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Quantum Supergroups III. Twistors

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Abstract: We establish direct connections at several levels between quantum groups and supergroups associated to bar-consistent anisotropic super Cartan datum by constructing an automorphism (called twistor) in the setting of covering quantum groups. The canonical bases of the halves of quantum groups and supergroups are shown to match under the twistor up to powers of $\sqrt{-1}$. We further show that the modified quantum group and supergroup are isomorphic over the rational function field adjoined with $\sqrt{-1}$, by constructing a twistor on the modified covering quantum group. An equivalence of categories of weight modules for quantum groups and supergroups follows.

Le plus court chemin entre deux vérités dans le
domaine réel passe par le domaine complexe.

—Jacques Hadamard

1. Introduction

1.1. A theory of quantum supergroups was developed systematically by Yamane [Y1,Y2] after the classical work of Drinfeld, Jimbo and Lusztig. Recently the interest in quantum supergroups has been revived (see [CW,CHW1,CHW2]) thanks to their categorification [HW] by Hill and one of the authors using the spin nilHecke and quiver Hecke superalgebras [W,EKL,KKT]. The work on quantum supergroups of *anisotropic type* (meaning no isotropic odd simple roots) has also motivated, in turn, further progress on categorification. The conjecture in [HW] that cyclotomic (spin) quiver Hecke superalgebras categorify the integrable modules of the supergroup has recently been proved by Kang, Kashiwara, and Oh [KKO]. The validity of this conjecture at rank one, in which case quiver Hecke superalgebras reduce to spin nilHecke algebras, was already noted in [HW] as an easy upgrading of the difficult categorification result of Ellis, Khovanov, and Lauda [EKL]. Yet another recent development is the categorification of the modified covering quantum group in rank one (see Ellis-Lauda [EL]).

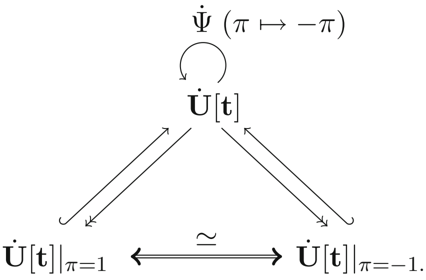
A basic observation in [HW] is that the parity functor Π categorifies a formal super sign π subject to $\pi^2 = 1$. This leads to the formulation of the so-called covering quantum group $\mathbf{U}$ in [HW,CW,CHW1], which allows a second formal parameter π such that $\pi^2 = 1$ besides the usual quantum parameter v. The specialization of $\mathbf{U}$ at $\pi = 1$, denoted by $\mathbf{U}|_{\pi=1}$, recovers the usual quantum group while the specialization of $\mathbf{U}$ at $\pi = -1$, denoted by $\mathbf{U}|_{\pi=-1}$, recovers the quantum supergroup of anisotropic type. In contrast to the versions of quantum supergroups over $\mathbb{C}(v)$ studied in literature, our covering (or super) quantum groups have a well-developed representation theory such as weight modules and integrable modules over $\mathbb{Q}(v)$, thanks to the enlarged Cartan subalgebras [CHW1]; moreover, they admit integral forms. Under a mild bar-consistent condition on the super Cartan datum, the half covering quantum group $\mathbf{f}$ ($\cong \mathbf{U}^-$) and the associated integrable modules admit a novel bar involution which sends $v \mapsto \pi v^{-1}$ and then admit canonical bases [CHW2].

The (covering) quantum supergroups are quantizations of Lie superalgebras associated to the anisotropic type super Cartan datum introduced in [Kac]. It has been known that Lie superalgebras associated to the super Cartan datum have representation theory similar to that of Kac-Moody algebras associated to the same super Cartan datum with $\mathbb{Z}_2$-grading forgotten; in particular, the character formulas for the integrable modules of these Lie algebras and superalgebras coincide. In the (only) finite type, this reduces to the well-known fact that the finite-dimensional modules of Lie superalgebra $\mathfrak{osp}(1|2n)$ and Lie algebra $\mathfrak{so}(2n+1)$ have the same characters. Such a similarity continues to hold at the quantum level. But a conceptual explanation for all these coincidences has been missing (see an earlier attempt [La] in finite type).

1.2. The goal of this paper is to establish (somewhat surprising) direct links at several levels between quantum groups and supergroups associated to bar-consistent super Cartan datum, which provide a conceptual explanation of the above coincidences.

We construct automorphisms (called twistors) denoted by Ψ, $\dot{\Psi}$ of the half covering quantum group $\mathbf{f}$ and the modified covering quantum group $\dot{\mathbf{U}}$, respectively. The construction of twistors requires an extension of scalars to include a square root of -1, denoted by $\mathbf{t}$ in this paper. The twistor switches π and $-\pi$, and hence specializes to an isomorphism between the half (and resp., modified) quantum group and its super counterpart. As an immediate consequence, we obtain an equivalence of categories of weight modules for quantum group $\mathbf{U}|_{\pi=1}$ and supergroup $\mathbf{U}|_{\pi=-1}$. We also formulate an *extended covering quantum group* with enlarged Cartan subalgebra and construct its twistor.

Symbolically, we summarize the role of the twistor in the case of modified covering quantum group in the following commutative diagram:



Alternatively, one can view the modified quantum group $\dot{\mathbf{U}}|_{\pi=1}$ and the modified quantum supergroup $\dot{\mathbf{U}}|_{\pi=-1}$ as two different rational forms of a common algebra $\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=1}$.

The two rational forms admit their own distinct integral forms. Remarkably the distinction between super and non-super algebras becomes blurred at the quantum level, even though a clear distinction exists between Lie algebras and Lie superalgebras (for example, there are "more" integrable modules for Lie algebras than for the corresponding Lie superalgebras [Kac]).

As an application, the twistor Ψ induces a transformation on the crystal lattice of $\mathbf{f}$ which behaves well with the crystal structure. By careful bookkeeping, we provide a purely algebraic proof of [CHW2, Proposition 6.7] that the crystal lattice of $\mathbf{f}$ is invariant under an anti-automorphism ϱ which fixes the Chevalley generators. Furthermore, the twistor Ψ is shown to match Lusztig-Kashiwara's canonical basis for $\mathbf{f}|_{\pi=1}$ [Lu1, K] with the canonical basis for the half quantum supergroup $\mathbf{f}|_{\pi=-1}$ constructed in [CHW2], up to integer powers of $\mathbf{t}$. Let us add that this does not give a new proof of the existence of the canonical basis for $\mathbf{f}$ or for the integrable modules of $\mathbf{U}$.

1.3. Although it is not very explicitly used in this paper, the connection between (one-parameter) quantum groups and two-parameter (v, t)-quantum groups developed by two of the authors [FL1] plays a basic role in our evolving understanding of the links between quantum groups and supergroups. A connection between (one-parameter) quantum groups and quantum supergroups can indeed be formulated by a "twisted lift" to two-parameter quantum groups which is followed by a "specialization" of the second parameter t to $\mathbf{t}$ with $\mathbf{t}^2 = -1$. But we have decided to adopt the more intrinsic and self-contained approach as currently formulated in this paper.

The isomorphism result on modified quantum (super)groups $\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=1}$ and $\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=-1}$ in this joint work was announced in [FL2], where the isomorphism in the rank one case was established somewhat differently from here.

A version of our equivalence of categories of weight modules for $\mathbf{U}|_{\pi=1}$ and $\mathbf{U}|_{\pi=-1}$ also appeared in [KKO] with a very different proof. Note that the notion of weight modules in *loc. cit.* is nonstandard and subtle, and the multi-parameter algebras formulated therein over $\mathbb{C}(v)$ or $\mathbb{C}(v)^\pi$ do not seem to admit rational forms or integral forms or modified counterparts as ours. Some construction similar to the twistor $\widehat{\Psi}$ for our extended covering quantum group (see Proposition 4.12) also appeared in [KKO]. In contrast to *loc. cit.*, our formula for $\widehat{\Psi}$ is very explicit; the twistor $\dot{\Psi}$ here preserves the integral forms (see Theorem 4.3), and this allows us to specialize v to be a root of unity without difficulty.

1.4. The paper is organized as follows.

In Sect. 2, after recalling some preliminaries, we formulate and establish a twistor Ψ of the half covering quantum group $\mathbf{f}[\mathbf{t}]$, which restricts to an isomorphism between the super and non-super half quantum groups. Here, we make crucial use of a new multiplication on $\mathbf{f}[\mathbf{t}]$ twisted by a distinguished bilinear form, and the general idea of such twisted multiplication goes back to [FL1].

In Sect. 3, we use the twistor Ψ to compare the crystal lattices between the $\pi = 1$ and $\pi = -1$ cases. In particular, we give an algebraic proof that the crystal lattice for $\mathbf{f}$ is preserved by an anti-involution ϱ. (This was stated in [CHW2, Proposition 6.7].) Then we show that the twistor Ψ matches the canonical basis elements of the half quantum supergroup $\mathbf{f}|_{\pi=-1}$ and those of half quantum group $\mathbf{f}|_{\pi=1}$, up to integer powers of $\mathbf{t}$.

In Sect. 4, we construct a twistor of the modified covering quantum group. This restricts to an isomorphism between the super and non-super modified quantum groups. An immediate corollary is an equivalence of categories of weight modules for the super and non-super quantum groups. A further consequence is an equivalence of BGG categories of modules for Kac-Moody Lie algebras and Lie superalgebras. Finally we construct an alternative twistor relating quantum groups to quantum supergroups upon enlarging the Cartan subalgebras.

2. The Twistor of Half Covering Quantum Group

2.1. The preliminaries. We review some basic definitions which can be found in [CHW1, CHW2] and references therein.

Definition 2.1. *A Cartan datum is a pair $(I, \cdot)$ consisting of a finite set I and a $\mathbb{Z}$-valued symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$ satisfying*

(a) $d_i = \frac{i \cdot i}{2} \in \mathbb{Z}_{>0}, \quad \forall i \in I$;
(b) $a_{ij} = 2\frac{i \cdot j}{i \cdot i} \in \mathbb{Z}_{\leq 0}$, *for* $i \neq j$ *in* I.

A Cartan datum is called a super Cartan datum of anisotropic type *if there is a partition* $I = I_{\bar{0}} \coprod I_{\bar{1}}$ *which satisfies the condition*

(c) $2\frac{i \cdot j}{i \cdot i} \in 2\mathbb{Z}$ *if* $i \in I_{\bar{1}}$ *and* $j \in I$.

A super Cartan datum of anisotropic type is called bar-consistent *if it additionally satisfies*

(d) $d_i \equiv p(i) \mod 2, \quad \forall i \in I$.

