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Wave Relations

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Abstract: The wave equation (free boson) problem is studied from the viewpoint of the relations on the symplectic manifolds associated to the boundary induced by solutions. Unexpectedly, there is still something to say about this simple, well-studied problem. In particular, boundaries which do not allow for a meaningful Hamiltonian evolution are not problematic from the viewpoint of relations. In the two-dimensional Minkowski case, these relations are shown to be Lagrangian. This result is then extended to a wide class of metrics and is conjectured to be true also in higher dimensions for nice enough metrics. A counterexample where the relation is not Lagrangian is provided by the Misner space.

Contents

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1. Introduction

In this note we study the wave equation from the point of view of evolution relations (as defined in [3,4]). In particular, we show that they are well behaved also in cases when the boundary does not allow for a meaningful Hamiltonian evolution. This is a case study for a simple well-studied problem (on which, unexpectedly, there was still something to say) supporting the relevance of the evolution relation approach. This note is self-contained and the relevant concepts from [3,4] are introduced when needed.

Fix a dimension m and a signature. To an m-dimensional compact oriented pseudo-Riemannian manifold (M, g), possibly with boundary, whose metric has the given signature, we associate a space of fields[1] $F_M := \mathcal{C}^\infty(M)$ and an action functional

$$S_{M,g}[\phi] := \frac{1}{2}\int_M d\phi \wedge *_g d\phi, \qquad \phi \in F_M, \tag{1}$$

where $*_g$ denotes the Hodge-$*$ operator induced by the metric g. More explicitly, writing the integrand in a local chart,

$$S_{M,g}[\phi] = \frac{1}{2}\int_M g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\,\sqrt{g}\,d^m x,$$

where $g = |\det(g_{\mu\nu})|$. According to the construction in [4], to an $(m-1)$-manifold Σ (with the extra structure of a function, a vector field, and a volume form) we can associate a space of boundary fields[2] (or phase space) Φ_Σ endowed with a symplectic structure $\omega_\Sigma = \delta\alpha_\Sigma$, where α_Σ is the 1-form on Φ_Σ arising as the boundary term of the variation of the action (1), such that for every M as above we get an epimorphism (and hence a surjective submersion) $\pi_M : F_M \to \Phi_{\partial M}$; moreover, $L_M := \pi_M(EL_M)$ is isotropic in $(\Phi_{\partial M}, \omega_{\partial M})$, where EL_M is the subset of solutions to the Euler–Lagrange (EL) equation $d *_g d\phi = 0$ or in local coordinates:

$$\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi) = 0. \tag{2}$$

[1] We mainly consider smooth functions in this note, which requires working in the setting of Fréchet spaces. For less regularity, and the corresponding Banach setting, see Sect. 5.5.1.

[2] Notice that F_M and Φ_Σ are Fréchet spaces and hence Fréchet manifolds.

Conjecture 1.1. *For any compact oriented pseudo-Riemannian manifold (M, g) with boundary, which can be isometrically embedded into some Euclidean space $\mathbb{R}^N$ equipped with constant metric, the subspace $L_M \subset \Phi_{\partial M}$ is Lagrangian.*

The Conjecture is easily proved in the case of Riemannian manifolds [3] from existence and uniqueness of solutions to the Dirichlet boundary problem $d * d\phi = 0$, $\phi|_{\partial M} = \phi_\partial$ where $\phi_\partial \in \mathcal{C}^\infty(\partial M)$ is a boundary condition for ϕ.

The Conjecture is also true if M is a pseudo-Riemannian manifold of the form $\Sigma \times I$, where $I = [t_0, t_1]$ is an interval, provided that the metric g pulls back to a non-degenerate metric g_t on $\Sigma \times \{t\}$ for any $t \in I$. This follows from existence and uniqueness for the initial value problem for the EL equation, with Cauchy data being a point of $\Phi_{\Sigma\times\{t_0\}}$, implying that L_M is the graph of an isomorphism $\Phi_{\Sigma\times\{t_0\}} \to \Phi_{\Sigma\times\{t_1\}}$. On the other hand, L_M is isotropic by a universal argument of [3,4], hence L_M is Lagrangian.

If the boundary of M is split into "incoming" and "outgoing" parts $\partial M = (\partial_{\text{in}} M)^{\text{op}} \sqcup \partial_{\text{out}} M$, then the subspace $L_M \subset \Phi_{\partial M} = \bar{\Phi}_{\partial_{\text{in}} M} \times \Phi_{\partial_{\text{out}} M}$ can be interpreted as a set-theoretic relation between $\Phi_{\partial_{\text{in}} M}$ and $\Phi_{\partial_{\text{out}} M}$—the "evolution relation"; the property that L_M is Lagrangian means that the evolution relation is a canonical relation between symplectic spaces. Here "op" stands for reversing the orientation and bar stands for changing the sign of the symplectic form. Gluing of manifolds along common boundary corresponds in this setting to set-theoretical composition of relations.

In this note, see Sect. 4, Theorem 4.2, we prove the following case of the Conjecture.

Theorem. *Let $m = 2$, M a compact domain with smooth boundary in the Minkowski plane, such that there are only finitely many boundary points with light-like tangent and such that the curvature of the boundary is nonzero at these points, and let g be the Minkowski metric restricted to M. Then L_M is Lagrangian.*

Notice that ∂M generally has several space-like and several time-like pieces separated by light-like points. A consequence of the theorem is that we can study the wave equation on compact domains: the appropriate boundary conditions consist in the choice of an affine Lagrangian subspace L' of $\Phi_{\partial M}$ that intersects L_M in one point and on which $\alpha_{\partial M} + \delta f$ vanishes for some local functional f on $\Phi_{\partial M}$. This also means that quantization on compact domains is possible, provided a suitable polarization of $\Phi_{\partial M}$ can be found.

It is tempting to extend the conjecture also to more general pseudo-Riemannian manifolds. This is expected to be the case for metrics that are nice enough, e.g., which do not differ much from the constant one (which is the case for small domains). However, this is not true in general and in Sect. 5.9 we show that the Misner space [8] provides a counterexample. From a different perspective, we may say that the condition that L is Lagrangian selects reasonable spacetimes.

If M is of the form $\Sigma \times [t_0, t_1]$, one can attempt to define the Hamiltonian evolution in the "time" parameter $t \in [t_0, t_1]$. The Hamiltonian H can be constructed in a standard way via the Legendre transform of the time-density of the action. Generally, for M pseudo-Riemannian and with no non-degeneracy condition on the metric pulled back to time-slices $\Sigma \times \{t\}$, H will be singular and one can employ the Gotay–Nester–Hinds algorithm [6] to construct a smaller phase space on which H and the associated Hamiltonian vector field are well-defined. However, generally the Hamiltonian vector field cannot be integrated to a finite-time Hamiltonian flow: both existence and uniqueness of solutions of the Hamilton's equations can fail. In Sect. 5.5 we study in detail an example of this situation: the radial evolution on an annulus on Minkowski plane. Despite the failure of the Hamiltonian picture in such cases, the formalism of canonical relations works perfectly and provides a more general framework for describing the evolution.

1.1. Plan of the paper.

- In Sect. 2 we review the construction of the boundary phase space for the classical field theory defined by action (1) on a general pseudo-Riemannian manifold with boundary.
- In Sect. 3 we specialize to the case of a domain M in the Minkowski plane and study several simple examples explicitly. In particular, we show that L_M is Lagrangian if M is a strip.[3] In the case when the boundary is light-like, we observe, however, that no choice of boundary condition leads to uniqueness of solutions. We also consider a diamond on the Minkowski plane with edges aligned in light-like directions and show that L_M for this domain is Lagrangian.
- Section 4 is central to this paper. Here we specialize further to the case of compact domains on the Minkowski plane bounded by a collection of smooth curves with only finitely many light-like points (with the technical requirement that the boundary should have non-zero curvature at the light-like points). We prove that L_M is Lagrangian for such domains (Theorem 4.2).
- In Sect. 5 we comment on several associated issues, in particular:
  - Problems with Dirichlet boundary conditions (non-transversality of the corresponding L' and L_M).
  - Constraint (Cauchy) subspaces of the phase space (constraints arising from the requirement of extendability of boundary fields to a solution of the wave equation in an open neighborhood of the boundary).
  - Conformal invariance of the problem. In particular, the result of Theorem 4.2 extends to domains with a non-flat Lorentzian metric conformally equivalent to the flat one.
  - The Hamiltonian formalism corresponding to radial evolution on the plane and issues with integrating the corresponding Hamiltonian vector field into a flow (both in the Fréchet and in the Banach setting).
  - The representation of the operad of little 2-disks by canonical relations coming from evolution relations L_M.
  - Interpretation of the property of being Lagrangian for the evolution relation for a general classical free field theory, possibly with gauge symmetry, in terms of (generalized) Lefschetz duality, and the specialization to the theory defined by the action (1).
  - Extension of the result of Theorem 4.2 to more general Lorentzian surfaces, satisfying certain constraints on the metric.
  - An example of a Lorentzian surface with a non-Lagrangian evolution relation—the Misner metric on a cylinder.

2. Classical Massless Free Boson on a Pseudo-Riemannian Manifold: Boundary Structures

The construction of [4] in case of the free massless boson, see action (1), on a pseudo-Riemannian manifold associates to a closed oriented $(m-1)$-manifold Σ endowed with a triple of a function, a vector field and a volume form $(\Gamma, u, \mu) \in \mathcal{C}^\infty(\Sigma) \times \mathfrak{X}(\Sigma) \times \Omega^{m-1}(\Sigma)$, a *pre-phase space*

$$\tilde{\Phi}_\Sigma = \mathcal{C}^\infty(\Sigma) \times \mathcal{C}^\infty(\Sigma) \tag{3}$$

[3] Since this is a noncompact manifold, appropriate restrictions on the behavior of fields at infinity are to be imposed.

with coordinates denoted by (ϕ, ϕ_n). The pre-phase space is endowed with the 1-form

$$\tilde{\alpha}_\Sigma = \int_\Sigma \mu\,(\Gamma\,\phi_n + u(\phi))\,\delta\phi \quad \in \Omega^1(\tilde{\Phi}_\Sigma) \tag{4}$$

More concretely, for each $(\phi, \phi_n) \in \tilde{\Phi}_\Sigma$, $\tilde{\alpha}_\Sigma(\phi, \phi_n)$ is the linear map $\tilde{\Phi}_\Sigma \to \mathbb{R}$,

$$(f, f_n) \mapsto \int_\Sigma \mu\,(\Gamma\,\phi_n + u(\phi))\,f$$

The presymplectic structure on the pre-phase space is defined as

$$\tilde{\omega}_\Sigma = \delta\tilde{\alpha}_\Sigma = \int_\Sigma \mu\,(\Gamma\,\delta\phi_n + u(\delta\phi))\,\delta\phi \quad \in \Omega^2(\tilde{\Phi}_\Sigma) \tag{5}$$

where δ in $\delta\tilde{\alpha}_\Sigma$ stands for de Rham differential on $\tilde{\Phi}_\Sigma$.[4] More concretely, $\tilde{\omega}_\Sigma$ is the skew symmetric bilinear map $\tilde{\Phi}_\Sigma \times \tilde{\Phi}_\Sigma \to \mathbb{R}$,

$$((f, f_n), (g, g_n)) \mapsto \int_\Sigma \mu\,(\Gamma\,f_n + u(f))\,g - \int_\Sigma \mu\,(\Gamma\,g_n + u(g))\,f$$

The phase space Φ_Σ is defined as the reduction of the pre-phase space by the kernel of the presymplectic form,

$$\Phi_\Sigma = \tilde{\Phi}_\Sigma / \ker(\tilde{\omega}_\Sigma)$$

The 2-form $\tilde{\omega}_\Sigma$ descends to a symplectic structure on the phase space, $\omega_\Sigma \in \Omega^2(\Phi_\Sigma)$.

Remark 2.1. The geometric data (Γ, u, μ) on Σ can be considered modulo equivalence $(\Gamma, u, \mu) \sim (c\Gamma, cu, c^{-1}\mu)$ for any nonvanishing $c \in \mathcal{C}^\infty(\Sigma)$. Also, the data (Γ, u, μ) up to this equivalence can be viewed as a section

$$(\Gamma + u)\mu \in \Gamma(\Sigma, (\mathbb{R} \oplus T\Sigma) \otimes \wedge^{m-1} T^*\Sigma)$$

where $\mathbb{R}$ stands for the trivial real line bundle over Σ.

In case when $\Sigma = \partial M$ is the boundary of an m-manifold M, the geometric data (Γ, u, μ) are inferred from the metric g on M as follows:

$$\Gamma(x) = g^{-1}(x)(n_x^*, n_x^*), \quad u(x) = g^{-1}(x)(n_x^*, \bullet) - \Gamma(x)n_x \quad \in T_x\partial M, \quad \mu = \iota_n\mu_g \tag{6}$$

Here we chose some vector field on the boundary[5] $n \in \Gamma(\partial M, i^*TM)$ transversal to the boundary everywhere (we denote $i : \partial M \hookrightarrow M$ the embedding of the boundary); $n^* \in \Gamma(\partial M, i^*T^*M)$ is the covector field on the boundary defined by $\langle n_x^*, n_x\rangle = 1$, $\langle n_x^*, T_x\partial M\rangle = 0$; $\mu_g = \sqrt{g}\,d^m x$ is the metric volume element on M; $\iota_\bullet$ stands for contraction of a form with a vector field.

The projection $\tilde{\pi}_M : F_M \to \tilde{\Phi}_{\partial M}$ sends $\phi \in \mathcal{C}^\infty(M)$ to $(\phi|_{\partial M}, \partial_n\phi|_{\partial M})$—values of ϕ at the boundary and derivative along n at the boundary.

