Commun. Math. Phys. 332, 759–781 (2014)
Digital Object Identifier (DOI) 10.1007/s00220-014-2064-3

Communications in
Mathematical
Physics

Singular Values of Products of Ginibre Random Matrices, Multiple Orthogonal Polynomials and Hard Edge Scaling Limits

Arno B. J. Kuijlaars[1], **Lun Zhang**[2]

[1] Department of Mathematics, KU Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium.
E-mail: arno.kuijlaars@wis.kuleuven.be
[2] School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, People's Republic of China. E-mail: lunzhang@fudan.edu.cn

Received: 5 August 2013 / Accepted: 19 December 2013
Published online: 9 May 2014 –

Abstract: Akemann, Ipsen and Kieburg recently showed that the squared singular values of products of M rectangular random matrices with independent complex Gaussian entries are distributed according to a determinantal point process with a correlation kernel that can be expressed in terms of Meijer G-functions. We show that this point process can be interpreted as a multiple orthogonal polynomial ensemble. We give integral representations for the relevant multiple orthogonal polynomials and a new double contour integral for the correlation kernel, which allows us to find its scaling limits at the origin (hard edge). The limiting kernels generalize the classical Bessel kernels. For $M = 2$ they coincide with the scaling limits found by Bertola, Gekhtman, and Szmigielski in the Cauchy–Laguerre two-matrix model, which indicates that these kernels represent a new universality class in random matrix theory.

1. Introduction

1.1. Products of Ginibre random matrices. Random matrix theory is a broad field with many applications in mathematics, physics, and beyond, as is witnessed by the survey volume [1] and the recent monographs [7,20,22,34]. Of particular importance for the development of the theory is the connection with determinantal point processes. Whenever the eigenvalues of a random matrix ensemble are a determinantal point process, one has explicit expressions for the eigenvalue distributions in terms of the correlation kernel. Tools from integrable systems may then be used to further analyze the correlation kernel in the large n limit, in order to establish, for example, universality of local eigenvalue correlations. It is a recent discovery that products of random matrices can fall in the framework of determinantal point processes.

The topic of products of random matrices can be traced back to the work of Furstenberg and Kesten [23], where the interest lies in the asymptotic behavior as the number of factors in the product tends to infinity. This work has been highly influential with important applications in Schrödinger operator theory [14] and in statistical physics relating to disordered and chaotic dynamical systems [18].

A more recent development is the study of eigenvalue and singular value distributions for the products of random matrices with a fixed number of factors, but allowing the size of the matrices to tend to infinity. With tools from free probability and diagrammatic expansions, one may find the limiting global eigenvalue distributions as in [8,15,16,33]. It turns out that, as in the theory of a single random matrix, the various limits exhibit a rich and interesting mathematical structure, which also show a large degree of universality, see e.g. [24,32]. Apart from physical applications, the study is also motivated by other fields like MIMO (multiple-input and multiple-output) networks in telecommunication [36].

Akemann and Burda [2] proved that the eigenvalues of products of complex Ginibre matrices are determinantal in the complex plane, see [25] for an extension to quaternionic Ginibre matrices. A similar determinantal structure holds for the eigenvalues of products of truncated unitary matrices [3]. The determinantal structure opens up the way to a more detailed analysis at the finite n level [3,6]. Very recently, Akemann et al. [5] found that the squared singular values of products of complex Ginibre matrices are a determinantal point process on the positive real line. This was further extended to the case of products of rectangular Ginibre matrices by Akemann et al. [4]. The correlation kernels in [2–5,25] are all expressed in terms of Meijer G-functions.

In this paper we follow [4]. We take $M \geq 1$ and let $X_1, X_2, \ldots, X_M$ be complex random matrices whose entries are independent with a complex Gaussian distribution, also known as Ginibre random matrices. We assume X_j has size $N_j \times N_{j-1}$ and form the product

$$Y_M = X_M X_{M-1} \cdots X_1. \tag{1.1}$$

Our interest lies in the squared singular values of Y_M, that is, the eigenvalues of $Y_M^* Y_M$, where the superscript * stands for conjugate transpose. We assume $N_0 = \min\{N_0, \ldots, N_M\}$, and write

$$\nu_j = N_j - N_0, \quad j = 0, \ldots, M, \quad n = N_0. \tag{1.2}$$

Thus $\nu_0 = 0$ and $Y_M^* Y_M$ is a square matrix of size n.

The case for the products of square matrices (i.e., $\nu_j = 0$ for every j) was considered by Akemann et al. [5], who showed that the squared singular values are distributed according to a determinantal point process with a correlation kernel that can be expressed in terms of Meijer G-functions. This was extended by Akemann et al. [4] to the general rectangular case. The determinantal point process is a biorthogonal ensemble [13] with joint probability density function (see [4, formula (18)])

$$P(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{j<k} (x_k - x_j) \det \left[w_{k-1}(x_j) \right]_{j,k=1,\ldots,n}, \tag{1.3}$$

where $x_j > 0$, $j = 1, \ldots, n$, are the squared singular values of Y_M,

$$w_k(x) = G^{M,0}_{0,M}\left({- \atop \nu_M, \nu_{M-1}, \ldots, \nu_2, \nu_1 + k} \middle| x \right), \tag{1.4}$$

and normalization constant (see [4, formula (21)])

$$Z_n = n! \prod_{i=1}^{n} \prod_{j=0}^{M} \Gamma(i + \nu_j).$$

The function w_k is a Meijer G-function (see e.g. [9,30] and the Appendix for a brief introduction) which can be written as a Mellin–Barnes integral

$$w_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s+\nu_1+k) \prod_{j=2}^{M} \Gamma(s+\nu_j) x^{-s}\, ds, \quad k = 0, 1, \ldots, \tag{1.5}$$

with $c > 0$. By the inversion formula for the Mellin transform we have

$$\int_0^\infty w_k(x) x^{s-1}\, dx = \Gamma(s+\nu_1+k) \prod_{j=2}^{M} \Gamma(s+\nu_j), \quad s > 0, \tag{1.6}$$

which in particular shows that the moments of w_k are given as products of Gamma functions.

By (1.5) and the functional equation of the Gamma function $\Gamma(z+1) = z\Gamma(z)$, we have

$$w_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s+\nu_1)_k \prod_{j=1}^{M} \Gamma(s+\nu_j) x^{-s}\, ds,$$

where the Pochhammer symbol

$$(s+\nu_1)_k = \frac{\Gamma(s+\nu_1+k)}{\Gamma(s+\nu_1)} = (s+\nu_1)(s+\nu_1+1)\cdots(s+\nu_1+k-1)$$

is a polynomial of degree k in the variable s. Then by taking linear combinations of the weights we could alternatively take

$$\widetilde{w}_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^k \prod_{j=1}^{M} \Gamma(s+\nu_j) x^{-s}\, ds, \tag{1.7}$$

in the definition of (1.3). This representation shows that (1.3) is fully symmetric in all parameters $\nu_1, \ldots, \nu_M$. Note that

$$\widetilde{w}_k(x) = \left(-x \frac{d}{dx}\right)^k w_0(x),$$

which can be easily obtained from (1.5).

1.2. Biorthogonal functions and the correlation kernel. From general properties of biorthogonal ensembles [13], it is known that (1.3) is a determinantal point process with correlation kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x^j (M_n^{-1})_{k,j} w_k(y), \tag{1.8}$$

where M_n is the matrix of moments of size $n \times n$,

$$M_n = \left(\int_0^\infty x^j w_k(x)\, dx \right)_{j,k=0,\ldots,n-1}. \tag{1.9}$$

In addition we have

$$K_n(x,y)=\sum_{k=0}^{n-1}P_k(x)Q_k(y), \tag{1.10}$$

where for each $k=0,1,\ldots,P_k$ is a monic polynomial of degree k and Q_k belongs to the linear span of $w_0,\ldots,w_k$ in such a way that

$$\int_0^\infty P_j(x)Q_k(x)\,\mathrm{d}x=\delta_{j,k}. \tag{1.11}$$

Thus the P_k and Q_k are biorthogonal functions that we consider for every non-negative integer k, not just for $k\le n-1$.

