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A Cohomology Theory of Grading-Restricted Vertex Algebras

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Abstract: We introduce a cohomology theory of grading-restricted vertex algebras. To construct the *correct* cohomologies, we consider linear maps from tensor powers of a grading-restricted vertex algebra to "rational functions valued in the algebraic completion of a module for the algebra," instead of linear maps from tensor powers of the algebra to a module for the algebra. One subtle complication arising from such functions is that we have to carefully address the issue of convergence when we compose these linear maps with vertex operators. In particular, for each $n \in \mathbb{N}$, we have an inverse system $\{H^n_m(V, W)\}_{m\in\mathbb{Z}_+}$ of nth cohomologies and an additional nth cohomology $H^n_\infty(V, W)$ of a grading-restricted vertex algebra V with coefficients in a V-module W such that $H^n_\infty(V, W)$ is isomorphic to the inverse limit of the inverse system $\{H^n_m(V, W)\}_{m\in\mathbb{Z}_+}$. In the case of $n = 2$, there is an additional second cohomology denoted by $H^2_{\frac{1}{2}}(V, W)$ which will be shown in a sequel to the present paper to correspond to what we call square-zero extensions of V and to first order deformations of V when $W = V$.

1. Introduction

Vertex (operator) algebras arose naturally in both mathematics and physics (see [BPZ, B1, FLM]) and are analogous to both Lie algebras and commutative associative algebras. In the studies of various algebraic structures, including in particular Lie algebras, associative algebras, commutative associative algebras, and their representations, the corresponding cohomology theories, such as Chevalley–Eilenberg cohomology of Lie algebras [CE], Hochschild cohomology of associative algebras [Ho] and Harrison or André–Quillen cohomology of commutative associative algebras [Ha, A, Q], play important roles. See, for example, the book [W] for an excellent introduction to these theories (except for the Harrison cohomology). These cohomologies describe naturally certain extensions of these algebras, extensions of their modules and also deformations of these algebras. Moreover, the powerful tool of homological algebra developed in the last sixty years has been used to obtain many old and new results on these algebras and their modules. Though much progress has been made in the theory of vertex (operator) algebras, especially in the case of simple vertex operator algebras satisfying certain finiteness and reductivity conditions, a *correct* cohomology theory of vertex (operator) algebras is urgently needed in order to have a better understanding of the structures of vertex (operator) algebras and their modules and to use the powerful tool of homological algebra.

In [KV], applying the general theory for algebras over operads developed by Ginzburg and Kapranov [GK], Kimura and Voronov introduced a cohomology theory of algebras over the operad of the moduli space of configurations of disjoint and ordered biholomorphic embeddings of the unit disk into the Riemann sphere. Motivated by the operadic and geometric formulation of vertex operator algebras by the author [Hu1, Hu2] and by Lepowsky and the author [HL1, HL2], Kimura and Voronov proposed in [KV] that their cohomology theory of algebras over the moduli space above also gives the cohomology theory of vertex operator algebras. Unfortunately, their proposal was based on the assumption that vertex operator algebras are in particular algebras over the operad of the moduli space mentioned above, while this assumption holds only for vertex operator algebras obtained from commutative associative algebras. In fact, given a vertex operator algebra

$$\left(V = \coprod_{n\in\mathbb{Z}} V_{(n)}, Y, \mathbf{1}, \omega\right)$$

that is not obtained from a commutative associative algebra, for $u, v \in V$ and a, b, $z \in \mathbb{C}^{\times}$,

$$Y(a^{L(0)}u, z)b^{L(0)}v$$

is in general not an element of V, even when a, b and z are chosen such that they give the configuration of three disjoint and ordered biholomorphic embeddings of the unit disk into the Riemann sphere. Instead, it is an element of the algebraic completion

$$\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$$

of V. So for $z \in \mathbb{C}^{\times}$, the map

$$Y(a^{L(0)}\cdot, z)b^{L(0)}\cdot : V \otimes V \to \overline{V}$$

in general does not belong to the endomorphism operad of the vector space V. Therefore, the vertex operator algebra V in general does not give an algebra over the operad of the moduli space mentioned above. In particular, the cohomology theory introduced by Kimura and Voronov cannot be used to give a cohomology theory of vertex operator algebras because of this subtle but crucial feature of the geometric and operadic formulation of vertex operator algebra. Moreover, vertex operator algebras satisfy an additional meromorphicity condition which one must take into account in any cohomology theory of vertex operator algebras.

In Section 11 of [B2], Borcherds also proposed a cohomology theory for general vertex algebras by using his categorical formulation of vertex algebra and an analogy with the Hochschild homology of associative algebras. However, the subtle details of this cohomology theory were not carried out and the basic properties that a cohomology theory must have were not discussed. More importantly, we are interested only in what we call grading-restricted vertex algebras and for these vertex algebras, a cohomology theory for general vertex algebras cannot be the *correct* one. Here is the reason: The notion of vertex algebra is too general to give properties strong enough for a good representation theory. The class of vertex algebras for which many substantial results in representation theory have been obtained is that of vertex operator algebras. Since we want to allow the deformations of the representation structures of the Virasoro algebra, especially the deformations of the central charges, we are interested in the slightly more general class of grading-restricted vertex algebras, for which conformal elements are not specified but $\mathbb{Z}$-gradings are still given and the grading-restriction condition is satisfied. For grading-restricted vertex algebras, a cohomology theory for general vertex algebras cannot be the *correct* one, because, for example, starting from a grading-restricted vertex algebra, the deformations corresponding to such a general cohomology theory in general will not give such a vertex algebra again.

In the present paper, we introduce a cohomology theory of grading-restricted vertex algebras (including vertex operator algebras). To overcome the difficulties in the proposal in [KV] mentioned above, our main new idea is to consider, instead of linear maps from the tensor powers of the vertex algebra to a module for the algebra, linear maps from the tensor powers of the vertex algebra to suitable spaces of "rational functions valued in the algebraic completion of the V-module" such that they are "composable" with m vertex operators in a natural sense and satisfy certain other natural properties. These linear maps form a chain complex but it is still not a correct chain complex for the grading-restricted vertex algebra because the commutativity property for the vertex algebra has not been taken into consideration. The correct chain complex for our cohomology is a subcomplex of this complex obtained by using shuffles in analogy with the construction of the Harrison cochain complex of a commutative associative algebra from its Hochschild cochain complex. One subtle complication arising from the functions mentioned above is that we have to carefully address the issue of convergence when we compose these linear maps with vertex operators. In particular, for each $n \in \mathbb{N}$, we have an inverse system $\{H^n_m(V, W)\}_{m\in\mathbb{Z}_+}$ of nth cohomologies and an additional nth cohomology $H^n_\infty(V, W)$ of a grading-restricted vertex algebra V with coefficients in a grading-restricted generalized V-module W such that $H^n_\infty(V, W)$ is isomorphic to the inverse limit of the inverse system $\{H^n_m(V, W)\}_{m\in\mathbb{Z}_+}$. In the case of $n = 2$, there is an additional second cohomology denoted by $H^2_{\frac{1}{2}}(V, W)$ which will be shown in a sequel [Hu3] to the present paper to correspond to what we call square-zero extensions of V and to first order deformations of V when $W = V$.

The ideas and constructions in the present paper can also be applied to grading-restricted open-string vertex algebras (see [HK1]) and grading-restricted full field algebras (see [HK2]) to introduce and study cohomologies for these algebras. We shall present these cohomology theories in future publications.

Note that for open-string vertex algebras and full field algebras, we have to work with complex variables, not formal variables. In particular, these algebras are defined over only the field of complex numbers. In this paper, we present our cohomology theory of grading-restricted vertex algebras only over the field of complex numbers so that it will be easy for us to generalize the definitions and results given in the present paper to these algebras. However, the cohomologies introduced in the present paper can be defined and studied for grading-restricted vertex algebras over an arbitrary field $\mathbb{F}$ of characteristic 0. In fact, to define and study these cohomologies for such grading-restricted vertex algebras over $\mathbb{F}$, we need only replace rational functions with only possible poles at $z_i = z_j$ for $i \neq j$ by the localization of the polynomial ring $\mathbb{F}[z_1, \ldots, z_n]$ by the first order polynomials $z_i - z_j$ for $i \neq j$ and replace series absolutely convergent to such rational functions in certain regions by series which are expansions of the elements of the localization corresponding to the expansions of such rational functions in the regions. See [FHL] for discussions on formal rational functions and their expansions over such a field $\mathbb{F}$.

In a sequel [Hu3] to the present paper, we shall show that for any $m \in \mathbb{Z}_+$, the first cohomology $H^1_m(V, W)$ of a grading-restricted vertex algebra V with coefficients in a grading-restricted generalized V-module W is linearly isomorphic to the space of derivations from V to W. We shall also show that the second cohomology $H^2_{\frac{1}{2}}(V, W)$ of V with coefficients in W corresponds bijectively to the set of equivalence classes of square-zero extensions of V by W and the second cohomology $H^2_{\frac{1}{2}}(V, V)$ of V with coefficients in V corresponds bijectively to the set of equivalence classes of first order deformations of V.

At this moment, the author still does not have any vanishing theorem or duality theorem for the cohomologies introduced in the present paper. It is not even clear whether $H^n_\infty(V, W)$ vanishes when n is large. These are important research topics for the future development and applications of this cohomology theory.

This paper is organized as follows: In Sect. 2, we recall the notions of grading-restricted vertex algebra and grading-restricted generalized module and also some useful results. In Sect. 3, we introduce and study $\overline{W}$-valued rational functions for a grading-restricted generalized module for a grading-restricted vertex algebra. These functions are crucial to our cohomology theory. We present our cohomology theory in Sect. 4.

The cohomologies introduced in the present paper were first presented in a talk by the author at the Cao Xi-Hua Algebra Forum at East China Normal University on June 1, 2010.

2. Grading-Restricted Vertex Algebras and Modules

In this section, we give the definitions of grading-restricted vertex algebra and grading-restricted generalized module and discuss their basic properties. As is mentioned in the introduction, we shall work only over the field $\mathbb{C}$ of complex numbers in this paper. In particular, all vector spaces are over $\mathbb{C}$.

A large part of the material in this section is from [FHL] but we shall use the duality properties instead of the Jacobi identity in this paper. Below we recall the definition of grading-restricted vertex algebra using the duality properties as the main axiom.

By a rational function of $z_1, \dots, z_n$, we mean a function of $z_1, \dots, z_n$ of the form

$$f(z_1, \dots, z_n) = \frac{P(z_1, \dots, z_n)}{Q(z_1, \dots, z_n)},$$

where $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ are polynomials in $z_1, \dots, z_n$. If the polynomials $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ have no common factors, then for a linear factor $g(z_1, \dots, z_n)$ of $Q(z_1, \dots, z_n)$, we say that $f(z_1, \dots, z_n)$ *has poles at the set of zeros of* $g(z_1, \dots, z_n)$ and the maximal power of $g(z_1, \dots, z_n)$ in $Q(z_1, \dots, z_n)$ is called the *order* of these poles. By a *rational function with the only possible poles at a set of points in* $\mathbb{C}^n$, we mean a rational function of the form above such that $P(z_1, \dots, z_n)$ and $Q(z_1, \dots, z_n)$ have no common factors, $Q(z_1, \dots, z_n)$ is a product of linear factors whose zeros are contained in that set of points in $\mathbb{C}^n$.

In the following definitions and in the rest of this paper, $x, x_1, x_2, \dots$ are formal commuting variables and $z, z_1, z_2, \dots$ are complex numbers or complex variables.

Definition 2.1. *A grading-restricted vertex algebra is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ equipped with a* vertex operator map

$$\begin{aligned} Y : V \otimes V &\to V[[x, x^{-1}]], \\ u \otimes v &\mapsto Y_V(u, x)v = \sum_{n\in\mathbb{Z}} (Y_V)_n(u)vx^{-n-1}, \end{aligned}$$

a vacuum $\mathbf{1} \in V_{(0)}$ *satisfying the following conditions:*

1. Grading restriction condition*: For $n \in \mathbb{Z}$,* $\dim V_{(n)} < \infty$ *and when n is sufficiently negative, $V_{(n)} = 0$.*
2. Lower-truncation condition for vertex operators*: For $u, v \in V$, $Y_V(u, x)v$ contain only finitely many negative power terms, that is, $Y_V(u, x)v \in V((x))$ (the space of formal Laurent series in x with coefficients in V and with finitely many negative power terms).*
3. Identity property*: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$.*
4. Creation property*: For $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and* $\lim_{x\to 0} Y_V(u, x)\mathbf{1} = u$.
5. Duality*: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$, the series*

$$\begin{gathered} \langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle, \\ \langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v\rangle, \\ \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \end{gathered}$$

   are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.
6. $L(0)$-bracket formula*: Let $L_V(0) : V \to V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then*

$$[L_V(0), Y_V(v, x)] = Y_V(L_V(0)v, x) + x\frac{d}{dx}Y_V(v, x)$$

   for $v \in V$.
7. $L(-1)$-derivative property*: Let $L_V(-1) : V \to V$ be the operator given by*

$$L_V(-1)v = \mathrm{Res}_x x^{-2} Y_V(v, x)\mathbf{1} = Y_{-2}(v)\mathbf{1}$$

   for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx}Y_V(u, x) = Y_V(L_V(-1)u, x) = [L_V(-1), Y_V(u, x)].$$

Definition 2.2. *A* grading-restricted generalized V-module *is a vector space W equipped with a vertex operator map*

$$\begin{aligned} Y_W : V \otimes W &\to W[[x, x^{-1}]], \\ u \otimes w &\mapsto Y_W(u, x)w = \sum_{n\in\mathbb{Z}} (Y_W)_n(u)wx^{-n-1} \end{aligned}$$

and linear operators $L_W(0)$ and $L_W(-1)$ on W satisfying the following conditions:

1. Grading restriction condition*: The vector space W is $\mathbb{C}$-graded, that is, $W = \coprod_{n\in\mathbb{C}} W_{(n)}$, such that $W_{(n)} = 0$ when the real part of n is sufficiently negative.*

2. Lower-truncation condition for vertex operators*: For* $u \in V$ *and* $w \in W$, $Y_W(u, x)w$ *contain only finitely many negative power terms, that is,* $Y_W(u, x)w \in W((x))$.
3. Identity property*: Let* 1_W *be the identity operator on* W*. Then* $Y_W(\mathbf{1}, x) = 1_W$.
4. Duality*: For* $u_1, u_2 \in V$, $w \in W$, $w' \in W' = \coprod_{n \in \mathbb{Z}} W^*_{(n)}$*, the series*

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle,$$
$$\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle,$$
$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$*, respectively, to a common rational function in* z_1 *and* z_2 *with the only possible poles at* $z_1, z_2 = 0$ *and* $z_1 = z_2$.
5. $L_W(0)$-bracket formula*: For* $v \in V$,

$$[L_W(0), Y_W(v, x)] = Y_W(L(0)v, x) + x\frac{d}{dx}Y_W(v, x).$$

6. $L(0)$-grading property*: For* $w \in W_{(n)}$*, there exists* $N \in \mathbb{Z}_+$ *such that* $(L_W(0) - n)^N w = 0$.
7. $L(-1)$-derivative property*: For* $v \in V$,

$$\frac{d}{dx}Y_W(u, x) = Y_W(L_V(-1)u, x) = [L_W(-1), Y_W(u, x)].$$

Since in this paper, we shall always consider grading-restricted generalized V-modules, for simplicity, we shall call them simply V-modules.

