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Partial Regularity of Suitable Weak Solutions to the Fractional Navier–Stokes Equations

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Abstract: In this paper, we study the partial regularity of fractional Navier–Stokes equations in $\mathbb{R}^3 \times (0, \infty)$ with $3/4 < s < 1$. We show that the suitable weak solution is regular away from a relatively closed singular set whose $(5-4s)$-dimentional Hausdorff measure is zero. The result is a generalization of the partial regularity for the classical Navier–Stokes system in Caffarelli et al. (Commun Pure Appl Math 35:771–831, 1982).

1. Introduction

The main purpose of this work is to develop a regularity theory for the incompressible fractional Navier–Stokes equation defined as follows:

$$\begin{cases} \partial_t u + (-\Delta)^s u + u \cdot \nabla u = -\nabla p + f, & x \in \mathbb{R}^3,\ t > 0, \\ \operatorname{div} u = 0, & x \in \mathbb{R}^3,\ t > 0, \end{cases} \tag{1.1}$$

where u is an unkown velocity vector field, p is pressure and f is a given force with $\nabla \cdot f = 0$. Throughout this article, we assume that $3/4 < s < 1$ and define $a = 1 - 2s$. As a convention in this work, $\overline{\nabla}$ and ∇ are the gradients on $\mathbb{R}^4_+$ and $\mathbb{R}^3$, respectively. Their associated divergence operators are denoted by Div and div, respectively.

If $0 < s < 1$, the fractional Navier–Stokes system (1.1) is an important mathematical model arising from the physical world. In [9], this equation is used to describe a fluid motion with internal friction interaction. In the viewpoint of the stochastic process, Zhang [19] showed that (1.1) can also be deduced via the stochastic Lagrangian particle approach.

For the classical Navier–Stokes equation ($s = 1$), it is well known that the global regularity of its solution is still an open problem and there are only some partial regularity results. In a series of work [11–13], Scheffer studied some class of weak solutions which satisfy the so-called local energy inequality. He proved that when $f = 0$, such weak solutions might have a singular set with finite 5/3-dimensional Hausdorff measure. Later in 1982, a prestigious improvement of Scheffer's results was made by Caffarelli et al. [1], where they showed that for any so-called suitable weak solution (u, p), the associated singular set has one-dimensional Hausdorff measure zero (also see Lin [7] for a simplified proof). As for the high-order fractional Navier–Stokes equations where $s > 1$, some important results on the regularity of solutions have been given since the 1960s. Firstly, Lions [8] showed that if $s \geq 5/4$, (1.1) has a unique global smooth solution for any prescribed smooth initial data (also see [6,16,18]). And for the case $1 < s < 5/4$, Katz and Pavlović showed in [6] that the Hausdorff dimension of the singular set at the time of first blow-up is at most $5 - 4s$. When $0 < s < 1$, (1.1) is completely different from the cases mentioned above. In our previous work [15], we firstly studied steady suitable weak solutions to (1.1). We showed that the solutions are regular away from a compact set whose (5 − 6s)-Hausdoff measure is zero if $1/2 < s < 5/6$. They are regular if $s \geq 5/6$. For more related works, we refer readers to [3,4].

Before proceeding, we give some definitions and notions which will be used in this article. Let $\dot{\mathrm{H}}^s_{\mathrm{div}}$ be the closure of the set $\left\{u \in \mathrm{C}^\infty_\mathrm{c}\left(\mathbb{R}^3; \mathbb{R}^3\right) : \operatorname{div} u = 0 \text{ in } \mathbb{R}^3\right\}$ under the norm defined by

$$\|u\|_{\dot{\mathrm{H}}^s} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

According to [2], for any $u \in \dot{\mathrm{H}}^s_{\mathrm{div}}$, there is an extension, denoted by u^*, of u on $\mathbb{R}^4_+$, such that

$$\begin{cases} \mathrm{Div}\left(\mathrm{y}^a\overline{\nabla}u^*\right) = 0, \quad (x, \mathrm{y}) \in \mathbb{R}^4_+ \\ \quad u^*(x, 0) = u(x), \quad x \in \mathbb{R}^3 \\ -\mathrm{C}_s \lim_{\mathrm{y}\to 0^+} \mathrm{y}^a \partial_\mathrm{y} u^* = (-\Delta)^s u(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where C_s is a constant depending only on s.With this extension, we define suitable weak solutions to (1.1) in the following:

Definition 1.1. Suppose that on an arbitary interval $I \subset (0, \infty)$, the force $f : \mathbb{R}^3 \times I \to \mathbb{R}^3$ is divergent free spatially in the sense of distribution. Moreover f belongs to $L^q(\mathbb{R}^3 \times I)$ for some $q > (9 + 6s)/(4s + 1)$. A pair (u, p) is called a suitable weak solution of (1.1) on $\mathbb{R}^3 \times I$ if the following conditions are satisfied:

(1) The functions $u : \mathbb{R}^3 \times I \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \times I \to \mathbb{R}$ satisfy

$$u \in L^\infty(I; L^2(\mathbb{R}^3)) \cap L^2(I; \dot{\mathrm{H}}^s_{\mathrm{div}}), \quad p \in L^{3/2}(\mathbb{R}^3 \times I).$$

(2) u, p and f satisfy the Eq. (1.1) in the distribution sense.
(3) For each nonnegative smooth function $\psi(x, \mathrm{y}, t)$ with compact support in $\mathbb{R}^3 \times I$ and $t_1, t \in I$ with $t_1 < t$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^3\times\{t\}} |u|^2\psi + 2\mathrm{C}_s \int_{t_1}^t \int_{\mathbb{R}^4_+} \psi\, \mathrm{y}^a|\overline{\nabla}u^*|^2 \\ &\leq \mathrm{C}_s \int_{t_1}^t \int_{\mathbb{R}^4_+} |u^*|^2\ \mathrm{Div}\left(\mathrm{y}^a\overline{\nabla}\psi\right) + \int_{t_1}^t \int_{\mathbb{R}^3} (u\cdot\nabla\psi)\left(2p + |u|^2\right) \\ &\quad + \int_{t_1}^t \int_{\mathbb{R}^3} |u|^2\{\psi_t + \mathrm{C}_s \lim_{\mathrm{y}\to 0^+}\left(\mathrm{y}^a\partial_\mathrm{y}\psi\right)\} + 2\int_{t_1}^t\int_{\mathbb{R}^3} \psi f\cdot u. \end{aligned} \tag{1.3}$$

In fact, if u is regular enough (smooth), then the equality in (1.3) holds from integration by parts.

Our main results are as follows. In Sect. 2, by an inductive argument, we prove

Theorem 1.2. *Suppose that* $3/4 < s < 1$ *and* $f \in L^2 \bigcap L^q$ *for some* $q > (9+6s)/(4s+1)$. *If* (u, p) *is a suitable weak solution of* (1.1)*, then there exists a positive small constant* ϵ_0 *so that for any* $(x_0, t_0) \in \mathbb{R}^3 \times I$ *with*

$$\limsup_{r\to 0^+} r^{-5+4s} \int_{Q_r^*(x_0,t_0)} y^a \, |\overline{\nabla} u^*|^2 < \epsilon_0, \tag{1.4}$$

u is regular in a neighborhood of (x_0, t_0)*, where* $Q_r^*(x_0, t_0) = B_r(x_0) \times (0, r) \times (t_0 - \frac{1}{2}r^{2s},\ t_0 + \frac{1}{2}r^{2s})$. ϵ_0 *depends on the parameter* s.

We call $(x_0, t_0) \in \mathbb{R}^3 \times I$ a regular point of u if there is a neighborhood of (x_0, t_0) where u is essentially bounded. The complement set of all regular points is called the singular set and denoted by $\mathrm{Sing}(u)$. In fact, for the singular set of u, using Theorem 1.2, we have

Theorem 1.3. *With the assumptions in Theorem* 1.2*, the singular set* $\mathrm{Sing}(u)$ *is a relatively closed set in* $\mathbb{R}^3 \times I$ *with* $\mathcal{H}^{5-4s}\,(\mathrm{Sing}(u)) = 0$.

Remark. It is obvious that Theorems 1.2 and 1.3 have generalized the well-known results for the classical Navier–Stokes system in Caffarelli–Kohn–Nirenberg [1] to the fractional case. It is worth pointing out that the condition $3/4 < s < 1$ is crucial and the method used here does not work for the case $0 < s \le 3/4$.

The remaining sections of this paper are organized as follows. Section 2 is devoted to the complete proof of Theorem 1.2. In Sect. 3 we prove Theorem 1.3 using the standard covering argument. Finally the proof of the existence of suitable weak solution is given in Sect. 4.

The following notations are also used in this article. Given two quantities a and b, we write $a \lesssim b$ if there is a universal positive constant C such that $a \le C\, b$. $a \lesssim_s b$ means that there is a positive constant C_s, depending only on s, such that $a \le C_s\, b$. For any measurable set $A \subset \mathbb{R}^n$ and some function $f \in L^1(A)$, $⨍_A f = \frac{1}{|A|}\int_A f$ is the average of f over A. For any $x_0 \in \mathbb{R}^3$ and $t_0 \ge 0$, we define

$$Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - \frac{1}{2}r^{2s},\ t_0 + \frac{1}{2}r^{2s}); \quad Q_{r,+}(x_0) = B_r(x_0) \times (0,\ r);$$
$$Q_r^*(x_0, t_0) = B_r(x_0) \times (0,\ r) \times (t_0 - \frac{1}{2}r^{2s},\ t_0 + \frac{1}{2}r^{2s}).$$

For simplicity, we denote $Q_r(0,0)$, $Q_r^*(0,0)$ and $Q_{r,+}(0)$ by Q_r, Q_r^* and $Q_{r,+}$, respectively. If $c > 0$ is a constant, we define

$$c\,B_r(x_0) = B_{cr}(x_0); \quad c\,Q_r(x_0, t_0) = c\,B_r(x_0) \times (t_0 - \frac{1}{2}c^{2s}r^{2s},\ t_0 + \frac{1}{2}c^{2s}r^{2s});$$
$$c\,Q_{r,+}(x_0) = c\,B_r(x_0) \times (0, cr);$$
$$cQ_r^*(x_0, t_0) = c\,Q_{r,+}(x_0) \times (t_0 - \frac{1}{2}c^{2s}r^{2s},\ t_0 + \frac{1}{2}c^{2s}r^{2s}).$$

Meanwhile, for any $n \in \mathbb{N}$, we let $r_n = 2^{-n}$ and define $B^n(x_0) = B_{r_n}(x_0) \subset \mathbb{R}^3$, $Q^n(x_0, t_0) = Q_{r_n}(x_0, t_0)$, $B_*^n(x_0) = Q_{r_n,+}(x_0)$, $Q_*^n(x_0, t_0) = Q_{r_n}^*(x_0, t_0)$. We simply denote $B^n(0)$, $Q^n(0, 0)$, $B_*^n(0)$ and $Q_*^n(0, 0)$ by B^n, Q^n, B_*^n and Q_*^n, respectively.

2. L^∞-Estimate

In this section, our main goal is to prove Theorem 1.2. The proof consists of three parts. In Sect. 2.1 the preliminary energy estimate is given. In Sect. 2.2 we give the proof of Proposition 2.3, which is the key to the proof of Theorem 1.2. We conclude the proof of Theorem 1.2 in Sect. 2.3.

2.1. Preliminary energy estimate. In this section, we show the following estimate, a straightforward result of the energy inequality (1.3):

Lemma 2.1. *For any given $(x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+$ and $k_0 \in \mathbb{N}$, we have, for all $k > k_0$, that*

$$\sup_{-r_k^{2s} \le t \le 0} \fint_{B^k(x_0)} |u|^2 + r_k^{-5} \int_{Q_*^k(x_0,t_0)} y^a |u^*|^2 + r_k^{-3} \int_{Q_*^k(x_0,t_0)} y^a \, |\overline{\nabla} u^*|^2$$

$$\lesssim_s \; r_{k_0}^{-5} \int_{Q_*^{k_0}(x_0,t_0)} y^a |u^*|^2 + \fint_{Q^{k_0}(x_0,t_0)} |u|^2 + \sum_{l=k_0}^{k} r_l^{2s} \fint_{Q^l(x_0,t_0)} |f| \, |u|$$

$$+ \sum_{l=k_0}^{k} r_l^{-a} \left[\fint_{Q^l(x_0,t_0)} |u|^3 + |u| \cdot | p - \overline{p}_l| \right],$$

where $\overline{p}_l$ is the averge of p over the ball $B^l(x_0)$ in $\mathbb{R}^3$.

Proof. Let χ_1 be a cut-off function satisfying:

$$\chi_1 \equiv 1 \quad \text{on} \quad Q^{k_0+1}(x_0, t_0); \qquad \chi_1 \equiv 0 \quad \text{off} \quad 3/2 \, Q^{k_0+1}(x_0, t_0);$$

$$0 \le \chi_1 \le 1; \qquad \sum_{i=1}^{2} r_{k_0}^i \|\nabla_x^i \chi_1\|_{L^\infty} + r_{k_0}^{2s} \, \|\partial_t \chi_1\|_{L^\infty} \le C.$$

χ_2 is another smooth function such that the following conditions hold:

$$\chi_2 \equiv 1 \text{ on } [0, r_{k_0+1}]; \qquad \chi_2(y) \equiv 0, \ \forall \, y \ge 3 r_{k_0+1} \big/ 2; \quad 0 \le \chi_2 \le 1;$$

$$\sum_{i=1}^{2} r_{k_0}^i \|\chi_2^{(i)}\|_{L^\infty} \le C.$$

Here $\chi_2^{(i)}$ denotes the ith order derivative of χ_2. Using χ_1 and χ_2, we define $\chi(x, y, t) = \chi_1(x, t)\chi_2(y)$. For $k \ge k_0$, we let

$$\phi_k(x, y, t) = (t_0 + r_k^{2s} - t)^{-3/(2s)} \exp\left\{ - \frac{3(|x - x_0|^{2s} + y^{2s} + r_k^{2s})}{4s^2 C_s (t_0 + r_k^{2s} - t)} \right\}.$$

Now we insert $\psi = \psi_k = \chi \phi_k$ into (1.3). The left-hand side of (1.3) can be bounded from below by

$$2 \, C_s \int_{-\infty}^{\infty} \int_{\mathbb{R}_+^4} \chi \, \phi_k \, y^a \, |\overline{\nabla} u^*|^2 \ge \; C_s \, r_k^{-3} \int_{Q_*^k(x_0,t_0)} y^a \, |\overline{\nabla} u^*|^2, \qquad \forall \, k > k_0. \tag{2.1}$$

In order to bound the left-hand side of (2.1) from above, we need to estimate the four terms on the right-hand side of (1.3) with $\psi = \psi_k = \chi \phi_k$. For convenience we denote these terms by A, B, C, D, respectively. Without loss of generality, in the following we assume that $(x_0, t_0) = (0, 0)$.