Akemann et al. [4,5] studied an extension of (1.3) to a two-matrix model and obtained in this framework that for certain polynomials $\widetilde{Q}_k$,

$$\int_0^\infty\int_0^\infty P_j(x)\widetilde{Q}_k(y)w_\nu^M(x,y)\,\mathrm{d}x\,\mathrm{d}y=h_j^M\delta_{j,k}, \tag{1.12}$$

with

$$w_\nu^M(x,y)=y^{\nu_1-1}e^{-y}\,G^{M-1,0}_{0,M-1}\left({- \atop \nu_M,\nu_{M-1},\ldots,\nu_2}\middle|\frac{x}{y}\right)$$

and

$$h_j^M=\prod_{m=0}^{M}(j+\nu_m)!;$$

see [4, formulas (25), (27) and (37)]. We emphasize that $\widetilde{Q}_k\neq Q_k$, since indeed Q_k is not a polynomial and $\widetilde{Q}_k$ is a multiple of the Laguerre polynomial $L_k^{(\nu_1)}$; see [4, formula (42)]. The biorthogonality (1.12) is related to (1.11), since

$$Q_k(x)=\frac{1}{h_k^M}\int_0^\infty\widetilde{Q}_k(y)w_\nu^M(x,y)\,\mathrm{d}y,$$

but we will not use this fact.

The starting point of this paper is the biorthogonality (1.11) and we first show that the polynomials P_k can be characterized as multiple orthogonal polynomials with respect to the first M weight functions $w_0,\ldots,w_{M-1}$. Hence, the point process (1.3) is a multiple orthogonal polynomial (MOP) ensemble in the sense of [28,29]. This further implies a representation of the correlation kernel K_n (1.10) in terms of the associated Riemann–Hilbert problem, which is helpful for future asymptotic analysis.

In [4] it is shown that the biorthogonal functions P_k and Q_k have integral representations as Meijer G-functions. We rederive these results in Sect. 3 using only the biorthogonality (1.11). The recurrence relations of the biorthogonal functions are explicitly given in Sect. 4. We turn to the study of the function K_n in Sect. 5. We derive a double contour integral representation of K_n, which allows us to find its scaling limit at the origin (hard edge). The limiting kernels generalize the classical Bessel kernel, and if $M=2$, it coincides with the limiting kernels in the Cauchy–Laguerre two-matrix model recently studied by Bertola, Gekhtman and Szmigielski in [12]. Universality suggests that the new limiting kernels should apply to more general situations for the products of independent complex random matrices, thus, representing a new universality class. Finally, we present the integrable form of the limiting kernels in the sense of Its et al. [27]. For the convenience of the reader, we include a short introduction to the Meijer G-function in the Appendix.

Remark 1.1. It is possible to consider the probability density function (1.3) for general parameters $\nu_1, \ldots, \nu_M > -1$. The condition $\nu_j > -1$ is needed in order to guarantee the existence of the moments in (1.9). All the constructions in this paper go through in that more general case.

However, we do not have a proof that (1.3) is a probability density function in the case of non-integer parameters, in particular we do not know that (1.3) is non-negative for all $x_1, \ldots, x_n$, although we strongly suspect that it will be the case.

2. Multiple Orthogonal Polynomial Ensemble

2.1. Multiple orthogonality. Our first result is that the point process (1.3) is a MOP ensemble [28,29] with M weight functions $w_0, \ldots, w_{M-1}$, where the w_k are defined in (1.4). This follows from the following lemma.

Lemma 2.1. *The linear span of the functions $w_0, w_1, \ldots, w_{n-1}$ is equal to the linear span of the functions*

$$x \mapsto x^j w_k(x), \quad k = 0, \ldots, M-1, \quad k + jM < n. \tag{2.1}$$

Proof. The linear span of $w_0, w_1, \ldots, w_{n-1}$ consists of all functions that can be written as

$$x \mapsto \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q(s) \prod_{j=1}^{M} \Gamma(s + \nu_j) x^{-s} \, \mathrm{d}s, \quad \deg q(s) \leq n - 1. \tag{2.2}$$

We have by (1.5) and a change of variables $s \mapsto s + j$,

$$\begin{aligned} x^j w_k(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s + \nu_1)_k \prod_{l=1}^{M} \Gamma(s + \nu_l) x^{j-s} \, \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s + \nu_1 + j)_k \prod_{l=1}^{M} (s + \nu_l)_j \prod_{l=1}^{M} \Gamma(s + \nu_l) x^{-s} \, \mathrm{d}s. \end{aligned}$$

This is of the form (2.2) with polynomial

$$q(s) = (s + \nu_1 + j)_k \prod_{l=1}^{M} (s + \nu_l)_j$$

of degree $k + jM$. Thus the functions (2.1) belong to the linear span of $w_0, \ldots, w_{n-1}$. It is readily seen that these are independent since they correspond to polynomials $q(s)$ that have different degrees. □

The polynomials P_k are therefore MOPs of type II with respect to the weights $w_0, \ldots, w_{M-1}$ and diagonal multiple indices, i.e.,

$$\int_0^\infty P_n(x) x^j w_k(x) \, \mathrm{d}x = 0, \quad j = 0, \ldots, \lceil \tfrac{n-k}{M} \rceil - 1, \quad k = 0, \ldots, M-1,$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$; see [26,37].

2.2. Riemann–Hilbert problem. As a consequence of Lemma 2.1, the polynomial P_n is characterized by the following Riemann–Hilbert problem. We look for a $(M+1)\times(M+1)$ matrix-valued function $Y : \mathbb{C}\setminus[0,\infty) \to \mathbb{C}^{(M+1)\times(M+1)}$ that is analytic with jump condition

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w_0(x) & \cdots & w_{M-1}(x) \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad x \in (0,\infty), \tag{2.3}$$

where Y_+ (Y_-) denotes the limiting value from the upper (lower) half-plane. As $z \to \infty$, we require

$$Y(z) = (I + O(1/z))\,\mathrm{diag}\left(z^n \; z^{-n_0} \; \cdots \; z^{-n_{M-1}}\right), \tag{2.4}$$

where $n_k = \lceil \frac{n-k}{M} \rceil$. Combined with appropriate local conditions near the origin that depend on the parameters $\nu_1, \nu_2, \ldots, \nu_M$, the Riemann–Hilbert problem (2.3)–(2.4) has a unique solution and the (1, 1) entry of Y is P_n; see [37]. Also, one has

$$K_n(x, y) = \frac{1}{2\pi i(x-y)}\left(0 \; w_0(y) \; \cdots \; w_{M-1}(y)\right) Y_+^{-1}(y)Y_+(x)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.5}$$

which is a manifestation of the Christoffel–Darboux formula for multiple orthogonal polynomials; see [19]. The representation (2.5) is potentially useful for asymptotic analysis although we will not pursue this here.

2.3. Special case $M = 2$. We now take a look at the case $M = 2$. If $M = 2$, then

$$w_0(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s+\nu_1)\Gamma(s+\nu_2)x^{-s}\,\mathrm{d}s.$$

This can be expressed in terms of the modified Bessel function of second kind (a.k.a. the Macdonald function). The formula 10.32.13 of [31] says that

$$2K_\nu(2\sqrt{x}) = \frac{x^{\nu/2}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(s-\nu)x^{-s}\,\mathrm{d}s, \quad c > \max(\nu, 0),$$

which after a change of variables $s \mapsto s + \nu + \alpha$ leads to

$$2K_\nu(2\sqrt{x}) = \frac{x^{-\nu/2-\alpha}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s+\nu+\alpha)\Gamma(s+\alpha)x^{-s}\,\mathrm{d}s, \quad c > \max(-\nu-\alpha, -\alpha).$$

We take $\alpha = \nu_2$, $\nu = \nu_1 - \nu_2$, to find that

$$w_0(x) = 2x^{(\nu_1+\nu_2)/2}K_{\nu_1-\nu_2}(2\sqrt{x}). \tag{2.6}$$

It will be convenient to assume that $\nu_1 \geq \nu_2$, which we can do without loss of generality.

Similarly,

$$w_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s+\nu_1+1)\Gamma(s+\nu_2)x^{-s}\,\mathrm{d}s = 2x^{(\nu_1+\nu_2+1)/2}K_{\nu_1+1-\nu_2}(2\sqrt{x}). \tag{2.7}$$

Thus if $\rho_\nu(x) = 2x^{\nu/2}K_\nu(2\sqrt{x})$, we have

$$w_0(x) = x^\alpha \rho_\nu(x), \quad w_1(x) = x^\alpha \rho_{\nu+1}(x).$$

Multiple orthogonal polynomials associated with the two weights (2.6)–(2.7) were considered by Van Assche and Yakubovich [38] for which they obtained four term recurrence relations; see also [17,40] for asymptotic results for these polynomials. In the random matrix context (i.e., the case where $\nu_j = N_j - N_0$ are integers), we have

$$\nu = N_1 - N_2, \quad \alpha = N_2 - N_0.$$

For the special case $\nu = \alpha = 0$ (i.e., the products of two square matrices), this relation was first observed in [39].

