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Blow Up Dynamics for Equivariant Critical Schrödinger Maps

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Abstract: For the Schrödinger map equation $u_t = u \times \triangle u$ in $\mathbb{R}^{2+1}$, with values in S^2, we prove for any $\nu > 1$ the existence of equivariant finite time blow up solutions of the form $u(x,t) = \phi(\lambda(t)x) + \zeta(x,t)$, where ϕ is a lowest energy steady state, $\lambda(t) = t^{-1/2-\nu}$ and $\zeta(t)$ is arbitrary small in $\dot{H}^1 \cap \dot{H}^2$.

1. Introduction

1.1. Setting of the problem and statement of the result. In this paper we consider the Schrödinger flow for maps from $\mathbb{R}^2$ to S^2:

$$\begin{aligned} &u_t = u \times \Delta u, \quad x = (x_1, x_2) \in \mathbb{R}^2, \ t \in \mathbb{R}, \\ &u|_{t=0} = u_0, \end{aligned} \tag{1.1}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in S^2 \subset \mathbb{R}^3$.

Equation (1.1) conserves the energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} dx |\nabla u|^2. \tag{1.2}$$

The problem is critical in the sense that both (1.1) and (1.2) are invariant with respect to the scaling $u(x,t) \to u(\lambda x, \lambda^2 t)$, $\lambda \in \mathbb{R}_+$.

To a finite energy map $u : \mathbb{R}^2 \to S^2$ one can associate the degree:

$$\deg(u) = \frac{1}{4\pi}\int_{\mathbb{R}^2} dx u_{x_1} \cdot J_u u_{x_2},$$

where J_u is defined by

$$J_u v = u \times v, \quad v \in \mathbb{R}^3.$$

It follows from (1.2) that

$$E(u) \geq 4\pi |\deg(u)|. \tag{1.3}$$

This inequality is saturated by the harmonic maps ϕ_m, $m \in \mathbb{Z}^+$:

$$\begin{aligned} &\phi_m(x) = e^{m\theta R} Q^m(r), \quad Q^m = (h_1^m, 0, h_3^m) \in S^2, \\ &h_1^m(r) = \frac{2r^m}{r^{2m}+1}, \quad h_3^m(r) = \frac{r^{2m}-1}{r^{2m}+1}. \end{aligned} \tag{1.4}$$

Here (r, θ) are polar coordinates in $\mathbb{R}^2$: $x_1 + ix_2 = e^{i\theta}r$, and R is the generator of the horizontal rotations:

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

or equivalently

$$Ru = \mathbf{k} \times u, \quad \mathbf{k} = (0, 0, 1).$$

One has

$$\deg \phi_m = m, \quad E(\phi_m) = 4\pi m.$$

Up to the symmetries ϕ_m are the only energy minimizers in their homotopy class.

Since ϕ_1 will play a central role in the analysis developed in this paper, we set $\phi = \phi_1$, $Q = Q_1$, $h_1 = h_1^1$, $h_3 = h_3^1$.

The local/global well-posedness of (1.1) has been extensively studied in past years. Local existence for smooth initial data goes back to [18], see also [14]. The case of small data of low regularity was studied in several works, the definite result being obtained by Bejenaru et al. in [3], where the global existence and scattering was proved for general $\dot{H}^1$ small initial data. Global existence for equivariant small energy initial data was proved earlier in [6] (by m-equivariant map $u : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$, $m \in \mathbb{Z}^+$ one means a map of the form $u(x) = e^{m\theta R} v(r)$, where $v : \mathbb{R}_+ \to S^2 \subset \mathbb{R}^3$, m-equivariance being preserved by the Schrödinger flow (1.1)). In the radial case $m = 0$, the global existence for H^2 data was established by Gustafson and Koo [11]. Very recently, Bejenaru et al. [4] proved global existence and scattering for equivariant data with energy less than 4π. The dynamics of m-equivariant Schrödinger maps with initial data close to ϕ_m was studied by Gustafson et al. [9,10,12] and later by Bejenaru and Tataru [5] in the case $m = 1$. The stability/instability results of these works strongly suggest a possibility of regularity breakdown in solutions of (1.1) via concentration of the lowest energy harmonic map ϕ. For a closely related model of wave maps this type of regularity breakdown was proved by Kriger et al. [13] and by Raphael and Rodnianski [17]. These authors showed the existence of 1- equivariant blow up solutions close to $\phi(\lambda(t)x)$ with $\lambda(t) \sim \frac{e^{\sqrt{|\ln(T^*-t)|}}}{T^*-t}$ as $t \to T^*$ [17], and with $\lambda(t) \sim \frac{1}{(T^*-t)^{1+\nu}}$ as $t \to T^*$ where $\nu > 1/2$ can be chosen arbitrarily [13] (here T^* is the blow up time). While the blow up dynamics exhibited in [17] is stable (in some strong topology), the continuum of blow up solutions constructed by Kriger et al. is believed to be non-generic. Recently, the results of [17] were generalized to the case of Schrödinger map equation (1.1) by Merle et al. in [15] where they proved the existence of 1-equivariant blow up solutions of (1.1) close to $\phi(\lambda(t)x)$ with $\lambda(t) \sim \frac{(\ln(T^*-t))^2}{T^*-t}$.

Our objective in this paper is to show that (1.1) also admits 1-equivariant Kriger–Schlag–Tataru type blow up solutions that correspond to certain initial data of the form

$$u_0 = \phi + \zeta_0,$$

where ζ_0 is 1-equivariant and can be chosen arbitrarily small in $\dot H^1 \cap \dot H^3$. Let us recall (see [5,9,10,12]) that such initial data result in unique local solutions of the same regularity, and as long as the solution exists it stays $\dot H^1$ close to a two parameter family of 1-equivariant harmonic maps $\phi^{\alpha,\lambda}$, $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, $\lambda \in \mathbb{R}_+$ generated from ϕ by rotations and scaling:

$$\phi^{\alpha,\lambda}(r,\theta) = e^{\alpha R}\phi(\lambda r, \theta).$$

The following theorem is the main result of this paper.

Theorem 1.1. *For any $\nu > 1$, $\alpha_0 \in \mathbb{R}$, and any $\delta > 0$ sufficiently small there exist $t_0 > 0$ and a 1-equivariant solution $u \in C((0, t_0], \dot H^1 \cap \dot H^3)$ of (1.1) of the form:*

$$u(x,t) = e^{\alpha(t)R}\phi(\lambda(t)x) + \zeta(x,t), \tag{1.5}$$

where

$$\lambda(t) = t^{-1/2-\nu}, \quad \alpha(t) = \alpha_0 \ln t, \tag{1.6}$$

$$\|\zeta(t)\|_{\dot H^1\cap \dot H^2} \le \delta, \quad \|\zeta(t)\|_{\dot H^3} \le C_{\nu,\alpha_0} t^{-1}, \quad \forall t \in (0, t_0]. \tag{1.7}$$

Furthermore, as $t \to 0$, $\zeta(t) \to \zeta^*$ in $\dot H^1 \cap \dot H^2$ with $\zeta^* \in H^{1+2\nu-}$.

Remark 1.2. In fact, using the arguments developed in this paper one can show that the same result remains valid with $\dot H^3$ replaced by $\dot H^{1+2s}$ for any $1 \le s < \nu$.

1.2. Strategy of the proof. The proof of Theorem 1.1 contains two main steps. The first step is a construction of approximate solutions $u^{(N)}$ that have the form (1.5), (1.6), (1.7), and solve (1.1) up to an arbitrarily high order error $O(t^N)$, very much in the spirit of the work of Kriger et al. [13].

The second step is to build the exact solution by solving the problem for the small remainder forward in time with zero initial data at $t = 0$. The control of remainder is achieved by means of suitable energy type estimates, see Sect. 3 for the details. The assumption $\nu > 1$ ensures that the approximate solutions that we have constructed, belong to $\dot H^1 \cap \dot H^3$, which allows us to work on the level of the H^3 well-posedness theory.

2. Approximate Solutions

2.1. Preliminaries. We consider (1.1) under the 1-equivariance assumption

$$u(x,t) = e^{\theta R}v(r,t), \quad v = (v_1, v_2, v_3) \in S^2 \subset \mathbb{R}^3. \tag{2.1}$$

Restricted to the 1-equivariant functions (1.1) takes the form

$$v_t = v \times (\triangle v + \frac{R^2}{r^2}v), \tag{2.2}$$

the energy being given by

$$E(u) = \pi \int_0^\infty dr r(|v_r|^2 + \frac{v_1^2 + v_2^2}{r^2}).$$

$Q = (h_1, 0, h_3)$ is a stationary solution of (2.2) and one has the relations

$$\partial_r h_1 = -\frac{h_1 h_3}{r}, \quad \partial_r h_3 = \frac{h_1^2}{r}, \tag{2.3}$$

$$\triangle Q + \frac{R^2}{r^2} Q = \kappa(r) Q, \quad \kappa(r) = -\frac{2h_1^2}{r^2}. \tag{2.4}$$

The goal of the present section is to prove the following result.

Proposition 2.1. *For any $\delta > 0$ sufficiently small and any N sufficiently large there exists an approximate solution $u^{(N)} : \mathbb{R}^2 \times \mathbb{R}_+^* \to S^2$ of (1.1) such that the following holds.*

(i) *$u^{(N)}$ is a C^∞ 1-equivariant profile of the form: $u^{(N)} = e^{\alpha(t)R}(\phi(\lambda(t)x) + \chi^{(N)}(\lambda(t)x, t))$, where $\chi^{(N)}(y,t) = e^{\theta R} Z^{(N)}(\rho, t)$, $\rho = |y|$, verifies*

$$\|\partial_\rho Z^{(N)}(t)\|_{L^2(\rho d\rho)}, \|\rho^{-1} Z^{(N)}(t)\|_{L^2(\rho d\rho)}, \|\rho \partial_\rho Z^{(N)}(t)\|_\infty \leq C\delta^{2\nu}, \tag{2.5}$$

$$\|\rho^{-l}\partial_\rho^k Z^{(N)}(t)\|_{L^2(\rho d\rho)} \leq C\delta^{2\nu-1} t^{1/2+\nu}, \quad k+l=2, \tag{2.6}$$

$$\|\rho^{-l}\partial_\rho^k Z^{(N)}(t)\|_{L^2(\rho d\rho)} \leq C t^{2\nu}, \quad k+l=3, \tag{2.7}$$

$$\|\partial_\rho Z^{(N)}(t)\|_\infty, \|\rho^{-1} Z^{(N)}(t)\|_\infty \leq C\delta^{2\nu-1} t^\nu, \tag{2.8}$$

$$\|\rho^{-l}\partial_\rho^k Z^{(N)}(t)\|_\infty \leq C t^{2\nu}, \quad 2 \leq l+k \leq 3, \tag{2.9}$$

for any $0 < t \leq T(N, \delta)$ with some $T(N,\delta) > 0$. The constants C here and below are independent of N and δ.
In addition, one has

$$\|\chi^{(N)}(t)\|_{\dot{W}^{4,\infty}} + \|\langle y\rangle^{-1}\chi^{(N)}(t)\|_{\dot{W}^{5,\infty}} \leq C t^{2\nu}, \tag{2.10}$$

and $\langle x\rangle^{2(\nu-1)}\nabla^4 u^{(N)}(t)$, $\langle x\rangle^{2(\nu-1)}\nabla^2 u_t^{(N)}(t) \in L^\infty(\mathbb{R}^2)$.
Furthermore, there exists $\zeta_N^* \in \dot{H}^1 \cap \dot{H}^{1+2\nu-}$ such that as $t \to 0$,

$e^{\alpha(t)R}\chi^{(N)}(\lambda(t)\cdot, t) \to \zeta_N^*$ in $\dot{H}^1 \cap \dot{H}^2$.

(ii) *The corresponding error $r^{(N)} = -u_t^{(N)} + u^{(N)} \times \Delta u^{(N)}$ verifies*

$$\|r^{(N)}(t)\|_{H^3} + \|\partial_t r^{(N)}(t)\|_{H^1} + \|\langle x\rangle r^{(N)}(t)\|_{L^2} \leq t^N, \quad 0 < t \leq T(\delta, N). \tag{2.11}$$

Remarks. 1. Note that estimates (2.5), (2.6) imply:

$$\|u^{(N)}(t) - e^{\alpha(t)R}\phi(\lambda(t)\cdot)\|_{\dot{H}^1\cap\dot{H}^2} \leq \delta^{2\nu-1}, \quad \forall t \in (0, T(N,\delta)]. \tag{2.12}$$

2. It follows from our construction that $\chi^{(N)}(t) \in \dot{H}^{1+2s}$ for any $s < \nu$ with the estimate $\|\chi^{(N)}(t)\|_{\dot{H}^{1+2s}(\mathbb{R}^2)} \leq C(t^{2\nu} + t^{s(1+2\nu)}\delta^{2\nu-2s})$.

3. The remainder $r^{(N)}$ verifies in fact, for any m, l, k,

$$\|\langle x\rangle^l \partial_t^m r^{(N)}(t)\|_{H^k} \leq C_{l,m,k} t^{N - C_{l,m,k}},$$

provided $N \geq C_{l,m,k}$.

We will give the proof of Proposition 2.1 in the case of ν irrational only, which allows us to slightly simplify the presentation. The extension to ν rational is straightforward.

To construct an arbitrarily good approximate solution we analyze separately the three regions that correspond to three different space scales: the inner region with the scale $r\lambda(t) \lesssim 1$, the self-similar region where $r = O(t^{1/2})$, and finally the remote region where $r = O(1)$. The inner region is the region where the blowup concentrates. In this region the solution will be constructed as a perturbation of the profile $e^{\alpha(t)R}Q(\lambda(t)r)$. The self-similar and remote regions are the regions where the solution is close to **k** and is described essentially by the corresponding linearized equation. In the self-similar region the profile of the solution will be determined uniquely by the matching conditions coming out of the inner region, while in the remote region the profile remains essentially a free parameter of the construction, only the limiting behavior at the origin is prescribed by the matching process, see Sects. 2.3 and 2.4 for the details, see also [1,2] for some closely related considerations in the context of the critical harmonic map heat flow.