(1) Estimate for the term A

Firstly we split the term A into four parts: $\mathrm{A} = \mathrm{A}.1 + \mathrm{A}.2 + \mathrm{A}.3 + \mathrm{A}.4$, where

$$\mathrm{A}.1 = a\,\mathrm{C}_s \int_{-\infty}^{\infty}\int_{\mathbb{R}^4_+} \mathrm{y}^a\, |u^*|^2\, \mathrm{y}^{-1}\, \chi_1\, (\chi_2'\,\phi_k + \chi_2 \partial_y \phi_k),$$

$$\mathrm{A}.2 = 2\,\mathrm{C}_s \int_{-\infty}^{\infty}\int_{\mathbb{R}^4_+} \mathrm{y}^a\, |u^*|^2\, \overline{\nabla}\chi \cdot \overline{\nabla}\phi_k,$$

$$\mathrm{A}.3 = \mathrm{C}_s \int_{-\infty}^{\infty}\int_{\mathbb{R}^4_+} \mathrm{y}^a |u^*|^2\, \phi_k\, \mathrm{Div}\left(\overline{\nabla}\chi\right), \quad \mathrm{A}.4 = \mathrm{C}_s \int_{-\infty}^{\infty}\int_{\mathbb{R}^4_+} |u^*|^2\, \chi\, \mathrm{Div}\left(\mathrm{y}^a\, \overline{\nabla}\phi_k\right).$$

The estimates for A.1–A.3 are similar. By the choices of the cut-off functions χ_1 and χ_2, all the derivatives of χ_1 and χ_2 are supported on $\mathrm{Q}^{k_0}\backslash \mathrm{Q}^{k_0+1}$ and $[r_{k_0+1}, r_{k_0}]$, respectively. Noticing that

$$\mathrm{A}.1 = a\,\mathrm{C}_s \int_{\mathrm{Q}^{k_0}}\int_{r_{k_0+1}}^{r_{k_0}} \mathrm{y}^a\, |u^*|^2\, \mathrm{y}^{-1}\, \chi_1\, (\chi_2'\,\phi_k + \chi_2 \partial_y \phi_k)$$

and $\phi_k \lesssim_s\, r_{k_0}^{-3}$, $\partial_y \phi_k \lesssim_s\, r_{k_0}^{-4}$, $\mathrm{y}^{-1} \le 2 r_{k_0}^{-1}$ if $y > r_{k_0+1}$, therefore it holds

$$\mathrm{A}.1 \lesssim_s\, r_{k_0}^{-5} \int_{\mathrm{Q}^{k_0}}\int_{r_{k_0+1}}^{r_{k_0}} \mathrm{y}^a\, |u^*|^2 \lesssim_s\, r_{k_0}^{-5} \int_{\mathrm{Q}^{k_0}_*} \mathrm{y}^a\, |u^*|^2.$$

Similarly, we can show that

$$\mathrm{A}.2 + \mathrm{A}.3 \lesssim_s\, r_{k_0}^{-5} \int_{\mathrm{Q}^{k_0}_*} \mathrm{y}^a\, |u^*|^2.$$

For the term A.4, the integral is on $\mathrm{Q}^{k_0}\backslash \mathrm{Q}^{k_0+1} \times [r_{k_0+1}, r_{k_0}]$. Hence by direct computation, we have

$$\mathrm{A}.4 \lesssim_s -r_k^{-5} \int_{\mathrm{Q}^k_*} \mathrm{y}^a\, |u^*|^2 + r_{k_0}^{-5} \int_{\mathrm{Q}^{k_0}_*} \mathrm{y}^a\, |u^*|^2.$$

Therefore we have, by applying the above estimates for A.1–A.4 to (1.3) and (2.1), that $\forall\, k > k_0$,

$$r_k^{-3} \int_{\mathrm{Q}^k_*} \mathrm{y}^a\, |\overline{\nabla} u^*|^2 + r_k^{-5} \int_{\mathrm{Q}^k_*} \mathrm{y}^a\, |u^*|^2 \lesssim_s\, r_{k_0}^{-5} \int_{\mathrm{Q}^{k_0}_*} \mathrm{y}^a\, |u^*|^2 + (\mathrm{B} + \mathrm{C} + \mathrm{D}). \tag{2.2}$$

(2) Estimate for the term B

We decompose B into two parts: $\mathrm{B} = \mathrm{B}.1 + \mathrm{B}.2$ where

$$\mathrm{B}.1 = \int_{Q^{k_0}} |u|^2 + u \cdot \nabla\psi_k \quad \text{and} \quad \mathrm{B}.2 = 2 \int_{Q^{k_0}} pu \cdot \nabla\psi_k.$$

By the definition of χ_1, B.1 can be estimated by

$$\mathrm{B}.1 \le \sum_{l=k_0}^{k-1} \int_{\mathrm{Q}^l\backslash \mathrm{Q}^{l+1}} |u|^3\, |\nabla\psi_k| + \int_{\mathrm{Q}^k} |u|^3\, |\nabla\psi_k|. \tag{2.3}$$

Recalling that $\psi_k = \chi\,\phi_k$ and $|\nabla\psi_k| \lesssim_s r_l^{-4}$ on $Q^l \backslash Q^{l+1}$, $\forall\, l = k_0, ..., k-1$, we can bound the right-hand side of (2.3) and show that

$$\mathrm{B.1} \le C_s^* \sum_{l=k_0}^{k} r_l^{2s-1} \fint_{Q^l} |u|^3. \tag{2.4}$$

On the other hand, we define a sequence of cut-off functions $\overline{\chi}_l$ ($l = 1, 2, \ldots$) such that $\overline{\chi}_l \equiv 1$ on $7/8Q^l$; $\overline{\chi}_l \equiv 0$ off Q^l; $\|\nabla\overline{\chi}_l\|_{L^\infty} \le C/r_l$. Then it holds

$$\int_{Q^{k_0}} p\,u\cdot\nabla\psi_k = \sum_{l=k_0}^{k-1} \int_{Q^{k_0}} p\,u\cdot\nabla\left(\left(\overline{\chi}_l - \overline{\chi}_{l+1}\right)\psi_k\right) + \int_{Q^{k_0}} p\,u\cdot\nabla\left(\overline{\chi}_k\psi_k\right).$$

Obviously, $\overline{\chi}_l - \overline{\chi}_{l+1}$ is compactly supported on B^l and $\overline{\chi}_k$ is compactly supported on B^k. Utilizing the incompressibility of u, we know that

$$\begin{aligned}\int_{Q^{k_0}} p\,u\cdot\nabla\psi_k = &\sum_{l=k_0}^{k-1} \int_{Q^l} \left(p - \overline{p}_l\right) u\cdot\nabla\left(\left(\overline{\chi}_l - \overline{\chi}_{l+1}\right)\psi_k\right)\\ &+ \int_{Q^k} \left(p - \overline{p}_k\right) u\cdot\nabla\left(\overline{\chi}_k\psi_k\right).\end{aligned} \tag{2.5}$$

Applying the facts that

$$\|\,\psi_k\,\|_{L^\infty(Q^l\backslash 7/16Q^l)} \lesssim_s r_l^{-3}, \quad \|\,\nabla\psi_k\,\|_{L^\infty(Q^l\backslash 7/16Q^l)} \lesssim_s r_l^{-4},$$

to the right-hand side of (2.5), we get

$$\mathrm{B.2} = 2\int_{Q^{k_0}} p\,u\cdot\nabla\psi_k \lesssim_s \sum_{l=k_0}^{k} r_l^{-a} \fint_{Q^l} |u|\left|p - \overline{p}_l\right|,$$

which, together with (2.4) yield the following estimate of the term B:

$$\mathrm{B} \lesssim_s \sum_{l=k_0}^{k} r_l^{-a}\left[\fint_{Q^l} |u|^3 + |u|\cdot|p - \bar{p}_l|\right], \qquad \forall\, k > k_0. \tag{2.6}$$

(3) Estimate for the term C

From our definition, the term C is given by

$$\mathrm{C} = \int_{Q^{k_0}} |u|^2 \lim_{y\to 0^+} [\chi\,\partial_t\phi_k + \chi_t\phi_k + C_s(y^a\,\phi_k\,\partial_y\chi + y^a\,\chi\,\partial_y\phi_k)].$$

In light that $\chi_2 \equiv 1$ in $[0, r_{k_0+1}]$ and $\chi\,\partial_t\phi_k + C_s y^a\,\chi\,\partial_y\phi_k \le 0$, the following estimate holds for C:

$$\mathrm{C} \le \int_{Q^{k_0}} |u|^2 \lim_{y\to 0^+} \{\chi_t\phi_k\} \lesssim_s \fint_{Q^{k_0}} |u|^2. \tag{2.7}$$

(4) Estimate for the term D

The term D, according to our definition, is given by

$$D = 2 \sum_{l=k_0}^{k-1} \int_{Q^l \setminus Q^{l+1}} \psi_k \, f \cdot u + 2 \int_{Q^k} \psi_k \, f \cdot u.$$

Noticing the support of ψ_k, we simply bound D by

$$D \lesssim_s \sum_{l=k_0}^{k} r_l^{2s} \fint_{Q^l} |f| \, |u|. \tag{2.8}$$

Finally, we complete the proof of Lemma 2.1 by applying (2.6)–(2.8) to the right-hand side of (2.2). □

Now we finish this section with an inequality, which plays a rather important role in the following sections:

Proposition 2.2. *Let u and u^* satisfy the condition* (1.2). *Then for any $0 < \gamma \le \frac{6}{3-2s}$, there exists some constant depending only on s and γ such that the following holds:*

$$\left(\fint_{3/4 \, B_\rho(x_0)} |\, u(x) - (u)_\rho \,|^\gamma \, dx \right)^{2/\gamma} \le C_s^* \;\; \rho^{2s-3} \int_{Q_{\rho,+}} y^a \, |\overline{\nabla} u^*|^2 dx dy, \;\; \forall \, \rho > 0.$$

Proof. By translation and scaling, we may assume that $x_0 = 0$ and $\rho = 1$. Let ξ^* be a smooth function on $\mathbb{R}^4_+$ such that $0 \le \xi^* \le 1$, $\xi^* \equiv 1$ on $3/4 \; Q_{1,+}$, $\xi^* \equiv 0$ outside $Q_{1,+}$ and $|\nabla \xi^*| \le 1$ on $\mathbb{R}^4_+$. Then $\xi^* u^* \in H^1(\mathbb{R}^4_+; y^a)$. Moreover, by [2], $\xi u \in H^s(\mathbb{R}^3)$ with $\xi(\cdot) = \xi^*(\cdot, 0)$. Since $(\xi u)^*|_{y=0} = \xi^* u^*|_{y=0} = \xi u$, $\int_{\mathbb{R}^4_+} y^a |\overline{\nabla}(\xi u)^*|^2 \le \int_{\mathbb{R}^4_+} y^a |\overline{\nabla}(\xi^* u^*)|^2$ in that $(\xi u)^*$ minimizes the weighted Dirichlet energy in $H^1(\mathbb{R}_4^+; y^a)$ with the prescribed data ξu on $y = 0$. By fractional Gagliardo–Nirenberg inequality in [10] and Hölder's inequality, we have

$$\begin{aligned} \|u\|^2_{L^\gamma(3/4B_1)} &\lesssim_{s,\gamma} \|\xi u\|^2_{L^{6/(3-2s)}(\mathbb{R}^3)} \lesssim_{s,\gamma} \|\xi u\|^2_{\dot{H}^s(\mathbb{R}^3)} = \int_{\mathbb{R}^4_+} y^a |\overline{\nabla}(\xi u)^*|^2 \\ &\lesssim_{s,\gamma} \int_{\mathbb{R}^4_+} y^a |\overline{\nabla}(\xi^* u^*)|^2 \lesssim_{s,\gamma} \left[\int_{Q_{1,+}} y^a |u^*|^2 + y^a |\overline{\nabla} u^*|^2 \right]. \end{aligned} \tag{2.9}$$

By the weighted Poincaré inequality in [5], it holds

$$\int_{Q_{1,+}} y^a |u^* - (u^*)_{Q_{1,+}}|^2 \lesssim_s \int_{Q_{1,+}} y^a |\overline{\nabla} u^*|^2, \tag{2.10}$$

where $(u^*)_{Q_{1,+}} = \frac{1}{|Q_{1,+}|} \int_{Q_{1,+}} y^a \, u^*(x, y) dx \, dy$. Thus by (2.9) and (2.10), we have

$$\|u - (u^*)_{Q_{1,+}}\|_{L^\gamma(3/4B_1)} \lesssim_{s,\gamma} \left(\int_{Q_{1,+}} y^a |\overline{\nabla} u^*|^2 \right)^{1/2}. \tag{2.11}$$

Since $u^*(x, \mathrm{y}) = u(x) + \int_0^{\mathrm{y}} \partial_{\mathrm{z}} u^* \, \mathrm{dz}$, one can show that

$$(u^*)_{\mathrm{Q}_{1,+}} - \fint_{\mathrm{B}_1} u = \mathrm{C}_s \int_{\mathrm{Q}_{1,+}} \mathrm{y}^a \int_0^{\mathrm{y}} \partial_{\mathrm{z}} u^* \, \mathrm{dz}.$$

Using Hölder's inequality and the fact $a = 1 - 2s < 0$, the left-hand side of the equality above can be estimated by

$$\left| (u^*)_{\mathrm{Q}_{1,+}} - \fint_{\mathrm{B}_1} u \right| \lesssim_s \left(\int_{\mathrm{Q}_{1,+}} \mathrm{y}^a \, |\overline{\nabla} u^*|^2 \right)^{1/2}. \tag{2.12}$$