If a meromorphic function $f(z_1, \ldots, z_n)$ on a region in C^n can be analytically extended to a rational function in $z_1, \ldots, z_n$, we shall use $R(f(z_1, \ldots, z_n))$ to denote this rational function.

Remark 2.3. Let V be a grading-restricted vertex algebra and W a V-module. Then the duality axiom can be rewritten as: For $u_1, u_2 \in V$, $w \in W$, $w' \in W'$,

$$\begin{aligned} R(\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w\rangle) &= R(\langle w', Y_W(u_2, z_2)Y_W(u_1, z_1)w\rangle) \\ &= R(\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w\rangle). \end{aligned}$$

The following result was proved in [FHL] (Proposition 3.5.1 in [FHL]):

Proposition 2.4. *For* $v_1, \ldots, v_n \in V$, $w \in W$ *and* $w' \in W'$,

$$\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w\rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ *to a rational function*

$$R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w\rangle)$$

in $z_1, \ldots, z_n$ *with the only possible poles at* $z_i = z_j$, $i \neq j$*, and* $z_i = 0$*. Moreover, the following commutativity holds: For* $\sigma \in S_n$,

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w\rangle) \\ &\quad = R(\langle w', Y_W(v_{\sigma(1)}, z_{\sigma(1)}) \cdots Y_W(v_{\sigma(n)}, z_{\sigma(n)})w\rangle). \end{aligned}$$

The following result, though not explicitly stated in [FHL], was implicitly given in Subsection 3.5 in [FHL]:

Proposition 2.5. *For $v_1, \ldots, v_n \in V$, $w \in W$, $w' \in W'$ and $i = 1, \ldots, n-1$,*

$$\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1}) \cdot Y_W(Y_V(v_i, z_i - z_{i+1})v_{i+1}, z_{i+1}) Y_W(v_{i+2}, z_{i+2}) \cdots Y_W(v_n, z_n)w\rangle$$

is absolutely convergent in the region given by $|z_1| > \cdots > |z_{i-1}| > |z_{i+1}| > \cdots > |z_n| > 0$, $|z_{i+1}| > |z_i - z_{i+1}| > 0$ and $|z_k - z_{i+1}| > |z_i - z_{i+1}| > 0$ for $k \neq i, i+1$ to a rational function

$$R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1}) \cdot Y_W(Y_V(v_i, z_i - z_{i+1})v_{i+1}, z_{i+1}) Y_W(v_{i+2}, z_{i+2}) \cdots Y_W(v_n, z_n)w\rangle)$$

with the only possible poles at $z_i = z_j$, $i \neq j$, and $z_i = 0$. Moreover, the following associativity holds:

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1}) \\ &\quad \cdot Y_W(v_i, z_i) Y_W(v_{i+1}, z_{i+1}) Y_W(v_{i+2}, z_{i+2}) \cdots Y_W(v_n, z_n)w\rangle) \\ &\quad = R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1}) \\ &\quad \cdot Y_W(Y_V(v_i, z_i - z_{i+1})v_{i+1}, z_{i+1}) Y_W(v_{i+2}, z_{i+2}) \cdots Y_W(v_n, z_n)w\rangle). \end{aligned}$$

Recall from Subsection 5.6 in [FHL] the linear map

$$\begin{aligned} Y_{WV}^W : W \otimes V &\to W[[z, z^{-1}]] \\ w \otimes v &\mapsto Y_{WV}^W(w, z)v \end{aligned}$$

defined by

$$Y_{WV}^W(w, z)v = e^{zL(-1)} Y_W(v, -z)w$$

for $v \in V$ and $w \in W$. The following result is a special case of Theorem 6.6.2 in [FHL]:

Proposition 2.6. *For $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n, v \in V$, $w \in W$ and $w' \in W'$,*

$$\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1}) Y_{WV}^W(w, z_i) Y_V(v_{i+1}, z_{i+1}) \cdots Y_V(v_n, z_n)v\rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ to a rational function

$$R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1}) Y_{WV}^W(w, z_i) Y_V(v_{i+1}, z_{i+1}) \cdots Y_V(v_n, z_n)v\rangle)$$

in $z_1, \ldots, z_n$ with the only possible poles at $z_i = z_j$, $i \neq j$, and $z_i = 0$. Moreover, the following commutativity holds: For $\sigma \in S_n$,

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1}) Y_{WV}^W(w, z_i) Y_V(v_{i+1}, z_{i+1}) \cdots Y_V(v_n, z_n)v\rangle) \\ &\quad = R(\langle w', Y_W(u_{\sigma(1)}, z_{\sigma(1)}) \cdots Y_W(u_{\sigma^{-1}(i)-1}, z_{\sigma^{-1}(i)-1}) \\ &\qquad \cdot Y_{WV}^W(w, z_i) Y_V(u_{\sigma(\sigma^{-1}(i)+1)}, z_{\sigma^{-1}(i)+1}) \cdots Y_W(v_{\sigma(n)}, z_{\sigma(n)})v\rangle). \end{aligned}$$

The following result, though not explicitly stated in [FHL], was implicitly given in Section 5.6 in [FHL]:

Proposition 2.7. *Let* $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n, v \in V$, $w \in W$ *and* $w' \in W'$. *Let* $u_k = v_k$ *for* $k = 1, \ldots, i-1, i+1, \ldots, n$, $u_i = w$ *and* $1 \leq j \leq n$. *Let* $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ *be* Y_W, Y_{WV}^W *or* Y_V *such that the expressions below are uniquely defined. Then*

$$\langle w', \mathcal{Y}_1(u_1, z_1) \cdots \mathcal{Y}_{j-1}(u_{j-1}, z_{j-1})$$
$$\cdot \mathcal{Y}_j(\mathcal{Y}_{j+1}(u_j, z_j - z_{j+1})u_{j+1}, z_{j+1})\mathcal{Y}_{j+2}(u_{j+2}, z_{j+2}) \cdots \mathcal{Y}_n(u_n, z_n)v\rangle$$

is absolutely convergent in the region given by $|z_1| > \cdots > |z_{j-1}| > |z_{j+1}| > \cdots > |z_n| > 0$, $|z_{j+1}| > |z_j - z_{j+1}| > 0$ *and* $|z_k - z_{j+1}| > |z_j - z_{j+1}| > 0$ *for* $k \neq j, j+1$ *to a rational function*

$$R(\langle w', \mathcal{Y}_1(u_1, z_1) \cdots \mathcal{Y}_{j-1}(u_{j-1}, z_{j-1})$$
$$\cdot \mathcal{Y}_j(\mathcal{Y}_{j+1}(u_j, z_j - z_{j+1})u_{j+1}, z_{j+1})\mathcal{Y}_{j+2}(u_{j+2}, z_{j+2}) \cdots \mathcal{Y}_n(u_n, z_n)v\rangle)$$

in $z_1, \ldots, z_n$ *with the only possible poles at* $z_i = z_j$, $i \neq j$, *and* $z_i = 0$. *Moreover, the following associativity holds: For* $j = 1, \ldots, i-2$ *or* $j = i+1, \ldots, n-1$,

$$\begin{aligned}
&R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_{i-1}, z_{i-1})\\
&\quad \cdot Y_{WV}^W(w, z_i)Y_V(v_{i+1}, z_{i+1}) \cdots Y_V(v_n, z_n)v\rangle)\\
&= R(\langle w', \mathcal{Y}_1(u_1, z_1) \cdots \mathcal{Y}_{j-1}(u_{j-1}, z_{j-1})\\
&\quad \cdot \mathcal{Y}_j(\mathcal{Y}_{j+1}(u_j, z_j - z_{j+1})u_{j+1}, z_{j+1})\mathcal{Y}_{j+2}(u_{j+2}, z_{j+2}) \cdots \mathcal{Y}_n(u_n, z_n)v\rangle).
\end{aligned}$$

Let $\overline{W}$ be the algebraic completion of W, that is, $\overline{W} = \prod_{n \in \mathbb{C}} W_{(n)} = (W')^*$. For $n \in \mathbb{Z}_+$, let $F_n\mathbb{C}$ be the configuration space of n points in $\mathbb{C}$, that is,

$$F_n\mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}.$$

For each $(z_1, \ldots, z_n, \zeta) \in F_{n+1}\mathbb{C}$, $v_1, \ldots, v_n \in V$, $w \in W$ and $w' \in W'$, we have an element

$$E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)Y_{WV}^W(w, \zeta)\mathbf{1}) \in \overline{W}$$

given by

$$\begin{aligned}
&\langle w', E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)Y_{WV}^W(w, \zeta)\mathbf{1})\rangle\\
&\quad = R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)Y_{WV}^W(w, \zeta)\mathbf{1}\rangle).
\end{aligned}$$

For $(z_1, \ldots, z_n, \zeta) \in F_{n+1}\mathbb{C}$, $v_1, \ldots, v_n \in V$ and $w \in W$, set

$$\begin{aligned}
&(E_W^{(n,1)}(v_1 \otimes \cdots \otimes v_n; w))(z_1, \ldots, z_n, \zeta)\\
&\quad = E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)Y_{WV}^W(w, \zeta)\mathbf{1}) \in \overline{W}.
\end{aligned}$$

We also define

$$E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w) : \{(z_1, \ldots, z_n) \in F_n\mathbb{C} \mid z_i \neq 0,\ i = 1, \ldots, n\} \to \overline{W}$$

by

$$\begin{aligned}
&(E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w))(z_1, \ldots, z_n)\\
&\quad = (E_W^{(n,1)}(v_1 \otimes \cdots \otimes v_n; w))(z_1, \ldots, z_n, 0)\\
&\quad = E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w).
\end{aligned}$$

The next result was in fact proved in [Hu1,Hu2] but was formulated using the geometry of spheres with punctures and local coordinates in a more general setting. Here we formulate it without using the language of geometry or operads and give a direct proof. Given a V-module $W = \coprod_{n\in\mathbb{C}} W_{(n)}$, let $P_n : W \to W_{(n)}$ for $n \in \mathbb{C}$ be the projection from W to $W_{(n)}$.

Proposition 2.8. *For $k, l_1, \ldots, l_{n+1} \in \mathbb{Z}_+$ and $v_1^{(1)}, \ldots, v_{l_1}^{(1)}, \ldots, v_1^{(n+1)}, \ldots, v_{l_{n+1}}^{(n+1)} \in V$, $w \in W$ and $w' \in W'$, the series*

$$\sum_{r_1,\ldots,r_n\in\mathbb{Z}, r_{n+1}\in\mathbb{C}} \langle w', (E_W^{(n,1)}(P_{r_1}((E_V^{(l_1)}(v_1^{(1)}\otimes\cdots\otimes v_{l_1}^{(1)};\mathbf{1}))(z_1^{(1)},\ldots,z_{l_1}^{(1)})) \otimes\cdots\otimes P_{r_n}((E_V^{(l_n)}(v_1^{(n)}\otimes\cdots\otimes v_{l_n}^{(n)};\mathbf{1}))(z_1^{(n)},\ldots,z_{l_n}^{(n)})); P_{r_{n+1}}((E_W^{(l_{n+1})}(v_1^{(n+1)}\otimes\cdots\otimes v_{l_{n+1}}^{(n+1)};w))(z_1^{(n+1)},\ldots,z_{l_{n+1}}^{(n+1)})))) (z_1^{(0)},\ldots,z_{n+1}^{(0)})\rangle \tag{2.1}$$

converges absolutely to

$$\langle w', (E_W^{(n)}(v_1^{(1)}\otimes\cdots\otimes v_{l_{n+1}}^{(n+1)};w))(z_1^{(1)}+z_1^{(0)},\ldots,z_{l_1}^{(1)}+z_1^{(0)}, \ldots, z_1^{(n+1)}+z_{n+1}^{(0)},\ldots,z_{l_{n+1}}^{(n+1)}+z_{n+1}^{(0)})\rangle. \tag{2.2}$$

when $0 < |z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$ for $i, j = 1, \ldots, n+1$, $i \neq j$, $p = 1, \ldots, l_i$, $q = 1, \ldots, l_j$.

