

# Tropical Coamoeba and Torus-Equivariant Homological Mirror Symmetry for the Projective Space

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**Abstract:** We introduce the notion of a tropical coamoeba which gives a combinatorial description of the Fukaya category of the mirror of a toric Fano stack. We show that the polyhedral decomposition of a real  $n$ -torus into  $n + 1$  permutohedra gives a tropical coamoeba for the mirror of the projective space  $\mathbb{P}^n$ , and we prove a torus-equivariant version of homological mirror symmetry for the projective space. As a corollary, we obtain homological mirror symmetry for toric orbifolds of the projective space.

## 1. Introduction

Let  $n$  be a natural number and  $\Delta$  be a convex lattice polytope in  $\mathbb{R}^n$ , i.e., the convex hull of a finite subset of  $\mathbb{Z}^n$ . We assume that the origin is in the interior of  $\Delta$ . One side of homological mirror symmetry for toric Fano stacks, conjectured by Kontsevich [Kon98], states that there is an equivalence

$$D^b \text{coh } X \cong D^b \text{Fuk } W \tag{1.1}$$

of two triangulated categories of geometric origins associated with  $\Delta$ . For the other side of homological mirror symmetry for toric manifolds, we refer the readers to the survey paper [FOOO12] and references therein.

The category on the left hand side is the derived category of coherent sheaves on the toric Fano stack  $X$ , defined as follows: Let  $\{v_i\}_{i=1}^r$  be the set of vertices of  $\Delta$  and take a simplicial stacky fan  $\Sigma$  such that the set of generators of one-dimensional cones is given by  $\{v_i\}_{i=1}^r$ . The associated toric stack is the quotient stack

$$X = [(\mathbb{C}^r \setminus \text{SR}(\Sigma))/K],$$

where the Stanley–Reisner locus  $\text{SR}(\Sigma)$  consists of points  $(z_1, \dots, z_r)$  such that there is no cone in  $\Sigma$  which contains all  $v_i$  for which  $z_i = 0$ , and

$$K = \text{Ker}(\phi \otimes \mathbb{C}^\times)$$

is the kernel of the tensor product with  $\mathbb{C}^\times$  of the map  $\phi : \mathbb{Z}^r \rightarrow \mathbb{Z}^n$  sending the  $i$ -th coordinate vector to  $v_i$  for  $i = 1, \dots, r$ . Although  $X$  depends not only on  $\Delta$  but also on  $\Sigma$ , the derived category  $D^b \text{coh } X$  is independent of this choice [Kaw05, Theorem 4.2] and depends only on  $\Delta$ .

On the right hand side, one takes a sufficiently general Laurent polynomial

$$W = \sum_{\omega \in \Delta \cap \mathbb{Z}^n} a_\omega x^\omega$$

whose Newton polytope coincides with  $\Delta$  as in [Giv95]. This defines an exact Lefschetz fibration

$$W : (\mathbb{C}^\times)^n \rightarrow \mathbb{C}$$

with respect to the standard cylindrical Kähler structure on  $(\mathbb{C}^\times)^n$ , and  $\mathfrak{Fuk} W$  is the directed Fukaya category defined by Seidel [Sei01, Sei08] whose set of objects is a distinguished basis of vanishing cycles and whose spaces of morphisms are Lagrangian intersection Floer complexes.

The equivalence (1.1) is proved for  $\mathbb{P}^2$  and  $\mathbb{P}^1 \times \mathbb{P}^1$  by Seidel [Sei01], weighted projective planes and Hirzebruch surfaces by Auroux et al. [AKO08], toric del Pezzo surfaces by Ueda [Ued06], and toric orbifolds of toric del Pezzo surfaces by Ueda and Yamazaki [UY13]. See also Auroux, Katzarkov and Orlov [AKO06] for homological mirror symmetry for not necessarily toric del Pezzo surfaces, Abouzaid [Abo06, Abo09] for an application of tropical geometry to homological mirror symmetry, Kerr [Ker08] for the behavior of homological mirror symmetry under weighted blowup of toric surfaces. Slightly different versions of homological mirror symmetry for toric stacks are proved by Fang et al. [Fan08, FLTZ11, FLTZ14] and Futaki and Ueda [FU10].

In this paper, we pass to the universal cover

$$\exp : \mathbb{C}^n \rightarrow (\mathbb{C}^\times)^n$$

of the torus and replace the Lefschetz fibration  $W$  with its pull-back

$$\tilde{W} = W \circ \exp : \mathbb{C}^n \rightarrow \mathbb{C}.$$

The fact that  $\tilde{W}$  has countably many critical points does not cause any problem, and one can formulate a torus-equivariant version of homological mirror symmetry for toric Fano stacks:

**Conjecture 1.1** *For a convex lattice polytope  $\Delta$  containing the origin in its interior, there is an equivalence*

$$D^b \text{coh}^{\mathbb{T}} X \cong D^b \mathfrak{Fuk} \tilde{W}$$

*of triangulated categories.*

Here  $\mathbb{T}$  is the  $n$ -dimensional torus acting on  $X$  and  $D^b \text{coh}^{\mathbb{T}} X$  is the derived category of  $\mathbb{T}$ -equivariant coherent sheaves on  $X$ . Our first main result is the proof of Conjecture 1.1 for the projective space:

**Theorem 1.2** *Conjecture 1.1 holds when  $X$  is the projective space.*

The case  $n = 3$  in Theorem 1.2 is a corollary of a result of Seidel [Sei11, Proposition 11.7] which describes the undirected Fukaya category of the fiber of  $W$ . Theorem 1.2 is an important step in the proof of homological mirror symmetry for the quintic threefold in [NU12]. Torus-equivariant homological mirror symmetry for  $X$  implies the ordinary homological mirror symmetry, not only for  $X$  but also for the quotient stack  $[X/A]$  for any finite subgroup  $A$  of the torus  $\mathbb{T}$  acting on  $X$ .

**Corollary 1.3** *For a convex lattice polytope  $\Delta$  which can be obtained from the polytope for  $\mathbb{P}^n$  by an integral linear transformation, one has an equivalence*

$$D^b \text{coh } X \cong D^b \mathfrak{F}\text{u}\mathfrak{k} W$$

*of triangulated categories.*

We introduce the notion of a *tropical coamoeba* of  $W$ , which consists of a decomposition

$$T = \bigcup_{i=1}^m P_i$$

of a real  $n$ -torus  $T = \mathbb{R}^n / \mathbb{Z}^n$  into the union of an ordered set of polytopes, together with a map

$$\text{deg} : F_1 \rightarrow \mathbb{Z}$$

from the set  $F_1$  of facets of  $P_i$  to  $\mathbb{Z}$  called the *degree*, and a map

$$\text{sgn} : F_2 \rightarrow \{1, -1\}$$

from the set  $F_2$  of codimension two faces of  $P_i$  called the *sign*, satisfying conditions in Definition 7.1. One can associate a directed  $A_\infty$ -category with a tropical coamoeba, and the conditions in Definition 7.1 ensure that this  $A_\infty$ -category is equivalent to  $\mathfrak{F}\text{u}\mathfrak{k} \tilde{W}$ . This enables us to divide Conjecture 1.1 into two steps:

**Conjecture 1.4** *Let  $\Delta$  be a convex lattice polytope in  $\mathbb{R}^n$  containing the origin in its interior. Then the following hold:*

- *There is a Laurent polynomial  $W : (\mathbb{C}^\times)^n \rightarrow \mathbb{C}$  such that*
  - *the Newton polytope of  $W$  coincides with  $\Delta$ , and*
  - *there exists a tropical coamoeba  $G$  of  $W$ .*

*This implies that the  $A_\infty$ -category  $\mathcal{A}_{\tilde{G}}$  associated with  $G$  is quasi-equivalent to  $\mathfrak{F}\text{u}\mathfrak{k} \tilde{W}$ ;*

$$\mathcal{A}_{\tilde{G}} \cong \mathfrak{F}\text{u}\mathfrak{k} \tilde{W}.$$

- *The derived category of the  $A_\infty$ -category  $\mathcal{A}_{\tilde{G}}$  is equivalent to the derived category of  $\mathbb{T}$ -equivariant coherent sheaves on the toric Fano stack  $X$  associated with  $\Delta$ ;*

$$D^b \mathcal{A}_{\tilde{G}} \cong D^b \text{coh}^{\mathbb{T}} X.$$

Our second main result is the proof of Conjecture 1.4 for the projective space:

**Theorem 1.5** *Conjecture 1.4 holds when  $X$  is the projective space. The tropical coamoeba in this case comes from a decomposition of a real  $n$ -torus into the union of  $n + 1$  permutohedra of order  $n + 1$ .*

A tropical coamoeba is a generalization of a dimer model to higher dimensions. The importance of dimer models in mirror symmetry is pointed out by Feng et al. [FHKV08] and elaborated in [UY11, UY, UY13]. The works of Bondal and Ruan [Bon06] and Fang et al. [Fan08, FLTZ11, FLTZ14] use constructible sheaves on a real torus and its universal cover to study equivariant homological mirror symmetry for toric stacks, and it is an interesting problem to explore the relationship between their approach and ours.

For any convex lattice polytope  $\Delta$  in  $\mathbb{R}^n$  and a vertex  $v$  of  $\Delta$ , one can obtain another polytope  $\Delta'$  by removing  $v$  from  $\Delta \cap \mathbb{Z}^n$  and taking the convex hull of the rest. On the complex side, this operation gives a birational map  $X_\Delta \dashrightarrow X_{\Delta'}$  between the corresponding toric stacks, which in turn gives a full and faithful functor

$$\Phi : D^b \text{coh } X_{\Delta'} \hookrightarrow D^b \text{coh } X_\Delta$$

by a result of Kawamata [Kaw05, Theorem 4.2]. On the symplectic side, one can choose a one-parameter family  $W_t$  of Laurent polynomials such that the Newton polytope of  $W_0$  is  $\Delta'$  and that of  $W_t$  for  $t \neq 0$  is  $\Delta$ , so that a result of Kerr [Ker08, Theorem 6] gives a full and faithful functor

$$\Psi : D^b \text{fũ} W_{\Delta'} \hookrightarrow D^b \text{fũ} W_\Delta.$$

It is clear that any lattice polytope can be embedded into a sufficiently large simplex, so that any lattice polytope can be obtained from a sufficiently large simplex by successively performing this operation. As a corollary, one obtains the following:

**Corollary 1.6** *For any lattice polytope  $\Delta$ , there exist a Laurent polynomial  $W'$  and a toric stack  $X'$  such that one has full and faithful functors  $F : D^b \text{coh } X \hookrightarrow D^b \text{fũ} W'$  and  $G : D^b \text{fũ} W \hookrightarrow D^b \text{coh } X'$ , where  $X$  is a toric stack associated with  $\Delta$  and  $W$  is a general Laurent polynomial whose Newton polytope is  $\Delta$ .*

The lattice polytope associated with  $X'$  and the Newton polytope of  $W'$  in Corollary 1.6 are sufficiently large simplexes containing  $\Delta$ . As the equivalence in Corollary 1.3 is given explicitly, one can in principle reduce homological mirror symmetry for a general toric stack to the problems of

- the behavior of the derived category of toric stacks under birational transformations, and
- the behavior of critical values of Laurent polynomials under deformations,

without any further Floer-theoretic computations on vanishing cycles. This is a special case of the relation between homological mirror symmetry and the minimal model program discussed in [BFK, DKK].

The organization of this paper is as follows: We collect basic definitions on Fukaya categories in Sect. 2. Symplectic Picard–Lefschetz theory and homological mirror symmetry for  $\mathbb{P}^2$  by Seidel are recalled in Sects. 3 and 4, respectively, which are used in Sect. 5 to prove homological mirror symmetry for  $\mathbb{P}^3$ . The Fukaya category of the mirror of  $\mathbb{P}^n$  for general  $n$  is computed in Sect. 6 by an induction on  $n$ . In Sect. 7, we define a tropical coamoeba as a combinatorial object which encode the information of the Fukaya category, and show that it allows one to summarize the result in Sect. 6 in a nice way.

## 2. Fukaya Categories

For a  $\mathbb{Z}$ -graded vector space  $N = \bigoplus_{j \in \mathbb{Z}} N^j$  and an integer  $i$ , the  $i$ -th shift of  $N$  to the left will be denoted by  $N[i]$ ;  $(N[i])^j = N^{i+j}$ .