We will always assume $I_{\bar{1}} \neq \emptyset$ without loss of generality. We note that (d) is almost always satisfied for super Cartan data of finite or affine type (with one exception which corresponds to a Dynkin diagram with two short roots of opposite parity at its both ends, called by $A^{(4)}(0, 2n)$). A super Cartan datum is always assumed to be bar-consistent in this paper. We note that a bar-consistent super Cartan datum satisfies

$$i \cdot j \in 2\mathbb{Z} \quad \text{for all } i, j \in I. \tag{2.1}$$

The $i \in I_{\bar{0}}$ are called even, $i \in I_{\bar{1}}$ are called odd. We define a parity function $p : I \to \{0, 1\}$ so that $i \in I_{\overline{p(i)}}$. We extend this function to the homomorphism $p : \mathbb{Z}[I] \to \mathbb{Z}_2$. Then p induces a *parity $\mathbb{Z}_2$-grading* on $\mathbb{Z}[I]$. We define the height function ht on $\mathbb{Z}[I]$ by letting $\mathrm{ht}(\sum_{i \in I} c_i i) = \sum_{i \in I} c_i$.

A *super root datum* associated to a super Cartan datum $(I, \cdot)$ consists of

(a) two finitely generated free abelian groups Y, X and a perfect bilinear pairing $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$;
(b) an embedding $I \subset X$ $(i \mapsto i')$ and an embedding $I \subset Y$ $(i \mapsto i)$ satisfying
(c) $\langle i, j' \rangle = \frac{2 i \cdot j}{i \cdot i}$ for all $i, j \in I$.

We will assume that the image of the imbedding $I \subset X$ (respectively, the image of the imbedding $I \subset Y$) is linearly independent in X (respectively, in Y); in the terminology of [Lu2], this means the datum is both X-regular and Y-regular.

If V is a vector space graded by $\mathbb{Z}[I]$, X, or Y, we will use the weight notation $|x| = \mu$ if $x \in V_\mu$. If V is a $\mathbb{Z}_2$-graded vector space, we will use the parity notation $p(x) = a$ if $x \in V_a$.

Let v and t be formal parameters, and let π be an indeterminate such that

$$\pi^2 = 1.$$

For a ring R with 1, we will form a new ring $R^\pi = R[\pi]/(\pi^2-1)$. Given an R^π-module (or algebra) M, the *specialization of* M *at* $\pi = \pm 1$ means the R-module (or algebra) $M|_{\pi=\pm 1} \overset{\text{def}}{=} R_\pm \otimes_{R^\pi} M$, where $R_\pm = R$ is viewed as a R^π-module on which π acts as ± 1.

Assume 2 is invertible in R; i.e. $\frac{1}{2} \in R$. We define

$$\varepsilon_+ = \frac{1+\pi}{2}, \qquad \varepsilon_- = \frac{1-\pi}{2}, \tag{2.2}$$

and note that $R^\pi = R\varepsilon_+ \oplus R\varepsilon_-$. In particular, since $\pi\varepsilon_\pm = \pm\varepsilon_\pm$ for an R^π-module M, we see that

$$M|_{\pi=\pm 1} \cong \varepsilon_\pm M.$$

Similarly, for an R-module M, we define

$$M[t^{\pm 1}] = R[t^{\pm 1}] \otimes_R M.$$

Let $\mathbf{t}^2 = -1 \in R$. Let us define the specialization of t at $\mathbf{t}$ to be

$$M[\mathbf{t}] = R[\mathbf{t}] \otimes_{R[t^{\pm 1}]} M[t^{\pm 1}] = R[\mathbf{t}] \otimes_R M.$$

(Note that the results herein may be reformulated in a context where $\mathbf{t}$ is replaced by an indeterminate solution to the equation $t^4 = 1$.)

Recall $\pi^2 = 1$. For $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$, we introduce a (v, π)-variant of quantum integers, quantum factorial and quantum binomial coefficients:

$$\begin{aligned}
[k]_{v,\pi} &= \frac{(\pi v)^k - v^{-k}}{\pi v - v^{-1}} \in \mathbb{Z}[v^{\pm 1}]^\pi, \\
[k]^!_{v,\pi} &= \prod_{l=1}^{k} [l]_{v,\pi} \in \mathbb{Z}[v^{\pm 1}]^\pi, \\
\begin{bmatrix} n \\ k \end{bmatrix}_{v,\pi} &= \frac{\prod_{l=n-k+1}^{n} \left((\pi v)^l - v^{-l}\right)}{\prod_{l=1}^{k} \left((\pi v)^l - v^{-l}\right)} \in \mathbb{Z}[v^{\pm 1}]^\pi.
\end{aligned} \tag{2.3}$$

We will use the notation

$$v_i = v^{d_i}, \quad t_i = t^{d_i}, \quad \pi_i = \pi^{d_i}, \quad \text{for } i \in I.$$

Let $(I, \cdot)$ be a super Cartan datum. The *half covering quantum group* $\mathbf{f}$ [CHW1, §1] is the $\mathbb{Q}(v)^\pi$-algebra with generators θ_i $(i \in I)$ and relations

$$\sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2}p(i)+kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i} \theta_i^{b_{ij}-k} \theta_j \theta_i^k = 0 \quad \text{for all } i \neq j \in I, \tag{2.4}$$

where

$$b_{ij} = 1 - a_{ij}.$$

As first noted in [HW], the $\mathbb{Q}$-algebra $\mathbf{f}$ admits a bar involution $^-$ such that

$$\overline{\theta_i} = \theta_i \ (\forall i \in I), \qquad \overline{\pi} = \pi, \qquad \overline{v} = \pi v^{-1}. \tag{2.5}$$

We define the divided powers

$$\theta_i^{(n)} = \frac{\theta_i^n}{[n]^!_{v_i,\pi_i}}. \tag{2.6}$$

These elements generate a $\mathbb{Z}[v^{\pm 1}]^\pi$-subalgebra of $\mathbf{f}$, denoted by ${}_{\mathbb{Z}}\mathbf{f}$. (In this paper, the notation $\mathbb{Z}[v^{\pm 1}]$ stands for the ring of Laurent polynomials in v.) Note that $\theta_i^{(n)}$ is bar invariant.

By specialization at $\pi = \pm 1$, we obtain the usual half quantum group $\mathbf{f}|_{\pi=1}$ and the half quantum supergroup $\mathbf{f}|_{\pi=-1}$, respectively. By leaving π as an indeterminate, we can simultaneously address both cases.

The algebra $\mathbf{f}$ has a $\mathbb{Z}[I] \times \mathbb{Z}_2$-grading obtained by setting $|\theta_i| = i$ and $p(\theta_i) = p(i)$, for $i \in I$. The algebra $\mathbf{f}$ is known [HW,CHW1] to be equipped with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ such that

$$(1, 1) = (\theta_i, \theta_i) = 1, \quad (\theta_i x, y) = (x, e_i'(y)),$$

where $e_i' : \mathbf{f} \to \mathbf{f}$ is the map satisfying

$$e_i'(1) = 0, \quad e_i'(\theta_j) = \delta_{ij}, \quad e_i'(xy) = e_i'(x)y + \pi^{p(i)p(x)} v^{-i\cdot|x|} x e_i'(y). \tag{2.7}$$

There exists [CHW2] a (non-super) algebra anti-automorphism of $\mathbf{f}$ such that

$$\varrho(\theta_i) = \theta_i \ (\forall i \in I), \qquad \varrho(xy) = \varrho(y)\varrho(x), \quad \forall x, y \in \mathbf{f}. \tag{2.8}$$

2.2. A twisted multiplication. Fix once and for all a total order $<$ on I. Recall the notation d_i, a_{ij} from Definition 2.1. Let $\phi : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ be the bilinear form defined by: for $i, j \in I$,

$$\phi(i, j) = \begin{cases} d_i a_{ij} & \text{if } j < i, \\ d_i & \text{if } j = i, \\ -2p(i)p(j) & \text{if } j > i. \end{cases} \tag{2.9}$$

Set

$$\delta_{i<j} = \begin{cases} 0, & \text{if } i \not< j, \\ 1, & \text{if } i < j. \end{cases}$$

By abuse of notation we regard $\mathbb{Z}_2 = \{0, 1\} \subset \mathbb{Z}$, and so by (2.1) we have

$$\phi(i, j) - \phi(j, i) = (-1)^{\delta_{i<j}} \big(i \cdot j + 2p(i)p(j)\big) \in 2\mathbb{Z}, \qquad \text{for } i \neq j.$$

In particular, we always have

$$\phi(i, j) - \phi(j, i) \equiv i \cdot j + 2p(i)p(j) \mod 4, \qquad \text{for } i \neq j. \tag{2.10}$$

Recall that $\mathbf{f}[t^{\pm 1}]$ denotes the $\mathbb{Q}(v)[t^{\pm 1}]^\pi$-algebra $\mathbb{Q}(v)[t^{\pm 1}]^\pi \otimes_{\mathbb{Q}(v)^\pi} \mathbf{f}$. Define a new multiplication $*$ on $\mathbf{f}[t^{\pm 1}]$ by setting

$$x * y = t^{\phi(|x|,|y|)} xy, \tag{2.11}$$

for homogeneous $x, y \in \mathbf{f}[t^{\pm 1}]$ and then extending it bilinearly. Since ϕ is bilinear, one verifies that $(\mathbf{f}[t^{\pm 1}], *)$ is a $\mathbb{Z}[I]$-graded associative algebra generated by θ_i. We will use the notation $x^{*n} = \underbrace{x * x * \ldots * x}_{n}$ for powers taken with respect to this product. We note that

$$\varrho(x * y) = t^{\phi(|x|,|y|)-\phi(|y|,|x|)}\varrho(y) * \varrho(x), \quad \forall x, y \text{ homogeneous.} \tag{2.12}$$

Proposition 2.2. *The algebra $(\mathbf{f}[t^{\pm 1}], *)$ has a presentation as the $\mathbb{Q}(v)[t^{\pm 1}]^{\pi}$-algebra with generators θ_i $(i \in I)$ and relations*

$$\sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2}p(i)+kp(i)p(j)} t^{k(b_{ij}-k)d_i+(b_{ij}-k)\phi(i,j)+k\phi(j,i)} \times \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i} \theta_i^{*\,b_{ij}-k} * \theta_j * \theta_i^{*k} = 0, \tag{2.13}$$

for all $i \neq j \in I$.

Proof. The relation (2.13) for $(\mathbf{f}[t^{\pm 1}], *)$ can be derived directly from the Serre relation (2.4) for $\mathbf{f}$, and vice versa. As the computation is straightforward, we skip the details. □

Remark 2.3. The twisted $*$-product on $\mathbf{f}[t^{\pm 1}]$ is a variant of the transformation defined in [FL1, §4] to relate one-parameter quantum group to two-parameter quantum group. The precise formula for the bilinear form ϕ is new, and it plays a crucial role in this paper.

2.3. The twistor Ψ. Recall that we set $\mathbf{t}^2 = -1$ and that $\mathbf{f}[\mathbf{t}]$ is the $\mathbb{Q}(v, \mathbf{t})^{\pi}$-algebra $\mathbb{Q}(v, \mathbf{t})^{\pi} \otimes_{\mathbb{Q}(v)[t^{\pm 1}]^{\pi}} \mathbf{f}[t^{\pm 1}]$. By specializing t and twisting v, we obtain the following $\mathbb{Q}(\mathbf{t})$-algebra isomorphism which plays a fundamental role in this paper.