Proof. By definition,

$$\langle w', (E_W^{(n)}(v_1^{(1)}\otimes\cdots\otimes v_{l_{n+1}}^{(n+1)};w))(z_1,\ldots,z_{l_1+\cdots+l_{n+1}})\rangle$$

is a rational function in $z_1, \ldots, z_{l_1+\cdots+l_{n+1}}$ with the only possible poles at $z_i = 0$ or $z_i = z_j$, $i \neq j$. Thus for fixed $z_p^{(i)} \in \mathbb{C}^\times$, $p = 1, \ldots, l_i$, $i = 1, \ldots, n+1$ and $z_i^{(0)}$ for $i = 1, \ldots, n+1$ satisfying $0 < |z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$ for $i, j = 1, \ldots, n+1$, $i \neq j$, $p = 1, \ldots, l_i$, $q = 1, \ldots, l_j$, the function

$$\langle w', (E_W^{(n)}(t_1^{L(0)}v_1^{(1)}\otimes\cdots\otimes t_1^{L(0)}v_{l_1}^{(1)}\otimes\cdots\otimes t_{n+1}^{L(0)}v_1^{(n+1)}\otimes\cdots\otimes t_{n+1}^{L(0)}v_{l_{n+1}}^{(n+1)};w)) (t_1z_1^{(1)}+z_1^{(0)},\ldots,t_1z_{l_1}^{(1)}+z_1^{(0)}, \ldots, t_{n+1}z_1^{(n+1)}+z_{n+1}^{(0)},\ldots,t_{n+1}z_{l_n}^{(n+1)}+z_{n+1}^{(0)})\rangle. \tag{2.3}$$

of $t_1, \ldots, t_{n+1}$ has an expansion as a Laurent series in $t_1, \ldots, t_{n+1}$ when $(t_1, \ldots, t_{n+1})$ are in the direct product of some annuli containing 1. Using induction and the associativity for V and W repeatedly, we see that the coefficients of this Laurent expansion are the same as the coefficients of the formal Laurent series

$$\sum_{r_1,\ldots,r_{n+1}\in\mathbb{Z}} \langle w', (E_W^{(n,1)}(P_{r_1}((E_V^{(l_1)}(v_1^{(1)}\otimes\cdots\otimes v_{l_1}^{(1)};\mathbf{1}))(z_1^{(1)},\ldots,z_{l_1}^{(1)})) \otimes\cdots\otimes P_{r_n}((E_V^{(l_n)}(v_1^{(n)}\otimes\cdots\otimes v_{l_n}^{(n)};\mathbf{1}))(z_1^{(n)},\ldots,z_{l_n}^{(n)})); P_{r_{n+1}}((E_W^{(l_{n+1})}(v_1^{(n+1)}\otimes\cdots\otimes v_{l_{n+1}}^{(n+1)};w))(z_1^{(n+1)},\ldots,z_{l_{n+1}}^{(n+1)})))) (z_1^{(0)},\ldots,z_{n+1}^{(0)})\rangle t_1^{r_1}\cdots t_n^{r_n}. \tag{2.4}$$

Thus (2.4) is absolutely convergent to (2.3) in the region where (2.3) has a Laurent expansion. In particular, when $t_1 = \cdots = t_{n+1} = 1$, we obtain that (2.1) is absolutely convergent to (2.2). □

For each $(\zeta, z_1, \ldots, z_n) \in F_{n+1}\mathbb{C}$, $v_1, \ldots, v_n \in V$, $w \in W$ and $w' \in W'$, we have an element

$$E(Y_{WV}^{W}(w, \zeta)Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)\mathbf{1}) \in \overline{W}$$

given by

$$\begin{aligned}&\langle w', E(Y_{WV}^{W}(w, \zeta)Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)\mathbf{1})\rangle\\&\quad = R(\langle w', Y_{WV}^{W}(w, \zeta)Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)\mathbf{1}\rangle).\end{aligned}$$

For $(\zeta, z_1, \ldots, z_n) \in F_{n+1}\mathbb{C}$, $v_1, \ldots, v_n \in V$ and $w \in W$, set

$$\begin{aligned}&(E_{WV}^{W;(1,n)}(w; v_1 \otimes \cdots \otimes v_n))(\zeta, z_1, \ldots, z_n)\\&\quad = E(Y_{WV}^{W}(w, \zeta)Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)\mathbf{1}) \in \overline{W}.\end{aligned}$$

We have:

Proposition 2.9. *For* $(\zeta, z_1, \ldots, z_n) \in F_{n+1}\mathbb{C}$, $v_1, \ldots, v_n \in V$ *and* $w \in W$,

$$\begin{aligned}&(E_{WV}^{W;(1,n)}(w; v_1 \otimes \cdots \otimes v_n))(\zeta, z_1, \ldots, z_n)\\&\quad = (E_{W}^{(n,1)}(v_1 \otimes \cdots \otimes v_n; w))(z_1, \ldots, z_n, \zeta).\end{aligned}$$

Proof. For $(\zeta, z_1, \ldots, z_n) \in F_{n+1}\mathbb{C}$, $v_1, \ldots, v_n \in V$, $w \in W$ and $w' \in W'$,

$$\begin{aligned}&\langle w', (E_{WV}^{W;(1,n)}(w; v_1 \otimes \cdots \otimes v_n))(\zeta, z_1, \ldots, z_n)\rangle\\&\quad = R(\langle w', Y_{WV}^{W}(w, \zeta)Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)\mathbf{1}\rangle)\\&\quad = R(\langle w', e^{\zeta L(-1)}Y_W(Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)\mathbf{1}, -\zeta)w\rangle)\\&\quad = R(\langle w', e^{\zeta L(-1)}Y_W(v_1, z_1 - \zeta) \cdots Y_W(v_n, z_n - \zeta)w\rangle)\\&\quad = R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)e^{\zeta L(-1)}w\rangle)\\&\quad = R(\langle w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)Y_{WV}^{W}(w, \zeta)\mathbf{1}\rangle)\\&\quad = \langle w', (E_{W}^{(n,1)}(v_1 \otimes \cdots \otimes v_n; w))(z_1, \ldots, z_n, \zeta)\rangle.\end{aligned}$$

□

We also define

$$E_{WV}^{W;(n)}(w; v_1 \otimes \cdots \otimes v_n) : \{(z_1, \ldots, z_n) \in F_n\mathbb{C} \mid z_i \neq 0,\ i = 1, \ldots, n\} \to \overline{W}$$

by

$$(E_{WV}^{W;(n)}(w; v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n) = (E_{WV}^{W;(1,n)}(w; v_1 \otimes \cdots \otimes v_n))(0, z_1, \ldots, z_n).$$

Then by Proposition 2.9,

$$E_{WV}^{W;(n)}(w; v_1 \otimes \cdots \otimes v_n) = E_{W}^{(n)}(v_1 \otimes \cdots \otimes v_n; w)$$

for $v_1, \ldots, v_n \in V$ and $w \in W$.

3. $\overline{W}$-Valued Rational Functions

Let V be a grading-restricted vertex algebra and W a V-module (recall our convention that a V-module means a grading-restricted generalized V-module in this paper). Recall the configuration spaces

$$F_n\mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$$

for $n \in \mathbb{Z}_+$.

Definition 3.1. *A* $\overline{W}$-valued rational function in $z_1, \dots, z_n$ with the only possible poles at $z_i = z_j$, $i \neq j$ *is a map*

$$\begin{aligned} f: \quad & F_n\mathbb{C} \to \overline{W} \\ (z_1, \dots, z_n) & \mapsto f(z_1, \dots, z_n) \end{aligned}$$

such that for any $w' \in W'$,

$$\langle w', f(z_1, \dots, z_n)\rangle$$

is a rational function in $z_1, \dots, z_n$ *with the only possible poles at* $z_i = z_j$, $i \neq j$.

For simplicity, we shall call the map that we just defined a $\overline{W}$*-valued rational function in* $z_1, \dots, z_n$ unless there might be other poles. Denote the space of all $\overline{W}$-valued rational functions in $z_1, \dots, z_n$ by $\widetilde{W}_{z_1,\dots,z_n}$. We define a left action of S_n on $\widetilde{W}_{z_1,\dots,z_n}$ by

$$(\sigma(f))(z_1, \dots, z_n) = f(z_{\sigma(1)}, \dots, z_{\sigma(n)})$$

for $f \in \widetilde{W}_{z_1,\dots,z_n}$.

Example 3.2. For $w \in W$, the $\overline{W}$-valued function $E^{(n)}_W(v_1 \otimes \cdots \otimes v_n; w)$ given by

$$(E^{(n)}_W(v_1 \otimes \cdots \otimes v_n; w))(z_1, \dots, z_n) = E(Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w)$$

in general might not be an element of $\widetilde{W}_{z_1,\dots,z_n}$, since there might be singularities at $z_i = 0$. But for $w \in W$ such that $Y_W(v, x)w \in W[[x]]$, $E^{(n)}_W(v_1 \otimes \cdots \otimes v_n; w)$ is indeed an element of $\widetilde{W}_{z_1,\dots,z_n}$. Since

$$E^{W;(n)}_{WV}(w; v_1 \otimes \cdots \otimes v_n) = E^{(n)}_W(v_1 \otimes \cdots \otimes v_n; w),$$

$E^{W;(n)}_{WV}(w; v_1 \otimes \cdots \otimes v_n)$ is also an element of $\widetilde{W}_{z_1,\dots,z_n}$. In particular, when $W = V$ and $w = \mathbf{1}$, $E^{(n)}_V(v_1 \otimes \cdots \otimes v_n; \mathbf{1}) \in \widetilde{V}_{z_1,\dots,z_n}$.

For $z \in \mathbb{C}^\times$, we shall use $\log z$ to denote $\log|z| + i \arg z$, $0 \leq \arg z < 2\pi$. Let $(L_W(0))_s$ be the semisimple part of $L_W(0)$, that is, $(L_W(0))_s w = nw$ for $w \in W_{(n)}$. Since W is a (grading-restricted generalized) V-module, for any $z \in \mathbb{C}^\times$,

$$\begin{aligned} z^{L_W(0)} &= e^{(\log z)L_W(0)} \\ &= e^{(\log z)(L_W(0))_s} e^{(\log z)(L_W(0)-(L_W(0))_s)} \end{aligned}$$

is a well-defined linear operator on $\overline{W}$.

Definition 3.3. *For $n \in \mathbb{Z}_+$, a linear map $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ is said to have the* $L(-1)$-derivative property *if (i)*

$$\frac{\partial}{\partial z_i}\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$
$$= \langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes L_V(-1)v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$

for $i = 1, \dots, n$, $v_1, \dots, v_n \in V$ and $w' \in W'$ and (ii)

$$\left(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n}\right)\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$
$$= \langle w', L_W(-1)(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$

and $v_1, \dots, v_n \in V$, $w' \in W'$. A linear map $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ is said to have the $L(0)$-conjugation property if for $v_1, \dots, v_n \in V$, $w' \in W'$, $(z_1, \dots, z_n) \in F_n\mathbb{C}$ and $z \in \mathbb{C}^\times$ so that $(zz_1, \dots, zz_n) \in F_n\mathbb{C}$,

$$\langle w', z^{L_W(0)}(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$
$$= \langle w', (\Phi(z^{L(0)}v_1 \otimes \cdots \otimes z^{L(0)}v_n))(zz_1, \dots, zz_n)\rangle.$$

Note that since $L_W(-1)$ is a weight-one operator on W, for any $z \in \mathbb{C}$, $e^{zL_W(-1)}$ is a well-defined linear operator on $\overline{W}$.

Proposition 3.4. *Let $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ be a linear map having the $L(-1)$-derivative property. Then for $v_1, \dots, v_n \in V$, $w' \in W'$, $(z_1, \dots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$ such that $(z_1 + z, \dots, z_n + z) \in F_n\mathbb{C}$,*

$$\langle w', e^{zL_W(-1)}(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle$$
$$= \langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1 + z, \dots, z_n + z)\rangle$$

and for $v_1, \dots, v_n \in V$, $w' \in W'$, $(z_1, \dots, z_n) \in F_n\mathbb{C}$, $z \in \mathbb{C}$ and $1 \le i \le n$ such that

$$(z_1, \dots, z_{i-1}, z_i + z, z_{i+1}, \dots, z_n) \in F_n\mathbb{C},$$

the power series expansion of

$$\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_{i-1}, z_i + z, z_{i+1}, \dots, z_n)\rangle \tag{3.1}$$

in z is equal to the power series

$$\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes e^{zL(-1)}v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle \tag{3.2}$$

in z. In particular, the power series (3.2) in z is absolutely convergent to (3.1) in the disk $|z| < \min_{i \neq j}\{|z_i - z_j|\}$.

Proof. This result follows immediately from the definition of $L(-1)$-derivative property and Taylor's theorem on power series expansions of analytic functions. □

We would like to take linear maps from tensor powers of V to $\widetilde{W}_{z_1,\dots,z_n}$ to be cochains in our cohomology theory. But to define the coboundary operator, we have to compose cochains with vertex operators. However, the images of vertex operator maps in general are not in the algebras or in the modules. They are in the algebraic completions of the algebras or modules. This is one of the most subtle features of the theory of grading-restricted vertex algebras or vertex operator algebras. Because of this subtlety, we cannot compose vertex operators directly. Instead, we first write a series by projecting an element of the algebraic completion of an algebra or a module to its homogeneous components, composing these homogeneous components with other vertex operators and then taking the formal sum. If this formal sum is absolutely convergent, then these operators can be composed and we shall use the usual notation to denote the composition obtained from the sums of these series. See [Hu2] for detailed discussions in the case of vertex operator algebras.

Since $\overline{W}$-valued rational functions above are valued in $\overline{W}$, not W, and for $z \in \mathbb{C}^{\times}$, $u, v \in V$, $w \in W$, $Y_V(u, z)v \in \overline{V}$ and $Y_W(u, z)v \in \overline{W}$, in general, we might not be able to compose a linear map from a tensor power of V to $\widetilde{W}_{z_1,\dots,z_n}$ with vertex operators. So we have to consider linear maps from tensor powers of V to $\widetilde{W}_{z_1,\dots,z_n}$ such that these maps can be composed with vertex operators in the sense mentioned above.