Therefore, the estimates (2.11) and (2.12) indicate that

$$\begin{aligned} &\left(\fint_{3/4\, \mathrm{B}_1} \left| u - \fint_{\mathrm{B}_1} u \right|^{\gamma} \right)^{1/\gamma} \\ &\leq \left(\fint_{3/4\mathrm{B}_1} \left| u - (u^*)_{\mathrm{Q}_{1,+}} \right|^{\gamma} \right)^{1/\gamma} + \left(\fint_{3/4\mathrm{B}_1} \left| (u^*)_{\mathrm{Q}_{1,+}} - \fint_{\mathrm{B}_1} u \right|^{\gamma} \right)^{1/\gamma} \\ &\lesssim_{s,\gamma} \left(\int_{\mathrm{Q}_{1,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2 \right)^{1/2}. \end{aligned}$$

□

2.2. Main assumptions for the inductive arguments. In this part, we deduce the necessary assumptions for the proof of L^{∞}-boundedness of u which are the starting points of the inductive arguments in next section. For any $r > 0$, we define :

$$\mathrm{A}(r) = r^{4s-5} \sup_{t \in \mathrm{I}_r} \int_{\mathrm{B}_r \times \{t\}} |u|^2, \quad \mathrm{G}(r) = r^{4s-6} \int_{\mathrm{Q}_r} |u|^3, \quad \mathrm{H}(r) = r^{4s-7} \int_{\mathrm{Q}_r^*} \mathrm{y}^a |u^*|^2,$$
$$\delta(r) = r^{4s-5} \int_{\mathrm{Q}_r^*} \mathrm{y}^a |\nabla u^*|^2, \quad \mathrm{F}(r) = r^{4s-9/2} \int_{\mathrm{Q}_r} |f|^{3/2},$$
$$\mathrm{P}_{\omega}(r) = r^{\alpha_{s,\omega}} \int_{\mathrm{I}_r} \left(\int_{\mathrm{B}_r} |p| \right)^{\omega},$$

where $\alpha_{s,\omega} = (4s - 5)\omega - 2s$. The main result in this section is

Proposition 2.3. *If the assumption* (1.4) *holds and* $(x_0, t_0) = (0, 0)$*, then*

$$\limsup_{r \to 0^+} \left[\mathrm{A}(r) + \mathrm{G}(r)^{2/3} + \mathrm{P}_{\omega_0}^{2/\omega_0}(r) \right] \lesssim_{s,\omega_0} \epsilon_0.$$

Here ϵ_0 *is the constant in* (1.4)*.* ω_0 *is a constant in the interval* $(4s/(6s - 3), 2)$*.*

The proof of this proposition consists of several lemmas.

Lemma 2.4. *If* $u \in \mathrm{H}^s$ *is a suitable weak solution of* (1.1)*, then for any* $\rho \geq 3r$ *and* $t \in \mathrm{I}_r = [-\frac{1}{2} r^{2s}, \ \frac{1}{2} r^{2s}]$*, we have that*

$$\int_{B_r}|u|^{3/s}\lesssim_s \frac{r^3}{\rho^{3(2s-1)/s}}A(\rho)^{3/(2s)}$$
$$+\rho^{\frac{3(5-4s)(4s-3)}{4s^2}}A(\rho)^{\frac{3(4s-3)}{4s^2}}\left(\int_{Q_{\rho,+}}y^a|\overline{\nabla}u^*|^2\right)^{\frac{3(3-2s)}{4s^2}}$$
$$+\frac{\rho^{(-s+\frac{5}{2})\frac{3}{2s}}}{r^{\frac{3(3-2s)}{2s}}}A(\rho)^{\frac{3}{4s}}\left(\int_{Q_{\rho,+}}y^a|\overline{\nabla}u^*|^2\right)^{\frac{3}{4s}}.$$

Proof. Let η be a cut-off function supported on $Q_{2r,+}$, which satisfies : $\eta\equiv 1$ on $Q_{r,+}$, $0\le\eta\le 1$ on $Q_{2r,+}$ and $\|\nabla\eta\|_{L^\infty}\lesssim 1/r$. By the fractional Gagliardo–Nirenberg inequality in [10] and the extension lemma of the fractional Laplacian in [2], we have

$$\|\eta_0 u\|_{L^{6/(3-2s)}(B_r)}\lesssim_s\|\eta_0 u\|_{\dot{H}^s}=\left(\int_{\mathbb{R}^4_+}y^a|\overline{\nabla}(\eta_0 u)^*|^2\right)^{1/2}.$$

Here $\eta_0(\cdot)=\eta(\cdot,0)$. $(\eta_0 u)^*$ is the extension of ηu to $\mathbb{R}^4_+$. Since ηu^* has the same boundary condition as $(\eta_0 u)^*$ on $y=0$, it holds

$$\|\eta_0 u\|_{L^{6/(3-2s)}(B_r)}\lesssim_s\left(\int_{\mathbb{R}^4_+}y^a|\overline{\nabla}(\eta_0 u)^*|^2\right)^{1/2}\lesssim\left(\int_{\mathbb{R}^4_+}y^a|\overline{\nabla}(\eta u^*)|^2\right)^{1/2}$$
$$\lesssim_s r^{-1}\left(\int_{Q_{2r,+}}y^a|u^*|^2\right)^{1/2}+\left(\int_{Q_{2r,+}}y^a|\overline{\nabla}u^*|^2\right)^{1/2}. \tag{2.13}$$

By (2.10) and (2.12), the first term on the right-hand side of (2.13) can by estimated by

$$\int_{Q_{2r,+}}y^a|u^*|^2\lesssim_s r^{2-2s}\int_{B_{2r}}|u|^2+r^2\int_{Q_{2r,+}}y^a|\overline{\nabla}u^*|^2. \tag{2.14}$$

In light of the support of η_0, (2.13)–(2.14) show that

$$\|u\|_{L^{6/(3-2s)}(B_r)}\lesssim_s r^{-s}\left(\int_{B_{2r}}|u|^2\right)^{1/2}+\left(\int_{Q_{2r,+}}y^a|\overline{\nabla}u^*|^2\right)^{1/2}. \tag{2.15}$$

By Hölder's inequality, it is easy to show that

$$\int_{B_r}|u|^{3/s}\le\left(\int_{B_r}|u|^2\right)^{\frac{3(4s-3)}{4s^2}}\left(\int_{B_r}|u|^{\frac{6}{3-2s}}\right)^{\frac{(3-2s)^2}{4s^2}}.$$

Applying (2.15) to the right-hand side of the inequality above, we have

$$\int_{B_r}|u|^{3/s}\lesssim_s\left(\int_{B_r}|u|^2\right)^{\frac{3(4s-3)}{4s^2}}\left(\int_{Q_{2r,+}}y^a|\overline{\nabla}u^*|^2\right)^{\frac{3(3-2s)}{4s^2}}+r^{-\frac{3(3-2s)}{2s}}\left(\int_{B_{2r}}|u|^2\right)^{3/(2s)}.$$

Therefore, for any $\rho\ge 2r$, we know that

$$\int_{B_r}|u|^{3/s}\lesssim_s\rho^{\frac{3(5-4s)(4s-3)}{4s^2}}A(\rho)^{\frac{3(4s-3)}{4s^2}}\left(\int_{Q_{\rho,+}}y^a|\overline{\nabla}u^*|^2\right)^{\frac{3(3-2s)}{4s^2}}$$

$$+r^{-\frac{3(3-2s)}{2s}}\left(\int_{B_{2r}}|u|^2\right)^{3/(2s)}. \tag{2.16}$$

We now turn to L^2-estimate of u on B_{2r}. At almost every time, we have

$$\int_{B_{2r}}|u|^2=\int_{B_{2r}}|u|^2-|u_\rho|^2+\int_{B_{2r}}|u_\rho|^2\lesssim\int_{B_{2r}}|u+u_\rho||u-u_\rho|+\left(\frac{r}{\rho}\right)^3\int_{B_\rho}|u|^2.$$

By Hölder's and Minkowski's inequalities, we get, for $\rho\geq 3r$, that

$$\int_{B_{2r}}|u|^2\lesssim\rho^s\left(\int_{B_\rho}|u|^2\right)^{1/2}\left(\int_{B_{3\rho/4}}|u-u_\rho|^{6/(3-2s)}\right)^{(3-2s)/6}+\left(\frac{r}{\rho}\right)^3\int_{B_\rho}|u|^2.$$

Applying Proposition 2.2 to the right-hand side above, it shows

$$\int_{B_{2r}}|u|^2\lesssim_s\rho^s\left(\int_{B_\rho}|u|^2\right)^{1/2}\left(\int_{Q_{\rho,+}}y^a|\overline{\nabla}u^*|^2\right)^{1/2}+\left(\frac{r}{\rho}\right)^3\int_{B_\rho}|u|^2.$$

Therefore, for almost every $t\in I_r$, we show that

$$\int_{B_{2r}}|u|^2\lesssim_s\rho^{-s+5/2}A(\rho)^{1/2}\left(\int_{Q_{\rho,+}}y^a|\overline{\nabla}u^*|^2\right)^{1/2}+\frac{r^3}{\rho^{4s-2}}A(\rho).$$

The proof can be fnished by applying the estimate above to (2.16). □

Lemma 2.5. *If u is the same as in Lemma 2.4, then for all $\rho\geq 3r$, it holds*

$$\begin{aligned}r^{4s-6}\int_{3/4Q_r}|u|\left||u|^2-|u_r|^2\right|\lesssim_s A(r)^{1/2}\Bigg[&\left(\frac{r}{\rho}\right)^{2s-1}A(\rho)^{1/2}\delta(r)^{1/2}\\&+\left(\frac{\rho}{r}\right)^{5-4s}A(\rho)^{\frac{4s-3}{4s}}\delta(\rho)^{\frac{3}{4s}}\\&+\left(\frac{\rho}{r}\right)^{\frac{10-7s}{2}}A(\rho)^{1/4}\delta(\rho)^{3/4}\Bigg].\end{aligned}$$

Proof. At almost every time t, by Hölder's inequality, we have

$$\begin{aligned}&\int_{3/4B_r}|u|\left||u|^2-|u_r|^2\right|\\&\leq\left(\int_{3/4B_r}|u|^{3/s}\right)^{s/3}\left(\int_{3/4B_r}|u+u_r|^2\right)^{1/2}\left(\int_{3/4B_r}|u-u_r|^{6/(3-2s)}\right)^{(3-2s)/6}.\end{aligned}$$

Applying Minkowski's inequality and Proposition 2.2 to the right-hand side above, we get

$$\int_{3/4B_r}|u|\left||u|^2-|u_r|^2\right|\lesssim_s\left(\int_{3/4B_r}|u|^{3/s}\right)^{s/3}\left(\int_{B_r}|u|^2\right)^{1/2}\left(\int_{Q_{r,+}}y^a|\overline{\nabla}u^*|^2\right)^{1/2}.$$

Integrating the above inequality with respect to time t over $\mathrm{I}_{3r/4}$, it holds

$$\int_{3/4\mathrm{Q}_r} |u| \left||u|^2 - |u_r|^2\right| \lesssim_s \sup_{t\in \mathrm{I}_r} \left(\int_{\mathrm{B}_r} |u|^2\right)^{1/2} \int_{\mathrm{I}_r} \left(\int_{\mathrm{B}_r} |u|^{3/s}\right)^{s/3} \cdot \left(\int_{\mathrm{Q}_{r,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{1/2} \mathrm{d}t$$

which shows that

$$r^{4s-6} \int_{3/4\mathrm{Q}_r} |u| \left||u|^2 - |u_r|^2\right| \lesssim_s r^{\frac{4s-7}{2}} \mathrm{A}(r)^{1/2} \int_{\mathrm{I}_r} \left(\int_{\mathrm{B}_r} |u|^{3/s}\right)^{s/3} \cdot \left(\int_{\mathrm{Q}_{r,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{1/2} \mathrm{d}t. \tag{2.17}$$

By Lemma 2.4, we know that for $\rho \geq 3r$ and $t \in \mathrm{I}_r$,

$$\left(\int_{\mathrm{B}_r} |u|^{3/s}\right)^{s/3} \left(\int_{\mathrm{Q}_{r,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{1/2} \lesssim_s \frac{r^s}{\rho^{2s-1}} \mathrm{A}(\rho)^{1/2} \left(\int_{\mathrm{Q}_{r,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{1/2} + \rho^{\frac{(5-4s)(4s-3)}{4s}} \mathrm{A}(\rho)^{\frac{(4s-3)}{4s}} \left(\int_{\mathrm{Q}_{\rho,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{\frac{3}{4s}} + \frac{\rho^{(-s+\frac{5}{2})\frac{1}{2}}}{r^{\frac{(3-2s)}{2}}} \mathrm{A}(\rho)^{\frac{1}{4}} \left(\int_{\mathrm{Q}_{\rho,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{\frac{3}{4}}.$$

Integrating the above estimate with respect to the time t over I_r, we get, by Hölder inequality, that

$$\int_{\mathrm{I}_r} \left(\int_{\mathrm{B}_r} |u|^{3/s}\right)^{s/3} \left(\int_{\mathrm{Q}_{r,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{1/2} \mathrm{d}t \lesssim_s \frac{r^{2s}}{\rho^{2s-1}} \mathrm{A}(\rho)^{1/2} \left(\int_{\mathrm{Q}_r^*} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{1/2} + r^{\frac{4s-3}{2}} \rho^{\frac{(5-4s)(4s-3)}{4s}} \mathrm{A}(\rho)^{\frac{4s-3}{4s}} \left(\int_{\mathrm{Q}_\rho^*} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{\frac{3}{4s}} + \frac{\rho^{\frac{5-2s}{4}}}{r^{\frac{3(1-s)}{2}}} \mathrm{A}(\rho)^{1/4} \left(\int_{\mathrm{Q}_\rho^*} \mathrm{y}^a |\overline{\nabla} u^*|^2\right)^{3/4},$$

which, together with the estimate (2.17), concludes the proof of Lemma 2.5. □

With Lemma 2.5, we can estimate the L^3-norm of u as follows:

Lemma 2.6. *For any $\rho \geq 3r$, we have*

$$\mathrm{G}(3/4r) \lesssim_s \left(\frac{r}{\rho}\right)^{4s-3} \mathrm{G}(\rho) + \left(\frac{\rho}{r}\right)^{6-4s} \delta(\rho)^{1/2} \left[\mathrm{A}(\rho) + \delta(\rho)\right].$$

Proof. By Lemma 2.5 and Young's inequality, for $\rho \geq 3r$, we have

$$\rho^{4s-6} \int_{3/4\mathrm{Q}_\rho} |u| \left| |u|^2 - |u_\rho|^2 \right| \lesssim_s \delta(\rho)^{1/2} \left[\mathrm{A}(\rho) + \delta(\rho) \right]$$

which shows that,

$$\begin{aligned} \int_{3/4\mathrm{Q}_r} |u|^3 &= \int_{3/4\mathrm{Q}_r} |u| \left(|u|^2 - |u_\rho|^2 \right) + \int_{3/4\mathrm{Q}_r} |u||u_\rho|^2 \\ &\lesssim_s \rho^{6-4s} \, \delta(\rho)^{1/2} \left[\mathrm{A}(\rho) + \delta(\rho) \right] + \int_{\mathrm{I}_{3r/4}} |u_\rho|^2 \left(\int_{\mathrm{B}_{3r/4}} |u| \mathrm{d}x \right) \mathrm{d}t. \end{aligned} \tag{2.18}$$

Using Hölder's and Young's inequalities, the second term on the right-hand term of (2.18) is bounded by

$$\begin{aligned} \int_{\mathrm{I}_{3r/4}} |u_\rho|^2 \left(\int_{\mathrm{B}_{3r/4}} |u| \mathrm{d}x \right) \mathrm{d}t &\lesssim \left(\frac{r}{\rho} \right)^2 \int_{\mathrm{I}_{3r/4}} \left(\int_{\mathrm{B}_\rho} |u|^3 \right)^{2/3} \left(\int_{\mathrm{B}_{3r/4}} |u|^3 \right)^{1/3} \mathrm{d}t \\ &\lesssim 2/3 \left(\frac{r}{\rho} \right)^3 \int_{\mathrm{Q}_\rho} |u|^3 + 1/3 \int_{3/4\mathrm{Q}_r} |u|^3. \end{aligned}$$

When the estimate above is applied to (2.18), we get

$$\mathrm{G}(3r/4) \lesssim_s \left(\frac{r}{\rho} \right)^{4s-3} \mathrm{G}(\rho) + \left(\frac{\rho}{r} \right)^{6-4s} \delta(\rho)^{1/2} \left[\mathrm{A}(\rho) + \delta(\rho) \right].$$

□

The next lemma is devoted to estimating the pressure term $\mathrm{P}_\omega(r)$.