2.2. Inner region $r\lambda(t) \lesssim 1$. We start by considering the inner region $0 \le r\lambda(t) \le 10t^{-\nu+\varepsilon_1}$, where $0 < \varepsilon_1 < \nu$ to be fixed later. Writing $v(r,t)$ as

$$v(r,t) = e^{\alpha(t)R}V(\lambda(t)r,t), \quad V = (V_1, V_2, V_3),$$

we get from (2.2)

$$t^{1+2\nu}V_t + \alpha_0 t^{2\nu} RV - t^{2\nu}(\nu + \frac{1}{2})\rho V_\rho = V \times (\triangle V + \frac{R^2}{\rho^2}V), \quad \rho = \lambda(t)r. \tag{2.13}$$

We look for a solution of (2.13) as a perturbation of the harmonic map profile $Q(\rho)$. Write

$$V = Q + Z,$$

and further decompose Z as

$$Z(\rho,t) = z_1(\rho,t)f_1(\rho) + z_2(\rho,t)f_2(\rho) + \gamma(\rho,t)Q(\rho),$$

where f_1, f_2 is the orthonormal frame on $T_Q S^2$ given by

$$f_1(\rho) = \begin{pmatrix} h_3(\rho) \\ 0 \\ -h_1(\rho) \end{pmatrix} \quad f_2(\rho) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

One has

$$\gamma = \sqrt{1 - |z|^2} - 1 = O(|z|^2), \quad z = z_1 + iz_2.$$

Note also the relations

$$\partial_\rho Q = -\frac{h_1}{\rho}f_1, \quad \partial_\rho f_1 = \frac{h_1}{\rho}Q, \quad f_2 = Q \times f_1,$$
$$\Delta f_1 + \frac{R^2}{\rho^2}f_1 = -\frac{1}{\rho^2}f_1 - \frac{2h_3h_1}{\rho^2}Q.$$

We now rewrite (2.13) in terms of z. One has

$$
\begin{aligned}
RV &= -h_3 z_2 f_1 + (h_3 z_1 + h_1(1+\gamma)) f_2 - h_1 z_2 Q,\\
\rho\partial_\rho V &= (\rho\partial_\rho z_1 - h_1(1+\gamma)) f_1 + \rho\partial_\rho z_2 f_2 + (h_1 z_1 + \rho\partial_\rho \gamma) Q.
\end{aligned}
\tag{2.14}
$$

We next compute the nonlinear term $V \times (\triangle V + \frac{R^2}{\rho^2} V)$. In the basis $\{f_1, f_2, Q\}$, the expression $\triangle V + \frac{R^2}{\rho^2} V$ can be written as follows:

$$
\begin{aligned}
\triangle V + \frac{R^2}{\rho^2} V = \big[\triangle z_1 - \frac{z_1}{\rho^2} - 2\frac{h_1}{\rho}\gamma_\rho\big] f_1 + \big[\triangle z_2 - \frac{z_2}{\rho^2}\big] f_2\\
+\big[\triangle \gamma + \kappa(\rho)(1+\gamma) + 2\frac{h_1}{\rho}\partial_\rho z_1 - 2\frac{h_1 h_3}{\rho^2} z_1\big] Q,
\end{aligned}
$$

which gives

$$
V\times(\triangle V+\frac{R^2}{\rho^2}V)=\big[(1+\gamma)Lz_2+F_1(z)\big]f_1-\big[(1+\gamma)Lz_1+F_2(z)\big]f_2+F_3(z)Q, \tag{2.15}
$$

where

$$
\begin{aligned}
L &= -\triangle + \frac{1-2h_1^2}{\rho^2},\\
F_1(z) &= z_2(\triangle\gamma - 2\frac{h_1 h_3}{\rho^2} z_1 + 2\frac{h_1}{\rho}\partial_\rho z_1),\\
F_2(z) &= z_1(\triangle\gamma - 2\frac{h_1 h_3}{\rho^2} z_1 + 2\frac{h_1}{\rho}\partial_\rho z_1) + \frac{2h_1}{\rho}(1+\gamma)\gamma_\rho\\
F_3(z) &= z_1\triangle z_2 - z_2 \triangle z_1 + \frac{2h_1}{\rho} z_2 \gamma_\rho.
\end{aligned}
\tag{2.16}
$$

Projecting (2.13) onto span$\{f_1, f_2\}$ and taking into account (2.14), (2.15), (2.16), we get the following reformulation of (2.13):

$$
\begin{aligned}
it^{1+2\nu} z_t - \alpha_0 t^{2\nu} h_3 z - i(\frac{1}{2}+\nu)t^{2\nu}\rho z_\rho &= Lz + F(z) + dt^{2\nu} h_1,\\
d &= \alpha_0 - i(\frac{1}{2}+\nu),\\
F(z) = \gamma L z + z(\triangle\gamma + \frac{2h_1}{\rho}\partial_\rho z_1 - \frac{2h_1 h_3 z_1}{\rho^2}) &+ \frac{2h_1}{\rho}(1+\gamma)\gamma_\rho + dt^{2\nu}\gamma h_1.
\end{aligned}
\tag{2.17}
$$

Note that F is at least quadratic in z.

We look for a solution of (2.17) as a power expansion in $t^{2\nu}$:

$$
z(\rho, t) = \sum_{k\geq 1} t^{2\nu k} z^k(\rho). \tag{2.18}
$$

Substituting (2.18) into (2.17) we get the following recurrent system for z^k, $k \geq 1$:

$$
Lz^1 = -dh_1, \tag{2.19}
$$

$$
Lz^k = \mathcal{F}_k, \quad k \geq 2, \tag{2.20}
$$

where $\mathcal{F}_k$ depends on $z^j, \quad j = 1, \dots, k-1$ only. We subject (2.19), (2.20) to zero initial conditions at $\rho = 0$:

$$z^k(0) = \partial_\rho z^k(0) = 0. \tag{2.21}$$

Lemma 2.2. *System (2.19), (2.20), (2.21) has a unique solution $(z^k)_{k\geq 1}$, with $z^k \in C^\infty(\mathbb{R}_+)$ for all $k \geq 1$. In addition, one has:*

(i) *z^k has an odd Taylor expansion at 0 that starts at order $2k+1$;*
(ii) *as $\rho \to \infty$, z^k has the following asymptotic expansion*

$$z^k(\rho) = \sum_{l=0}^{2k} \sum_{j\leq k-(l-1)/2} c^k_{j,l}\rho^{2j-1}(\ln\rho)^l, \tag{2.22}$$

with some constants $c^k_{j,l}$. The asymptotic expansion (2.22) can be differentiated any number of times with respect to ρ.

Proof. First note that the equation $Lf = 0$ has two explicit solutions: $h_1(\rho)$ and $h_2(\rho) = \frac{\rho^4+4\rho^2\ln\rho-1}{\rho(\rho^2+1)}$.

Consider the case $k = 1$:

$$Lz^1 = -dh_1,$$

$$z^1(0) = \partial_\rho z^1(0) = 0.$$

One has

$$\begin{aligned} z^1(\rho) &= -\frac{d}{4}\int_0^\rho ds s(h_1(\rho)h_2(s) - h_1(s)h_2(\rho))h_1(s) \\ &= -\frac{d\rho}{(1+\rho^2)}\int_0^\rho ds\frac{s(s^4+4s^2\ln s-1)}{(1+s^2)^2} + \frac{d(\rho^4+4\rho^2\ln\rho-1)}{\rho(\rho^2+1)}\int_0^\rho ds\frac{s^3}{(1+s^2)^2} \end{aligned} \tag{2.23}$$

Since h_1 is a C^∞ function that has an odd Taylor expansion at $\rho = 0$ with a linear leading term, one can easily write an odd Taylor series for z^1 with a cubic leading term, which proves (i) for $k = 1$.

The asymptotic behavior of z^1 at infinity can be obtained directly from the representation (2.23). As claimed, one has

$$z^1(\rho) = c^1_{1,0}\rho + c^1_{1,1}\rho\ln\rho + \sum_{j\leq 0}\sum_{l=0,1,2} c^1_{j,l}\rho^{2j-1}(\ln\rho)^l,$$

with $c^1_{1,0} = -c^1_{1,1} = -d$.

Consider $k > 1$. Assume that $z^j, \ j \leq k-1$, verify (i) and (ii). Then, using (2.17), one can easily check that $\mathcal{F}_k$ is an odd C^∞ function vanishing at $\rho = 0$ at order $2k-1$, with the following asymptotic expansion as $\rho \to \infty$:

$$\begin{aligned} \mathcal{F}_k &= \sum_{j=1}^{k-1}\sum_{l=0}^{2k-2j-1} \alpha^k_{j,l}\rho^{2j-1}(\ln\rho)^l + \sum_{l=0}^{2k-2}\alpha^k_{0,l}\rho^{-1}(\ln\rho)^l \\ &\quad + \sum_{l=0}^{2k-1}\alpha^k_{-1,l}\rho^{-3}(\ln\rho)^l + \sum_{j\leq -2}\sum_{l=0}^{2k}\alpha^k_{j,l}\rho^{2j-1}(\ln\rho)^l. \end{aligned}$$

As a consequence, $z^k(\rho) = \frac{1}{4}\int_0^\rho dss(h_1(\rho)h_2(s) - h_1(s)h_2(\rho))\mathcal{F}_k(s)$ is a C^∞ function with an odd Taylor series at zero starting at order $2k+1$ and as $\rho \to \infty$,

$$z^k(\rho) = \sum_{l=0}^{2k} \sum_{j \le k-(l-1)/2} c_{j,l}^k (\ln \rho)^l \rho^{2j-1},$$

as required. This concludes the proof of Lemma 2.2. □

Returning to v we get a formal solution of (2.2) of the form

$$v(r,t) = e^{\alpha(t)R} V(\lambda(t)r, t), \quad V(\rho, t) = Q + \sum_{k\ge 1} t^{2\nu k} Z^k(\rho), \tag{2.24}$$

$Z^k = (Z_1^k, Z_2^k, Z_3^k)$, where Z_i^k, $i = 1, 2$, are smooth odd functions of ρ vanishing at 0 at order $2k+1$, and Z_3^k is an even function vanishing at zero at order $2k+2$. As $\rho \to \infty$, one has

$$\begin{aligned} Z_i^k(\rho) &= \sum_{l=0}^{2k} \sum_{j \le k-(l-1)/2} c_{j,l}^{k,i} (\ln \rho)^l \rho^{2j-1}, \quad i = 1, 2, \\ Z_3^k(\rho) &= \sum_{l=0}^{2k} \sum_{j \le k+1-l/2} c_{j,l}^{k,3} (\ln \rho)^l \rho^{2j-2}, \end{aligned} \tag{2.25}$$

with some real coefficients $c_{j,l}^{k,i}$ verifying

$$c_{k+1,0}^{k,3} = 0, \quad \forall k \ge 1.$$

The asymptotic expansions (2.25) can be differentiated any number of times with respect to ρ.

Note that in the limit $\rho \to \infty$, $y \equiv rt^{-1/2} \to 0$, expansion (2.24), (2.25), rewritten in terms of y, give at least formally

$$\begin{aligned} V_i(\lambda(t)r, t) &= \sum_{j\ge 0} t^{\nu(2j+1)} \sum_{l=0}^{2j+1} (\ln y - \nu \ln t)^l V_i^{j,l}(y), \quad i = 1, 2, \\ V_3(\lambda(t)r, t) &= 1 + \sum_{j\ge 1} t^{2\nu j} \sum_{l=0}^{2j} (\ln y - \nu \ln t)^l V_3^{j,l}(y), \\ V_i^{j,l}(y) &= \sum_{k \ge -j+l/2} c_{k,l}^{k+j,i} y^{2k-1}, \quad i = 1, 2, \\ V_3^{j,l}(y) &= \sum_{k \ge -j+l/2} c_{k+1,l}^{k+j,3} y^{2k}, \end{aligned} \tag{2.26}$$

where the coefficients $c_{j,l}^{k,i}$ with $k \ne 0$ are defined by (2.25) and $c_{j,0}^{0,i}$ come from the expansion of Q as $\rho \to \infty$:

$$h_1(\rho) = \sum_{j \le 0} c_{j,0}^{0,1} \rho^{2j-1}, \quad h_3(\rho) = 1 + \sum_{j\le 0} c_{j,0}^{0,3} \rho^{2j-2}, \quad c_{j,0}^{0,2} = 0.$$

The role of y-expansion (2.26) will become clear in the next subsection where we will use it to perform the transition from the inner region to the self-similar region.

For $N \geq 2$ define

$$z_{\text{in}}^{(N)} = \sum_{k=1}^{N} t^{2\nu k} z^k, \quad z_{\text{in}}^{(N)} = z_{in,1}^{(N)} + i z_{in,2}^{(N)}.$$

Then $z_{\text{in}}^{(N)}$ solves (2.17) up to the error $X_N = -it^{1+2\nu}\partial_t z_{\text{in}}^{(N)} + \alpha_0 t^{2\nu} h_3 z_{\text{in}}^{(N)} + i(\frac{1}{2} + \nu)t^{2\nu}\rho\partial_\rho z_{\text{in}}^{(N)} + dt^{2\nu}h_1 + Lz_{\text{in}}^{(N)} + F(z_{\text{in}}^{(N)})$. Using the fact that z^k are defined recursively it is not difficult to check that the error X_N verifies

$$|\rho^{-l}\partial_\rho^k\partial_t^m X_N| \leq C_{k,l,m} t^{2\nu N-m}\langle\rho\rangle^{2N-1-l-k}\ln(2+\rho), \tag{2.27}$$

for any $k, m \in \mathbb{N}$, $0 \leq l \leq (2N+1-k)_+$, $0 \leq \rho \leq 10t^{-\nu+\varepsilon_1}$, $0 < t \leq T(N)$, with some $T(N) > 0$.