For a V-module $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ and $m \in \mathbb{C}$, let $P_m : \overline{W} \to W_{(m)}$ be the projection from $\overline{W}$ to $W_{(m)}$.

Definition 3.5. *Let $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ be a linear map. For $m \in \mathbb{N}$, Φ is said to be* composable with m vertex operators *if the following conditions are satisfied:*

1. *Let $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = m + n$, $v_1, \dots, v_{m+n} \in V$ and $w' \in W'$. Set*

$$\Psi_i = (E_V^{(l_i)}(v_{l_1+\cdots+l_{i-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{i-1}+l_i}; \mathbf{1})) (z_{l_1+\cdots+l_{i-1}+1} - \zeta_i, \dots, z_{l_1+\cdots+l_{i-1}+l_i} - \zeta_i) \tag{3.3}$$

for $i = 1, \dots, n$. Then there exist positive integers $N(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \dots, k$, $i \neq j$ such that the series

$$\sum_{r_1,\dots,r_n\in\mathbb{Z}} \langle w', (\Phi(P_{r_1}\Psi_1 \otimes \cdots \otimes P_{r_n}\Psi_n))(\zeta_1, \dots, \zeta_n)\rangle,$$

is absolutely convergent when

$$|z_{l_1+\cdots+l_{i-1}+p} - \zeta_i| + |z_{l_1+\cdots+l_{j-1}+q} - \zeta_i| < |\zeta_i - \zeta_j|$$

for $i, j = 1, \dots, k$, $i \neq j$ and for $p = 1, \dots, l_i$ and $q = 1, \dots, l_j$. and the sum can be analytically extended to a rational function in $z_1, \dots, z_{m+n}$, independent of $\zeta_1, \dots, \zeta_n$, with the only possible poles at $z_i = z_j$ of order less than or equal to $N(v_i, v_j)$ for $i, j = 1, \dots, k$, $i \neq j$.

2. *For $v_1, \dots, v_{m+n} \in V$, there exist positive integers $N(v_i, v_j)$ depending only on v_i and v_j for $i, j = 1, \dots, k$, $i \neq j$ such that for $w' \in W'$,*

$$\sum_{q\in\mathbb{C}} \langle w', (E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; P_q((\Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))(z_{m+1}, \dots, z_{m+n})))(z_1, \dots, z_m)\rangle$$

is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_k| > 0$ *for* $i = 1, \dots, m$ *and* $k = m+1, \dots, m+n$ *and the sum can be analytically extended to a rational function in* $z_1, \dots, z_{m+n}$ *with the only possible poles at* $z_i = z_j$ *of orders less than or equal to* $N(v_i, v_j)$ *for* $i, j = 1, \dots, k$, $i \neq j$,.

Remark 3.6. In the first version of the present paper, we did not assume the existence of the positive integers $N(v_i, v_j)$ for $i, j = 1, \dots, k$, $i \neq j$. We did get a cohomology theory without such an assumption. On the other hand, since the correlation functions for grading-restricted vertex algebras do have this property, here we add this assumption. (In fact, the existence of $N(v_i, v_{i+1})$ can be seen immediately from Proposition 2.5 and the fact that $Y_V(v_i, z_i - z_{i+1})v_{i+1}$ contains only finitely many negative power terms in $z_i - z_{i+1}$ (the lower-truncation condition). The existence of $N(v_i, v_j)$ then follows from the existence of $N(v_i, v_{i+1})$ and Proposition 2.4.) But we remark that the cohomology theory without this assumption might still be important in the future studies. We might call the cohomology theory without this assumption of the paper the *cohomology theory without upper bounds on orders of poles*.

We shall denote the rational functions in Conditions 1 and 2 of Definition 3.5 by

$$R(\langle w', \Phi(E_V^{(l_1)}(v_1 \otimes \cdots \otimes v_{l_1}; \mathbf{1}) \otimes \\ \cdots \otimes E_V^{(l_n)}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}; \mathbf{1}))(z_1, \dots, z_{m+n})\rangle)$$

and

$$R(\langle w', (E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))(z_1, \dots, z_{m+n})\rangle),$$

respectively.

Example 3.7. For $w \in W$ satisfying $Y_W(v, x)w \in W[[x]]$, the $\overline{W}$-valued rational function $E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w)$ for $v_1, \dots, v_n \in V$ give a linear map

$$\begin{aligned} E^{(n)}_{W;\, w} : \quad & V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n} \\ & v_1 \otimes \cdots \otimes v_n \mapsto E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w). \end{aligned}$$

This linear map has the $L(-1)$-derivative property, the $L(0)$-conjugation property and by Proposition 2.8 is composable with m vertex operators for any $m \in \mathbb{Z}_+$. Moreover, let f be a homogeneous rational functions of degree 0 in $z_1, \dots, z_n$ with the only possible poles at $z_i = z_j$, $i \neq j$, then $f E^{(n)}_{W;\, w} : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ defined by

$$\begin{aligned} &((f E^{(n)}_{W;\, w})(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n) \\ &\quad = f(z_1, \dots, z_n)(E_W^{(n)}(v_1 \otimes \cdots \otimes v_n; w))(z_1, \dots, z_n) \end{aligned}$$

for $v_1, \dots, v_n \in V$ has the $L(0)$-conjugation property and is composable with m vertex operators for any $m \in \mathbb{Z}_+$. In particular, $f E^{(n)}_{V;\, \mathbf{1}}$ has the $L(0)$-conjugation property and is composable with m vertex operators for any $m \in \mathbb{Z}_+$.

Let $\Phi: V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ be composable with m vertex operators. Then for $l_1, \dots, l_n \in \mathbb{Z}_+$ such that $l_1 + \cdots + l_n = m+n$, $v_1, \dots, v_{m+n} \in V$ and $w \in W$, we have an element

$$E(\Phi(E_V^{(l_1)}(v_1 \otimes \cdots \otimes v_{l_1}; \mathbf{1}) \otimes \cdots \otimes E_V^{(l_n)}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}; \mathbf{1})))$$

of $\widetilde{W}_{z_1,\dots,z_{m+n-1}}$ given by

$$\begin{aligned}&\langle w', (E(\Phi(E_V^{(l_1)}(v_1 \otimes \cdots \otimes v_{l_1}; \mathbf{1}) \otimes \\ &\quad \cdots \otimes E_V^{(l_n)}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}; \mathbf{1}))))(z_1, \dots, z_{m+n})\rangle \\ &= R(\langle w', \Phi(E_V^{(l_1)}(v_1 \otimes \cdots \otimes v_{l_1}; \mathbf{1}) \otimes \\ &\quad \cdots \otimes E_V^{(l_n)}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}; \mathbf{1}))(z_1, \dots, z_{m+n})\rangle)\end{aligned}$$

For $v_1, \dots, v_{m+n} \in V$, we have an element

$$E(E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n})))$$

of $\widetilde{W}_{z_1,\dots,z_{m+n}}$ given by

$$\begin{aligned}&\langle w', (E(E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))))(z_1, \dots, z_{m+n})\rangle \\ &= R(\langle w', (E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \\ &\qquad (\Phi(v_{m+1} \otimes \cdots \otimes v_{m+n}))(z_{m+1}, \dots, z_{m+n}))(z_1, \dots, z_m)\rangle)).\end{aligned}$$

Also for $v_1, \dots, v_{n+m} \in V$, since by Proposition 2.9,

$$\begin{aligned}&\sum_{q\in\mathbb{C}} \langle w', (E_{WV}^{W;(m)}(P_q((\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)); \\ &\qquad\qquad v_{n+1} \otimes \cdots \otimes v_{n+m}))(z_{n+1}, \dots, z_{n+m})\rangle \\ &= \sum_{q\in\mathbb{C}} \langle w', (E_W^{(m)}(v_{n+1} \otimes \cdots \otimes v_{n+m}; \\ &\qquad\qquad P_q((\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n))))(z_{n+1}, \dots, z_{n+m})\rangle,\end{aligned} \tag{3.4}$$

the left-hand side of (3.4) is absolutely convergent and can be analytically expended to a rational function in $z_1, \dots, z_n$ with the only possible poles at $z_i = z_j$, $i \neq j$ if and only if the same conclusions hold for the right-hand side of (3.4). Since Φ is composable with m vertex operators, the right-hand side is indeed absolutely convergent and can be analytically expended to a rational function $z_1, \dots, z_n$ with the only possible poles at $z_i = z_j$, $i \neq j$. Thus the same conclusions hold for the left-hand side. Denote the corresponding rational function by

$$R(\langle w', (E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m}))(z_1, \dots, z_{n+m})\rangle).$$

We obtain an element

$$E(E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m}))$$

of $\widetilde{W}_{z_1,\dots,z_{n+m}}$ given by

$$\begin{aligned}&\langle w', E(E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m}))(z_1, \dots, z_{n+m})\rangle \\ &= R(\langle w', (E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m}))(z_1, \dots, z_{n+m})\rangle).\end{aligned}$$

By Proposition 2.9, we have

$$
\begin{aligned}
&E(E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m}))(z_1, \ldots, z_{n+m}) \\
&\quad = (E(E_W^{(m)}(v_{n+1} \otimes \cdots \otimes v_{n+m}; \Phi(v_1 \otimes \cdots \otimes v_n)))(z_{n+1}, \ldots, z_{n+m}, z_1, \cdots z_n).
\end{aligned}
\tag{3.5}
$$

We now define

$$
\begin{aligned}
\Phi \circ (E_{V;\,\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\,\mathbf{1}}^{(l_n)}) &: V^{\otimes m+n} \to \widetilde{W}_{z_1,\ldots,z_{m+n}}, \\
E_W^{(m)} \circ_{m+1} \Phi &: V^{\otimes m+n} \to \widetilde{W}_{z_1,\ldots,z_{m+n-1}}
\end{aligned}
$$

and

$$
E_{WV}^{W;(m)} \circ_0 \Phi : V^{\otimes m+n} \to \widetilde{W}_{z_1,\ldots,z_{m+n-1}}
$$

by

$$
\begin{aligned}
&(\Phi \circ (E_{V;\,\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\,\mathbf{1}}^{(l_n)}))(v_1 \otimes \cdots \otimes v_{m+n-1}) \\
&\quad = E(\Phi(E_{V;\mathbf{1}}^{(l_1)}(v_1 \otimes \cdots \otimes v_{l_1}) \otimes \cdots \\
&\qquad\qquad \otimes E_{V;\mathbf{1}}^{(l_n)}(v_{l_1+\cdots+l_{n-1}+1} \otimes \cdots \otimes v_{l_1+\cdots+l_{n-1}+l_n}))), \\
&(E_W^{(m)} \circ_{m+1} \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) \\
&\quad = E(E_W^{(m)}(v_1 \otimes \cdots \otimes v_m; \Phi(v_{m+1} \otimes \cdots \otimes v_{m+n})))
\end{aligned}
$$

and

$$
\begin{aligned}
&(E_{WV}^{W;(m)} \circ_0 \Phi)(v_1 \otimes \cdots \otimes v_{m+n}) \\
&\quad = E(E_{WV}^{W;(m)}(\Phi(v_1 \otimes \cdots \otimes v_n); v_{n+1} \otimes \cdots \otimes v_{n+m})),
\end{aligned}
$$

respectively. In the case that $l_1 = \cdots = l_{i-1} = l_{i+1} = 1$ and $l_i = m - n - 1$ for some i, for simplicity, we shall also use $\Phi \circ_i E_{V;\,\mathbf{1}}^{(l_i)}$ to denote $\Phi \circ (E_{V;\,\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\,\mathbf{1}}^{(l_n)})$.

We define an action of S_n on the space $\mathrm{Hom}(V^{\otimes n}, \widetilde{W}_{z_1,\ldots,z_n})$ of linear maps from $V^{\otimes n}$ to $\widetilde{W}_{z_1,\ldots,z_n}$ by

$$
(\sigma(\Phi))(v_1 \otimes \cdots \otimes v_n) = \sigma(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}))
$$

for $\sigma \in S_n$ and $v_1, \ldots, v_n \in V$.

We shall use the notation $\sigma_{i_1,\ldots,i_n} \in S_n$ to denote the permutation given by

$$
\sigma_{i_1,\ldots,i_n}(j) = i_j
$$

for $j = 1, \ldots, n$. We have

Proposition 3.8. *For $m \in \mathbb{Z}_+$,*

$$
E_{WV}^{W;(m)} \circ_0 \Phi = \sigma_{n+1,\ldots,n+m,1,\ldots,n}(E_W^{(m)} \circ_{m+1} \Phi). \tag{3.6}
$$

Proof. The equality (3.6) follows from (3.5) and the definition of the action of S_{m+n} on $\widetilde{W}_{z_1,\ldots,z_{m+n}}$. □

We also have:

Proposition 3.9. *The subspace of* $\mathrm{Hom}(V^{\otimes n}, \widetilde{W}_{z_1,\dots,z_n})$ *consisting of linear maps having the* $L(-1)$*-derivative property, having the* $L(0)$*-conjugation property or being composable with* m *vertex operators is invariant under the action of* S_n.