[4] For more details on "local" differential forms on spaces of fields, see e.g. [5].

[5] We do not require any compatibility of n with the metric on M.

Remark 2.2. Choosing a different transversal vector field at the boundary, $n' = an + w$ with nonvanishing $a \in \mathcal{C}^\infty(\partial M)$ and with $w \in \mathfrak{X}(\partial M)$ a tangent vector field on ∂M, results in different induced geometric data on the boundary:

$$(\Gamma', u', \mu') = (a^{-2}\Gamma, a^{-1}u - a^{-2}w, a\mu)$$

The new projection $\tilde{\pi}'_M : F_M \to \tilde{\Phi}_\Sigma$, corresponding to n', sends $\phi \in \mathcal{C}^\infty(M)$ to $(\phi|_{\partial M}, \partial_{n'}\phi|_{\partial M}) \in \tilde{\Phi}_{\partial M}$ and can be viewed as the old one, composed with a linear isomorphism of the pre-phase space $\tilde{\Phi}_{\partial M} \to \tilde{\Phi}_{\partial M}$ sending $(\phi, \phi_n) \mapsto (\phi, a\phi_n + w(\phi))$.

The pull-back of the 1-form $\tilde{\alpha}_{\partial M}$ to the space of fields F_M is

$$\tilde{\pi}_M^* \tilde{\alpha}_{\partial M} = \int_{\partial M} (*_g d\phi \cdot \delta\phi)|_{\partial M} = \int_{\partial M} (\iota_{g^{-1}(d\phi)}\mu_g \cdot \delta\phi)|_{\partial M} \tag{7}$$

It arises as the boundary term of the variation of the action (1):

$$\delta S = (-1)^{m-1} \int_M d\delta\phi \wedge *_g d\phi = -\int_M (d *_g d\phi) \cdot \delta\phi + \tilde{\pi}_M^* \tilde{\alpha}_{\partial M}$$

Remark 2.3. According to the construction of [3,4], one associates to an $(m-1)$-manifold Σ with a pseudo-Riemannian metric on a cylinder $M_\epsilon = \Sigma \times [0, \epsilon]$ the space $\tilde{\Phi}_\Sigma$ of 1-jets[6] of functions on M_ϵ at $\Sigma \times \{0\}$. The one-form $\tilde{\alpha}_\Sigma \in \Omega^1(\tilde{\Phi}_\Sigma)$ arises as the part of the boundary term of the variation of S on M_ϵ corresponding to the contribution of the boundary component $\Sigma \times \{0\}$. The geometric data (Γ, u, μ) introduced above constitute the part of the metric on M_ϵ necessary to define the 1-form $\tilde{\alpha}_\Sigma$. The transversal vector field n arises from the 1-jet of the embedding of the cylinder $M_\epsilon \hookrightarrow M$ as a neighborhood of the boundary of M.

3. Two-Dimensional Minkowski Case

Consider the Minkowski plane $\mathbb{R}^{1,1}$ with coordinates (x, y) and metric $g = dx^2 - dy^2$. Let D be a domain[7] of $\mathbb{R}^{1,1}$ with smooth boundary, with metric given by restriction of the Minkowski metric on $\mathbb{R}^{1,1}$ to D. As above, $F_D = \mathcal{C}^\infty(D)$ and the action (1) is

$$S_{D,g}[\phi] = \frac{1}{2} \int_D [(\partial_x\phi)^2 - (\partial_y\phi)^2]\, dx\, dy$$

Unless otherwise stated, in case of an unbounded domain D, we assume that kth derivatives of fields have asymptotics

$$\partial^k\phi \sim O((x^2 + y^2)^{-\frac{\eta+k}{2}}) \tag{8}$$

at infinity, where $k = 0, 1, 2, \ldots$ and $\eta > 0$ is some constant.

The corresponding Euler–Lagrange equation is just the wave equation

$$\partial_x^2\phi - \partial_y^2\phi = 0$$

[6] Only 1-jets are required since the density of the action S is of second order in the field derivatives.

[7] By domain here we mean the closure of an open subset.

3.1. Examples of boundary structures. In this section we consider D a half-space in $\mathbb{R}^{1,1}$ with space-like, time-like or light-like boundary $\Sigma = \partial D \simeq \mathbb{R}$.

Consider the case $D = \mathbb{R} \times [y_0, \infty)$ with space-like boundary $\partial D = \mathbb{R} \times \{y_0\}$. Using the construction of Sect. 2, we choose the transversal vector field at the boundary to be $n = \partial_y$ and obtain the geometric structure (6) on the boundary $(\Gamma, u, \mu) = (-1, 0, -dx)$. The pre-phase space is $\tilde{\Phi}_{\partial D} = \mathcal{C}^\infty(\mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R}) \ni (\phi, \phi_n)$ and the projection $\tilde{\pi}_D : F_D \to \tilde{\Phi}_{\partial D}$ sends $\phi \in \mathcal{C}^\infty(D)$ to $(\phi|_{y=y_0}, \partial_y\phi|_{y=y_0})$. The 1-form (4) on the pre-phase space is

$$\tilde{\alpha} = \int_{\mathbb{R}} dx\, \phi_n\, \delta\phi \quad \in \Omega^1(\tilde{\Phi}_{\partial D}) \tag{9}$$

and its differential

$$\tilde{\omega} = \int_{\mathbb{R}} dx\, \delta\phi_n \wedge \delta\phi \quad \in \Omega^2(\tilde{\Phi}_{\partial D}) \tag{10}$$

is weakly non-degenerate, i.e. $\ker \tilde{\omega} = 0$. Thus, there is no symplectic reduction and the phase space coincides with the pre-phase space, $\Phi_{\partial D} = \tilde{\Phi}_{\partial D}$, with symplectic structure $\omega = \tilde{\omega}$.

Similarly, for $D = [x_0, \infty) \times \mathbb{R}$ with time-like boundary $\partial D = \{x_0\} \times \mathbb{R}$ we pick $n = \partial_x$, which induces geometric data $(\Gamma, u, \mu) = (1, 0, dy)$ on the boundary. The projection π_D sends $\phi \in \mathcal{C}^\infty(D)$ to $(\phi|_{x=x_0}, \partial_x\phi|_{x=x_0})$. The 1-form $\tilde{\alpha}$ and its differential $\tilde{\omega}$ are again given by formulae (9, 10). Again, the non-degeneracy of $\tilde{\omega}$ implies that $\Phi_{\partial D} = \tilde{\Phi}_{\partial D}$, $\omega = \tilde{\omega}$.

Next, consider a half-space on $\mathbb{R}^{1,1}$ with light-like boundary. Using coordinates $\sigma_+ = y + x$, $\sigma_- = y - x$ on $\mathbb{R}^{1,1}$, we set $D = \{(\sigma_+, \sigma_-) \in \mathbb{R}^{1,1} \mid \sigma_- \geq \sigma_-^0\}$ for some $\sigma_-^0 \in \mathbb{R}$. Introducing coordinate vector fields $\partial_\pm = \frac{1}{2}(\partial_y \pm \partial_x)$, we set $n = \partial_-$. This choice yields the boundary geometric data[8] $(\Gamma, u, \mu) = (0, -2\partial_+, -\frac{1}{2}d\sigma_+)$, therefore

$$\tilde{\alpha} = \int_{\mathbb{R}} d\sigma_+\, \partial_+\phi\, \delta\phi, \qquad \tilde{\omega} = \delta\tilde{\alpha} = \int_{\mathbb{R}} d\sigma_+\, (\partial_+\delta\phi) \wedge \delta\phi$$

Using the linear structure on the pre-phase space, we can regard the presymplectic structure $\tilde{\omega}$ as an anti-symmetric bilinear form on $\tilde{\Phi}$ given by

$$\tilde{\omega}((\phi, \phi_n), (\psi, \psi_n)) = \int_{\mathbb{R}} d\sigma_+\, ((\partial_+\phi)\, \psi - (\partial_+\psi)\, \phi) \tag{11}$$

The kernel of $\tilde{\omega}$ and hence the symplectic reduction depend on the allowed behavior of ϕ at $\sigma_+ \to \infty$. For instance, we have the following.

(i) If we require $\lim_{\sigma_+\to\infty} \phi(\sigma_+) = 0$ then the presymplectic form (11) becomes

$$\tilde{\omega}((\phi, \phi_n), (\psi, \psi_n)) = 2 \int_{\mathbb{R}} d\sigma_+\, (\partial_+\phi)\, \psi \tag{12}$$

[8] It is useful to note that in coordinates $\sigma_\pm$, the metric, its inverse and the metric volume element on D are, respectively, $g = -d\sigma_+ \cdot d\sigma_-$, $g^{-1} = -4\,\partial_+ \cdot \partial_-$, $\mu_g = \frac{1}{2}\, d\sigma_+ \wedge d\sigma_-$. Here $\cdot$ stands for the symmetrized tensor product.

So, $(\phi, \phi_n) \in \ker \tilde{\omega}$ iff $\partial_+\phi = 0$, but by the vanishing requirement at $\sigma_+ \to \infty$ this implies $\phi = 0$. Hence, $\ker \tilde{\omega} = \{0\} \times \mathcal{C}^\infty(\mathbb{R}) \subset \tilde{\Phi}$ and the phase space is

$$\Phi = \tilde{\Phi}/\ker \tilde{\omega} = \mathcal{C}^\infty(\mathbb{R}) \ni \phi \tag{13}$$

with (non-degenerate) symplectic structure given by r.h.s. of (11).

(ii) Requiring that ϕ has some (possibly, different) limits at $\sigma_+ \to \pm\infty$, we get a boundary term, integrating by parts in (11): $\tilde{\omega}((\phi, \phi_n), (\psi, \psi_n)) = -|\phi\psi|_{-\infty}^{+\infty} + 2\int_{\mathbb{R}} d\sigma_+ (\partial_+\phi)\,\psi$. Thus $(\phi, \phi_n) \in \ker \tilde{\omega}$ iff $\partial_+\phi = 0$ and $\phi(\pm\infty) = 0$, which again implies $\phi = 0$. So, $\ker \tilde{\omega}$ is the same as in case of vanishing condition at $\sigma_+ \to \infty$ and the phase space is again given by (13) (though now we impose different asymptotical conditions on ϕ).

(iii) Imposing periodic asymptotics $\phi(+\infty) = \phi(-\infty)$, we get back to (12) but now the kernel becomes bigger:

$$\ker \tilde{\omega} = \{(\phi = C, \phi_n \in \mathcal{C}^\infty(\mathbb{R})) \mid C \in \mathbb{R}\} \subset \tilde{\Phi}$$

Thus the phase space is $\Phi = \mathcal{C}^\infty(\mathbb{R})/\mathbb{R}$ where we consider functions differing by a constant shift as equivalent. We can choose the section of this quotient e.g. by requiring $\phi(0) = 0$. In this case the projection $\pi_D : F_D \to \Phi$ maps $\phi \in \mathcal{C}^\infty(D)$ to $\psi(\sigma_+) = \phi(\sigma_+, \sigma_-^0) - \phi(0, \sigma_-^0)$.

3.2. Canonical relations. Related to examples of the previous section, with $D \subset \mathbb{R}^{1,1}$ a half-space with boundary $\partial D = \Sigma$ a line in $\mathbb{R}^{1,1}$, are cases when $D \subset \mathbb{R}^{1,1}$ is a strip with boundary $\Sigma \sqcup \Sigma^{\mathrm{op}}$ where op denotes the opposite orientation. In all these cases L_D is Lagrangian as we presently prove.

For Σ space-like, consider $D = \mathbb{R} \times [y_0, y_1]$. Denote $\pi := \partial_y\phi$. Then the 1-form on the phase space

$$\Phi_{\partial D} = \mathcal{C}^\infty(\mathbb{R})^{\times 4} \ni (\phi_0, \pi_0, \phi_1, \pi_1)$$

is

$$\alpha = \int_{\mathbb{R}} (\pi_1 \delta\phi_1 - \pi_0 \delta\phi_0)\, dx$$

where subscript i corresponds to boundary components $y = y_i$ of the strip, $i = 0, 1$ (and we are still assuming asymptotics (8) for fields π, ϕ). The Euler–Lagrange equation can be rewritten as a system

$$\begin{aligned} \partial_y\pi &= \partial_x^2\phi \\ \partial_y\phi &= \pi \end{aligned}$$

The system is Hamiltonian with respect to the symplectic form $\int_{\mathbb{R}} \delta\pi \wedge \delta\phi\, dx$ and to the Hamiltonian function $H = \frac{1}{2}\int_{\mathbb{R}} (\pi^2 + (\partial_x\phi)^2)\, dx$. Since L_D is the graph of the corresponding Hamiltonian flow from time y_0 to time y_1, it is Lagrangian.

Similarly one proves that L_D is Lagrangian for Σ time-like, for the strip $D = [x_0, x_1] \times \mathbb{R}$.

Finally, consider the case when Σ is light-like. Passing to coordinates $\sigma_\pm$, we consider the strip $D = \{(\sigma_+, \sigma_-) \in \mathbb{R}^{1,1} \,|\, \sigma_-^0 \le \sigma_- \le \sigma_-^1\}$. The Euler–Lagrange equation becomes $\partial_+\partial_-\phi = 0$, which has general solution

$$\phi(\sigma_+, \sigma_-) = f(\sigma_+) + g(\sigma_-) \tag{14}$$

with f and g arbitrary functions. Therefore, for any σ_-^0 and σ_-^1, L_D is the diagonal in $\bar{\Phi}_\Sigma \times \Phi_\Sigma$, where bar denotes opposite symplectic structure, so it is Lagrangian. Observe however that g cannot be determined by boundary conditions. As a consequence, on such strips we cannot have uniqueness of solutions.