Set

$$\begin{aligned} \gamma_{\text{in}}^{(N)} &= \sqrt{1-|z_{\text{in}}^{(N)}|^2} - 1, \\ Z_{\text{in}}^{(N)} &= z_{in,1}^{(N)} f_1 + z_{in,2}^{(N)} f_2 + \gamma_{\text{in}}^{(N)} Q, \\ V_{\text{in}}^{(N)} &= Q + Z_{\text{in}}^{(N)} \in S^2. \end{aligned}$$

Then $V_{\text{in}}^{(N)}$ solves

$$t^{1+2\nu}\partial_t V_{\text{in}}^{(N)} + \alpha_0 t^{2\nu} R V_{\text{in}}^{(N)} - t^{2\nu}(\nu+\frac{1}{2})\rho\partial_\rho V_{\text{in}}^{(N)} = V_{\text{in}}^{(N)} \times (\triangle_\rho V_{\text{in}}^{(N)} + \frac{R^2}{\rho^2} V_{\text{in}}^{(N)}) + \mathcal{R}_{\text{in}}^{(N)}, \tag{2.28}$$

with $\mathcal{R}_{\text{in}}^{(N)} = \text{im}\, X_N f_1 - \text{re}\, X_N f_2 + \frac{\text{im}(\bar{X}_N z^{(N)})}{1+\gamma^{(N)}} Q$ admitting the same estimate as X_N. Note also that it follows from our analysis that for $0 \leq \rho \leq 10t^{-\nu+\varepsilon_1}$, $0 < t \leq T(N)$,

$$|\rho^{-l}\partial_\rho^k Z_{\text{in}}^{(N)}| \leq C_{k,l} t^{2\nu}\langle\rho\rangle^{1-l-k}\ln(2+\rho), \quad k \in \mathbb{N}, \quad l \leq (3-k)_+. \tag{2.29}$$

As a consequence, we obtain the following result.

Lemma 2.3. *There exists $T(N) > 0$ such that for any $0 < t \leq T(N)$ the following holds.*

(i) *The profile $Z_{\text{in}}^{(N)}(\rho, t)$ verifies*

$$\|\partial_\rho Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho d\rho, 0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} \leq Ct^\nu, \tag{2.30}$$

$$\|\rho^{-1} Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho d\rho, 0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} \leq Ct^\nu, \tag{2.31}$$

$$\|Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} + \|\rho\partial_\rho Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} \leq Ct^\nu, \tag{2.32}$$

$$\|\rho^{-l}\partial_\rho^k Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho d\rho, 0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} \leq Ct^{2\nu}(1+|\ln t|), \quad k+l=2, \tag{2.33}$$

$$\|\rho^{-l}\partial_\rho^k Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho d\rho, 0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} \leq Ct^{2\nu}, \quad k+l \geq 3,\ l \leq (3-k)_+, \tag{2.34}$$

$$\|\partial_\rho Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} + \|\rho^{-1} Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} \leq Ct^{2\nu}(1+|\ln t|), \tag{2.35}$$

$$\|\rho^{-l}\partial_\rho^k Z_{\text{in}}^{(N)}(t)\|_{L^\infty(0\leq\rho\leq 10t^{-\nu+\varepsilon_1})} \leq Ct^{2\nu}, \quad 2 \leq l+k,\ l \leq (3-k)_+. \tag{2.36}$$

(ii) *The error* $\mathcal{R}_{\text{in}}^{(N)}$ *admits the estimates*

$$\begin{aligned} \|\rho^{-l}\partial_\rho^k \mathcal{R}_{\text{in}}^{(N)}(t)\|_{L^2(\rho d\rho, 0\le\rho\le 10t^{-\nu+\varepsilon_1})} &\le t^{N\varepsilon_1}, \quad 0\le l+k\le 3,\\ \|\rho^{-l}\partial_\rho^k \partial_t \mathcal{R}_{\text{in}}^{(N)}(t)\|_{L^2(\rho d\rho, 0\le\rho\le 10t^{-\nu+\varepsilon_1})} &\le t^{N\varepsilon_1}, \quad 0\le k+l\le 1, \end{aligned} \tag{2.37}$$

provided $N > \varepsilon_1^{-1}$.

2.3. Self-similar region $rt^{-1/2} \lesssim 1$. We next consider the self-similar region $\frac{1}{10}t^{\varepsilon_1} \le rt^{-1/2} \le 10t^{-\varepsilon_2}$, where $0 < \varepsilon_2 < 1/2$ to be fixed later. In this region we expect the solution to be close to **k**. In this regime it will be convenient to use the stereographic representation of (2.2):

$$(v_1, v_2, v_3) = v \to w = \frac{v_1 + iv_2}{1+v_3} \in \mathbb{C}\cup\{\infty\}.$$

Equation (2.2) is equivalent to

$$iw_t = -\triangle w + r^{-2}w + G(w,\bar{w},w_r), \quad G(w,\bar{w},w_r) = \frac{2\bar{w}}{1+|w|^2}(w_r^2 - r^{-2}w^2). \tag{2.38}$$

Slightly more generally, if $w(r,t)$ is a solution of

$$iw_t = -\triangle w + r^{-2}w + G(w,\bar{w},w_r) + A, \tag{2.39}$$

then $v = \left(\frac{2\,\mathrm{re}\,w}{1+|w|^2}, \frac{2\,\mathrm{im}\,w}{1+|w|^2}, \frac{1-|w|^2}{1+|w|^2}\right) \in S^2$ solves

$$v_t = v \times (\triangle v + \frac{R^2}{r^2}v) + \mathcal{A}, \tag{2.40}$$

with $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ given by

$$\mathcal{A}_1 + i\mathcal{A}_2 = -2i\frac{A + w^2\bar{A}}{(1+|w|^2)^2}, \quad \mathcal{A}_3 = \frac{4\,\mathrm{im}(w\bar{A})}{(1+|w|^2)^2}.$$

Consider (2.38). Write w as

$$w(r,t) = e^{i\alpha(t)}W(y,t), \quad y = rt^{-1/2}.$$

Then (2.38) becomes

$$itW_t - \alpha_0 W = \mathcal{L}W + G(W,\bar{W},W_y), \tag{2.41}$$

where

$$\mathcal{L} = -\triangle + y^{-2} + i\frac{1}{2}y\partial_y.$$

Note that as $y\to 0$, (2.26) gives the following expansion:

$$W(y,t) = \sum_{j\ge 0}\sum_{l=0}^{2j+1}\sum_{i\ge -j+l/2} \alpha(j,i,l)t^{\nu(2j+1)}(\ln y - \nu\ln t)^l y^{2i-1}, \tag{2.42}$$

where the coefficients $\alpha(j,i,l)$ can be expressed explicitly in terms of $c^{k,i'}_{j',l'}$, $1 \le k \le j+i$, $j' \le i$, $0 \le l' \le l$. This suggests the following ansatz for W:

$$W(y,t) = \sum_{j\ge 0}\sum_{l=0}^{2j+1} t^{\nu(2j+1)}(\ln y - \nu \ln t)^l W_{j,l}(y). \tag{2.43}$$

Substituting (2.43) into (2.41) one gets the following recurrent system for $W_{j,l}$, $0 \le l \le 2j+1$, $j \ge 0$:

$$\begin{cases} (\mathcal{L}-\mu_0)W_{0,1} = 0, \\ (\mathcal{L}-\mu_0)W_{0,0} = -i(1/2+\nu)W_{0,1} + 2y^{-1}\partial_y W_{0,1}, \end{cases} \tag{2.44}$$

$$\begin{cases} (\mathcal{L}-\mu_j)W_{j,2j+1} = \mathcal{G}_{j,2j+1}, \\ (\mathcal{L}-\mu_j)W_{j,2j} = \mathcal{G}_{j,2j} - i(2j+1)(1/2+\nu)W_{j,2j+1} + 2(2j+1)y^{-1}\partial_y W_{j,2j+1}, \\ (\mathcal{L}-\mu_j)W_{j,l} = \mathcal{G}_{j,l} - i(l+1)(1/2+\nu)W_{j,l+1} \\ \quad +2(l+1)y^{-1}\partial_y W_{j,l+1} + (l+1)(l+2)y^{-2}W_{j,l+2}, \quad 0 \le l \le 2j-1. \end{cases} \tag{2.45}$$

Here $\mu_j = -\alpha_0 + i\nu(2j+1)$, and $\mathcal{G}_{j,l}$ is the contribution of the nonlinear term $G(W,\bar W, W_r)$, that depends only on $W_{i,n}$, $i \le j-1$:

$$G(W,\bar W,W_r) = -\sum_{j\ge 1}\sum_{l=0}^{2j+1} t^{(2j+1)\nu}(\ln y - \nu\ln t)^l \mathcal{G}_{j,l}(y),$$
$$\mathcal{G}_{j,l}(y) = \mathcal{G}_{j,l}(y; W_{i,n},\ 0 \le n \le 2i+1,\ 0 \le i \le j-1).$$

One has

Lemma 2.4. *Given coefficients a_j, b_j, $j \ge 0$, there exists a unique solution of* (2.44), (2.45)*, $W_{j,l} \in C^\infty(\mathbb{R}^*_+)$, $0 \le l \le 2j+1$, $j \ge 0$, such that as $y \to 0$, $W_{j,l}$ has the following asymptotic expansion*

$$W_{j,l}(y) = \sum_{i \ge -j+l/2} d_i^{j,l} y^{2i-1}, \tag{2.46}$$

with

$$d_1^{j,1} = a_j, \quad d_1^{j,0} = b_j. \tag{2.47}$$

The asymptotic expansion (2.46) *can be differentiated any number of times with respect to y.*

Proof. First note that equation $(\mathcal{L}-\mu_j)f = 0$ has a basis of solutions $\{e_j^1, e_j^2\}$ such that

(i) e_j^1 is a C^∞ odd function, $e_j^1(y) = y + O(y^3)$ as $y \to 0$;
(ii) $e_j^2 \in C^\infty(\mathbb{R}^*_+)$ and admits the representation:

$$e_j^2(y) = y^{-1} + \kappa_j e_j^1(y)\ln y + \tilde e_j^2(y), \quad \kappa_j = -\frac{i}{4} - \frac{\mu_j}{2},$$

where $\tilde e_j^2$ is a C^∞ odd function, $\tilde e_j^2(y) = O(y^3)$ as $y \to 0$.

Consider (2.44). From $(\mathcal{L}-\mu_0)W_{0,1}=0$ and (2.46), (2.47), we get

$$W_{0,1}=a_0e_0^1.$$

Consider the equation for $W_{0,0}$:

$$(\mathcal{L}-\mu_0)W_{0,0}=-i(1/2+\nu)W_{0,1}+2y^{-1}\partial_yW_{0,1}.$$

The right hand side has the form: $2a_0y^{-1}$ + a C^∞ odd function. Therefore, the equation has a unique solution $W_{0,0}^0$ of the form

$$W_{0,0}^0(y)=d_0y^{-1}+\tilde{W}_{0,0}^0(y),$$

where $d_0=\frac{a_0}{k_j}$ and $\tilde{W}_{0,0}^0$ is a C^∞ odd function, $\tilde{W}_{0,0}^0(y)=O(y^3)$ as $y\to 0$. Together with (2.46), (2.47), this gives:

$$W_{0,0}=W_{0,0}^0+b_0e_0^1.$$

Consider the case $j\geq 1$. We have

$$(\mathcal{L}-\mu_j)W_{j,l}=\mathcal{F}_{j,l},\quad 0\leq l\leq 2j+1, \tag{2.48}$$

where

$$\begin{aligned}\mathcal{F}_{j,2j+1}&=\mathcal{G}_{j,2j+1},\\ \mathcal{F}_{j,2j}&=\mathcal{G}_{j,2j}-i(2j+1)(1/2+\nu)W_{j,2j+1}+2(2j+1)y^{-1}\partial_yW_{j,2j+1},\\ \mathcal{F}_{j,l}&=\mathcal{G}_{j,l}-i(l+1)(1/2+\nu)W_{j,l+1}\\ &\quad+2(l+1)y^{-1}\partial_yW_{j,l+1}+(l+1)(l+2)y^{-2}W_{j,l+2},\quad 0\leq l\leq 2j-1.\end{aligned} \tag{2.49}$$

The resolution of (2.48) is based on the following ODE lemma whose proof is left to the reader.

Lemma 2.5. *Let F be a $C^\infty(\mathbb{R}_+^*)$ function of the form*

$$F(y)=\sum_{j=k}^{0}F_jy^{2j-1}+\tilde{F}(y),$$

where $\tilde{F}$ is a C^∞ odd function and $k\leq -1$. Then there exists a unique constant A such that the equation $(\mathcal{L}-\mu_j)u=F+Ay^{-3}$ has a solution $u\in C^\infty(\mathbb{R}_+^*)$ with the following behavior as $y\to 0$:

$$u(y)=\sum_{j\geq k+1}u_jy^{2j-1},\quad u_1=0.$$

More precisely, we proceed as follows. Assume that $W_{i,n}$, $0 \le n \le 2i+1$, $i \le j-1$ has the prescribed behavior (2.46), (2.47). Then it is not difficult to check that $\mathcal{G}_{j,l}$ admit the following expansion as $y \to 0$:

$$\begin{aligned}
\mathcal{G}_{j,2j+1}(y) &= \sum_{i\ge 1} g^i_{j,2j+1} y^{2i-1},\\
\mathcal{G}_{j,2j}(y) &= \sum_{i\ge 0} g^i_{j,2j} y^{2i-1},\\
\mathcal{G}_{j,l}(y) &= \sum_{i\ge -j+l/2-1} g^i_{j,l} y^{2i-1}, \quad l \le 2j-1.
\end{aligned} \tag{2.50}$$

Consider $W_{j,2j+1}$. From $(\mathcal{L}-\mu_j)W_{j,2j+1} = \mathcal{G}_{j,2j+1}$ we get

$$W_{j,2j+1} = W^0_{j,2j+1} + c_0 e^1_j, \tag{2.51}$$

where $W^0_{j,2j+1}$ is a unique C^∞ odd solution of $(\mathcal{L}-\mu_j)f = \mathcal{G}_{j,2j+1}$ that satisfies $W^0_{j,2j+1}(y) = O(y^3)$ as $y \to 0$. The constant c_0 remains undetermined at this stage.