**Definition 2.1** An  $A_\infty$ -category  $\mathcal{A}$  consists of

- the set  $\mathfrak{Ob}(\mathcal{A})$  of objects,
- for  $c_1, c_2 \in \mathfrak{Ob}(\mathcal{A})$ , a  $\mathbb{Z}$ -graded vector space  $\text{hom}_{\mathcal{A}}(c_1, c_2)$  called the space of morphisms, and
- operations

$$m_l : \text{hom}_{\mathcal{A}}(c_{l-1}, c_l) \otimes \cdots \otimes \text{hom}_{\mathcal{A}}(c_0, c_1) \longrightarrow \text{hom}_{\mathcal{A}}(c_0, c_l)$$

of degree  $2 - l$  for  $l = 1, 2, \dots$  and  $c_i \in \mathfrak{Ob}(\mathcal{A})$ ,  $i = 0, \dots, l$ , satisfying the  $A_\infty$ -relations

$$\sum_{i=0}^{l-1} \sum_{j=i+1}^l (-1)^{\deg a_1 + \cdots + \deg a_i - i} m_{l+i-j+1}(a_l \otimes \cdots \otimes a_{j+1} \otimes m_{j-i}(a_j \otimes \cdots \otimes a_{i+1}) \otimes a_i \otimes \cdots \otimes a_1) = 0, \quad (2.1)$$

for any positive integer  $l$ , any sequence  $c_0, \dots, c_l$  of objects of  $\mathcal{A}$ , and any sequence of morphisms  $a_m \in \text{hom}_{\mathcal{A}}(c_{m-1}, c_m)$  for  $m = 1, \dots, l$ .

The  $A_\infty$ -relations (2.1) for  $l = 1, 2$ , and  $3$  show that  $m_1$  squares to zero and  $m_2$  defines an associative operation on the cohomology of  $m_1$ . The resulting non-unital category is called the *cohomological category* of  $\mathcal{A}$ . An  $A_\infty$ -category satisfying  $m_k = 0$  for  $k \geq 3$  corresponds to a *differential graded category* (i.e. a category whose spaces of morphisms are complexes such that the differential  $d$  satisfies the Leibniz rule with respect to the composition) by

$$d(a) = (-1)^{\deg a} m_1(a), \quad a_2 \circ a_1 = (-1)^{\deg a_1} m_2(a_2, a_1).$$

The derived category of an  $A_\infty$ -category is defined using *twisted complexes*, which are introduced by Bondal and Kapranov [BK90] for differential graded categories and generalized to  $A_\infty$ -categories by Kontsevich [Kon95]. Here we follow the exposition of Seidel [Sei08] closely. For an  $A_\infty$ -category  $\mathcal{A}$ , its *additive enlargement*  $\Sigma\mathcal{A}$  is the  $A_\infty$ -category whose set of object consists of formal direct sums

$$X = \bigoplus_{i \in I} V^i \otimes X^i$$

where  $I$  is a finite set,  $\{X^i\}_{i \in I}$  is a family of objects of  $\mathcal{A}$ , and  $\{V^i\}_{i \in I}$  is a family of graded vector spaces. The space of morphisms is given by

$$\text{hom}_{\Sigma\mathcal{A}} \left( \bigoplus_{i \in I_0} V_0^i \otimes X_0^i, \bigoplus_{i \in I_1} V_1^i \otimes X_1^i \right) = \bigoplus_{i,j} \text{hom}_{\mathbb{C}}(V_0^i, V_1^j) \otimes \text{hom}_{\mathcal{A}}(X_0^i, X_1^j)$$

and the  $A_\infty$ -operations are

$$m_d^{\Sigma\mathcal{A}}(a_d, \dots, a_1)^{i_d, i_0} = \sum_{i_1, \dots, i_d} (-1)^\dagger \phi_d^{i_d, i_{d-1}} \circ \dots \circ \phi_1^{i_1, i_0} \otimes \mu_d^{\mathcal{A}}(x_d^{i_d, i_{d-1}}, \dots, x_1^{i_1, i_0}),$$

where  $\dagger = \sum_{p < q} \deg \phi_p^{i_p, i_{p-1}} \cdot (\deg x_q^{i_q, i_{q-1}} - 1)$  and  $a_k = (a_k^{j_i}) = (\phi_k^{j_i} \otimes x_k^{j_i})$ .

A twisted complex is a pair

$$\left( X = \bigoplus_{i \in I} V^i \otimes X^i, \delta_X = (\delta_X^{j_i}) \right)$$

of an object  $X$  of  $\Sigma\mathcal{A}$  and a morphism  $\delta \in \text{hom}_{\Sigma\mathcal{A}}^1(X, X)$ , satisfying the *Maurer–Cartan equation*

$$\sum_{i=1}^{\infty} m_r^{\Sigma\mathcal{A}}(\delta_X, \dots, \delta_X) = 0.$$

Twisted complexes constitute an  $A_\infty$ -category  $\mathcal{T}w\mathcal{A}$ , whose  $A_\infty$ -operations are given by

$$m_d^{\mathcal{T}w\mathcal{A}}(a_d, \dots, a_1) = \sum_{i_0, \dots, i_d} m_{d+i_0+\dots+i_d}^{\Sigma\mathcal{A}} \left( \overbrace{(\delta_{X_d}, \dots, \delta_{X_d})}^{i_d}, a_d, \underbrace{(\delta_{X_{d-1}}, \dots, \delta_{X_{d-1}})}_{i_{d-1}}, a_{d-1}, \dots, a_1, \underbrace{(\delta_{X_0}, \dots, \delta_{X_0})}_{i_0} \right),$$

where the sum is over all  $i_0, \dots, i_d \geq 0$ . The  $A_\infty$ -relations in  $\mathcal{T}w\mathcal{A}$  comes from that of  $\mathcal{A}$  and the Maurer–Cartan equation. The cohomological category  $D^b\mathcal{A}$  of  $\mathcal{T}w\mathcal{A}$  is triangulated, and the mapping cone of a closed morphism  $c \in \text{hom}_{\mathcal{T}w\mathcal{A}}^0(X_0, X_1)$  is defined by

$$\left( C = \mathbb{C}[1] \otimes X_0 \oplus \mathbb{C} \otimes X_1, \delta_C = \begin{pmatrix} \mathbf{1}_{1,1} \otimes \delta_{X_0} & 0 \\ -\mathbf{1}_{1,0} \otimes c & \mathbf{1}_{0,0} \otimes \delta_{X_1} \end{pmatrix} \right)$$

where  $\mathbf{1}_{i,j} \in \text{hom}_{\mathbb{C}}(\mathbb{C}[i], \mathbb{C}[j])$  is the identity morphism of degree  $i - j$ .

The Fukaya category  $\mathfrak{Fuk} M$  of a symplectic manifold  $(M, \omega)$  is an  $A_\infty$ -category whose objects are Lagrangian submanifolds of  $M$  (together with additional structures such as gradings, spin structures and flat  $U(1)$  bundles on them) and whose spaces of morphisms are Lagrangian intersection Floer complexes [Fuk93, FOOO09, Sei08]: For two objects  $L_1$  and  $L_2$  intersecting transversely,  $\text{hom}(L_1, L_2)$  is a graded vector space spanned by intersection points  $L_1 \cap L_2$ . For a positive integer  $k$ , a sequence  $(L_0, \dots, L_k)$  of objects, and morphisms  $p_\ell \in L_{\ell-1} \cap L_\ell$  for  $\ell = 1, \dots, k$ , the  $A_\infty$ -operation  $m_k$  is given by counting the virtual number of holomorphic disks with Lagrangian boundary conditions;

$$m_k(p_k, \dots, p_1) = \sum_{p_0 \in L_0 \cap L_k} \# \overline{\mathcal{M}}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k) p_0.$$

Here,  $\overline{\mathcal{M}}_{k+1}(L_0, \dots, L_k; p_0, \dots, p_k)$  is the stable compactification of the moduli space of holomorphic maps  $\phi : D^2 \rightarrow M$  from the unit disk  $D^2$  with  $k + 1$  marked points

$(z_0, \dots, z_k)$  on the boundary respecting the cyclic order, with the following boundary condition: Let  $\partial_l D^2 \in \partial D^2$  be the interval between  $z_l$  and  $z_{l+1}$ , where we set  $z_{k+1} = z_0$ . Then  $\phi(\partial_l D^2) \subset L_\ell$  and  $\phi(z_l) = p_l$  for  $\ell = 0, \dots, k$ .

Let  $M$  be a symplectic manifold and  $p : \tilde{M} \rightarrow M$  be a regular covering with the covering transformation group  $G$ , so that there is an exact sequence

$$1 \rightarrow \pi_1(\tilde{M}) \xrightarrow{p_*} \pi_1(M) \rightarrow G \rightarrow 1$$

of groups. Let  $i : L \hookrightarrow M$  be a Lagrangian submanifold. If the image of  $i_* : \pi_1(L) \rightarrow \pi_1(M)$  is contained in the image of  $p_*$ , then the set of connected components of  $\tilde{L} = p^{-1}(L)$  forms a torsor over  $G$ , so that one has

$$\tilde{L} = \coprod_{g \in G} \tilde{L}_g$$

for a choice of a connected component  $\tilde{L}_e \subset \tilde{L}$ . Given a pair  $(L, L')$  of such Lagrangian submanifolds, one has an isomorphism

$$\text{hom}_{\mathfrak{F}\text{ut} M}(L, L') \cong \bigoplus_{g \in G} \text{hom}_{\mathfrak{F}\text{ut} \tilde{M}}(\tilde{L}_e, \tilde{L}'_g),$$

which is compatible with the  $A_\infty$ -operations.

### 3. Symplectic Picard–Lefschetz Theory

Let  $\pi : E \rightarrow \mathbb{C}$  be a holomorphic function on a Kähler manifold  $E$ , whose Kähler form is exact. We assume that  $E$  is complete as a Riemannian manifold, and  $\|\nabla\pi\|$  is a proper function on  $E$ . The mirrors of toric Fano stacks satisfy these conditions [Sei10, Example 6.1]. The map  $\pi$  is said to be an *exact Lefschetz fibration* if all the critical points of  $\pi$  are non-degenerate. This means that for any critical point  $p \in E$ , one can choose a holomorphic local coordinate  $(x_1, \dots, x_n)$  of  $E$  around  $p$  such that

$$\pi(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2 + w, \quad (3.1)$$

where  $w$  is the critical value of  $\pi$ . For the moment, we assume that all the critical values are distinct and 0 is a regular value of  $\pi$ . We choose the origin as the base point and write

$$E_0 = \pi^{-1}(0).$$

A *vanishing path* is an embedded path  $\gamma : [0, 1] \rightarrow \mathbb{C}$  such that

- $\gamma(0) = 0$ ,
- $\gamma(1)$  is a critical value of  $\pi$ , and
- $\gamma(t)$  is not a critical value of  $\pi$  for  $t \in (0, 1)$ .

A *distinguished set of vanishing paths* is an ordered set  $(\gamma_i)_{i=1}^m$  of vanishing paths  $\gamma_i : [0, 1] \rightarrow \mathbb{C}$  such that

- $\{\gamma_i(1)\}_{i=1}^m$  is the set of critical values of  $\pi$ ,
- images of  $\gamma_i$  and  $\gamma_j$  for  $i \neq j$  intersect only at the origin,
- $\gamma'_i(0) \neq 0$  for  $i = 1, \dots, m$ , and
- $\arg \gamma'_1(0) > \dots > \arg \gamma'_m(0)$  for a suitable choice of a branch of the argument map.

Let  $\gamma$  be a vanishing path and  $y$  be the critical point of  $\pi$  above  $\gamma(1)$ . The *vanishing cycle* along  $\gamma$  is the cycle of  $E_0$  which collapses to the critical point  $y$  by the symplectic parallel transport along  $\gamma$ ;

$$V_\gamma = \left\{ x \in E_0 \mid \lim_{t \rightarrow 1} \tilde{\gamma}_x(t) = y \right\}.$$

Here, the horizontal lift  $\tilde{\gamma}_x : [0, 1] \rightarrow E$  of  $\gamma : [0, 1] \rightarrow \mathbb{C}$  starting from  $x \in E_0$  is defined by the condition that the tangent vector of the curve  $\tilde{\gamma}$  is orthogonal to the tangent space of the fiber with respect to the Kähler form.

The vanishing cycle  $V_\gamma$  is a Lagrangian  $(n-1)$ -sphere  $E_0$ . The trajectory

$$\Delta_\gamma = \bigcup_{x \in V_\gamma} \text{Im } \tilde{\gamma}_x$$

of the vanishing cycle is called the *Lefschetz thimble*. It is a Lagrangian ball in  $E$  whose boundary is the corresponding vanishing cycle;

$$\partial \Delta_\gamma = V_\gamma.$$

For a distinguished set  $(\gamma_i)_{i=1}^m$  of vanishing paths, the ordered set

$$\mathbf{V} = (V_{\gamma_1}, \dots, V_{\gamma_n})$$

is called the *distinguished basis of vanishing cycles*.

To define the Fukaya category of the Lefschetz fibration, let

$$\beta : \tilde{E} = \{(x, y) \in E \times \mathbb{C} \mid \pi(x) = y^2\} \rightarrow E$$

be the double cover of  $E$  branched along the fiber  $E_0 = \pi^{-1}(0)$  over the origin. Then the covering transformation  $\iota : (x, y) \mapsto (x, -y)$  defines a  $\mathbb{Z}/2\mathbb{Z}$ -action on  $\tilde{E}$ , which induces a  $\mathbb{Z}/2\mathbb{Z}$ -action on the Fukaya category  $\mathfrak{Fuk} \tilde{E}$  of  $\tilde{E}$ . Roughly speaking, the Fukaya category  $\mathfrak{F}(\pi)$  of the Lefschetz fibration  $\pi$  is defined as the  $\iota$ -invariant part of  $\mathfrak{Fuk} \tilde{E}$ ; objects of  $\mathfrak{F}(\pi)$  are  $\iota$ -invariant Lagrangian submanifolds of  $\tilde{E}$ , and the space of morphisms in  $\mathfrak{F}(\pi)$  are  $\iota$ -invariant part of morphisms in  $\mathfrak{Fuk} \tilde{E}$ . The precise definition is given in [Sei08, Section 18].