For general M, there is an $M+2$ term recurrence relation (this follows from general theory of MOP, cf. [26, Section 23.1.4]) and we will determine the recurrence coefficients explicitly in Sect. 4.

3. Integral Representations

Integral representations for the biorthogonal polynomials P_k and their dual functions Q_k are given in [4] where they were derived from a two matrix model. We rederive these results directly from the biorthogonality (1.11).

3.1. Integral representation for Q_k. Recall the biorthogonality (1.11). The biorthogonal function Q_k has the form

$$Q_k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} q_k(s)\prod_{j=1}^{M}\Gamma(s+\nu_j)x^{-s}\,\mathrm{d}s,$$

where q_k is a polynomial of degree k. The biorthogonality (1.11) then says that

$$\frac{1}{2\pi i}\int_0^\infty \int_{c-i\infty}^{c+i\infty} P_l(x)q_k(s)\prod_{j=1}^{M}\Gamma(s+\nu_j)x^{-s}\,\mathrm{d}s\,\mathrm{d}x = \delta_{l,k}.$$

It turns out that we can write down q_k explicitly as stated in the following proposition.

Proposition 3.1. *We have*

$$Q_k(x) = \frac{(-1)^k}{\prod_{j=0}^{M}\Gamma(k+1+\nu_j)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(x^k w_0(x)\right), \tag{3.1}$$

and

$$q_k(s) = \frac{(s-k)_k}{\prod_{j=0}^{M}\Gamma(k+1+\nu_j)}. \tag{3.2}$$

Proof. It is easy to see after applying an integration by parts k times that

$$\int_0^\infty x^l \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (x^k w_0(x))\,\mathrm{d}x = 0, \quad \text{for } l < k.$$

Note that integrated terms do not contribute, since

$$w_0(x) = O(x^\alpha (\log x)^{r-1}), \quad \text{as } x \to 0+, \tag{3.3}$$

with $\alpha = \min(\nu_1, \ldots, \nu_M) > -1$ and $r = \#\{j \mid \nu_j = \alpha\}$, which can be deduced from properties of the Mellin transform (1.6); see e.g. [21, Theorem 4], and since for $x \to +\infty$, we have

$$w_0(x) = O\left(x^\theta e^{-Mx^{1/M}}\right), \quad \theta = \frac{1}{M}\left(\tfrac{1}{2}(1-M) + \sum_{j=1}^M \nu_j\right);$$

see [30, Theorem 5.7.5].

Similarly,

$$\begin{aligned}\int_0^\infty x^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k (x^k w_0(x))\,\mathrm{d}x &= (-1)^k k! \int_0^\infty x^k w_0(x)\,\mathrm{d}x \\ &= (-1)^k \prod_{j=0}^M \Gamma(k+1+\nu_j),\end{aligned}$$

where we recall (1.6) and the fact that $\nu_0 = 0$. Thus if Q_k is defined by (3.1), then we have

$$\int_0^\infty x^l Q_k(x)\,\mathrm{d}x = \delta_{l,k}, \quad \text{for } l = 0, 1, \ldots, k. \tag{3.4}$$

Since

$$x^k w_0(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \prod_{j=1}^M \Gamma(s+\nu_j) x^{k-s}\,\mathrm{d}s,$$

we find by taking k derivatives that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(x^k w_0(x)\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (-1)^k (s-k)_k \prod_{j=1}^M \Gamma(s+\nu_j) x^{-s}\,\mathrm{d}s.$$

Thus

$$Q_k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} q_k(s) \prod_{j=1}^M \Gamma(s+\nu_j) x^{-s}\,\mathrm{d}s, \tag{3.5}$$

with q_k as in (3.2). This proves that Q_k belongs to the linear span of $w_0, \ldots, w_{k-1}$ and (3.4) shows that it is indeed the biorthogonal function. □

Note that (3.1) is a Rodrigues-type formula for Q_k. Note also that (3.5) is an integral representation, which because of (3.2) we may also write as

$$Q_k(x) = \frac{1}{2\pi i \prod_{j=0}^{M} \Gamma(k+\nu_j+1)} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=0}^{M} \Gamma(s+\nu_j)}{\Gamma(s-k)} x^{-s} \, \mathrm{d}s. \tag{3.6}$$

By (A.1), we can identify (3.6) as a Meijer G-function:

$$Q_k(x) = \frac{1}{\prod_{j=0}^{M} \Gamma(k+\nu_j+1)} G^{M+1,0}_{1,M+1}\left({-k \atop \nu_0, \nu_1, \ldots, \nu_M} \middle| x \right). \tag{3.7}$$

Up to a multiplicative constant and an easy transformation of the Meijer G-function, (3.7) is the same as [4, formula (49)].

3.2. Integral representation for P_n. There is a similar integral representation for P_n.

Proposition 3.2. *We have for $x > 0$,*

$$P_n(x) = \frac{\prod_{j=0}^{M} \Gamma(n+\nu_j+1)}{2\pi i} \oint_{\Sigma} \frac{\Gamma(t-n)}{\prod_{j=0}^{M} \Gamma(t+\nu_j+1)} x^t \, \mathrm{d}t, \tag{3.8}$$

where Σ is a closed contour that encircles $0, 1, \ldots, n$ once in the positive direction.

Proof. In the proof we assume that P_n is given by (3.8) and we show that P_n is a monic polynomial of degree n satisfying

$$\int_0^\infty P_n(x) \widetilde{w}_k(x) \, \mathrm{d}x = 0, \quad k = 0, \ldots, n-1, \tag{3.9}$$

where $\widetilde{w}_k$ is defined in (1.7).

The integrand in the right-hand side of (3.8) is meromorphic on $\mathbb{C}$ with simple poles at $0, 1, \ldots, n$ (the poles of the numerator at the negative integers are cancelled by the poles of the factor $\Gamma(t+1)$ in the denominator). Thus by the residue theorem

$$P_n(x) = \prod_{j=0}^{M} \Gamma(n+\nu_j+1) \sum_{l=0}^{n} \operatorname*{Res}_{t=l} \left(\frac{\Gamma(t-n)}{\prod_{j=0}^{M} \Gamma(t+\nu_j+1)} \right) x^l.$$

We can evaluate the residues to obtain

$$P_n(x) = \sum_{l=0}^{n} \frac{(-1)^{n-l}}{(n-l)!} \frac{\prod_{j=0}^{M} \Gamma(n+\nu_j+1)}{\prod_{j=0}^{M} \Gamma(l+\nu_j+1)} x^l, \tag{3.10}$$

which shows that P_n is a monic polynomial of degree n.

To verify (3.9) we use

$$\int_0^\infty x^t \widetilde{w}_k(x) \, \mathrm{d}x = (t+1)^k \prod_{j=1}^{M} \Gamma(t+\nu_j+1),$$

which follows from (1.7) and the inversion formula for Mellin transforms. Then we can compute by (3.8) and an interchange of integrals,

$$\begin{aligned}
&\int_0^\infty P_n(x)\widetilde{w}_k(x)\,\mathrm{d}x \\
&= \frac{\prod_{j=0}^M \Gamma(n+\nu_j+1)}{2\pi i} \oint_\Sigma \frac{\Gamma(t-n)}{\prod_{j=0}^M \Gamma(t+\nu_j+1)} (t+1)^k \prod_{j=1}^M \Gamma(t+\nu_j+1)\,\mathrm{d}t \\
&= \frac{\prod_{j=0}^M \Gamma(n+\nu_j+1)}{2\pi i} \oint_\Sigma \frac{\Gamma(t-n)(t+1)^k}{\Gamma(t+1)}\,\mathrm{d}t \\
&= \frac{\prod_{j=0}^M \Gamma(n+\nu_j+1)}{2\pi i} \oint_\Sigma \frac{(t+1)^k}{t(t-1)\cdots(t-n)}\,\mathrm{d}t.
\end{aligned}$$

The remaining integrand is a rational function that behaves like $O(t^{k-n-1})$ as $t \to \infty$. The contour Σ encircles all the poles once in the positive direction. Thus by moving the contour to infinity, we find that the integral vanishes for $k \leq n-1$, which is the required biorthogonality (3.9). □

The formula (3.10) shows that P_n is a hypergeometric polynomial

$$P_n(x) = (-1)^n \prod_{j=1}^M \frac{\Gamma(n+\nu_j+1)}{\Gamma(\nu_j+1)} \, {}_1F_M\left(\begin{array}{c} -n \\ 1+\nu_1, \ldots, 1+\nu_M \end{array} \middle| x \right),$$

as in [4, formula (44)]. We can also identify P_n in (3.8) as a Meijer G-function:

$$P_n(x) = -\prod_{j=0}^M \Gamma(n+\nu_j+1)\, G^{0,1}_{1,M+1}\left(\begin{array}{c} n+1 \\ -\nu_0, -\nu_1, \ldots, -\nu_{M-1}, -\nu_M \end{array} \middle| x \right), \quad (3.11)$$

which is equivalent to [4, formula (45)].