3.3. Light-like diamond. Consider a diamond in Minkowski plane with piecewise light-like boundary,[9]

$$D = \{(\sigma_+, \sigma_-) \in \mathbb{R}^{1,1} \mid \sigma_+^0 \le \sigma_+ \le \sigma_+^1,\ \sigma_-^0 \le \sigma_- \le \sigma_-^1\}$$

We label the four vertices of the diamond as

$$a = (\sigma_+^0, \sigma_-^0),\ b = (\sigma_+^1, \sigma_-^0),\ c = (\sigma_+^1, \sigma_-^1),\ d = (\sigma_+^0, \sigma_-^1)$$

Proceeding as in Sect. 3.1, we obtain the pre-phase space[10]

$$\Phi_{\partial D} = \{\phi \in C^0(\partial D) \text{ smooth on edges of } \partial D\}$$

We denote restrictions of ϕ to the four edges of the diamond by $\phi^{ab}, \phi^{dc} \in \mathcal{C}^\infty[\sigma_+^0, \sigma_+^1]$, $\phi^{ad}, \phi^{bc} \in \mathcal{C}^\infty[\sigma_-^0, \sigma_-^1]$ respectively.

The pre-symplectic 2-form induced on $\Phi_{\partial D}$ is

$$\omega = \int_{\partial D} \epsilon\, d\delta\phi \wedge \delta\phi \tag{15}$$

where $\epsilon = +1$ on two edges parallel to ∂_+ and $\epsilon = -1$ on the other two. Viewed as an anti-symmetric bilinear pairing $\Phi_{\partial D} \otimes \Phi_{\partial D} \to \mathbb{R}$, the pre-symplectic structure is

$$\omega(\phi, \psi) = 2\int_{\partial D} \epsilon\, d\phi \cdot \psi + 2\,(\phi_a\psi_a - \phi_b\psi_b + \phi_c\psi_c - \phi_d\psi_d) \tag{16}$$

where we used integration by parts to transfer derivatives from ψ to ϕ. Subscript a, b, c, d stands here for evaluation of ϕ or ψ at the corresponding vertex of the diamond.

It follows from (16) that $\phi \in \ker\omega$ implies $\phi = C \in \mathbb{R}$—a constant on the whole ∂D. On the other hand $\omega(C, \psi) = 2C(\psi_a - \psi_b + \psi_c - \psi_d)$, hence $\ker\omega = 0$. Thus ω is actually non-degenerate and $(\Phi_{\partial D}, \omega)$ is the symplectic phase space, with no further symplectic reduction required.

[9] The construction of Sect. 2 extends naturally to the case of manifolds with piecewise smooth boundary. In this case, for the pre-phase space (3) one takes pairs of piecewise smooth continuous functions (smooth where the boundary is smooth).

[10] We are not including the normal derivative ϕ_n in our description of $\Phi_{\partial D}$, since it does not appear in the 2-form (15) and would be eliminated by symplectic reduction anyway.

3.3.1. Evolution relation Using the general ansatz (14) for solutions of the wave equation, the evolution relation $L \subset \Phi_{\partial D}$ can be described as

$$L = \{\underbrace{\phi(\sigma_+, \sigma_-)}_{\in \partial D} = f(\sigma_+) + g(\sigma_-) \mid f \in \mathcal{C}^\infty[\sigma_+^0, \sigma_+^1],\ g \in \mathcal{C}^\infty[\sigma_-^0, \sigma_-^1]\}$$

To show that $L \subset \Phi_{\partial D}$ is a Lagrangian subspace (and thus verify Conjecture 1.1 in this case), we check isotropicity and coisotropicity of L. For isotropicity, we have

$$\begin{aligned}\omega|_L &= \int_{[\sigma_+^0, \sigma_+^1]} d\delta f \wedge (\delta g(\sigma_-^0) - \delta g(\sigma_-^1)) + \int_{[\sigma_-^0, \sigma_-^1]} d\delta g \wedge (\delta f(\sigma_+^0) - \delta f(\sigma_+^1)) \\ &= (\delta f(\sigma_+^1) - \delta f(\sigma_+^0)) \wedge (\delta g(\sigma_-^0) - \delta f(\sigma_-^1)) + (\delta g(\sigma_-^1) - \delta g(\sigma_-^0)) \wedge (\delta f(\sigma_+^0) - \delta f(\sigma_+^1)) = 0\end{aligned}$$

Thus L is indeed isotropic. For coisotropicity, (16) implies that for $\phi \in L$ and ψ arbitrary,

$$\begin{aligned}\omega(\phi, \psi) = 2 \int_{[\sigma_+^0, \sigma_+^1]} df\ (\psi^{ab} - \psi^{dc}) + 2 \int_{[\sigma_-^0, \sigma_-^1]} dg\ (\psi^{ad} - \psi^{bc}) \\ + \text{contributions of corners}\end{aligned}$$

Thus $\psi \in L^\perp$ implies (by setting $f(\sigma_+^0) = g(\sigma_-^0) = 0$ and taking df or dg to be the difference of two bump 1-forms localized near two points, so that the total integral vanishes) $\psi^{ab} - \psi^{dc} = C$, $\psi^{ad} - \psi^{bc} = C'$ where $C, C' \in \mathbb{R}$ are two constants. This implies in turn that $\psi \in L$, with corresponding $f_\psi(\sigma_+)$, $g_\psi(\sigma_-)$ given by

$$f_\psi(\sigma_+) = \psi^{ab}(\sigma_+) - \psi^{ab}(\sigma_+^0), \quad g_\psi(\sigma_-) = \psi^{ad}(\sigma_-)$$

This proves coisotropicity of L and hence L is indeed Lagrangian.

3.3.2. Hamilton–Jacobi action. Restriction of the action (1) to solutions of Euler–Lagrange equation is in general

$$S|_{EL} = \frac{1}{2} \int_M d\phi \wedge *d\phi = \underbrace{-\frac{1}{2} \int_M \phi \wedge d * d\phi}_{0 \text{ on } EL} + \frac{1}{2} \int_{\partial M} (\phi \wedge *d\phi)|_{\partial M}$$

Since this expression is given by a boundary term, it descends to a function on L (at least as a subspace of the pre-phase space, in the general case).

In case of the diamond we have

$$S|_{EL} = \frac{1}{2} \int_{\partial D} \epsilon\ \phi\, d\phi = \frac{1}{2} (-\phi_a^2 + \phi_b^2 - \phi_c^2 + \phi_d^2) \quad \in \mathcal{C}^\infty(L)$$

Note that this Hamilton–Jacobi action depends only on the values of ϕ at the vertices of the diamond.

4. Wave Equation on Compact Domains in Minkowski Plane

Let $D \subset \mathbb{R}^{1,1}$ be a connected compact domain in the Minkowski plane. We make the following assumptions about its boundary $\gamma = \partial D$.

(A) Each connected component γ_k, $1 \le k \le N$, of the boundary γ is a smooth simple closed curve.
(B) There are finitely many points on γ with light-like tangent; we denote this set of points I.
(C) The curvature of γ (as a multi-component plane curve) at points of I is non-zero.

Assume that each curve γ_k is parameterized by $t \in \mathbb{R}/(T_k \cdot \mathbb{Z})$, with $T_k \in \mathbb{R}$ the period. We assume that the orientation of γ_k induced from the parametrization agrees with the one induced from the orientation of D. Define $\theta : \gamma_k \to \mathbb{R}/(\pi \cdot \mathbb{Z})$ and $v : \gamma_k \to \mathbb{R}_{>0}$ by $\theta(t) = \arctan(\frac{\dot y}{\dot x}) + \frac{\pi}{2}$, $v(t) = (\dot x^2 + \dot y^2)^{1/2}$.

4.1. Phase space, symplectic structure. The phase space[11] (the space of boundary fields) associated to γ is $\Phi_\gamma = \{(\phi, \phi_n) \in \mathcal{C}^\infty(\gamma) \times \mathcal{C}^\infty(\gamma)\}$ The projection $\pi : F_D \to \Phi_\gamma$ sends $\phi \in \mathcal{C}^\infty(D)$ to its restriction to γ and the normal derivative at a point on γ; "normal" means an outward pointing unit normal vector to the boundary with respect to *Euclidean* metric on the plane.

The geometric data (6) on γ, associated to the choice of the Euclidean normal vector field $n = \cos\theta\, \partial_x + \sin\theta\, \partial_y$, is: $(\Gamma, u, \mu) = (\cos(2\theta), -\frac{1}{v}\sin(2\theta)\partial_t, v\, dt)$, which yields the following boundary 1-form (4) on Φ_γ:

$$\alpha = \int_\gamma dt\ (v\cos(2\theta)\partial_n - \sin(2\theta)\partial_t)\phi\ \delta\phi$$

where $\partial_n\phi := \phi_n$ is a notation. It generates a constant 2-form on Φ_γ

$$\omega = \delta\alpha = \int_\gamma dt\ (v\cos(2\theta)\partial_n - \sin(2\theta)\partial_t)\delta\phi\ \wedge \delta\phi$$

Using the linear structure on Φ_γ, we can view ω as an anti-symmetric pairing $\Phi_\gamma \otimes \Phi_\gamma \to \mathbb{R}$,

$$\omega((\phi, \phi_n), (\psi, \psi_n)) = \int_\gamma dt \cdot (v\cos(2\theta)\phi_n\ \psi - \sin(2\theta)\partial_t\phi\ \psi - \phi\ v\cos(2\theta)\psi_n + \phi\ \sin(2\theta)\partial_t\psi) \qquad (17)$$

Proposition 4.1. *Two-form ω is non-degenerate on Φ_γ.*

Proof. Indeed, by (17), a pair $(\phi, \phi_n) \in \Phi_\gamma$ is in the kernel of ω if and only if

$$\begin{cases} -v\cos(2\theta)\phi = 0 \\ -\partial_t(\sin(2\theta)\phi) + v\cos(2\theta)\phi_n - \sin(2\theta)\partial_t\phi = 0 \end{cases} \Leftrightarrow \begin{cases} \phi = 0 \\ \phi_n = 0 \end{cases}$$

where we use that, by assumption (B), $\cos(2\theta)$ vanishes in isolated points. □

[11] In the terminology of Sect. 2, we should be calling it the pre-phase space. Below (cf. Proposition 4.1) we will show that the presymplectic form on Φ_γ is in fact symplectic, so that no further symplectic reduction is needed. Thus the terminology is justified.

4.2. Evolution relation: main theorem. Set $EL_D = \{\phi \in \mathcal{C}^\infty(D) \mid d * d\phi = 0\} \subset F_D$—the space of solutions to the wave equation in D and also set

$$L = \pi(EL_D) \subset \Phi_\gamma$$

—the *evolution relation*.

Theorem 4.2. *The evolution relation L is a Lagrangian subspace of Φ_γ.*

4.2.1. Evolution relation in the simply connected case and involutions $E_\pm$ on the boundary. In case when D is *simply connected* (N = 1), the space of solutions of the wave equation in the bulk EL_D is given by

$$EL_D = \{\phi = F + G \mid F, G \in \mathcal{C}^\infty(D),\ \partial_- F = \partial_+ G = 0\} \tag{18}$$

Note that globally $\partial_- F = 0$ does not imply $F = F(\sigma_+)$, e.g. if D is not convex.

The two distributions $\partial_\pm$ on D induce two equivalence relations $\epsilon_\pm$ on points of D, where two points in D are considered equivalent if they can be connected by a light-like segment with tangent $\partial_\pm$ lying inside D. In turn, $\epsilon_\pm$ induce equivalence relations $\mathcal{E}_\pm$ on points of γ.

Denote $I_\pm = \{p \in I \mid \theta(p) = \mp\pi/4\}$, so that $I = I_+ \sqcup I_-$.

By assumptions (A, B), an equivalence class of $\mathcal{E}_\pm$ of order 1 is necessarily a point of $I_\pm$ and an equivalence class of order $n \geq 3$ necessarily contains $n - 2$ points of $I_\pm$. Thus there is only a finite set of points $I'_\pm \subset \gamma$ with equivalence class of $\mathcal{E}_\pm$ of order $\neq 2$.

Therefore, equivalence relations $\mathcal{E}_\pm$ induce two orientation-reversing smooth involutions $E_\pm : (\gamma - I'_\pm) \to (\gamma - I'_\pm)$, i.e. for a point $p \in \gamma - I'$, $E_\pm(p)$ is the point on γ where one of the two light-like lines in D starting at p hits γ second time.[12]

Denote

$$\mathcal{C}^\infty(\gamma)^{E_\pm} = \{f \in \mathcal{C}^\infty(\gamma) \mid f \circ E_\pm = f \text{ on } \gamma - I'_\pm\}$$

To describe the evolution relation $L_D = \pi(EL_D) \subset \Phi_\gamma$, we need the following two decompositions for the unit (Euclidean) normal vector ∂_n at a point on γ:

$$\partial_n = -\frac{1}{v}\cot(\theta - \pi/4)\partial_t + \sqrt{2}\frac{1}{\sin(\theta - \pi/4)}\partial_-$$
$$\partial_n = -\frac{1}{v}\cot(\theta + \pi/4)\partial_t + \sqrt{2}\frac{1}{\sin(\theta + \pi/4)}\partial_+$$

If we denote $f = F|_\gamma, g = G|_\gamma \in \mathcal{C}^\infty(\gamma)$, then $\partial_- F = \partial_+ G = 0$ implies

$$\partial_n F = -\frac{1}{v}\cot(\theta - \pi/4)\ \partial_t f, \qquad \partial_n G = -\frac{1}{v}\cot(\theta + \pi/4)\ \partial_t g$$

Thus, for D *simply connected*, we may describe L as

$$L = \Big\{(\phi, \phi_n) = \Big(f + g, -\frac{1}{v}(\cot(\theta - \pi/4)\ \partial_t f + \cot(\theta + \pi/4)\ \partial_t g)\Big) \mid f \in \mathcal{C}^\infty(\gamma)^{E_-},\ g \in \mathcal{C}^\infty(\gamma)^{E_+}\Big\} \tag{19}$$

[12] The reader is referred to Sect. 5.2 for explicit formulae for $E_\pm$ in some examples.