Theorem 2.4. *There is a $\mathbb{Q}(\mathbf{t})$-algebra isomorphism $\Psi : \mathbf{f}[\mathbf{t}] \to (\mathbf{f}[\mathbf{t}], *)$ satisfying*

$$\Psi(\theta_i) = \theta_i \ (i \in I), \quad \Psi(v) = \mathbf{t}^{-1}v, \quad \Psi(\pi) = -\pi, \quad \Psi(xy) = \Psi(x) * \Psi(y). \tag{2.14}$$

The transformation Ψ is called the *twistor* on $\mathbf{f}[\mathbf{t}]$.

Proof. Set

$$S_{ij} := \sum_{k=0}^{b_{ij}} (-1)^k (-\pi)^{\binom{k}{2}p(i)+kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{\mathbf{t}_i^{-1}v_i,(-\pi)_i} \theta_i^{*\,b_{ij}-k} * \theta_j * \theta_i^{*k}.$$

To show such a $\mathbb{Q}(\mathbf{t})$-linear map Ψ exists, it suffices to show that the images of the generators satisfy (2.4) with respect to $*$; that is,

$$S_{ij} = 0 \quad \text{for all } i \neq j \in I. \tag{2.15}$$

To that end, fix $i \neq j \in I$. Unraveling the definition of $*$, we have

$$S_{ij} = \sum_{k=0}^{b_{ij}} (-1)^k (-\pi)^{\binom{k}{2}p(i)+kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{\mathbf{t}_i^{-1}v_i,(-\pi)_i} \\ \times \mathbf{t}^{(\binom{k}{2}+\binom{b_{ij}-k}{2}+k(b_{ij}-k))d_i+(b_{ij}-k)\phi(i,j)+k\phi(j,i)} \theta_i^{b_{ij}-k}\theta_j\theta_i^k.$$

One verifies that $\binom{k}{2} + \binom{b_{ij}-k}{2} = \binom{b_{ij}}{2} - k(b_{ij}-k)$ and $\begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{\mathbf{t}_i^{-1}v_i,(-\pi)_i} = \mathbf{t}^{k(b_{ij}-k)d_i} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i}$. Using these identities, we rewrite the above identity for S_{ij} as

$$\begin{aligned} &\mathbf{t}^{-\binom{b_{ij}}{2}d_i} S_{ij} \\ &= \sum_{k=0}^{b_{ij}} (-1)^k (-\pi)^{\binom{k}{2}p(i)+kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{\mathbf{t}_i^{-1}v_i,(-\pi)_i} \mathbf{t}^{(b_{ij}-k)\phi(i,j)+k\phi(j,i)} \theta_i^{b_{ij}-k}\theta_j\theta_i^k \\ &= \sum_{k=0}^{b_{ij}} (-1)^k (-\pi)^{\binom{k}{2}p(i)+kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i} \mathbf{t}^{\clubsuit} \theta_i^{b_{ij}-k}\theta_j\theta_i^k, \end{aligned} \tag{2.16}$$

where

$$\clubsuit = k(b_{ij}-k)d_i + (b_{ij}-k)\phi(i,j) + k\phi(j,i). \tag{2.17}$$

Now let us consider $\clubsuit$. First assume that $i < j$. Then we find that

$$\begin{aligned} \clubsuit &= k(b_{ij}-k)d_i - 2(b_{ij}-k)p(i)p(j) + kd_i a_{ij} \\ &= -2\binom{k}{2}d_i + 2kp(i)p(j) - 2b_{ij}p(i)p(j). \end{aligned}$$

Next assume that $i > j$. Then we have

$$\begin{aligned} \clubsuit &= k(b_{ij}-k)d_i + (b_{ij}-k)d_i a_{ij} - 2kp(i)p(j) \\ &= -2\binom{k}{2}d_i + a_{ij}(b_{ij}-2k)d_i - 2kp(i)p(j). \end{aligned}$$

Note that $2a_{ij}d_i \equiv 0 \mod 4$, thanks to (2.1). In either case when $i < j$ or $i > j$, we see that

$$\clubsuit = 2\binom{k}{2}d_i + 2kp(i)p(j) + c(i,j) \mod 4,$$

where

$$c(i,j) = \begin{cases} 2b_{ij}p(i)p(j), & \text{if } i < j, \\ -d_i\binom{a_{ij}}{2}, & \text{if } i > j. \end{cases}$$

Recall $\mathbf{t}^2 = -1$. By the bar-consistent condition we have $2d_i = 2p(i) \mod 4$, and thus $\mathbf{t}^{\clubsuit} = \mathbf{t}^{c(i,j)}(-1)^{\binom{k}{2}p(i)+kp(i)p(j)}$. Then we can rewrite (2.16) and apply the Serre relation (2.4) for $\mathbf{f}$ to conclude that

$$\mathbf{t}^{-\binom{b_{ij}}{2}d_i - c(i,j)} S_{ij} = \sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2}p(i)+kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i} \theta_i^{b_{ij}-k}\theta_j\theta_i^k = 0.$$

Therefore, (2.15) is verified and Ψ is well defined.

Finally, to see that Ψ is an isomorphism, we note that a similar argument can be used to show that a map $\Phi : (\mathbf{f}[\mathbf{t}], *) \to \mathbf{f}[\mathbf{t}]$ satisfying

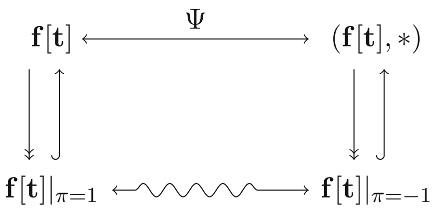
$$\Phi(\theta_i) = \theta_i, \quad \Phi(v) = \mathbf{t}v, \quad \Phi(\pi) = -\pi, \quad \Phi(x * y) = \Phi(x)\Phi(y),$$

is well defined as well; clearly Φ is the inverse of Ψ. □

Theorem 2.4 provides a way to compare the super and non-super half quantum groups via Ψ. Indeed, recall the idempotents $\varepsilon_\pm$ from (2.2). Then from $\Psi(\pi) = -\pi$, we see that $\Psi(\varepsilon_\pm) = \varepsilon_\mp$. In particular, $\Psi(\varepsilon_\pm \mathbf{f}[\mathbf{t}]) = \varepsilon_\mp \mathbf{f}[\mathbf{t}]$, in effect swapping the super and non-super specializations at $\pi = -1$ and $\pi = 1$.

Corollary 2.5. *There is a $\mathbb{Q}(\mathbf{t})$-linear isomorphism $\Psi : \mathbf{f}[\mathbf{t}]|_{\pi=1} \to \mathbf{f}[\mathbf{t}]|_{\pi=-1}$.*

Using the identification $\mathbf{f}[\mathbf{t}]|_{\pi=\pm 1} \cong \varepsilon_\pm \mathbf{f}[\mathbf{t}]$, we have inclusions $\mathbf{f}[\mathbf{t}]|_{\pi=\pm 1} \hookrightarrow \mathbf{f}[\mathbf{t}]$. Theorem 2.4 and Corollary 2.5 can be summarized symbolically in the following diagram:



For $i_1, \ldots, i_n \in I$, we denote

$$\mathbf{N}(i_1 + \ldots + i_n) = \sum_{1 \le r < s \le n} i_r \cdot i_s,$$
$$\mathbf{p}(i_1 + \ldots + i_n) = \sum_{1 \le r < s \le n} p(i_r)p(i_s).$$

By convention, $\mathbf{N}(i_1) = \mathbf{p}(i_1) = 0$. Note that $\mathbf{N}(\cdot)$ is always an even integer by (2.1).

The following proposition on the $\mathbb{Q}(\mathbf{t})$-linear involution ϱ of $\mathbf{f}[\mathbf{t}]$ will be used in the next section.

Proposition 2.6. *The involutions $\Psi\varrho\Psi^{-1}$ and ϱ on $\mathbf{f}[\mathbf{t}]$ are equal up to a sign on each weight space. More precisely, we have*

$$\Psi\varrho\Psi^{-1}(x) = (-1)^{\frac{\mathbf{N}(\nu)}{2} + \mathbf{p}(\nu)}\varrho(x), \quad \textit{for } x \in \mathbf{f}_\nu. \tag{2.18}$$

Proof. We prove the formula (2.18) by induction on the height $\mathrm{ht}(|x|)$.

The formula clearly holds when $\mathrm{ht}(|x|) \leq 1$.

Now assume that the formula holds for x with $\mathrm{ht}(|x|) \geq 1$ and for y with $\mathrm{ht}(|y|) \geq 1$. Recall $\mathbf{t}^2 = -1$. Then by (2.8), (2.10), (2.11), (2.12), and (2.14), we have

$$\begin{aligned}\Psi\varrho\Psi^{-1}(x * y) &= \Psi\big(\varrho(\Psi^{-1}(y))\,\varrho(\Psi^{-1}(x))\big)\\ &= \Psi\varrho\Psi^{-1}(y) * \Psi\varrho\Psi^{-1}(x)\\ &= (-1)^{\frac{\mathbf{N}(|y|)}{2}+\mathbf{p}(|y|)+\frac{\mathbf{N}(|x|)}{2}+\mathbf{p}(|x|)}\varrho(y) * \varrho(x)\\ &= (-1)^{\frac{\mathbf{N}(|y|)}{2}+\mathbf{p}(|y|)+\frac{\mathbf{N}(|x|)}{2}+\mathbf{p}(|x|)}\mathbf{t}^{\phi(|y|,|x|)-\phi(|x|,|y|)}\varrho(x * y)\\ &= (-1)^{\frac{\mathbf{N}(|x*y|)}{2}+\mathbf{p}(|x*y|)}\varrho(x * y).\end{aligned}$$

Hence the formula (2.18) holds for $x * y$. This completes the induction.

Since $\mathbf{N}$ and $\mathbf{p}$ only depend on the weight, $\Psi\varrho\Psi^{-1}$ and ϱ are proportional on each weight space. The proposition is proved. □

3. Comparison of Crystal Lattices and Canonical Bases

3.1. Comparing crystal lattices. For $x \in \mathbf{f}_\nu$, there is a unique decomposition of the form

$$x = \sum_{n\geq 0} \theta_i^{(n)} x_n, \tag{3.1}$$

such that $x_n = 0$ for all but finitely many n, $x_n \in \mathbf{f}_{\nu-ni}$, and $e_i'(x_n) = 0$ for all n. We will refer to this as its $\mathbf{i}$-*string decomposition*. Then we define Kashiwara operators

$$\tilde{e}_i x = \sum_{n\geq 1} \theta_i^{(n-1)} x_n,$$
$$\tilde{f}_i x = \sum_{n\geq 0} \theta_i^{(n+1)} x_n.$$

Let $\mathcal{A} \subset \mathbb{Q}(v)$ be the ring of rational functions with no poles at $v = 0$ and so $\mathcal{A}^\pi = \mathcal{A}[\pi] \subset \mathbb{Q}(v)^\pi$. The crystal lattice $\mathcal{L}$ of $\mathbf{f}$ is the $\mathcal{A}^\pi$-lattice generated by

$$B = \left\{\tilde{f}_{i_1} \ldots \tilde{f}_{i_n} 1 \mid \forall i_1, \ldots, i_n \in I, \forall n\right\}.$$

According to [CHW2], the set $\mathcal{B} := (B \cup \pi B) + v\mathcal{L}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/v\mathcal{L}$, called the (maximal) crystal basis for $\mathbf{f}$.