4. Recurrence Relations

By Lemma 2.1 and general theory of MOPs (cf. [26, Chapter 23]), it follows that the polynomials P_n satisfy an $M+2$ term recurrence relation

$$x P_n(x) = P_{n+1}(x) + \sum_{k=0}^M a_{k,n} P_{n-k}(x). \quad (4.1)$$

There is a dual recurrence relation

$$x Q_n(x) = Q_{n-1}(x) + \sum_{k=0}^M b_{k,n} Q_{n+k}(x), \quad (4.2)$$

where because of the biorthogonality (1.11),

$$a_{k,n} = \int_0^\infty x P_n(x)\, Q_{n-k}(x)\,\mathrm{d}x, \quad b_{k,n} = \int_0^\infty P_{n+k}(x)\, x\, Q_n(x)\,\mathrm{d}x.$$

Therefore

$$a_{k,n} = b_{k,n-k}. \quad (4.3)$$

It is the aim of this section to calculate these recurrence coefficients explicitly.

4.1. Coefficients $b_{k,n}$.

Proposition 4.1. *We have for $k = 0, \ldots, M$,*

$$b_{k,n} = \left(\prod_{j=0}^{M} (n + \nu_j + 1)_k \right) \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{j!(k+1-j)!} \prod_{i=0}^{M} (n + j + \nu_i). \tag{4.4}$$

Proof. We have from (3.5), after a change of variable $s \mapsto s + 1$,

$$\begin{aligned} x Q_n(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q_n(s) \prod_{j=1}^{M} \Gamma(s + \nu_j) x^{-s+1} \, ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q_n(s+1) \prod_{j=1}^{M} \Gamma(s + \nu_j + 1) x^{-s} \, ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q_n(s+1) \prod_{j=1}^{M} (s + \nu_j) \prod_{j=1}^{M} \Gamma(s + \nu_j) x^{-s} \, ds. \end{aligned}$$

Then $q_n(s+1) \prod_{j=1}^{M} (s + \nu_j)$ is a polynomial in s of degree $n + M$ and it is our task to show that

$$q_n(s+1) \prod_{j=1}^{M} (s + \nu_j) = q_{n-1}(s) + \sum_{k=0}^{M} b_{k,n} q_{n+k}(s) \tag{4.5}$$

with $b_{k,n}$ given by (4.4).

By (3.2) we have that all terms in (4.5) are zero for $s = 1, \ldots, n-1$, i.e., all terms are divisible by $q_{n-1}(s)$. If we do this division and use (3.2) then we find that we have to prove

$$\prod_{j=0}^{M} \frac{s + \nu_j}{n + \nu_j} = 1 + \sum_{k=0}^{M} \frac{b_{k,n}}{\prod_{j=0}^{M} (n + \nu_j)_{k+1}} (s - n - k)_{k+1}.$$

Write $s = t + n$. Then we have to prove

$$f(t) = \prod_{j=0}^{M} (n + \nu_j) + \sum_{k=0}^{M} \frac{b_{k,n}}{\prod_{j=0}^{M} (n + \nu_j + 1)_k} (t - k)_{k+1}, \tag{4.6}$$

as an identity for polynomials in t, where

$$f(t) = \prod_{j=0}^{M} (t + n + \nu_j). \tag{4.7}$$

Both sides of (4.6) have degree $M + 1$ and for $t = 0$ the identity (4.6) is valid. The polynomials $t \mapsto (t - k)_{k+1}$ for $k = 0, \ldots, m$ are a basis for the vector space of polynomials of degree $\leq M + 1$ that vanish at $t = 0$. Then it is clear that there exists coefficients $b_{k,n}$ such that (4.6) holds.

By contour integration we obtain from (4.6)

$$\frac{b_{k,n}}{\prod_{j=0}^{M}(n+\nu_j+1)_k} = \frac{1}{2\pi i}\oint_{\Sigma}\frac{f(t)}{(t-k-1)_{k+2}}\,\mathrm{d}t, \quad k=0,\ldots,M, \tag{4.8}$$

where Σ is a closed contour that encircles the points $0,\ldots,k$ once in the positive direction. This leads by the residue theorem to

$$b_{k,n} = \left(\prod_{j=0}^{M}(n+\nu_j+1)_k\right)\sum_{j=0}^{k+1}(-1)^{k+1-j}\frac{f(j)}{j!(k+1-j)!},$$

which gives (4.4) in view of the definition (4.7) of $f(t)$. □

4.2. Coefficients $a_{k,n}$. Because of (4.3) we immediately find an expression for the recurrence coefficients $a_{k,n}$.

Corollary 4.2. *We have for $k=0,\ldots,M$,*

$$a_{k,n} = \left(\prod_{j=0}^{M}(n-k+\nu_j+1)_k\right)\sum_{j=0}^{k+1}(-1)^{k+1-j}\frac{\prod_{i=0}^{M}(n-k+j+\nu_i)}{j!(k+1-j)!}. \tag{4.9}$$

Reversing the order of summation we also have

$$a_{k,n} = \left(\prod_{j=0}^{M}(n-k+\nu_j+1)_k\right)\sum_{j=0}^{k+1}(-1)^{j}\frac{\prod_{i=0}^{M}(n+1-j+\nu_i)}{j!(k+1-j)!}.$$

Proof. Use (4.3), (4.4) and reverse the order of summation. □

From (4.9) we see that $a_{k,n}$ is a polynomial expression in n, which seems to be of degree $k(M+1)+M+1=(k+1)(M+1)$. However there is a cancellation in the leading order terms and $a_{k,n}$ is actually a polynomial in n of degree $(k+1)M$.

Lemma 4.3. *For every k we have*

$$a_{k,n} = \binom{M+1}{k+1}n^{(k+1)M} + O\left(n^{(k+1)M-1}\right).$$

Proof. From (4.3) and the contour integral representation (4.8) for $b_{k,n}$ we find

$$a_{k,n} = \left(\prod_{j=0}^{M}(n-k+\nu_j+1)_k\right)\frac{1}{2\pi i}\oint_{\Sigma}\frac{g_n(t)}{(t-k-1)_{k+2}}\,\mathrm{d}t, \tag{4.10}$$

where

$$g_n(t) = \prod_{j=0}^{M}(t+n-k+\nu_j) = \sum_{l=0}^{M+1}p_l(t)n^l$$

is a polynomial of degree $M+1$ in n. The coefficient $p_l(t)$ is a polynomial in t of degree $\deg p_l(t) = M+1-l$. Thus

$$\frac{1}{2\pi i}\oint_\Sigma \frac{g_n(t)}{(t-k-1)_{k+2}}\,\mathrm{d}t = \sum_{l=0}^{M+1}\left(\frac{1}{2\pi i}\oint_\Sigma \frac{p_l(t)}{(t-k-1)_{k+2}}\,\mathrm{d}t\right)n^l.$$

The integral vanishes if p_l is a polynomial of degree $\leq k$ since in that case the integrand is $O(t^{-2})$, and we can move the contour to infinity. This happens for $l \geq M-k+1$. For $l = M-k$, we have

$$p_{M-k}(t) = \binom{M+1}{k+1}t^{k+1} + O(t^k), \quad \text{as } t \to \infty,$$

and by a residue calculation at infinity we obtain

$$\frac{1}{2\pi i}\oint_\Sigma \frac{p_{M-k}(t)}{(t-k-1)_{k+2}}\,\mathrm{d}t = \binom{M+1}{k+1}.$$

Thus the second factor in the right-hand side of (4.10) is a polynomial of degree $M-k$ in n with leading coefficient $\binom{M+1}{k+1}$.

The other factor is a monic polynomial in n of degree $k(M+1)$. Thus $a_{k,n}$ has degree $k(M+1)+M-k = (k+1)M$ with leading coefficient $\binom{M+1}{k+1}$, as claimed in the lemma. □

Let's finally write down (4.10) for small values of M.

Case $M=1$. For $M=1$ we have a three term recurrence

$$xP_n(x) = P_{n+1}(x) + a_{0,n}P_n(x) + a_{1,n}P_{n-1}(x)$$

with

$$a_{0,n} = 2n + \nu_1 + 1, \quad a_{1,n} = n(n+\nu_1).$$

This is the recurrence relation for monic Laguerre polynomials with parameter ν_1.