Note that for this description we implicitly use the property that the maps

$$\{F \in \mathcal{C}^\infty(D) \mid \partial_\pm F = 0\} \quad \stackrel{\pi}{\to} \quad \mathcal{C}^\infty(\gamma)^{E_\pm}$$

are surjective, for which assumption (C) is essential. Note also that the expression $\cot(\theta - \frac{\pi}{4})\partial_t f + \cot(\theta + \frac{\pi}{4})\partial_t g$ in (19) is smooth on the whole γ.

4.2.2. Evolution relation in the non-simply connected case. In general, when D is not necessarily simply connected, the r.h.s. of (18) is valid as a *local* description of the space of solutions, but globally F, G may fail to exist as single valued functions on D. One global description of EL_D is as follows:

$$EL_D = \{\phi \in \mathcal{C}^\infty(D) \mid d\phi = \kappa + \lambda, \text{ where } \kappa, \lambda \in \Omega^1_{\text{closed}}(D),\ \iota_{\partial_-}\kappa = \iota_{\partial_+}\lambda = 0\} \tag{20}$$

where $\iota_{\partial_\pm}$ is the contraction with the vector field $\partial_\pm$.

For D non-simply connected (note that the involutions $E_\pm$ still make perfect sense, though now they may relate pairs of points in different connected components of γ), the r.h.s. of (19) defines a subspace $L^{\text{glob}} \subset L$ corresponding to solutions of the wave equation with single valued F, G: $L^{\text{glob}} = \pi(EL_D^{\text{glob}})$ where EL_D^{glob} is given by r.h.s. of (18).

Lemma 4.3.

$$\dim(L/L^{\text{glob}}) = N - 1 \tag{21}$$

Proof. In D we have a short exact sequence

$$EL_D^{\text{glob}} \hookrightarrow EL_D \twoheadrightarrow H^1(D) \tag{22}$$

where $H^1(D)$ is the de Rham cohomology of D in degree 1; the second arrow sends $\phi \mapsto [\kappa] \in H^1(D)$ where we use description (20). Surjectivity of the second map follows from surjectivity of the map $\{\kappa \in \Omega^1_{\text{closed}}(D) \mid \iota_{\partial_-}\kappa = 0\} \to H^1(D)$ sending $\kappa \mapsto [\kappa]$. To prove the latter, note that we can reorder boundary components so that for any $1 \le i < N$ there exists an open subset $U_i \subset \gamma_i - I'$ such that $E_-(U_i) \subset \gamma_j$ for some $j > i$. For every i, take $\psi_i \in \Omega^1(U_i)$ a bump 1-form supported on U_i, and construct a closed ∂_--horizontal 1-form on D as $\kappa_i = p_-^*(\psi_i + E_-^*\psi_i)$ where $p_- : D \to \gamma/E_-$ is the projection to the boundary along ∂_-. It easy to see, by looking at periods along γ_i, that restrictions to the boundary $\{\kappa_i|_\gamma\}_{i=1}^{N-1}$ span the kernel of $H^1(\gamma) \to \mathbb{R}$ (pairing with the fundamental class of γ). Therefore $\{\kappa_i\}$ span $H^1(D)$.

It follows from (22) that $\dim(EL_D/EL_D^{\text{glob}}) = N - 1$ and since $\pi : EL_D \to L$ is an isomorphism,[13] we have $\dim(L/L^{\text{glob}}) = N - 1$. □

[13] Surjectivity follows from the definition of L. Injectivity can be seen as follows: restrictions to γ of the 1-forms κ, λ of (20) can be explicitly and uniquely recovered from $(\phi, \phi_n) \in L$ by formulae (24,25) below. Hence $(\phi, \phi_n) = 0$ implies $\kappa|_\gamma = \lambda|_\gamma = 0$, which in turn implies, by $\partial_\mp$-horizontality of κ, λ, that $\kappa = \lambda = 0$ in D. Hence, there can be no non-zero point of EL_D inducing zero on the boundary.

4.3. Proof of Theorem 4.2.

Lemma 4.4. *$L \subset \Phi_\gamma$ is isotropic.*

Proof. Indeed, due to (7) and using Stokes' theorem, for $(\phi, \phi_n), (\psi, \psi_n) \in L$ we have

$$\begin{aligned}\omega((\phi, \phi_n), (\psi, \psi_n)) &= \int_\gamma ((*d\tilde{\phi})\ \tilde{\psi} - (*d\tilde{\psi})\ \tilde{\phi})|_\gamma \\ &= \int_D d((*d\tilde{\phi})\ \tilde{\psi} - (*d\tilde{\psi})\ \tilde{\phi}) \\ &= \int_D (d*d\tilde{\phi})\ \tilde{\psi} - (d*d\tilde{\psi})\ \tilde{\phi} = 0\end{aligned}$$

where $\tilde{\phi}, \tilde{\psi} \in EL_D$ are extensions of $(\phi, \phi_n), (\psi, \psi_n)$ into the bulk D as solutions of the wave equation. □

This proof is a specialization of a general argument, applicable to any classical field theory, cf. [3].

Note that Lemma 4.4 implies that L^{glob} is isotropic in Φ_γ.

Lemma 4.5.

$$\dim \frac{(L^{\mathrm{glob}})^\perp}{L^{\mathrm{glob}}} = 2\,(N-1)$$

Proof. Let us calculate the symplectic complement of L^{glob} in Φ_γ. For $(\psi, \psi_n) \in L^{\mathrm{glob}}$, with f, g denoting the $E_\mp$-invariant parts as in (19), we have

$$\begin{aligned}\omega((\phi, \phi_n), (\psi, \psi_n)) &= -\int_\gamma dt\ (\phi\ \partial_t(f-g) - (f+g)\ (v\cos(2\theta)\phi_n - \sin(2\theta)\partial_t\phi)) \\ &= -\int_\gamma dt\ f\ (-(1-\sin(2\theta))\partial_t\phi - v\cos(2\theta)\phi_n) \\ &\quad -\int_\gamma dt\ g\ ((1+\sin(2\theta))\partial_t\phi - v\cos(2\theta)\phi_n) \qquad (23)\end{aligned}$$

Therefore $(L^{\mathrm{glob}})^\perp$ consists of pairs $(\phi, \phi_n) \in \Phi_\gamma$ for which the 1-forms

$$\alpha = -\frac{1}{2}\, dt\ (-(1-\sin(2\theta))\partial_t\phi - v\cos(2\theta)\phi_n) \in \Omega^1(\gamma), \qquad (24)$$

$$\beta = \frac{1}{2}\, dt\ ((1+\sin(2\theta))\partial_t\phi - v\cos(2\theta)\phi_n) \in \Omega^1(\gamma) \qquad (25)$$

are E_-- and E_+-invariant, respectively.

The inverse of (24,25) is given by

$$d\phi = \alpha + \beta, \quad dt\ \phi_n = -\frac{1}{v}\,(\cot(\theta - \pi/4)\,\alpha + \cot(\theta + \pi/4)\,\beta)$$

The map $\rho : \Phi_\gamma \to \Omega^1(\gamma) \times \Omega^1(\gamma)$ sending $(\phi, \phi_n) \mapsto (\alpha, \beta)$, as defined by (24, 25), has image

$$\begin{aligned}\mathrm{im}(\rho) = \{(\alpha, \beta) \in \Omega^1(\gamma) \times \Omega^1(\gamma) \mid \alpha + \beta \in \Omega^1_{\mathrm{exact}}(\gamma),\ \alpha \text{ vanishes on } I_-, \\ \beta \text{ vanishes on } I_+\} \qquad (26)\end{aligned}$$

and kernel

$$\ker(\rho) = \{(\phi, \phi_n) \in \Omega^0_{\text{closed}}(\gamma) \times \{0\}\} \tag{27}$$

On the other hand, the value of ρ on $(\phi, \phi_n) \in L^{\text{glob}}$ is $(\alpha, \beta) = (df, dg)$, where f, g are the $E_\mp$-invariant parts of ϕ as in r.h.s. of (19). Thus for the restriction of ρ to L^{glob} we have

$$\text{im}(\rho|_{L^{\text{glob}}}) = \Omega^1_{\text{exact}}(\gamma)^{E_-} \times \Omega^1_{\text{exact}}(\gamma)^{E_+} \tag{28}$$

and the kernel is

$$\ker(\rho|_{L^{\text{glob}}}) = \ker(\rho) \cap L^{\text{glob}} = \{(\phi, \phi_n) = (C, 0) \mid C \in \mathbb{R}\}$$

By (23),

$$(L^{\text{glob}})^\perp = \rho^{-1}(\Omega^1(\gamma)^{E_-} \times \Omega^1(\gamma)^{E_+}) \tag{29}$$

in particular, due to (26),

$$\text{im}(\rho|_{(L^{\text{glob}})^\perp}) = \{(\alpha, \beta) \in \Omega^1(\gamma)^{E_-} \times \Omega^1(\gamma)^{E_+} \mid \alpha + \beta \in \Omega^1_{\text{exact}}(\gamma)\} \tag{30}$$

and $\ker(\rho|_{(L^{\text{glob}})^\perp}) = \ker(\rho)$, cf. (27). Therefore, the quotient $(L^{\text{glob}})^\perp / L^{\text{glob}}$ fits into the short exact sequence

$$\Omega^0_{\text{closed}}(\gamma)/\{\text{constants}\} \hookrightarrow (L^{\text{glob}})^\perp / L^{\text{glob}} \overset{\rho}{\twoheadrightarrow} \rho((L^{\text{glob}})^\perp)/\rho(L^{\text{glob}}) \tag{31}$$

The space on the left here is $(N-1)$-dimensional. To find $\dim((L^{\text{glob}})^\perp / L^{\text{glob}})$, we need to find the dimension of the space on the right.

Define the map $\sigma : \Omega^1(\gamma) \times \Omega^1(\gamma) \to \mathbb{R}^{2N}$ sending two 1-forms on γ to the set of their periods around the connected components of γ,

$$(\alpha, \beta) \mapsto \left(\oint_{\gamma_1} \alpha, \cdots, \oint_{\gamma_N} \alpha, \oint_{\gamma_1} \beta, \cdots, \oint_{\gamma_N} \beta \right)$$

The kernel of σ is $\ker(\sigma) = \Omega^1_{\text{exact}}(\gamma) \times \Omega^1_{\text{exact}}(\gamma)$. Note that by (28,30), this implies $\ker(\sigma) \cap \rho((L^{\text{glob}})^\perp) = \rho(L^{\text{glob}})$. Thus σ induces an injective map $\sigma : \rho((L^{\text{glob}})^\perp)/\rho(L^{\text{glob}}) \hookrightarrow \mathbb{R}^{2N}$. Its image is

$$\begin{aligned} &\sigma(\rho((L^{\text{glob}})^\perp)/\rho(L^{\text{glob}})) \\ &= \left\{(a_1, \ldots, a_N, b_1, \ldots, b_N) \mid \sum_{i=1}^N a_i = \sum_{i=1}^N b_i = 0,\ a_1 + b_1 = 0, \ldots, a_N + b_N = 0\right\} \end{aligned} \tag{32}$$

Here the relations $\sum_i a_i = \sum_i b_i = 0$ arise because $\int_\gamma \alpha = 0$ for $\alpha \in \Omega^1(\gamma)^{E_\pm}$, since the involutions $E_\pm$ are orientation-reversing. The relations $a_i + b_i = 0$ arise because of the relation $\alpha + \beta \in \Omega^1_{\text{exact}}(\gamma)$ in (30). The dimension of the right hand side of (32) is $2N - (N+2) + 1 = N - 1$ (since there are $N+2$ relations and one relation between relations, $\left(\sum_i a_i\right) + \left(\sum_i b_i\right) - \sum_i (a_i + b_i) = 0$). Hence, $\dim \rho((L^{\text{glob}})^\perp)/\rho(L^{\text{glob}}) = N-1$ and, by (31), $\dim((L^{\text{glob}})^\perp / L^{\text{glob}}) = 2(N-1)$. □

Lemma 4.6. (i) *The quotient* $(L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}}$ *inherits a non-degenerate symplectic pairing from* Φ_γ.
(ii) *The symplectic double orthogonal to* L^{glob} *in* Φ_γ *is* $(L^{\mathrm{glob}})^{\perp\perp} = L^{\mathrm{glob}}$.

Proof. It follows from the proof of Lemma 4.5 that $(L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}}$ fits into the following exact sequence:

$$\mathbb{R} \to H^0(\gamma) \to (L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}} \to H^1(\gamma) \to \mathbb{R} \tag{33}$$

Here the maps, going from left to right, are:

- realization of constants as constant functions on γ,
- realization of locally constant functions on γ as elements of $(L^{\mathrm{glob}})^{\perp}$ (with vanishing ϕ_n),
- map $\sigma \circ \rho : (L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}} \to H^1(\gamma) \times H^1(\gamma)$ composed with projection to the first factor,
- pairing with fundamental class of γ.