Lemma 2.7. *Suppose that ω and q_0 are constants satisfying*

$$\omega > s/(2s-1) \qquad \text{and} \qquad 1 < q_0 \leq \frac{3}{3-2s} \tag{2.19}$$

and ν_1, ν_2, ν_1^*, ν_2^* are positive constants such that

$$\nu_1^* + \nu_2^* = 1, \quad \nu_1 + \nu_2 = 1, \quad \omega\nu_2^* \leq 1, \quad \omega\nu_2 \leq 1, \quad \text{and} \quad 3 - 3q_0 + 2s\,q_0\,\nu_2 \geq 0. \tag{2.20}$$

Then we have, for any $\rho \geq 4r$, that

$$\begin{aligned} \mathrm{P}_\omega^{1/\omega}(r) \lesssim_{q_0,\omega} \left(\frac{r}{\rho} \right)^{(4s-2)-2s/\omega} \mathrm{P}_\omega^{1/\omega}(\rho) + \left(\frac{r}{\rho} \right)^{4s-2-2s\nu_2^*} \mathrm{A}(\rho)^{\nu_1^*} \delta(\rho)^{\nu_2^*} \\ + \left(\frac{\rho}{r} \right)^{2-4s+2s\nu_2+3/q_0} \mathrm{A}(\rho)^{\nu_1} \delta(\rho)^{\nu_2}. \end{aligned}$$

Proof. Let ϕ be a cut-off function compactly supported in $\mathrm{B}_{3\rho}$ with the property: $\phi \equiv 1$ on $\mathrm{B}_{2\rho}$, $0 \le \phi \le 1$, $|\nabla\phi| \le 1/\rho$ and $|\nabla^2\phi| \le 1/\rho^2$. For all $x \in \mathrm{B}_\rho$, we can decompose p as follows: $p = p_1 + p_2$ where

$$\begin{aligned} p_1(x) &= \frac{3}{4\pi}\int_{\mathbb{R}^3} \frac{p(y)}{|x-y|}\,\Delta\phi(y) + \frac{3}{2\pi}\int_{\mathbb{R}^3} \frac{p(y)}{|x-y|^3}\,(x-y)\cdot\nabla_i\phi(y), \\ p_2(x) &= \frac{3}{4\pi}\int_{\mathbb{R}^3} \nabla_{ij}\left(\phi(y)\frac{1}{|x-y|}\right)\left(u^i - u^i_{4\rho}\right)\left(u^j - u^j_{4\rho}\right). \end{aligned} \tag{2.21}$$

For the term p_1, we know that $|p_1(x)| \lesssim ⨍_{4\mathrm{B}_\rho} |p|, \quad \forall\, x \in \mathrm{B}_\rho$. Therefore for any $r \le \rho$, we get

$$\int_{\mathrm{B}_r} |p_1| \lesssim r^3\, \|p_1\|_{L^\infty(\mathrm{B}_r)} \lesssim \left(\frac{r}{\rho}\right)^3 \int_{4\mathrm{B}_\rho} |p|$$

which indicates that

$$r^{\alpha_{s,\omega}} \int_{\mathrm{I}_r} \left(\int_{\mathrm{B}_r} |p_1|\right)^{\omega} \lesssim_\omega \left(\frac{r}{\rho}\right)^{(4s-2)\omega-2s} \mathrm{P}_\omega(4\rho). \tag{2.22}$$

As for the second term p_2, we have

$$\begin{aligned} p_2 = p_{2,1} + p_{2,2} = \frac{3}{4\pi}\int_{\mathbb{R}^3} & \frac{1}{|x-y|}\nabla_{ij}\phi\left(u^i - u^i_{4\rho}\right)(u^j - u^j_{4\rho}) \\ & + 2\nabla_i\left(\frac{1}{|x-y|}\right)\nabla_j\phi\left(u^i - u^i_{4\rho}\right)(u^j - u^j_{4\rho}) \\ & + \frac{3}{4\pi}\int_{\mathbb{R}^3} \nabla_{ij}\left(\frac{1}{|x-y|}\right)\phi\left(u^i - u^i_{4\rho}\right)(u^j - u^j_{4\rho}). \end{aligned}$$

By the choice of cut-off function ϕ, $p_{2,1}$ can be estimated by

$$|p_{2,1}(x)| \lesssim ⨍_{3\mathrm{B}_\rho} |u - u_{4\rho}|^2, \qquad \forall\, x \in \mathrm{B}_\rho.$$

Then for all $r \le \rho$ and $x \in B_\rho$, we have, by applying Hölder's inequality and Proposition 2.2 to the right-hand side above, that

$$\begin{aligned} |p_{2,1}(x)| &\lesssim \rho^{2s\,\nu_2^*-3}\left(\int_{\mathrm{B}_{4\rho}} |u|^2\right)^{\nu_1^*}\left(\int_{\mathrm{B}_{3\rho}} |u - u_{4\rho}|^{\frac{6}{3-2s}}\right)^{\frac{3-2s}{6}\cdot 2\,\nu_2^*} \\ &\lesssim \rho^{2s\,\nu_2^*-3}\left(\int_{\mathrm{B}_{4\rho}} |u|^2\right)^{\nu_1^*}\left(\int_{\mathrm{Q}_{4\rho,+}} \mathrm{y}^a|\overline{\nabla} u^*|^2\right)^{\nu_2^*}. \end{aligned}$$

This shows, by integrating over $\mathrm{B_r}$ and $\mathrm{I_r}$, that

$$\begin{aligned} \int_{\mathrm{I}_r}\left(\int_{\mathrm{B}_r} |p_{2,1}|\right)^{\omega} &\lesssim_\omega r^{3\omega}\,\rho^{[2s\nu_2^*-3+(5-4s)\nu_1^*]\omega}\,\mathrm{A}(4\rho)^{\omega\nu_1^*}\int_{\mathrm{I}_r}\left(\int_{\mathrm{Q}_{4\rho,+}} \mathrm{y}^a|\overline{\nabla} u^*|^2\right)^{\omega\nu_2^*} \\ &\lesssim_\omega r^{3\omega+2s(1-\omega\nu_2^*)}\,\rho^{(2s\nu_2^*+2-4s)\,\omega}\,\mathrm{A}(4\rho)^{\omega\nu_1^*}\,\delta(4\rho)^{\omega\nu_2^*}. \end{aligned}$$

Multiplying both sides above by $r^{\alpha_{s,\omega}}$, it holds

$$r^{\alpha_{s,\omega}} \int_{\mathrm{I}_r} \left(\int_{\mathrm{B}_r} |p_{2,1}| \right)^{\omega} \lesssim_{\omega} \left(\frac{r}{\rho} \right)^{(4s-2-2s\nu_2^*)\,\omega} \mathrm{A}(4\rho)^{\omega\nu_1^*}\, \delta(4\rho)^{\omega\nu_2^*}. \tag{2.23}$$

Finally, we give the estimate on $p_{2,2}$. By Hölder's inequality and Calderon–Zygmund estimate (see [14]), we have, for all q_0 satisfying the hypothesis in the lemma, that

$$\int_{\mathrm{B}_r} |p_{2,2}| \lesssim r^{3\,(1-\frac{1}{q_0})} \, \|p_{2,2}\|_{L^{q_0}} \lesssim_{q_0} r^{3\,(1-\frac{1}{q_0})} \left\| \phi\, |u - u_{4\rho}|^2 \right\|_{L^{q_0}}.$$

By the assumptions on q_0, ν_1 and ν_2, we know that $2q_0\nu_1 \le 2$ and $2q_0\nu_2 \le 6/(3-2s)$. Therefore using Hölder's and Minkowski's inequalities, we have

$$\int_{\mathrm{B}_r} |p_{2,2}| \lesssim_{q_0} r^{3\,(1-\frac{1}{q_0})} \left(\int_{\mathrm{B}_{4\rho}} |u|^2 \right)^{\nu_1} \left(\int_{\mathrm{B}_{3\rho}} |u - u_{4\rho}|^{6/(3-2s)} \right)^{\frac{3-2s}{6}\cdot 2\nu_2} \rho^{2s\nu_2 - 3 + 3/q_0}.$$

Applying Proposition 2.2 to the right-hand side above, the following inequality holds :

$$\int_{\mathrm{B}_r} |p_{2,2}| \lesssim_{q_0} r^{3\,(1-\frac{1}{q_0})} \left(\int_{\mathrm{B}_{4\rho}} |u|^2 \right)^{\nu_1} \left(\int_{\mathrm{Q}_{4\rho,+}} \mathrm{y}^a |\overline{\nabla} u^*|^2 \right)^{\nu_2} \rho^{2s\nu_2 - 3 + 3/q_0}$$

which leads to the estimate

$$r^{\alpha_{s,\omega}} \int_{\mathrm{I}_r} \left(\int_{\mathrm{B}_r} |p_{2,2}| \right)^{\omega} \lesssim_{q_0,\omega} \left(\frac{\rho}{r} \right)^{\omega(2-4s+2s\nu_2+3/q)} \mathrm{A}(4\rho)^{\omega\nu_1} \delta(4\rho)^{\omega\nu_2} \tag{2.24}$$

From the above estimates (2.22)–(2.24), we get that

$$\begin{aligned} \mathrm{P}_{\omega}^{1/\omega}(r) \lesssim_{q_0,\omega} & \left(\frac{r}{\rho} \right)^{(4s-2)-2s/\omega} \mathrm{P}_{\omega}^{1/\omega}(4\rho) + \left(\frac{r}{\rho} \right)^{4s-2-2s\nu_2^*} \mathrm{A}(4\rho)^{\nu_1^*} \delta(4\rho)^{\nu_2^*} \\ & + \left(\frac{\rho}{r} \right)^{2-4s+2s\nu_2+3/q_0} \mathrm{A}(4\rho)^{\nu_1} \delta(4\rho)^{\nu_2}. \end{aligned}$$

□

The following lemma gives the estimate for $\mathrm{A}(r)$.

Lemma 2.8. *Let u be the same as before and $\rho_0 > 0$ be a constant suitably small such that*

$$\delta(\rho) < \epsilon_0, \quad \forall\ \rho < \rho_0.$$

Let $\omega > 4s/(6s-3)$ be a positive constant. Then for all $\rho \ge 6r$ and $\rho < \rho_0$, we have that

$$\begin{aligned} \mathrm{A}(r) \lesssim_s & \left(\frac{r}{\rho} \right)^{\tau(s,\omega)} \left(\mathrm{A}(\rho) + \mathrm{G}(\rho)^{2/3} + \mathrm{P}_{\omega}^{2/\omega}(\rho) \right) + \left(\frac{\rho}{r} \right)^{6-6s} \mathrm{A}(\rho)\, \delta(\rho)^{1/2} \\ & + \left(\frac{\rho}{r} \right)^{\beta(s,\omega)} \delta(\rho) + \left(\frac{\rho}{r} \right)^{7-4s} \mathrm{F}(\rho)^{4/3}. \end{aligned}$$

Here ϵ_0 is the constant in (1.4). $\tau(s,\omega)$ and $\beta(s,\omega)$ are some positive constants depending on s and ω.

Proof. Firstly, we introduce the three cut-off functions ξ_i $(i = 1, 2, 3)$: (1) $\xi_1 \in C_0^\infty(\mathbb{R}^3)$, $\xi_1 \equiv 0$ outside $\mathrm{B}_{3/2r}$, $\xi_1 \equiv 1$ on B_r, $0 \le \xi_1 \le 1$ and $r\,\|\nabla \xi_1\|_{L^\infty} + r^2 \left\|\nabla^2 \xi_1\right\|_{L^\infty} \lesssim 1$; (2) $\xi_2 \in C^\infty(\mathbb{R}_+)$, $\xi_2(y) \equiv 0$ if $y \ge 3/2r$, $\xi_2 \equiv 1$ on $(0,\ r)$, $0 \le \xi_2 \le 1$ and $r\ \left\|\xi_2'\right\|_{L^\infty} + r^2 \left\|\xi_2''\right\|_{L^\infty} \lesssim 1$; (3) $\xi_3 \in C_0^\infty(\mathbb{R})$, $\xi_3 \equiv 0$ outside $\mathrm{I}_{3/2r}$, $\xi_3 \equiv 1$ on I_r, $0 \le \xi_3 \le 1$ and $r^{2s}\ \left\|\xi_3'\right\|_{L^\infty} \lesssim 1$. Letting $\psi(x, y, t)$ in (1.3) be $\xi_1(x)\xi_2(y)\xi_3(t)$, we know that

$$\int_{\mathrm{B}_r \times \{t\}} |u|^2 \lesssim_s r^{-2} \int_{\mathrm{Q}_{2r}^*} \mathrm{y}^a |u^*|^2 + r^{-2s} \iint_{\mathrm{Q}_{2r}} |u|^2 + r^{-1} \int_{\mathrm{Q}_{3r/2}} |p||u| + r^{-1} \int_{\mathrm{Q}_{3r/2}} |u| \left||u|^2 - |u_{2r}|^2\right| + \iint_{\mathrm{Q}_{2r}} |f||u|.$$

which, by (2.14), implies that

$$\int_{\mathrm{B}_r \times \{t\}} |u|^2 \lesssim_s \int_{\mathrm{Q}_{2r}^*} \mathrm{y}^a |\overline{\nabla} u^*|^2 + r^{-2s} \int_{\mathrm{Q}_{2r}} |u|^2 + r^{-1} \int_{\mathrm{Q}_{3r/2}} |p||u| + r^{-1} \int_{\mathrm{Q}_{3r/2}} |u| \left||u|^2 - |u_{2r}|^2\right| + \int_{\mathrm{Q}_{2r}} |f||u|. \tag{2.25}$$

We now denote by I.1 to I.4 the last four terms on the right-hand side of (2.25), respectively.