Case $M=2$. For $M=2$ we have a four term recurrence

$$xP_n(x) = P_{n+1}(x) + a_{0,n}P_n(x) + a_{1,n}P_{n-1}(x) + a_{2,n}P_{n-2}(x)$$

with

$$\begin{aligned}
a_{0,n} &= 3n^2 + (3+2\nu_1+2\nu_2)n + (1+\nu_1+\nu_2+\nu_1\nu_2),\\
a_{1,n} &= n(n+\nu_1)(n+\nu_2)(3n+\nu_1+\nu_2),\\
a_{2,n} &= n(n-1)(n+\nu_1)(n+\nu_1-1)(n+\nu_2)(n+\nu_2-1).
\end{aligned}$$

This agrees with the recurrence coefficients given in [38, Theorem 4] if we use $\alpha = \nu_2$, $\nu = \nu_1 - \nu_2$.

5. Double Integral Representation and Large n Limit of K_n

In this section, we are concerned with the correlation kernel $K_n(x,y)$ defined in (1.10).

5.1. Double integral formula for K_n. The correlation kernel admits a double contour integral representation.

Proposition 5.1. *We have*

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} \mathrm{d}s \oint_\Sigma \mathrm{d}t \prod_{j=0}^{M} \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} \frac{x^t y^{-s-1}}{s-t}, \tag{5.1}$$

where Σ is a closed contour going around $0, 1, \ldots, n$ in the positive direction and $\operatorname{Re} t > -1/2$ *for* $t \in \Sigma$.

Proof. The correlation kernel (1.10) can be written as a double integral

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \mathrm{d}s \oint_\Sigma \mathrm{d}t \prod_{j=0}^{M} \frac{\Gamma(s+\nu_j)}{\Gamma(t+\nu_j+1)} \sum_{k=0}^{n-1} \frac{\Gamma(t-k)}{\Gamma(s-k)} x^t y^{-s}, \tag{5.2}$$

where we used the integral representation (3.8) for P_k and (3.6) for Q_k. From the functional equation $\Gamma(z+1) = z\Gamma(z)$, one can easily check that

$$(s-t-1)\frac{\Gamma(t-k)}{\Gamma(s-k)} = \frac{\Gamma(t-k)}{\Gamma(s-k-1)} - \frac{\Gamma(t-k+1)}{\Gamma(s-k)},$$

which means that there is a telescoping sum

$$(s-t-1)\sum_{k=0}^{n-1} \frac{\Gamma(t-k)}{\Gamma(s-k)} = \frac{\Gamma(t-n+1)}{\Gamma(s-n)} - \frac{\Gamma(t+1)}{\Gamma(s)}. \tag{5.3}$$

We are going to make sure that $s-t-1 \neq 0$ when $s \in c + i\mathbb{R}$ and $t \in \Sigma$. We do this by taking $c = 1/2$ and let Σ go around $0, 1, \ldots, n$ but with $\operatorname{Re} t > -1/2$ for $t \in \Sigma$. Then we insert (5.3) into (5.2) and get

$$\begin{aligned} K_n(x,y) &= \frac{1}{(2\pi i)^2} \int_{1/2-i\infty}^{1/2+i\infty} \mathrm{d}s \oint_\Sigma \mathrm{d}t \prod_{j=0}^{M} \frac{\Gamma(s+\nu_j)}{\Gamma(t+\nu_j+1)} \frac{\Gamma(t-n+1)}{\Gamma(s-n)} \frac{x^t y^{-s}}{s-t-1} \\ &\quad - \frac{1}{(2\pi i)^2} \int_{1/2-i\infty}^{1/2+i\infty} \mathrm{d}s \oint_\Sigma \mathrm{d}t \prod_{j=1}^{M} \frac{\Gamma(s+\nu_j)}{\Gamma(t+\nu_j+1)} \frac{x^t y^{-s}}{s-t-1}. \end{aligned}$$

The t-integral in the second double integral vanishes by Cauchy's theorem, since the integrand does not have any singularities inside Σ. We change $s \mapsto s+1$ in the first double integral and we obtain (5.1). □

We can rewrite the kernel in terms of Meijer G-functions

Corollary 5.2. *We have*

$$\begin{aligned} K_n(x,y) &= \int_0^1 G^{0,1}_{1,M+1}\left({n \atop -\nu_0, \ldots, -\nu_M} \middle| ux \right) G^{M,1}_{M+1,0}\left({-n \atop \nu_0, \ldots, \nu_M} \middle| uy \right) \mathrm{d}u \\ &= -\prod_{j=1}^{M} (n+\nu_j) \int_0^1 P_{n-1}(ux) Q_n(uy)\, \mathrm{d}u. \end{aligned} \tag{5.4}$$

Proof. Note that

$$\frac{x^t y^{-s-1}}{s-t} = -\int_0^1 (ux)^t (uy)^{-s-1}\,\mathrm{d}u. \tag{5.5}$$

The kernel (5.1) then is

$$K_n(x,y) = -\int_0^1 \left(\frac{1}{2\pi i} \oint_{\Sigma} \frac{\Gamma(t-n+1)}{\prod_{j=0}^M \Gamma(t+\nu_j+1)} (ux)^t\,\mathrm{d}t \right) \times \left(\frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\prod_{j=0}^M \Gamma(s+\nu_j+1)}{\Gamma(s-n+1)} (uy)^{-s-1}\,\mathrm{d}s \right) \mathrm{d}u. \tag{5.6}$$

By the definition (A.1) and change of variables $t \mapsto -t$, $s \mapsto s+1$, both factors in the u integral can be identified as Meijer G-functions and the first identity in (5.4) follows.

The second identity in (5.4) follows from (3.7) and (3.11). □

5.2. Microscopic limit of K_n at the hard edge. With the help of the contour integral representation (5.1) for K_n, we derive its scaling limit near the origin (hard edge). The limiting kernels are denoted by K_ν^M, where ν stands for the collection of parameters $\nu_1, \dots, \nu_M$.

Theorem 5.3. *With $\nu_1, \dots, \nu_M$ being fixed, we have*

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(\frac{x}{n}, \frac{y}{n}\right) = K_\nu^M(x,y),$$

uniformly for x, y in compact subsets of the positive real axis, where

$$\begin{aligned} K_\nu^M(x,y) &= \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} \mathrm{d}s \int_\Sigma \mathrm{d}t \prod_{j=0}^M \frac{\Gamma(s+\nu_j+1)}{\Gamma(t+\nu_j+1)} \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s-t} \\ &= \int_0^1 G^{1,0}_{0,M+1}\left({- \atop -\nu_0, -\nu_1, \dots, -\nu_M} \middle| ux \right) \\ &\quad \times G^{M,0}_{0,M+1}\left({- \atop \nu_1, \dots, \nu_M, \nu_0} \middle| uy \right) \mathrm{d}u, \end{aligned} \tag{5.7}$$

and where Σ is a contour starting from $+\infty$ in the upper half plane and returning to $+\infty$ in the lower half plane which encircles the positive real axis and $\mathrm{Re}\, t > -1/2$ *for $t \in \Sigma$; see Fig. 1 for an illustration.*

Proof. The reflection formula of the Gamma function says that

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t}, \tag{5.8}$$

which means that

$$\frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} = \frac{\Gamma(n-s)}{\Gamma(n-t)} \frac{\sin \pi s}{\sin \pi t}. \tag{5.9}$$

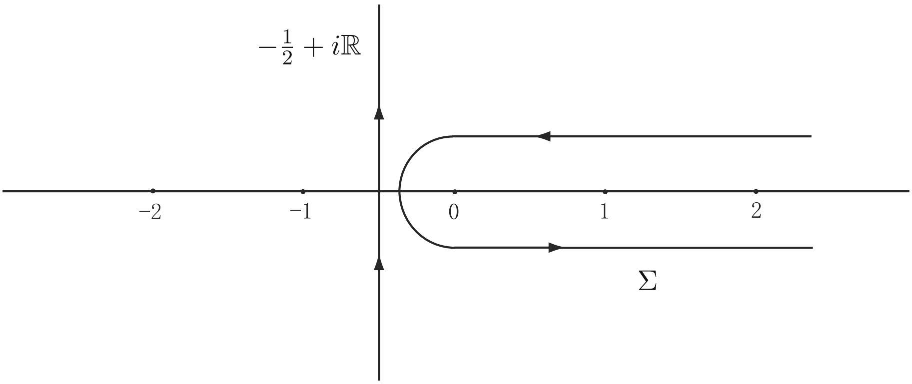

Fig. 1. The two contours of the double integral in (5.7)

As $n \to \infty$, we have the following ratio asymptotics of Gamma functions (cf. [31, formula 5.11.13])

$$\frac{\Gamma(n-s)}{\Gamma(n-t)} = n^{t-s}\left(1+O(n^{-1})\right), \tag{5.10}$$

which can be easily verified using Stirling's formula. By modifying the contour Σ in (5.1) from a closed contour around $0, 1, \ldots, n$ to a two sided unbounded contour as in Fig. 1 and applying (5.9) and (5.10), we readily obtain the first identity in (5.7), provided that we can take the limit inside of the integral.