Proof. This result follows directly from the definitions. □

We know that compositions of maps are associative. But for maps whose compositions are defined using sums of absolutely convergent series as we have discussed above, even if all the compositions involved exist, we still might not have associativity in general because iterated sums in different orders might not be equal to each other in general. However, when such compositions are analytic in some sense, associativity does hold. In particular, for the maps considered in this paper, we do have the following proposition that gives in particular some associativity results:

Proposition 3.10. *Let* $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ *be composable with* m *vertex operators. Then we have:*

1. *For* $p \le m$, Φ *is composable with* p *vertex operators and for* $p, q \in \mathbb{Z}_+$ *such that* $p+q \le m$ *and* $l_1, \dots, l_n \in \mathbb{Z}_+$ *such that* $l_1+\cdots+l_n = p+n$, $\Phi\circ(E^{(l_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}})$ *and* $E^{(p)}_W \circ_{p+1} \Phi$ *are composable with* q *vertex operators.*
2. *For* $p, q \in \mathbb{Z}_+$ *such that* $p+q \le m$, $l_1, \dots, l_n \in \mathbb{Z}_+$ *such that* $l_1+\cdots+l_n = p+n$ *and* $k_1, \dots, k_{p+n} \in \mathbb{Z}_+$ *such that* $k_1+\cdots+k_{p+n} = q+p+n$, *we have*

$$(\Phi\circ(E^{(l_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}}))\circ(E^{(k_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(k_{p+n})}_{V;\,\mathbf{1}})$$
$$= \Phi\circ(E^{(k_1+\cdots+k_{l_1})}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(k_{l_1+\cdots+l_{n-1}+1}+\cdots+k_{p+n})}_{V;\,\mathbf{1}}).$$

3. *For* $p, q \in \mathbb{Z}_+$ *such that* $p+q \le m$ *and* $l_1, \dots, l_n \in \mathbb{Z}_+$ *such that* $l_1+\cdots+l_n = p+n$, *we have*

$$E^{(q)}_W \circ_{q+1} (\Phi\circ(E^{(l_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}})) = (E^{(q)}_W \circ_{q+1} \Phi)\circ(E^{(l_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}}).$$

4. *For* $p, q \in \mathbb{Z}_+$ *such that* $p+q \le m$, *we have*

$$E^{(p)}_W \circ_{p+1} (E^{(q)}_W \circ_{q+1} \Phi) = E^{(p+q)}_W \circ_{p+q+1} \Phi.$$

Proof. Conclusion 1 is clear from the definition.

Let $v^{(i)}_j \in V$ for $i = 1, \dots, p+n$, $j = 1, \dots, k_i$. Then

$$\begin{aligned}
&((\Phi\circ(E^{(l_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}}))\circ(E^{(k_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(k_{p+n})}_{V;\,\mathbf{1}}))(v^{(1)}_1\otimes\cdots\otimes v^{(p+n)}_{k_{p+n}})\\
&\quad= E((\Phi\circ(E^{(l_1)}_{V;\,\mathbf{1}}\otimes\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}}))(E^{(k_1)}_{V;\,\mathbf{1}}(v^{1)}_1\otimes\cdots\otimes v^{(1)}_{k_1})\otimes\\
&\qquad\qquad\cdots\otimes E^{(k_{p+n})}_{V;\,\mathbf{1}}(v^{(p+n)}_1\otimes\cdots\otimes v^{(p+n)}_{k_{p+n}})))\\
&\quad= E(\Phi(E^{(l_1)}_{V;\,\mathbf{1}}(E^{(k_1)}_{V;\,\mathbf{1}}(v^{(1)}_1\otimes\cdots\otimes v^{(1)}_{k_1})\otimes\cdots\otimes E^{(k_{l_1})}_{V;\,\mathbf{1}}(v^{(l_1)}_1\otimes\cdots\otimes v^{(l_1)}_{k_{l_1}}))\otimes\\
&\qquad\cdots\otimes E^{(l_n)}_{V;\,\mathbf{1}}(E^{(k_{l_1+\cdots+l_{n-1}+1})}_{V;\,\mathbf{1}}(v^{(l_1+\cdots+l_{n-1}+1)}_1\otimes\cdots\otimes v^{(l_1+\cdots+l_{n-1}+1)}_{k_{l_1+\cdots+l_{n-1}+1}})\otimes\\
&\qquad\qquad\cdots\otimes E^{(k_{p+n})}_{V;\,\mathbf{1}}(v^{(p+n)}_1\otimes\cdots\otimes v^{(p+n)}_{k_{p+n}})))).
\end{aligned}\tag{3.7}$$

By Proposition 2.8, the right-hand side of (3.7) is equal to

$$\begin{aligned}
&E(\Phi(E_{V;\,\mathbf{1}}^{(k_1+\cdots+k_{l_1})}(v_1^{(1)}\otimes\cdots\otimes v_{k_{l_1}}^{(l_1)})\otimes\\
&\quad\cdots\otimes E_{V;\,\mathbf{1}}^{(k_{l_1+\cdots+l_{n-1}+1}+\cdots+k_{p+n})}(v_1^{(l_1+\cdots+l_{n-1}+1)}\otimes\cdots\otimes v_{k_{p+n}}^{(p+n)})))\\
&=(\Phi\circ(E_{V;\,\mathbf{1}}^{(k_1+\cdots+k_{l_1})}\otimes\cdots\otimes E_{V;\,\mathbf{1}}^{(k_{l_1+\cdots+l_{n-1}+1}+\cdots+k_{p+n})}))(v_1^{(1)}\otimes\cdots\otimes v_{k_{p+n}}^{(p+n)}).
\end{aligned}\tag{3.8}$$

From (3.7) and (3.8), we obtain Conclusion 2.

Conclusions 3 and 4 can be proved similarly. □

4. Chain Complexes and Cohomologies

Let V be a vertex operator algebra and W a V-module. For $n\in\mathbb{Z}_+$, let $\widehat{C}_0^n(V,W)$ be the vector space of all linear maps from $V^{\otimes n}$ to $\widetilde{W}_{z_1,\ldots,z_n}$ satisfying the $L(-1)$-derivative property and the $L(0)$-conjugation property. For $m,n\in\mathbb{Z}_+$, let $\widehat{C}_m^n(V,W)$ be the vector spaces of all linear maps from $V^{\otimes n}$ to $\widetilde{W}_{z_1,\ldots,z_n}$ composable with m vertex operators and satisfying the $L(-1)$-derivative property and the $L(0)$-conjugation property. Also, let $\widehat{C}_m^0(V,W)=W$. Then we have

$$\widehat{C}_m^n(V,W)\subset\widehat{C}_{m-1}^n(V,W)$$

for $m\in\mathbb{Z}_+$. Let

$$\widehat{C}_\infty^n(V,W)=\bigcap_{m\in\mathbb{N}}\widehat{C}_m^n(V,W).$$

By Example 3.7, $\widehat{C}_\infty^n(V,V)$ is nonempty.

For $n\in\mathbb{N}$ and $m\in\mathbb{Z}_+$, we define a coboundary operator

$$\hat{\delta}_m^n:\widehat{C}_m^n(V,W)\to\widehat{C}_{m-1}^{n+1}(V,W)$$

by

$$\begin{aligned}
\hat{\delta}_m^n(\Phi)&=E_W^{(1)}\circ_2\Phi+\sum_{i=1}^n(-1)^i\Phi\circ_i E_{V;\mathbf{1}}^{(2)}+(-1)^{n+1}E_{WV}^{W;(1)}\circ_0\Phi\\
&=E_W^{(1)}\circ_2\Phi+\sum_{i=1}^n(-1)^i\Phi\circ_i E_{V;\mathbf{1}}^{(2)}+(-1)^{n+1}\sigma_{n+1,1,\ldots,n}(E_W^{(1)}\circ_2\Phi)
\end{aligned}\tag{4.1}$$

for $\Phi\in\widehat{C}_m^n(V,W)$, where the second equality is obtained by using (3.6). Explicitly, for $v_1,\ldots,v_{n+1}\in V$, $w'\in W'$ and $(z_1,\ldots,z_{n+1})\in F_{n+1}\mathbb{C}$,

$$\begin{aligned}
&\langle w',((\hat{\delta}_m^n(\Phi))(v_1\otimes\cdots\otimes v_{n+1}))(z_1,\ldots,z_{n+1})\rangle\\
&\quad=R(\langle w',Y_W(v_1,z_1)(\Phi(v_2\otimes\cdots\otimes v_{n+1}))(z_2,\ldots,z_{n+1})\rangle)\\
&\qquad+\sum_{i=1}^n(-1)^iR(\langle w',(\Phi(v_1\otimes\cdots\otimes v_{i-1}
\end{aligned}$$

$$\otimes(Y_V(v_i, z_i - \zeta_i)Y_V(v_{i+1}, z_{i+1} - \zeta_i)\mathbf{1})$$
$$\otimes v_{i+2} \otimes \cdots \otimes v_{n+1}))(z_1, \ldots, z_{i-1}, \zeta_i, z_{i+2}, \ldots, z_{n+1})\rangle)$$
$$+(-1)^{n+1} R(\langle w', Y_W(v_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle),$$

which is in fact independent of ζ_i. In particular, when we take $\zeta_i = z_{i+1}$ for $i = 1, \ldots, n$, we obtain

$$\begin{aligned}
&\langle w', ((\hat{\delta}^n_m(\Phi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \ldots, z_{n+1})\rangle \\
&\quad = R(\langle w', Y_W(v_1, z_1)(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, \ldots, z_{n+1})\rangle) \\
&\quad\quad + \sum_{i=1}^{n} (-1)^i R(\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - z_{i+1})v_{i+1} \\
&\quad\quad\quad \otimes \cdots \otimes v_{n+1}))(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1})\rangle) \\
&\quad\quad + (-1)^{n+1} R(\langle w', Y_W(v_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle).
\end{aligned}$$

By Proposition 3.10, $\hat{\delta}^n_m(\Phi)$ is composable with $m - 1$ vertex operators and has the $L(-1)$-derivative property and the $L(0)$-conjugation property. So $\hat{\delta}^n_m(\Phi) \in \widehat{C}^{n+1}_{m-1}(V, W)$ and $\hat{\delta}^n_m$ is indeed a map whose image is in $\widehat{C}^{n+1}_{m-1}(V, W)$.

In the definition of $\hat{\delta}^n_m$ above, we require $m \in \mathbb{Z}_+$ so that each term in the right-hand sides of the first and second equalities of (4.1) is well defined. However, in the case $n = 2$, there is a subspace of $\widehat{C}^2_0(V, W)$ containing $\widehat{C}^2_m(V, W)$ for all $m \in \mathbb{Z}_+$ such that $\hat{\delta}^2_m$ is still defined on this subspace.

Let $\widehat{C}^2_{\frac{1}{2}}(V, W)$ be the subspace of $\widehat{C}^2_0(V, W)$ consisting of elements Φ such that for $v_1, v_2, v_3 \in V$, $w' \in W'$,

$$\sum_{r \in \mathbb{C}} \big(\langle w', E^{(1)}_W(v_1; P_r((\Phi(v_2 \otimes v_3))(z_2 - \zeta, z_3 - \zeta)))(z_1, \zeta)\rangle$$
$$+\langle w', (\Phi(v_1 \otimes P_r((E^{(2)}_V(v_2 \otimes v_3; \mathbf{1}))(z_2 - \zeta, z_3 - \zeta))))(z_1, \zeta)\rangle\big)$$

and

$$\sum_{r \in \mathbb{C}} \big(\langle w', (\Phi(P_r((E^{(2)}_V(v_1 \otimes v_2; \mathbf{1}))(z_1 - \zeta, z_2 - \zeta)) \otimes v_3))(\zeta, z_3)\rangle$$
$$+\langle w', E^{W;(1)}_{WV}(P_r((\Phi(v_1 \otimes v_2))(z_1 - \zeta, z_2 - \zeta)); v_3))(\zeta, z_3)\rangle\big)$$

are absolutely convergent in the regions $|z_1 - \zeta| > |z_2 - \zeta|, |z_2 - \zeta| > 0$ and $|\zeta - z_3| > |z_1 - \zeta|, |z_2 - \zeta| > 0$, respectively, and can be analytically extended to rational functions in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$. Note that here we do not require the individual series

$$\sum_{r \in \mathbb{C}} \big(\langle w', E^{(1)}_W(v_1; P_r((\Phi(v_2 \otimes v_3))(z_2 - \zeta, z_3 - \zeta)))(z_1, \zeta)\rangle,$$
$$\sum_{r \in \mathbb{C}} \langle w', (\Phi(v_1 \otimes P_r((E^{(2)}_V(v_2 \otimes v_3; \mathbf{1}))(z_2 - \zeta, z_3 - \zeta))))(z_1, \zeta)\rangle,$$
$$\sum_{r \in \mathbb{C}} \langle w', (\Phi(P_r((E^{(2)}_V(v_1 \otimes v_2; \mathbf{1}))(z_1 - \zeta, z_2 - \zeta)) \otimes v_3))(\zeta, z_3)\rangle,$$
$$\sum_{r \in \mathbb{C}} \langle w', E^{W;(1)}_{WV}(P_r((\Phi(v_1 \otimes v_2))(z_1 - \zeta, z_2 - \zeta)); v_3))(\zeta, z_3)\rangle$$

to be absolutely convergent. We denote the corresponding rational functions by

$$R(\langle w', (E_W^{(1)}(v_1; \Phi(v_2 \otimes v_3))(z_1, z_2, z_3)\rangle$$
$$+\langle w', (\Phi(v_1 \otimes E_V^{(2)}(v_2 \otimes v_3; \mathbf{1})))(z_1, z_2, z_3)\rangle)$$

and

$$R(\langle w', (\Phi(E_V^{(2)}(v_1 \otimes v_2; \mathbf{1})) \otimes v_3))(z_1, z_2, z_3)\rangle$$
$$+\langle w', (E_{WV}^{W;(1)}(\Phi(v_1 \otimes v_2); v_3))(z_1, z_2, z_3)\rangle).$$

Clearly, $\widehat{C}_m^2(V, W) \subset \widehat{C}_{\frac{1}{2}}^2(V, W)$ for $m \in \mathbb{Z}_+$. We define a coboundary operator

$$\hat{\delta}_{\frac{1}{2}}^2 : \widehat{C}_{\frac{1}{2}}^2(V, W) \to \widehat{C}_0^3(V, W)$$

by

$$\begin{aligned}
&\langle w', ((\hat{\delta}_{\frac{1}{2}}^2(\Phi))(v_1 \otimes v_2 \otimes v_3))(z_1, z_2, z_3)\rangle\\
&\quad = R(\langle w', (E_W^{(1)}(v_1; \Phi(v_2 \otimes v_3))(z_1, z_2, z_3)\rangle\\
&\qquad + \langle w', (\Phi(v_1 \otimes E_V^{(2)}(v_2 \otimes v_3; \mathbf{1})))(z_1, z_2, z_3)\rangle)\\
&\quad - R(\langle w', (\Phi(E_V^{(2)}(v_1 \otimes v_2; \mathbf{1})) \otimes v_3))(z_1, z_2, z_3)\rangle\\
&\qquad + \langle w', (E_{WV}^{W;(1)}(\Phi(v_1 \otimes v_2); v_3))(z_1, z_2, z_3)\rangle)
\end{aligned}$$

for $w' \in W'$, $\Phi \in \widehat{C}_{\frac{1}{2}}^2(V, W)$, $v_1, v_2, v_3 \in V$ and $(z_1, z_2, z_3) \in F_3\mathbb{C}$.