1. Estimate of I.1. By Hölder's inequality, it holds

$$\mathrm{I.1} = r^{-2s} \int_{\mathrm{Q}_{2r}} |u|^2 \lesssim r^{3-4s} \int_{\mathrm{I}_{2r}} \left(\int_{2\mathrm{B}_r} |u|^{3/s} \right)^{2s/3}.$$

From Lemma 2.4, we have, for all $\rho \ge 6r$, that

$$\mathrm{I.1} \lesssim_s \frac{r^3}{\rho^{4s-2}}\,\mathrm{A}(\rho) + \rho^{5-4s}\,\mathrm{A}(\rho)^{\frac{4s-3}{2s}}\,\delta(\rho)^{\frac{3-2s}{2s}} + \frac{\rho^{5-3s}}{r^s}\,\mathrm{A}(\rho)^{\frac{1}{2}}\,\delta(\rho)^{\frac{1}{2}}. \tag{2.26}$$

2. Estimate of I.2. By the decomposition (2.21), we have

$$\mathrm{I.2} \lesssim r^{-1} \int_{\mathrm{Q}_{3r/2}} |p_1||u| + r^{-1} \int_{\mathrm{Q}_{3r/2}} |p_{2,1}||u| + r^{-1} \int_{\mathrm{Q}_{3r/2}} |p_{2,2}||u|. \tag{2.27}$$

We denote the three terms on the right-hand side above by I.2.1–I.2.3, respectively. Utilizing (2.21) and Hölder inequality, it holds:

$$\mathrm{I.2.1} \lesssim r^{-1} \int_{\mathrm{I}_{3r/2}} \|p_1(\cdot, t)\|_{L^\infty(\mathrm{B}_{3r/2})} \int_{\mathrm{B}_{3r/2}} |u| \lesssim r^{-1} \rho^{-3} \int_{\mathrm{I}_{3r/2}} \int_{\mathrm{B}_\rho} |p| \int_{\mathrm{B}_{3r/2}} |u| \lesssim_s r^{2-s} \rho^{-3} \int_{\mathrm{I}_{3r/2}} \int_{\mathrm{B}_\rho} |p| \left(\int_{\mathrm{B}_{3r/2}} |u|^{3/s} \right)^{s/3}.$$

Applying Lemma 2.4 to the inequality above, we get

$$\begin{aligned}\mathrm{I.2.1} \lesssim_s\ & \frac{r^2}{\rho^{2s+2}}\,\mathrm{A}^{1/2}(\rho)\int_{\mathrm{I}_{3r/2}}\int_{\mathrm{B}_\rho}|p| \\ & + r^{2-s}\rho^{\frac{(5-4s)(4s-3)}{4s}-3}\,\mathrm{A}(\rho)^{\frac{4s-3}{4s}}\int_{\mathrm{I}_{3r/2}}\int_{\mathrm{B}_\rho}|p|\left(\int_{\mathrm{Q}_{\rho,+}}\mathrm{y}^a|\overline{\nabla}u^*|^2\right)^{\frac{3-2s}{4s}} \\ & + \frac{r^{1/2}}{\rho^{\frac{2s+7}{4}}}\,\mathrm{A}(\rho)^{1/4}\int_{\mathrm{I}_{3r/2}}\int_{\mathrm{B}_\rho}|p|\left(\int_{\mathrm{Q}_{\rho,+}}\mathrm{y}^a|\overline{\nabla}u^*|^2\right)^{1/4}.\end{aligned}$$

Since $\omega > 4s/(6s-3)$, by Hölder's inequality, the right-hand side above can be bounded by

$$\begin{aligned}\mathrm{I.2.1} \lesssim_{s,\omega}\ & \frac{r^{2+2s-2s/\omega}}{\rho^{6s-3-2s/\omega}}\,\mathrm{A}(\rho)^{1/2}\,\mathrm{P}_\omega^{1/\omega}(\rho) \\ & + r^{2s+1/2-2s/\omega}\rho^{-6s+9/2+2s/\omega}\,\mathrm{A}(\rho)^{\frac{4s-3}{4s}}\,\mathrm{P}_\omega^{1/\omega}(\rho)\,\delta(\rho)^{\frac{3-2s}{4s}} \\ & + \frac{r^{(3s+1)/2-2s/\omega}}{\rho^{11s/2-9/2-2s/\omega}}\mathrm{A}(\rho)^{1/4}\,\mathrm{P}_\omega^{1/\omega}(\rho)\,\delta(\rho)^{1/4}. \end{aligned} \tag{2.28}$$

Now wo go to the term I.2.2. Let $\nu_1^* = 0$, $\nu_2^* = 1$ in (2.20). By Hölder's inequality, we know that

$$\begin{aligned}\mathrm{I.2.2}\ & \lesssim r^{-1}\int_{\mathrm{I}_{3r/2}}\|p_{2,1}\|_{L^\infty(\mathrm{B}_{3r/2})}\int_{\mathrm{B}_{3r/2}}|u| \\ & \lesssim_s r^{-1}\rho^{2s-3}\int_{\mathrm{I}_{3r/2}}\left(\int_{\mathrm{Q}_{\rho,+}}\mathrm{y}^a|\overline{\nabla}u^*|^2\right)\int_{\mathrm{B}_{3r/2}}|u| \\ & \lesssim_s r^{1/2}\rho^{-4s+9/2}\,\mathrm{A}(\rho)^{1/2}\,\delta(\rho). \end{aligned} \tag{2.29}$$

For I.2.3, by Hölder's inequality, it holds that

$$\begin{aligned}\mathrm{I.2.3} & \lesssim r^{-1}\int_{\mathrm{I}_{3r/2}}\left(\int_{\mathrm{B}_{3r/2}}|p_{2,2}|^2\right)^{1/2}\left(\int_{\mathrm{B}_{3r/2}}|u|^2\right)^{1/2} \\ & \lesssim r^{-1}\,\rho^{\frac{5-4s}{2}}\,\mathrm{A}(\rho)^{1/2}\int_{\mathrm{I}_{3r/2}}\|p_{2,2}\|_{L^2}. \end{aligned}$$

By Calderon–Zygmund estimate (see [14]), the L^2 norm of $p_{2,2}$ on $\mathrm{B}_{3r/2}$ can be estimated for almost every $t \in \mathrm{I}_{3r/2}$. Therefore from the estimate above, we see that

$$\mathrm{I.2.3} \lesssim_s r^{-1}\,\rho^{\frac{5-4s}{2}}\,\mathrm{A}(\rho)^{1/2}\int_{\mathrm{I}_{3r/2}}\left(\int_{\mathrm{B}_{3\rho/4}}|u-u_\rho|^4\right)^{1/2}.$$

Since $s > 3/4$, then $6/(3-2s) > 4$. Therefore applying Proposition 2.2 to the estimate above, we can show

$$\mathrm{I.2.3} \lesssim_s \left(\frac{\rho}{r}\right)\mathrm{A}(\rho)^{1/2}\int_{\mathrm{I}_{3r/2}}\int_{\mathrm{Q}_{\rho,+}}\mathrm{y}^a|\overline{\nabla}u^*|^2 \lesssim_s \frac{\rho^{6-4s}}{r}\,\mathrm{A}(\rho)^{1/2}\,\delta(\rho). \tag{2.30}$$

Using (2.28)–(2.30), I.2 can be bounded by

$$\begin{aligned} \mathrm{I.2} \lesssim_{s,\omega} & \frac{r^{2+2s-2s/\omega}}{\rho^{6s-3-2s/\omega}} \mathrm{A}(\rho)^{1/2} \mathrm{P}_{\omega}^{1/\omega}(\rho) \\ & + r^{2s+1/2-2s/\omega} \rho^{-6s+9/2+2s/\omega} \mathrm{A}(\rho)^{\frac{4s-3}{4s}} \mathrm{P}_{\omega}^{1/\omega}(\rho)\, \delta(\rho)^{\frac{3-2s}{4s}} \\ & + \frac{r^{(3s+1)/2-2s/\omega}}{\rho^{11s/2-9/2-2s/\omega}} \mathrm{A}(\rho)^{1/4} \mathrm{P}_{\omega}^{1/\omega}(\rho)\, \delta(\rho)^{1/4} \\ & + r^{1/2} \rho^{-4s+9/2} \mathrm{A}(\rho)^{1/2} \delta(\rho) + \frac{\rho^{6-4s}}{r} \mathrm{A}(\rho)^{1/2} \delta(\rho). \end{aligned} \tag{2.31}$$

3. Estimate of I.3. By Lemma 2.5, it is clear that

$$\begin{aligned} \mathrm{I.3} \lesssim_s r^{5-4s} \Bigg[& \left(\frac{\rho}{r}\right)^{6-6s} \mathrm{A}(\rho)\, \delta(\rho)^{1/2} + \left(\frac{\rho}{r}\right)^{3(5-4s)/2} \mathrm{A}(\rho)^{\frac{6s-3}{4s}} \delta(\rho)^{\frac{3}{4s}} \\ & + \left(\frac{\rho}{r}\right)^{\frac{15-11s}{2}} \mathrm{A}(\rho)^{\frac{3}{4}} \delta(\rho)^{\frac{3}{4}} \Bigg]. \end{aligned} \tag{2.32}$$

4. Estimate of I.4. By Hölder inequality, we have

$$\mathrm{I.4} \lesssim \left(\int_{\mathrm{Q}_\rho} |f|^{3/2}\right)^{2/3} \left(\int_{\mathrm{Q}_\rho} |u|^3\right)^{1/2} \lesssim \rho^{5-4s} \mathrm{F}(\rho)^{2/3} \mathrm{G}(\rho)^{1/3}. \tag{2.33}$$

Applying (2.26) and (2.31)–(2.33) to (2.25), we have

$$\begin{aligned} \mathrm{A}(r) \lesssim_{s,\omega} & \left(\frac{r}{\rho}\right)^{4s-2} \mathrm{A}(\rho) + \left(\frac{\rho}{r}\right)^{5-4s} \delta(\rho) \\ & + \left(\frac{\rho}{r}\right)^{5-4s} \mathrm{A}(\rho)^{\frac{4s-3}{2s}} \delta(\rho)^{\frac{3-2s}{2s}} + \left(\frac{\rho}{r}\right)^{5-3s} \mathrm{A}(\rho)^{\frac{1}{2}} \delta(\rho)^{\frac{1}{2}} \\ & + \left(\frac{r}{\rho}\right)^{6s-3-2s/\omega} \mathrm{A}(\rho)^{1/2} \mathrm{P}_{\omega}^{1/\omega}(\rho) \\ & + \left(\frac{\rho}{r}\right)^{-6s+9/2+2s/\omega} \mathrm{A}(\rho)^{\frac{4s-3}{4s}} \mathrm{P}_{\omega}^{1/\omega}(\rho)\, \delta(\rho)^{\frac{3-2s}{4s}} \\ & + \left(\frac{r}{\rho}\right)^{(11s-9)/2-2s/\omega} \mathrm{A}(\rho)^{1/4} \mathrm{P}_{\omega}^{1/\omega}(\rho)\, \delta(\rho)^{1/4} + \left(\frac{r}{\rho}\right)^{4s-9/2} \mathrm{A}(\rho)^{1/2} \delta(\rho) \\ & + \left(\frac{\rho}{r}\right)^{6-4s} \mathrm{A}(\rho)^{1/2} \delta(\rho) + \left(\frac{\rho}{r}\right)^{6-6s} \mathrm{A}(\rho)\, \delta(\rho)^{1/2} \\ & + \left(\frac{\rho}{r}\right)^{3(5-4s)/2} \mathrm{A}(\rho)^{\frac{6s-3}{4s}} \delta(\rho)^{\frac{3}{4s}} + \left(\frac{\rho}{r}\right)^{\frac{15-11s}{2}} \mathrm{A}(\rho)^{\frac{3}{4}} \delta(\rho)^{\frac{3}{4}} \\ & + \left(\frac{\rho}{r}\right)^{5-4s} \mathrm{F}(\rho)^{2/3} \mathrm{G}(\rho)^{1/3}. \end{aligned}$$

Finally, the proof can be finished by Young's inequality. □

In the end we go to the proof of Proposition 2.3:

Proof of Proposition 2.3. By Lemma 2.5 and Young's inequality, for all $\rho \geq 4r$, it holds that

$$\begin{aligned} \mathrm{G}^{2/3}(r) &\lesssim_s \left(\frac{r}{\rho}\right)^{\frac{2(4s-3)}{3}} \mathrm{G}^{2/3}(\rho) + \left(\frac{\rho}{r}\right)^{\frac{4(3-2s)}{3}} \left[\delta^{1/3}(\rho)\mathrm{A}^{2/3}(\rho) + \delta(\rho)\right] \\ &\lesssim_s \left(\frac{r}{\rho}\right)^{\frac{2(4s-3)}{3}} \mathrm{G}^{2/3}(\rho) + \left(\frac{r}{\rho}\right)^{4s-2} \mathrm{A}(\rho) + \left(\frac{\rho}{r}\right)^{8} \delta(\rho). \end{aligned} \tag{2.34}$$

We set the constants in Lemma 2.7 as follows:

$$\begin{aligned} &\omega = \omega_0, \quad q_0 = \frac{3}{3 - 2s/\omega_0}, \quad \nu_1 = \nu_1^* = 1 - 1/\omega_0, \quad \nu_2 = \nu_2^* = 1/\omega_0, \\ &\omega_0 \in (4s/(6s-3), 2). \end{aligned} \tag{2.35}$$