The symplectic structure ω on Φ_γ induces a well defined pairing $\underline{\omega}$ on $(L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}}$. Using the truncation of sequence (33)

$$H^0(\gamma)/\mathbb{R} \to (L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}} \to H^1(\gamma)|_{\int_\gamma = 0} \tag{34}$$

and the fact that symplectic structure (17) can be written as

$$\omega((\phi, \phi_n), (\psi, \psi_n)) = \int_\gamma \phi(-\alpha_\psi + \beta_\psi) - \psi(-\alpha_\phi + \beta_\phi)$$

we see that, choosing some splitting of (34) from the right, we can write the block matrix of $\underline{\omega}$ as

$$\begin{pmatrix} 0 & -2\langle , \rangle \\ 2\langle , \rangle & * \end{pmatrix} \tag{35}$$

where the first and second row/column correspond to the left and right terms of (34) respectively; $\langle , \rangle$ is the non-degenerate pairing between the left and right terms of (34) induced from Poincaré duality $H^0(\gamma) \otimes H^1(\gamma) \to \mathbb{R}$; the lower right block is dependent on the choice of splitting of (34). Ansatz (35) implies that the anti-symmetric pairing $\underline{\omega}$ on $L^{\perp}/L$ is *non-degenerate*. Thus $(L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}}$ is the symplectic reduction of $L^{\perp}$ and $\underline{\omega}$ is the induced symplectic structure on reduction. Non-degeneracy of $\underline{\omega}$ also immediately implies that $(L^{\mathrm{glob}})^{\perp\perp} = L^{\mathrm{glob}}$. □

Proof of Theorem 4.2. The map $\rho : F_\gamma \to \Omega^1(\gamma) \times \Omega^1(\gamma)$ defined in the proof of Lemma 4.5 sends $(\phi, \phi_n) \in L$ to $(\kappa|_\gamma, \lambda|_\gamma)$, where κ, λ are closed $\partial_{\mp}$-horizontal 1-forms corresponding to (ϕ, ϕ_n) by (20). Thus the image of ρ on L is

$$\rho(L) = \{(\alpha, \beta) \in \Omega^1(\gamma)^{E_-} \times \Omega^1(\gamma)^{E_+} \mid \alpha + \beta \in \Omega^1_{\mathrm{exact}}(\gamma)\} \tag{36}$$

Hence, by (29), $L \subset (L^{\mathrm{glob}})^{\perp}$. Taking into account isotropicity of L, we have a sequence of inclusions

$$L^{\mathrm{glob}} \subset L \subset L^{\perp} \subset (L^{\mathrm{glob}})^{\perp}$$

Passing to the symplectic reduction (quotient by L^{glob}) we get

$$L/L^{\text{glob}} \subset L^{\perp}/L^{\text{glob}} \subset (L^{\text{glob}})^{\perp}/L^{\text{glob}}$$

By (21) and Lemma 4.5, L/L^{glob} is an $(N-1)$-dimensional isotropic subspace in a $2(N-1)$-dimensional symplectic space, hence L/L^{glob} is Lagrangian. Hence, $L/L^{\text{glob}} = L^{\perp}/L^{\text{glob}}$ and therefore $L = L^{\perp}$. This finishes the proof that L is Lagrangian. □

5. Remarks

Unless stated otherwise, in this section we are assuming the setup of Sect. 4.

5.1. Dirichlet polarization. It is interesting that $\dim((L^{\text{glob}})^{\perp}/L^{\text{glob}})$ depends only on the *topology* of the domain D, at least as long as the mild assumptions A, B, C hold. On the other hand, L itself is sensitive to the *geometry* of the boundary γ, in particular to dynamics on points of γ defined by joint action of involutions E_+, E_-. In particular, for the map $\mathcal{D} : L \to \mathcal{C}^{\infty}(\gamma)$, sending $(\phi, \phi_n) \mapsto \phi$, we have the following (we assume for simplicity that D is simply connected).

- If there is a point on the boundary $p \in \gamma$ and a number $n \geq 1$ such that

$$(E_+E_-)^n p = p \tag{37}$$

  then by (19) on L we have $\sum_{i=0}^{n-1} \phi((E_+E_-)^i p) - \phi(E_-(E_+E_-)^i p) = 0$, hence $\mathcal{D}$ is not surjective (equivalently, in general there is no existence for Dirichlet boundary problem for the wave equation on D).
- If there is an open subset of the boundary $U \subset \gamma - I$ such that (37) holds for every $p \in U$ for some fixed $n \geq 1$, then $\mathcal{D}$ is not injective (no uniqueness for Dirichlet problem): for ψ_U a bump function supported on U, we define

$$f = \sum_{i=0}^{n-1} \left((E_+^* E_-^*)^i \psi_U + E_-^* (E_+^* E_-^*)^i \psi_U \right) = \sum_{i=0}^{n-1} \left((E_-^* E_+^*)^i \psi_U + E_+^* (E_-^* E_+^*)^i \psi_U \right)$$

  Then f is simultaneously E_+- and E_--invariant, hence by (19),

$$(0, -\frac{1}{v}(\cot(\theta - \pi/4) - \cot(\theta + \pi/4))\ \partial_t f) \in L$$

  is a non-zero vector in L lying in kernel of $\mathcal{D}$.
- If there is a point $p \in \gamma$, such that its orbit under the joint action of E_+ and E_- is *dense* in γ, then $\mathcal{D}$ is injective (there is uniqueness for Dirichlet problem): by (19), to have a vector in L lying in kernel of $\mathcal{D}$, we need a function $f \in \mathcal{C}^{\infty}(\gamma)$ which is both E_+- and E_--invariant. But f has to be constant on the dense $E_{\pm}$-orbit in γ, thus f is a constant and gives zero vector in L.

5.2. Explicit examples of involutions $E_\pm$: disk and annulus. First consider a unit disk on $\mathbb{R}^{1,1}$, defined in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ by $r \le 1$ with the boundary unit circle parameterized by the angular coordinate $t = \theta \in \mathbb{R}/(2\pi\mathbb{Z})$.[14] The four light-like points on the boundary are:

$$I = \{\underbrace{\pi/4, -3\pi/4}_{I_-}, \underbrace{-\pi/4, 3\pi/4}_{I_+}\}$$

and the involutions $E_\pm$ on the boundary circle are:

$$E_- : \theta \leftrightarrow \pi/2 - \theta, \qquad E_+ : \theta \leftrightarrow -\pi/2 - \theta$$

Next, consider the annulus defined by $r_1 \le r \le r_2$. We consider both inner and outer circle parameterized by the angular coordinate θ. We will put superscripts "in", "out" to indicate to which boundary component a point belongs. The eight light-like boundary points are:

$$I = \{\underbrace{(\pi/4)^{\text{in}}, (-3\pi/4)^{\text{in}}, (\pi/4)^{\text{out}}, (-3\pi/4)^{\text{out}}}_{I_-}, \underbrace{(-\pi/4)^{\text{in}}, (3\pi/4)^{\text{in}}, (-\pi/4)^{\text{out}}, (3\pi/4)^{\text{out}}}_{I_+}\}$$

The involutions are:

$$E_\pm : \theta^{\text{in}} \leftrightarrow \left(\mp\frac{\pi}{4} + \arccos\left(\frac{r_1}{r_2}\cos(\theta \pm \frac{\pi}{4})\right)\right)^{\text{out}} \tag{38}$$

$$E_\pm : \theta^{\text{out}} \leftrightarrow \left(\mp\frac{\pi}{2} - \theta\right)^{\text{out}} \text{ for}$$

$$\theta^{\text{out}} \in \left(\mp\frac{\pi}{4} - \theta_0, \mp\frac{\pi}{4} + \theta_0\right)^{\text{out}} \cup \left(\pm\frac{3\pi}{4} - \theta_0, \pm\frac{3\pi}{4} + \theta_0\right)^{\text{out}} \tag{39}$$

where $\theta_0 = \arccos\frac{r_1}{r_2}$ and the sign of arccos in (38) is chosen in such a way that in the limit $r_1 \to r_2$ we get the involution $\theta^{\text{in}} \leftrightarrow \theta^{\text{out}}$. For each choice of the sign $\pm$, the equivalence relation $\mathcal{E}_\pm$ has two equivalence classes of order 1: $\{(\mp\pi/4)^{\text{out}}\}$, $\{(\pm 3\pi/4)^{\text{out}}\}$ and two equivalence classes of order 3:

$$\left\{\left(\mp\frac{\pi}{4}\right)^{\text{in}}, \left(\mp\frac{\pi}{4} - \theta_0\right)^{\text{out}}, \left(\mp\frac{\pi}{4} + \theta_0\right)^{\text{out}}\right\},$$
$$\left\{\left(\pm\frac{3\pi}{4}\right)^{\text{in}}, \left(\pm\frac{3\pi}{4} - \theta_0\right)^{\text{out}}, \left(\pm\frac{3\pi}{4} + \theta_0\right)^{\text{out}}\right\}$$

Elements of the latter classes correspond to points of the boundary where involution $E_\pm$ is discontinuous. All the other equivalence classes are of order 2.

[14] Note that this convention agrees with conventions introduced in the beginning of Sect. 4, but now θ is to be considered modulo 2π, not modulo π.

5.3. Constraint (Cauchy) subspace of the phase space. Fix a closed curve $\gamma \subset \mathbb{R}^{1,1}$ subject to assumptions A, B, C of Sect. 4. Denote D_{in} the compact domain of $\mathbb{R}^{1,1}$ bounded by γ and denote D_{out} the complement of D_{in} in $\mathbb{R}^{1,1}$.

By specializing a general construction of [4], one can associate to γ two subspaces of the phase space $C_{\text{in}}, C_{\text{out}} \subset \Phi_\gamma$ consisting of pairs $(\phi, \phi_n) \in \mathcal{C}^\infty(\gamma)^{\times 2}$ extendable as solutions of the wave equation into some open neighborhood of γ in D_{in} or D_{out} respectively. Note that one can view $C_{\text{in}}, C_{\text{out}}$ as being associated to the two orientations of γ: $C_{\text{in}} = C(\gamma) \subset (\Phi_\gamma, \omega_\gamma), C_{\text{out}} = C(\gamma^{\text{op}}) \subset (\Phi_{\gamma^{\text{op}}} = \Phi_\gamma, \omega_{\gamma^{\text{op}}} = -\omega_\gamma)$, where γ is understood as coming with counterclockwise orientation by default and "op" denotes orientation reversal.

Remark 5.1. A related concept to the Cauchy subspaces C_{in}, C_{out} introduced above is the subspace C of Φ_γ consisting of pairs (ϕ, ϕ_n) extendable as solutions of the wave equation into a tubular neighborhood of γ in $\mathbb{R}^{1,1}$ (as opposed to an open neighborhood in the relative topology of D_{in} or D_{out}). Obviously, $\mathsf{C} = C_{\text{in}} \cap C_{\text{out}}$.

We split light-like points of γ into those where D_{in} is convex and those where D_{out} is convex: $I = I^{\text{in}} \sqcup I^{\text{out}}$. We also introduce involutions $E^{\text{in}}_\pm$, $E^{\text{out}}_\pm$ on points of γ, induced by following light-like lines in D_{in} or D_{out} respectively. Note that since D_{out} is non-compact, a light-like line starting at a point on γ may run to infinity, thus involutions $E^{\text{out}}_\pm$ are only defined on some subsets of γ. In particular, $E^{\text{out}}_\pm$ is defined in an open neighborhood of points of $I^{\text{out}}_\pm$ (with the same $\pm$).

Proposition 5.2. (1) *The subspace $C_{\text{in}} \subset \Phi_\gamma$ consists of pairs $(\phi, \phi_n) \in \mathcal{C}^\infty(\gamma)^{\times 2}$ such that:*

(a) *For every point $z \in I^{\text{in}}_\pm$ there is an $E^{\text{in}}_\pm$-invariant open neighborhood $z \in U_z \subset \gamma$ such that the restriction of the 1-form $\rho_\pm \in \Omega^1(\gamma)$ to U_z is $E^{\text{in}}_\pm$-invariant. Here $\rho_+ := \beta$, $\rho_- := \alpha$ are the two 1-forms on γ defined by (24,25).*

(b) *For every point $z \in I^{\text{out}}_\pm$ the ∞-jet of the 1-form $\rho_\pm \in \Omega^1(\gamma)$ at z is $E^{\text{out}}_\pm$-invariant. The second subspace $C_{\text{out}} \subset \Phi_\gamma$ is described similarly where we should interchange superscripts "in" and "out" in the description of constraints (1a,1b) above.*

(2) *Subspaces $C_{\text{in}}, C_{\text{out}} \subset \Phi_\gamma$ are symplectic w.r.t. symplectic form ω on Φ_γ. Symplectic orthogonals to $C_{\text{in}}, C_{\text{out}}$ in Φ_γ are zero.*

Proof. To prove necessity of constraints (1a, 1b), assume that a pair $(\phi, \phi_n) \in \Phi_\gamma$ comes from a solution $\tilde{\phi}$ of the wave equation on an open neighborhood V of γ in D_{in}. We can fit into V a topological annulus $D \subset V$ with boundary $\partial D = \gamma' \sqcup \gamma$. The associated involutions $E_\pm(D)$ on ∂D coincide with $E^{\text{in}}_\pm(\gamma)$ on some neighborhoods of points $z \in I^{\text{in}}_\pm(\gamma)$, which implies constraint (1a) by (36). To see (1b), fix a sign $\pm$ and fix a point $z \in I^{\text{out}}_\pm(\gamma)$. We can choose the annulus D in such a way that the equivalence class of z under equivalence relation $\mathcal{E}_\pm(D)$ is $\{x, z, y\}$ with $x, y \in \gamma'$. Denote $U' \subset \gamma'$ an open interval on γ' bounded by points x, y (among the two possible intervals we choose the $E_\pm(D)$-invariant one). Also fix a neighborhood U of z in γ; point z splits U into two intervals, U_1 and U_2. Condition (1b) on the jet of $\rho_\pm$ at z arises from necessity to smoothly sew an $E_\pm$-invariant 1-form $\rho_\pm$ on U' with $E^*_\pm(\rho_\pm|_{U_1})$ at point x and with $E^*_\pm(\rho_\pm|_{U_2})$ at point y.

