gauge theories such as 4-dimensional BF theory and BFCG theory, where fake flatness holds only 'on shell.' We will not do this here.

4.5. Cartan 2-geometry. We now have a gravity action for a $\mathbf{Tel}(p,q)$ 2-connection, invariant under gauge transformations of $\mathcal{F}$. However, is it invariant under (strict) gauge transformations of the principal $\mathbf{Tel}(p,q)$ 2-bundle $\mathbf{Tel}(\mathcal{F})$? Such a transformation is the same as a gauge transformation of the bundle of objects, the principal $\mathrm{IO}(p,q)$ bundle $\mathbf{I}\mathcal{F}$. Locally, these act on e and ω to give

$$\begin{aligned} \omega &\mapsto h\omega h^{-1} + h\, dh^{-1} =: \omega' \\ e &\mapsto he + d_{\omega'} v \end{aligned}$$

where h and v are functions with values in $\mathrm{O}(p,q)$ and $\mathbb{R}^{p,q}$, respectively. This amounts to an $\mathrm{O}(p,q)$ gauge transformation followed by shifting e by an arbitrary covariantly exact 1-form. The action is not invariant under all such transformations, but only under those for which $v = 0$. These transformations are precisely the gauge transformations of the fake 2-frame 2-bundle $\mathbf{2}\mathcal{F}$. To see this, recall that the teleparallel 2-bundle can be built from the fake 2-frame 2-bundle via

$$\mathbf{Tel}(\mathcal{F}) \cong \mathbf{2}\mathcal{F} \times_{\mathbf{Poinc}(p,q)} \mathbf{Tel}(p,q).$$

Thus, any gauge transformation of $\mathbf{2}\mathcal{F}$ gives one of $\mathbf{Tel}(\mathcal{F})$. These are precisely the transformations with $v = 0$, since the gauge 2-group of $\mathbf{2}\mathcal{F}$ has only Lorentz transformations as objects, not translations.

This strongly parallels the typical situation in gauge-theoretic descriptions of gravity and related theories: the fields fit neatly together into a connection for one group, but gauge invariance is maintained only for a subgroup. A simple example is the **Palatini action** for general relativity in $n = p + q$ dimensions. This action, which depends on a connection ω on a fake tangent bundle, with curvature R, together with a coframe field e, can be written:

$$S_{\mathrm{Pal}}[e,\omega] = \int \star\big(\underbrace{e \wedge \cdots \wedge e}_{n-2} \wedge R\big)$$

The star operator $\star$ on the exterior bundle $\Lambda\mathcal{T}$ turns the $\Lambda^n\mathcal{T}$-valued n-form in parentheses into an ordinary real-valued n-form. Using Proposition 27, we can view the Palatini action as a function of a connection on the *extended* fake frame bundle $\mathbf{I}\mathcal{F} = \mathcal{F} \times_{\mathrm{O}(p,q)} \mathrm{IO}(p,q)$. However, the action is still invariant only under gauge transformations of the subbundle $\mathcal{F}$.

The deep geometric reason for this type of apparently broken gauge symmetry in gravitational gauge theories is the subject of 'Cartan geometry'. Cartan geometry [40] is the study of spaces that look infinitesimally like homogeneous spaces. It is thus a 'differential' extension of Klein's Erlangen program [26] for understanding geometry using homogeneous spaces. The Erlangen program already uses a kind of 'broken symmetry' to describe geometry.

In Klein's theory two groups play vital roles: a Lie group G of symmetries of a homogeneous space X, and a closed subgroup $G' \subseteq G$, the stabilizer of an arbitrarily specified point, which allows us to identify X with G/G'. So, a **Klein geometry** is technically a pair (G, G') consisting of a Lie group and a closed subgroup, but we think of these as a tool for studying the geometry of the homogeneous space G/G'.

A Cartan geometry is then a space that is infinitesimally 'modeled on' G/G'. We will not need a precise definition of Cartan geometry here; what is important for our purposes is that it involves a principal G' bundle P together with a connection on the associated principal G bundle $P \times_{G'} G$. Moreover, gauge transformations of P give isomorphic Cartan geometries, while more general gauge transformations of $P \times_{G'} G$ can severely deform the geometry.