Therefore by Lemma 2.7, we have that

$$\mathrm{P}_{\omega_0}^{2/\omega_0}(r) \lesssim_{\omega_0} \left(\frac{r}{\rho}\right)^{4(2s-1)-4s/\omega_0} \mathrm{P}_{\omega_0}^{2/\omega_0}(\rho) + \left(\frac{\rho}{r}\right)^{2(5-4s)} \mathrm{A}(\rho)^{2-2/\omega_0} \delta(\rho)^{2/\omega_0}, \quad \forall\, \rho \geq 4r.$$

Since we assume in (2.35) that $\omega_0 < 2$, it holds $2 - 2/\omega_0 < 1$. Therefore by Young's inequality, for any $\rho \geq 4r$, the above estimate can be reduced to

$$\begin{aligned} \mathrm{P}_{\omega_0}^{2/\omega_0}(r) &\lesssim_{\omega_0} \left(\frac{r}{\rho}\right)^{4(2s-1)-4s/\omega_0} \mathrm{P}_{\omega_0}^{2/\omega_0}(\rho) + \left(\frac{\rho}{r}\right)^{2(5-4s)} \mathrm{A}(\rho)^{2-2/\omega_0} \delta(\rho)^{2/\omega_0} \\ &\lesssim_{\omega_0} \left(\frac{r}{\rho}\right)^{4(2s-1)-4s/\omega_0} \mathrm{P}_{\omega_0}^{2/\omega_0}(\rho) + \left(\frac{r}{\rho}\right)^{4s-2} \mathrm{A}(\rho) + \left(\frac{\rho}{r}\right)^{\beta(s,\omega_0)} \delta(\rho), \end{aligned} \tag{2.36}$$

where $\beta(s, \omega_0)$ is chosen suitably large so that (2.36) and Lemma 2.8 hold for the same $\beta(s, \omega_0)$. Moreover we can choose a suitably small $\tau(s, \omega_0)$ such that the following holds:

$$\mathrm{P}_{\omega_0}^{2/\omega_0}(r) \lesssim_{\omega_0} \left(\frac{r}{\rho}\right)^{\tau(s,\omega_0)} \left(\mathrm{P}_{\omega_0}^{2/\omega_0}(\rho) + \mathrm{A}(\rho)\right) + \left(\frac{\rho}{r}\right)^{\beta(s,\omega_0)} \delta(\rho), \qquad \forall\, \rho \geq 4r. \tag{2.37}$$

We can also choose $\tau(s, \omega_0)$ suitably small so that (2.37) and Lemma 2.8 hold with the same exponent $\tau(s, \omega_0)$. Using (2.34), (2.37) and Lemma 2.8, we have, for suitably small $\tau(s, \omega_0)$ and suitably large $\beta(s, \omega_0)$, that

$$\begin{aligned} \mathrm{M}(r) \leq\; &\mathrm{C}^*_{s,\omega_0} \left(\frac{r}{\rho}\right)^{\tau(s,\omega_0)} \mathrm{M}(\rho) + \mathrm{C}^*_{s,\omega_0} \left(\frac{\rho}{r}\right)^{\beta(s,\omega_0)} \delta(\rho) \\ &+ \mathrm{C}^*_{s,\omega_0} \left(\frac{\rho}{r}\right)^{6-6s} \mathrm{A}(\rho)\, \delta(\rho)^{1/2} + \mathrm{C}^*_{s,\omega_0} \left(\frac{\rho}{r}\right)^{7-4s} \mathrm{F}(\rho)^{4/3}, \end{aligned} \tag{2.38}$$

where $\mathrm{M}(r) = \mathrm{A}(r) + \mathrm{G}(r)^{2/3} + \mathrm{P}_{\omega_0}^{2/\omega_0}(r)$. We choose a $\gamma_0 < 1/6$ so that $\mathrm{C}^*_{s,\omega_0}\gamma_0^{\tau(s,\omega_0)} \leq 1/4$, then by (2.36), it holds:

$$\mathrm{M}(\gamma_0\rho) \leq \frac{1}{4}\,\mathrm{M}(\rho) + \mathrm{C}^*_{s,\omega_0}\gamma_0^{-\beta(s,\omega_0)}\,\epsilon_0 + \mathrm{C}^*_{s,\omega_0}\gamma_0^{6s-6}\epsilon_0^{1/2}\,\mathrm{A}(\rho) + \mathrm{C}^*_{s,\omega_0}\gamma_0^{4s-7}\mathrm{F}(\rho)^{4/3}, \qquad \forall\, \rho < \rho_0,$$

where ρ_0 and ϵ_0 satisfy the hypothesis in Lemma 2.8. Now we fix the γ_0 and choose ϵ_0 small enough such that $\mathrm{C}^*_{s,\omega_0}\gamma_0^{6s-6}\epsilon_0^{1/2} \leq 1/4$, then the above estimate implies that

$$\mathrm{M}(\gamma_0\rho) \leq \frac{1}{2}\,\mathrm{M}(\rho) + \mathrm{C}^*_{s,\omega_0}\gamma_0^{-\beta(s,\omega_0)}\,\epsilon_0 + \mathrm{C}^*_{s,\omega_0}\gamma_0^{4s-7}\mathrm{F}(\rho)^{4/3}, \quad \forall\, \rho < \rho_0. \tag{2.39}$$

In light of the integrability condition of f, we have that $\lim_{\rho\to 0^+}\mathrm{F}(\rho) = 0$. Therefore we can find a constant $\mathrm{C}^{**}_{s,\omega_0}$ such that

$$\mathrm{M}(\gamma_0\rho) \leq \frac{1}{2}\,\mathrm{M}(\rho) + \mathrm{C}^{**}_{s,\omega_0}\epsilon_0, \qquad \forall\, \rho < \rho_0.$$

Standard iteration argument shows that

$$\lim_{\rho\to 0}\mathrm{M}(\rho) \leq \mathrm{C}^{**}_{s,\omega_0}\epsilon_0.$$

□

Now we set up the assumptions for our inductive arguments in Sect. 2.3. For any $r > 0$, we rescale (u, p) by defining,

$$u_r(x) = r^{2s-1}\,u(rx, r^{2s}t), \quad p_r(x) = r^{4s-2}\,p(rx, r^{2s}t).$$

(u_r, p_r) solve (1.1) with the force $f_r(x) = r^{4s-1}f(rx, r^{2s}t)$. The Caffarelli-Silvestre extension in [2] of u_r is given by $u_r^*(x, y, t) = r^{2s-1}\,u^*(rx, ry, r^{2s}t)$.

By the small energy condition (1.4), we get that $\limsup_{r\to 0^+}\int_{\mathrm{Q}_1^*}\mathrm{y}^a\,|\overline{\nabla}u_r^*|^2 \leq \epsilon_0$. Since $q > (9+6s)/(4s+1)$ and $s > 3/4$, it holds

$$\lim_{r\to 0}\int_{\mathrm{Q}_1}|f_r(x,t)|^q = \lim_{r\to 0}\, r^{(4s-1)q-(3+2s)}\int_{\mathrm{Q}_r}|f(x,t)|^q = 0.$$

Therefore by Proposition 2.3, we have, for $r > 0$ small enough, that

$$\int_{\mathrm{Q}_1^*}\mathrm{y}^a\,|\overline{\nabla}u_r^*|^2 + \|f_r\|_{L^q(\mathrm{Q}_1)} + \left(\int_{\mathrm{Q}_1}|u_r(x,t)|^3\right)^{2/3} + \sup_{t\in \mathrm{I}_1}\int_{\mathrm{B}_1\times\{t\}}|u_r|^2 + \left[\int_{\mathrm{I}_1}\left(\int_{\mathrm{B}_1}|p_r|\mathrm{d}x\right)^{\omega_0}\mathrm{d}t\right]^{2/\omega_0} \lesssim_s \epsilon_0. \tag{2.40}$$

Fixing a small $r > 0$ such that (2.40) holds, we study (u_r, p_r, f_r) in what follows. Without ambiguity, we still use (u, p, f) to denote (u_r, p_r, f_r). By (2.14) and (2.40), we have

$$\int_{\mathrm{Q}_1^*}\mathrm{y}^a\,|u^*|^2 + \|f\|_{L^q(\mathrm{Q}_1)} + \left(\int_{\mathrm{Q}_1}|u(x,t)|^3\right)^{2/3}$$

$$+ \sup_{t\in I_1}\int_{B_1\times\{t\}}|u|^2 + \left[\int_{I_1}\left(\int_{B_1}|p|\mathrm{d}x\right)^{\omega_0}\mathrm{d}t\right]^{1/\omega_0}$$
$$\lesssim_s \left(\epsilon_0 + \epsilon_0^{1/2}\right). \tag{2.41}$$

Choosing ϵ_0, depending on s, small enough, we get, by (2.41), that

$$\int_{Q_1^*} y^a\,|u^*|^2 + \|f\|_{L^q(Q_1)} + \left(\int_{Q_1}|u(x,t)|^3\right)^{2/3}$$
$$+ \sup_{t\in I_1}\int_{B_1\times\{t\}}|u|^2 + \left[\int_{I_1}\left(\int_{B_1}|p|\mathrm{d}x\right)^{\omega_0}\mathrm{d}t\right]^{1/\omega_0} \le \epsilon_0^{1/3} := \epsilon_1. \tag{2.42}$$

This is the assumption for the inductive argument in Sect. 2.3.

2.3. Inductive arguments and L^∞ *- estimate of* u. In this section, we show that

Proposition 2.9. *There exists some small constant* $\epsilon_1 = \epsilon_1(s)$ *such that if the condition* (2.42) *holds, then*

$$⨍_{Q^k(x_0,t_0)}|u|^3 + r_k^{\alpha_0}\,|u|\cdot|\,p-\overline{p}_k\,| \le \epsilon_1^{2/3}, \quad \forall\, k\ge 3, \quad (x_0,t_0)\in Q^3$$

where $\overline{p}_k$ *denotes the average of* p *on the ball* $B^k(x_0) = B_{2^{-k}}(x_0)$ *and* $\alpha_0 = \max\{\frac{4s}{3} - 1,\ \frac{2s}{\omega_0} - 1\}$.

Remark. By Lebesgue differentiation Theorem (see [14]), Proposition 2.9 indicates the desired L^∞ - boundedness of u in Q^3.

Proof of Proposition 2.9. Using the same decomposition of p as in Lemma 3.2 of [1] (taking $r = 1/8$ and $\rho = 1/2$ there), we have, by (2.42), that

$$\int_{Q^3(x_0,t_0)}|u|\cdot|p-\overline{p}_3| \lesssim \left(\int_{Q^2}|u|^3\right)^{1/3}\left(\int_{Q^1}|u|^3\right)^{2/3}$$
$$+ \left(\int_{Q^2}|u|^3\right)^{1/3}\left(\sup_{t\in I_1}\int_{B_1}|u|^2\right)$$
$$+ \left(\int_{Q^2}|u|^3\right)^{1/3}\left(\int_{Q_1}|u|^3\right)^{2/3}$$
$$+ \left(\sup_{t\in I^2}\int_{B^2}|u|^2\right)^{1/2}\left(\int_{I_1}(\int_{B_1}|p|)^{\omega_0}\right)^{1/\omega_0}$$
$$\lesssim \epsilon_1^{3/2}.$$

Here we used the facts that $Q^3(x_0,t_0)\subset Q^2$, $Q^2(x_0,t_0)\subset Q^1$ and $Q^1(x_0,t_0)\subset Q^1$. Utilizing this estimate and (2.42), we know that

$$⨍_{Q^3(x_0,t_0)}|u|^3 + r_3^{\alpha_0}\,|u|\cdot|\,p-\overline{p}_3\,| \le C\left(\epsilon_1 + \epsilon_1^{3/2}\right) \le \epsilon_1^{2/3}, \tag{2.43}$$

provided that ϵ_1 is small enough.

Inductively we assume

$$\fint_{Q^l(x_0,t_0)} |u|^3 + r_l^{\alpha_0} |u| \cdot |p - \overline{p}_l| \le \epsilon_1^{2/3}, \qquad \forall\ 3 \le l \le k. \tag{2.44}$$

By Lemma 2.1 and (2.42), we have, for all $3 \le i \le k$, that

$$\sup_{t \in I^i(t_0)} \fint_{B^i(x_0)} |u|^2 + r_i^{-3} \int_{Q_*^i(x_0,t_0)} y^a |\overline{\nabla} u^*|^2$$
$$\le C_s^* \epsilon_1 + C_s^* \sum_{l=3}^{i} r_l^{2s} \fint_{Q^l(x_0,t_0)} |f|\,|u| + r_l^{-a} \left[\fint_{Q^l(x_0,t_0)} |u|^3 + |u| \cdot |p - \overline{p}_l| \right]. \tag{2.45}$$

For simplicity we define

$$G_i = \sum_{l=3}^{i} r_l^{2s} \fint_{Q^l(x_0,t_0)} |f|\,|u|, \qquad H_i = \sum_{l=3}^{i} r_l^{-a} \left[\fint_{Q^l(x_0,t_0)} |u|^3 + |u| \cdot |p - \overline{p}_l| \right],$$
$$\forall 3 \le i \le k.$$

For the term G_i, we have, by Young's inequality, that

$$G_i \le \sum_{l=3}^{i} r_l^{2s} \fint_{Q^l(x_0,t_0)} \left(r_l^{1/2} |f|^{3/2} + |u|^3\, r_l^{-1} \right) \le \sum_{l=3}^{i} r_l^{2s+1/2} \fint_{Q^l(x_0,t_0)} |f|^{3/2}$$
$$+ \sum_{l=3}^{i} r_l^{-a} \fint_{Q^l(x_0,t_0)} |u|^3.$$

The last term above can be absorbed by H_i. As for the term with the force f, we apply the integrability condition on f and show, by Hölder's inequality, that

$$\sum_{l=3}^{i} r_l^{2s+1/2} \fint_{Q^l(x_0,t_0)} |f|^{3/2} \lesssim_q \sum_{l=3}^{i} r_l^{2s+1/2-(9+6s)/(2q)} \left(\int_{Q_1} |f|^q \right)^{3/(2q)} \lesssim_{q,s} \epsilon_1^{3/2}.$$