The t-integral in (5.7) converges since $\Gamma(t+\nu_j+1)$ increases if we go to infinity along Σ and

$$|\sin \pi t| \geq |\sinh \pi \operatorname{Im} t|.$$

Also the s integral converges since

$$|\,\Gamma(x+iy)\,| \sim \sqrt{2\pi}|y|^{x-(1/2)}e^{-\pi|y|/2},$$

as $y \to \pm\infty$ for bounded real value of x; see [31, formula 5.11.9]. Therefore, $\Gamma(s+\nu_j+1)$ tends to 0 at an exponential rate if $|s| \to \infty$ with $\operatorname{Re} s = -1/2$. We can then indeed justify the interchange of limit and integrals for every M by the dominated convergence theorem.

By (5.8), we see

$$\frac{\sin \pi s}{\sin \pi t} = \frac{\Gamma(1+t)\Gamma(-t)}{\Gamma(1+s)\Gamma(-s)},$$

and using the trick (5.5) as in the proof of Proposition 5.1, we obtain

$$K_\nu^M(x,y) = -\int_0^1 \left(\frac{1}{2\pi i}\int_\Sigma \frac{\Gamma(-t)}{\prod_{j=1}^M \Gamma(t+\nu_j+1)}(ux)^t \,\mathrm{d}t\right) \times \left(\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\prod_{j=1}^M \Gamma(s+\nu_j+1)}{\Gamma(-s)}(uy)^{-s-1}\,\mathrm{d}s\right)\mathrm{d}u.$$

The change of variables $t \mapsto -t$ and $s \mapsto s+1$ takes both integrals into the form (A.1) of a Meijer G-function, and the second identity in (5.7) follows. □

It is known that the limiting mean distribution of the squared singular values for the products of M Ginibre matrices blows up with a rate $x^{-M/(M+1)}$ near the origin (see [16,33]). Extending the notion of universality at the hard edge, we are led to the expectation that the kernels described in Theorem 5.3 should appear in more general situations of the products of independent complex random matrices, and possibly in other models of random matrix theory.

5.3. Special case $M = 1$. Let's now take a closer look at the limiting kernels $K_\nu^M(x, y)$ for special values of M. If $M = 1$ and $\nu_1 = \nu$, one has since $\nu_0 = 0$ (we drop the superscript $M = 1$)

$$K_\nu(x, y) = \int_0^1 G_{0,2}^{1,0}\left({- \atop 0, -\nu} \middle| ux \right) G_{0,2}^{1,0}\left({- \atop \nu, 0} \middle| uy \right) \mathrm{d}u.$$

Since

$$G_{0,2}^{1,0}\left({- \atop 0, -\nu} \middle| ux \right) = (ux)^{-\nu/2} J_\nu(2\sqrt{ux}),$$
$$G_{0,2}^{1,0}\left({- \atop \nu, 0} \middle| uy \right) = (uy)^{\nu/2} J_\nu(2\sqrt{uy}),$$

where J_ν denotes the Bessel function of the first kind of order ν (see [31, formula 10.9.23]), it then follows that

$$\begin{aligned} K_\nu(x, y) &= \left(\frac{y}{x}\right)^{\nu/2} \int_0^1 J_\nu(2\sqrt{ux}) J_\nu(2\sqrt{uy})\, \mathrm{d}u \\ &= 4\left(\frac{y}{x}\right)^{\nu/2} K^{\mathrm{Bes},\nu}(4x, 4y), \end{aligned}$$

where

$$K^{\mathrm{Bes},\nu}(x, y) = \frac{J_\nu(\sqrt{x})\sqrt{y} J_\nu'(\sqrt{y}) - \sqrt{x} J_\nu'(\sqrt{x}) J_\nu(\sqrt{y})}{2(x - y)}, \quad \nu > -1,$$

is the Bessel kernel of order ν that appears as the scaling limit of the Laguerre or Jacobi unitary ensembles at the hard edge [35], as expected.

5.4. Special case $M = 2$. If $M = 2$, one has from (5.7) that (we drop the superscript $M = 2$)

$$K_{\nu_1,\nu_2}(x, y) = \int_0^1 G_{0,3}^{1,0}\left({- \atop 0, -\nu_1, -\nu_2} \middle| ux \right) G_{0,3}^{2,0}\left({- \atop \nu_1, \nu_2, 0} \middle| uy \right) \mathrm{d}u. \qquad (5.11)$$

It is interesting that these kernels appeared earlier in another random matrix model, namely in the Cauchy two-matrix model with linear potentials, see [10,12].

The Cauchy two matrix model is defined by the probability measure

$$\frac{1}{\mathcal{Z}_n} \frac{\det(M_1)^a \det(M_2)^b e^{-\mathrm{Tr}(V_1(M_1)+V_2(M_2))}}{\det(M_1 + M_2)^n} \mathrm{d}M_1\, \mathrm{d}M_2, \quad a, b > -1,\ a + b > -1,$$

defined on the space of two $n \times n$ positive semidefinite Hermitian matrices M_1 and M_2, with two scalar potentials V_1, V_2 defined on the positive real axis that grow sufficiently fast as $x \to +\infty$.

The eigenvalues of M_1 and M_2 form a determinantal point process with a correlation kernel which is defined in terms of the Cauchy biorthogonal polynomials [11] $p_l(x)$ and $q_m(y)$ satisfying

$$\int_0^\infty \int_0^\infty \frac{x^a y^b e^{-V_1(x)-V_2(y)}}{x+y} p_l(x) q_m(y)\, \mathrm{d}x\, \mathrm{d}y = \delta_{l,m}.$$

For the linear case $V_1(x) = x$ and $V_2(y) = y$, it was established in [12, Theorem 2.2] that the correlation kernel for the eigenvalues of M_1 has a scaling limit at the origin given by

$$\int_0^1 G^{1,0}_{0,3}\left({- \atop a, 0, -b} \middle| ux \right) G^{2,0}_{0,3}\left({- \atop b, 0, -a} \middle| uy \right) \mathrm{d}u. \tag{5.12}$$

This is slightly different from (5.11), since we cannot freely permute the parameters $\nu_1, \nu_2, 0$ in (5.11).

However, from (A.2) we see that

$$G^{1,0}_{0,3}\left({- \atop a, 0, -b} \middle| ux \right) = (ux)^a G^{1,0}_{0,3}\left({- \atop 0, -a, -b-a} \middle| ux \right),$$
$$G^{2,0}_{0,3}\left({- \atop b, 0, -a} \middle| uy \right) = (uy)^{-a} G^{2,0}_{0,3}\left({- \atop b+a, a, 0} \middle| uy \right).$$

Hence,

$$\int_0^1 G^{1,0}_{0,3}\left({- \atop a, 0, -b} \middle| ux \right) G^{2,0}_{0,3}\left({- \atop b, 0, -a} \middle| uy \right) \mathrm{d}u = \left(\frac{x}{y}\right)^a K_{a+b,a}(x,y).$$

The prefactor $\left(\frac{x}{y}\right)^a$ is irrelevant in a kernel for a determinantal point process as it does not change the determinants that give the point correlations. Therefore we see that the limiting kernels (5.12) in the Cauchy two matrix models are the same kernels as the limiting kernels for squared singular values of products of two complex Ginibre matrices. This supports our conjecture that the kernels (5.7) have a universal character and appear in a wider context.

5.5. Integrable form of the limiting kernels. An integral operator with kernel $K(x, y)$ is called integrable if

$$K(x,y) = \frac{\sum_{i=1}^n f_i(x) g_i(y)}{x-y}, \quad \text{with} \quad \sum_{i=1}^n f_i(x) g_i(x) = 0,$$

for some $n \in \{2, 3, \ldots\}$, and certain functions f_i and g_i. Integral operators of this form benefit from the fact that there is a Riemann–Hilbert setting for the study of the associated resolvent kernels, determinants, etc.; see [27]. The kernels of standard universality classes (sine, Airy, Bessel) encountered in random matrix theory all belong to the class of integrable operators. The representation (2.5) of K_n in terms of the solution of a Riemann–Hilbert problem is also of the integrable form.

We conclude this paper by giving the integrable form of the limiting kernels derived in Theorem 5.3. Our argument follows [12, Section 5], where this was shown for the case $M = 2$.