In short: in a physical theory based on Cartan geometry, we expect to see a G connection for some Lie group G, but gauge invariance only under some closed subgroup G'. In the Palatini example we have $G = \mathrm{IO}(p, q)$ and $G' = \mathrm{O}(p, q)$, but this general pattern is ubiquitous in attempts to describe gravity as a gauge theory by combining the connection and coframe field into a larger connection, such as MacDowell–Mansouri gravity [42] and related theories [41].

One might question whether such symmetry-broken gravitational theories are 'true' gauge theories. We have no intention of trying to settle this question. Rather, we take the view that these theories are useful and geometrically interesting regardless of the answer. Teleparallel gravity has a similarly 'broken' gauge symmetry, but with an interesting difference. Here we have a 2-connection for some Lie 2-group, but symmetry only under a sub-2-group of this. This suggests that teleparallel gravity is based on a form of geometry analogous to Cartan geometry, *but with groups replaced by 2-groups*.

The present paper is not the place for extensive study of 'Cartan 2-geometry'. However, we would like to consider what a straightforward reading of this analogy seems to imply for our teleparallel gravity action. As we have seen, in teleparallel gravity the $\mathbf{Poinc}(p, q)$ 2-connection can be combined with the coframe field e to give a $\mathbf{Tel}(p, q)$ 2-connection $((\omega, e), d_\omega e)$, but the theory remains invariant only under $\mathbf{Poinc}(p, q)$ gauge transformations. This suggests interpreting the theory in terms 'Cartan 2-geometry' modeled on a 'homogeneous 2-space' given as the quotient $\mathbf{Tel}(p, q)/\mathbf{Poinc}(p, q)$. But what does this 2-space look like?

First, having done everything 'strictly', it is easy to define a *strict* quotient of strict 2-groups:

Definition 34. Let $\mathbf{G}$, be strict Lie 2-group with strict Lie sub-2-group $\mathbf{G}'$ (i.e. the groups of objects and morphisms of $\mathbf{G}'$ are Lie subgroups of those of $\mathbf{G}$, and the maps are all restrictions of the corresponding maps in the definition of $\mathbf{G}$). The **strict quotient** $\mathbf{G}/\mathbf{G}'$ is the Lie groupoid with

- $\mathbf{G}_0/\mathbf{G}'_0$ as objects
- $\mathbf{G}_1/\mathbf{G}'_1$ as morphisms
- source, target, composition and identity-assigning maps induced from those in $\mathbf{G}$.

It is straightforward to check that the maps in $\mathbf{G}$ induce the corresponding maps in a well-defined way on the quotient, and that the result is indeed a Lie groupoid.

There is a natural left action of $\mathbf{G}$ on the Lie groupoid $\mathbf{G}/\mathbf{G}'$, induced by the left action of $\mathbf{G}$ on its underlying Lie groupoid (see Example 11). By analogy with the 1-group case, we may also refer to the strict quotient $\mathbf{G}/\mathbf{G}'$ as a 'homogeneous 2-space' for the 2-group $\mathbf{G}$. More generally, a quick way to define a **homogeneous G 2-space** is to say it is any strict $\mathbf{G}$ 2-space isomorphic to one of the form $\mathbf{G}/\mathbf{G}'$, though it is not hard to give a more intrinsic definition. Continuing with the analogy to the ordinary case, we may also refer to the pair $(\mathbf{G}, \mathbf{G}')$ as a (**strict**) **Klein 2-geometry**.

Proposition 35. $\mathbf{Tel}(p, q)/\mathbf{Poinc}(p, q)$ *is isomorphic as a* $\mathbf{Tel}(p, q)$ *2-space to the space* $\mathbb{R}^{p,q}$.

Proof. Taking the strict quotient $\mathbf{Tel}(p, q)/\mathbf{Poinc}(p, q)$, we obtain a Lie groupoid with:

- $\mathrm{IO}(p, q)/\mathrm{O}(p, q) \cong \mathbb{R}^{p,q}$ as objects,
- $(\mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q})/\mathrm{IO}(p, q) \cong \mathbb{R}^{p,q}$ as morphisms,
- source and target maps $\mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ both the identity.