Here we used the condition that $q > (9 + 6s)/(4s + 1)$, by which the summation in the first inequality above is uniformly bounded by a finite number independent of k. The last two estimates above show that $G_i \le C_{q,s} \epsilon_1^{3/2} + H_i$, for all $3 \le i \le k$. In light of (2.44), H_i can be bounded from above by $C_s \epsilon_1^{2/3}$. Thus if ϵ_1 is small enough, then we have

$$G_i + H_i \lesssim_s \epsilon_1^{2/3}, \quad \forall 3 \le i \le k.$$

Applying the last estimate above to (2.45), we have

$$\sup_{t \in I^i(t_0)} \fint_{B^i(x_0)} |u|^2 + r_i^{-3} \int_{Q_*^i(x_0,t_0)} y^a |\overline{\nabla} u^*|^2 \lesssim_s \epsilon_1^{2/3}, \quad \forall\ 3 \le i \le k. \tag{2.46}$$

In the following, we show that (2.44) for $l = k+1$. By Hölder's inequality and Proposition 2.2, we have

$$\int_{B^{k+1}(x_0)} |u|^3 \le \left(\int_{B^{k+1}(x_0)} |u|^2\right)^{-3a/(4s)} \left(\int_{B^{k+1}(x_0)} |u|^{6/(3-2s)}\right)^{\frac{3-2s}{6}\cdot\frac{6}{4s}}$$
$$\lesssim_s \left(\int_{B^k(x_0)} |u|^2\right)^{-3a/(4s)} \left\{ r_k^{-s} \left(\int_{B^k(x_0)} |u|^2\right)^{1/2} + \left(\int_{Q_+^k(x_0)} y^a |\overline{\nabla} u^*|^2\right)^{1/2}\right\}^{6/(4s)}$$

which shows that

$$\fint_{Q^{k+1}(x_0,t_0)} |u|^3 \lesssim_s \left(\sup_{t\in I_k(t_0)} \fint_{B^k(x_0)} |u|^2\right)^{-3a/(4s)} \cdot \left\{\left(\sup_{t\in I_k(t_0)} \fint_{B^k(x_0)} |u|^2\right)^{3/(4s)} + \left(r_k^{-3} \int_{Q_*^k(x_0,t_0)} y^a |\overline{\nabla} u^*|^2\right)^{3/(4s)}\right\}.$$

Applying (2.46) to the right-hand side above, we get, for sufficiently small ϵ_1, that

$$\fint_{Q^{k+1}(x_0,t_0)} |u|^3 \lesssim_s \epsilon_1 \le 1/2\, \epsilon_1^{2/3}. \tag{2.47}$$

On the other hand, we decompose p by setting $p = p_1 + p_2 + p_3 + p_4$, same as the decomposition used in Lemma 3.2 of [1] (taking $r = r_{k+1}$ and $\rho = 1$ there). Then by (2.42), (2.45)–(2.46) and similar arguments for the proof of Lemma 3.2 in [1], we have the following estimates:

$$\fint_{Q^{k+1}(x_0,t_0)} |u|\cdot|p_1 - (p_1)_{B^{k+1}}| \lesssim \left(\fint_{Q^{k+1}(x_0,t_0)} |u|^3\right)^{1/3} \left(\fint_{Q^k(x_0,t_0)} |u|^3\right)^{2/3} \lesssim_s \epsilon_1^{7/9},$$

$$\fint_{Q^{k+1}(x_0,t_0)} |u|\cdot|p_2 - (p_2)_{B^{k+1}}| \lesssim r_{k+1} \left(\fint_{Q^{k+1}(x_0,t_0)} |u|^3\right)^{1/3} \sum_{l=0}^{k} r_l^{-1} \sup_{t\in I^l(t_0)} \fint_{B^l(x_0)} |u|^2 \lesssim_s \epsilon_1^{4/3},$$

$$\fint_{Q^{k+1}(x_0,t_0)} |u|\cdot|p_3 - (p_3)_{B^{k+1}}| \le r_{k+1}^{1-4s/3} \left(\fint_{Q^{k+1}(x_0,t_0)} |u|^3\right)^{1/3} \left(\int_{Q_1} |u|^3\right)^{2/3} \lesssim_s r_{k+1}^{1-4s/3}\, \epsilon_1,$$

$$\fint_{Q^{k+1}(x_0,t_0)} |u|\cdot|p_4 - (p_4)_{B^{k+1}}| \lesssim r_{k+1}^{1-2s/\omega_0} \left(\sup_{t\in I^{k+1}(t_0)} \fint_{B^{k+1}(x_0)} |u|^2\right)^{1/2}$$

$$\times \left(\int_{\mathrm{I}_1} \Big(\int_{\mathrm{B}_1} |p| \Big)^{\omega_0} \right)^{1/\omega_0}$$

$$\lesssim_s \; r_{k+1}^{1-2s/\omega_0} \epsilon_1^{3/2}.$$

Adding the four estimates above together, we get that

$$r_{k+1}^{\alpha_0} \fint_{\mathrm{Q}^{k+1}(x_0,t_0)} |u| \cdot |p - \overline{p}_{k+1}| \lesssim_s \left(\epsilon_1^{7/9} + \epsilon_1^{4/3} + \epsilon_1 + \epsilon_1^{3/2} \right).$$

Therefore when ϵ_1 small enough, it holds

$$r_{k+1}^{\alpha_0} \fint_{\mathrm{Q}^{k+1}(x_0,t_0)} |u| \cdot |p - \overline{p}_{k+1}| \leq 1/2\, \epsilon_1^{2/3}. \tag{2.48}$$

From (2.47) and (2.48), we see that (2.44) is true for $l = k + 1$. □

3. Estimate of the Singular Set

By Theorem 1.2, $\mathrm{Sing}(u)$ is a relatively closed set. In the following we prove

$$\mathcal{H}^{5-4s}(\mathrm{Sing}(u)) = 0. \tag{3.1}$$

Here $\mathcal{H}^{5-4s}$ is the $(5-4s)$-dimensional Hausdorff measure. Fixing a $\delta > 0$, then for any $(x, t) \in \mathrm{Sing}(u)$, we can find a $r_{x,t} < \delta$ such that

$$r_{x,t}^{4s-5} \int_{\mathrm{Q}^*_{r_{x,t}}(x,t)} \mathrm{y}^a\, |\overline{\nabla} u^*|^2 \geq \epsilon_0/2. \tag{3.2}$$

Here $\mathrm{Q}^*_{r_{x,t}}(x,t) = B_{r_{x,t}}(x) \times (0, r_{x,t}) \times (t - \frac{1}{2} r_{x,t}^{2s}, t + \frac{1}{2} r_{x,t}^{2s})$. Using $\mathrm{Q}_{r_{x,t}}(x,t) = B_{r_{x,t}}(x) \times (t - \frac{1}{2} r_{x,t}^{2s}, t + \frac{1}{2} r_{x,t}^{2s})$, we define the family $\mathcal{F} = \{\mathrm{Q}_{r_{x,t}}(x,t) : (x,t) \in \mathrm{Sing}(u)\}$, which forms a covering of $\mathrm{Sing}(u)$. By Lemma 6.1 in [1], there exists a sequence of cylinders $\{\mathrm{Q}_{r_{x_i,t_i}}(x_i, t_i)\} \subset \mathcal{F}$ so that these cylinders are mutually disjoint and satisfy $\mathrm{Sing}(u) \subset \cup_i \mathrm{Q}_{5r_{x_i,t_i}}(x_i, t_i)$. For simplicity we denote $\mathrm{Q}_{r_{x_i,t_i}}(x_i, t_i)$, $\mathrm{Q}^*_{r_{x_i,t_i}}(x_i, t_i)$ by $\mathrm{Q}_{r_{x_i,t_i}}$ and $\mathrm{Q}^*_{r_{x_i,t_i}}$, respectively. Using (3.2), we get

$$\begin{aligned} \sum_i (5r_i)^{5-4s} &= 5^{5-4s} \sum_i r_i^{5-4s} \lesssim_s \epsilon_0^{-1} \sum_i \int_{\mathrm{Q}^*_{r_i,x_i}} \mathrm{y}^a\, |\overline{\nabla} u^*|^2 \\ &= \epsilon_0^{-1} \int_{\bigcup_i \mathrm{Q}^*_{r_i,x_i}} \mathrm{y}^a\, |\overline{\nabla} u^*|^2. \end{aligned} \tag{3.3}$$

Still by (3.2), the following estimate holds

$$\sum_i r_i^{4+2s} \lesssim \epsilon_0^{-1} \sum_i r_i^{6s-1} \int_{\mathrm{Q}^*_{r_i,x_i}} \mathrm{y}^a\, |\overline{\nabla} u^*|^2 \lesssim \epsilon_0^{-1}\, \delta^{6s-1} \int_{\bigcup_i \mathrm{Q}^*_{r_i,x_i}} \mathrm{y}^a\, |\overline{\nabla} u^*|^2.$$

This estimate shows that the Lebesgue measure of $\bigcup_i Q^*_{r_{x_i,t_i}}$ can be estimated by

$$\left| \bigcup_i Q^*_{r_{x_i,t_i}} \right| \lesssim \sum_i r_{x_i,t_i}^{4+2s} \lesssim \epsilon_0^{-1} \delta^{6s-1} \int_{\cup_i Q^*_{r_{x_i,t_i}}} y^a\, \left|\overline{\nabla} u^*\right|^2.$$

Since $y^a|\overline{\nabla} u^*|^2$ is integrable, we can always choose $\delta > 0$ small enough so that $\bigcup_i Q^*_{r_{x_i,t_i}}$ has small Lebesgue measure. This fact, together with the integrability of $y^a|\bar{\nabla} u^*|^2$, implies that the most right-hand side of (3.3) can be bounded from above by an arbitarily small constant $\epsilon > 0$, provided that δ is small enough. With this δ, we can show that

$$\mathcal{H}^{5-4s}(\mathrm{Sing}(u)) \le \mathcal{H}^{5-4s}_\delta(\mathrm{Sing}(u)) \lesssim \sum_i (5r_{x_i,t_i})^{5-4s} \le \epsilon.$$

Therefore (3.1) holds by taking $\epsilon \to 0$ in the above estimate.

4. Appendix: Existence of Suitable Weak Solution

In this part, we use the following function spaces:

$$\begin{aligned}
\mathcal{V} &= \{v \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3) : \ \mathrm{div}\ v = 0\};\\
\mathrm{H} &= \text{the closure of } V \text{ in } L^2(\mathrm{R}^3);\\
\mathrm{V} &= \dot{\mathrm{H}}^s_{\mathrm{div}}, \quad \text{the closure of } \mathcal{V} \text{ under the norm}\\
\|u\|^2_{\dot{\mathrm{H}}^s} &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y.\\
\mathrm{V}' &= \text{the dual space of } \mathrm{V}.
\end{aligned}$$

Our main result is as follows:

Theorem 4.1. *Suupose that $u_0 \in \mathrm{H}$ and $f \in L^2(0,T;\mathrm{V}')$. Then there exists a weak solution (u,p) of (1.1) satisfying the following conditions: $u \in L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{V})$ and $u(t) \to u_0$ weakly in H and for each nonnegative smooth function $\psi(x,\mathrm{y},t)$ with compact support and $t \in (0,T)$,*

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\{t\}} |u|^2\psi + 2\mathrm{C}_s\int_0^t\int_{\mathbb{R}^4_+} \psi\, \mathrm{y}^a|\overline{\nabla} u^*|^2\\
&\le \mathrm{C}_s\int_0^t\int_{\mathbb{R}^4_+} |u^*|^2\ \mathrm{Div}\left(\mathrm{y}^a\overline{\nabla}\psi\right) + \int_0^t\int_{\mathbb{R}^3} (u\cdot\nabla\psi)\left(2p+|u|^2\right)\\
&\quad + \int_0^t\int_{\mathbb{R}^3} |u|^2\{\psi_t + \mathrm{C}_s \lim_{\mathrm{y}\to 0^+}\left(\mathrm{y}^a\partial_\mathrm{y}\psi\right)\} + 2\int_0^t\int_{\mathbb{R}^3}\psi f\cdot u. \qquad (4.1)
\end{aligned}$$

where u^* is the extension of u satisfying (1.2).

We split the proof of Theorem 4.1 into several parts. We consider the following hypervisicosity perturbation of the Navier–Stokes equations:

$$\begin{cases}
\frac{d}{dt}u_\epsilon + (-\Delta)^s u_\epsilon - \epsilon\Delta u_\epsilon + u_\epsilon\cdot\nabla u_\epsilon + \nabla p_\epsilon = f,\ \forall\,\epsilon > 0.\\
u_\epsilon \in L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{V}\cap\mathrm{V}_1),\\
u_\epsilon(0) = u_0.
\end{cases} \qquad (4.2)$$

where $\mathrm{V}_1 =$ the closure of $\mathcal{V}$ under the norm $\|u\|_{\dot{\mathrm{H}}^1} = \left(\int_{\mathbb{R}^3}|\nabla u(x)|^2\,\mathrm{d}x\right)^{1/2}$.

In order to study (4.2), we need the following lemma:

Lemma 4.2. *Let u_0 and f satisfy the conditions in Theorem 4.1 and $w \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot w = 0$. Then there exists (u, p) such that $u \in L^\infty(0, T; \mathrm{H}) \cap L^2(0, T; \mathrm{V} \cap \mathrm{V}_1)$, $u(0) = u_0$ and it solves*

$$u_t + w \cdot \nabla u + (-\Delta)^s u - \epsilon \Delta u + \nabla p = f \tag{4.3}$$

in the weak sense. And also for each nonnegative smooth function $\psi(x, \mathrm{y}, t)$ with compact support, the following equality holds:

$$\begin{aligned}&\int_{\mathbb{R}^3\times\{t\}} |u|^2\psi + 2\mathrm{C}_s \int_0^t \int_{\mathbb{R}^4_+} \psi\, \mathrm{y}^a |\overline{\nabla} u^*|^2 + 2\epsilon \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \psi \\ &= \epsilon \int_0^t \int_{\mathbb{R}^3} |u|^2 \Delta\psi + \mathrm{C}_s \int_0^t \int_{\mathbb{R}^4_+} |u^*|^2 \ \mathrm{Div}\left(\mathrm{y}^a \overline{\nabla}\psi\right) + \int_0^t \int_{\mathbb{R}^3} \left(2pu + |u|^2 w\right) \cdot \nabla\psi \\ &\quad + \int_0^t \int_{\mathbb{R}^3} |u|^2 \{\psi_t + \mathrm{C}_s \lim_{\mathrm{y}\to 0^+} \left(\mathrm{y}^a \partial_\mathrm{y} \psi\right)\} + 2 \int_0^t \int_{\mathbb{R}^3} \psi f \cdot u.\end{aligned} \tag{4.4}$$

Proof. The existence of weak solution to (4.3) may be proved by using the Faedo-Galerkin method, and the argument is similar to the proof of Theorem 1.1 in Chapter III of Temam [17] (also see Lemma A3 and A7 in [1]). We omit the details.