We note the following useful properties of $\mathcal{L}$ (with the same proof as usual [K]).

Lemma 3.1. *Let $x = \sum_{n\geq 0} \theta_i^{(n)} x_n$ be the i-string decomposition of $x \in \mathbf{f}$. Then,*

(1) *$x \in \mathcal{L}$ if and only if $x_n \in \mathcal{L}$ for all n.*
(2) *If $x + v\mathcal{L} \in \mathcal{B}$, then $x = \theta_i^{(n)} x_n$ mod $v\mathcal{L}$ for some n and $x_n + v\mathcal{L} \in \mathcal{B}$.*
(3) *If $\tilde{e}_j x = 0$ for all $j \in I$ then $x = 0$; if $\tilde{e}_j x \neq 0$ then $\tilde{f}_j \tilde{e}_j x = x$.*

To take advantage of Theorem 2.4, we need to extend scalars to include $\mathbf{t}$. We let $\mathcal{A}[\mathbf{t}] = \mathbb{Q}(\mathbf{t}) \otimes_{\mathbb{Q}} \mathcal{A}$, the subring of $\mathbb{Q}(v, \mathbf{t})$ of rational functions with no poles at $v = 0$. Then set $\mathcal{L}[\mathbf{t}] = \mathcal{A}[\mathbf{t}]^{\pi} \otimes_{\mathcal{A}^{\pi}} \mathcal{L}$.

The isomorphism Ψ in Theorem 2.4, which sends $v \mapsto \mathbf{t}^{-1}v$ and $\pi \mapsto -\pi$, clearly preserves the $\mathbb{Q}(\mathbf{t})$-algebra $\mathcal{A}[\mathbf{t}]^{\pi}$.

Lemma 3.2. *The following properties hold:*

(1) $\Psi(\theta_i^{(n)}) = \theta_i^{(n)}$ *for* $n \geq 1$*;*
(2) $e_i'(\Psi(x)) = \mathbf{t}^{\phi(i,|x|-i)}\Psi(e_i'(x))$ *for all homogeneous* $x \in \mathbf{f}[\mathbf{t}]$ *and* $i \in I$*;*
(3) *Let* $x \in \mathbf{f}[\mathbf{t}]_\nu$ *with its* i*-string decomposition* (3.1) *for a given* $i \in I$*. Then* $\Psi(x)$ *has the following* i*-string decomposition*

$$\Psi(x) = \sum_{n \geq 0} \mathbf{t}^{\phi(ni,\nu)-n^2 d_i} \theta_i^{(n)} \Psi(x_n).$$

Proof. Recall the definitions (2.3) of $[n]_{v,\pi}$ and (2.14) of Ψ. We have

$$\Psi\big([n]_{v,\pi}\big) = [n]_{\mathbf{t}^{-1}v,-\pi} = \mathbf{t}^{n-1}\,[n]_{v,\pi}\,.$$

We prove (1) by induction on n. The case when $n = 1$ is clear. Assume $\Psi(\theta_i^{(n-1)}) = \theta_i^{(n-1)}$. By definition of the divided power (2.6), we have

$$\begin{aligned}\Psi(\theta_i^{(n)}) &= \Psi\big([n]_{v_i,\pi_i}^{-1}\,\theta_i\theta_i^{(n-1)}\big)\\ &= \mathbf{t}_i^{1-n}\,[n]_{v_i,\pi_i}^{-1}\,\Psi(\theta_i) * \Psi\big(\theta_i^{(n-1)}\big)\\ &= \mathbf{t}_i^{1-n}\,[n]_{v_i,\pi_i}^{-1}\,\mathbf{t}_i^{n-1}\theta_i\theta_i^{(n-1)} = \theta_i^{(n)}.\end{aligned}$$

Now let us verify (2). It is trivial if $\mathrm{ht}|x| \leq 1$. Otherwise, it suffices to show that if (2) holds for $x, y \in \mathbf{f}[\mathbf{t}]$, then it holds for xy. By (2.7) we compute

$$\begin{aligned}&e_i'(\Psi(xy))\\ &= \mathbf{t}^{\phi(|x|,|y|)}e_i'(\Psi(x)\Psi(y))\\ &= \mathbf{t}^{\phi(|x|,|y|)}\big(e_i'(\Psi(x))\Psi(y) + \pi^{p(i)p(x)}v^{-i\cdot|x|}\Psi(x)e_i'(\Psi(y))\big)\\ &= \mathbf{t}^{\phi(i,|y|)}e_i'(\Psi(x)) * \Psi(y) + \pi^{p(i)p(x)}v^{-i\cdot|x|}\mathbf{t}^{\phi(|x|,i)}\Psi(x) * e_i'(\Psi(y))\\ &\overset{(\star)}{=} \mathbf{t}^{\phi(i,|y|)+\phi(i,|x|-i)}\Psi(e_i'(x)y) + \pi^{p(i)p(x)}v^{-i\cdot|x|}\mathbf{t}^{\phi(|x|,i)+\phi(i,|y|-i)}\Psi(xe_i'(y))\\ &\overset{(\star\star)}{=} \mathbf{t}^{\phi(i,|y|)+\phi(i,|x|-i)}\Psi(e_i'(x)y) + (-\pi)^{p(i)p(x)}(\mathbf{t}^{-1}v)^{-i\cdot|x|}\mathbf{t}^{\phi(i,|x|)+\phi(i,|y|-i)}\Psi(xe_i'(y))\\ &= \mathbf{t}^{\phi(i,|x|+|y|-i)}\Psi\big(e_i'(x)y + \pi^{p(i)p(x)}v^{-i\cdot|x|}xe_i'(y)\big)\\ &= \mathbf{t}^{\phi(i,|xy|-i)}\Psi(e_i'(xy)),\end{aligned}$$

where the equation $(\star)$ follows from the inductive assumption and (2.14) and $(\star\star)$ follows from (2.10).

Finally, we prove (3). Such an identity for $\Psi(x)$ follows by the definition of Ψ and (1), and the claim that this is an i-string decomposition follows from (2). □

Proposition 3.3. *The isomorphism* Ψ *preserves the lattice* $\mathcal{L}[\mathbf{t}]$, *i.e.*, $\Psi(\mathcal{L}[\mathbf{t}]) = \mathcal{L}[\mathbf{t}]$. *Furthermore,* Ψ *induces an isomorphism* Ψ_0 *on* $\mathcal{L}[\mathbf{t}]/v\mathcal{L}[\mathbf{t}]$ *such that*

$$\Psi_0(x) = \mathbf{t}^{\ell(x)}x \qquad \forall x \in \mathcal{B},$$

where $\ell(x)$ *is some integer depending on* x.

Proof. We first observe that $\Psi(\mathcal{L}[\mathbf{t}]) \subseteq \mathcal{L}[\mathbf{t}]$, as this follows from using induction on height along with Lemma 3.1(1) and (3), and Lemma 3.2(3). On the other hand, Lemma 3.2 can be rewritten in terms of Ψ^{-1} (essentially by replacing $\mathbf{t}$ with $\mathbf{t}^{-1}$ in (2) and (3)) and so a similar argument shows $\Psi^{-1}(\mathcal{L}[\mathbf{t}]) \subseteq \mathcal{L}[\mathbf{t}]$. Therefore $\Psi(\mathcal{L}[\mathbf{t}]) = \mathcal{L}[\mathbf{t}]$.

Let $x + v\mathcal{L}[\mathbf{t}] \in \mathcal{B}$. We proceed by induction on the height of x. First note that $\Psi_0(1 + v\mathcal{L}[\mathbf{t}]) = 1 + v\mathcal{L}[\mathbf{t}]$ and $\Psi_0(\pi + v\mathcal{L}[\mathbf{t}]) = -\pi + v\mathcal{L}[\mathbf{t}]$, so the proposition holds with $\ell(1 + v\mathcal{L}[\mathbf{t}]) = 0$ and $\ell(\pi + v\mathcal{L}[\mathbf{t}]) = 2$.

If $\mathrm{ht}|x| \geq 1$, then by Lemma 3.1(2) and (3), there is an $i \in I$ such that we can write $x + v\mathcal{L}[\mathbf{t}] = \theta_i^{(n)}x_n + v\mathcal{L}[\mathbf{t}]$ with $x_n + v\mathcal{L}[\mathbf{t}] \in \mathcal{B}$ and $n > 0$. Then by induction on the height and Lemma 3.2(3), we have

$$\Psi_0(x + v\mathcal{L}[\mathbf{t}]) = \mathbf{t}^{\phi(ni,\nu)-n^2d_i+\ell(x_n+v\mathcal{L}[\mathbf{t}]^{\pi})}x + v\mathcal{L}[\mathbf{t}].$$

The proposition is proved. □

It was stated in [CHW2, Proposition 6.7] that $\mathcal{L}$ is ϱ-invariant. In contrast to the non-super setting (as done by Lusztig and Kashiwara), this is not easy to verify algebraically using the tools in *loc. cit.* because the bilinear form on $\mathcal{L}/v\mathcal{L}$ is not positive definite. Here we are in a position to furnish an algebraic proof of [CHW2, Proposition 6.7].

Proposition 3.4. *The involution* ϱ *preserves* $\mathcal{L}$, *i.e.*, $\varrho(\mathcal{L}) = \mathcal{L}$.

Proof. Since $\frac{1}{2} \in \mathcal{A}$, we note that

$$\mathcal{L} = \varepsilon_+\mathcal{L} \oplus \varepsilon_-\mathcal{L} \cong \mathcal{L}|_{\pi=1} \oplus \mathcal{L}|_{\pi=-1}.$$

We similarly have a decomposition $\varrho = \varrho_+ \oplus \varrho_-$ where $\varrho_\pm(x) = \varrho(\varepsilon_\pm x)$, and by definition we see that under the isomorphism $\varepsilon_\pm\mathbf{f}[\mathbf{t}] \cong \mathbf{f}[\mathbf{t}]|_{\pi=\pm1}$, $\varrho_\pm$ corresponds to $\varrho|_{\pi=\pm1}$.

Since it is known [K,Lu2] that $\varrho_{\pi=1}(\mathcal{L}|_{\pi=1}) = \mathcal{L}|_{\pi=1}$, it suffices to show that

$$\varrho|_{\pi=-1}(\mathcal{L}|_{\pi=-1}) = \mathcal{L}|_{\pi=-1}.$$

Since $\Psi(\pi) = -\pi$, we have $\Psi(\mathcal{L}[\mathbf{t}]|_{\pi=1}) = \mathcal{L}[\mathbf{t}]|_{\pi=-1}$. Let $x \in \mathcal{L}|_{\pi=-1}$. Since $x \in \mathcal{L}|_{\pi=-1} \subset \mathcal{L}[\mathbf{t}]|_{\pi=-1}$, by Proposition 2.6 we have

$$\varrho|_{\pi=-1}(x) = (-1)^{\frac{\mathbf{N}(|x|)}{2}+\mathbf{p}(|x|)}\Psi\varrho|_{\pi=1}\Psi^{-1}(x) \in \mathcal{L}[\mathbf{t}]|_{\pi=-1}.$$

On the other hand, by definition we have $\varrho(x) \in \mathbf{f}|_{\pi=-1}$, and hence

$$\varrho|_{\pi=-1}(x) \in \mathcal{L}[\mathbf{t}]|_{\pi=-1} \cap \mathbf{f}|_{\pi=-1} = \mathcal{L}|_{\pi=-1}.$$

The proposition is proved. □

3.2. Comparing canonical bases. The bar involution on $\mathbf{f}$ in (2.5) extends trivially to an involution $^-$ of $\mathbf{f}[t^{\pm 1}]$ and $\mathbf{f}[\mathbf{t}]$ by letting $\bar{t} = t$ and $\bar{\mathbf{t}} = \mathbf{t}$ respectively.