There are two important classes of  $\iota$ -invariant Lagrangian submanifolds in  $\tilde{E}$ . One of them, called of type (U), is the inverse image

$$\tilde{L} = \beta^{-1}(L) = \tilde{L}_+ \amalg \tilde{L}_-$$

of a Lagrangian submanifold  $L$  whose image by  $\pi$  is contained in a simply-connected domain inside  $\mathbb{C}^\times$  (i.e.,  $\mathbb{C}$  minus the base point). It is the disjoint union of two connected components  $\tilde{L}_+$  and  $\tilde{L}_-$ . The other, called of type (B), is the inverse image

$$\tilde{\Delta}_\gamma = \beta^{-1}(\Delta_\gamma)$$

of the Lefschetz thimble  $\Delta_\gamma$  for a vanishing path  $\gamma$ . It is a Lagrangian  $n$ -sphere in  $\tilde{E}$ .

For type (U) Lagrangian submanifolds  $\tilde{L}_0$  and  $\tilde{L}_1$  of  $\tilde{E}$ , their intersections are two disjoint copies of intersections between  $L_0$  and  $L_1$  in  $E$ . By taking  $\iota$ -invariant, one can show that there is a natural isomorphism

$$\text{hom}_{\mathfrak{F}(\pi)}(\tilde{L}_0, \tilde{L}_1) \cong \text{hom}_{\mathfrak{Fuk} E}(L_0, L_1)$$



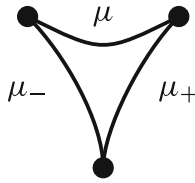


Fig. 1. A matching path

of vector spaces, which lifts to a cohomologically full and faithful  $A_\infty$ -functor

$$\mathfrak{Fuk} E \rightarrow \mathfrak{F}(\pi).$$

For type (B) Lagrangian submanifolds, the situation is a little more complicated, but the conclusion is that the full  $A_\infty$ -subcategory of  $\mathfrak{F}(\pi)$  consisting of  $\tilde{\Delta} = (\tilde{\Delta}_{\gamma_1}, \dots, \tilde{\Delta}_{\gamma_m})$  for a distinguished set  $(\gamma_i)_{i=1}^m$  of vanishing paths is quasi-isomorphic to the *directed subcategory*  $\mathfrak{Fuk}^{\rightarrow}(V)$  of  $\mathfrak{Fuk} E_0$ , whose set of objects is the distinguished basis  $V = (V_{\gamma_1}, \dots, V_{\gamma_m})$  of vanishing cycles, whose spaces of morphisms are given by

$$\text{hom}_{\mathfrak{Fuk}^{\rightarrow}(V)}(V_{\gamma_i}, V_{\gamma_j}) = \begin{cases} \mathbb{C} \cdot \text{id}_{V_{\gamma_i}} & i = j, \\ \text{hom}_{\mathfrak{Fuk} E_0}(V_{\gamma_i}, V_{\gamma_j}) & i < j, \\ 0 & \text{otherwise,} \end{cases}$$

and non-trivial  $A_\infty$ -operations coincide with those in  $\mathfrak{Fuk} E_0$ . We write this  $A_\infty$ -category as  $\mathfrak{Fuk} \pi$ . Although  $\mathfrak{Fuk} \pi$  depends on the choice of a distinguished set of vanishing paths, the derived category  $D^b \mathfrak{Fuk} \pi$  is independent of this choice and gives an invariant of the Lefschetz fibration  $\pi$ .

Let  $\mu : [-1, 1] \rightarrow \mathbb{C}$  be an embedded path in  $\mathbb{C}$  such that  $\mu^{-1}(\text{Critv}(\pi)) = \{-1, 1\}$ . One can deform  $\mu$  and split it into two pieces  $\mu_{\pm}(t) = \mu(\pm t)$  to obtain a pair of vanishing paths as shown in Fig. 1. If the vanishing cycles  $V_{\mu_-}$  and  $V_{\mu_+}$  are isotopic as exact framed Lagrangian  $(n - 1)$ -spheres in  $E_0$ , then  $\mu$  is called a *matching path*. In this case, one can perturb  $\Delta_{\mu_+} \cup \Delta_{\mu_-}$  to obtain a Lagrangian  $n$ -sphere  $\Sigma_\mu$  in  $E$  called the *matching cycle*.

Symplectic Picard–Lefschetz theory describes the action of the symplectic Dehn-twist along a Lagrangian sphere on the derived Fukaya category. It follows that the type (U) Lagrangian submanifold  $\tilde{\Sigma}_\mu = \beta^{-1}(\Sigma_\mu)$  of  $\tilde{E}$  coming from a matching path  $\mu$  is isomorphic to the mapping cone over the (unique up to scalar) non-trivial morphism from  $\tilde{\Delta}_{\mu_-}$  to  $\tilde{\Delta}_{\mu_+}$  in the derived Fukaya category  $D^b \mathfrak{F}(\pi)$  of the Lefschetz fibration;

$$\tilde{\Sigma}_\mu \cong \text{Cone}(\tilde{\Delta}_{\mu_-} \rightarrow \tilde{\Delta}_{\mu_+}).$$

This is important since it allows one to reduce Floer-theoretic computation for matching cycles in  $\mathfrak{Fuk} E$  to that for vanishing cycles in  $\mathfrak{Fuk} E_0$ . By iterating this process, one ends up with the case of symplectic 2-manifolds, where Lagrangian submanifolds are simple closed curves and the problem of counting holomorphic disks is purely combinatorial.

A natural source of matching paths is a *Lefschetz bifibration*. It is a diagram

$$\begin{array}{ccccc} & & \Psi = \psi \circ \varpi & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{E} & \xrightarrow{\varpi} & \mathbb{C}^2 & \xrightarrow{\psi} & \mathbb{C} \end{array}$$

with certain genericity conditions, which implies that for any critical point of  $\Psi$ , there are local holomorphic coordinates of  $\mathcal{E}$  and  $\mathbb{C}^2$  such that

$$\varpi(x_1, \dots, x_{2n}) = (x_1^2 + x_2^2 + \dots + x_{2n}^2, x_1), \quad \psi(y_1, y_2) = y_1.$$

Then the map

$$\mathcal{E}_w \xrightarrow{\varpi_w} \mathcal{S}_w$$

from  $\mathcal{E}_w = \Psi^{-1}(w)$  to  $\mathcal{S}_w = \psi^{-1}(w)$  for a general  $w \in \mathbb{C}$  is a Lefschetz fibration, and by chasing the trajectory of critical values of  $\varpi_w$  as  $w$  varies along a vanishing path  $\gamma$ , one obtains a matching path  $\mu$  in  $\mathcal{S}_0$  such that the matching cycle  $\Sigma_\mu$  is Hamiltonian isotopic to the vanishing cycle  $V_\gamma$ .

### 4. Homological Mirror Symmetry for $\mathbb{P}^2$

We recall homological mirror symmetry for  $\mathbb{P}^2$  proved by Seidel [Sei01] in this section. The mirror of  $\mathbb{P}^2$  is given by the Laurent polynomial

$$W(x, y) = x + y + \frac{1}{xy},$$

which has critical points  $(x, y) = (1, 1), (\omega, \omega), (\omega^2, \omega^2)$  with critical values  $3, 3\omega, 3\omega^2$ . Here  $\omega = \exp(2\pi\sqrt{-1}/3)$  is a primitive cubic root of unity. Let  $(\gamma_i)_{i=1}^3$  be the distinguished set of vanishing paths obtained as the straight line segments from the origin to the critical values of  $W$  as shown in Fig. 2. The corresponding vanishing cycles are denoted by  $(C_i)_{i=1}^3$ .

Consider the Lefschetz bifibration

$$\begin{array}{ccc}
 & \xrightarrow{W = \psi \circ \varpi} & \\
 (\mathbb{C}^\times)^2 & \xrightarrow{\varpi} \mathbb{C} \times \mathbb{C}^\times \xrightarrow{\psi} & \mathbb{C}
 \end{array} \tag{4.1}$$

where

$$\varpi(x, y) = \left( x + y + \frac{1}{xy}, y \right)$$

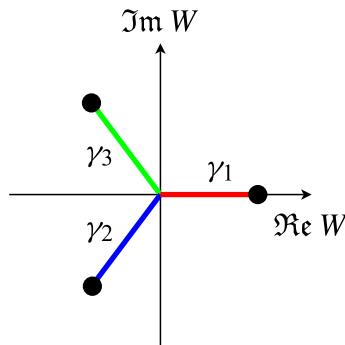


Fig. 2. A path on the  $W$ -plane

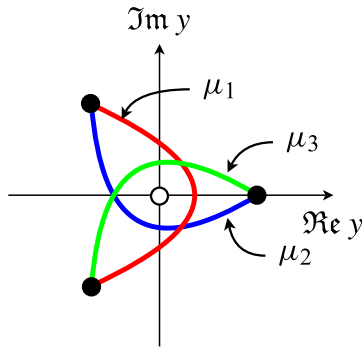


Fig. 3. Matching paths on the  $y$ -plane

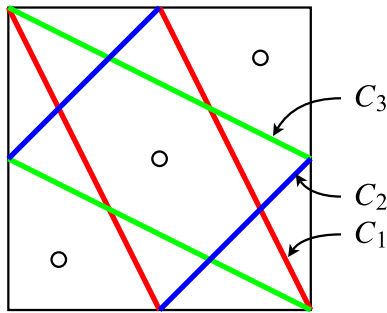


Fig. 4. Vanishing cycles on  $W^{-1}(0)$

and

$$\psi(u, v) = u.$$

The critical points of

$$\varpi_t : W^{-1}(t) \rightarrow \psi^{-1}(t) \cong \text{Spec } \mathbb{C}[y, y^{-1}]$$

are given by

$$2x + \frac{1}{x^2} = t,$$

with critical values

$$y = \frac{1}{x^2}.$$

The critical values are given by  $y = (-2)^{2/3}$  at  $t = 0$ , which moves as shown in Fig. 3 along the vanishing paths  $(\gamma_i)_{i=1}^3$ . These trajectories  $(\mu_i)_{i=1}^3$  are matching paths corresponding to  $(C_i)_{i=1}^3$ . The fiber  $W^{-1}(0)$  can be compactified to an elliptic curve by adding one point over  $y = 0$  and two points over  $y = \infty$ . The vanishing cycles on  $W^{-1}(0)$  are shown in Fig. 4.

On the mirror side, one has the full exceptional collection

$$(E_1, E_2, E_3) = (\Omega_{\mathbb{P}^2}^2(2)[2], \Omega_{\mathbb{P}^2}(1)[1], \mathcal{O}_{\mathbb{P}^2})$$

by Beilinson [Bei78]. The endomorphism algebra is given by

$$\text{hom}(E_i, E_j) = \begin{cases} \wedge^{j-i} V & i \leq j, \\ 0 & \text{otherwise,} \end{cases}$$

where  $V$  is a 3-dimensional vector space. This endomorphism algebra is formal as an  $A_\infty$ -algebra with respect to the standard enhancement of  $D^b \text{coh } \mathbb{P}^2$ . One can easily see that there is an isomorphism

$$\mathfrak{Fut}^{\rightarrow} W^{-1}(0) \rightarrow D^b \text{coh } \mathbb{P}^2$$

of  $A_\infty$ -categories sending  $C_i$  to  $E_i$  for  $i = 1, 2, 3$ . Indeed, one can see in Fig. 4 that  $C_i$  and  $C_j$  for  $i \neq j$  intersect at three points, and six triangles bounded by  $C_1, C_2$  and  $C_3$  correspond to six non-zero compositions in

$$\text{hom}(E_2, E_3) \otimes \text{hom}(E_1, E_2) \rightarrow \text{hom}(E_1, E_3).$$

The torus-equivariant version of homological mirror symmetry for  $\mathbb{P}^2$  (and more generally for toric del Pezzo surfaces) is discussed in detail in [UY13].