In the following, we will prove the equality (4.4). Writing $F = f - w \cdot \nabla u$, then

$$u_t + (-\Delta)^s u - \epsilon \Delta u + \nabla p = F. \tag{4.5}$$

Mollifying (in $\mathbb{R}^3 \times \mathbb{R}$) each term of (4.5), we get sequences of smooth functions $\{u^m\}$, $\{p^m\}$ and $\{F^m\}$ such that the following holds:

$$\frac{\mathrm{d}u_m}{\mathrm{d}t} + (-\Delta)^s u_m - \epsilon \Delta u_m + \nabla p_m = F_m, \quad \mathrm{div}\, u_m = 0. \tag{4.6}$$

and as $m \to \infty$, we know that

$$\begin{cases} u_m \to u \quad \text{in} \quad L^{2+4s/3}(\mathbb{R}^3 \times (0, T)); \quad u_m^* \to u^* \quad \text{in } L^{2+4s/3}(\mathbb{R}^4_+ \times (0, T), \mathrm{y}^a); \\ \overline{\nabla} u_m^* \to \overline{\nabla} u^* \quad \text{in } L^2(\mathbb{R}^4_+ \times (0, T), \mathrm{y}^a); \\ p_m \to p \quad \text{in } L^{1+2s/3}(\mathbb{R}^3 \times (0, T)); \quad F_m \to F \quad \text{in } L^2(\mathbb{R}^3 \times (0, T)). \end{cases} \tag{4.7}$$

Multiplying both sides of (4.6) with $u_m\psi$ and integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^3\times\{t\}} |u_m|^2\psi + 2\mathrm{C}_s \int_0^t \int_{\mathbb{R}^4_+} \psi\, \mathrm{y}^a |\overline{\nabla} u_m^*|^2 + 2\epsilon \int_0^t \int_{\mathbb{R}^3} |\nabla u_m|^2 \psi \\ &= \epsilon \int_0^t \int_{\mathbb{R}^3} |u|^2 \Delta\psi + \mathrm{C}_s \int_0^t \int_{\mathbb{R}^4_+} |u_m^*|^2 \ \mathrm{Div}\left(\mathrm{y}^a \overline{\nabla}\psi\right) + \int_0^t \int_{\mathbb{R}^3} 2p_m u_m \cdot \nabla\psi \\ &\quad + \int_0^t \int_{\mathbb{R}^3} |u_m|^2 \{\psi_t + \mathrm{C}_s \lim_{\mathrm{y}\to 0^+} \left(\mathrm{y}^a \partial_\mathrm{y} \psi\right)\} + 2 \int_0^t \int_{\mathbb{R}^3} \psi F_m \cdot u_m.\end{aligned} \tag{4.8}$$

which, by (4.7), shows (4.4). □

Let $\{\Psi_\delta\}_{\delta>0}$ be the retarded mollifier defined in [1]. Then by direct computation, we have the following lemma (here the details of its proof is omitted):

Lemma 4.3. *For any* $u \in L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{V})$, *we have that* $\nabla \cdot \Psi_\delta(u) = 0$, *and*

$$\sup_{0\le t\le T} \int_{\mathbb{R}^3} |\Psi_\delta(u)|^2 \mathrm{d}x \le \mathrm{C}_s \sup_{0\le t\le T} \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x,$$

$$\int_0^T \|\Psi_\delta(u)\|^2_{\dot{\mathrm{H}}^s} \mathrm{d}t \le \mathrm{C}_s \|u\|^2_{L^2(0,T;\mathrm{V})}$$

where C_s *is a positive constant depending only on* s.

For the problem (4.2), we have:

Lemma 4.4. *Suupose that* $u_0 \in \mathrm{H}$ *and* $f \in L^2(\mathbb{R}^3 \times (0,T))$. *Then there is a weak solution* (u_ϵ, p_ϵ) *to* (4.2) *such the following energy inequality holds: for each nonnegative smooth function* $\psi(x,\mathrm{y},t)$ *with compact support and for any* $t \in (0,T)$, $\epsilon > 0$

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\{t\}} |u_\epsilon|^2\psi + 2\mathrm{C}_s \int_0^t \int_{\mathbb{R}^4_+} \psi\, \mathrm{y}^a |\overline{\nabla} u^*_\epsilon|^2 + 2\epsilon \int_0^t \int_{\mathbb{R}^3} \psi\, |\nabla u_\epsilon|^2 \\
&\le \epsilon \int_0^t \int_{\mathbb{R}^3} \psi\, |u_\epsilon|^2 + \mathrm{C}_s \int_0^t \int_{\mathbb{R}^4_+} |u^*_\epsilon|^2 \ \mathrm{Div}\left(\mathrm{y}^a \overline{\nabla} \psi\right) \\
&\quad + \int_0^t \int_{\mathbb{R}^3} (u_\epsilon \cdot \nabla \psi) \left(2p_\epsilon + |u_\epsilon|^2\right) \\
&\quad + \int_0^t \int_{\mathbb{R}^3} |u_\epsilon|^2 \{\psi_t + \mathrm{C}_s \lim_{\mathrm{y}\to 0^+} \left(\mathrm{y}^a \partial_\mathrm{y} \psi\right)\} + 2\int_0^t \int_{\mathbb{R}^3} \psi f \cdot u_\epsilon.
\end{aligned} \tag{4.9}$$

Proof. For any large N, let $\delta = T/N$ and we solve the following system:

$$\begin{cases} \frac{d}{dt} u_N + (-\Delta)^s u_N - \epsilon \Delta u_N + \Psi_\delta(u_N) \cdot \nabla u_N + \nabla p_N = f, \\ u_N \in L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{V}) \ \text{ and } \ u_N(0) = u_0. \end{cases}$$

Such u_N and p_N exist by applying Lemma 4.2 inductively on each time interval $(m\delta, (m+1)\delta)$, $0 \le m \le N-1$. Obviously, for any $0 < t < T$, it holds

$$\int_{\mathbb{R}^3\times\{t\}} |u_N|^2 + \mathrm{C}_s \int_0^t \|u_N\|^2_{\dot{\mathrm{H}}^s} + \epsilon \int_0^t \|u_N\|^2_{\dot{\mathrm{H}}^1} = \int_{\mathbb{R}^3} |u_0|^2 + 2\int_0^t \int_{\mathbb{R}^3} f \cdot u_N$$

which indicates that for any fixed $\epsilon > 0$ $\{u_N\}$ is bounded in $L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{V} \cap \mathrm{V}_1)$. Let V_2 be the closure of $\mathcal{V}$ in $H^2(\mathbb{R}^3)$ and V_2' be its dual space. Obviously, $\frac{d}{dt} u_N$ is bounded in $L^2(0,T;\mathrm{V}_2')$. By Theorem 2.1 in Chapter III of Temam [17], we have that

$$\{u_N\} \text{ stays in a compact subset of } L^2(\mathbb{R}^3 \times (0,T)). \tag{4.10}$$

Since $\{u_N\}$ is bounded in $L^\infty(0,T;\mathrm{H}) \cap L^2(0,T;\mathrm{V})$, by the interpolation inequality, we get

$$\begin{cases} \{u_N\} \ \text{is bounded in } L^{2+4s/3}(\mathbb{R}^3 \times (0,T)), \\ \{p_N\} \ \text{is bounded in } L^{1+2s/3}(\mathbb{R}^3 \times (0,T)). \end{cases} \tag{4.11}$$

Therefore there are subsequences of (u_N, p_N), for simplicity, still denoted by $\{u_N\}$ converging to (u_ϵ, p_ϵ) as $N \to \infty$:

$$\begin{cases} u_N \to u_\epsilon \ \text{ weakly in } L^2(0,T;\mathrm{V}\cap\mathrm{V}_1) \text{ and } \ u_N \to u_\epsilon \ \text{ weak-star in } L^\infty(0,T;\mathrm{H}); \\ u_N \to u_\epsilon \ \text{ strongly in } L^\alpha(\mathbb{R}^3\times(0,T)), \ \text{where } 2 \le \alpha < 2+4s/3; \\ \Psi_\delta(u_N) \to u_\epsilon \ \text{ strongly in } L^\alpha(\mathbb{R}^3\times(0,T)), \ \text{where } 2 \le \alpha < 2+4s/3; \\ p_N \to p_\epsilon \ \text{ weakly in } L^{1+2s/3}(\mathbb{R}^3\times(0,T)). \end{cases} \tag{4.12}$$

It is easy to check that (u_ϵ, p_ϵ) is a weak solution of (4.2). We only need to verify the energy inequality for (u_ϵ, p_ϵ). By Lemma 4.2, we have

$$\begin{aligned} &\int_{\mathbb{R}^3\times\{t\}} |u_N|^2\psi + 2\mathrm{C}_s\int_0^t\int_{\mathbb{R}^4_+}\psi\, \mathrm{y}^a|\overline{\nabla}u_N^*|^2 + \epsilon\int_0^t\int_{\mathbb{R}^3}\psi\,|\nabla u_N|^2 \\ &= \epsilon\int_0^t\int_{\mathbb{R}^3}\psi\,|u_N|^2 + \int_0^t\int_{\mathbb{R}^3}\Big(2p_N u_N + |u_N|^2\Psi_\delta(u_N)\Big)\cdot\nabla\psi \\ &\quad + \mathrm{C}_s\int_0^t\int_{\mathbb{R}^4_+}|u_N^*|^2\ \mathrm{Div}\left(\mathrm{y}^a\overline{\nabla}\psi\right) + \int_0^t\int_{\mathbb{R}^3}|u_N|^2\{\psi_t + \mathrm{C}_s\lim_{\mathrm{y}\to 0^+}\left(\mathrm{y}^a\partial_\mathrm{y}\psi\right)\} \\ &\quad + 2\int_0^t\int_{\mathbb{R}^3}\psi f\cdot u_N \end{aligned} \tag{4.13}$$

For u_N^*, by potential theory, it satisfies the following properties:

$$\begin{cases} u_N^* \to u_\epsilon^* \text{ strongly in } \ L^\alpha(\mathrm{y}^a,\mathbb{R}^4_+\times(0,T)), \ 2\le\alpha<2+4s/3; \\ \overline{\nabla}u_N^* \to \overline{\nabla}u_\epsilon^* \text{ weakly in } \ L^2(\mathrm{y}^a,\mathbb{R}^4_+\times(0,T)) \end{cases} \tag{4.14}$$

Applying (4.12) and (4.14) to (4.13), we get the energy inequality (4.9). □

Finally we go back to the proof of Theorem 4.1

Proof of Theorem 4.1. Firstly from (4.2), we have

$$\int_{\mathbb{R}^3\times\{t\}}|u_\epsilon|^2 + \mathrm{C}_s\int_0^t\|u_\epsilon\|^2_{\dot{\mathrm{H}}^s} + \epsilon\int_0^t\|u_\epsilon\|^2_{\dot{\mathrm{H}}^1} = \int_{\mathbb{R}^3}|u_0|^2 + 2\int_0^t\int_{\mathbb{R}^3} f\cdot u_\epsilon$$

By fractional Nirenberg-Gargliardo inequality in [10] and Hölder inequality, we get

$$\int_{\mathbb{R}^3\times\{t\}}|u_\epsilon|^2 + \mathrm{C}_s\int_0^t\left(\int_{\mathbb{R}^3}|u_\epsilon|^{6/(3-2s)}\right)^{\frac{3-2s}{3}} \le \int_{\mathbb{R}^3\times\{t\}}|u_\epsilon|^2 + \mathrm{C}_s\int_0^t\|u_\epsilon\|^2_{\dot{\mathrm{H}}^s} \le \ C \tag{4.15}$$

where C is a positive universal constant not depending on ϵ.

Using the similar arguments as in Lemma 4.4, we conclude the existence of subsequences, still denoted by (u_ϵ, p_ϵ) converging to (u, p) as $\epsilon \to 0$;

$$\begin{cases} u_\epsilon \to u \ \text{ weakly in } L^2(0,T;\mathrm{V}) \text{ and } \ u_\epsilon \to u \ \text{ weak-star in } L^\infty(0,T;\mathrm{H}); \\ u_\epsilon \to u \ \text{ strongly in } L^\alpha(\mathbb{R}^3\times(0,T)) \text{ and } u_\epsilon^* \to u^* \ \text{ strongly in } \ L^\alpha(\mathrm{y}^a,\mathbb{R}^4_+\times(0,T)), \\ \qquad\qquad \forall\, 2\le\alpha<2+4s/3; \\ p_\epsilon \to p \ \text{ weakly in } L^{1+2s/3}(\mathbb{R}^3\times(0,T)); \ \ \overline{\nabla} u_\epsilon^* \to \overline{\nabla} u^* \\ \qquad\qquad \text{weakly in } \ L^2(\mathrm{y}^a,\mathbb{R}^4_+\times(0,T)). \end{cases} \tag{4.16}$$

For any test function ϕ with compact support in time and space, we have, by (4.16), we have that $\lim_{\epsilon\to 0}\epsilon\int_0^t\int_{\mathbb{R}^3}\Delta u_\epsilon\phi = \lim_{\epsilon\to 0}\epsilon\int_{t_0}^t\int_{\mathbb{R}^3}u_\epsilon\Delta\phi = 0$. Applying this and (4.16) to (4.2), we show that $(u,\,p)$ solves (1.1) in the weak sense.

Finally, we check the energy inequality for $(u,\,p)$. By (4.16), we have $\lim_{\epsilon\to 0}\epsilon\int_0^t\int_{\mathbb{R}^3}|u_\epsilon|^2\psi=0$. When $\epsilon\to 0$, $\int_0^t\int_{\mathbb{R}^4_+}\psi\mathrm{y}^a|\overline{\nabla}u_\epsilon^*|^2$ is lower semicontinuous and $\int_0^t\int_{\mathbb{R}^3}\psi|\nabla u_\epsilon|^2$ is nonnegative. Meanwhile, by (4.16), each other term in (4.9) converges to the corresponding term involving $u,\ p,\ u^*$. This proves the energy inequality (4.1). □

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