Conversely, to check sufficiency of (1a, 1b), fix $(\phi, \phi_n) \in \mathcal{C}^\infty(\gamma)^{\times 2}$ satisfying (1a, 1b) and fix an annulus $D \subset D_{\text{in}}$ with boundary $\partial D = \gamma \sqcup \gamma'$, thin enough, so that for every $z \in I^{\text{in}}_\pm(\gamma)$, the neighborhood U_z where we have $E^{\text{in}}_\pm$-invariance of $\rho_\pm$ contains the maximal $E_\pm(D)$-invariant neighborhood of z. Then (1a) ensures that 1-form $E^*_\pm(\rho_\pm|_\gamma)$

is smooth on the image of γ in γ' under $E_\pm$ and (1b) ensures that it can be extended to a smooth $E_\pm^{\text{out}}(\gamma')$-invariant 1-form on γ'. Thus we obtain a pair $(\alpha, \beta) \in \Omega^1(\partial D)^{E_-} \times \Omega^1(\partial D)^{E_+}$ which restricts to (ρ_-, ρ_+) on γ. Then we construct the solution $\tilde{\phi}$ of wave equation in D as

$$\tilde{\phi}(\zeta) = \phi(\zeta_0) + \int_{\zeta_0}^{\zeta} (p_-^*\alpha + p_+^*\beta), \quad \zeta \in D \tag{40}$$

where ζ_0 is some arbitrary chosen point on γ, $p_\pm : D \to \partial D/E_\pm$ are projections from D to the boundary along light-like lines. Integration path from ζ_0 to ζ in D is chosen arbitrarily (the integrand is exact since it is closed and restricts to an exact 1-form on one of the two boundary components). By construction, $\tilde{\phi}$ induces back (ϕ, ϕ_n) on γ.

The case of C_{out} is treated similarly.

The calculation of the symplectic orthogonal to C_{in} (case of C_{out} is analogous) in Φ_γ follows the proof of Proposition 4.1. We can choose in (17) $(\psi, \psi_n) \in C_{\text{in}}$ with ψ_n a bump function in neighborhood of any point $z \in \gamma - I$ and vanishing in some open neighborhood of every point of I and $\psi = 0$. This proves that $(\phi, \phi_n) \in C_{\text{in}}^\perp$ has $\phi(z) = 0$. Next, choosing ψ a bump function as above and $\psi_n = 0$ we prove that $\phi_n(z) = 0$. Thus $C_{\text{in}}^\perp = 0$. This also implies that C_{in} is symplectic. □

Remark 5.3. Note that $C_{\text{in}}, C_{\text{out}}$ cannot be described in intrinsic terms of γ, using only the geometric data (Γ, u, μ) as introduced in Sect. 2: we need more detailed information on the behavior of the metric near γ (since we need to know the involutions $E_\pm^{\text{in,out}}$ near light-like points of γ).

Remark 5.4. Let $D \subset \mathbb{R}^{1,1}$ be a (topological) annulus bounded by $\partial D = \gamma = \gamma_1 \sqcup \gamma_2$ subject to conditions of Sect. 4. We assume that γ_1 is the inner boundary component and γ_2 the outer one. Then by Theorem 4.2, the corresponding L is a canonical relation $L \subset C_{\text{out}}(\gamma_1) \times C_{\text{in}}(\gamma_2)$. Denoting $p_{1,2}$ the projections to the first and second factors in $C_{\text{out}}(\gamma_1) \times C_{\text{in}}(\gamma_2)$, p_1 is never injective[15] on L and p_2 is never surjective.[16] Moreover, p_1 is surjective and p_2 is injective if and only if the following condition holds:

$$I^{\text{out}}(\gamma_1) = \varnothing$$

Note that L cannot be a graph of a map $C_{\text{out}}(\gamma_1) \to C_{\text{in}}(\gamma_2)$ (nor in the opposite direction).

5.4. Conformal invariance. In case $\dim M = 2$, the action (1) is invariant under Weyl transformations—local rescaling of metric $g_{\mu\nu}(x) \to \Omega(x) \cdot g_{\mu\nu}(x)$ with $\Omega \in \mathcal{C}^\infty(M)$, $\Omega > 0$. Hence for $F : (M, g) \to (M', g')$ a conformal diffeomorphism (i.e. $F^*g' = \Omega \cdot g$ with $\Omega > 0$) of 2-dimensional pseudo-Riemannian manifolds, we have $S_{M',g'}(\phi') = S_{M,g}(F^*\phi')$. Thus F induces a symplectomorphism of phase spaces $F^* : \Phi_{\partial M'} \to \Phi_{\partial M}$ which takes $L_{M'}$ to L_M.

[15] Indeed, we can take an open subset $U \subset \gamma_2$ such that $E_+(U) \subset \gamma_2$, and a bump function ψ_U on U. Then we construct a nonzero element $\psi_U + E_+^*\psi_U \in L \cap \ker p_1$.

[16] The reason is that $p_2(L)$ is given by $E_\pm$-invariance constraint for 1-forms $\rho_\pm|_{\gamma_2}$ on certain *finite* open subsets $U_\pm$ of γ_2, whereas $C_{\text{in}}(\gamma_2)$ is given by the constraint (1a) of Proposition 5.2 on *arbitrarily small* neighborhoods of light-like points of γ_2.

In particular, in case of domains D in $\mathbb{R}^{1,1}$, pairs of a symplectic manifold and an Lagrangian submanifold $(\Phi_{\partial D}, L_D)$ are canonically isomorphic for domains D related by a conformal transformation of $\mathbb{R}^{1,1}$, e.g. a translation, a Lorentz boost or a rescaling.

Also, Theorem 4.2 implies that for $D \subset \mathbb{R}^2$ a compact domain on the plane endowed with some Lorentzian metric g_D, conformally equivalent to a domain $D' \subset \mathbb{R}^{1,1}$ with Minkowski metric, $L_D \subset \Phi_{\partial D}$ is Lagrangian.

5.5. Hamiltonian for a circle. Consider polar coordinates[17] $(\xi, \theta) \in \mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$ on $\mathbb{R}^{1,1}$, $x = e^{\xi}\cos\theta$, $y = e^{\xi}\sin\theta$. Phase spaces $\Phi_{S^1_{\xi_0}}$ for circles given by $\xi = \xi_0$ are canonically symplectomorphic for different values of ξ_0 by conformal invariance.

For a circle centered at the origin, define a function on the phase space

$$H = \frac{1}{2}\oint_{S^1} d\theta\ \cos(2\theta)\ ((\phi_n)^2 + (\partial_\theta\phi)^2) \quad \in \mathcal{C}^\infty(\Phi_{S^1}) \tag{41}$$

where[18] $\phi_n = \partial_\xi\phi|_{S^1}$. It generates a Hamiltonian vector field $\check{H}$ defined by $\iota_{\check{H}}\omega = -\delta H$. Explicitly:

$$\check{H} = \oint_{S^1} \phi_n\ \frac{\delta}{\delta\phi} + \frac{1}{\cos(2\theta)}(\sin(2\theta)\partial_\theta\phi_n + \partial_\theta(\sin(2\theta)\phi_n + \cos(2\theta)\partial_\theta\phi))\ \frac{\delta}{\delta\phi_n} \tag{42}$$

Then the infinitesimal evolution in ξ is given by the flow equation for $\check{H}$:

$$\partial_\xi\phi = \check{H}\circ\phi, \quad \partial_\xi\phi_n = \check{H}\circ\phi_n$$

—this is just an equivalent restatement of the wave equation (2) in coordinates (ξ, θ).

One way to get the function (41) is to consider the radial density $\mathcal{L} \in \mathcal{C}^\infty(\Phi_{S^1})$ of action (1) in an annulus $\mathrm{Ann}_{\xi_0}^{\xi_1}$ defined by $\xi_0 \le \xi \le \xi_1$,

$$S = \int_{\xi_0}^{\xi_1} d\xi\ \mathcal{L}(\phi|_{S^1_\xi}, \partial_\xi\phi|_{S^1_\xi}),$$
$$\mathcal{L} = \oint_{S^1} d\theta\ \frac{1}{2}\left(\cos(2\theta)\ (\phi_n^2 - (\partial_\theta\phi)^2) - \sin(2\theta)\ \phi_n\,\partial_\theta\phi\right)$$

Then one defines

$$H = \oint_{S^1} d\theta\ \phi_n\ \frac{\delta\mathcal{L}}{\delta\phi_n} - \mathcal{L} \tag{43}$$

which yields (41). Note that (43) is indeed the formula for Legendre transform, but we do not switch to canonical momenta $p = \frac{\delta\mathcal{L}}{\delta\phi_n} = \cos(2\theta)\phi_n - \sin(2\theta)\partial_\theta\phi$.

The Hamiltonian vector field $\check{H}$ (42) is only well-defined on a subspace

$$C_0 = \{(\phi, \phi_n) \in \mathcal{C}^\infty(S^1)^{\times 2} \mid (\partial_\theta\phi_n - \partial_\theta\phi)|_{\theta\in\{\pm\frac{\pi}{4}, \pm\frac{3\pi}{4}\}} = 0\} \quad \subset \Phi_{S^1}$$

[17] We are using an unconventional radial coordinate, since this choice makes rescaling a translation in ξ. In this section we will sometimes refer to ξ as the "time", as the parameter of Hamiltonian dynamics.

[18] This is a different normalization of the transversal vector field than in Sect. 4. The reason for this choice is that the isomorphism $\Phi_{S^1_{\xi_0}} \simeq \Phi_{S^1_{\xi_1}}$ coming from conformal invariance in these coordinates is just $(\phi, \phi_n) \mapsto (\phi, \phi_n)$.

due to δH not being in the image of the map of vector bundles $\omega^\# : T\Phi_{S^1} \to T^*\Phi_{S^1}$ (which is injective by weak non-degeneracy of ω, but not an isomorphism) unless one restricts the base to $C_0 \subset \Phi_{S^1}$. More precisely, $\check{H}$ is defined as a section of the pullback of the tangent bundle $T\Phi_{S^1}$ to C_0, but it is not generally tangent to C_0. However, one may further restrict $\check{H}$ to a smaller subspace $C_1 \subset C_0$ where it is tangent to C_0; subspace C_1 is given by certain restrictions on 3-jets of (ϕ, ϕ_n) at light-like points on S^1. To find the maximal subspace of Φ_{S^1} on which $\check{H}$ is defined as a tangent vector field, one can iterate this process: cf. the Gotay–Nester–Hinds (GNH) geometric constraint algorithm [1,6]. This way one finds a sequence of subspaces $\Phi_{S^1} \supset C_0 \supset C_1 \supset C_2 \supset \cdots$ where $\check{H}$ on C_{k+1} is tangent to C_k, with C_k given by constraints on $(2k+1)$-jets of boundary fields at light-like points of S^1. The process does not stabilize at a finite step, and the maximal subspace where $\check{H}$ is defined as a tangent vector field is $C_\infty = \cap_k C_k$ which coincides with $C_{\mathrm{out}}(S^1)$ given by constraint (1b) of Proposition 5.2.

Integrating the vector field $\check{H}$ to a *flow* on $C_{\mathrm{out}}(S^1)$ is equivalent to writing the evolution relation $L \subset \bar{\Phi}_{S^1} \times \Phi_{S^1}$ for the geometric annulus Ann_0^ξ (we are assuming $\xi > 0$) as the graph of a map $F_\xi : C_{\mathrm{out}}(S^1) \to C_{\mathrm{out}}(S^1)$. This is impossible due to issues with existence/uniqueness for the initial value problem for the wave equation on the annulus (cf. Remark 5.4). Specifically, projections $p_{1,2} : \Phi_{S^1} \times \Phi_{S^1} \to \Phi_{S^1}$ restricted to L yield a diagram

$$C_{\mathrm{out}}(S^1) \overset{p_1}{\twoheadleftarrow} L \overset{p_2}{\hookrightarrow} \underbrace{C_{\mathrm{in}}(S^1)}_{\subset C_{\mathrm{out}}(S^1)} \tag{44}$$

where neither map is an isomorphism.