This is nothing but the space $\mathbb{R}^{p,q}$ thought of as a Lie groupoid with only identity morphisms. $\mathbf{Tel}(p, q)$ acts via its group of objects $\mathrm{IO}(p, q)$, via the usual action of $\mathrm{IO}(p, q)$ on $\mathbb{R}^{p,q}$. □

4.6. Weakening. In this paper, we have dealt entirely with *strict* constructions. This approach eases the transition to higher gauge theory, since it is built directly on constructions familiar from ordinary gauge theory. For example, as we have seen:

- a strict 2-group has an ordinary group of objects;
- a strict principal 2-bundle has an ordinary principal bundle of objects;
- a strict 2-connection consists of an ordinary connection together with a 2-form with values in an associated vector bundle;
- strict gauge transformations are induced by ordinary gauge transformations.

However, even when one tries to keep everything 'strict', as we have done here, 'weak' ideas can sneak in unexpectedly. An example is in the higher gauge theory interpretation of four-dimensional **BF theory**. As mentioned in the introduction, this is a gauge theory for some Lie group G, with action given by

$$S(A, B) = \int \mathrm{tr}\big(B \wedge F\big).$$

Here, A is a connection on a principal G bundle P, F it its curvature, B is a $(P \times_G \mathfrak{g})$-valued 2-form, and G is assumed semisimple, so that the Killing form 'tr' is nondegenerate. This theory can be viewed as a higher gauge theory for the **tangent 2-group** $\mathbf{T}G$, whose crossed module is $(G, \mathfrak{g}, \mathrm{Ad}, 0)$: the fields (A, B) are the appropriate sort of ingredients for a 2-connection on the strict 2-bundle $\mathbf{P} \times_G \mathbf{T}G$, and the field equations imply the fake flatness condition.

However, while BF theory viewed as a higher gauge theory can be built entirely on the *strict* constructions defined in this paper, it has an extra symmetry that does not come from strict gauge transformations of the principal $\mathbf{T}G$ 2-bundle. If we shift B by a covariantly exact 2-form:

$$B \mapsto B + d_A a$$

the action changes only by a boundary term, thanks to the Bianchi identity $d_A F = 0$. Remarkably, this additional symmetry implies BF theory is invariant under *weak* gauge transformations.

Given this lesson from BF theory, it is interesting to ask how teleparallel gravity behaves under weak gauge transformations when we regard it as a $\mathbf{Tel}(p, q)$ higher gauge theory, even though we have done everything 'strictly' so far. For this, we need to know a bit about how weak gauge transformations act on 2-connections.

General (weak) gauge symmetries of principal 2-bundles have been described by Bartels [13]. For our purposes, it suffices to recall, in the case where our principal 2-bundle is equipped with a 2-connection, how the 2-connection data transform under gauge transformations [10,16]. If $\mathbf{G}$ is a 2-group with crossed module (G, H, t, α), then gauge transformations on a trivial principal $\mathbf{G}$ 2-bundle act on a 2-connection (A, B) to give a new 2-connection (A', B') with:

$$\begin{aligned} A' &= gAg^{-1} + g\,dg^{-1} + \underline{t}(a) \\ B' &= \alpha'(g)(B) + d_{A'}a + a \wedge a \end{aligned} \tag{5}$$

where g is a G-valued function and a is an $\mathfrak{h}$-valued 1-form. In the second equation, the covariant differential is defined by $d_{A'}a = da + \underline{\alpha}(A') \wedge a$. Strict gauge transformations correspond to the case where $a = 0$; the fully general ones are called **weak**.

Theorem 36. *If* $\pi_1(M) = 1$*, then all 2-connections on the trivial* $\mathbf{Tel}(p, q)$ *2-bundle over* M *are equivalent under weak* $\mathbf{Tel}(p, q)$ *gauge transformations.*

Proof. On a trivial $\mathbf{Tel}(p, q)$ 2-bundle, a 2-connection $(A, B) = ((\omega, e), d_\omega e)$ consists of an $\mathfrak{o}(p, q)$-valued 1-form ω, an $\mathbb{R}^{p,q}$-valued 1-form e, and the 2-form $B = d_\omega e$. Specializing the formula (5) for weak gauge transformations, we find that these data transform to give a new 2-connection $(A', B') = ((\omega', e'), d_{\omega'} e')$ with:

$$\begin{aligned} \omega' &= h\omega h^{-1} + h\,dh^{-1} \\ e' &= he + d_{\omega'}v + a \\ B' &= d_{\omega'}e' \end{aligned} \tag{6}$$

where $h\colon M \to \mathrm{O}(p, q)$, $v\colon M \to \mathbb{R}^{p,q}$ are smooth maps, and $a\colon TM \to \mathbb{R}^{p,q}$ is an $\mathbb{R}^{p,q}$-valued 1-form on M. In obtaining these equations we have used that $a \wedge a$ vanishes in this case: it is defined using the Lie bracket, which vanishes on $\mathbb{R}^{p,q} \subseteq \mathfrak{io}(p, q)$.