Proposition 4.1. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+ + 1$, $\hat{\delta}_{m-1}^{n+1} \circ \hat{\delta}_m^n = 0$. We also have $\hat{\delta}_{\frac{1}{2}}^2 \circ \hat{\delta}_2^1 = 0$.*

Proof. Let $\Phi \in \widehat{C}^n(V, W)$. Then

$$\begin{aligned}
&(\hat{\delta}_{m-1}^{n+1} \circ \hat{\delta}_m^n)(\Phi)\\
&\quad = E_W^{(1)} \circ_2 (\hat{\delta}_m^n(\Phi)) + \sum_{i=1}^{n+1}(-1)^i (\hat{\delta}_m^n(\Phi)) \circ_i \otimes E_{V;\mathbf{1}}^{(2)}\\
&\qquad + (-1)^{n+2}\sigma_{n+2,1,\dots,n+1}(E_W^{(1)} \circ_2 (\hat{\delta}_m^n(\Phi)))\\
&\quad = E_W^{(1)} \circ_2 (E_W^{(1)} \circ_2 \Phi) + \sum_{j=1}^{n}(-1)^j E_W^{(1)} \circ_2 (\Phi \circ_{j+1} E_{V;\mathbf{1}}^{(2)})\\
&\qquad + (-1)^{n+1} E_W^{(1)} \circ_2 (\sigma_{n+1,1,\dots,n}(E_W^{(1)} \circ_2 \Phi))\\
&\qquad - (E_W^{(1)} \circ_2 \Phi) \circ_1 E_{V;\mathbf{1}}^{(2)} + \sum_{i=2}^{n+1}(-1)^i (E_W^{(1)} \circ_2 \Phi) \circ_i E_{V;\mathbf{1}}^{(2)}\\
&\qquad + \sum_{i=1}^{n+1}(-1)^i \sum_{j=1}^{i-2}(-1)^j (\Phi \circ_j E_{V;\mathbf{1}}^{(2)}) \circ_i E_{V;\mathbf{1}}^{(2)}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=2}^{n+1}(-1)^{i}(-1)^{i-1}(\Phi \circ_{i-1} E^{(2)}_{V;\mathbf{1}}) \circ_{i} E^{(2)}_{V;\mathbf{1}}\\
&+ \sum_{i=1}^{n}(-1)^{i}(-1)^{i}(\Phi \circ_{i} E^{(2)}_{V;\mathbf{1}}) \circ_{i} E^{(2)}_{V;\mathbf{1}}\\
&+ \sum_{i=1}^{n+1}(-1)^{i}\sum_{j=i+2}^{n+1}(-1)^{j-1}(\Phi \circ_{j} E^{(2)}_{V;\mathbf{1}}) \circ_{i} E^{(2)}_{V;\mathbf{1}}\\
&+ (-1)^{n+1}\sum_{i=1}^{n}(-1)^{i}(\sigma_{n+1,1,\dots,n}(E^{(1)}_{W} \circ_{2} \Phi)) \circ_{i} E^{(2)}_{V;\mathbf{1}}\\
&+ (\sigma_{n+1,1,\dots,n}(E^{(1)}_{W} \circ_{2} \Phi)) \circ_{n+1} E^{(2)}_{V;\mathbf{1}}\\
&+(-1)^{n+2}\sigma_{n+2,1,\dots,n+1}(E^{(1)}_{W} \circ_{2} (E^{(1)}_{W} \circ_{2} \Phi))\\
&+ (-1)^{n+2}\sum_{j=1}^{n}(-1)^{j}\sigma_{n+2,1,\dots,n+1}(E^{(1)}_{W} \circ_{2} (\Phi \circ_{j} E^{(2)}_{V;\mathbf{1}}))\\
&- \sigma_{n+2,1,\dots,n+1}(E^{(1)}_{W} \circ_{2} \sigma_{n+1,1,\dots,n}(E^{(1)}_{W} \circ_{2} \Phi)).
\end{aligned}
\tag{4.2}
$$

We now prove that in the right-hand side of (4.2), (i) the first and the fourth terms, (ii) the second and the fifth terms, (iii) the third and the twelfth terms, (iv) the sixth and the ninth term, (v) the seventh and the eighth terms, (vi) the tenth and the thirteenth terms, (vii) the eleventh and fourteenth terms cancel with each other, and thus the right-hand side of (4.1) is equal to 0, proving the proposition. In the proofs of these cancellations below, we actually also have to switch the order of absolutely convergent iterated sums. But just as in the proof of Proposition 3.10, since all the iterated sums are absolutely convergent to rational functions, the corresponding multisums are all absolutely convergent and thus all iterated sums are equal. Because of this general fact, we shall omit the discussion of the orders of the iterated sums.

(i) The first and the fourth terms: For $w' \in W'$, $v_1, \dots, v_{n+2} \in V$ and $(z_1, \dots, z_{n+2}) \in F_{n+2}\mathbb{C}$, applying the first term to $v_1 \otimes \cdots \otimes v_{n+2}$, evaluating the result at $(z_1, \dots, z_{n+2})$ and then pairing the result with w', we obtain

$$
\begin{aligned}
&\langle w', (E(E^{(1)}_{W}(v_1; E(E^{(1)}_{W}(v_2; \Phi(v_3 \otimes \cdots \otimes v_{n+2}))))))(z_1, \dots, z_{n+1})\rangle\\
&\quad = R(\langle w', (E^{(1)}_{W}(v_1; E(E^{(1)}_{W}(v_2; \Phi(v_3 \otimes \cdots \otimes v_{n+2})))))(z_1, \dots, z_{n+1})\rangle)\\
&\quad = R(\langle w', Y_W(v_1, z_1)E(E^{(1)}_{W}(v_2; \Phi(v_3 \otimes \cdots \otimes v_{n+2})))(z_1, \dots, z_{n+1})\rangle)\\
&\quad = R(\langle w', Y_W(v_1, z_1)Y_W(v_2, z_2)(\Phi(v_3 \otimes \cdots \otimes v_{n+2}))(z_3, \dots, z_{n+2})\rangle)\\
&\quad = R(\langle w', Y_W(Y_V(v_1, z_1 - z_2)v_2, z_2)(\Phi(v_3 \otimes \cdots \otimes v_{n+2}))(z_3, \dots, z_{n+2})\rangle),
\end{aligned}
\tag{4.3}
$$

where in the last step, we have used the associativity of the module W. On the other hand, applying the fourth term to $v_1 \otimes \cdots \otimes v_{n+2}$, evaluating the result at $(z_1, \dots, z_{n+2})$ and then pairing the result with w', we obtain

$$
\begin{aligned}
&\langle w', -((E_W^{(1)} \circ_2 \Phi)(E_V^{(2)}(v_1 \otimes v_2; \mathbf{1}) \otimes v_3 \otimes \cdots \otimes v_{n+2}))(z_1, \ldots, z_{n+2})\rangle\\
&\quad = -\langle w', (E(E_W^{(1)}(E_V^{(2)}(v_1 \otimes v_2; \mathbf{1}); \Phi(v_3 \otimes \cdots \otimes v_{n+2})))))(z_1, \ldots, z_{n+2})\rangle\\
&\quad = -R(\langle w', (E_W^{(1)}(E_V^{(2)}(v_1 \otimes v_2; \mathbf{1}); \Phi(v_3 \otimes \cdots \otimes v_{n+2})))(z_1, \ldots, z_{n+2})\rangle)\\
&\quad = -R(\langle w', Y_W(Y_V(v_1, z_1 - z_2)v_2, z_2)(\Phi(v_3 \otimes \cdots \otimes v_{n+2}))(z_3, \ldots, z_{n+2})\rangle).
\end{aligned} \tag{4.4}
$$

Since the right-hand sides of (4.3) and (4.4) differ only by a sign for $w' \in W'$, $v_1, \ldots, v_{n+2} \in V$ and $(z_1, \ldots, z_{n+2}) \in F_{n+2}\mathbb{C}$, these two terms indeed cancel with each other.

(ii) The second and the fifth terms: By Proposition 3.10, these two terms differ only by a sign and thus cancel with each other.

(iii) The third and the twelfth terms: For $w' \in W'$, $v_1, \ldots, v_{n+2} \in V$ and $(z_1, \ldots, z_{n+2}) \in F_{n+2}\mathbb{C}$, applying the third term to $v_1 \otimes \cdots \otimes v_{n+2}$, evaluating the result at $(z_1, \ldots, z_{n+2})$ and then pairing the result with w', we obtain

$$
\begin{aligned}
&\langle w', (-1)^{n+1}(E(E_W^{(1)}(v_1;\\
&\qquad \sigma_{n+1,1,\ldots,n}(E(E_W(v_{n+2}; \Phi(v_2 \otimes \cdots \otimes v_{n+1})))))))(z_1, \ldots, z_{n+2})\rangle\\
&= (-1)^{n+1} R(\langle w', (E_W^{(1)}(v_1;\\
&\qquad \sigma_{n+1,1,\ldots,n}(E(E_W(v_{n+2}; \Phi(v_2 \otimes \cdots \otimes v_{n+1}))))))(z_1, \ldots, z_{n+2})\rangle)\\
&= (-1)^{n+1} R(\langle w', Y_W(v_1, z_1)\\
&\qquad \cdot\sigma_{n+1,1,\ldots,n}(E(E_W(v_{n+2}; \Phi(v_2 \otimes \cdots \otimes v_{n+1}))))(z_2, \ldots, z_{n+2})\rangle)\\
&= (-1)^{n+1} R(\langle w', Y_W(v_1, z_1)\\
&\qquad \cdot(E(E_W(v_{n+2}; \Phi(v_2 \otimes \cdots \otimes v_{n+1}))))(z_{n+2}, z_2 \ldots, z_{n+1})\rangle)\\
&= (-1)^{n+1} R(\langle w', Y_W(v_1, z_1)Y_W(v_{n+2}, z_{n+2})\\
&\qquad\qquad \cdot(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2 \ldots, z_{n+1})\rangle)\\
&= (-1)^{n+1} R(\langle w', Y_W(v_{n+2}, z_{n+2})Y_W(v_1, z_1)\\
&\qquad\qquad \cdot(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2 \ldots, z_{n+1})\rangle),
\end{aligned} \tag{4.5}
$$

where in the last step, we have used the commutativity of the module W. On the other hand, applying the twelfth term to $v_1 \otimes \cdots \otimes v_{n+2}$, evaluating the result at $(z_1, \ldots, z_{n+2})$ and then pairing the result with w', we obtain

$$
\begin{aligned}
&\langle w', (-1)^{n+2}\sigma_{n+2,1,\ldots,n+1}(E(E_W^{(1)}(v_{n+2}; E(E_W^{(1)}(v_1;\\
&\qquad\qquad \Phi(v_2 \otimes \cdots \otimes v_{n+1}))))))(z_1, \ldots, z_{n+2})\rangle\\
&= (-1)^{n+2} R(\langle w', (E_W^{(1)}(v_{n+2}; E(E_W^{(1)}(v_1;\\
&\qquad\qquad \Phi(v_2 \otimes \cdots \otimes v_{n+1})))))(z_{n+2}, z_1, \ldots, z_{n+1})\rangle)\\
&= (-1)^{n+2} R(\langle w', Y_W(v_{n+2}, z_{n+2})Y_W(v_1, z_1)\\
&\qquad\qquad \cdot(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, \ldots, z_{n+1})\rangle).
\end{aligned} \tag{4.6}
$$

Since the right-hand sides of (4.5) and (4.6) differ only by a sign for $w' \in W'$, $v_1, \ldots, v_{n+2} \in V$ and $(z_1, \ldots, z_{n+2}) \in F_{n+2}\mathbb{C}$, these two terms indeed cancel with each other.

(iv) The sixth and the ninth terms: By Proposition 3.10, when $i \neq j$, we have

$$
(\Phi \circ_j E_{V;\mathbf{1}}^{(2)}) \circ_i E_{V;\mathbf{1}}^{(2)} = \Phi \circ (E_{V;\mathbf{1}}^{(l_1)} \otimes \cdots \otimes E_{V;\mathbf{1}}^{(l_n)}),
$$

where $l_k = 1$ if $k \neq i, j$ and $l_k = 2$ if $k = i$ or $k = j$. Since for any $a_{ij} \in C^{n+2}(V, W)$,

$$\sum_{j=1}^{n+1}\sum_{i=1}^{j-2} a_{ij} = \sum_{i=1}^{n+1}\sum_{j=i+2}^{n+1} a_{ij},$$

we see that these two terms differ by a sign and thus cancel with each other.

(v) The seventh and the eighth terms: By Proposition 3.10, we have

$$(\Phi \circ_{i-1} E^{(2)}_{V;\mathbf{1}}) \circ_i E^{(2)}_{V;\mathbf{1}} = \Phi \circ_{i-1} E^{(3)}_{V;\mathbf{1}}$$

for $i = 2, \ldots, n+1$ and

$$(\Phi \circ_i E^{(2)}_{V;\mathbf{1}}) \circ_i E^{(2)}_{V;\mathbf{1}} = \Phi \circ_i E^{(3)}_{V;\mathbf{1}}$$

for $i = 1, \ldots, n$. Thus these two terms differ by a sign and cancel with each other.