Lemma 3.5. *The map* Ψ *commutes with the bar map on* $\mathbf{f}[\mathbf{t}]$*, i.e.,* $^- \circ \Psi = \Psi \circ {}^-$.

Proof. By the definition of Ψ given in Theorem 2.4, the only nontrivial thing to check is the commutativity when acting on v. Indeed, recalling $\mathbf{t}^4 = 1$, we have

$$\overline{\Psi(v)} = \mathbf{t}^{-1}\pi v^{-1} = -\pi(\mathbf{t}^{-1}v)^{-1} = \Psi(\bar{v}).$$

The lemma is proved. □

As proven in [CHW2] (generalizing the approach of [K]), there exists a globalization map $G : \mathcal{L}[\mathbf{t}]/v\mathcal{L}[\mathbf{t}] \to \mathcal{L}[\mathbf{t}] \cap \overline{\mathcal{L}}[\mathbf{t}]$ such that for each $b \in \mathcal{B}$, $G(b)$ is the unique bar-invariant vector in $\mathcal{L}[\mathbf{t}]$ such that $G(b) + v\mathcal{L}[\mathbf{t}] = b$. The set $\{G(b) : b \in \mathcal{B}\}$ is called the canonical π-basis for $\mathbf{f}$.

Specializing $\pi = 1$ yields the usual canonical basis of Lusztig and Kashiwara, while specializing $\pi = -1$ yields a (signed) canonical basis for the half quantum supergroup. Even though we have established a connection on the level of crystal lattices and crystal bases, it is somewhat surprising to see that Ψ allows us to establish a direct and precise link between the canonical bases for the two specializations. Recall $\ell(\cdot)$ from Proposition 3.3, which is integer-valued but may not be even-integer-valued in general.

Theorem 3.6. *For any* $b \in \mathcal{B}$*, we have*

$$\Psi(G(b)) = \mathbf{t}^{\ell(b)} G(b).$$

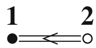
In particular, $\Psi(G(b)|_{\pi=1})$ *is proportional to* $G(b)|_{\pi=-1}$.

Proof. It follows by Lemma 3.5 that $\Psi(G(b))$ is bar-invariant. It follows by the definition of the maps and Proposition 3.3 that

$$\Psi(G(b)) + v\mathcal{L}[\mathbf{t}] = \Psi(b) = \mathbf{t}^{\ell(b)} b.$$

Therefore, $\mathbf{t}^{-\ell(b)}\Psi(G(b)) = G(b)$ and thus $\Psi(\varepsilon_+ G(b)) = \varepsilon_- \mathbf{t}^{\ell(b)} G(b)$. □

Example 3.7. Let $(I, \cdot)$ be the super Cartan datum associated to $\mathfrak{osp}(1|4)$ with $I = \{\mathbf{1}, \mathbf{2}\}$ (where $\mathbf{1}$ is the odd simple root) and Dynkin diagram given by



Then

$$p(\mathbf{1}) = 1, \quad p(\mathbf{2}) = 0;$$
$$\mathbf{1}\cdot\mathbf{1} = 2, \quad \mathbf{1}\cdot\mathbf{2} = \mathbf{2}\cdot\mathbf{1} = -2, \quad \mathbf{2}\cdot\mathbf{2} = 4;$$
$$\phi(\mathbf{1},\mathbf{1}) = 1, \quad \phi(\mathbf{1},\mathbf{2}) = 0, \quad \phi(\mathbf{2},\mathbf{1}) = -2, \quad \phi(\mathbf{2},\mathbf{2}) = 2.$$

It is an easy computation that

$$\tilde{f}_{\mathbf{1}}\tilde{f}_{\mathbf{2}}\tilde{f}_{\mathbf{1}}1 = \theta_{\mathbf{1}}(\theta_{\mathbf{2}}\theta_{\mathbf{1}} - v^2\theta_{\mathbf{1}}\theta_{\mathbf{2}}) + v^2\theta_{\mathbf{1}}^{(2)}\theta_{\mathbf{2}}.$$

In particular, $G(\tilde{f}_{\mathbf{1}}\tilde{f}_{\mathbf{2}}\tilde{f}_{\mathbf{1}}1 + v\mathcal{L}) = \theta_{\mathbf{1}}\theta_{\mathbf{2}}\theta_{\mathbf{1}}$, and $\Psi\big(G(\tilde{f}_{\mathbf{1}}\tilde{f}_{\mathbf{2}}\tilde{f}_{\mathbf{1}}1 + v\mathcal{L})\big) = \mathbf{t}^{-1}\theta_{\mathbf{1}}\theta_{\mathbf{2}}\theta_{\mathbf{1}}$.

4. The Twistor of Modified Covering Quantum Group

4.1. The modified covering quantum group. To facilitate the definition of modified covering quantum group next, we recall the definition of the covering quantum group $\mathbf{U}$. We recall that $b_{ij} = 1 - a_{ij}$.

Definition 4.1 [CHW1]. *The covering quantum group* $\mathbf{U}$ *associated to a super root datum* $(Y, X, I, \cdot)$ *is the* $\mathbb{Q}(v)^\pi$*-algebra with generators* E_i, F_i, K_μ, *and* J_μ, *for* $i \in I$ *and* $\mu \in Y$, *subject to the relations:*

$$J_\mu J_\nu = J_{\mu+\nu}, \quad K_\mu K_\nu = K_{\mu+\nu}, \quad K_0 = J_0 = J_\nu^2 = 1, \quad J_\mu K_\nu = K_\nu J_\mu, \tag{4.1}$$

$$J_\mu E_i = \pi^{\langle \mu, i' \rangle} E_i J_\mu, \quad J_\mu F_i = \pi^{-\langle \mu, i' \rangle} F_i J_\mu, \tag{4.2}$$

$$K_\mu E_i = v^{\langle \mu, i' \rangle} E_i K_\mu, \quad K_\mu F_i = v^{-\langle \mu, i' \rangle} F_i K_\mu, \tag{4.3}$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{ij} \frac{J_{d_i i} K_{d_i i} - K_{-d_i i}}{\pi_i v_i - v_i^{-1}}, \tag{4.4}$$

$$\sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2} p(i) + kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i, \pi_i} E_i^{b_{ij}-k} E_j E_i^k = 0 \ \ (i \neq j), \tag{4.5}$$

$$\sum_{k=0}^{b_{ij}} (-1)^k \pi^{\binom{k}{2} p(i) + kp(i)p(j)} \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i, \pi_i} F_i^{b_{ij}-k} F_j F_i^k = 0 \ \ (i \neq j), \tag{4.6}$$

for $i, j \in I$ *and* $\mu, \nu \in Y$.

Again by specialization at $\pi = \pm 1$, we obtain the usual quantum group $\mathbf{U}|_{\pi=1}$ (with extra central elements) and the super quantum group $\mathbf{U}|_{\pi=-1}$. We extend scalars and set $\mathbf{U}[t^{\pm 1}] = \mathbb{Q}(v)[t^{\pm 1}]^\pi \otimes_{\mathbb{Q}(v)^\pi} \mathbf{U}$.

We endow $\mathbf{U}$ with a $\mathbb{Z}[I]$-grading by setting

$$|E_i| = i, \quad |F_i| = -i, \quad |J_\mu| = |K_\mu| = 0, \tag{4.7}$$

and also endow $\mathbf{U}$ with a $\mathbb{Z}_2$-grading by setting

$$p(E_i) = p(F_i) = p(i), \quad p(J_\mu) = p(K_\mu) = 0. \tag{4.8}$$

The definition of the covering quantum group $\mathbf{U}$ is also internally coherent with the notion of the modified covering quantum group $\dot{\mathbf{U}}$, which we now introduce.

Definition 4.2. *The modified covering quantum group* $\dot{\mathbf{U}}$ *associated to the root datum* $(Y, X, I, \cdot)$ *is defined to be the associative* $\mathbb{Q}(v)^\pi$*-algebra without unit which is generated by the symbols* 1_λ, $E_i 1_\lambda$ *and* $F_i 1_\lambda$, *for* $\lambda \in X$ *and* $i \in I$, *subject to the relations:*

$$1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda, \tag{4.9}$$

$$(E_i 1_\lambda) 1_{\lambda'} = \delta_{\lambda, \lambda'} E_i 1_\lambda, \quad 1_{\lambda'} (E_i 1_\lambda) = \delta_{\lambda', \lambda + i'} E_i 1_\lambda, \tag{4.10}$$

$$(F_i 1_\lambda) 1_{\lambda'} = \delta_{\lambda, \lambda'} F_i 1_\lambda, \quad 1_{\lambda'} (F_i 1_\lambda) = \delta_{\lambda', \lambda - i'} F_i 1_\lambda, \tag{4.11}$$

$$(E_i F_j - \pi^{p(i)p(j)} F_j E_i) 1_\lambda = \delta_{ij} \left[\langle i, \lambda \rangle \right]_{v_i, \pi_i} 1_\lambda, \tag{4.12}$$

$$\sum_{k=0}^{b_{ij}}(-1)^k\pi^{\binom{k}{2}p(i)+kp(i)p(j)}\begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i} E_i^{b_{ij}-k}E_jE_i^k1_\lambda = 0 \quad (i \neq j), \tag{4.13}$$

$$\sum_{k=0}^{b_{ij}}(-1)^k\pi^{\binom{k}{2}p(i)+kp(i)p(j)}\begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i} F_i^{b_{ij}-k}F_jF_i^k1_\lambda = 0 \quad (i \neq j), \tag{4.14}$$

where $i, j \in I$, $\lambda, \lambda' \in X$, *and we use the notation* $xy1_\lambda = (x1_{\lambda+|y|})(y1_\lambda)$ *for* $x, y \in \mathbf{U}$.

As with the half covering quantum groups, if we set $\pi = 1$ then $\dot{\mathbf{U}}|_{\pi=1}$ is the modified quantum group of Lusztig, whereas if $\pi = -1$ then $\dot{\mathbf{U}}|_{\pi=-1}$ is the modified quantum supergroup.