Given a pair of 2-connections (A, B) and (A', B') on the trivial $\mathbf{Tel}(p, q)$ 2-bundle over M, we wish to solve the above equations for h, v, and a. It is worth noting first that v and a do not act in independent ways: we may clearly absorb $d_{\omega'}v$ into the definition of a, and hence without loss of generality assume $v = 0$.

Now ω and ω' are *flat* connections on the trivial $\mathrm{O}(p, q)$ bundle over M. But the moduli space of flat connections on any fixed $\mathrm{O}(p, q)$ bundle is contained in $\hom(\pi_1(M), \mathrm{O}(p, q))$, so $\pi_1(M) = 1$ implies ω and ω' must be related by a gauge transformation $h\colon M \to \mathrm{O}(p, q)$; that is, $h\omega h^{-1} + hdh^{-1} = \omega'$. This same h changes e to he. But, choosing $a = e' - he$, gives a gauge transformation mapping $\omega \mapsto \omega'$ and $e \mapsto e'$. An automatic consequence is that $B \mapsto B'$. □

This theorem actually gives a 'physics proof' that teleparallel gravity, viewed as a $\mathbf{Tel}(p, q)$ higher gauge theory, cannot be invariant under arbitrary weak gauge transformations. If it were, the theorem would imply the action was locally *independent* of the fields. This cannot be true since teleparallel gravity has local degrees of freedom:

indeed, it is locally equivalent to general relativity. Of course, one can also check more directly that the action is not invariant under the weak gauge transformations given by Eq. (6).

We found before that teleparallel gravity is invariant under strict **Poinc**(p, q) gauge transformations. The obvious question now is whether it is also invariant under *weak* **Poinc**(p, q) gauge transformations. The answer to this question is already implicit in the proof of Theorem 36, which really shows a bit more than what the theorem states:

Theorem 37. *If* $\pi_1(M) = 1$*, then all 2-connections on the trivial* **Tel**(p, q) *2-bundle over* M *are equivalent under weak* **Poinc**(p, q) *gauge transformations.*

Proof. In the proof of Theorem 36, we noted that in the transformations (6), we could take $v = 0$ without loss of generality, and hence all **Tel**(p, q) 2-connections are related by transformations of the form

$$\begin{aligned} \omega' &= h\omega h^{-1} + h\, dh^{-1} \\ e' &= he + a \\ B' &= d_{\omega'} e' \end{aligned}$$

These are just gauge transformations coming from the Poincaré 2-group **Poinc**(p, q). □

In terms of teleparallel gravity, an immediate consequence of this result is that we can parametrize **Tel**(p, q) 2-connections by picking a fiducial 2-connection $(\omega_0, e_0, d_{\omega_o} e_0)$ and considering all weak **Poinc**(p, q) gauge transformations of it. This lets us write the action, a function of **Tel**(p, q) 2-connections, instead as a function of **Poinc**(p, q) gauge transformations:

$$S[h, a] = S[\omega(h, a), e(h, a), d_\omega e]$$

where $\omega = h\omega_0 h^{-1} + h\, dh^{-1}$ and $e = he_0 + a$.

Summarizing our observations so far, consider this sequence of 2-groups:

$$\mathrm{O}(p, q) \longrightarrow \mathbf{Poinc}(p, q) \longrightarrow \mathbf{Tel}(p, q)$$

The 2-connection of teleparallel gravity is a **Tel**(p, q) 2-connection. Considering strict gauge transformations, the action is invariant only under **Poinc**(p, q) transformations, and this led us to the suggestion that teleparallel gravity should be about 'Cartan 2-geometry' based on the homogeneous 2-space **Tel**(p, q)/**Poinc**$(p, q) \cong \mathbb{R}^{p,q}$. On the other hand, considering weak gauge transformations, the action is invariant only under $\mathrm{O}(p, q)$ transformations. This suggests 'Cartan 2-geometry' based instead on the homogeneous 2-space **Tel**$(p, q)/\mathrm{O}(p, q)$. We shall now see that this 2-space is the same as the fundamental groupoid of $\mathbb{R}^{p,q}$.