Consider $\mathcal{F}_{j,2j}$. It has the form: $(g^0_{j,2j} + 2(2j+1)c_0)y^{-1}$ + a C^∞ odd function. Therefore, for $W_{j,2j}$ we obtain

$$W_{j,2j} = W^0_{j,2j} + c_1 e^1_j, \tag{2.52}$$

where $W^0_{j,2j}$ is a unique solution of $(\mathcal{L}-\mu_j)f = \mathcal{F}_{j,2j}$, that satisfies as $y \to 0$,

$$W^0_{j,2j} = d_1 y^{-1} + O(y^3), \quad d_1 = \frac{g^0_{j,2j} + 2(2j+1)c_0}{2k_j}. \tag{2.53}$$

Similarly to c_0, the constant c_1 is arbitrary here.

Consider $\mathcal{F}_{j,2j-1}$. It follows from (2.49), (2.50), (2.51), (2.52), (2.53) that

$$\mathcal{F}_{j,2j-1} = (g^{-1}_{j,2j-1} - 4jd_1)y^{-3} + const\ y^{-1} + \text{an } C^\infty \text{ odd function.}$$

The equation $(\mathcal{L}-\mu_j)W_{j,2j-1} = \mathcal{F}_{j,2j-1}$ has a solution of form (2.46) iff

$$g^{-1}_{j,2j-1} - 4jd_1 = 0,$$

which gives

$$c_0 = \frac{k_j g_{j,2j-1}{-1} - 2jg^0_{j,2j}}{4j(2j+1)}.$$

With this choice of c_0 one gets

$$W_{j,2j-1} = W^0_{j,2j-1} + c_2 e^1_j,$$

where $W^0_{j,2j-1}$ is a unique solution of $(\mathcal{L}-\mu_j)f = \mathcal{F}_{j,2j-1}$, that satisfies as $y \to 0$,

$$W^0_{j,2j-1} = const\ y^{-1} + O(y^3).$$

Continuing the procedure one successively finds $W_{j,2j-2}, \dots, W_{j,0}$ in the form $W_{j,2j+1-k} = W^0_{j,2j+1-k} + c_k e^1_j$, $k \le 2j+1$, where $W^0_{j,2j+1-k}$ is an unique solution of $(\mathcal{L} - \mu_j) f = \mathcal{F}_{j,2j+1-k}$, that as $y \to 0$ has an asymptotic expansion of the form (2.46) with vanishing coefficients $d_1^{j,l}$. The constant c_k, $k \le 2j-1$, is determined uniquely by the solvability condition of the equation for $W_{j,2j-k-1}$ (see lemma 2.5). Finally, c_{2j+1}, c_{2j+2} are given by (2.47):

$$c_{2j+1} = a_j, \quad c_{2j+2} = b_j.$$

□

We denote by $W^{ss}_{j,l}(y)$, $0 \le l \le 2j+1$, $j \ge 0$, the solution of (2.44), (2.45) given by Lemma 2.4 with $a_j = \alpha(j,1,1)$, $b_j = \alpha(j,1,0)$, see (2.42). Since expansion (2.42) is a solution of (2.41), the uniqueness part of Lemma 2.4 ensures that

$$W^{ss}_{j,l}(y) = \sum_{i \ge -j+l/2} \alpha(j,i,l) y^{2i-1}, \quad \text{as } y \to 0. \tag{2.54}$$

We next study the behavior of $W^{ss}_{j,l}$, $0 \le l \le 2j+1$, $j \ge 0$, at infinity. One has

Lemma 2.6. *Given coefficients $a_{j,l}$, $b_{j,l}$, $0 \le l \le 2j+1$, $j \ge 0$, there exists a unique solution of* (2.44), (2.45) *of the following form.*

$$W_{0,l} = W^0_{0,l} + W^1_{0,l}, \quad l = 0, 1, \tag{2.55}$$

$$W_{j,l} = W^0_{j,l} + W^1_{j,l} + W^2_{j,l}, \quad 0 \le l \le 2j+1, \; j \ge 1, \tag{2.56}$$

where $(W^i_{j,l})_{\substack{0 \le l \le 2j+1 \\ j \ge 1}}$, $i = 0, 1$, are two solutions of (2.44), (2.45) *that, as $y \to \infty$, have the following asymptotic expansion*

$$\begin{aligned} \sum_{l=0}^{2j+1} (\ln y - \nu \ln t)^l W^i_{j,l}(y) &= \sum_{l=0}^{2j+1} (\ln y + (-1)^i \ln t/2)^l \hat{W}^i_{j,l}(y), \quad i = 0, 1, \\ \hat{W}^0_{j,l}(y) &= y^{2i\alpha_0 + 2\nu(2j+1)} \sum_{k \ge 0} \hat{w}_k^{j,l,0} y^{-2k}, \\ \hat{W}^1_{j,l}(y) &= e^{iy^2/4} y^{-2i\alpha_0 - 2 - 2\nu(2j+1)} \sum_{k \ge 0} \hat{w}_k^{j,l,-1} y^{-2k}, \end{aligned} \tag{2.57}$$

with

$$\hat{w}_0^{j,l,0} = a_{j,l}, \quad \hat{w}_0^{j,l,-1} = b_{j,l}. \tag{2.58}$$

Finally, the interaction part $W^2_{j,l}$ can be written as

$$W^2_{j,l}(y) = \sum_{-j-1 \le m \le j} e^{-imy^2/4} y^{2i\alpha_0(2m+1)} W_{j,l,m}(y), \tag{2.59}$$

where $W_{j,l,m}$ have the following asymptotic expansion as $y \to \infty$:

$$W_{j,l,m}(y) = \sum_{k \geq m+2} \sum_{\substack{m-j \leq i \leq j-m \\ j-m-i \in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,m} y^{2\nu(2i+1)-2k} (\ln y)^s, \quad m \geq 1,$$

$$W_{j,l,m}(y) = \sum_{k \geq -m} \sum_{\substack{-j-m-2 \leq i \leq j+m \\ j-m-i \in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,m} y^{2\nu(2i+1)-2k} (\ln y)^s, \quad m \leq -2,$$

$$W_{j,l,0}(y) = \sum_{k \geq 1} \sum_{\substack{-j \leq i \leq j-2 \\ j-i \in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,0} y^{2\nu(2i+1)-2k} (\ln y)^s, \tag{2.60}$$

$$W_{j,l,-1}(y) = \sum_{k \geq 1} \sum_{\substack{-j+1 \leq i \leq j-1 \\ j-i+1 \in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} w_{k,i,s}^{j,l,-1} y^{2\nu(2i+1)-2k} (\ln y)^s.$$

The asymptotic expansion (2.57)*,* (2.60) *can be differentiated any number of times with respect to* y*.*

Any solution of (2.44)*,* (2.45) *has form* (2.55)*,* (2.56)*,* (2.57)*,* (2.59)*,* (2.60)*.*

Proof. First note that equation $(\mathcal{L} - \mu_j) f = 0$ has a basis of solutions $\{f_j^1, f_j^2\}$ with the following behavior at infinity:

$$f_j^1(y) = y^{2i\alpha_0 + 2\nu(2j+1)} \sum_{k \geq 0} f_{j,1}^k y^{-2k}, \quad f_j^2(y) = e^{iy^2/4} y^{-2i\alpha_0 - 2\nu(2j+1) - 2} \sum_{k \geq 0} f_{j,2}^k y^{-2k},$$

$f_{j,1}^0 = f_{j,2}^0 = 1$. As a consequence, the homogeneous system

$$\begin{aligned} (\mathcal{L} - \mu_j) g_{2j+1} &= 0, \\ (\mathcal{L} - \mu_j) g_{2j} &= -i(2j+1)(1/2+\nu) g_{2j+1} + 2(2j+1) y^{-1} \partial_y g_{2j+1}, \\ (\mathcal{L} - \mu_j) g_l &= -i(l+1)(1/2+\nu) g_{l+1} \\ &\quad + 2(l+1) y^{-1} \partial_y g_{l+1} + (l+1)(l+2) y^{-2} g_{l+2}, \quad 0 \leq l \leq 2j-1. \end{aligned} \tag{2.61}$$

has a basis of solutions $\{\mathbf{g}_j^{i,m}\}_{\substack{i=1,2, \\ m=0,\dots,2j+1}}$,

$$\mathbf{g}_j^{i,m} = (g_{j,0}^{i,m}, \dots, g_{j,2j+1}^{i,m}), \quad 0 \leq m \leq 2j+1, \ i = 1, 2,$$

defined by

$$\sum_{l=0}^{2j+1} (\ln y - \nu \ln t)^l g_{j,l}^{i,m}(y) = \sum_{l=0}^{2j+1} (\ln y + (-1)^{i-1} \ln t/2)^l \xi_{j,l}^{i,m}(y), \tag{2.62}$$

where $(\xi_{j,l}^{i,m})_{l=0,\dots,2j+1}$ is the unique solution of

$$\begin{aligned} (\mathcal{L} - \mu_j) \xi_{2j+1} &= 0, \\ (\mathcal{L} - \mu_j) \xi_{2j} &= -i(2j+1)(i-1) \xi_{2j+1} + 2(2j+1) y^{-1} \partial_y \xi_{2j+1}, \\ (\mathcal{L} - \mu_j) \xi_l &= -i(l+1)(i-1) \xi_{l+1} + 2(l+1) y^{-1} \partial_y \xi_{l+1} \\ &\quad + (l+1)(l+2) y^{-2} \xi_{l+2}, \quad 0 \leq l \leq 2j-1, \end{aligned} \tag{2.63}$$

verifying

$$\begin{aligned}
\xi_{j,l}^{i,m}(y) &= 0, \quad l > 2j+1-m, \\
\xi_{j,2j+1-m}^{i,m}(y) &= f_j^i(y), \\
\xi_{j,l}^{1,m}(y) &= y^{2i\alpha_0+2\nu(2j+1)} \sum_{k\geq 2j+1-l-m} \xi_{l,k}^1 y^{-2k}, \quad y \to +\infty, \\
\xi_{j,l}^{2,m}(y) &= e^{iy^2/4} y^{-2i\alpha_0-2\nu(2j+1)-2} \sum_{k\geq 2j+1-l-m} \xi_{l,k}^2 y^{-2k} \quad y \to +\infty.
\end{aligned} \tag{2.64}$$

Consider $W_{0,l}$, $l = 0, 1$. We have

$$(\mathcal{L} - \mu_0) W_{0,1} = 0,$$
$$(\mathcal{L} - \mu_0) W_{0,0} = -i(1/2+\nu) W_{0,1} + 2y^{-1}\partial_y W_{0,1},$$

which gives

$$W_{0,l}(y) = \sum_{\substack{i=1,2,\\ m=0,1}} A_{i,m} g_{0,l}^{i,m}(y), \quad l = 0, 1,$$

with some constants $A_{i,m}$, $i = 1, 2$, $m = 0, 1$. It follows from(2.62), (2.64) that $W_{0,l}$, $l = 0, 1$ have the form (2.55), (2.57) with $\hat{w}_0^{j,l,0} = A_{1,1-l}$, $\hat{w}_0^{j,l,-1} = A_{2,1-l}$, $l = 0, 1$, which together with (2.58) gives $A_{1,m} = a_{0,1-m}$, $A_{2,m} = b_{0,1-m}$, $m = 0, 1$.