However, the flow of $\check{H}$ in *negative* time $-\xi < 0$ exists as a map

$$F_{-\xi} = p_1(p_2^{-1}(\bullet) \cap L) : \quad C(-\xi) \to C_{\mathrm{out}}(S^1)$$

where $C(-\xi)$ is the subspace of $C_{\mathrm{out}}(S^1)$ defined as

$$\begin{aligned} C(-\xi) = p_2(L) = \{(\phi, \phi_n) \in \mathcal{C}^\infty(S^1)^{\times 2} \mid \\ \alpha(\phi, \phi_n) \text{ is } E_-\text{-invariant on } \underbrace{\left(\frac{\pi}{4} - \theta_0, \frac{\pi}{4} + \theta_0\right) \cup \left(-\frac{3\pi}{4} - \theta_0, -\frac{3\pi}{4} + \theta_0\right)}_{U_-(\xi) \subset S^1}, \\ \beta(\phi, \phi_n) \text{ is } E_+\text{-invariant on } \underbrace{\left(-\frac{\pi}{4} - \theta_0, -\frac{\pi}{4} + \theta_0\right) \cup \left(\frac{3\pi}{4} - \theta_0, \frac{3\pi}{4} + \theta_0\right)}_{U_+(\xi) \subset S^1}\} \end{aligned}$$

where p_2 and L are as in the diagram (44), $\theta_0 = \arccos(e^{-\xi})$ and involutions on S^1 are $E_- : \theta \leftrightarrow \pi/2 - \theta$, $E_+ : \theta \leftrightarrow -\pi/2 - \theta$; α and β are the 1-forms defined by (24, 25). Note that $C(-\xi) \subset \Phi_{S^1}$ is *not* a symplectic subspace; also $F_{-\xi}$ is not injective. What happens instead is that $C(-\xi) \subset \Phi_{S^1}$ is coisotropic, with

$$\begin{aligned} C(-\xi)^\perp &= \ker F_{-\xi} = p_2(\ker p_1 \cap L) \\ &= \{(\phi, \phi_n) \in \mathcal{C}^\infty(S^1)^{\times 2} \text{ such that } \alpha|_{S^1 - U_-(\xi)} = 0,\ \beta|_{S^1 - U_+(\xi)} = 0, \end{aligned}$$

$$\alpha|_{U_-(\xi)} \text{ is } E_-\text{-invariant},\ \beta|_{U_-(\xi)} \text{ is } E_+\text{-invariant},\ \phi(\pi/4) + \int_{\pi/4}^{\pi/4+\theta_0} \alpha = 0\} \tag{45}$$

Formula (45) for $\ker F_{-\xi}$ follows from restricting the solution (40) to the inner boundary circle. Coincidence of the kernel of $F_{-\xi}$ with the symplectic orthogonal to $C(-\xi)$ follows from Theorem 4.2:

$$\begin{aligned} C(-\xi)^\perp &= \left\{u = (\phi, \phi_n) \subset \Phi_{S^1} \mid \forall s \in L,\ \langle \underbrace{(0,u)}_{\in \bar\Phi_{S^1}\times\Phi_{S^1}}, s\rangle_{\bar\Phi_{S^1}\times\Phi_{S^1}}\right\} \\ &= p_2(L^\perp \cap\ 0 \times \Phi_{S^1}) = p_2(L \cap\ 0 \times \Phi_{S^1}) = \ker F_{-\xi} \end{aligned}$$

Thus $F_{-\xi}$ descends to the symplectic reduction $\underline{C}(-\xi) = C(-\xi)/C(-\xi)^\perp$ and yields an isomorphism of symplectic spaces

$$\underline{F}_{-\xi}:\quad \underline{C}(-\xi) \xrightarrow{\sim} C_{\text{out}}(S^1)$$

which is a symplectomorphism, since before reduction $F_{-\xi}$ pulls back the symplectic structure on $C_{\text{out}}(S^1)$ to the presymplectic structure on $C(-\xi)$, as follows from isotropicity of L, the graph of $F_{-\xi}$: for any pair of elements $u, v \in C(-\xi)$ we have

$$\langle F_{-\xi}(u), F_{-\xi}(v)\rangle_{\Phi_{S^1}} - \langle u, v\rangle_{\Phi_{S^1}} = -\langle \underbrace{(F_{-\xi}(u), u)}_{\in L}, \underbrace{(F_{-\xi}(v), v)}_{\in L}\rangle_{\bar\Phi_{S^1}\times\Phi_{S^1}} = 0$$

With some abuse of terminology, one may call $\underline{F}_{-\xi}$ the "reduced flow" of the Hamiltonian vector field $\check H$ in negative time $-\xi < 0$. Then it is reasonable to define the reduced flow in positive time $\xi > 0$ to be the inverse map:

$$\underline{F}_{+\xi} = (\underline{F}_{-\xi})^{-1}:\quad C_{\text{out}}(S^1) \xrightarrow{\sim} \underline{C}(-\xi)$$

Note that the reduced Hamiltonian flow does not satisfy the usual semigroup law $\underline{F}_{\xi+\xi'} = \underline{F}_{\xi'} \circ \underline{F}_{\xi}$, since the range of $\underline{F}_{\xi}$ and the domain of $\underline{F}_{\xi'}$ do not match. Instead we have the following composition law. First consider flows in negative time. The map $\underline{F}_{-\xi}: C(-\xi)/C(-\xi)^\perp \to C_{\text{out}}(S^1)$ can be restricted to a subspace $C(-\xi-\xi')/C(-\xi)^\perp$; this restriction is an isomorphism $C(-\xi-\xi')/C(-\xi)^\perp \xrightarrow{\sim} C(-\xi')$. The latter induces an isomorphism of quotients $\underline{F}_{-\xi,-\xi'}:\ \underline{C}(-\xi-\xi') \xrightarrow{\sim} \underline{C}(-\xi')$. Then the composition law is:

$$\underline{F}_{-\xi-\xi'} = \underline{F}_{-\xi'} \circ \underline{F}_{-\xi,-\xi'} \tag{46}$$

In other words, we take the symplectic reduction of the three spaces in the upper row of the diagram

$$\begin{array}{ccccc} C(-\xi-\xi') & \xrightarrow{F_{-\xi}|_{C(-\xi-\xi')}} & C(-\xi') & \xrightarrow{F_{-\xi'}} & C_{\text{out}}(S^1) \\ \downarrow & & \downarrow & & \\ C(-\xi) & \xrightarrow{F_{-\xi}} & C_{\text{out}}(S^1) & & \end{array} \tag{47}$$

by $C(-\xi-\xi')^\perp$ (done in two steps: reduction by $C(-\xi)^\perp$ and then by $C(-\xi-\xi')^\perp/C(-\xi)^\perp$), $C(-\xi')^\perp$ and $\{0\}$, respectively. Vertical arrows in (47) are inclusions of subspaces of Φ_{S^1}; composition of the two arrows in the upper row is $F_{-\xi-\xi'}$.

For the composition of reduced flows in positive time, we take the inverse of (46) and interchange $\xi \leftrightarrow \xi'$, obtaining

$$\underline{F}_{\xi+\xi'} = \underline{F}_{\xi',\xi} \circ \underline{F}_{\xi} \tag{48}$$

where $\underline{F}_{\xi',\xi} = \underline{F}^{-1}_{-\xi',-\xi} : \underline{C}(-\xi) \xrightarrow{\sim} \underline{C}(-\xi-\xi')$ is the reduction of the restriction $\underline{F}_{\xi'}|_{C(-\xi)} : C(-\xi) \xrightarrow{\sim} C(-\xi-\xi')/C(-\xi')^\perp$ by $C(-\xi)^\perp$.

Remark 5.5. The Hamiltonian (41) descends to the symplectic reduction $\underline{C}(-\xi)$. To see this, note that one can rewrite (41) in terms of 1-forms (24, 25) as

$$H = \oint_{S^1} -\cot\left(\theta-\frac{\pi}{4}\right)\ \iota_{\partial_\theta}\alpha\cdot\alpha + \cot\left(\theta+\frac{\pi}{4}\right)\ \iota_{\partial_\theta}\beta\cdot\beta$$

Applying this to a point $u+v \in \Phi_{S^1}$ with $u \in C(-\xi)$ and $v \in C(-\xi)^\perp$ we obtain

$$\begin{aligned}
&H(u+v) - H(u)\\
&= \oint_{S^1} -\cot\left(\theta-\frac{\pi}{4}\right)\ \iota_{\partial_\theta}(2\alpha_u+\alpha_v)\cdot\alpha_v + \cot\left(\theta+\frac{\pi}{4}\right)\ \iota_{\partial_\theta}(2\beta_u+\beta_v)\cdot\beta_v\\
&= \int_{U_-(\xi)} \underbrace{-\cot\left(\theta-\frac{\pi}{4}\right)\ \iota_{\partial_\theta}(2\alpha_u+\alpha_v)\cdot\alpha_v}_{E_--\text{invariant}} + \int_{U_+(\xi)} \underbrace{\cot\left(\theta+\frac{\pi}{4}\right)\ \iota_{\partial_\theta}(2\beta_u+\beta_v)\cdot\beta_v}_{E_+-\text{invariant}}\\
&= 0
\end{aligned}$$

Thus H does indeed descend to $\underline{C}(-\xi)$. Moreover, the Hamiltonian vector field $\check{H}$ descends to the reduction too. This follows from the explicit formulae for the action of $\check{H}$ on the 1-forms α, β:

$$\check{H}\alpha = -\partial_\theta\left(\cot\left(\theta-\frac{\pi}{4}\right)\cdot\alpha\right), \qquad \check{H}\beta = -\partial_\theta\left(\cot\left(\theta+\frac{\pi}{4}\right)\cdot\beta\right)$$

which imply that for $\check{H}$ viewed as a linear map $C_{\text{out}} \to C_{\text{out}}$, both subspaces $C(-\xi)$ and $C(-\xi)^\perp$ are invariant.

5.5.1. Banach vs. Fréchet. The impossibility to integrate the vector field $\check{H}$ into a flow on $C_{\text{out}}(S^1)$ comes from the fact that since we required from the start that fields are smooth, $\Phi_{S^1} = \mathcal{C}^\infty(S^1)^{\times 2}$ is naturally equipped with Fréchet (but not Banach) topology and hence the Picard–Lindelöf theorem for existence and uniqueness of integral trajectories for $\check{H}$ does not apply. We could have chosen a different model for the space of fields from the start, e.g., setting the space of fields to be $F_D = \mathcal{C}^2(D)$ and requiring only $\mathcal{C}^2$-differentiability for the boundary ∂D in case of a general domain. The phase space then is $\Phi_{\partial D} = \mathcal{C}^2(\partial D)\times\mathcal{C}^1(\partial D) \ni (\phi, \phi_n)$, equipped with standard Banach topology. The proof of Theorem 4.2 goes through in this setting without any change and, being Lagrangian, $L_D \subset \Phi_{\partial D}$ is automatically closed, and hence a Banach (complete) subspace. In this setting we can try to pass to the Hamiltonian formalism on annuli, with H and $\check{H}$ still given by (41, 42). Then proceeding with the GNH construction as above, we

construct a sequence of subspaces $\Phi_{S^1} \supset C_0 \supset C_1 \supset \cdots$ where C_k becomes a subset of $\mathcal{C}^{k+3}(S^1) \times \mathcal{C}^{k+2}(S^1)$ (since an application of $\check{H}$, viewed as a linear map $C_k \to C_{k-1}$, decreases the regularity by 1 due to the derivatives appearing in 42), with constraints on $(2k+1)$-jets at light-like points of S^1, as before. In the end, the maximal subspace C_∞ of Φ_{S^1}, where $\check{H}$ is defined and to which it is tangent, is $C_\infty = \cap_k C_k$. Note that $C_\infty \subset \Phi_{S^1}$ is not a complete subspace (already C_0 is not), hence again the Picard–Lindelöf theorem does not apply. Note also that in the Banach setting $C_\infty \neq C_{\text{out}}(S^1)$ since the r.h.s., defined as in Sect. 5.3, has only $\mathcal{C}^2 \times \mathcal{C}^1$ regularity (with constraints on the 1-jets of the 1-forms α, β at light-like points, as opposed to ∞-jets arising in the Fréchet setting, cf. Proposition 5.21b).

5.6. Relational representation of the little 2-disks operad. Let E_2 be the operad of little 2-disks [7], with $E_2(n)$ the configuration space of n numbered disjoint (geometric) disks inside a disk of radius 1 centered at the origin in Euclidean $\mathbb{R}^2$; these configurations can be viewed as domains $D \subset \mathbb{R}^2$ obtained by cutting n small disks out of a unit disk. Composition $\circ_i : E_2(m) \times E_2(n) \to E_2(m+n-1)$ for $1 \leq i \leq m$ consists in shrinking an element of $E_2(n)$ and gluing it into an element of $E_2(m)$ instead of the ith disk of the latter.

Part of the data of classical field theory defined by action (1) on Minkowski plane is the morphism of operads

$$Z : E_2 \to \text{IsoRel}(\Phi) \tag{49}$$

where $\Phi = \Phi_{S^1}$ is the phase space for the unit circle in $\mathbb{R}^{1,1}$ (radius and origin are in fact irrelevant due to conformal invariance). For a symplectic space V we denote $\text{IsoRel}(V)$ the operad of isotropic relations,

$$\begin{aligned}\text{IsoRel}(n) &= \{\underbrace{V \times \cdots \times V}_{n} \nrightarrow V\} \\ &= \{L \subset \underbrace{\bar{V} \times \cdots \bar{V}}_{n} \times V \mid L \text{ an isotropic subspace}\}\end{aligned}$$

where $\nrightarrow$ is the symbol for an isotropic relation, bar stands for changing the sign of symplectic form. Composition in IsoRel is the set theoretic composition of relations. Morphism Z sends an element of $E_2(n)$, viewed as a compact domain $D \subset \mathbb{R}^{1,1}$ with n "incoming" boundary circles and one "outgoing" boundary circle, to the corresponding evolution relation $L_D \subset \Phi_{\partial D} \simeq \bar{\Phi}^{\times n} \times \Phi$, which is canonical (Lagrangian) by Theorem 4.2. The fact that Z is indeed a morphism of operads, i.e. is consistent w.r.t. the operadic composition, is an expression of the general gluing property of classical field theory (here it simply amounts to the fact that a function ϕ on a glued domain $D_1 \cup D_2$ solves the wave equation iff its restrictions to $D_{1,2}$ solve the wave equation).

More generally, one can introduce a colored operad $\tilde{E}_2$, with colors being closed curves on $\mathbb{R}^{1,1}$ modulo conformal transformations and elements of $\tilde{E}_2(n)$ being general compact domains with $n+1$ boundary components, with composition defined (when the colors match) by conformal transformation of one domain and gluing in the hole in another domain. Then we have a morphism of colored operads from $\tilde{E}_2$ to the colored operad of isotropic relations $\Phi_{\gamma_1} \times \cdots \times \Phi_{\gamma_n} \nrightarrow \Phi_{\gamma_{n+1}}$ with the same set of colors: conformal classes of curves $\gamma_1, \ldots, \gamma_{n+1}$.