### 5. Homological Mirror Symmetry for $\mathbb{P}^3$

The mirror of the projective space  $\mathbb{P}^3$  is given by the Laurent polynomial

$$W(x, y, z) = x + y + z + \frac{1}{xyz}$$

with critical points  $x = y = z = \pm 1, \pm\sqrt{-1}$  and critical values  $\pm 4, \pm 4\sqrt{-1}$ . Choose a distinguished set of vanishing paths  $(\gamma_i)_{i=1}^4$  as the straight line segments from the origin to the critical values as shown in Fig. 5, and let  $(C_i)_{i=1}^4$  be the corresponding

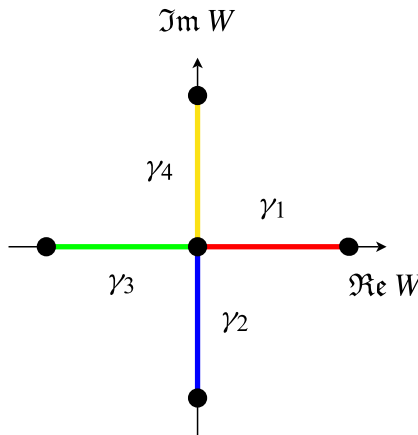


Fig. 5. A distinguished set of vanishing paths

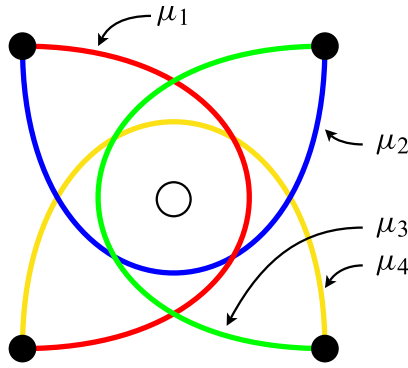


Fig. 6. Matching paths on the  $z$ -plane

distinguished basis of vanishing cycles. To use Picard–Lefschetz theory, consider the Lefschetz bifibration

$$\begin{array}{ccc}
 & W = \psi \circ \varpi & \\
 & \curvearrowright & \\
 (\mathbb{C}^\times)^3 & \xrightarrow{\varpi} \mathbb{C} \times \mathbb{C}^\times & \xrightarrow{\psi} \mathbb{C}
 \end{array} \tag{5.1}$$

where

$$\varpi(x, y, z) = \left( x + y + z + \frac{1}{xyz}, z \right)$$

and

$$\psi(u, v) = u.$$

The critical points of

$$\varpi_t : W^{-1}(t) \rightarrow \psi^{-1}(t) \cong \text{Spec } \mathbb{C}[z, z^{-1}]$$

are given by

$$x = y, \quad 3x + \frac{1}{x^3} = t,$$

with critical values

$$z = \frac{1}{x^3}.$$

The critical values are given by  $z = (-3)^{3/4}$  at  $t = 0$ , which moves as shown in Fig. 6 along the vanishing paths  $(c_i)_{i=1}^4$ . These trajectories  $(\mu_i)_{i=1}^4$  are matching paths corresponding to  $(C_i)_{i=1}^4$ . Take  $z = 1$  as a base point and choose a distinguished set  $(\delta_i)_{i=1}^4$  of vanishing paths for  $\varpi_0$  as straight line segments from the base point as shown in Fig. 7. The fiber  $\varpi_0^{-1}(z)$  is a branched double cover of  $\mathbb{C}^\times$  by the  $y$ -projection

$$\begin{array}{ccc}
 \pi_z : \varpi_0^{-1}(z) & \rightarrow & \mathbb{C}^\times \\
 \cup & & \cup \\
 (x, y, z) & \mapsto & y.
 \end{array}$$

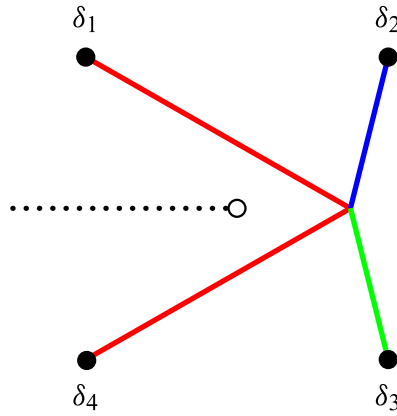


Fig. 7. A distinguished set of vanishing paths

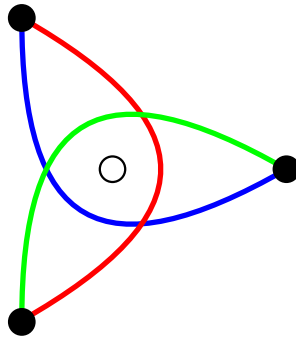


Fig. 8. Matching paths on the  $y$ -plane

Figure 8 shows the behavior of these branch points along vanishing paths  $(\delta_i)_{i=1}^4$ , which can be considered as matching paths coming from the Lefschetz bifibration

$$\begin{array}{ccc}
 & \varpi_0 = \varphi \circ \pi & \\
 W^{-1}(0) & \xrightarrow{\pi} & \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{\varphi} \mathbb{C}^\times \\
 & & 
 \end{array} \tag{5.2}$$

where  $\pi(x, y, z) = (z, y)$  and  $\varphi(z, y) = z$ . Note that one has an inductive structure here, as (5.2) is almost identical to (4.1).

One can see that the number of intersection points of  $C_i$  and  $C_j$  for  $i < j$  is equal to the dimension of  $\wedge^{j-i} V$ , where  $V$  is a vector space of dimension four. As an example, consider the intersection of  $C_1$  and  $C_2$ . The matching paths  $\mu_1$  and  $\mu_2$  intersect at one critical value of  $\varpi_0$  and one regular value of  $\varpi_0$ . The intersection of  $C_1$  and  $C_2$  over the critical value of  $\varpi_0$  consists of one point, i.e., the critical point of  $\varpi_0$ . The intersection of  $C_1$  and  $C_2$  over the regular value of  $\varpi_0$  consist of three points, as one can see from Fig. 8 (cf. also Figs. 3, 4). As for the intersection of  $C_1$  and  $C_3$ , the corresponding matching paths intersect at two regular points of  $\varpi_0$ , and the intersection over each of them consists of three points.

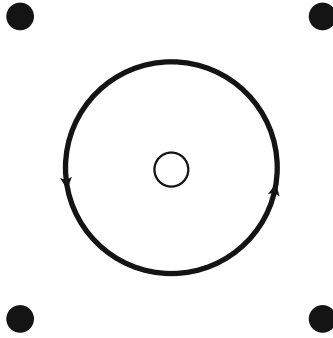


Fig. 9. A loop in the  $z$ -plane

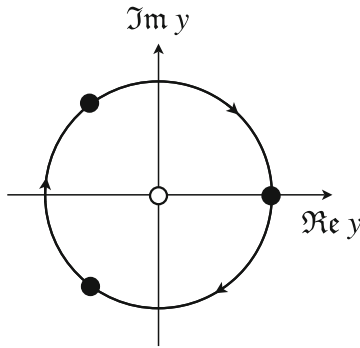


Fig. 10. The behavior of branch points of  $\pi_z$

To use Picard–Lefschetz theory to do computations in the Fukaya category of  $W$ , consider the pull-back

$$\begin{array}{ccc}
 W^{-1}(0)^\sim & \xrightarrow{\tilde{\varpi}_0} & \mathbb{C} \\
 \downarrow & & \downarrow \text{exp} \\
 W^{-1}(0) & \xrightarrow{\varpi_0} & \mathbb{C}^\times
 \end{array}$$

of  $\varpi_0$  by the universal cover of the algebraic torus. The existence of infinitely many critical points for a given critical value does not cause any problem, since the corresponding vanishing cycles do not intersect. The passage from  $W^{-1}(0)$  to  $W^{-1}(0)^\sim$  can be taken into account by noting that as one goes counterclockwise around the origin in the  $z$ -plane as shown in Fig. 9, the branch points of  $\pi_z$  rotates clockwise by  $2\pi/3$  as in Fig. 10.

The universal cover of the  $z$ -plane is obtained by cutting the  $z$ -plane along the dashed line in Fig. 7 and gluing infinitely-many copies of it. We set the point  $z = 1$  on the zero-th sheet as the base point  $*$  and take a distinguished set of vanishing paths for  $\tilde{\varpi}_0$  as in Fig. 11.

Let  $\overline{\Delta}_1, \overline{\Delta}_2,$  and  $\overline{\Delta}_3$  be the vanishing cycles of  $\varpi_0$  along the vanishing paths  $\delta_1, \delta_2$  and  $\delta_3$  respectively. We write the vanishing cycles of  $\tilde{\varpi}_0$  along the vanishing paths  $\delta_i$  in Fig. 11 as  $\Delta_i$  for  $i \in \mathbb{Z}$ . Let further  $\mathcal{B}$  be the Fukaya category of  $\varpi_0^{-1}(1)$  consisting of

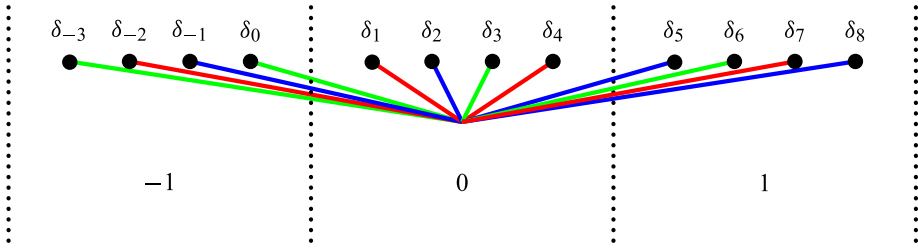


Fig. 11. Vanishing paths on the universal cover

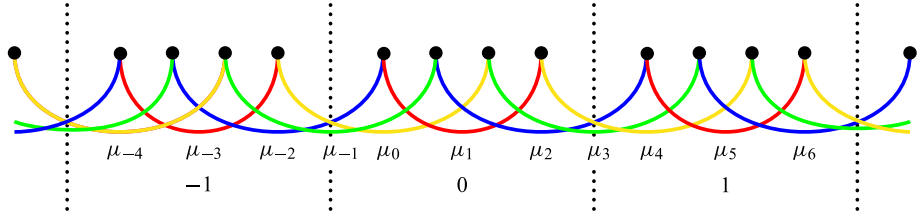


Fig. 12. Matching paths on the universal cover

$\{\bar{\Delta}_i\}_{i=1}^3$  and  $\tilde{\mathcal{B}}$  be the Fukaya category of  $\tilde{\omega}_0^{-1}(*)$  consisting of  $\{\Delta_i\}_{i \in \mathbb{Z}}$ . Then one has a quasi-equivalence

$$\tilde{\mathcal{B}} \xrightarrow{\sim} \mathcal{B}$$

of  $A_\infty$ -categories sending  $\Delta_i$  to  $\bar{\Delta}_{\bar{i}}$ , where  $\bar{i}$  is  $i$  modulo 3. We write the directed subcategory of  $\tilde{\mathcal{B}}$  with respect to the order

$$\Delta_i < \Delta_j, \quad i < j$$

as  $\tilde{\mathcal{A}}$ . The spaces of morphisms between  $\Delta_i$  can be written as

$$\text{hom}_{\tilde{\mathcal{A}}}(\Delta_i, \Delta_j) = \begin{cases} \mathbb{C} \cdot \text{id}_i & i = j, \\ \mathbb{C} \cdot \text{id}_{i,j} \oplus \mathbb{C} \cdot \text{id}_{i,j}^\vee & i < j \text{ and } j \equiv i \pmod{3}, \\ \bar{V} & i < j \text{ and } j \equiv i + 1 \pmod{3}, \\ \wedge^2 \bar{V} & i < j \text{ and } j \equiv i + 2 \pmod{3}, \\ 0 & i > j, \end{cases}$$

where  $\text{id}_i$  is the unit and

$$\bar{V} = \text{span} \{e_1, e_2, e_3\}$$

is a vector space of dimension three. One can show, by direct counting of triangles just as in Sect. 4, that the  $A_\infty$ -operation  $m_2$  on the spaces of morphisms is given by the wedge product, where  $\text{id}_{i,j}$  and  $\text{id}_{i,j}^\vee$  are identified with the elements  $1 \in \wedge^0 \bar{V}$  and  $e_1 \wedge e_2 \wedge e_3 \in \wedge^3 \bar{V}$  respectively. Higher  $A_\infty$ -operations on  $\tilde{\mathcal{A}}$  are irrelevant for the argument below.

Let  $C_i$  for  $i \in \mathbb{Z}$  be the lift to  $W^{-1}(0)^\sim$  of a vanishing cycle on  $W^{-1}(0)$ , which corresponds to the matching path  $\mu_i$  obtained by concatenating  $\delta_i$  and  $\delta_{i+3}$  as in Fig. 12.