We next consider $j \geq 1$. Assume that $W_{i,n}$, $0 \leq n \leq 2i+1$, $i \leq j-1$ has the prescribed behavior (2.56), (2.57),(2.59), (2.60). Then it is not difficult to check that $\mathcal{G}_{j,l}$ has the form

$$\mathcal{G}_{j,l}(y) = \sum_{-j-1\leq m\leq j} e^{-imy^2/4} y^{2i\alpha_0(2m+1)} \mathcal{G}_{j,l}^m(y), \tag{2.65}$$

where $\mathcal{G}_{j,l}^m$, $m = 0, -1$, are given by

$$\begin{aligned}
\mathcal{G}_{j,l}^m(y) &= \mathcal{G}_{j,l}^{m,0}(y) + \mathcal{G}_{j,l}^{m,1}(y), \quad m = 0, -1, \\
\mathcal{G}_{j,l}^{0,0}(y) &= \mathcal{G}_{j,l}(y;\, W_{i,n}^0, 0 \leq n \leq 2i+1, 0 \leq i \leq j-1), \\
e^{iy^2/4}\mathcal{G}_{j,l}^{-1,0}(y) &= \mathcal{G}_{j,l}(y;\, W_{i,n}^1, 0 \leq n \leq 2i+1, 0 \leq i \leq j-1),
\end{aligned} \tag{2.66}$$

and admit the following asymptotic expansions and as $y \to \infty$:

$$\begin{aligned}
\mathcal{G}_{j,l}^{0,0}(y) &= \sum_{k\geq 1} \sum_{s=0}^{2j+1-l} T_{k,j,s}^{j,l,0} y^{2\nu(2j+1)-2k} (\ln y)^s, \\
\mathcal{G}_{j,l}^{0,1}(y) &= \sum_{k\geq 2} \sum_{\substack{-j\leq i\leq j-2\\ j-i\in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} T_{k,i,s}^{j,l,0} y^{2\nu(2i+1)-2k} (\ln y)^s,
\end{aligned} \tag{2.67}$$

$$\mathcal{G}_{j,l}^{-1,0}(y) = \sum_{k\geq 2} \sum_{s=0}^{2j+1-l} T_{k,-j-1,s}^{j,l,-1} y^{-2\nu(2j+1)-2k} (\ln y)^s,$$
$$\mathcal{G}_{j,l}^{-1,1}(y) = \sum_{k\geq 1} \sum_{\substack{-j+1\leq i\leq j-1\\ j-i+1\in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} T_{k,i,s}^{j,l,-1} y^{2\nu(2i+1)-2k} (\ln y)^s. \tag{2.68}$$

Finally, $\mathcal{G}_{j,l}^m$, $m \neq 0, -1$, have the following behavior as $y \to \infty$

$$\mathcal{G}_{j,l}^{m}(y) = \sum_{k\geq m+1} \sum_{\substack{m-j\leq i\leq j-m\\ j-m-i\in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} T_{k,i,s}^{j,l,m} y^{2\nu(2i+1)-2k} (\ln y)^s, \quad m\geq 1,$$
$$\mathcal{G}_{j,l}^{m}(y) = \sum_{k\geq |m|-1} \sum_{\substack{-j-m-2\leq i\leq j+m\\ j+m-i\in \mathbb{Z}}} \sum_{s=0}^{2j+1-l} T_{k,i,s}^{j,l,m} y^{2\nu(2i+1)-2k} (\ln y)^s, \quad m\leq -2. \tag{2.69}$$

Therefore, integrating (2.45), one gets

$$W_{j,l} = \tilde{W}_{j,l} + \sum_{\substack{i=1,2\\ m=0,\dots,2j+1}} A_{i,m}\mathbf{g}_{j,l}^{i,m},$$
$$\tilde{W}_{j,l}(y) = \sum_{-j-1\leq m\leq j} e^{-imy^2/4} y^{2i\alpha_0(2m+1)} \tilde{W}_{j,l}^m(y), \tag{2.70}$$

where $e^{-imy^2/4}y^{2i\alpha_0(2m+1)}\tilde{W}_{j,l}^m$ is a unique solution of (2.45) with $\mathcal{G}_{j,l}$ replaced by $e^{-imy^2/4}y^{2i\alpha_0(2m+1)}\mathcal{G}_{j,l}^m(y)$, that has the following behavior as $y \to +\infty$:

$$\tilde{W}_{j,l}^{m}(y) = \sum_{k\geq m+2} \sum_{\substack{m-j\leq i\leq j-m\\ j-m-i\in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} \tilde{w}_{k,i,s}^{j,l,m} y^{2\nu(2i+1)-2k} (\ln y)^s, \quad m\geq 1,$$
$$\tilde{W}_{j,l}^{m}(y) = \sum_{k\geq -m} \sum_{\substack{-j-m-2\leq i\leq j+m\\ j-m-i\in 2\mathbb{Z}}} \sum_{s=0}^{2j+1-l} \tilde{w}_{k,i,s}^{j,l,m} y^{2\nu(2i+1)-2k} (\ln y)^s, \quad m\leq -2. \tag{2.71}$$

Finally for $m = 0, -1$ one has:

$$\tilde{W}_{j,l}^{0}(y) = \tilde{W}_{j,l}^{0,0}(y) + \tilde{W}_{j,l}^{0,1}(y),$$
$$\tilde{W}_{j,l}^{-1}(y) = \tilde{W}_{j,l}^{-1,0}(y) + \tilde{W}_{j,l}^{-1,1}(y), \tag{2.72}$$

where $\tilde{W}^{0,i}_{j,l}$ and $e^{iy^2/4}y^{-2i\alpha_0}\tilde{W}^{-1,i}_{j,l}$ are solutions of (2.45) with $\mathcal{G}_{j,l}$ replaced by $\mathcal{G}^{0,i}_{j,l}$ and $y^{-2i\alpha_0}e^{iy^2/4}\mathcal{G}^{-1,i}_{j,l}$ respectively, with the following asymptotics as $y \to \infty$:

$$\begin{aligned}
\tilde{W}^{0,0}_{j,l}(y) &= \sum_{k\geq 1}\sum_{s=0}^{2j+1-l} \tilde{w}^{j,l,0}_{k,j,s} y^{2\nu(2j+1)-2k}(\ln y)^s,\\
\tilde{W}^{0,1}_{j,l}(y) &= \sum_{k\geq 1}\sum_{\substack{-j\leq i\leq j-2\\ j-i\in 2\mathbb{Z}}}\sum_{s=0}^{2j+1-l} \tilde{w}^{j,l,0}_{k,i,s} y^{2\nu(2i+1)-2k}(\ln y)^s,\\
\tilde{W}^{-1,0}_{j,l}(y) &= \sum_{k\geq 2}\sum_{s=0}^{2j+1-l} \tilde{w}^{j,l,-1}_{k,-j-1,s} y^{-2\nu(2j+1)-2k}(\ln y)^s,\\
\tilde{W}^{-1,1}_{j,l}(y) &= \sum_{k\geq 1}\sum_{\substack{-j+1\leq i\leq j-1\\ j-i\in 2\mathbb{Z}}}\sum_{s=0}^{2j+1-l} \tilde{w}^{j,l,-1}_{k,i,s} y^{2\nu(2i+1)-2k}(\ln y)^s.
\end{aligned} \tag{2.73}$$

Clearly, $W^0_{j,l} = \tilde{W}^{0,0}_{j,l} + \sum_{m=0}^{2j+1} A_{1,m}\mathbf{g}^{1,m}_{j,l}$, and $W^1_{j,l} = e^{-imy^2/4}\tilde{W}^{-1,0}_{j,l} + \sum_{m=0}^{2j+1} A_{2,m}\mathbf{g}^{2,m}_{j,l}$ are solutions of (2.45) with $\mathcal{G}_{j,l}$ replaced by $\mathcal{G}^{0,0}_{j,l} = \mathcal{G}_{j,l}(W^0_{i,n},\ i \leq j-1)$ and $e^{iy^2/4}\mathcal{G}^{-1,0}_{j,l} = \mathcal{G}_{j,l}(W^1_{i,n},\ i \leq j-1)$ respectively. As a consequence, $W^i_{j,l}$, $i = 0, 1$, $0 \leq l \leq 2j+1$, have the form (2.57) with $\hat{w}^{j,l,i}_0 = A^{1-i}_{2j+1-l}$, $i = 0, -1$, $l = 0, \dots, 2j+1$, which together with (2.58) gives $A_{1,m} = a_{j,2j+1-m}$, $A_{2,m} = b_{j,2j+1-m}$, $m = 0, \dots, 2j+1$. □

Let $W^{(N)}_{\text{in}}(y,t)$ be the the stereographic representation of $V^{(N)}_{\text{in}}(t^{-\nu}y,t) = (V^{(N)}_{in,1}(t^{-\nu}y,t), V^{(N)}_{in,2}(t^{-\nu}y,t), V^{(N)}_{in,3}(t^{-\nu}y,t))$:

$$W^{(N)}_{\text{in}}(y,t) = \frac{V^{(N)}_{in,1}(t^{-\nu}y,t) + iV^{(N)}_{in,2}(t^{-\nu}y,t)}{1 + V^{(N)}_{in,3}(t^{-\nu}y,t)}.$$

For $N \geq 2$ define

$$\begin{aligned}
W^{(N)}_{ss}(y,t) &= \sum_{j=0}^{N}\sum_{l=0}^{2j+1} t^{\nu(2j+1)}(\ln\rho)^l W^{ss}_{j,l}(y),\\
A^{(N)}_{ss} &= -it\partial_t W^{(N)}_{ss} + \alpha_0 W^{(N)}_{ss} + \mathcal{L}W^{(N)}_{ss} + G(W^{(N)}_{ss}, \bar{W}^{(N)}_{ss}, \partial_y W^{(N)}_{ss}),\\
V^{(N)}_{ss}(\rho,t) &= \Big(\frac{2\,\mathrm{re}\, W^{(N)}_{ss}}{1+|W^{(N)}_{ss}|^2}, \frac{2\,\mathrm{im}\, W^{(N)}_{ss}}{1+|W^{(N)}_{ss}|^2}, \frac{1-|W^{(N)}_{ss}|^2}{1+|W^{(N)}_{ss}|^2}\Big), \quad \rho = t^{-\nu}y,\\
Z^{(N)}_{ss}(\rho,t) &= V^{(N)}_{ss}(\rho,t) - Q(\rho).
\end{aligned}$$

Fix $\varepsilon_1 = \frac{\nu}{2}$. Then, as a direct consequence of the previous analysis, we obtain the following result.

Lemma 2.7. *For* $0 < t \leq T(N)$ *the following holds.*

(i) *For any* $k,\ l,$*and* $\frac{1}{10}t^{\varepsilon_1} \leq y \leq 10t^{\varepsilon_1}$, *one has*

$$|y^{-l}\partial_y^k\partial_t^i(W_{ss}^{(N)} - W_{\text{in}}^{(N)})| \leq C_{k,l,i}t^{\nu(N+1-\frac{l+k}{2})-i}, \quad i = 0, 1. \tag{2.74}$$

(ii) *The profile* $Z_{ss}^{(N)}$ *verifies*

$$\|\partial_\rho Z_{ss}^{(N)}(t)\|_{L^2(\rho d\rho,\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^\eta, \tag{2.75}$$

$$\|\rho^{-1}Z_{ss}^{(N)}(t)\|_{L^2(\rho d\rho,\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^\eta, \tag{2.76}$$

$$\|Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^\eta, \tag{2.77}$$

$$\|\rho\partial_\rho Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^\eta, \tag{2.78}$$

$$\|\rho^{-l}\partial_\rho^k Z_{ss}^{(N)}(t)\|_{L^2(\rho d\rho,\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^{\nu+\frac{1}{2}+\eta}, \quad k+l = 2, \tag{2.79}$$

$$\|\rho^{-l}\partial_\rho^k Z_{\text{in}}^{(N)}(t)\|_{L^2(\rho d\rho,\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^{2\nu}, \quad k+l \geq 3, \tag{2.80}$$

$$\|\rho^{-l}\partial_\rho^k Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^{\nu+\eta}, \quad k+l = 1, \tag{2.81}$$

$$\|\rho^{-l}\partial_\rho^k Z_{ss}^{(N)}(t)\|_{L^\infty(\frac{1}{10}t^{-\nu+\varepsilon_1}\leq\rho\leq 10t^{-\nu-\varepsilon_2})} \leq Ct^{2\nu}, \quad 2 \leq l+k. \tag{2.82}$$

Here and below η *stands for small positive constants depending on* ν *and* ε_2*, that may change from line to line.*

(iii) *The error* $A_{ss}^{(N)}$ *admits the estimate*

$$\|y^{-l}\partial_y^k\partial_t^i A_{ss}^{(N)}(t)\|_{L^2(ydy,\frac{1}{10}t^{\varepsilon_1}\leq y\leq 10t^{-\varepsilon_2})} \leq t^{\nu N(1-2\varepsilon_2)-i}, \quad 0 \leq l+k \leq 4, \quad i = 0, 1. \tag{2.83}$$

2.4. Remote region $r \sim 1$. We next consider the remote region $t^{-\varepsilon_2} \leq rt^{-1/2}$. Consider the formal solution $\sum_{j\geq 0}\sum_{l=0}^{2j+1} t^{\nu(2j+1)}(\ln y - \nu\ln t)^l W_{j,l}^{ss}(y)$ constructed in the previous subsection. By Lemma 2.6, it has form (2.55), (2.56), (2.57), (2.59), (2.60), with some coefficients $\hat{w}_k^{j,l,i}$, $w_{k,i,s}^{j,l,m}$. Note that in the limit $y \to \infty$, $r \to 0$, the main order terms of the expansion $\sum_{j\geq 0}\sum_{l=0}^{2j+1} t^{\nu(2j+1)+i\alpha_0}(\ln y - \nu\ln t)^l W_{j,l}^{ss}(t^{-1/2}r)$ are given by

$$\begin{aligned}
&\sum_{j\geq 0}\sum_{l=0}^{2j+1} t^{\nu(2j+1)+i\alpha_0}(\ln y - \nu\ln t)^l W_{j,l}^{ss}(t^{-1/2}r)\\
&\sim \sum_{k\geq 0}\frac{t^k}{r^{2k}}\sum_{j\geq 0}\sum_{l=0}^{2j+1}\hat{w}_k^{j,l,0}(\ln r)^l r^{2i\alpha_0+2\nu(2j+1)}\\
&+\frac{e^{\frac{ir^2}{4t}}}{t}\sum_{k\geq 0}\frac{t^k}{r^{2k}}\sum_{j\geq 0}\sum_{l=0}^{2j+1}\hat{w}_k^{j,l,-1}\left(\frac{r}{t}\right)^{-2i\alpha_0-2\nu(2j+1)-2}\left(\ln\left(\frac{r}{t}\right)\right)^l,
\end{aligned} \tag{2.84}$$

which means that in the region $t^{-\varepsilon_2} \le rt^{-1/2}$ we have to look for the solution of (2.38) as a perturbation of the time independent profile

$$\sum_{j\ge 0}\sum_{l=0}^{2j+1} \beta_0(j,l)(\ln r)^l r^{2\nu(2j+1)},$$

with $\beta_0(j,l) = \hat{w}_0^{j,l,0}$.