Definition 38. Let X be a manifold. The **fundamental groupoid** of X, denoted $\Pi_1(X)$, is the Lie groupoid whose objects are points of X and whose morphisms are homotopy classes of paths in X.

For $\mathbb{R}^{p,q}$, any two points are connected by a unique homotopy class of paths, so $\Pi_1(\mathbb{R}^{p,q})$ is particularly simple. It is clearly isomorphic to the Lie groupoid for which:

- $\mathbb{R}^{p,q}$ is the group of objects,
- $\mathbb{R}^{p,q} \times \mathbb{R}^{p,q}$ is the group of morphisms,

- the source of the morphism (v, w) is v,
- the target of the morphism (v, w) is $v + w$,
- the composite $(v, w) \circ (v', w')$, when defined, is $(v', w + w')$.

Identifying $\Pi_1(\mathbb{R}^{p,q})$ with this Lie groupoid, we can turn $\Pi_1(\mathbb{R}^{p,q})$ into a $\mathbf{Tel}(p, q)$ 2-space, defining an action

$$\mathbf{Tel}(p, q) \times \Pi_1(\mathbb{R}^{p,q}) \to \Pi_1(\mathbb{R}^{p,q})$$

given on objects by

$$\begin{aligned} \mathrm{IO}(p, q) \times \mathbb{R}^{p,q} &\to \mathbb{R}^{p,q} \\ ((h, v), w) &\mapsto hw + v \end{aligned}$$

and given on morphisms as follows:

$$\begin{aligned} (\mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q}) \times (\mathbb{R}^{p,q} \times \mathbb{R}^{p,q}) &\to \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \\ \big(((h, v), w), (v', w')\big) &\mapsto (hv' + v, hw' + w). \end{aligned}$$

We then have:

Theorem 39. *As* $\mathbf{Tel}(p, q)$ *2-spaces,* $\mathbf{Tel}(p, q)/\mathrm{O}(p, q) \cong \Pi_1(\mathbb{R}^{p,q})$.

Proof. To form the strict quotient, we simply take the quotient of groups at both the object and the morphism level, obtaining a Lie groupoid with:

- $\mathrm{IO}(p, q)/\mathrm{O}(p, q) \cong \mathbb{R}^{p,q}$ as objects, and
- $(\mathrm{IO}(p, q) \ltimes \mathbb{R}^{p,q})/\mathrm{O}(p, q) \cong \mathbb{R}^{p,q} \times \mathbb{R}^{p,q}$ as morphisms.

Comparing to the description of $\mathbf{Tel}(p, q)$ in Definition 24, it is clear that the above description of $\Pi_1(\mathbb{R}^{p,q})$ results from ignoring the $\mathrm{O}(p, q)$ parts at both object and morphism levels, and that the action of $\mathbf{Tel}(p, q)$ on $\Pi_1(\mathbb{R}^{p,q})$ just comes from the left action of $\mathbf{Tel}(p, q)$ on itself. □

5. Outlook

We have seen that the geometry of teleparallel gravity is closely related to the Poincaré 2-group, and to the teleparallel 2-group. Previously, the Poincaré 2-group $\mathbf{Poinc}(p, q)$ seemed mathematically natural but without much physical justification: why should we treat Lorentz transformations as *objects* but translations as *morphisms*? Our answer is that if we do this, the Weitzenböck connection and its torsion fit together into a flat $\mathbf{Poinc}(p, q)$ 2-connection. Moreover, by extending $\mathbf{Poinc}(p, q)$ to the teleparallel 2-group $\mathbf{Tel}(p, q)$, we can also include the coframe field as part of the 2-connection, allowing us to write an action for teleparallel gravity as a function of just a $\mathbf{Tel}(p, q)$ 2-connection.

On the other hand, we have seen that this action is invariant only under gauge transformations living in a sub-2-group: $\mathbf{Poinc}(p, q)$ if we consider only strict gauge transformations, or $\mathrm{O}(p, q)$ if we consider weak ones. We have discussed how this is analogous to many other approaches to gravity [41,42] where Cartan geometry [40] provides the geometric foundation, and suggested that 'Cartan 2-geometry' may play an analogous role in teleparallel gravity and other higher gauge theories.