Note that this discussion is very general: we only used the general gluing property of field theory, conformal invariance [which is specific for dimension $m = 2$ in case of action (1)] and the fact that evolution relations are Lagrangian (and in particular isotropic).

5.7. Free field theories and Lefschetz duality. An abstract way to view a free classical field theory, natural from the standpoint of the Batalin–Vilkovisky formalism on manifolds with boundary [3], is as follows. One associates to an m-manifold M (possibly endowed with some geometric data, depending on the field theory model in question) a complex of vector spaces $F_M^\bullet$ with differential Q_M equipped with a degree -1 non-degenerate pairing $\omega_M^{(k)} : F_M^k \otimes F_M^{1-k} \to \mathbb{R}$, satisfying $\omega_M^{(k)}(X, Y) = \omega_M^{(1-k)}(Y, X)$ for $X, Y \in F_M$, and to a closed $(m-1)$-manifold Σ a cochain complex $\Phi_\Sigma^\bullet$ with differential $Q_{\partial\Sigma}$, equipped with degree 0 symplectic structure—a non-degenerate pairing $\omega_\Sigma^{(k)} : \Phi_\Sigma^k \otimes \Phi_\Sigma^{-k} \to \mathbb{R}$ satisfying $\omega_\Sigma^{(k)}(x, y) = -(-1)^k \omega_\Sigma^{(k)}(y, x)$ for $x, y \in \Phi_\Sigma$. To the inclusion of the boundary $\Sigma = \partial M \hookrightarrow M$ the field theory associates a chain projection $\pi_M : F_M^\bullet \to \Phi_{\partial M}^\bullet$ intertwining the differentials Q_M and $Q_{\partial M}$. The differential Q_M, the projection π_M and the pairings $\omega_M, \omega_{\partial M}$ are required to satisfy the following coherence condition:

$$\omega_M(Q_M X, Y) - (-1)^{\deg X} \omega_M(X, Q_M Y) = \omega_{\partial M}(\pi_M(X), \pi_M(Y)) \tag{50}$$

for $X, Y \in F_M$.

The short exact sequence

$$\ker \pi_M \hookrightarrow F_M^\bullet \overset{\pi_M}{\twoheadrightarrow} \Phi_{\partial M}^\bullet$$

induces a long exact sequence in Q-cohomology:

$$\cdots \to H_{Q_M}^k(\ker \pi_M) \to H_{Q_M}^k \overset{\pi_*}{\to} H_{Q_{\partial M}}^k \overset{\beta}{\to} H_{Q_M}^{k+1}(\ker \pi_M) \to \cdots \tag{51}$$

The pairings $\omega_M, \omega_{\partial M}$ induce well-defined pairings on cohomology

$$()_M : H_{Q_M}^k \otimes H_{Q_M}^{1-k}(\ker \pi) \to \mathbb{R}, \tag{52}$$

$$(,)_{\partial M} : H_{Q_{\partial M}}^k \otimes H_{Q_{\partial M}}^{-k} \to \mathbb{R} \tag{53}$$

In many cases [3] these pairings can be proven to be non-degenerate. In particular, for abelian Chern–Simons theory, $(,)_M$ is the Lefschetz duality between de Rham cohomology of a 3-manifold and cohomology relative to the boundary, whereas $(,)_{\partial M}$ is the Poincaré duality for de Rham cohomology of the boundary 2-manifold.

The non-degeneracy of the pairing (52) in the second argument can be shown[19] to be equivalent to the property of being Lagrangian for $\mathrm{im}(\pi_*) \subset H_{Q_{\partial M}}^\bullet$.

In the case of the theory defined by the action (1), the space of fields F_M is a two-term complex (owing to the absence of gauge symmetry) with $F_M^0 = C^\infty(M) \ni \phi$, $F_M^1 = \Omega^m(M) \ni \phi^+$, differential $Q_M : \phi \mapsto d * d\phi$ and pairing $\omega_M(\phi, \phi^+) = \int_M \phi \wedge \phi^+$.

[19] Indeed, one has $\mathrm{im}(\pi_*)^\perp = \{[x] \in H_{Q_{\partial M}}^\bullet \mid (\pi_*[Y], [x])_{\partial M} = 0 \ \forall\, [Y] \in H_{Q_M}^\bullet\}$. Using the property $(\pi_*[Y], [x])_{\partial M} = (-1)^{\deg[Y]+1}([Y], \beta[x])_M$ following from (50), we see that $\mathrm{im}(\pi_*)^\perp = \beta^{-1} \ker_2(,)_M$, where we denoted $\ker_2(,)_M$ the kernel of the map $H_{Q_M}^\bullet(\ker \pi) \to (H_{Q_M}^{1-\bullet})^*$ induced by the pairing (52). Thus $\mathrm{im}(\pi_*)^\perp = \ker \beta = \mathrm{im}(\pi_*)$ if and only if $\ker_2(,)_M$ vanishes.

The boundary phase space is a one-term complex $\Phi^0_{\partial M} = \Phi_{\partial M}$ with zero differential and symplectic structure $\omega_{\partial M}$ described in Sect. 2. The exact sequence (51) becomes in this case

$$0 \to \{\phi \;\Big|\; \begin{matrix} d*d\phi = 0, \\ \pi_M(\phi) = 0 \end{matrix} \} \to EL_M \overset{\pi_* = \pi_M}{\longrightarrow} \Phi_{\partial M} \overset{\beta}{\longrightarrow} \frac{\Omega^m(M)}{\{d*d\phi \mid \pi_M(\phi) = 0\}} \to \frac{\Omega^m(M)}{\{d*d\phi\}} \to 0 \tag{54}$$

and $\mathrm{im}(\pi_*) = L_M \subset \Phi_{\partial M}$. Thus, whenever Conjecture 1.1 holds for M, the "Lefschetz duality" (52)

$$EL_M \otimes \frac{\Omega^n(M)}{\{d*d\phi \mid \pi_M(\phi) = 0\}} \to \mathbb{R}$$

is non-degenerate (non-degeneracy in the first term is trivial, whereas for the second term one really needs that L_M is Lagrangian). The pairing between the rightmost and the leftmost terms of (54),

$$\frac{\Omega^m(M)}{\{d*d\phi\}} \otimes \{\phi \mid d*d\phi = 0,\ \pi_M(\phi) = 0\} \to \mathbb{R} \tag{55}$$

is trivially non-degenerate in the second factor, whereas non-degeneracy in the first factor is non-obvious and constitutes a natural extension of Conjecture 1.1. In the case of Riemannian signature, (55) becomes, by the Hodge–Morrey decomposition theorem [2], the pairing

$$H^m(M) \otimes H^0(M, \partial M) \to \mathbb{R}$$

which is a special case of the standard Lefschetz duality and is indeed non-degenerate. On the other hand, for M a compact domain in the Minkowski plane as in Theorem 4.2, one can easily show that both outmost terms of (54) vanish.

5.8. More general Lorentzian surfaces. By inspection of its proof, Theorem 4.2 generalizes straightforwardly to the case of a compact surface M with smooth boundary endowed with a Lorentzian metric g smooth up to the boundary, if the following conditions hold:

(a) The two null-distributions $\partial_+ \subset TM$, $\partial_- \subset TM$ of the metric g induce, as in Sect. 4.2.1, two piecewise smooth involutions $E_\pm$ on the boundary ∂M with finitely many points removed.
(b) For each choice of the sign $\pm$, the restriction map $C^\infty(M)^{\partial_\pm} \to C^\infty(\partial M)^{E_\pm}$ is surjective. Here $C^\infty(M)^{\partial_\pm}$ stands for the space of smooth functions on M, constant along the distribution ∂_+ or ∂_-, respectively.
(c) The first Betti number of the cohomology of M relative to the boundary vanishes, $\dim H^1(M, \partial M) = 0$.

Remark 5.6. i. Obviously, conditions (a, b,c) hold if (M, g) is conformally equivalent to a domain $D \subset \mathbb{R}^{1,1}$ in the Minkowski plane satisfying conditions (A, B, C) of Sect. 4.

ii. For M a domain $D \subset \mathbb{R}^{1,1}$ in the Minkowski plane, condition (b) is equivalent to assumption (C) of Sect. 4, i.e. the assumption that lightlike points of the boundary are neither inflection, nor undulation points.
iii. The presence of a family of null-curves originating at ∂M and asymptotically approaching a limiting closed null-curve in M spoils both conditions (a) and (b), see the example in Sect. 5.9.
iv. For (M, g) a general Lorentzian surface, if $M' \subset M$ is a sufficiently small disk cut out of M, conditions (a, b, c) hold for $(M', g|_{M'})$ and thus the corresponding evolution relation $L_{M'} \subset \Phi_{\partial M'}$ is Lagrangian.

5.9. An example where L is not Lagrangian: the Misner space. Consider the following Lorentzian manifold (the Misner space [8]): $M = S^1 \times [-1, 1]$—a cylinder with coordinates $x \in \mathbb{R}/2\pi\mathbb{Z}$, $y \in [-1, 1]$—endowed with the Lorentzian metric

$$g = dx\, dy - y\, dx^2$$

The corresponding null-distributions on M are:

$$\partial_+ = \partial_y, \quad \partial_- = -\partial_x - y\, \partial_y$$

In particular, the "in-boundary" $S^1 \times \{-1\}$ is spacelike and the "out-boundary" $S^1 \times \{1\}$ is timelike. Moreover, the circle $S^1 \times \{0\}$ is a leaf of the distribution ∂_-, i.e. a closed null-curve.

The equations for the integral curves of distributions $\partial_\pm$ (the null-curves) are

$$\frac{dx}{dy} = \frac{1}{y}, \quad \frac{dx}{dy} = 0$$

for the ∂_-- and ∂_+-curves, respectively. In particular, all ∂_--curves originating at either boundary circle asymptotically approach the null-cycle $S^1 \times \{0\}$. On the other hand, the ∂_+-curves are simply the vertical lines $\{x\} \times [-1, 1]$, for any $\{x\} \in S^1$.

The phase space associated to the boundary of M by the construction of Sect. 2 is

$$\Phi_{\partial M} = \underbrace{C^\infty(S^1) \times C^\infty(S^1)}_{\Phi_{\partial_{\text{in}} M}} \times \underbrace{C^\infty(S^1) \times C^\infty(S^1)}_{\Phi_{\partial_{\text{out}} M}} \ni (\phi^{\text{in}}, \phi^{\text{in}}_n, \phi^{\text{out}}, \phi^{\text{out}}_n)$$

where we have chosen the transversal vector field to be $n = 2\partial_y - \partial_x$ at the in-boundary and $n = 2\partial_y + \partial_x$ at the out-boundary. The symplectic form (5) on the phase space is

$$\omega = \oint_{S^1} dx\, (\delta\phi^{\text{in}} \wedge \delta\phi^{\text{in}}_n + \delta\phi^{\text{out}} \wedge \delta\phi^{\text{out}}_n)$$

For the evolution relation, consider first the "global" Euler-Lagrange space (in the sense of Sect. 4.2.2):

$$EL^{\text{glob}} = \{\phi = F + G \ \in C^\infty(M) \mid F, G \in C^\infty(M),\ \partial_- F = \partial_+ G = 0\}$$

Since all ∂_--curves asymptotically approach the single closed null-curve $S^1 \times \{0\}$, the function F is forced by continuity to be constant (which can be absorbed into G). Thus the restriction to the "global part" of the evolution relation is

$$\begin{aligned} L^{\text{glob}} &= \pi(EL^{\text{glob}}) \\ &= \{(\phi^{\text{in}} = g(x),\ \phi^{\text{in}}_n = -\partial_x g(x),\ \phi^{\text{out}} = g(x),\ \phi^{\text{out}}_n = \partial_x g(x)) \\ &\quad \in \Phi_{\partial M} \mid g \in C^\infty(S^1)\} \end{aligned}$$

The symplectic orthogonal to L^{glob} is readily calculated to be

$$(L^{\mathrm{glob}})^{\perp} = \big\{(\phi^{\mathrm{in}},\ \phi_n^{\mathrm{in}},\ \phi^{\mathrm{out}},\ \phi_n^{\mathrm{out}}) \quad \in \Phi_{\partial M} \\ \mid \partial_x \phi^{\mathrm{in}}(x) - \phi_n^{\mathrm{in}}(x) - \partial_x \phi^{\mathrm{out}}(x) - \phi_n^{\mathrm{out}}(x) = 0\ \forall x \in \mathbb{R}/2\pi\mathbb{Z}\big\}$$

which implies that L^{glob} is isotropic and

$$\dim\,(L^{\mathrm{glob}})^{\perp}/L^{\mathrm{glob}} = \infty$$

(since in $(L^{\mathrm{glob}})^{\perp}$ one can choose ϕ^{in}, ϕ_n^{in}, ϕ^{out} as independent functions, whereas in L^{glob} they are all expressed in terms of a single function g).

The true Euler–Lagrange space, where the possible multivaluedness of F, G is taken into account, is given by (20). In the case of the Misner geometry, $\iota_{\partial_-}\kappa = 0$ implies that $\int_{S^1\times\{0\}} \kappa = 0$, hence κ defines zero cohomology class in $H^1(M)$ and therefore κ (and hence λ too) is exact. This implies that there is no distinction between EL and EL^{glob} in the case at hand. Thus $L = L^{\mathrm{glob}}$ and, by the discussion above, *the evolution relation L is isotropic, but not Lagrangian.*

It is easy to check that also the two halves of the Misner cylinder considered above, $M_1 = S^1 \times [-1, 0]$ and $M_2 = S^1 \times [0, 1]$, produce non-Lagrangian evolution relations.

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