The algebra $\dot{\mathbf{U}}$ has a (left) $\mathbf{U}$-action given by

$$E_i \cdot x1_\lambda = (E_i1_{\lambda+|x|})x1_\lambda, \quad F_i \cdot x1_\lambda = (F_i1_{\lambda+|x|})x1_\lambda,$$
$$K_\nu \cdot x1_\lambda = v^{\langle\nu,\lambda+|x|\rangle}x1_\lambda, \quad J_\nu \cdot x1_\lambda = \pi^{\langle\nu,\lambda+|x|\rangle}x1_\lambda.$$

There is also a similar right $\mathbf{U}$-action on $\dot{\mathbf{U}}$. Then as in [Lu1], $\dot{\mathbf{U}}$ can be identified with the $\mathbb{Q}(v)^\pi$-algebra on the symbols $x1_\lambda$ for $x \in \mathbf{U}$ and $\lambda \in X$ satisfying

$$x1_\lambda y1_\mu = \delta_{\lambda,\mu+|y|}xy1_\mu, \quad K_\nu1_\lambda = v^{\langle\nu,\lambda\rangle}1_\lambda, \quad J_\nu1_\lambda = \pi^{\langle\nu,\lambda\rangle}1_\lambda. \tag{4.15}$$

Denote by ${}_{\mathbb{Z}}\dot{\mathbf{U}}$ the $\mathbb{Z}[v^{\pm1}]^\pi$-subalgebra of $\dot{\mathbf{U}}$ generated by 1_λ, $E_i^{(n)}1_\lambda$ and $F_i^{(n)}1_\lambda$, for $n \geq 1$, $\lambda \in X$ and $i \in I$ (here we recall the definition of divided powers (2.6)). Then ${}_{\mathbb{Z}}\dot{\mathbf{U}}$ is a $\mathbb{Z}[v^{\pm1}]^\pi$-form of $\dot{\mathbf{U}}$.

4.2. The twistor $\dot{\Psi}$. Recall the bilinear form $\phi(\cdot,\cdot) : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ from (2.9), and that we have an embedding $\mathbb{Z}[I] \hookrightarrow X$ given by $i \mapsto i'$. Fix once and for all a transversal $C \subset X$ for the coset representatives of $X/\mathbb{Z}[I]$. Then we define the bilinear pairing $\dot{\phi}(\cdot,\cdot) : \mathbb{Z}[I] \times X \to \mathbb{Z}$ by

$$\dot{\phi}(\nu, \mu' + \lambda) = \phi(\nu, \mu), \quad \text{for all } \nu, \mu \in \mathbb{Z}[I],\ \lambda \in C. \tag{4.16}$$

The map $\dot{\Psi}$ in the following theorem can be viewed as a counterpart in the setting of modified covering quantum group of the isomorphism Ψ in Theorem 2.4. Note that we do not need to use a twisted multiplication in this setting as for Ψ. By base changes we set as usual $\dot{\mathbf{U}}[\mathbf{t}] = \mathbb{Q}(v,\mathbf{t})^\pi \otimes_{\mathbb{Q}(v)^\pi} \dot{\mathbf{U}}$ and ${}_{\mathbb{Z}}\dot{\mathbf{U}}[\mathbf{t}] = \mathbb{Z}[v^{\pm1},\mathbf{t}]^\pi \otimes_{\mathbb{Z}[v^{\pm1}]^\pi} {}_{\mathbb{Z}}\dot{\mathbf{U}}$.

Theorem 4.3. (1) *There is an automorphism* $\dot{\Psi}$ *of the* $\mathbb{Q}(\mathbf{t})$*-algebra* $\dot{\mathbf{U}}[\mathbf{t}]$ *of order* 4 *such that, for all* $i \in I$ *and* $\lambda \in X$,

$$\dot{\Psi}(1_\lambda) = 1_\lambda, \quad \dot{\Psi}(E_i1_\lambda) = \mathbf{t}^{d_i\langle i,\lambda\rangle - \dot{\phi}(i,\lambda)}E_i1_\lambda, \quad \dot{\Psi}(F_i1_\lambda) = \mathbf{t}^{\dot{\phi}(i,\lambda)}F_i1_\lambda,$$
$$\dot{\Psi}(\pi) = -\pi, \quad \dot{\Psi}(v) = \mathbf{t}^{-1}v.$$

(2) *The automorphism* $\dot{\Psi}$ *preserves the* $\mathbb{Z}[v^{\pm1},\mathbf{t}]^\pi$*-form* ${}_{\mathbb{Z}}\dot{\mathbf{U}}[\mathbf{t}]$.

Proof. (1) Once we verify that the endomorphism $\dot{\Psi}$ is well defined, it is clearly an automorphism of order four by checking the images of the generators. To verify that $\dot{\Psi}$

is well defined, it suffices to check that the images of the generators satisfy the relations. It is straightforward to verify (4.9)–(4.11), and we leave that as an exercise to the reader.

Let us check (4.12). We compute

$$\begin{aligned}
&(\mathbf{t}^{d_i\langle i,\lambda-j'\rangle-\dot\phi(i,\lambda-j')}E_i1_{\lambda-j'})(\mathbf{t}^{\dot\phi(j,\lambda)}F_j1_\lambda)\\
&\quad-(-\pi)^{p(i)p(j)}(\mathbf{t}^{\dot\phi(j,\lambda+i')}F_j1_{\lambda+i'})(\mathbf{t}^{d_i\langle i,\lambda\rangle-\dot\phi(i,\lambda)}E_i1_\lambda)\\
&=\mathbf{t}^{d_i\langle i,\lambda-j'\rangle-\dot\phi(i,\lambda-j')+\dot\phi(j,\lambda)}(E_iF_j-(-\pi)^{p(i)p(j)}\mathbf{t}^{i\cdot j+\phi(j,i)-\phi(i,j)}F_jE_i)1_\lambda\\
&=\mathbf{t}^{d_i\langle i,\lambda-j'\rangle-\dot\phi(i,\lambda-j')+\dot\phi(j,\lambda)}(E_iF_j-\pi^{p(i)p(j)}F_jE_i)1_\lambda\\
&=\delta_{ij}\mathbf{t}_i^{\langle i,\lambda\rangle-1}\,[\langle i,\lambda\rangle]_{v_i,\pi_i}\\
&=\delta_{ij}\,[\langle i,\lambda\rangle]_{\mathbf{t}_i^{-1}v_i,-\pi_i}\,.
\end{aligned}$$

Next, let us check the Serre relations. As the proof of (4.14) are similar, we will only check (4.13). Let us set

$$E_{ij}(k)=\dot\Psi(E_i^{b_{ij}-k}E_jE_i^k1_\lambda).$$

We want to verify that

$$\sum_{k=0}^{b_{ij}}(-1)^k(-\pi)^{\binom{k}{2}p(i)+kp(i)p(j)}\begin{bmatrix}b_{ij}\\k\end{bmatrix}_{\mathbf{t}^{-1}v_i,-\pi_i}E_{ij}(k)=0. \tag{4.17}$$

First note that

$$\Psi(E_i^s1_\lambda)=\prod_{t=1}^{s}\mathbf{t}^{d_i\langle i,\lambda+(t-1)i'\rangle-\dot\phi(i,\lambda+(t-1)i')}E_i1_{\lambda+(t-1)i'}=\mathbf{t}^{\binom{s}{2}d_i+s(d_i\langle i,\lambda\rangle-\dot\phi(i,\lambda))}E_i^s1_\lambda.$$

By using the factorization $E_i^{b_{ij}-k}E_jE_i^k1_\lambda=E_i^{b_{ij}-k}1_{\lambda+j'+ki'}E_j1_{\lambda+ki'}E_i^k1_\lambda$ and the identity $\binom{k}{2}+\binom{b_{ij}-k}{2}+k(b_{ij}-k)=\binom{b_{ij}}{2}$, we compute that

$$E_{ij}(k)=\mathbf{t}^{\spadesuit_{ij}(k)+\heartsuit_{ij}}E_i^{b_{ij}-k}E_jE_i^k1_\lambda,$$

where

$$\begin{aligned}
\heartsuit_{ij}&=b_{ij}(i\cdot j+d_i\langle i,\lambda\rangle-\dot\phi(i,\lambda))+d_j\langle j,\lambda\rangle-\dot\phi(j,\lambda)+\binom{b_{ij}}{2},\\
\spadesuit_{ij}(k)&=-k\phi(j,i)-(b_{ij}-k)\phi(i,j).
\end{aligned}$$

Then

$$\begin{bmatrix}b_{ij}\\k\end{bmatrix}_{\mathbf{t}^{-1}v_i,-\pi_i}E_{ij}(k)=\mathbf{t}^{\heartsuit_{ij}+\spadesuit_{ij}(k)+k(b_{ij}-k)d_i}\begin{bmatrix}b_{ij}\\k\end{bmatrix}_{v_i,\pi_i}E_i^{b_{ij}-k}E_jE_i^k1_\lambda.$$

Recall $\clubsuit$ from (2.17). Since $\phi(i_1,i_2)\in2\mathbb{Z}$ for $i_1\neq i_2\in I$, we see that

$$\spadesuit_{ij}(k)+k(b_{ij}-k)d_i\equiv\clubsuit\equiv2\binom{k}{2}+2kp(i)p(j)+c(i,j)\mod 4.$$

Then we see that

$$\sum_{k=0}^{b_{ij}}(-1)^k(-\pi)^{\binom{k}{2}p(i)+kp(i)p(j)}\begin{bmatrix}b_{ij}\\k\end{bmatrix}_{\mathbf{t}^{-1}v_i,-\pi_i}E_{ij}(k)$$
$$=\mathbf{t}^{\heartsuit_{ij}+c(i,j)}\sum_{k=0}^{b_{ij}}(-1)^k\pi^{\binom{k}{2}p(i)+kp(i)p(j)}\begin{bmatrix}b_{ij}\\k\end{bmatrix}_{v_i,\pi_i}E_i^{b_{ij}-k}E_jE_i^k1_\lambda=0.$$

This finishes the verification of the Serre relations, whence (1).

Part (2) follows immediately from (1) by noting that $\dot{\Psi}$ preserves the divided powers up to some integer power of $\mathbf{t}$. □

Since $\dot{\Psi}(\pi)=-\pi$, we obtain the following variant of Theorem 4.3.

Theorem 4.4. *The automorphism $\dot{\Psi}$ of $\dot{\mathbf{U}}[\mathbf{t}]$ induces an isomorphism of $\mathbb{Q}(\mathbf{t})$-algebras $\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=1}\cong\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=-1}$ and an isomorphism of $\mathbb{Z}[\mathbf{t}]$-algebras ${}_{\mathbb{Z}}\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=1}\cong{}_{\mathbb{Z}}\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=-1}$. In particular, we have embeddings $\dot{\mathbf{U}}|_{\pi=\pm1}\hookrightarrow\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=\mp1}$ and ${}_{\mathbb{Z}}\dot{\mathbf{U}}|_{\pi=\pm1}\hookrightarrow{}_{\mathbb{Z}}\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=\mp1}$.*

Hence, one may view the algebras $\dot{\mathbf{U}}|_{\pi=1}$ and $\dot{\mathbf{U}}|_{\pi=-1}$ as two different rational forms of the algebra $\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=1}$ (or equivalently, of $\dot{\mathbf{U}}[\mathbf{t}]|_{\pi=-1}$). We shall refer to the automorphism $\dot{\Psi}$ in Theorem 4.3 as a *twistor* on $\dot{\mathbf{U}}[\mathbf{t}]$.