(vi) The tenth and the thirteenth terms: Since i or j are less than or equal to n, by definition, we have

$$(\sigma_{n+1,1,\ldots,n}(E^{(1)}_W \circ_2 \Phi)) \circ_i E^{(2)}_{V;\mathbf{1}} = \sigma_{n+2,1,\ldots,n+1}(E^{(1)}_W \circ_2 (\Phi \circ_i E^{(2)}_{V;\mathbf{1}}).$$

Then by Proposition 3.10, these two terms differ by a sign and thus cancel with each other.

(vii) The eleventh and the fourteenth terms: For $w' \in W'$, $v_1, \ldots, v_{n+2} \in V$ and $(z_1, \ldots, z_{n+2}) \in F_{n+2}\mathbb{C}$, applying the eleventh term to $v_1 \otimes \cdots \otimes v_{n+2}$, evaluating the result at $(z_1, \ldots, z_{n+2})$ and then pairing the result with w', we obtain

$$\begin{aligned}
&\langle w', ((\sigma_{n+1,1,\ldots,n}(E^{(1)}_W \circ_2 \Phi))(v_1 \otimes \cdots \otimes v_n \\
&\qquad\qquad \otimes E^{(2)}_V(v_{n+1} \otimes v_{n+2}; \mathbf{1})))(z_1, \ldots, z_{n+2})\rangle \\
&\quad = \langle w', ((E^{(1)}_W \circ_2 \Phi)(E^{(2)}_V(v_{n+1} \otimes v_{n+2}; \mathbf{1}) \\
&\qquad\qquad \otimes v_1 \otimes \cdots \otimes v_n))(z_{n+1}, z_{n+2}, z_1, \ldots, z_n)\rangle \\
&\quad = \langle w', (E(E^{(1)}_W(E^{(2)}_V(v_{n+1} \otimes v_{n+2}; \mathbf{1}); \\
&\qquad\qquad \Phi(v_1 \otimes \cdots \otimes v_n))))(z_{n+1}, z_{n+2}, z_1, \ldots, z_n)\rangle \\
&\quad = R(\langle w', (E^{(1)}_W(E^{(2)}_V(v_{n+1} \otimes v_{n+2}; \mathbf{1}); \\
&\qquad\qquad \Phi(v_1 \otimes \cdots \otimes v_n)))(z_{n+1}, z_{n+2}, z_1, \ldots, z_n)\rangle) \\
&\quad = R(\langle w', Y_W(Y_V(v_{n+1}, z_{n+1} - z_{n+2})v_{n+2}, z_{n+2}) \\
&\qquad\qquad \cdot(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle) \\
&\quad = R(\langle w', Y_W(v_{n+1}, z_{n+1})Y_W(v_{n+2}, z_{n+2})(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle) \\
&\quad = R(\langle w', Y_W(v_{n+2}, z_{n+2})Y_W(v_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n)\rangle),
\end{aligned} \tag{4.7}$$

where in the last two steps, we have used the associativity and commutativity of the V-module W. On the other hand, applying the fourteenth term to $v_1 \otimes \cdots \otimes v_{n+2}$, evaluating the result at $(z_1, \dots, z_{n+2})$ and then pairing the result with w', we obtain

$$\begin{aligned}
&\langle w', ((E_W^{(1)} \circ_2 \sigma_{n+1,1,\dots,n}(E_W^{(1)} \circ_2 \Phi))(v_{n+2} \otimes v_1 \otimes \\
&\qquad\qquad\qquad \cdots \otimes v_{n+1}))(z_{n+2}, z_1, \dots, z_{n+1})\rangle \\
&= -\langle w', (E(E_W^{(1)}(v_{n+2}; \\
&\qquad ((\sigma_{n+1,1,\dots,n}(E_W^{(1)} \circ_2 \Phi))(v_1 \otimes \cdots \otimes v_{n+1})))))(z_{n+2}, z_1, \dots, z_{n+1})\rangle \\
&= -R(\langle w', Y_W(v_{n+2}, z_{n+2}) \\
&\qquad \cdot((\sigma_{n+1,1,\dots,n}(E_W^{(1)} \circ_2 \Phi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \dots, z_{n+1})\rangle) \\
&= -R(\langle w', Y_W(v_{n+2}, z_{n+2}) \\
&\qquad \cdot((E_W^{(1)} \circ_2 \Phi)(v_{n+1} \otimes v_1 \cdots \otimes v_n))(z_{n+1}, z_1 \dots, z_n)\rangle) \\
&= -R(\langle w', Y_W(v_{n+2}, z_{n+2}) \\
&\qquad \cdot(E(E_W^{(1)}(v_{n+1}; \Phi(v_1 \otimes \cdots \otimes v_n))))(z_{n+1}, z_1, \dots, z_n)\rangle) \\
&= -R(\langle w', Y_W(v_{n+2}, z_{n+2})Y_W(v_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, \dots, z_n)\rangle).
\end{aligned} \tag{4.8}$$

Since the right-hand sides of (4.7) and (4.8) differ only by a sign for $w' \in W'$, $v_1, \dots, v_{n+2} \in V$ and $(z_1, \dots, z_{n+2}) \in F_{n+2}\mathbb{C}$, these two terms indeed cancel with each other.

The second conclusion follows from the first conclusion. In fact since

$$\hat{\delta}_2^1(\widehat{C}_2^1(V, W)) \subset \widehat{C}_1^2(V, W) \subset \widehat{C}_{\frac{1}{2}}^2(V, W),$$
$$\hat{\delta}_{\frac{1}{2}}^2 \circ \hat{\delta}_2^1 = \hat{\delta}_1^2 \circ \hat{\delta}_2^1 = 0.$$

□

By Proposition 4.1, we have complexes

$$0 \longrightarrow \widehat{C}_m^0(V, W) \xrightarrow{\hat{\delta}_m^0} \widehat{C}_{m-1}^1(V, W) \xrightarrow{\hat{\delta}_{m-1}^1} \cdots \xrightarrow{\hat{\delta}_1^{m-1}} \widehat{C}_0^m(V, W) \longrightarrow 0 \tag{4.9}$$

for $m \in \mathbb{N}$ and

$$0 \longrightarrow \widehat{C}_3^0(V, W) \xrightarrow{\hat{\delta}_3^0} \widehat{C}_2^1(V, W) \xrightarrow{\hat{\delta}_2^1} \widehat{C}_{\frac{1}{2}}^2(V, W) \xrightarrow{\hat{\delta}_{\frac{1}{2}}^2} \widehat{C}_0^3(V, W) \longrightarrow 0, \tag{4.10}$$

where the first and last arrows are the trivial embeddings and projections. But these complexes are not yet the chain complexes for V. We have to consider certain subcomplexes of these complexes. To define these subcomplexes, we need to use shuffles.

For $n \in \mathbb{N}$ and $1 \leq p \leq n-1$, let $J_{n;p}$ be the set of elements of S_n which preserve the order of the first p numbers and the order of the last $n-p$ numbers, that is,

$$J_{n,p} = \{\sigma \in S_n \mid \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(n)\}.$$

Elements of $J_{n;p}$ are called *shuffles*. Let $J_{n;p}^{-1} = \{\sigma \mid \sigma \in J_{n;p}\}$. For $m, n \in \mathbb{N}$ and or $m = \frac{1}{2}$, $n = 2$, let $C_m^n(V, W)$ be the subspace of $\widehat{C}_m^n(V, W)$ consisting of maps Φ such that

$$\sum_{\sigma \in J_{n;p}^{-1}} (-1)^{|\sigma|}\sigma(\Phi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})) = 0.$$

We also let

$$C^n_\infty(V, W) = \bigcap_{m\in\mathbb{N}} C^n_m(V, W).$$

Theorem 4.2. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, $\hat{\delta}^n_m(C^n_m(V, W)) \subset C^{n+1}_{m-1}(V, W)$. Also $\hat{\delta}^2_{\frac{1}{2}}(C^2_{\frac{1}{2}}(V, W)) \subset C^3_0(V, W)$.*

Proof. Let $\Phi \in C^n_m(V, W)$. We need to prove $\hat{\delta}^n_m(\Phi) \in C^{n+1}_{m-1}(V, W)$. By definition, we have

$$\begin{aligned}
&\sum_{\sigma\in J^{-1}_{n+1;p}} (-1)^{|\sigma|}\sigma(\hat{\delta}^n_m(\Phi))\\
&\quad= \sum_{\sigma\in J^{-1}_{n+1;p}} (-1)^{|\sigma|}\sigma(E^{(1)}_W \circ_2 \Phi)\\
&\quad\quad+ \sum_{\sigma\in J^{-1}_{n+1;p}} (-1)^{|\sigma|} \sum_{i=1}^{n}(-1)^i\sigma(\Phi \circ_i E^{(2)}_{V;\mathbf{1}})\\
&\quad\quad+(-1)^{n+1} \sum_{\sigma\in J^{-1}_{n+1;p}} (-1)^{|\sigma|}\sigma(\sigma_{n+1,1\dots,n}(E^{(1)}_W \circ_2 \Phi)).
\end{aligned} \tag{4.11}$$

Note that for any $\sigma \in J^{-1}_{n+1;p}$, $\sigma(1)$ is either 1 or $p + 1$. So the first term in the right-hand side of (4.11) is equal to

$$\sum_{\substack{\sigma \in J^{-1}_{n+1;p}\\ \sigma(1)=1}} (-1)^{|\sigma|}\sigma(E^{(1)}_W \circ_2 \Phi) + \sum_{\substack{\sigma \in J^{-1}_{n+1;p}\\ \sigma(1)=p+1}} (-1)^{|\sigma|}\sigma(E^{(1)}_W \circ_2 \Phi). \tag{4.12}$$

The subset $\{\sigma \in J^{-1}_{n+1;p} \mid \sigma(1) = 1\}$ of S_{n+1} is the image of $J^{-1}_{n;p}$ under the embedding from S_n to S_{n+1} given by mapping an element of S_n to an element of S_{n+1} permuting only the last n numbers. Since $\Phi \in C^n_m(V, W)$, the first term in (4.12) is equal to 0. Similarly, the subset $\{\sigma \in J^{-1}_{n+1;p} \mid \sigma(1) = p+1\}$ of S_{n+1} is the image of $J^{-1}_{n;p}$ under the embedding from S_n to S_{n+1} given by mapping an element of S_n to an element of S_{n+1} permuting only the numbers $1, \dots, p, p+2, \dots, n+1$. So the second term in (4.12) is also equal to 0. Thus the first term in the right-hand side of (4.11) is equal to 0. Similarly, with $n + 1$ and p playing the role of 1 and $p + 1$ in the argument above, the last term in the right-hand side of (4.11) is also equal to 0.

The second term in the right-hand side of (4.11) is equal to

$$\begin{aligned}
&\sum_{i=1}^{p}(-1)^i \sum_{\substack{\sigma \in J^{-1}_{n+1;p}\\ 1\leq\sigma(i),\ \sigma(i+1)\leq p}} (-1)^{|\sigma|}\sigma(\Phi \circ_i E^{(2)}_{V;\mathbf{1}})\\
&\quad+\sum_{i=1}^{p}(-1)^i \sum_{\substack{\sigma \in J^{-1}_{n+1;p}\\ p+1\leq\sigma(i),\ \sigma(i+1)\leq n}} (-1)^{|\sigma|}\sigma(\Phi \circ_i E^{(2)}_{V;\mathbf{1}})
\end{aligned}$$

$$+\sum_{i=p+1}^{n}(-1)^i \sum_{\substack{\sigma\in J_{n+1;p}^{-1}\\ 1\le\sigma(i)\le p\\ p+1\le\sigma(i+1)\le n}} (-1)^{|\sigma|}\sigma(\Phi\circ_i E^{(2)}_{V;\mathbf{1}})$$

$$+\sum_{i=p+1}^{n}(-1)^i \sum_{\substack{\sigma\in J_{n+1;p}^{-1}\\ p+1\le\sigma(i)\le n\\ 1\le\sigma(i+1)\le p}} (-1)^{|\sigma|}\sigma(\Phi\circ_i E^{(2)}_{V;\mathbf{1}}). \tag{4.13}$$

For $\sigma \in J_{n+1;p}^{-1}$, if $1 \le \sigma(i), \sigma(i+1) \le p$ or $p+1 \le \sigma(i), \sigma(i+1) \le n$, then $\sigma(i+1) = \sigma(i)+1$. In these cases, the permutation of the numbers $1, \dots, i-1, i+1, \dots, n$ induced from σ^{-1} is in $J_{n;p-1}^{-1}$ or $J_{n;p}^{-1}$ and every element of $J_{n;p-1}^{-1}$ or $J_{n;p}^{-1}$ is obtained uniquely in this way. Since $\Phi \in C_m^n(V, W)$, the first two terms in (4.13) are equal to 0. Let $\sigma \in J_{n+1;p}^{-1}$ satisfying $p+1 \le \sigma(i) \le n,\ 1 \le \sigma(i+1) \le p$. Let $\sigma_{i+1,i}$ be the transposition exchanging i and $i+1$. Then $\tau = \sigma \circ \sigma_{i+1,i}$ is an element of $J_{n+1;p}^{-1}$ satisfying $1 \le \tau(i) \le p,\ p+1 \le \tau(i+1) \le n$. Moreover, $|\sigma \circ \sigma_{i+1,i}| = |\sigma| + 1$. Also, by the commutativity of V,

$$(\sigma\circ\sigma_{i+1,i})(\Phi\circ_i E^{(2)}_{V;\mathbf{1}}) = \sigma(\Phi\circ_i E^{(2)}_{V;\mathbf{1}}).$$

Thus the third term and the fourth term in (4.13) cancel with each other. The calculations above show that (4.13) is equal to 0.

Now we have proved that the right-hand side of (4.11) is equal to 0. By (4.11), $\hat{\delta}_m^n(\Phi) \in C_{m-1}^{n+1}(V, W)$.