Let $\theta \in C_0^\infty(\mathbb{R})$, $\theta(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2. \end{cases}$ For $N \ge 2$, and $\delta > 0$ we define

$$f_0(r) \equiv f_0^{(N)}(r) = \theta(\delta^{-1}r)\sum_{j=0}^{N}\sum_{l=0}^{2j+1} \beta_0(j,l)(\ln r)^l r^{2i\alpha_0+2\nu(2j+1)}.$$

Note that $e^{i\theta} f_0 \in H^{1+2\nu-}$ and

$$\|e^{i\theta} f_0\|_{\dot{H}^s} \le C\delta^{1+2\nu-s}, \quad \forall\, 0 \le s < 1+2\nu. \tag{2.85}$$

Write $w(r,t) = f_0(r) + \chi(r,t)$. Then χ solves

$$\begin{aligned}
&i\chi_t = -\triangle\chi + r^{-2}\chi + \mathcal{V}_0\partial_r\chi + \mathcal{V}_1\chi + \mathcal{V}_2\bar{\chi} + \mathcal{N} + \mathcal{D}_0,\\
&\mathcal{V}_0 = \frac{4\bar{f}_0\partial_r f_0}{1+|f_0|^2}, \quad \mathcal{V}_1 = -\frac{2|f_0|^2(2+|f_0|^2)}{r^2(1+|f_0|^2)^2} - \frac{2\bar{f}_0^2(\partial_r f_0)^2}{(1+|f_0|^2)^2},\\
&\mathcal{V}_2 = \frac{2(r^2(\partial_r f_0)^2 - f_0^2)}{r^2(1+|f_0|^2)^2},\\
&\mathcal{D}_0 = (-\triangle + r^{-2})f_0 + G(f_0, \bar{f}_0, \partial_r f_0).
\end{aligned} \tag{2.86}$$

Finally, $\mathcal{N}$ contains the terms that are at least quadratic in χ and it has the form

$$\begin{aligned}
\mathcal{N} &= N_0(\chi,\bar{\chi}) + \chi_r N_1(\chi,\bar{\chi}) + \chi_r^2 N_2(\chi,\bar{\chi}),\\
N_0(\chi,\bar{\chi}) &= G(f_0+\chi, \bar{f}_0+\bar{\chi}, \partial_r f_0) - G(f_0, \bar{f}_0, \partial_r f_0) - \mathcal{V}_1\chi - \mathcal{V}_2\bar{\chi},\\
N_1(\chi,\bar{\chi}) &= \frac{4\partial_r f_0(\bar{f}_0+\bar{\chi})}{1+|f_0+\chi|^2} - \mathcal{V}_0,\\
N_2(\chi,\bar{\chi}) &= \frac{2(\bar{f}_0+\bar{\chi})}{1+|f_0+\chi|^2}.
\end{aligned} \tag{2.87}$$

Accordingly to (2.55), (2.56), (2.57), (2.59), (2.60), we look for χ as

$$\chi(r,t) = \sum_{\substack{q\ge 0\\ k\ge 1}} t^{2\nu q+k} \sum_{\substack{-\min\{k,q\}\le m\le \min\{(k-2)_+,q\}\\ q-m\in 2\mathbb{Z}}} \sum_{s=0}^{q} e^{-im\Phi}(\ln r - \ln t)^s g_{k,q,m,s}(r), \tag{2.88}$$

where

$$\Phi = \frac{r^2}{4t} + 2\alpha_0 \ln t + \varphi(r),$$

with φ to be chosen later.

Substituting this ansatz to the expressions $-i\chi_t - \triangle\chi + r^{-2}\chi + \mathcal{V}_0\partial_r\chi + \mathcal{V}_1\chi + \mathcal{V}_2\bar{\chi}$, N, we get

$$
\begin{aligned}
&-i\chi_t + \triangle\chi - r^{-2}\chi + \mathcal{V}_0\partial_r\chi + \mathcal{V}_1\chi + \mathcal{V}_2\bar{\chi} \\
&\quad = \sum_{\substack{q\geq 0\\ k\geq 2}} t^{2\nu q+k-2} \sum_{\substack{-\min\{k,q\}\leq m\leq \min\{(k-2)_+,q\}\\ q-m\in 2\mathbb{Z}}} \sum_{s=0}^{q} e^{-im\Phi}(\ln r - \ln t)^s \Psi^{lin}_{k,q,m,s}, \\
&N_0(\chi,\bar{\chi}) = \sum_{\substack{q\geq 0\\ k\geq 4}} t^{2\nu q+k-2} \sum_{\substack{-\min\{k,q\}\leq m\leq \min\{(k-2)_+,q\}\\ q-m\in 2\mathbb{Z}}} \sum_{s=0}^{q} e^{-im\Phi}(\ln r - \ln t)^s \Psi^{nl,0}_{k,q,m,s}, \\
&\chi_r N_1(\chi,\bar{\chi}) = \sum_{\substack{q\geq 0\\ k\geq 3}} t^{2\nu q+k-2} \sum_{\substack{-\min\{k,q\}\leq m\leq \min\{(k-2)_+,q\}\\ q-m\in 2\mathbb{Z}}} \sum_{s=0}^{q} e^{-im\Phi}(\ln r - \ln t)^s \Psi^{nl,1}_{k,q,m,s}, \\
&(\chi_r)^2 N_2(\chi,\bar{\chi}) = \sum_{\substack{q\geq 0\\ k\geq 2}} t^{2\nu q+k-2} \sum_{\substack{-\min\{k,q\}\leq m\leq \min\{(k-2)_+,q\}\\ q-m\in 2\mathbb{Z}}} \sum_{s=0}^{q} e^{-im\Phi}(\ln r - \ln t)^s \Psi^{nl,2}_{k,q,m,s},
\end{aligned}
$$

Here

$$
\Psi^{lin}_{k,q,m,s} = \frac{m(m+1)r^2}{4} g_{k,q,m,s} + \Psi^{lin,1}_{k,q,m,s} + \Psi^{lin,2}_{k,q,m,s}, \tag{2.89}
$$

with $\Psi^{lin,1}_{k,q,m,s}$ and $\Psi^{lin,2}_{k,q,m,s}$ depending respectively on $g_{k-1,q,m,s'}$, $s' = s, s+1$ and $g_{k-2,q,m,s'}$, $s' = s, s+1, s+2$ only:

$$
\begin{aligned}
\Psi^{lin,1}_{k,q,m,s} &= -i(2\nu q + k - 1 - m - 2im\alpha_0) g_{k-1,q,m,s} + i(m+1)(s+1) g_{k-1,q,m,s+1} \\
&\quad + imr(\partial_r - im\varphi'(r) - \frac{1}{2}\mathcal{V}_0(r)) g_{k-1,q,m,s},
\end{aligned} \tag{2.90}
$$

$$
\begin{aligned}
\Psi^{lin,2}_{k,q,m,s} &= -e^{im\varphi}\triangle(e^{-im\varphi} g_{k-2,q,m,s}) - \frac{2(s+1)}{r} e^{im\varphi}\partial_r(e^{-im\varphi} g_{k-2,q,m,s+1}) \\
&\quad - \frac{(s+1)(s+2)}{r^2} g_{k-2,q,m,s+2} + \mathcal{V}_0 e^{im\varphi}\partial_r(e^{-im\varphi} g_{k-2,q,m,s}) \\
&\quad + (r^{-2} + \mathcal{V}_1) g_{k-2,q,m,s} + \mathcal{V}_2 \bar{g}_{k-2,q,-m,s}.
\end{aligned} \tag{2.91}
$$

Here and below we use the convention $g_{k,q,m,s} = 0$ if $(k,q,m,s) \notin \Omega$, where

$$
\Omega = \{k \geq 1, q \geq 0, 0 \leq s \leq q, q - m \in 2\mathbb{Z}, -\min\{k,q\} \leq m \leq \min\{k-1,q\}\}.
$$

The nonlinear terms $\Psi^{nl,i}_{k,q,m,s}$, $i = 0, 1$, depend only on $g_{k',q',m',s'}$ with $k' \leq k-2$. More precisely,

$$
\Psi^{nl,0}_{k,q,m,s} = \Psi^{nl,0}_{k,q,m,s}(r; g_{k',q',m',s'},\ k' \leq k-3),
$$

$$
\Psi^{nl,1}_{k,q,m,s} = \Psi^{nl,1}_{k,q,m,s}(r; g_{k',q',m',s'},\ k' \leq k-2).
$$

Finally, $\Psi^{nl,2}_{k,q,m,s}$ has the following structure

$$\begin{aligned}
\Psi^{nl,2}_{2,q,m,s} &= -\delta_{m,-2}\frac{r^2\bar{f}_0}{2(1+|f_0|^2)}\sum_{\substack{q_1+q_2=q\\ s_1+s_2=s}} g_{1,q_1,-1,s_1}g_{1,q_2,-1,s_2},\\
\Psi^{nl,2}_{k,q,m,s} &= \Psi^{nl,2,0}_{k,q,m,s}+\tilde{\Psi}^{nl,2}_{k,q,m,s}, \quad k\geq 3,\\
\Psi^{nl,2,0}_{k,q,m,s} &= \frac{(m+1)r^2\bar{f}_0}{1+|f_0|^2}\sum_{\substack{q_1+q_2=q\\ s_1+s_2=s}} g_{1,q_1,-1,s_1}g_{k-1,q_2,m+1,s_2},
\end{aligned} \tag{2.92}$$

with $\tilde{\Psi}^{nl,2}_{k,q,m,s}$ depending on $g_{k',q',m',s'}$, $k'\leq k-2$ only:

$$\tilde{\Psi}^{nl,2}_{k,q,m,s} = (r;\, g_{k',q',m',s'},\ k'\leq k-2).$$

Note that

$$\Psi^{nl,2,0}_{k,q,-1,s}=0, \quad \forall k,q,s.$$

Equation (2.86) is equivalent to

$$\begin{cases}
\Psi^{lin}_{2,0,0,0}+\mathcal{D}_0=0,\\
\Psi^{lin}_{k,q,m,s}+\Psi^{nl}_{k,q,m,s}=0, \quad (k,q,m,s)\in\Omega,\ (k,q,m,s)\neq(2,0,0,0),
\end{cases} \tag{2.93}$$

Here $\Psi^{nl}_{k,q,m,s}=\Psi^{nl,0}_{k,q,m,s}+\Psi^{nl,1}_{k,q,m,s}+\Psi^{nl,2}_{k,q,m,s}$.

We view (2.93) as a recurrent system with respect to $k\geq 1$ of the form

$$\begin{cases}
\Psi^{lin}_{2,0,0,0}+\mathcal{D}_0=0,\\
\Psi^{lin}_{2,2j,0,s}=0, \quad (j,s)\neq(0,0),\\
\Psi^{lin}_{2,2j+1,1,s}=0,
\end{cases} \tag{2.94}$$

and

$$\begin{cases}
\Psi^{lin}_{k+1,q,m,s}+\Psi^{nl}_{k+1,q,m,s}=0, \quad m=0,1\\
\Psi^{lin}_{k,q,m,s}+\Psi^{nl}_{k,q,m,s}=0, \quad m\neq 0,1
\end{cases}, \quad k\geq 2. \tag{2.95}$$

Consider (2.94). Choosing φ as

$$\varphi(r)=-i\int_0^r ds\,\frac{\bar{f}_0(s)\partial_s f_0(s)-f_0(s)\partial_s\bar{f}_0(s)}{1+|f_0(s)|^2}, \tag{2.96}$$

we can rewrite (2.94) in the following form

$$\begin{cases}
(4\nu j+1)g_{1,2j,0,s}-(s+1)g_{1,2j,0,s+1}=0, \quad (j,s)\neq(0,0),\\
g_{1,0,0,0}=-iD_0,\\
r\partial_r g_{1,2j+1,-1,s}+(2\nu(2j+1)+2+2i\alpha_0-r(\ln(1+|f_0|^2))')g_{1,2j+1,-1,s}=0.
\end{cases} \tag{2.97}$$

Accordingly to (2.84), we solve this system as follows:

$$
\begin{aligned}
g_{1,2j,0,s} &= 0, \quad (j,s) \neq (0,0),\\
g_{1,0,0,0} &= -iD_0,\\
g_{1,2j+1,-1,s} &= \beta_1(j,s)(1+|f_0|^2)r^{-2i\alpha_0-2\nu(2j+1)-2}, \;\; 0 \le s \le 2j+1, \;\; 0 \le j \le N,\\
g_{1,2j+1,-1,s} &= 0, \quad j > N,
\end{aligned}
\tag{2.98}
$$

where $\beta_1(j,s) = \hat{w}_0^{j,l,-1}$.

Consider (2.95). We will solve it with the "zero boundary conditions" at zero. To formulate the result we need to introduce some notations. For $m \in \mathbb{Z}$, we denote by $\mathcal{A}_m$ the space of continuous functions $a : \mathbb{R}_+ \to \mathbb{C}$ such that

(i) $a \in C^\infty(\mathbb{R}_+^*)$, $\operatorname{supp} a \subset \{r \le 2\delta\}$;
(ii) for $0 \le r < \delta$, a has an absolutely convergent expansion of the form

$$
a(r) = \sum_{\substack{n \ge K(m)\\ n-m-1 \in 2\mathbb{Z}}} \sum_{l=0}^{n} \alpha_{n,l} (\ln r)^l r^{2\nu n},
$$

where $K(m) = m+1$ if $m \ge 0$, and $K(m) = |m|-1$ if $m \le -1$. For $k \ge 1$ we define $\mathcal{B}_k$ as the space of continuous functions $b : \mathbb{R}_+ \to \mathbb{C}$ such that

(i) $b \in C^\infty(\mathbb{R}_+^*)$;
(ii) for $0 \le r < \delta$, b has an absolutely convergent expansion of the form

$$
b(r) = \sum_{n=0}^{\infty} \sum_{l=0}^{2n} \beta_{n,l} r^{4\nu n} (\ln r)^l,
$$

(iii) for $r \ge 2\delta$, b is a polynome of degree $k-1$.