Remark 4.5. (1) The definition of the covering quantum group $\mathbf{U}$ and the modified covering quantum group $\dot{\mathbf{U}}$ makes sense for super Cartan datum without the bar-consistent condition (d) in Definition 2.1. But the above theorems require the bar-consistent condition.

(2) The integer $\dot{\phi}(i,\lambda)$ admits a geometric interpretation (compare the integers $e_{\mu,n\alpha_i}$ and $f_{\mu,n\alpha_i}$ in [Li, 5.1]).

4.3. Category equivalences. Recall that a $\dot{\mathbf{U}}$-module M over $\mathbb{Q}(v)$ is called *unital* if each $m\in M$ is a finite sum of the form $m=\sum_{\lambda\in X}1_\lambda m$. When specializing π to ±1, we obtain the definition of a unital $\dot{\mathbf{U}}|_{\pi=\pm1}$-module over $\mathbb{Q}(v)$. We denote the categories of unital modules over $\mathbb{Q}(v)$ of $\dot{\mathbf{U}}$ (resp., $\dot{\mathbf{U}}|_{\pi=1}$, $\dot{\mathbf{U}}|_{\pi=-1}$) by $\dot{\mathcal{C}}$ (and resp., $\dot{\mathcal{C}}_{\pi=1}$, $\dot{\mathcal{C}}_{\pi=-1}$). We have $\dot{\mathcal{C}}=\dot{\mathcal{C}}_{\pi=1}\oplus\dot{\mathcal{C}}_{\pi=-1}$. We denote the category of unital modules over the field $\mathbb{Q}(v,\mathbf{t})$ of $\dot{\mathbf{U}}$ (and resp., $\dot{\mathbf{U}}|_{\pi=1}$, $\dot{\mathbf{U}}|_{\pi=-1}$) by $\dot{\mathcal{C}}^{\mathbf{t}}$ (and resp., $\dot{\mathcal{C}}^{\mathbf{t}}_{\pi=1}$, $\dot{\mathcal{C}}^{\mathbf{t}}_{\pi=-1}$).

The following is an immediate consequence of Theorem 4.4.

Proposition 4.6. *The twistor $\dot{\Psi}$ induces a category equivalence between $\dot{\mathcal{C}}^{\mathbf{t}}_{\pi=1}$ and $\dot{\mathcal{C}}^{\mathbf{t}}_{\pi=-1}$.*

The main novelty in the definition of the covering quantum group $\mathbf{U}$ is the additional generators J_μ in the Cartan subalgebra, which lead to a natural formulation of the integral form of $\mathbf{U}$ and weight modules of $\mathbf{U}$ (see [CHW1]). A $\mathbf{U}$-module over $\mathbb{Q}(v)$ (and resp., $\mathbf{U}[\mathbf{t}]$-module over $\mathbb{Q}(v,\mathbf{t})$) M is called a *weight module* of $\mathbf{U}$ (and resp., $\mathbf{U}[\mathbf{t}]$) if $M=\oplus_{\lambda\in X}M_\lambda$ with

$$M_\lambda=\left\{m\in M\mid K_\nu m=v^{\langle\nu,\lambda\rangle}m,\ J_\nu m=\pi^{\langle\nu,\lambda\rangle}m,\ \ \forall\,\nu\in Y\right\}.$$

We denote the category of weight modules of $\mathbf{U}$ (and resp., $\mathbf{U}[\mathbf{t}]$) over the respective fields as above by $\mathcal{C}$ (and resp., $\mathcal{C}^{\mathbf{t}}$). Similarly, we have categories of weight modules of $\mathbf{U}[\mathbf{t}]|_{\pi=\pm1}$ over $\mathbb{Q}(v, \mathbf{t})$ denoted by $\mathcal{C}^{\mathbf{t}}_{\pi=\pm1}$.

One can suitably formulate the BGG category $\mathcal{O}^{\mathbf{t}}$, $\mathcal{O}^{\mathbf{t}}_{\pi=1}$, $\mathcal{O}^{\mathbf{t}}_{\pi=-1}$ as subcategories of $\mathcal{C}^{\mathbf{t}}$, $\mathcal{C}^{\mathbf{t}}_{\pi=1}$ and $\mathcal{C}^{\mathbf{t}}_{\pi=-1}$, respectively.

Recall the definition of the highest weight $\mathbf{U}$-modules $V(\lambda)$ over $\mathbb{Q}(v)$, for $\lambda \in X$; see [CHW1, Proposition 2.6.5]. Then $V(\lambda)_{\pi=\pm1}$ is a simple $\mathbf{U}|_{\pi=\pm1}$-module. Let $X^+ = \{\lambda \in X \mid \langle i, \lambda\rangle \in \mathbb{Z}_{\geq 0}, \forall i \in I\}$ be the set of dominant weights. Then $\{V(\lambda)_{\pi=1} | \lambda \in X^+\}$ and $\{V(\lambda)_{\pi=-1} | \lambda \in X^+\}$ form a complete list of pairwise non-isomorphic simple integrable modules of $\mathbf{U}|_{\pi=1}$ and of $\mathbf{U}|_{\pi=-1}$, respectively.

Proposition 4.7. (1) *The categories $\mathcal{C}^{\mathbf{t}}_{\pi=1}$ and $\mathcal{C}^{\mathbf{t}}_{\pi=-1}$ are equivalent.*
(2) *The characters of the integrable $\mathbf{U}|_{\pi=-1}$-module $V(\lambda)_{\pi=-1}$ and the integrable $\mathbf{U}|_{\pi=1}$-module $V(\lambda)_{\pi=1}$ coincide, for each $\lambda \in X^+$.*

Proof. The equivalence in (1) follows by a sequence of category equivalences:

$$\mathcal{C}^{\mathbf{t}}_{\pi=1} \cong \dot{\mathcal{C}}^{\mathbf{t}}_{\pi=1} \cong \dot{\mathcal{C}}^{\mathbf{t}}_{\pi=-1} \cong \mathcal{C}^{\mathbf{t}}_{\pi=-1},$$

where the second equivalence follows by Proposition 4.6, the first equivalence is easy [Lu2, §23.1.4], and the third equivalence is completely analogous.

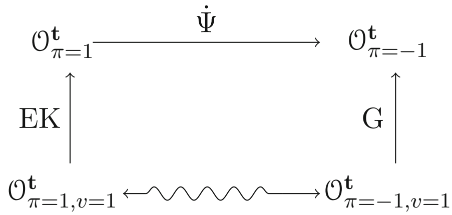
The category equivalences above always send each object M to M (where the weight structure remains unchanged), and moreover, the equivalence from $\mathcal{C}^{\mathbf{t}}_{\pi=1}$ to $\mathcal{C}^{\mathbf{t}}_{\pi=-1}$ sends $V(\lambda)_{\pi=1}$ to $V(\lambda)_{\pi=-1}$ while preserving weight space decompositions. The proposition is proved. □

Corollary 4.8. *The BGG categories $\mathcal{O}^{\mathbf{t}}_{\pi=1}$ of modules over the quantum group $\mathbf{U}[\mathbf{t}]|_{\pi=1}$ and $\mathcal{O}^{\mathbf{t}}_{\pi=-1}$ over the quantum supergroup $\mathbf{U}[\mathbf{t}]|_{\pi=-1}$ are equivalent via $\dot{\Psi}$.*

Remark 4.9. Thanks to the isomorphism of the integral forms ${}_{\mathbb{Z}}\dot{\mathbf{U}}|_{\pi=1}$ and ${}_{\mathbb{Z}}\dot{\mathbf{U}}|_{\pi=-1}$ in Theorem 4.4, a version of category equivalence similar to Propositions 4.7 holds when specializing v to be a root of unity.

Remark 4.10. Proposition 4.7(2) was stated in [CHW1] without proof, and there has been another proof given in [KKO]. A version of Proposition 4.7(1) on the equivalence of the weight modules of somewhat different algebras over $\mathbb{C}(v)^{\pi}$ also appeared in [KKO] with a very different proof. Note that the notion of weight modules in *loc. cit.* is nonstandard and subtle, and the algebras formulated therein over $\mathbb{C}(v)$ (or $\mathbb{C}(v)^{\pi}$) do not seem to admit rational forms or integral forms or modified forms as ours; in particular, their formulation does not make sense when v is a root of unity.

Remark 4.11. Let $X^{\mathrm{ev}} = \{\lambda \in X \mid \langle i, \lambda\rangle \in 2\mathbb{Z}, \forall i \in I_{\bar{1}}\}$. Denote by $\mathcal{O}^{\mathbf{t}}_{\pi=1,v=1}$ (and resp., $\mathcal{O}^{\mathbf{t}}_{\pi=-1,v=1}$) the BGG category of X^{ev}-weighted modules over the Lie algebra (and resp., Lie superalgebra) associated to the super root datum $(Y, X, I, \cdot)$. Using the technique of quantization of Lie bialgebras, Etingof–Kazhdan [EK] established an equivalence of categories between $\mathcal{O}^{\mathbf{t}}_{\pi=1,v=1}$ and $\mathcal{O}^{\mathbf{t}}_{\pi=1}$. As a super analogue, Geer [G] similarly established a quantization of Lie bisuperalgebras (Geer's super analogue was formulated for the finite type basic Lie superalgebras, but it makes sense for Kac-Moody as done by Etingof–Kazhdan.) This leads to an equivalence of categories between $\mathcal{O}^{\mathbf{t}}_{\pi=-1,v=1}$ and $\mathcal{O}^{\mathbf{t}}_{\pi=-1}$ (where the restriction to the weights in X^{ev} is necessary; see the classification of integrable modules in [K]). When combining with our category equivalence in Corollary 4.8, we obtain an equivalence of highest weight categories between BGG categories for Lie algebras and superalgebras. This equivalence provides an irreducible character formula in $\mathcal{O}^{\mathbf{t}}_{\pi=-1,v=1}$ whenever the corresponding irreducible module of $\mathcal{O}^{\mathbf{t}}_{\pi=1,v=1}$ admits a solution of the Kazhdan-Lusztig conjecture (by Beilinson-Bernstein, Brylinski-Kashiwara, Kashiwara-Tanisaki).