Let further  $\mathfrak{Fuk} W^{-1}(0)^\sim$  be the Fukaya category of  $W^{-1}(0)^\sim$  consisting of  $\{C_i\}_{i \in \mathbb{Z}}$  and  $\mathfrak{Fuk}^\rightarrow W^{-1}(0)^\sim$  be its directed subcategory with respect to the order  $C_i < C_j$  for  $i < j$ . By symplectic Picard–Lefschetz theory recalled in Sect. 3, there is a cohomologically full and faithful functor

$$\mathfrak{Fuk} W^{-1}(0)^\sim \rightarrow D^b \tilde{\mathcal{A}},$$

which maps the objects as

$$C_i \mapsto \text{Cone} \left( \Delta_i \xrightarrow{\text{id}_{i,i+3}} \Delta_{i+3} \right).$$

On the mirror side, the passage to the universal cover of the  $z$ -plane corresponds to working equivariantly with respect to the subgroup

$$\mathbb{T}_3 = \{(\alpha, \beta, \gamma) \in \mathbb{T} \mid \alpha = \beta = 1\}$$

of the torus  $\mathbb{T} \cong (\mathbb{C}^\times)^3$  acting on  $\mathbb{P}^3$  by

$$\begin{aligned} \mathbb{T} \ni (\alpha, \beta, \gamma) : \quad & \begin{array}{ccc} \mathbb{P}^3 & \rightarrow & \mathbb{P}^3 \\ \Psi & & \Psi \end{array} \\ & [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : \alpha x_1 : \beta x_2 : \gamma x_3]. \end{aligned}$$

The full exceptional collection

$$(E_1, E_2, E_3, E_4) = (\Omega_{\mathbb{P}^3}^3(3)[3], \Omega_{\mathbb{P}^3}^2(2)[2], \Omega_{\mathbb{P}^3}^1(1)[1], \mathcal{O}_{\mathbb{P}^3})$$

admits a natural  $\mathbb{T}$ -linearization, so that the endomorphism algebra is given by

$$\text{hom}(E_i, E_j) = \begin{cases} \wedge^{j-i} V & i \leq j, \\ 0 & \text{otherwise,} \end{cases}$$

with the natural  $\mathbb{T}$ -action. Moreover, this endomorphism algebra is formal as an  $A_\infty$ -algebra with respect to a standard enhancement of  $D^b \text{coh}^{\mathbb{T}} \mathbb{P}^3$ . Now it is easy to see that there is an  $A_\infty$ -functor

$$\mathfrak{Fuk}^\rightarrow W^{-1}(0)^\sim \rightarrow D^b \text{coh}^{\mathbb{T}_3} \mathbb{P}^3$$

sending  $C_{i+4j}$  to  $E_i \otimes \rho_j$ , where  $\rho_j : \mathbb{T}_3 \rightarrow \mathbb{C}^\times$  for  $j \in \mathbb{Z}$  is the one-dimensional representation sending  $(1, 1, \gamma) \in \mathbb{T}_3$  to  $\gamma^j$ ; for example, one has

$$\begin{aligned} \text{hom}(E_1 \otimes \rho_i, E_2 \otimes \rho_j) &= \begin{cases} \mathbb{C} \cdot e_4 & j = i - 1, \\ \bar{V} & j = i, \\ 0 & \text{otherwise,} \end{cases} \\ \text{hom}(E_1 \otimes \rho_i, E_3 \otimes \rho_j) &= \begin{cases} \bar{V} \wedge e_4 & j = i - 1, \\ \wedge^2 \bar{V} & j = i, \\ 0 & \text{otherwise,} \end{cases} \\ \text{hom}(E_1 \otimes \rho_i, E_4 \otimes \rho_j) &= \begin{cases} (\wedge^2 \bar{V}) \wedge e_4 & j = i - 1, \\ \wedge^3 \bar{V} & j = i, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

which exactly matches the computation in the Fukaya category, as we show for general  $n$  in Sect. 6. This suffices to show the equivalence

$$D^b \mathfrak{Fuk} \rightarrow W^{-1}(0) \sim \cong D^b \text{coh}^{\mathbb{T}^3} \mathbb{P}^3,$$

which induces the equivalence

$$D^b \mathfrak{Fuk} W \cong D^b \text{coh} \mathbb{P}^3$$

by passing to the non-equivariant situation.

### 6. Inductive Description of the Fukaya Category

The mirror of the projective space  $\mathbb{P}^n$  is given by the Laurent polynomial

$$W(x_1, \dots, x_n) = x_1 + \dots + x_n + \frac{1}{x_1 \cdots x_n}, \tag{6.1}$$

with critical points

$$x_1 = \dots = x_n = \zeta^{1-i}, \quad \zeta = \exp(2\pi \sqrt{-1}/(n+1)), \quad i = 1, \dots, n+1$$

and critical values  $(n+1)\zeta^{1-i}$ . Choose a distinguished set of vanishing paths  $(\gamma_i)_{i=1}^{n+1}$  as the straight line segments from the origin to the critical values, so that  $\gamma_i(1) = \zeta^{1-i}$ . The Fukaya category of  $W$  consisting of vanishing cycles  $C_i$  along  $\gamma_i$  for  $i = 1, \dots, n+1$  will be denoted by  $\mathfrak{Fuk} W$ .

**Theorem 6.1** *The spaces of morphisms in  $\mathfrak{Fuk} W$  are given by*

$$\text{hom}(C_i, C_j) = \begin{cases} \mathbb{C} \cdot \text{id}_{C_i} & i = j, \\ \wedge^{j-i} V & i < j, \\ 0 & \text{otherwise,} \end{cases}$$

where  $V$  is an  $(n+1)$ -dimensional vector space and an element of  $\wedge^i V$  has degree  $i$ . The  $A_\infty$ -operations  $\mathfrak{m}_k$  are given by the wedge product for  $k = 2$ , and vanish for  $k \neq 2$ .

*Proof.* Consider the Lefschetz bifibration

$$\begin{array}{ccc} & W = \psi \circ \varpi & \\ & \curvearrowright & \\ (\mathbb{C}^\times)^n & \xrightarrow{\varpi} \mathbb{C} \times \mathbb{C}^\times \xrightarrow{\psi} & \mathbb{C} \end{array} \tag{6.2}$$

where

$$\varpi(x_1, \dots, x_n) = \left( x_1 + \dots + x_n + \frac{1}{x_1 \cdots x_n}, x_n \right)$$

and

$$\psi(u, v) = u.$$

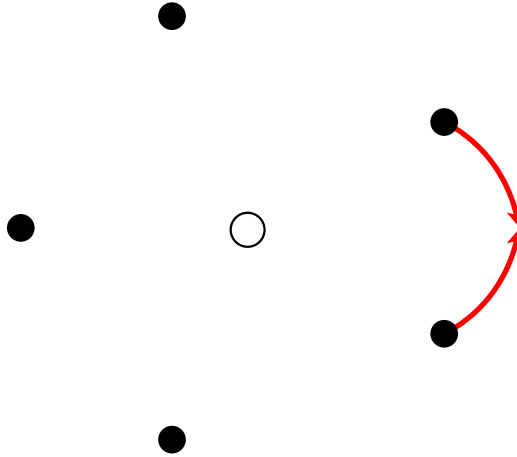


Fig. 13. The behavior of solutions of (6.3)

The critical points of

$$\varpi_t : W^{-1}(t) \rightarrow \psi^{-1}(t) \cong \text{Spec } \mathbb{C}[x_n, x_n^{-1}]$$

are given by

$$x_1 = \dots = x_{n-1}, \quad nx_1^{n+1} - tx_1^n + 1 = 0$$

with critical values

$$x_n = \frac{1}{x_1^n}.$$

As one varies  $t$  along the vanishing path  $\gamma_1$  from  $t = 0$  to  $t = n + 1$ , two points  $x = \exp(\pm\pi/(n + 1)\sqrt{-1})/\sqrt[n+1]{n}$  from the set of solutions of

$$nx^{n+1} - tx^n + 1 = 0 \tag{6.3}$$

at  $t = 0$  collide at  $x = 1$  and  $t = n + 1$ , while the absolute values of other points remains to be smaller than these two points, so that their behavior is as shown in Fig. 13. Here and below, all figures are for  $n = 4$ , but the general case is completely parallel. The corresponding trajectory of the critical values of  $\varpi_t$  is shown in Fig. 14.

Now consider the Lefschetz bifibration

$$\begin{array}{ccc} & \varpi_0 = \varphi \circ \pi & \\ & \curvearrowright & \\ W^{-1}(0) & \xrightarrow{\pi} \mathbb{C}^\times \times \mathbb{C}^\times & \xrightarrow{\varphi} \mathbb{C}^\times \end{array} \tag{6.4}$$

where  $\pi(x_1, \dots, x_n) = (x_n, x_{n-1})$  and  $\phi(x_n, x_{n-1}) = x_n$ . Take  $x_n = 1$  as a base point and choose a distinguished set  $(\delta_i)_{i=1}^{n+1}$  of vanishing paths for  $\varpi_0$  as the straight line segments from the base point as shown in Fig. 15. Consider the pull-back

$$\begin{array}{ccc} W^{-1}(0) \sim & \xrightarrow{\tilde{\varpi}_0} & \mathbb{C} \\ \downarrow & & \downarrow \text{exp} \\ W^{-1}(0) & \xrightarrow{\varpi_0} & \mathbb{C}^\times \end{array}$$

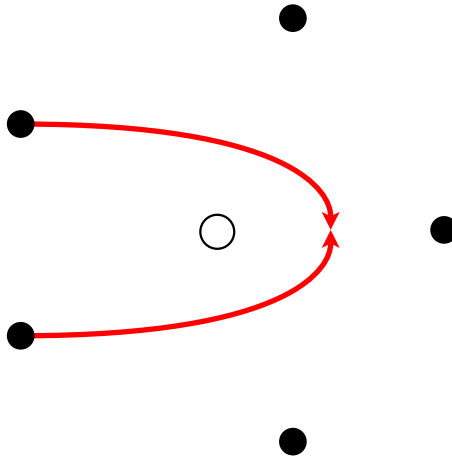


Fig. 14. Matching paths on the  $x_n$ -plane

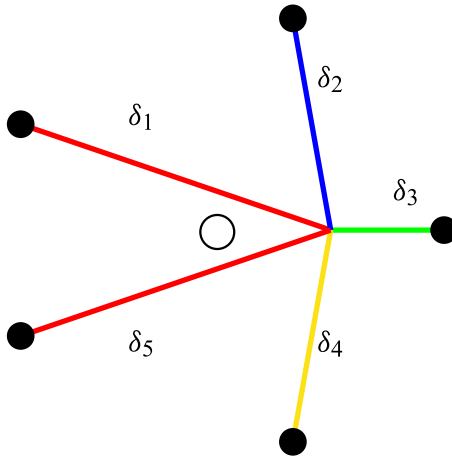


Fig. 15. Vanishing paths for  $w_0$

of  $w_0$  by the universal cover of the  $x_n$ -plane. The  $j$ -th lift of the vanishing cycle  $C_i \subset W^{-1}(0)$  to  $W^{-1}(0)^\sim$  will be denoted by  $C_{i+(n+1)j}$  for  $i = 1, \dots, n + 1$  and  $j \in \mathbb{Z}$ . We write the Fukaya category of  $W^{-1}(0)^\sim$  consisting of  $\{C_i\}_{i \in \mathbb{Z}}$  as  $\mathfrak{Fuk} W^{-1}(0)^\sim$ .

The universal cover of the  $x_n$ -plane is obtained by gluing infinitely many copy of the  $x_n$ -plane cut along the negative real axis. We take the point  $x_n = 1$  on the zeroth sheet as the base point  $*$  and take a distinguished set  $(\delta)_{i \in \mathbb{Z}}$  of vanishing paths as in Fig. 17. The vanishing cycle along  $\delta_i$  will be denoted by  $\Delta_i$ . We write the directed Fukaya category of  $\tilde{w}_0$  consisting of  $(\Delta_i)_{i \in \mathbb{Z}}$  as  $\tilde{\mathcal{A}}$ . The matching path corresponding to  $C_i$  for  $i \in \mathbb{Z}$  is obtained by concatenating  $\delta_i$  and  $\delta_{i+n}$  as in Fig. 18.