Proposition 5.4. *With $K_\nu^M(x, y)$ defined in (5.7), we have*

$$K_\nu^M(x, y) = \frac{\mathcal{B}\left(G^{1,0}_{0,M+1}\left({- \atop -\nu_0, -\nu_1, \ldots, -\nu_M}\middle| x\right), G^{M,0}_{0,M+1}\left({- \atop \nu_1, \ldots, \nu_M, \nu_0}\middle| y\right)\right)}{x - y}, \tag{5.13}$$

where $\mathcal{B}(\cdot, \cdot)$ is a bilinear operator defined by

$$\mathcal{B}(f(x), g(y)) = (-1)^{M+1} \sum_{j=0}^{M} (-1)^j (\Delta_x)^j f(x) \left(\sum_{i=0}^{M-j} a_{i+j} (\Delta_y)^i g(y)\right), \tag{5.14}$$

with $\Delta_x = x\frac{\mathrm{d}}{\mathrm{d}x}$ and $\Delta_y = y\frac{\mathrm{d}}{\mathrm{d}y}$. The constants a_i in (5.14) are determined by

$$\prod_{i=1}^{M}(x - \nu_i) = \sum_{i=0}^{M} a_i x^i, \tag{5.15}$$

that is,

$$a_i = (-1)^i e_{M-i}(\nu_1, \ldots, \nu_M) \tag{5.16}$$

with $e_i(\nu_1, \ldots, \nu_M)$ being the elementary symmetric polynomial.

The bilinear operator $\mathcal{B}$ is called a point-split bilinear concomitant in [12].

Proof. We set

$$f(x) = G^{1,0}_{0,M+1}\left({- \atop -\nu_0, -\nu_1, \ldots, -\nu_M}\middle| x\right), \tag{5.17}$$

$$g(y) = G^{M,0}_{0,M+1}\left({- \atop \nu_1, \ldots, \nu_M, \nu_0}\middle| y\right). \tag{5.18}$$

By (5.7), our aim is then to evaluate the integral

$$K_\nu^M(x, y) = \int_0^1 f(tx)g(ty)\,\mathrm{d}t. \tag{5.19}$$

Note that the Meijer-G function satisfies the differential equation (A.3). For f and g given by (5.17) and (5.18), this implies that for every t,

$$g(ty) \prod_{j=0}^{M}(\Delta_x + \nu_j) f(tx) = -txf(tx)g(ty), \tag{5.20}$$

$$f(tx) \prod_{j=0}^{M}(\Delta_y - \nu_j) g(ty) = (-1)^M tyf(tx)g(ty). \tag{5.21}$$

If M is odd we subtract these two identities, while if M is even we add them together. Since the arguments in both cases are similar, we restrict to the case where M is odd.

Subtracting (5.20) from (5.21) we obtain

$$\begin{aligned}
&(x-y)f(tx)g(ty)\\
&\quad=\frac{1}{t}\left(f(tx)\prod_{j=0}^{M}(\Delta_y-\nu_j)g(ty)-g(ty)\prod_{j=0}^{M}(\Delta_x+\nu_j)f(tx)\right)\\
&\quad=\frac{1}{t}\sum_{i=0}^{M}a_i\left(f(tx)(\Delta_y)^{i+1}g(ty)+(-1)^i g(ty)(\Delta_x)^{i+1}f(tx)\right),
\end{aligned}\tag{5.22}$$

where the constants a_i are defined in (5.15) and (5.16). We next observe that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\sum_{j=0}^{i}(-1)^j\,(\Delta_x)^j\,f(tx)\,\left(\Delta_y\right)^{i-j}\,g(ty)\right)\\
&\quad=\frac{1}{t}\left(f(tx)(\Delta_y)^{i+1}g(ty)+(-1)^i g(ty)(\Delta_x)^{i+1}f(tx)\right),
\end{aligned}$$

which by (5.14) and (5.22) implies that

$$(x-y)f(tx)g(ty)=\frac{\partial}{\partial t}\mathcal{B}(f(tx),g(ty)).\tag{5.23}$$

Using (5.23) in (5.19) we find

$$(x-y)K_\nu^M(x,y)=\mathcal{B}(f(x),g(y))-\lim_{t\to 0+}\mathcal{B}(f(tx),g(ty)).$$

It thus remains to show that

$$\lim_{t\to 0+}\mathcal{B}(f(tx),g(ty))=0,\tag{5.24}$$

and to do this we need to understand the behavior of f and g at the origin.

First of all, we have by [31, formula 16.18.1]) and (5.17) that f is a hypergeometric function

$$f(x)=\frac{1}{\prod_{j=1}^{M}\Gamma(1-\nu_j)}\,{}_0F_M\left(\begin{matrix}-\\1-\nu_1,\ldots,1-\nu_M\end{matrix}\middle|-x\right),\tag{5.25}$$

so that f is analytic at the origin. Next by (5.18), the definition of (A.1), and the properties of the Mellin transform (see e.g. [21]), we find

$$\int_0^\infty(\Delta_y)^i g(y)y^{s-1}\,\mathrm{d}y=(-s)^i\frac{\prod_{j=1}^{M}\Gamma(s+\nu_j)}{\Gamma(1-s)}.$$

Then it follows in the same way as we obtained (3.3) that

$$(\Delta y)^i g(y)=O(y^\alpha(\log y)^{r-1})\quad\text{as } y\to 0+,\tag{5.26}$$

with $\alpha=\min(\nu_1,\ldots,\nu_M)>-1$ and $r=\#\{j\mid\nu_j=\alpha\}$.

Now we look at the $j = 0$ term in (5.14) which is

$$f(x)\sum_{i=0}^{M} a_i(\Delta_y)^i g(y) = f(x)\prod_{i=0}^{M}(\Delta_y - \nu_i)g(y) = -f(x)yg(y),$$

where in the last step we used (5.21) with $t = 1$. Replacing $x \mapsto tx$, $y \mapsto ty$, we find by (5.25) and (5.26) that the limit is 0 as $t \to 0+$. For $j \geq 1$ we have

$$(\Delta_x)^j f(x) = O(x) \quad \text{as } x \to 0,$$

and then it follows from (5.26) that the terms in (5.14) with $j \geq 1$ are all $O(x)O(y^\alpha(\log y)^{r-1})$ as $x, y \to 0+$. Replacing $x \mapsto tx$, $y \mapsto ty$, we then find that these terms tend to 0 as well as $t \to 0+$. This proves (5.24) and it completes the proof of Proposition 5.4. $\square$

Acknowledgements. We thank Gernot Akemann, Jesper R. Ipsen and Mario Kieburg for interesting discussions and for providing us with an early copy of the preprint [4]. The first author is supported by KU Leuven Research Grant OT/12/073, the Belgian Interuniversity Attraction Pole P07/18, FWO Flanders projects G.0641.11 and G.0934.13, and by Grant No. MTM2011-28952-C02 of the Spanish Ministry of Science and Innovation. The second author was a Postdoctoral Fellow of the Fund for Scientific Research - Flanders (Belgium), and is also supported by The Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning (No. SHH1411007) and by Grant SGST 12DZ 2272800 from Fudan University.

A. The Meijer G-function

We give a brief introduction to the Meijer G-function in this appendix. By definition, the Meijer G-function is given by the following contour integral in the complex plane:

$$\begin{aligned} G^{m,n}_{p,q}\left({a_1,\ldots,a_p \atop b_1,\ldots,b_q}\middle| z\right) &= G^{m,n}_{p,q}\left({\mathbf{a_p} \atop \mathbf{b_q}}\middle| z\right) \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\prod_{j=1}^{m}\Gamma(b_j+u)\prod_{j=1}^{n}\Gamma(1-a_j-u)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-u)\prod_{j=n+1}^{p}\Gamma(a_j+u)} z^{-u}\,\mathrm{d}u, \end{aligned} \tag{A.1}$$

where Γ denotes the usual gamma function and the branch cut of z^{-u} is taken along the negative real axis. It is also assumed that

- $0 \leq m \leq q$ and $0 \leq n \leq p$, where m, n, p and q are integer numbers;
- The real or complex parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ satisfy the conditions

  $$a_k - b_j \neq 1, 2, 3, \ldots, \quad \text{for } k = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m,$$

  i.e., none of the poles of $\Gamma(b_j + u)$, $j = 1, 2, \ldots, m$ coincides with any poles of $\Gamma(1 - a_k - u)$, $k = 1, 2, \ldots, n$.