Finally, we set $\mathcal{B}_k^0 = \{b \in \mathcal{B}_k, b(0) = 0\}$.

Clearly, for any m, k, one has $r\partial_r \mathcal{A}_m \subset \mathcal{A}_m$, $r\partial_r \mathcal{B}_k \subset \mathcal{B}_k$, $\mathcal{B}_k \mathcal{A}_m \subset \mathcal{A}_m$. Note also that

$$
\begin{gathered}
f_0 \in r^{2i\alpha_0}\mathcal{A}_0, \quad \varphi \in \mathcal{B}_1^0, \quad g_{1,0,0,0} \in r^{2i\alpha_0-2}\mathcal{A}_0,\\
g_{1,2j+1,-1,s} \in r^{-2i\alpha_0-2\nu(2j+1)-2}\mathcal{B}_1, \quad 0 \le s \le 2j+1.
\end{gathered}
$$

Furthermore, one checks easily that if for all $(k,q,m,s) \in \Omega$, $g_{k,q,m,s} \in r^{2i\alpha_0(1+2m)-2\nu q-2k}\mathcal{A}_m$ if $m \neq -1$ and $g_{k,q,-1,s} \in r^{-2i\alpha_0-2\nu q-2k}\mathcal{B}_k$, then

$$
\begin{aligned}
\Psi^{lin,i}_{k,q,m,s}, \Psi^{nl,j}_{k,q,m,s}, \tilde{\Psi}^{nl,2}_{k,q,m,s} &\in r^{2i\alpha_0(1+2m)-2\nu q-2(k-1)}\mathcal{A}_m, \quad m \neq -1,\\
\Psi^{lin,2}_{k,q,-1,s}, \Psi^{nl,j}_{k,q,-1,s}, \tilde{\Psi}^{nl,2}_{k,q,-1,s} &\in r^{-2i\alpha_0-2\nu q-2(k-1)}\mathcal{B}_{k-2},
\end{aligned}
\tag{2.99}
$$

$i = 1,2, \;\; j = 0,1,2.$

Consider (2.95). Using (2.89), (2.90), (2.91), (2.92), (2.96), one can rewrite it as

$$
\begin{cases}
\frac{1}{4}m(m+1)r^2 g_{k,q,m,s} = B_{k,q,m,s}, \quad m \neq 0,-1,\\
r\partial_r g_{k,q,m,s} + \left(2\nu q + k + 1 + 2i\alpha_0 - \frac{r(\bar{f}_0\partial_r f_0 + f_0\partial_r \bar{f}_0)}{1+|f_0|^2}\right) g_{k,q,-1,s} = C_{k,q,-1,s},\\
(2\nu q + k)g_{k,q,0,s} - (s+1)g_{k,q,0,s+1} = C_{k,q,0,s} + D_{k,q,s},
\end{cases}
\tag{2.100}
$$

where $B_{k,q,m,s}$, $C_{k,q,m,s}$ depend on $g_{k',q',m',s'}$, $k' \le k-1$ only:

$$B_{k,q,m,s} = B_{k,q,m,s}(r;\, g_{k',q',m',s'},\ k' \le k-1), \quad m \neq 0, -1,$$
$$C_{k,q,m,s} = C_{k,q,m,s}(r;\, g_{k',q',m',s'},\ k' \le k-1), \quad m = 0, -1,$$

and have the following form

$$\begin{aligned} B_{k,q,m,s} &= -\Psi^{lin,1}_{k,q,m,s} - \Psi^{lin,2}_{k,q,m,s} - \Psi^{nl}_{k,q,m,s}, \quad m \neq 0, -1 \\ C_{k,q,m,s} &= -i\Psi^{lin,2}_{k+1,q,m,s} - i\tilde{\Psi}^{nl}_{k+1,q,m,s}, \quad m = 0, -1. \end{aligned} \tag{2.101}$$

Finally $D_{k,q,s}$ depend only on $g_{k,q,1,s}$ and is given by

$$D_{k,q,s} = -i\Psi^{nl,2,0}_{k+1,q,0,s} = -i\frac{r^2 \bar{f}_0}{1+|f_0|^2} \sum_{\substack{q_1+q_2=q \\ s_1+s_2=s}} g_{1,q_1,-1,s_1} g_{k,q_2,1,s_2}. \tag{2.102}$$

Note that $D_{2,q,s} = 0$.

Remark 2.8. It is not difficult to check that if

$$g_{k,q,m,s} = 0, \quad \forall q > (2N+1)(2k-2),\ m \neq 0, 1,$$
$$g_{k,q,m,s} = 0, \quad \forall q > (2N+1)(2k-1),\ m = 0, 1,$$

then

$$B_{k,q,m,s} = 0, \quad \forall q > (2N+1)(2k-2),\ m \neq 0, 1,$$
$$C_{k,q,m,s} = 0, \quad \forall q > (2N+1)(2k-1),\ m = 0, 1,$$
$$D_{k,q,s} = 0, \quad \forall q > (2N+1)(2k-1).$$

We are now in position to prove the following result.

Lemma 2.9. *There exists a unique solution* $(g_{k,q,m,s})_{\substack{(k,q,m,s)\in\Omega \\ k\ge 2}}$ *of* (2.100) *verifying*

$$\begin{aligned} g_{k,q,m,s} &\in r^{2i\alpha_0(2m+1)-2\nu q-2k}\mathcal{A}_m, \quad m \neq -1, \\ g_{k,q,-1,s} &\in r^{-2i\alpha_0-2\nu q-2k}\mathcal{B}_k. \end{aligned} \tag{2.103}$$

In addition, one has

$$\begin{aligned} g_{k,q,m,s} &= 0, \quad \forall q > (2N+1)(2k-2),\ m \neq 0, 1, \\ g_{k,q,m,s} &= 0, \quad \forall q > (2N+1)(2k-1),\ m = 0, 1, \end{aligned} \tag{2.104}$$

Proof. For $k = 2$ (2.100), (2.101), (2.92) give

$$\frac{1}{2}r^2 g_{2,2j,-2,s} = B_{2,2j,-2,s}, \quad 0 \le s \le 2j,\ 1 \le j, \tag{2.105}$$

$$r\partial_r g_{2,2j+1,-1,s} + \left(2\nu(2j+1) + 3 + 2i\alpha_0 - \frac{r(\bar{f}_0\partial_r f_0 + f_0\partial_r \bar{f}_0)}{1+|f_0|^2}\right) g_{2,2j+1,-1,s}$$
$$= C_{2,2j+1,-1,s}, \quad 0 \le s \le 2j+1,\ 0 \le j, \tag{2.106}$$

$$(4\nu j + 2)g_{2,2j,0,s} - (s+1)g_{2,2j,0,s+1} = C_{2,2j,0,s}, \quad 0 \le s \le 2j,\ 0 \le j. \tag{2.107}$$

Recall that $B_{2,q,m,s}$, $C_{2,q,m,s}$ depend only on $g_{1,q',m',s'}$ and therefore, are known by now. By (2.99), (2.101) and Remark 2.8 they verify

$$\begin{aligned} B_{2,q,-2,s} &\in r^{-6i\alpha_0-2\nu q-2}\mathcal{A}_{-2}, \quad m \neq 0,-1\\ C_{2,q,0,s} &\in r^{2i\alpha_0-2\nu q-4}\mathcal{A}_0, \quad C_{2,q,-1,s} \in r^{-2i\alpha_0-2\nu q-4}\mathcal{B}_1,\\ B_{2,q,-2,s} &= 0, \quad q > 2(2N+1),\\ C_{2,q,m,s} &= 0, \quad q > 3(2N+1), \;\; m = 0,1. \end{aligned}$$

Therefore, we get from (2.105), (2.106),

$$\begin{aligned} g_{2,2j,-2,s} &= \frac{2}{r^2}B_{2,2j,-2,s} \in r^{-6i\alpha_0-4\nu j-4}\mathcal{A}_{-2}, \quad 0 \le s \le 2j, \;\; 1 \le j,\\ g_{2,2j,0,2j} &= \frac{1}{4j\nu+2}C_{2,2j,0,2j} \in r^{2i\alpha_0-4\nu j-4}\mathcal{A}_0, \;\; 0 \le j,\\ g_{2,2j,0,s} &= \frac{1}{4j\nu+2}C_{2,2j,0,s} + \frac{s+1}{4j\nu+2}g_{2,2j,0,s+1} \in r^{2i\alpha_0-4\nu j-4}\mathcal{A}_0, \;\; 0 \le s \le 2j,\\ g_{2,2j,-2,s} &= 0, \quad j > 2N+1,\\ g_{2,2j,0,s} &= 0, \quad j \ge 3N+2, \end{aligned} \tag{2.108}$$

Consider (2.107). Write

$$g_{2,2j+1,-1,s} = r^{-2i\alpha_0-3-2\nu(2j+1)}(1+|f_0|^2)\hat{g}_{2,2j+1,-1,s}.$$

Then $\hat{g}_{2,2j+1,-1,s}$ solves

$$\partial_r \hat{g}_{2,2j+1,-1,s} = r^{-2}\hat{C}_{2,2j+1,-1,s}, \tag{2.109}$$

where

$$\hat{C}_{2,2j+1,-1,s} = r^{2i\alpha_0+4+2\nu(2j+1)}(1+|f_0|^2)^{-1}C_{2,2j+1,-1,s}.$$

Since $C_{2,2j+1,-1,s} \in r^{-2i\alpha_0-2\nu(2j+1)-4}\mathcal{B}_1$, we have:

(i) for $0 \le r < \delta$, $\hat{C}_{2,2j+1,-1,s}$ admits an absolutely convergent expansion of the form

$$\hat{C}_{2,2j+1,-1,s} = \sum_{n=0}^{\infty}\sum_{l=0}^{2n}\beta_{n,l}r^{4\nu n}(\ln r)^l,$$

(ii) for $r \ge 2\delta$, $\hat{C}_{2,2j+1,-1,s}$ is a constant.
Clearly, there exists a unique solution $\hat{g}_{2,2j+1,-1,s}$ of (2.109) such that $\hat{g}_{2,2j+1,-1,s} \in r^{-1}\mathcal{B}_2$. It is given by

$$\hat{g}_{2,2j+1,-1,s}(r) = \int_0^r d\rho\rho^{-2}(\hat{C}_{2,2j+1,-1,s}(\rho) - \beta_{0,0}) - \beta_{0,0}r^{-1},$$
$$0 \le s \le 2j+1, \;\; 0 \le j.$$

Finally, since $C_{2,q,-1,s} = 0$ for $q > 3(2N+1)$, one has

$$g_{2,2j+1,-1,s} = 0, \quad j > 3N+1.$$

We next proceed by induction. Suppose we have solved (2.100) with $k = 2, \ldots, l-1$, $l \geq 3$, and have found $(g_{k,q,m,s})_{\substack{(k,q,m,s)\in\Omega \\ 2\leq k\leq l-1}}$ verifying (2.103) and (2.104). Consider $k = l$. From the first line in (2.100) we have:

$$\frac{1}{4}m(m+1)r^2 g_{l,q,m,s} = B_{l,q,m,s}, \quad m \neq 0, -1,$$

where $B_{l,q,m,s}$ are known by now and, by (2.99), (2.101) and Remark 2.8, satisfy

$$B_{l,q,m,s} \in r^{2i\alpha_0(2m+1)-2\nu q-2(l-1)}\mathcal{A}_m,$$
$$B_{l,q,m,s} = 0, \quad q > 2(2N+1)(2l-2).$$

As a consequence, one obtains for $m \neq 0, -1$:

$$\begin{aligned} g_{l,q,m,s} &= \frac{4}{m(m+1)r^2}B_{l,q,m,s} \in r^{2i\alpha_0(2m+1)-2\nu q-2l}\mathcal{A}_m, \\ g_{l,q,m,s} &= 0, \quad q > 2(2N+1)(2l-2). \end{aligned} \tag{2.110}$$

We next consider the equations for $g_{l,2j,0,s}$:

$$(4\nu j+l)g_{l,2j,0,s} - (s+1)g_{l,2j,0,s+1} = C_{l,2j,0,s} + D_{l,2j,s}, \quad 0 \leq s \leq 2j, \ 0 \leq j. \tag{2.111}$$

The right hand side $C_{l,2j,0,s} + D_{l,2j,s}$ depends only on $g_{l,q_1,1,s_1}$ and g_{k,q_2,m_2,s_2}, $k \leq l-1$, and by (2.99), (2.101), (2.110) and Remark 2.8, satisfies

$$C_{l,2j,0,s} + D_{l,2j,s} \in r^{2i\alpha_0-4\nu j-2l}\mathcal{A}_0,$$
$$C_{l,2j,0,s} + D_{l,2j,s} = 0, \quad j > (2N+1)(2l-1).$$

Therefore, the solution of (2.111) verifies

$$g_{l,2j,0,s} \in r^{2i\alpha_0-4\nu j-2l}\mathcal{A}_0, \quad 0 \leq s \leq 2j, \ 0 \leq j,$$
$$g_{l,2j,0,s} = 0, \quad j > (2N+1)(2l-1).$$