Note that the fiber of  $w_0$  is isomorphic to the fiber of

$$\begin{aligned} \overline{W} : \quad & (\mathbb{C}^\times)^{n-1} \quad \rightarrow \quad \mathbb{C} \\ & \cup \quad \quad \quad \cup \\ & (x_1, \dots, x_{n-1}) \mapsto x_1 + \dots + x_{n-1} + \frac{1}{x_1 \cdots x_{n-1}} \end{aligned}$$

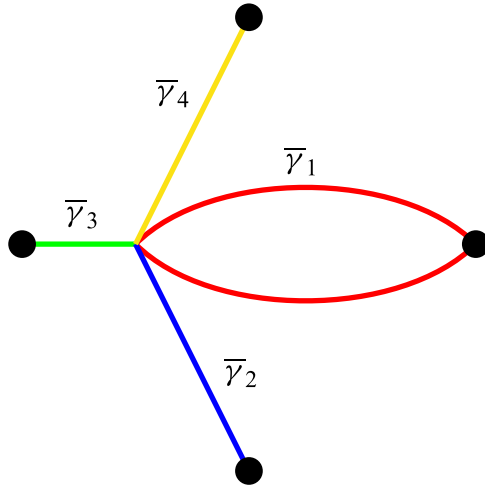


Fig. 16. Vanishing paths for  $\bar{W}$

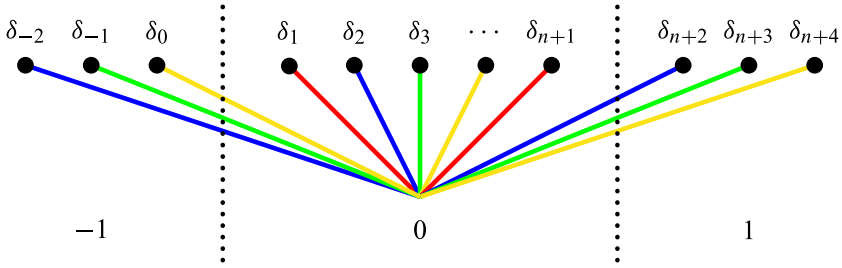


Fig. 17. Vanishing paths on the universal cover

by

$$\begin{aligned} \varpi_0^{-1}(x_n) &\xrightarrow{\psi} \bar{W}^{-1}\left(-x_n^{(n+1)/n}\right) \\ (x_1, \dots, x_n) &\mapsto x_n^{1/n}(x_1, \dots, x_{n-1}). \end{aligned}$$

As  $x_n$  varies along the vanishing paths in Fig. 15, its image by the map  $x \mapsto -x^{(n+1)/n}$  behaves as in Fig. 16, which are homotopic to the vanishing paths for  $\bar{W}$ . The fiber of  $\pi_1 : \varpi_0^{-1}(1) \rightarrow \mathbb{C}^\times$  at  $x_n = 1$  can be identified with the fiber of  $\bar{W}$  at  $t = -1$ , which in turn can be identified with the fiber of  $\bar{W}$  at the origin by symplectic parallel transport. Under this identification, the vanishing paths  $\delta_i$  in Fig. 16 can be identified with the vanishing paths  $\bar{\gamma}_{\bar{i}}$  for  $\bar{W}$ , where  $\bar{i}$  is  $i$  modulo  $n$ . It follows that the vanishing cycle  $\Delta_i$  along  $\delta_i$  corresponds to the vanishing cycle  $\bar{C}_{\bar{i}}$  along  $\bar{\gamma}_{\bar{i}}$ .

Assume that the assertion of Theorem 6.1 holds for  $\bar{W}$ , so that one has

$$\text{hom}_{\mathfrak{S} \text{ut } \bar{W}}(\bar{C}_i, \bar{C}_j) = \begin{cases} \mathbb{C} \cdot \text{id}_i & i = j \\ \wedge^{j-i} \bar{V} & i < j, \\ 0 & \text{otherwise,} \end{cases}$$

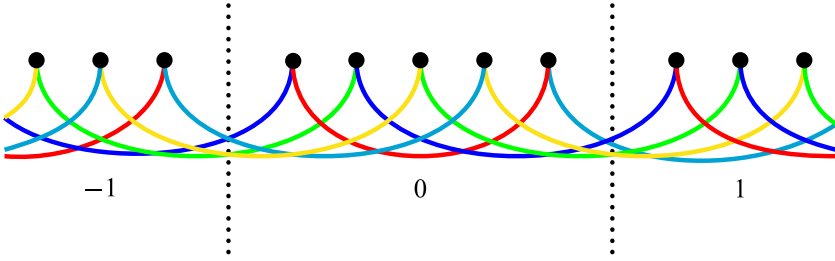


Fig. 18. Matching paths on the universal cover

where

$$\overline{V} = \text{span}\{e_1, \dots, e_n\}$$

is an  $n$ -dimensional vector space, an element of  $\wedge^k \overline{V}$  has degree  $k$ , and the  $A_\infty$ -operation is given by the wedge product. Then one has

$$\text{hom}_{\tilde{\mathcal{A}}}(\Delta_i, \Delta_j) = \begin{cases} \mathbb{C} \cdot \text{id}_i & i = j \\ \wedge^0 \overline{V} \oplus \wedge^n \overline{V} & i < j \text{ and } j \equiv i \pmod{n}, \\ \wedge^{\overline{j-i}} \overline{V} & i < j \text{ and } j \not\equiv i \pmod{n}, \\ 0 & \text{otherwise} \end{cases}$$

as a vector space, where  $0 \leq \overline{j-i} < n$  is a representative of  $[j-i] \in \mathbb{Z}/n\mathbb{Z}$ . The gradings of  $\Delta_i$  are chosen so that an element of  $\wedge^k V$  has degree  $k$ . The  $A_\infty$ -operations  $m_0$  and  $m_1$  vanish, and  $m_2$  is given by the wedge product as

$$m_2(\sigma, \tau) = (-1)^{\text{deg } \tau} \sigma \wedge \tau.$$

We write the elements of  $\text{hom}(\Delta_i, \Delta_{i+n})$  corresponding to  $1 \in \wedge^0 \overline{V}$  and  $e_1 \wedge \dots \wedge e_n$  as  $\text{id}_{i,i+n}$  and  $\text{id}_{i,i+n}^\vee$  respectively.

By symplectic Picard–Lefschetz theory recalled in Sect. 3, there is a cohomologically full and faithful functor

$$\mathfrak{F}ut W^{-1}(0)^\sim \rightarrow D^b \tilde{\mathcal{A}},$$

which maps the objects as

$$C_i \mapsto \left\{ \Delta_i \xrightarrow{\text{id}_{i,i+n}} \Delta_{i+n} \right\}.$$

Then one has

$$\begin{aligned} \text{hom}(C_i, C_j) &= \text{hom} \left( \left\{ \Delta_i \xrightarrow{\text{id}_{i,i+n}} \Delta_{i+n} \right\}, \left\{ \Delta_j \xrightarrow{\text{id}_{j,j+n}} \Delta_{j+n} \right\} \right) \\ &= \left\{ \begin{array}{ccc} \text{hom}(\Delta_{i+n}, \Delta_j) & \xrightarrow{(-1)^{\text{deg } \bullet - 1} m_2(\bullet, \text{id}_{i,i+n})} & \text{hom}(\Delta_i, \Delta_j) \\ \downarrow -m_2(\text{id}_{j,j+n}, \bullet) & & m_2(\text{id}_{j,j+n}, \bullet) \downarrow \\ \text{hom}(\Delta_{i+n}, \Delta_{j+n}) & \xrightarrow{(-1)^{\text{deg } \bullet - 1} m_2(\bullet, \text{id}_{i,i+n})} & \text{hom}(\Delta_i, \Delta_{j+n}) \end{array} \right\}, \end{aligned}$$

where the last line denotes the total complex of the double complex. If  $j < i - 3$ , then every term in the last line of the right hand side is trivial. If  $i - n \leq j \leq i - 1$ , then the right hand side is given by

$$\left\{ \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \wedge^{j-i+n} \overline{V} \end{array} \right\}$$

which is spanned by

$$\left( \begin{array}{ccc} & \{\Delta_i & \longrightarrow & \Delta_{i+n}\} \\ & \tau \downarrow & & \\ \{\Delta_j & \longrightarrow & \Delta_{j+n}\} & \end{array} \right) \in \text{hom}^1(C_i, C_j)$$

for  $\tau \in \wedge^{j-i+n} \overline{V}$ . If  $i = j$ , then the complex on the right hand side is given by

$$\left\{ \begin{array}{ccc} 0 & \longrightarrow & \mathbb{C} \cdot \text{id}_{\Delta_i} \\ \downarrow & & \downarrow \\ \mathbb{C} \cdot \text{id}_{\Delta_{i+n}} & \longrightarrow & \mathbb{C} \cdot \text{id}_{i,i+n} \oplus \mathbb{C} \cdot \text{id}_{i,i+n}^\vee \end{array} \right\}$$

whose cohomology group is spanned by

$$\left( \begin{array}{ccc} \{\Delta_i & \longrightarrow & \Delta_{i+n}\} \\ \text{id}_{\Delta_i} \downarrow & & \text{id}_{\Delta_{i+n}} \downarrow \\ \{\Delta_i & \longrightarrow & \Delta_{i+n}\} \end{array} \right) \in \text{hom}^0(C_i, C_j)$$

and

$$\left( \begin{array}{ccc} & \{\Delta_i & \longrightarrow & \Delta_{i+n}\} \\ & \text{id}_{i+n,i}^\vee \downarrow & & \\ \{\Delta_i & \longrightarrow & \Delta_{i+n}\} & \end{array} \right) \in \text{hom}^1(C_i, C_j).$$

If  $i + 1 \leq j \leq i + n - 1$ , then the complex on the right hand side is given by

$$\left\{ \begin{array}{ccc} 0 & \longrightarrow & \wedge^{j-i} \overline{V} \\ \downarrow & & \downarrow \\ \wedge^{j-i} \overline{V} & \longrightarrow & \wedge^{j-i} \overline{V} \end{array} \right\}$$

whose cohomology group is spanned by

$$\begin{array}{ccc} \{\Delta_i & \longrightarrow & \Delta_{i+n}\} \\ \tau \downarrow & & (-1)^{j-i} \tau \downarrow \\ \{\Delta_i & \longrightarrow & \Delta_{i+n}\} \end{array}$$

for  $\tau \in \wedge^{j-i} \bar{V}$ . If  $j = i + n$ , then the complex on the right hand side is given by

$$\left\{ \begin{array}{ccc} \mathbb{C} \cdot \text{id}_{\Delta_{i+n}} & \longrightarrow & \mathbb{C} \cdot \text{id}_{i+n, i+2n} \oplus \mathbb{C} \cdot \text{id}_{i+n, i+2n}^{\vee} \\ \downarrow & & \downarrow \\ \mathbb{C} \cdot \text{id}_{i+n, i+n} \oplus \mathbb{C} \cdot \text{id}_{i+n, i+2n}^{\vee} & \longrightarrow & \mathbb{C} \cdot \text{id}_{i, i+2n} \oplus \mathbb{C} \cdot \text{id}_{i, i+2n}^{\vee} \end{array} \right\},$$

whose cohomology group is spanned by

$$\begin{array}{ccc} \{\Delta_i & \longrightarrow & \Delta_{i+n}\} \\ \text{id}_{i, i+n}^{\vee} \downarrow & & \downarrow (-1)^{n-1} \text{id}_{i+n, i+2n}^{\vee} \\ \{\Delta_{i+n} & \longrightarrow & \Delta_{i+2n}\}. \end{array}$$

If  $j > i + n$ , then the complex on the right hand side is acyclic.

If we write

$$C_{i,j} = C_{i+(n+1)j}, \quad i = 1, \dots, n + 1 \text{ and } j \in \mathbb{Z},$$

then the above calculation can be summarized as

$$\text{hom}(C_{i,j}, C_{i',j'}) = (\wedge^{i'-i} V \otimes \rho_{j'-j})^{\mathbb{T}_n}, \quad 1 \leq i < i' \leq n + 1,$$

where  $\mathbb{T}_n = \mathbb{C}^\times$  is an algebraic torus,

$$\begin{array}{ccc} \rho_i : \mathbb{T}_n & \rightarrow & \mathbb{C}^\times \\ \psi & & \psi \\ \alpha & \mapsto & \alpha^i \end{array}$$

is an irreducible representation of  $\mathbb{T}_n$ ,

$$V = \rho_0 \oplus \dots \oplus \rho_0 \oplus \rho_1,$$

is an  $(n + 1)$ -dimensional representation of  $\mathbb{T}_n$ , and  $\bullet^{\mathbb{T}_n}$  denotes the subspace of  $\mathbb{T}_n$ -invariants.

By descending from  $W^{-1}(0)^\sim$  to  $W^{-1}(0)$  and taking the directed subcategory, one obtains

$$\text{hom}_{\mathfrak{F}\text{u}\mathfrak{k} W}(C_i, C_j) = \begin{cases} \mathbb{C} \cdot \text{id}_{C_i} & i = j, \\ \wedge^{j-i} V & i > j, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that the  $A_\infty$ -operation  $m_2$  on  $\mathfrak{F}\text{u}\mathfrak{k} W$  is given by wedge product. One can also show, either by direct calculation or for degree reasons, that  $A_\infty$ -operations  $m_k$  for  $k \neq 2$  on  $\mathfrak{F}\text{u}\mathfrak{k} W$  vanishes, and Theorem 6.1 is proved.  $\square$

In the proof of Theorem 6.1, we have thrown away the extra information obtained by lifting from  $W^{-1}(0)$  to its  $\mathbb{Z}$ -cover  $W^{-1}(0)^\sim$  at each step of the induction. One can also keep this information, and the resulting category can be described as follows:



**Theorem 6.2** *Let*

$$\tilde{W} = W \circ \exp : \mathbb{C}^n \rightarrow \mathbb{C}$$

*be the pull-back of the mirror  $W$  of  $\mathbb{P}^n$  by the  $\mathbb{Z}^n$ -covering given by the exponential map*

$$\exp : \mathbb{C}^n \rightarrow (\mathbb{C}^\times)^n.$$

*Let  $C_{i,\mathbf{j}}$  denote the  $\mathbf{j}$ -th lift of  $C_i$  for  $i = 1, \dots, n+1$  and  $\mathbf{j} \in \mathbb{Z}^n$ . Then one has*

$$\text{hom}(C_{i,\mathbf{j}}, C_{i',\mathbf{j}'}) = \begin{cases} \mathbb{C} \cdot \text{id}_{C_{i,\mathbf{j}}} & i = i' \text{ and } \mathbf{j} = \mathbf{j}', \\ (\wedge^{i'-i} V \otimes \rho_{\mathbf{j}'-\mathbf{j}})^\mathbb{T} & i < i', \\ 0 & \text{otherwise,} \end{cases}$$

*where  $V$  is an  $(n+1)$ -dimensional vector space with an action of an algebraic torus  $\mathbb{T} = (\mathbb{C}^\times)^n$  given by*

$$\begin{array}{ccc} \mathbb{T} \ni (\alpha_1, \dots, \alpha_n) : & \mathbb{C}^{n+1} & \rightarrow & \mathbb{C}^{n+1} \\ & \downarrow \Psi & & \downarrow \Psi \\ & (x_0, x_1, \dots, x_n) & \mapsto & (x_0, \alpha_1 x_1, \dots, \alpha_n x_n), \end{array}$$

*and*

$$\begin{array}{ccc} \rho_{\mathbf{j}} : & \mathbb{T} & \rightarrow & \mathbb{C}^\times \\ & \downarrow \Psi & & \downarrow \Psi \\ & (\alpha_1, \dots, \alpha_n) & \mapsto & (\alpha_1^{j_1}, \dots, \alpha_n^{j_n}) \end{array}$$

*is a one-dimensional representation of  $\mathbb{T}$  for  $\mathbf{j} = (j_1, \dots, j_n)$ .*

The proof of Theorem 6.2 is completely parallel to that of Theorem 6.1.