4.4. Extended covering quantum groups. We first work over a formal parameter t. Let $\mathbf{T}$ be the group algebra (in multiplicative form) of the group $\mathbb{Z}[I] \times Y$, that is, the $\mathbb{Q}(v)[t^{\pm 1}]^{\pi}$-algebra with generators T_μ, Υ_ν, for $\mu \in Y$ and $\nu \in \mathbb{Z}[I]$, and relations

$$T_\mu T_{\mu'} = T_{\mu+\mu'}, \quad \Upsilon_\nu \Upsilon_{\nu'} = \Upsilon_{\nu+\nu'}, \quad T_\mu \Upsilon_\nu = \Upsilon_\nu T_\mu, \quad T_0 = \Upsilon_0 = 1. \tag{4.18}$$

We define an action of $\mathbf{T}$ on $\mathbf{U}[t^{\pm 1}]$ by

$$T_\mu \cdot x = t^{\langle \mu, \eta' \rangle} x, \quad \Upsilon_\nu \cdot x = t^{\phi(\nu,\eta)} x \quad \text{for all } x \in \mathbf{U}[t^{\pm 1}]_\eta. \tag{4.19}$$

Then we form the semi-direct $\mathbb{Q}(v)[t^{\pm 1}]^{\pi}$-algebra $\widehat{\mathbf{U}}[t^{\pm 1}] = \mathbf{T} \ltimes \mathbf{U}[t^{\pm 1}]$ with respect to the above action of $\mathbf{T}$; that is, $TxT^{-1} = T \cdot x$ for all $T \in \mathbf{T}$ and $x \in \mathbf{U}[t^{\pm 1}]$. By specialization, we obtain a $\mathbb{Q}(v, \mathbf{t})^{\pi}$-algebra $\widehat{\mathbf{U}}[\mathbf{t}]$, which is called the *extended covering quantum group*.

Proposition 4.12. *There is a $\mathbb{Q}(\mathbf{t})$-algebra automorphism $\widehat{\Psi}$ on $\widehat{\mathbf{U}}[\mathbf{t}]$ such that*

$$\widehat{\Psi}(E_i) = \mathbf{t}_i^{-1} \Upsilon_i^{-1} T_{d_i i} E_i, \quad \widehat{\Psi}(F_i) = F_i \Upsilon_i, \quad \widehat{\Psi}(K_\nu) = T_{-\nu} K_\nu, \quad \widehat{\Psi}(J_\nu) = T_\nu^2 J_\nu,$$
$$\widehat{\Psi}(T_\nu) = T_\nu, \quad \widehat{\Psi}(\Upsilon_\nu) = \Upsilon_\nu, \quad \widehat{\Psi}(v) = \mathbf{t}^{-1} v, \quad \widehat{\Psi}(\pi) = -\pi.$$

The automorphism $\widehat{\Psi}$ will be called the *twistor* on $\widehat{\mathbf{U}}[\mathbf{t}]$.

Proof. We first show that such a map is well defined by showing that relations (4.1)–(4.6) and (4.18) are satisfied by the images of the generators. The relations (4.1)–(4.3) and (4.18) are straightforward to verify, and we leave this to the reader.

Let us verify (4.4). On one hand, we have

$$\begin{aligned}
&\widehat{\Psi}(E_i)\widehat{\Psi}(F_j) - \widehat{\Psi}(\pi)^{p(i)p(j)} \widehat{\Psi}(F_j)\widehat{\Psi}(E_i) \\
&= \mathbf{t}_i^{-1} \Upsilon_i^{-1} T_{d_i i} E_i F_j \Upsilon_j - (-\pi)^{p(i)p(j)} F_j \Upsilon_j \mathbf{t}_i^{-1} \Upsilon_i^{-1} T_{d_i i} E_i \\
&= \mathbf{t}^{-d_i + d_j - \phi(j,i)} \Upsilon_i^{-1} \Upsilon_j T_{d_i i} \Big(E_i F_j - \mathbf{t}^{i \cdot j + \phi(j,i) - \phi(i,j)} (-\pi)^{p(i)p(j)} F_j E_i \Big) \\
&= \mathbf{t}^{-d_i + d_j - \phi(j,i)} \Upsilon_i \Upsilon_j^{-1} T_{d_i i} (E_i F_j - \pi^{p(i)p(j)} F_j E_i),
\end{aligned} \tag{4.20}$$

where the last equality follows from (2.10) and $\mathbf{t}^2 = -1$. On the other hand,

$$\delta_{ij}\frac{\widehat{\Psi}(J_{d_i i})\widehat{\Psi}(K_{d_i i}) - \widehat{\Psi}(K_{-d_i i})}{\widehat{\Psi}(\pi_i)\widehat{\Psi}(v_i) - \widehat{\Psi}(v_i)^{-1}} = \delta_{ij}\frac{T_{d_i i}J_{d_i i}K_{d_i i} - T_{d_i i}K_{-d_i i}}{(-\mathbf{t})^{-d_i}\pi_i v_i - \mathbf{t}_i v_i^{-1}}$$
$$= \delta_{ij}\mathbf{t}_i^{-1}T_{d_i i}\frac{J_{d_i i}K_{d_i i} - K_{-d_i i}}{\pi_i v_i - v_i^{-1}}. \quad (4.21)$$

Then comparing (4.20) and (4.21), we see that they are equal for all $i, j \in I$, whence (4.4).

It remains to check the Serre relations (4.5) and (4.6). As these computations are entirely similar, let us prove (4.6). Then recalling (2.17), we see that

$$(F_i\Upsilon_i)^{b_{ij}-k}(F_j\Upsilon_j)(F_i\Upsilon_i)^k = \mathbf{t}^{\binom{b_{ij}}{2}-k(b_{ij}-k)d_i+\clubsuit}F_i^{b_{ij}-k}F_jF_i^k\Upsilon_{b_{ij}i+j}.$$

Hence as in the proof of Theorem 2.4, we have

$$\sum_{k=0}^{b_{ij}}(-1)^k(-\pi)^{\binom{k}{2}p(i)+kp(i)p(j)}\begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{\mathbf{t}^{-1}v_i,-\pi_i}(F_i\Upsilon_i)^{b_{ij}-k}(F_j\Upsilon_j)(F_i\Upsilon_i)^k$$
$$= \left(\sum_{k=0}^{b_{ij}}(-1)^k\pi^{\binom{k}{2}p(i)+kp(i)p(j)}\begin{bmatrix} b_{ij} \\ k \end{bmatrix}_{v_i,\pi_i}F_i^{b_{ij}-k}F_jF_i^k\right)\mathbf{t}^{\binom{b_{ij}}{2}+c(i,j)}\Upsilon_{b_{ij}i+j} = 0.$$

The proposition is proved. □

Remark 4.13. Here is a heuristic way of thinking about the extended covering quantum group and its twistor. The algebra $\mathbf{U}$ acts on $\dot{\mathbf{U}}$ via

$$1 \mapsto \sum_{\lambda\in X}1_\lambda,\ E_i \mapsto \sum_{\lambda\in X}E_i1_\lambda,\ F_i \mapsto \sum_{\lambda\in X}F_i1_\lambda,\ K_\nu \mapsto \sum_{\lambda\in X}v^{\langle\nu,\lambda\rangle}1_\lambda,\ J_\nu \mapsto \sum_{\lambda\in X}\pi^{\langle\nu,\lambda\rangle}1_\lambda.$$

Then $\dot{\Psi}$ induces an alternate $\mathbf{U}$-module structure on $\dot{\mathbf{U}}$ via

$$1 \mapsto \sum 1_\lambda, \quad E_i \mapsto \sum_{\lambda\in X}\mathbf{t}^{d_i\langle i,\lambda\rangle-\dot{\phi}(i,\lambda)}E_i1_\lambda, \quad F_i \mapsto \sum_{\lambda\in X}\mathbf{t}^{\dot{\phi}(i,\lambda)}F_i1_\lambda,$$
$$K_\nu \mapsto \sum_{\lambda\in X}(\mathbf{t}^{-1}v)^{\langle\nu,\lambda\rangle}1_\lambda, \quad J_\nu \mapsto \sum_{\lambda\in X}(-\pi)^{\langle\nu,\lambda\rangle}1_\lambda.$$

Merging these two actions leads to the introduction of new semisimple elements T_ν and Υ_μ such that $T_\nu \mapsto \sum_{\lambda\in X}t^{\langle\nu,\lambda\rangle}1_\lambda$ and $\Upsilon_\mu \mapsto \sum_{\lambda\in X}t^{\dot{\phi}(\mu,\lambda)}1_\lambda$.

Remark 4.14. Some construction similar to the twistor $\widehat{\Psi}$ as in Proposition 4.12 appeared in [KKO]. In contrast to *loc. cit.*, our formula for $\widehat{\Psi}$ is very explicit.

By specialization, the twistor on $\widehat{\mathbf{U}}[\mathbf{t}]$ leads to an isomorphism between the extended super and non-super quantum groups.

Corollary 4.15. *The $\mathbb{Q}(\mathbf{t})$-algebras $\widehat{\mathbf{U}}[\mathbf{t}]|_{\pi=1}$ and $\widehat{\mathbf{U}}[\mathbf{t}]_{\pi=-1}$ are isomorphic under $\widehat{\Psi}$.*

The twistor $\Psi : \mathbf{f}[\mathbf{t}] \to (\mathbf{f}[\mathbf{t}], *)$ in Theorem 2.4 is intimately related to the twistor $\widehat{\Psi} : \widehat{\mathbf{U}}[\mathbf{t}] \to \widehat{\mathbf{U}}[\mathbf{t}]$ in Proposition 4.12, as we shall describe.

There is an injective $\mathbb{Q}(v)[t^{\pm 1}]^{\pi}$-algebra homomorphism (see [CHW1, §2.1])

$$(\cdot)^{-} : \mathbf{f}[t^{\pm 1}] \longrightarrow \mathbf{U}[t^{\pm 1}], \tag{4.22}$$

such that $\theta_i^{-} = F_i$ for all $i \in I$.

Lemma 4.16. *There is an injective* $\mathbb{Q}(v)[t^{\pm 1}]^{\pi}$*-algebra homomorphism*

$$\chi : (\mathbf{f}[t^{\pm 1}], *) \longrightarrow \widehat{\mathbf{U}}[t^{\pm 1}]$$

such that

$$\chi(x) = x^{-}\Upsilon_{\nu}, \qquad \forall x \in \mathbf{f}[t^{\pm 1}]_{\nu}.$$

Proof. One checks by definition that, for $x, y \in \mathbf{f}[t^{\pm 1}]$ homogeneous,

$$(x * y)^{-}\Upsilon_{|x|+|y|} = x^{-}\Upsilon_{|x|}y^{-}\Upsilon_{|y|}.$$

The lemma is proved. □

Now specializing t to $\mathbf{t}$ for χ and $(\cdot)^{-}$ above, we obtain an injective $\mathbb{Q}(v, \mathbf{t})^{\pi}$-algebra homomorphism $\chi : (\mathbf{f}[\mathbf{t}], *) \longrightarrow \widehat{\mathbf{U}}[\mathbf{t}]$, and an injective $\mathbb{Q}(v, \mathbf{t})^{\pi}$-algebra homomorphism $(\cdot)^{-} : \mathbf{f}[\mathbf{t}] \longrightarrow \mathbf{U}[\mathbf{t}]$. The following proposition can be verified by definitions, which we leave to the reader.

Proposition 4.17. *We have a commutative diagram of* $\mathbb{Q}(\mathbf{t})$*-algebra homomorphisms:*

$$\begin{array}{ccc} \mathbf{f}[\mathbf{t}] & \xhookrightarrow{(\cdot)^{-}} & \mathbf{U}[\mathbf{t}] \\ \Psi \downarrow & & \downarrow \widehat{\Psi} \\ (\mathbf{f}[\mathbf{t}], *) & \xhookrightarrow{\chi} & \widehat{\mathbf{U}}[\mathbf{t}] \end{array}$$

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