Let $\Phi \in C_{\frac{1}{2}}^2(V, W)$. Note that $J_{3;2} = \sigma_{3,1,2}(J_{3;1})$. To prove $\hat{\delta}^2_{\frac{1}{2}}(\Phi) \in C_0^3(V, W)$, we need only prove

$$\sum_{\sigma\in J_{3;1}^{-1}} (-1)^{|\sigma|}\sigma(\hat{\delta}^2_{\frac{1}{2}}(\Phi)) = 0. \tag{4.14}$$

By definition, for $v_1, v_2, v_3 \in V$, $w' \in W'$, $(z_1, z_2, z_3) \in F_3\mathbb{C}$ and $\zeta \in \mathbb{C}$ such that $(z_1-\zeta, z_2-\zeta)$, $(z_2-\zeta, z_3-\zeta)$, $(z_1-\zeta, z_3-\zeta)$, (z_1, ζ), (z_2, ζ), $(z_3, \zeta) \in F_2\mathbb{C}$, we have

$$\begin{aligned}
&\sum_{\sigma\in J_{3;1}^{-1}} (-1)^{|\sigma|}\langle w', \sigma(\hat{\delta}^2_{\frac{1}{2}}(\Phi))(v_1\otimes v_2\otimes v_3))(z_1, z_2, z_3)\rangle\\
&\quad= \sum_{\sigma\in J_{3;1}^{-1}} (-1)^{|\sigma|}\langle w', (\hat{\delta}^2_{\frac{1}{2}}(\Phi))(v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes v_{\sigma(3)}))(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})\rangle\\
&\quad= \langle w', (\hat{\delta}^2_{\frac{1}{2}}(\Phi))(v_1\otimes v_2\otimes v_3))(z_1, z_2, z_3)\rangle\\
&\qquad - \langle w', (\hat{\delta}^2_{\frac{1}{2}}(\Phi))(v_2\otimes v_1\otimes v_3))(z_2, z_1, z_3)\rangle\\
&\qquad + \langle w', (\hat{\delta}^2_{\frac{1}{2}}(\Phi))(v_2\otimes v_3\otimes v_1))(z_2, z_3, z_1)\rangle\\
&\quad= R(\langle w', (E^{(1)}_W(v_1; \Phi(v_2\otimes v_3)))(z_1, z_2, z_3)\rangle\\
&\qquad\quad + \langle w', (\Phi(v_1\otimes (E^{(2)}_V(v_2\otimes v_3; \mathbf{1})))(z_1, z_2, z_3)\rangle)
\end{aligned}$$

$$
\begin{aligned}
&- R(\langle w', (\Phi(E_V^{(2)}(v_1 \otimes v_2; \mathbf{1}) \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\quad + \langle w', (E_{WV}^{W;(1)}(\Phi(v_1 \otimes v_2); v_3))(z_1, z_2, z_3)\rangle) \\
&- R(\langle w', (E_W^{(1)}(v_2; \Phi(v_1 \otimes v_3)))(z_1, z_2, z_3)\rangle \\
&\quad + \langle w', (\Phi(v_2 \otimes E_V^{(2)}(v_1 \otimes v_3; \mathbf{1}))(z_1, z_2, z_3)\rangle) \\
&+ R(\langle w', (\Phi(E_V^{(2)}(v_2 \otimes v_1; \mathbf{1}) \otimes v_3))(z_1, z_2, z_3)\rangle \\
&\quad + \langle w', (E_{WV}^{W;(1)}(\Phi(v_2 \otimes v_1); v_3))(z_1, z_2, z_3)\rangle) \\
&+ R(\langle w', (E_W^{(1)}(v_2; \Phi(v_3 \otimes v_1)))(z_1, z_2, z_3)\rangle \\
&\quad + \langle w', (\Phi(v_2 \otimes (E_V^{(2)}(v_3 \otimes v_1; \mathbf{1})))(z_1, z_2, z_3)\rangle) \\
&- R(\langle w', (\Phi(E_V^{(2)}(v_2 \otimes v_3; \mathbf{1}) \otimes v_1))(z_1, z_2, z_3)\rangle \\
&\quad - \langle w', (E_{WV}^{W;(1)}(\Phi(v_2 \otimes v_3); v_1))(z_1, z_2, z_3)\rangle)
\end{aligned} \tag{4.15}
$$

Since $\Phi \in C_{\frac{1}{2}}^2(V, W)$ and $E_{V;\mathbf{1}}^{(2)} \in C_{\frac{1}{2}}^2(V, V)$, we have

$$
(\Phi(u_1 \otimes u_2))(\zeta_1, \zeta_2) = (\Phi(u_2 \otimes u_1))(\zeta_2, \zeta_1),
$$
$$
(E_V^{(2)}(u_1 \otimes u_2; \mathbf{1}))(\zeta_1, \zeta_2) = (E_V^{(2)}(u_2 \otimes u_1; \mathbf{1}))(\zeta_2, \zeta_1)
$$

for $u_1, u_2 \in V$ and $(\zeta_1, \zeta_2) \in F_2\mathbb{C}$. Also

$$
E_W^{(1)}(u; w) = E_{WV}^{W;(1)}(w; u)
$$

for $u \in V$ and $w \in W$. From these formulas, we see that the first and sixth terms, the second and fourth terms and the third and fifth terms in (4.15) cancel with each other, proving (4.14). □

Let $\delta_m^n = \hat{\delta}_m^n|_{C_m^n(V,W)}$ for $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$ or $m = \frac{1}{2}$, $n = 2$. By Theorem 4.2, the image of δ_m^n is in $C_{m-1}^{n+1}(V, W)$. So we obtain a linear map

$$
\delta_m^n : C_m^n(V, W) \to C_{m-1}^{n+1}(V, W)
$$

for each pair $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$ or

$$
\delta_{\frac{1}{2}}^2 : C_{\frac{1}{2}}^2(V, W) \to C_0^3(V, W).
$$

For $m \in \mathbb{Z}_+$, we have a subcomplex

$$
0 \longrightarrow C_m^0(V, W) \xrightarrow{\delta_m^0} C_{m-1}^1(V, W) \xrightarrow{\delta_{m-1}^1} \cdots \xrightarrow{\delta_1^{m-1}} C_0^m(V, W) \longrightarrow 0 \tag{4.16}
$$

of the complex (4.9) and also a subcomplex

$$
0 \longrightarrow C_3^0(V, W) \xrightarrow{\delta_3^0} C_2^1(V, W) \xrightarrow{\delta_2^1} C_{\frac{1}{2}}^2(V, W) \xrightarrow{\delta_{\frac{1}{2}}^2} C_0^3(V, W) \longrightarrow 0 \tag{4.17}
$$

of the complex (4.10).

Since $C_{\infty}^{n}(V, W) \subset C_{m}^{n}(V, W)$ for any $m \in \mathbb{Z}_{+}$ and $C_{m_2}^{n}(V, W) \subset C_{m_1}^{n}(V, W)$, for $m_1, m_2 \in \mathbb{Z}_{+}$ satisfying $m_1 \leq m_2$, $\delta_{m}^{n}\Big|_{C_{\infty}^{n}(V,W)}$ is independent of m. Let

$$\delta_{\infty}^{n} = \delta_{m}^{n}\Big|_{C_{\infty}^{n}(V,W)} : C_{\infty}^{n}(V, W) \to C_{\infty}^{n+1}(V, W).$$

We obtain a complex

$$0 \longrightarrow C_{\infty}^{0}(V, W) \xrightarrow{\delta_{\infty}^{0}} C_{\infty}^{1}(V, W) \xrightarrow{\delta_{\infty}^{1}} C_{\infty}^{2}(V, W) \xrightarrow{\delta_{\infty}^{2}} \cdots . \tag{4.18}$$

Using the complexes (4.16), (4.17) and (4.18), we now introduce the cohomology spaces of V.

Definition 4.3. *For $m \in \mathbb{Z}_{+}$ and $n \in \mathbb{N}$, we define the* nth cohomology $H_{m}^{n}(V, W)$ **of** V with coefficient in W and composable with m vertex operators *to be*

$$H_{m}^{n}(V, W) = \ker \delta_{m}^{n} / \mathrm{im}\ \delta_{m+1}^{n-1}.$$

We also define

$$H_{\frac{1}{2}}^{2}(V, W) = \ker \delta_{\frac{1}{2}}^{2} / \mathrm{im}\ \delta_{2}^{1}$$

and

$$H_{\infty}^{n}(V, W) = \ker \delta_{\infty}^{n} / \mathrm{im}\ \delta_{\infty}^{n-1}$$

for $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ and $m_1, m_2 \in \mathbb{Z}_{+}$, since $C_{m_2}^{n}(V, W) \subset C_{m_1}^{n}(V, W)$, we have

$$\delta_{m_2+1}^{n-1}(C_{m_2+1}^{n-1}(V, W)) \subset \delta_{m_1+1}^{n-1}(C_{m_2+1}^{n-1}(V, W)) \cap C_{m_2}^{n}(V, W),$$
$$\ker \delta_{m_2}^{n} \subset \ker \delta_{m_1}^{n} \cap C_{m_2}^{n}(V, W).$$

Thus we have an injective linear map $f_{m_1 m_2} : H_{m_2}^{n}(V, W) \to H_{m_1}^{n}(V, W)$ given by

$$f_{m_1 m_2}(\Phi + \ker \delta_{m_2}^{n}) = \Phi + \ker \delta_{m_1}^{n}.$$

Proposition 4.4. *For $n \in \mathbb{N}$, $(H_{m}^{n}(V, W), f_{m_1 m_2})$ is an inverse system. Moreover, their inverse limits are linearly isomorphic to $H_{\infty}^{n}(V, W)$ for $n \in \mathbb{N}$.*

Proof. This follows straightforwardly from the definitions. □

Proposition 4.5. *Let V be a grading-restricted vertex algebra and W a generalized V-module. Then $H_{m}^{0}(V, W) = W$ for any $m \in \mathbb{Z}_{+}$.*

Proof. By definition, for any $m \in \mathbb{Z}_{+}$, $C_{m}^{0}(V, W) = W$ and $\delta_{m}^{0}(C_{m}^{0}(V, W)) = 0$. So $H_{m}^{0}(V, W) = C_{m}^{0}(V, W) = W$. □

We shall discuss the first and second cohomologies in [Hu3].

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References

[A] André, M.: *Méthode simpliciale en algèbre homologique et algèbre commutative*. In: Lecture notes mathematics, Vol. **32**, Berlin: Springer, 1967
[BPZ] Belavin, A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetries in two-dimensional quantum field theory. Nucl. Phys. B **241**, 333–380 (1984)
[B1] Borcherds, R.: Vertex algebras, kac-moody algebras, and the monster. Proc. Natl. Acad. Sci. USA **83**, 3068–3071 (1986)
[B2] Borcherds, R.: *Vertex algebras*. In: Kashiwara, A., Matsuo, A., Saito, K., Satake, I. (eds.) Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., Vol. **160**, Boston, MA: Birkhäuser Boston, 1998, pp. 35–77
[CE] Chevalley, C., Eilenberg, S.: Cohomology theory of Lie groups and Lie algebras. Trans. Amer. Math. 930 Soc. **63**, 85–124 (1948)
[FHL] Frenkel, I., Huang, Y.-Z., Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules. Memoirs American Math. Soc. **104**, 1–64 (1993)
[FLM] Frenkel, I., Lepowsky, J., Meurman, A.: *Vertex operator algebras and the monster*. In: Pure and Applied Mathematics, Vol. **134**, UK: Academic Press, 1988
[GK] Ginzburg, V., Kapranov, M.: Koszul duality for operads. Duke Math. J. **76**, 203–272 (1994)
[Ha] Harrison, D.K.: Commutative algebras and cohomology. Trans. Am. Math. Soc. **104**, 191–204 (1962)
[Ho] Hochschild, G.: On the cohomology groups of an associative algebra. Ann. Math. (2) **46**, 58–67 (1945)
[Hu1] Huang, Y.-Z.: Geometric interpretation of vertex operator algebras. Proc. Natl. Acad. Sci. USA **88**, 9964–9968 (1991)
[Hu2] Huang, Y.-Z.: Two-dimensional conformal geometry and vertex operator algebras. In: Progress in Mathematics, Vol **148**. Boston: Birkhäuser, 1997
[Hu3] Huang, Y.-Z.: The first and second cohomologies of grading-restricted vertex algebras. Comm. Math. Phys. (2014, to appear)
[HK1] Huang, Y.-Z., Kong, L.: Open-string vertex algebras, tensor categories and operads. Comm. Math. Phys. **250**, 433–471 (2004)
[HK2] Huang, Y.-Z., Kong, L.: Full field algebras. Comm. Math. Phys. **272**, 345–396 (2007)
[HL1] Huang, Y.-Z., Lepowsky, J.: *Vertex operator algebras and operads*. In: Corwin, L., Gelfand, I., Lepowsky, J. (eds.) The Gelfand Mathematical Seminars, 1990–1992. Boston: Birkhäuser, 1993, pp. 145–161
[HL2] Huang, Y.-Z., Lepowsky, J.: *Operadic formulation of the notion of vertex operator algebra*. In: Proceeding of the 1992 Joint Summer Research Conference on Conformal Field Theory, Topological Field Theory and Quantum Groups, Mount Holyoke, 1992. Contemp. Math., Vol. **175**, Providence, RI: American Mathematical Society, 1994, pp. 131–148
[KV] Kimura, T., Voronov, A.A.: *The cohomology of algebras over moduli spaces*. In: The moduli spaces of curves (Texel Island, 1994). Progress in Mathematics, Vol. **129**, Boston: Birkhäuser Boston, 1995, pp. 305–334
[Q] Quillen, D.: Homology of Commutative Rings, Mimeographed Notes. Cambridge: MIT, 1968
[W] Weibel, C.: *An introduction to homological algebras*. In: Cambridge Studies in Advanced Mathematics, Vol. **38**, Cambridge: Cambridge University Press, 1994

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