Indeed, we expect 'Cartan 2-geometry' should be an interesting subject in its own right, with much broader applications than those suggested here. In Cartan geometry, to each homogeneous space G/G', there corresponds a type of geometry that can be put on a more general manifold. Similarly, in Cartan 2-geometry each homogeneous 2-space $\mathbf{G}/\mathbf{G}'$ should give a type of geometry—or rather, a type of '2-geometry'—that can be put on a more general Lie groupoid.

Our work suggests several things to be done toward developing a general theory of Cartan 2-geometry. We would like to touch on two key points for this effort.

First, while we have focussed on 2-bundles for which the 'base 2-space' is actually just a manifold, Cartan 2-geometry should in general involve 2-connections on bundles over interesting Lie groupoids, so many of the ideas we have discussed here deserve to be generalized to that case.

Second, despite our pragmatic use of *strict* constructions throughout most of this paper, we ultimately expect the weak analogs of concepts we have described to play a more fundamental role; weak constructions in category theory are the most natural, and often the most interesting. In the teleparallel gravity case, we have seen that we naturally get an interesting Lie groupoid as our 'model 2-space' only if we consider weak gauge transformations. But 'weakening' one aspect of a theory tends to suggest, if not demand, weakening other aspects. For example, once we allow weak gauge transformations, we can use them to assemble 'weak 2-bundles' where the transition functions on overlaps satisfy the usual equations only up to an isomorphism. In fact, even if we try building strict 2-bundles initially, weak gauge transformations do not preserve strictness, so we're essentially forced to use weak bundles. Once we have weak bundles, strict 2-connections no longer make sense. And so on.

The theory of 2-bundles [13] has been developed in considerable generality, including weak principal bundles for *weak* 2-groups, as well as the possibility of very general 'base 2-spaces'—that is, base spaces that are Lie groupoids rather than mere manifolds. On the other hand, the theory of 2-connections, while understood in a weak context [8, 10] is so far best understood in the case where the base space is just a manifold. Thus, the theory of 2-connections on 2-bundles over 2-spaces deserves further study. Understanding 'Cartan 2-connections' may be aided by studying the concrete examples presented in this paper.

In the meantime, the heuristic picture is perhaps clear enough to venture a guess on some details of 'Cartan 2-geometry', in the case arising from teleparallel gravity, where the model 'Klein 2-geometry' is

$$\mathbf{Tel}(p,q)/\mathrm{O}(p,q) \cong \Pi_1(\mathbb{R}^{p,q}).$$

For this, we would like a 2-connection on a principal $\mathbf{Tel}(p,q)$ 2-bundle that reduces to a principal $\mathrm{O}(p,q)$ 2-bundle. It also seems reasonable to take $\Pi_1(M)$ as the base 2-space, where M is a $(p+q)$-dimensional manifold. In fact, there is an easy way to get an $\mathrm{O}(p,q)$ 2-bundle over this Lie groupoid, and extend it to a $\mathbf{Tel}(p,q)$ 2-bundle. Start with a fake frame bundle $\mathcal{F} \to M$, equipped with a flat connection ω. Form the 2-space $\Pi_1^{\mathrm{hor}}(\mathcal{F},\omega)$ whose objects are points in $\mathcal{F}$ and whose morphisms are homotopy classes of *horizontal* paths in $\mathcal{F}$. There is an obvious projection

$$\Pi_1^{\mathrm{hor}}(\mathcal{F},\omega) \to \Pi_1(M),$$

and it is easy to see that the object and morphism maps are both principal $\mathrm{O}(p,q)$ bundles; in fact, this is a principal $\mathrm{O}(p,q)$ 2-bundle. Generalizing our associated 2-bundle construction to allow for a base 2-space, the extension $\mathrm{O}(p,q) \to \mathbf{Tel}(p,q)$ gives a $\mathbf{Tel}(p,q)$ 2-bundle. Carrying on with the analogy to ordinary Cartan geometry, we expect our Cartan 2-geometry to involve a 2-connection on this 2-bundle, subject to certain 'nondegeneracy' conditions. However, we leave the details for further work.

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