The contour γ is chosen in such a way that all the poles of $\Gamma(b_j + u)$, $j = 1, \ldots, m$ are on the left of the path, while all the poles of $\Gamma(1 - a_k - u)$, $k = 1, \ldots, n$ are on the right, which is usually taken to go from $-i\infty$ to $i\infty$. In particular, it can be a loop starting and ending at $+\infty$ if $p > q$, or a loop beginning and ending at $-\infty$ if $p < q$. Most of the known special functions can be viewed as special cases of the Meijer G-functions, we refer to [30,31] for more details. We end this appendix with several formulas used in this paper.

- From the definition (A.1), it is easily seen that

$$z^{\rho} G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a_p}\\ \mathbf{b_q}\end{matrix}\middle| z\right) = G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a_p}+\rho\\ \mathbf{b_q}+\rho\end{matrix}\middle| z\right). \tag{A.2}$$

- The Meijer G-function $G^{m,n}_{p,q}\left(\begin{smallmatrix}\mathbf{a_p}\\ \mathbf{b_q}\end{smallmatrix}\middle| z\right)$ satisfies the following linear differential equation of order $\max(p, q)$:

$$\left[(-1)^{p-m-n} z \prod_{j=1}^{p}\left(z\frac{\mathrm{d}}{\mathrm{d}z} - a_j + 1\right) - \prod_{j=1}^{q}\left(z\frac{\mathrm{d}}{\mathrm{d}z} - b_j\right)\right] G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a_p}\\ \mathbf{b_q}\end{matrix}\middle| z\right) = 0; \tag{A.3}$$

see [31, formula 16.21.1].

References

1. Akemann, G., Baik, J., Di Francesco, P. (eds.): The Oxford Handbook of Random Matrix Theory. Oxford University Press, Oxford (2011)
2. Akemann, G., Burda, Z.: Universal microscopic correlation functions for products of independent Ginibre matrices. J. Phys. A Math. Theor. **45**, 465201 (2012)
3. Akemann, G., Burda, Z., Kieburg, M., Nagao, T.: Universal microscopic correlation functions for products of truncated unitary matrices, preprint. arXiv:1310.6395
4. Akemann, G., Ipsen, J.R., Kieburg, M.: Products of rectangular random matrices: singular values and progressive scattering. Phys. Rev. E **88**, 052118 (2013)
5. Akemann, G., Kieburg, M., Wei, L.: Singular value correlation functions for products of Wishart random matrices. J. Phys. A Math. Theor. **46**, 275205 (2013)
6. Akemann, G., Strahov, E.: Hole probabilities and overcrowding estimates for products of complex Gaussian matrices. J. Stat. Phys. **151**, 987–1003 (2013)
7. Anderson, G.W., Guionnet, A., Zeitouni, O.: An Introduction to Random Matrices. Cambridge University Press, Cambridge (2009)
8. Banica, T., Belinschi, S., Capitaine, M., Collins, B.: Free Bessel laws. Can. J. Math. **63**, 3–37 (2011)
9. Beals, R., Szmigielski, J.: Meijer G-functions: a gentle introduction. Not. Am. Math. Soc. **60**, 866–872 (2013)
10. Bertola, M., Gekhtman, M., Szmigielski, J.: The Cauchy two-matrix model. Comm. Math. Phys. **287**, 983–1014 (2009)
11. Bertola, M., Gekhtman, M., Szmigielski, J.: Cauchy biorthogonal polynomials. J. Approx. Theory **162**, 832–867 (2010)
12. Bertola, M., Gekhtman, M., Szmigielski, J.: Cauchy–Laguerre two-matrix model and the Meijer-G random point field. Commun. Math. Phys. **326**(1), 111–144 (2014)
13. Borodin, A.: Biorthogonal ensembles. Nucl. Phys. B **536**, 704–732 (1999)
14. Bougerol, P., Lacroix, J.: Products of random matrices with applications to Schrödinger operators. In: Huber, P., Rosenblatt, M. (eds) Progress in Probability and Statistics, vol. 8. Birkhäuser, Boston (1985)
15. Burda, Z., Janik, R.A., Waclaw, B.: Spectrum of the product of independent random Gaussian matrices. Phys. Rev. E **81**, 041132 (2010)
16. Burda, Z., Jarosz, A., Livan, G., Nowak, M.A., Swiech, A.: Eigenvalues and singular values of products of rectangular Gaussian random matrices. Phys. Rev. E **82**, 061114 (2010) (the extended version Acta Phys. Polon. B 42, 939–985 (2011))
17. Coussement, E., Coussement, J., Van Assche, W.: Asymptotic zero distribution for a class of multiple orthogonal polynomials. Trans. Am. Math. Soc. **360**, 5571–5588 (2008)
18. Crisanti, A., Paladin, G., Vulpiani, A.: Products of Random Matrices in Statistical Physics. Springer Series in Solid-State Sciences, vol. 104. Springer, Heidelberg (1993)
19. Daems, E., Kuijlaars, A.B.J.: A Christoffel–Darboux formula for multiple orthogonal polynomials. J. Approx. Theory **130**, 188–200 (2004)

20. Deift, P.: Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert approach. Courant Lecture Notes in Mathematics, vol. 3. American Mathematical Society, Providence (1999)
21. Flajolet, P., Gourdon, X., Dumas, P.: Mellin transforms and asymptotics: harmonic sums. Theor. Comput. Sci. **144**, 3–58 (1995)
22. Forrester, P.J.: Log-Gases and Random Matrices. Princeton University Press, Princeton (2010)
23. Furstenberg, H., Kesten, H.: Products of random matrices. Ann. Math. Stat. **31**, 457–469 (1960)
24. Götze, F., Tikhomirov, A.: On the asymptotic spectrum of products of independent random matrices, preprint. arXiv:1012.2710
25. Ipsen, J.R.: Products of independent quaternion Ginibre matrices and their correlation functions. J. Phys. A Math. Theor. **46**, 265201 (2013)
26. Ismail, M.E.H.: Classical and Quantum Orthogonal Polynomials in One Variable. Encyclopedia of Mathematics and its Applications, vol. 98. Cambridge University Press, London (2005)
27. Its, A.R., Izergin, A.G., Korepin, V.E., Slavnov, N.A.: Differential equations for quantum correlation functions. Int. J. Mod. Phys. B **4**, 1003–1037 (1990)
28. Kuijlaars, A.B.J.: Multiple orthogonal polynomial ensembles. In: Arvesú, J., Marcellán, F., Martínez-Finkelshtein, A. (eds.) Recent Trends in Orthogonal Polynomials and Approximation Theory. Contemporary Mathematics, vol. 507, pp. 155–176 (2010)
29. Kuijlaars, A.B.J.: Multiple orthogonal polynomials in random matrix theory. In: Bhatia, R. (ed.) Proceedings of the International Congress of Mathematicians, vol. III, Hyderabad, India, pp. 1417–1432 (2010)
30. Luke, Y.L.: The Special Functions and their Approximations. Academic Press, New York (1969)
31. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.): NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge. (2010) (Print companion to [DLMF])
32. O'Rourke, S., Soshnikov, A.: Products of independent non-Hermitian random matrices. Electron. J. Probab. **81**, 2219–2245 (2011)
33. Penson, K.A., Życzkowski, K.: Product of Ginibre matrices: Fuss-Catalan and Raney distributions. Phys. Rev. E **83**, 061118 (2011)
34. Tao, T.: Topics in Random Matrix Theory. Graduate Studies in Mathematics, vol. 132. American Mathematical Society, Providence (2012)
35. Tracy, C., Widom, H.: Level-spacing distributions and the Bessel kernel. Commun. Math. Phys. **161**, 289–309 (1994)
36. Tulino, A.M., Verdú, S.: Random Matrix Theory and Wireless Communications. Foundations and Trends in Communications and Information Theory, vol. 1, pp. 1–182. Now Publisher, Hanover (2004)
37. Van Assche, W., Geronimo, J.S., Kuijlaars, A.B.J.: Riemann–Hilbert problems for multiple orthogonal polynomials. In: Bustoz J. et al. (eds.) Special Functions 2000: Current Perspectives and Future Directions. Kluwer, Dordrecht, pp. 23–59 (2001)
38. Van Assche, W., Yakubovich, S.B.: Multiple orthogonal polynomials associated with Macdonald functions. Integral Transforms Spec. Funct. **9**, 229–244 (2000)
39. Zhang, L.: A note on the limiting mean distribution of singular values for products of two Wishart random matrices. J. Math. Phys. **54**, 083303, 8 pp. (2013)
40. Zhang, L., Román, P.: The asymptotic zero distribution of multiple orthogonal polynomials associated with Macdonald functions. J. Approx. Theory **163**, 143–162 (2011)

Communicated by P. Deift