## 7. Tropical Coamoeba

We introduce the notion of a tropical coamoeba and prove Theorem 1.5 in this section. A tropical coamoeba is a generalization of a pair of a dimer model and an internal perfect matching on it to higher dimensions. See [UY11, UY, UY13, FU10] and references therein for dimer models and its application to homological mirror symmetry.

**Definition 7.1** *A tropical coamoeba  $G = ((P_i)_{i=1}^m, \text{deg}, \text{sgn})$  of a Laurent polynomial  $W : (\mathbb{C}^\times)^n \rightarrow \mathbb{C}$  consists of*

- a polyhedral decomposition

$$T = \bigcup_{i=1}^m P_i,$$

*of a real  $n$ -torus  $T = \mathbb{R}^n / \mathbb{Z}^n$  into an ordered set  $(P_i)_{i=1}^m$  of polytopes,*

- a map

$$\text{deg} : F_1 \rightarrow \mathbb{Z}$$

*from the set  $F_1$  of facets to  $\mathbb{Z}$  called the *degree*, and*

- a map

$$\text{sgn} : F_2 \rightarrow \{1, -1\}$$

from the set  $F_2$  of codimension two faces called the *sign*,

satisfying the following:

- There is a CW complex  $Y$  in  $W^{-1}(0)$  and a deformation retraction

$$F : W^{-1}(0) \times [0, 1] \rightarrow W^{-1}(0),$$

$$F(\bullet, 0) = \text{id}_{W^{-1}(0)}, \quad \text{Im } F(\bullet, 1) = Y, \quad F(\bullet, 1)|_Y = \text{id}_Y,$$

such that the restriction of  $F(\bullet, 1)$  to the union of a distinguished basis  $(C_i)_{i=1}^m$  of vanishing cycles is a surjection onto  $Y$ .

- The argument map  $\text{Arg} : (\mathbb{C}^\times)^n \rightarrow T$  induces a homeomorphism  $Y \xrightarrow{\sim} \bigcup_{f \in F_1} f$  into the union of facets.
- The boundary of the polytope  $P_i$  is the image of the vanishing cycle  $C_i$ ;

$$\text{Arg}(F(C_i, 1)) = \partial P_i, \quad i = 1, \dots, m.$$

- There is a natural one-to-one correspondence between the set of common facets of  $P_i$  and  $P_j$  and intersection points of  $C_i$  and  $C_j$ , and the degree function is given by the Maslov index of the intersection with respect to suitable gradings of  $W^{-1}(0)$  and  $(C_i)_{i=1}^m$ .
- For each codimension two face  $e \in F_2$ , one has an  $A_\infty$ -operation

$$m_k(f_1, \dots, f_k) = \text{sgn}(e) f_0 \tag{7.1}$$

in the Fukaya category  $\mathfrak{Fuk} W$ , where  $(f_0, f_1, \dots, f_k)$  is the set of facets around  $e$ , identified with intersections of vanishing cycles as above. Moreover, any non-trivial  $A_\infty$ -operation in  $\mathfrak{Fuk} W$  comes from a codimension two face of  $P_i$  in this way.

- Let  $\tilde{W} = W \circ \text{exp}$  be the pull-back of  $W$  by the universal covering map  $\text{exp} : \mathbb{C}^n \rightarrow (\mathbb{C}^\times)^n$ . Then the pull-back  $\tilde{G}$  of  $G$  to the universal cover  $\mathbb{R}^n \rightarrow T$  gives a tessellation of  $\mathbb{R}^n$ , which encodes the information of  $\mathfrak{Fuk} \tilde{W}$  in just the same way as above, so that polytopes, facets, and codimension two faces correspond to vanishing cycles of  $\tilde{W}$ , their intersection points, and  $A_\infty$ -operations respectively.

It follows from the definition that if  $G$  is a tropical coamoeba of  $W$ , then one can associate a directed  $A_\infty$ -categories  $\mathcal{A}_G$  whose set of objects, a basis of the space of morphisms, and non-trivial  $A_\infty$ -operations on this basis are given by the set of polytopes, the set of facets, and the set of codimension two faces respectively, which satisfies

$$\mathfrak{Fuk} W \cong \mathcal{A}_G.$$

Moreover, the directed  $A_\infty$ -category  $\mathcal{A}_{\tilde{G}}$  associated with the pull-back  $\tilde{G}$  of  $G$  to the universal cover is equivalent to the Fukaya category associated with  $\tilde{W}$ ;

$$\mathfrak{Fuk} \tilde{W} \cong \mathcal{A}_{\tilde{G}}.$$

Now we prove Theorem 1.5. We first discuss the case of  $\mathbb{P}^2$  along the lines of [UY13]. The mirror of  $\mathbb{P}^2$  is given by

$$W(x, y) = x + y + \frac{1}{xy},$$

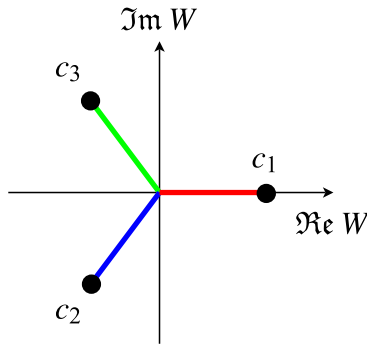


Fig. 19. A distinguished set of vanishing paths

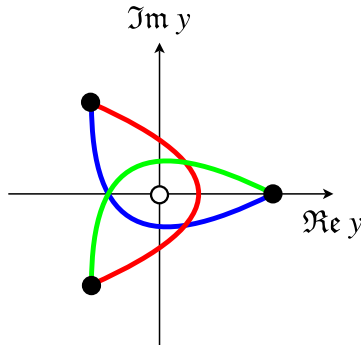


Fig. 20. The trajectories of the branch points

which has three critical values  $3, 3\omega$  and  $3\omega^2$ . Choose a distinguished set  $(c_i)_{i=1}^3$  of vanishing paths as the straight line segments from the origin to each critical values as in Fig. 19. The  $y$ -projection

$$\begin{array}{ccc} \varpi_t : W^{-1}(t) & \rightarrow & \mathbb{C}^\times \\ \downarrow & & \downarrow \\ (x, y) & \mapsto & y \end{array}$$

has three branch points, which moves as shown in Fig. 20 along the vanishing paths. The trajectories of these branch points are images of vanishing cycles by  $\varpi = \varpi_0$ . There are six disks in  $W^{-1}(0)$  bounded by these vanishing cycles, which are projected onto three triangles in Fig. 20. By contracting these six disks, one obtains a graph on  $W^{-1}(0)$  whose  $\pi$  projection is shown in Fig. 21. Figure 22 shows a schematic picture of the image of this graph by the argument map. Here, the color scheme in Figs. 21 and 22 is not a continuation of the scheme introduced in Figs. 19 and 20. The horizontal and the vertical axes in Fig. 22 correspond to  $\arg y$  and  $\arg x$  respectively. The inverse image of the circle on the  $y$ -plane in Fig. 21 by  $\varpi_0$  is a non-trivial double cover of it, which maps to a cycle in the class  $(2, -1) \in H_1(T, \mathbb{Z}) \cong \mathbb{Z}^2$  shown in black in Fig. 22. Three legs in Fig. 21 connect two branches of the double cover  $\varpi_0$ , which map to vertical line segments in Fig. 22. As a result, one obtains the division of  $T$  into three hexagons as shown in Fig. 23. It is easy to see that the set of edges in Fig. 23 corresponds to the set of intersection points of vanishing cycles, and the set of nodes corresponds to holomorphic

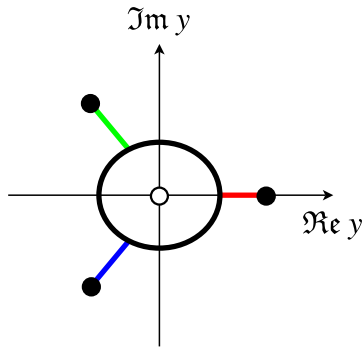


Fig. 21. Contracting  $W^{-1}(0)$

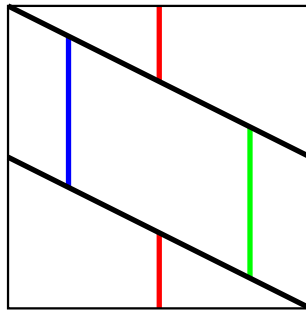


Fig. 22. Image of the contraction by the argument map

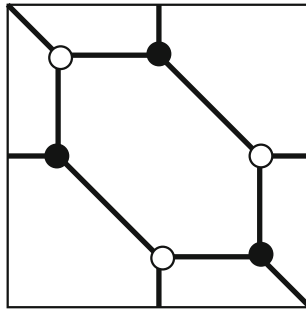


Fig. 23. The honeycomb tiling

disks bounded by vanishing cycles. The colors of the nodes correspond to the signs of the  $A_\infty$ -operations.

Now we discuss the case of  $\mathbb{P}^3$ . By contracting the matching paths in Fig. 6, one obtains a circle with four legs shown in Fig. 24. The fiber of  $\varpi_0$  over a point on this circle is symplectomorphic to  $\overline{W}^{-1}(0)$ , which can be contracted to the honeycomb graph in Fig. 23 as explained above. As one goes around the circle, this honeycomb graph undergoes a monodromy (Fig. 25)

$$D_1 \mapsto D_2 \mapsto D_3 \mapsto D_1$$

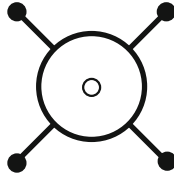


Fig. 24. Contraction on the  $z$ -plane

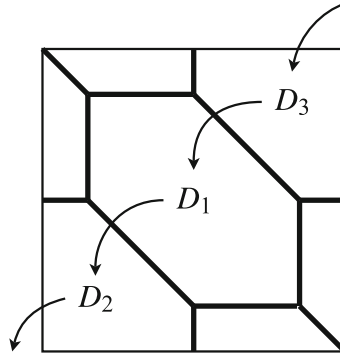


Fig. 25. The monodromy around the origin

of order three, where  $D_i$  is the face in the honeycomb graph corresponding to the  $i$ -th vanishing cycle of  $\varpi_0$ . The image by the argument map of this honeycomb graph bundle over the circle on the  $z$ -plane divides the 3-torus  $T$  into an obliquely-embedded hexagonal cylinder. Four legs in Fig. 24 give four faces perpendicular to the  $\arg z$ -axis, which cut this hexagonal cylinder into four truncated octahedra  $(P_i)_{i=1}^4$ .

A *truncated octahedron* is the polytope with fourteen faces, thirty-six edges and twenty-four vertices, which is obtained from an octahedron by truncating at its six vertices. One of the four truncated octahedra in  $T$  is shown in Fig. 26, where we have chosen to draw  $\arg x$  and  $\arg y$  horizontally, and  $\arg z$  vertically. By pulling back this division of  $T$  into four truncated octahedra to the universal cover  $\mathbb{R}^3 \rightarrow T$ , one obtains the *bitruncated cubic honeycomb*, which is the Voronoi tessellation for the body-centered cubic lattice.

It is straightforward to see that intersections of vanishing cycles and  $A_\infty$ -operations in Fukaya category correspond to faces and edges of truncated octahedra respectively, so that the decomposition of  $T$  into four truncated octahedra, together with a suitable choice of the functions  $\mu$  and  $\text{sgn}$ , gives a tropical coamoeba of  $\tilde{W}$ : Matching paths are contracted as in Figs. 27 and 28 shows the intersections of the matching path  $\mu_1$  for  $C_1$  with three other matching paths. These intersections correspond to faces of  $P_1$  shown in Figs. 29, 30, and 31, which can be seen to be in natural bijection with intersection points of  $C_1$  with  $C_2, C_3$  and  $C_4$  by comparing with the discussion in Sect. 5. It is also straightforward to see that the edges of  $P_i$  corresponds to  $A_\infty$ -operations in  $\mathfrak{Fuc} \tilde{W}$ ; for example, twelve edges corresponding to

$$m_2 : \text{hom}^1(C_2, C_3) \otimes \text{hom}^1(C_1, C_2) \rightarrow \text{hom}^2(C_1, C_3)$$

are shown in Fig. 32.

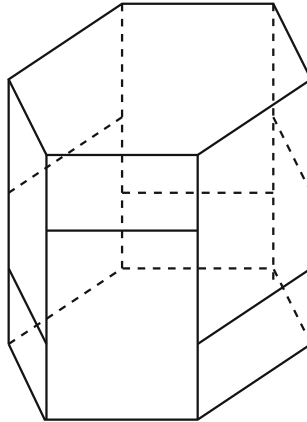


Fig. 26. A truncated octahedron

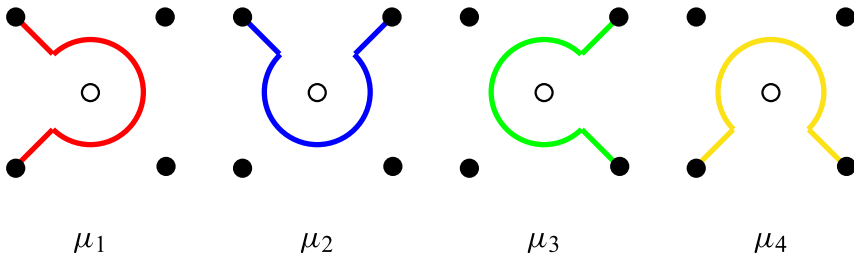


Fig. 27. Contractions of the matching paths

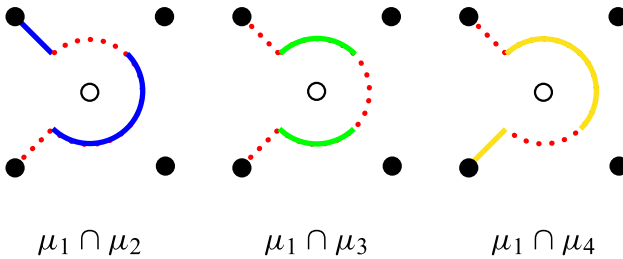


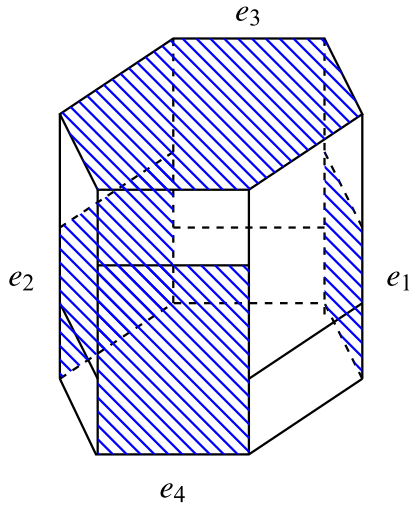
Fig. 28. Intersections of contracted matching paths

Now we discuss the general case. The *permutohedron* of order  $n + 1$  is an  $n$ -dimensional polytope lying on the hyperplane

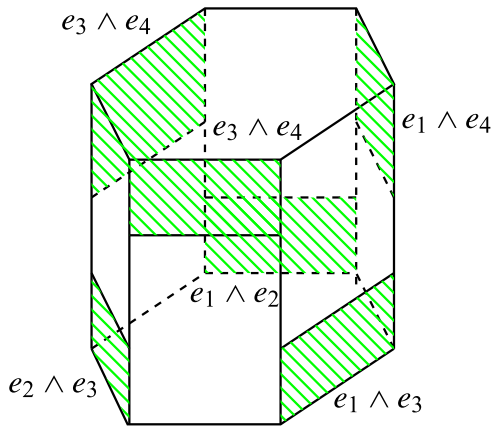
$$H = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \dots + x_{n+1} = \frac{n(n+1)}{2} \right\},$$

defined as the convex hull of the orbit of  $(1, 2, \dots, n + 1) \in \mathbb{R}^{n+1}$  under the action of the symmetric group  $\mathfrak{S}_{n+1}$  by permutations of coordinates. Note that the permutohedron of order three is a hexagon, and the permutohedron of order four is a truncated octahedron. A facet of a permutohedron of order  $n$  corresponds to a division

$$B_1 \sqcup B_2 = \{1, 2, \dots, n + 1\}$$



**Fig. 29.** The facets of  $P_1$  adjacent to  $P_2$



**Fig. 30.** The facets of  $P_1$  adjacent to  $P_3$

of the set  $\{1, 2, \dots, n + 1\}$  into the disjoint union of two subsets, and a codimension two face corresponds to a division

$$B_1 \sqcup B_2 \sqcup B_3 = \{1, 2, \dots, n + 1\}$$

into the disjoint union of three subsets. The facet corresponding to the division  $B_1 \sqcup B_2$  is given by

$$\sum_{i \in B_1} x_i = 1 + 2 + \dots + \#B_1,$$

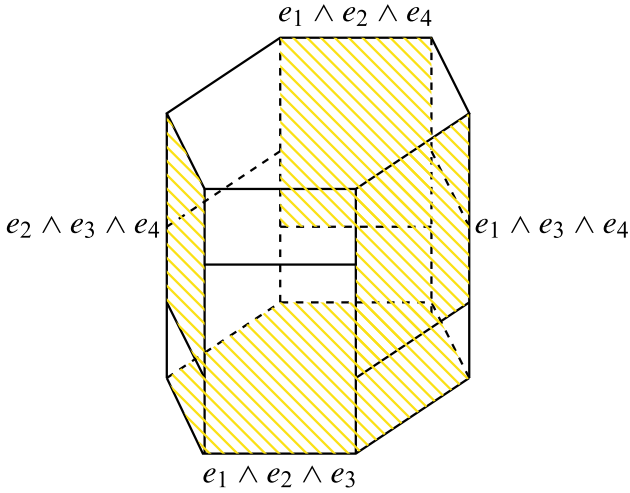


Fig. 31. The facets of  $P_1$  adjacent to  $P_4$

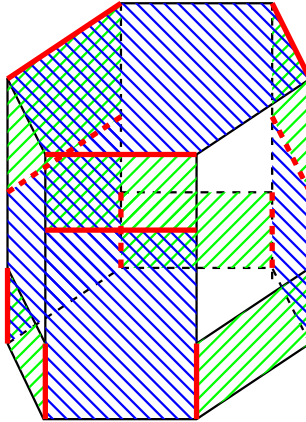


Fig. 32. The edges of  $P_1$  adjacent to  $P_2$  and  $P_3$

and the codimension two face corresponding to the division  $B_1 \sqcup B_2 \sqcup B_3$  is given by

$$\sum_{i \in B_1} x_i = 1 + 2 + \dots + \#B_1,$$

$$\sum_{i \in B_1 \sqcup B_2} x_i = 1 + 2 + \dots + \#(B_1 \sqcup B_2),$$

so that the inclusion of a face into a facet corresponds to a subdivision of a division of length two into a division of length three. The translations of the permutohedron of order  $n + 1$  by the lattice of rank  $n$  generated by

$$\ell_i = (n + 1)e_i - (e_1 + \dots + e_{n+1}), \quad i = 1, \dots, n + 1,$$

where  $e_i$  is the  $i$ -th coordinate vector, tessellates the hyperplane  $H$ . The polytope adjacent to the permutohedron through the facet corresponding to the division  $B_1 \sqcup B_2$  is the



translate of the permutohedron by

$$\sum_{i \in B_2} \ell_i.$$

Every codimension two face of this tessellation is adjacent to three facets, corresponding to  $B_1 \sqcup B_2$ ,  $B'_1 \sqcup B'_2$  and  $B''_1 \sqcup B''_2$  such that  $B''_2 = B_2 \sqcup B'_2$ .

The set of facets of the permutohedron of order  $n + 1$  maps bijectively to a basis of  $\wedge^\bullet V / (\wedge^0 V \oplus \wedge^{n+1} V)$  by

$$B_1 \sqcup B_2 \mapsto \wedge_{i \in B_2} e_i.$$

Under this correspondence, the translates of three facets share a codimension two face if and only if they correspond to  $u, v$  and  $w$  in  $\wedge^\bullet V / (\wedge^0 V \oplus \wedge^{n+1} V)$  such that  $w = \pm u \wedge v$ .

Now we inductively show that the quotient of the above tessellation by the lattice  $\Lambda \cong \mathbb{Z}^n$  generated by

$$\ell_i + (\ell_1 + \cdots + \ell_n), \quad i = 1, \dots, n$$

is a tropical coamoeba for the mirror of  $\mathbb{P}^n$ . By contracting the union of the matching paths for  $\varpi_0$  on the  $x_n$ -plane, one obtains a circle  $S$  with  $n + 1$  legs  $\{l_1, \dots, l_{n+1}\}$ , numbered clockwise. The fiber over a point on  $S$  can be contracted to the union of  $n$  permutohedra  $\{\overline{P}_i\}_{i=1}^n$  of order  $n$  by induction hypothesis, which undergoes the cyclic monodromy

$$\overline{P}_i \mapsto \overline{P}_{i+1}, \quad i = 1, \dots, n$$

as one goes around the circle. Its image by the argument map gives a division of  $T^n$  into an oblique cylinder over  $\overline{P}_1$ , which is divided into  $n + 1$  permutohedra  $\{P_i\}_{i=1}^{n+1}$  of order  $n + 1$  by the  $n + 1$  facets coming from  $n + 1$  legs: Let us call the direction of  $\arg x_n$  vertical and other directions horizontal. The  $x_n$ -projection of the contracted vanishing cycle consists of two legs  $l_i, l_{i+n}$  and the part of the circumference between them. The horizontal facets corresponding to  $l_i$  and  $l_{i+n}$  corresponds to  $e_{n+1}$  and  $e_1 \wedge \cdots \wedge e_n$  respectively. There are  $2^n - 2$  vertical facets of the cylinder, and the one corresponding to

$$e_{i_1} \wedge \cdots \wedge e_{i_r}$$

is divided into two, one corresponding to

$$e_{i_1} \wedge \cdots \wedge e_{i_r}$$

and the other corresponding to

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e_{n+1}.$$

As a whole, one obtains  $2^{n+1} - 2$  facets, and  $P_1$  can be identified with the permutohedron of order  $n + 1$ . Under this identification,  $P_i$  can be identified with the translation of  $P_1$  by  $\ell_{n+1}$ , and the union  $\bigcup_{i=1}^{n+1} P_i$  is a fundamental region of the lattice  $\Lambda$ . The degree function takes the value  $|B_2|$  on the facet corresponding to the division  $B_1 \sqcup B_2$ , and the value  $\text{sgn}(B'_2, B_2)$  of the sign function on the codimension two face  $f$  of  $P_i$ , where the

facet of  $P_i$  corresponding to  $B_1 \sqcup B_2$  intersects  $P_j$  and the facet of  $P_j$  corresponding to the division  $B'_1 \sqcup B'_2$  intersects  $P_k$  for  $i < j < k$ , is given by

$$\wedge_{i \in B_2 \sqcup B'_2} e_i = \text{sgn}(B'_2, B_2) \cdot (-1)^{|B_2|} (\wedge_{i \in B'_2} e_i) \wedge (\wedge_{i \in B_2} e_i).$$

The  $A_\infty$ -category  $\mathcal{A}_G$  associated with the tropical coamoeba

$$G = ((P_i)_{i=1}^{n+1}, \text{deg}, \text{sgn})$$

defined above is quasi-equivalent to the full subcategory of a standard differential graded enhancement of  $D^b \text{coh } \mathbb{P}^n$  consisting of

$$(E_1, E_2, \dots, E_{n+1}) = (\Omega_{\mathbb{P}^n}^n(n)[n], \Omega_{\mathbb{P}^n}^{n-1}(n-1)[n-1], \dots, \mathcal{O}_{\mathbb{P}^n}).$$

This implies the equivalence

$$D^b \mathcal{A}_G \cong D^b \text{coh } \mathbb{P}^n$$

of triangulated categories, since  $(E_1, \dots, E_{n+1})$  is a full exceptional collection by Beilinson [Bei78]. It is clear that this equivalence lifts to the equivalence

$$D^b \mathcal{A}_{\tilde{G}} \cong D^b \text{coh}^{\mathbb{T}} \mathbb{P}^n$$

by sending the object of  $\mathcal{A}_{\tilde{G}}$  corresponding to the  $\mathbf{j}$ -th lift of  $P_i$  for  $\mathbf{j} \in \Lambda \cong \mathbb{Z}^n$  to  $E_i \otimes \rho_{\mathbf{j}}$ , and Theorem 1.5 is proved. Theorem 1.2 is an immediate consequence of Theorem 1.5, which in turn implies Corollary 1.3 just as in the two-dimensional case [UY13].

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