Finally for $g_{l,2j+1,-1,s}$, $0 \leq s \leq 2j+1$, $0 \leq j$ we have

$$\begin{aligned} &r\partial_r g_{l,2j+1,m,s} + \left(2\nu(2j+1) + l + 1 + 2i\alpha_0 - \frac{r(\bar{f}_0\partial_r f_0 + f_0\partial_r \bar{f}_0)}{1+|f_0|^2}\right)g_{l,2j+1,-1,s} \\ &\quad = C_{l,2j+1,-1,s}, \end{aligned} \tag{2.112}$$

with $C_{l,2j+1,-1,s} \in r^{-2i\alpha_0-2\nu(2j+1)-2l}\mathcal{B}_{l-1}$ such that

$$C_{l,2j+1,-1,s} = 0, \quad 2j+1 > (2N+1)(2l-1). \tag{2.113}$$

Equation (2.112) has a unique solution $g_{l,2j+1,-1,s}$ verifying $g_{l,2j+1,-1,s} \in r^{-2i\alpha_0-2\nu(2j+1)-2l}\mathcal{B}_l$, which is given by

$$\begin{aligned} g_{l,2j+1,-1,s} &= r^{-2i\alpha_0-2\nu(2j+1)-l-1}(1+|f_0|^2)\hat{g}_{l,2j+1,-1,s}, \\ \hat{g}_{l,2j+1,-1,s} &= \int_0^r d\rho\rho^{-l}\Big(\hat{C}_{l,2j+1,-1,s} - \sum_{0\leq n\leq\frac{l-1}{4\nu}}\sum_{p=0}^{2n}\beta_{n,p}\rho^{4\nu n}(\ln\rho)^p\Big) \\ &\quad - \int_r^\infty d\rho\rho^{-l}\sum_{0\leq n\leq\frac{l-1}{4\nu}}\sum_{p=0}^{2n}\beta_{n,p}\rho^{4\nu n}(\ln\rho)^p, \end{aligned}$$

where

$$\hat{C}_{l,2j+1,-1,s} = r^{2i\alpha_0+2\nu(2j+1)+2l}(1+|f_0|^2)^{-1}C_{l,2j+1,-1,s},$$
$$\hat{C}_{l,2j+1,-1,s} = \sum_{n=0}^{\infty}\sum_{p=0}^{2n}\beta_{n,p}r^n(\ln r)^p, \quad r<\delta.$$

By (2.113),

$$g_{l,2j+1,-1,s}=0, \quad 2j+1>(2N+1)(2l-1).$$

Let us define

$$w_{\text{rem}}^{(N)}(r,t) = f_0(r) + \sum_{(k,q,m,s)\in\Omega,\, k\le N} t^{k+2\nu q}e^{-im\Phi}(\ln r - \ln t)^s g_{k,q,m,s}(r),$$
$$A_{\text{rem}}^{(N)} = -i\partial_t w_{\text{rem}}^{(N)} - \Delta w_{\text{rem}}^{(N)} + r^{-2}w_{\text{rem}}^{(N)} + G(w_{\text{rem}}^{(N)}, \bar{w}_{\text{rem}}^{(N)}, \partial_r w_{\text{rem}}^{(N)})$$
$$W_{\text{rem}}^{(N)}(y,t) = e^{-i\alpha(t)}w_{\text{rem}}^{(N)}(rt^{-1/2},t).$$

As a direct consequence of the previous analysis we get:

Lemma 2.10. *There exists $T(N,\delta)>0$ such that for $0<t\le T(N,\delta)$ the following holds.*

(i) *For any $0\le l,k\le 4$, $i=0,1$ and $\frac{1}{10}t^{-\varepsilon_2}\le y\le 10t^{-\varepsilon_2}$, one has*

$$|y^{-l}\partial_y^k\partial_t^i(W_{ss}^{(N)} - W_{\text{rem}}^{(N)})| \le t^{\nu(1-2\varepsilon_2)N} + t^{\varepsilon_2 N}, \tag{2.114}$$

provided N is sufficiently large (depending on ε_2).

(ii) *The profile $w_{\text{rem}}^{(N)}(r,t)$ verifies*

$$\|r^{-l}\partial_r^k(w_{\text{rem}}^{(N)}(t) - f_0)\|_{L^2(rdr,r\ge\frac{1}{10}t^{1/2-\varepsilon_2})} \le Ct^{\eta}, \quad 0\le k+l\le 3, \tag{2.115}$$
$$\|r\partial_r w_{\text{rem}}^{(N)}(t)\|_{L^\infty(r\ge\frac{1}{10}t^{1/2-\varepsilon_2})} \le C\delta^{2\nu}, \tag{2.116}$$
$$\|r^{-l}\partial_r^k w_{\text{rem}}^{(N)}(t)\|_{L^\infty(r\ge\frac{1}{10}t^{1/2-\varepsilon_2})} \le C(\delta^{2\nu-k-l} + t^{\nu-(k+l)/2+\eta}), \quad 0\le k+l\le 4, \tag{2.117}$$
$$\|r^{-l-1}\partial_r^k w_{\text{rem}}^{(N)}(t)\|_{L^\infty(r\ge\frac{1}{10}t^{1/2-\varepsilon_2})} \le C(\delta^{2\nu-6} + t^{\nu-3+\eta}), \quad k+l=5 \tag{2.118}$$

(iii) *The error $A_{\text{rem}}^{(N)}(r,t)$ admits the estimate*

$$\|r^{-l}\partial_r^k\partial_t^i A_{\text{rem}}^{(N)}(t)\|_{L^2(rdr,r\ge\frac{1}{10}t^{1/2-\varepsilon_2})} \le t^{\varepsilon_2 N}, \quad 0\le l+k\le 3,\ i=0,1, \tag{2.119}$$

provided N is sufficiently large.

2.4.1. Proof of Proposition 2.1. We are now in position to finish the proof of Proposition 2.1. Fix ε_2 verifying $0 < \varepsilon_2 < \frac{1}{2}$. For $N \geq 2$, define

$$\hat{W}_{\rm ex}^{(N)}(\rho,t) = \theta(t^{\nu-\varepsilon_1}\rho)W_{\rm in}^{(N)}(t^\nu\rho,t) + (1-\theta(t^{\nu-\varepsilon_1}\rho))\theta(t^{\nu+\varepsilon_2}\rho)W_{ss}^{(N)}(t^\nu\rho,t) + (1-\theta(t^{\nu+\varepsilon_2}\rho))e^{-i\alpha(t)}w_{\rm rem}^{(N)}(t^{\nu+1/2}\rho,t),$$

$$V_{\rm ex}^{(N)}(\rho,t) = \Big(\frac{2\,{\rm re}\,\hat{W}_{\rm ex}^{(N)}}{1+|\hat{W}_{\rm ex}^{(N)}|^2}, \frac{2\,{\rm im}\,\hat{W}_{\rm ex}^{(N)}}{1+|\hat{W}_{\rm ex}^{(N)}|^2}, \frac{1-|\hat{W}_{\rm ex}^{(N)}|^2}{1+|\hat{W}_{\rm ex}^{(N)}|^2}\Big).$$

Clearly, $V_{\rm ex}^{(N)}(\rho,t)$ is well defined for ρ is sufficiently large, and for $\rho < t^{-\nu+\varepsilon_1}$ $V_{\rm ex}^{(N)}(\rho,t)$ coincides with $V_{\rm in}^{(N)}(\rho,t)$. Therefore, setting

$$V^{(N)}(\rho,t) = \begin{cases} V_{\rm in}^{(N)}(\rho,t), & \rho \leq \frac{1}{2}t^{-\nu+\varepsilon_1}, \\ V_{\rm ex}^{(N)}(\rho,t) & \rho \geq \frac{1}{2}t^{-\nu+\varepsilon_1}. \end{cases}$$

$$u^{(N)}(x,t) = e^{(\alpha(t)+\theta)R}V^{(N)}(\lambda(t)|x|,t),$$

we get a C^∞ 1- equivariant profile $u^{(N)} : \mathbb{R}^2 \times \mathbb{R}_+^* \to S^2$ that, by Lemmas 2.3 (i), 2.7 (ii), 2.10 (ii), for any $N \geq 2$ verifies part (i) of Proposition 2.1, ζ_N^* being given by

$$\zeta_N^*(x) = e^{\theta R}\hat{\zeta}_N^*(|x|), \quad \hat{\zeta}_N^* = \Big(\frac{2\,{\rm re}\, f_0}{1+|f_0|^2}, \frac{2\,{\rm im}\, f_0}{1+|f_0|^2}, \frac{1-|f_0|^2}{1+|f_0|^2}\Big).$$

By Lemmas 2.3 (ii), 2.7 (i), (iii) and 2.10 (i), (iii), for N sufficiently large the error $r^{(N)} = -u_t^{(N)} + u^{(N)} \times \Delta u^{(N)}$ satisfies

$$\|r^{(N)}(t)\|_{H^3} + \|\partial_t r^{(N)}(t)\|_{H^1} + \|\langle x\rangle r^{(N)}(t)\|_{L^2} \leq t^{\eta N}, \quad t \leq T(N,\delta),$$

with some $\eta = \eta(\nu,\varepsilon_2) > 0$. Re-denoting $N = \frac{N}{\eta}$ we obtain a family of approximate solutions $u^{(N)}(t)$ verifying Proposition 2.1.

3. Proof of the Theorem

3.1. Main proposition. The proof of Theorem 1.1 will be achieved by compactness arguments that rely on the following result. Let $u^{(N)}$, $T = T(N,\delta)$ be as in Proposition 2.1. Consider the Cauchy problem

$$u_t = u \times \Delta u, \quad t \geq t_1, \qquad u|_{t=t_1} = u^{(N)}(t_1), \tag{3.1}$$

with $0 < t_1 < T$.

One has

Proposition 3.1. *For N sufficiently large there exists $0 < t_0 < T$ such that for any $t_1 \in (0,t_0)$ the solution $u(t)$ of (3.1) verifies:*

(i) *$u - u^{(N)}$ is in $C([t_1,t_0],H^3)$ and*

$$\|u(t) - u^{(N)}(t)\|_{H^3} \leq t^{N/2}, \quad \forall t_1 \leq t \leq t_0. \tag{3.2}$$

(ii) *Furthermore,* $\langle x\rangle(u(t)-u^{(N)}(t))\in L^2$ *and*

$$\|\langle x\rangle(u(t)-u^{(N)}(t))\|_{L^2}\leq t^{N/2},\quad \forall t_1\leq t\leq t_0. \tag{3.3}$$

Proof. The proof is by a bootstrap argument. Write

$$\begin{aligned} u^{(N)}(x,t)&=e^{\alpha(t)R}U^{(N)}(\lambda(t)x,t),\quad r^{(N)}(x,t)=\lambda^2(t)e^{\alpha(t)R}R^{(N)}(\lambda(t)x,t)\\ u(x,t)&=e^{\alpha(t)R}U(\lambda(t)x,t),\quad U(y,t)=U^{(N)}(y,t)+S(y,t),\\ U^{(N)}(y,t)&=\phi(y)+\chi^{(N)}(y,t). \end{aligned}$$

Then $S(t)$ solves

$$t^{1+2\nu}S_t+\alpha_0t^{2\nu}RS-(\nu+\frac{1}{2})y\cdot\nabla S=S\times\Delta U^{(N)}+U^{(N)}\times\Delta S+S\times\Delta S+R^{(N)}(t). \tag{3.4}$$

Assume that

$$\|S\|_{L^\infty(\mathbb{R}^2)}\leq\delta_1, \tag{3.5}$$

with δ_1 sufficiently small. Note that since S is 1-equivariant and

$$(\phi,S)+(\chi^{(N)},S)+|S|^2=0 \tag{3.6}$$

where $\|\chi^{(N)}\|_{L^\infty(\mathbb{R}^2)}\leq C\delta^{2\nu}$ (see (2.5)), the bootstrap assumption (3.5) implies

$$\|S\|_{L^\infty(\mathbb{R}^2)}\leq C\|\nabla S\|_{L^2(\mathbb{R}^2)}. \tag{3.7}$$

3.1.1. Energy control. We will first derive a bootstrap control of the energy norm:

$$J_1(t)=\int_{\mathbb{R}^2}dy(|\nabla S|^2+\kappa(\rho)|S|^2),\quad \rho=|y|.$$

It follows from (3.4) that

$$t^{1+2\nu}\frac{d}{dt}\int dy|\nabla S|^2=-2\int dy(S\times\Delta U^{(N)},\Delta S)+2\int dy\big(\nabla R^{(N)},\nabla S\big), \tag{3.8}$$

$$\begin{aligned} t^{1+2\nu}\frac{d}{dt}\int dy\kappa(\rho)|S|^2=&-(\frac{1}{2}+\nu)t^{2\nu}\int dy(2\kappa+\rho\kappa')(S,S)\\ &+2\int dy\kappa(U^{(N)}\times\Delta S,S)+2\int dy\kappa(R^{(N)},S). \end{aligned} \tag{3.9}$$

Recall that $U^{(N)}=\phi+\chi^{(N)}$, with ϕ solving $\Delta\phi=\kappa\phi$, which means that

$$(S\times\Delta\phi,\Delta S)-\kappa(\phi\times\Delta S,S)=0.$$

Therefore, combining (3.8), (3.9), we get

$$t^{1+2\nu}\frac{d}{dt}J_1(t)=\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3+\mathcal{E}_4,$$

where

$$\mathcal{E}_1 = -2\int dy(S\times\Delta\chi^{(N)}, \Delta S),$$
$$\mathcal{E}_2 = 2\int dy\kappa(\chi^{(N)}\times\Delta S, S),$$
$$\mathcal{E}_3 = -(\frac{1}{2}+\nu)t^{2\nu}\int dy(2\kappa+\rho\kappa')(S,S),$$
$$\mathcal{E}_4 = 2\int dy\big[(\nabla R^{(N)}, \nabla S)+\kappa(R^{(N)}, S)\big].$$

From Proposition 2.1 we have

$$|\mathcal{E}_j| \le Ct^{2\nu}\|S\|^2_{H^1}, \quad j=1,\dots,3,$$
$$|\mathcal{E}_4| \le Ct^{N+\nu+1/2}\|\nabla S\|_{L^2}.$$

Combining these inequalities we obtain

$$\Big|\frac{d}{dt}J_1(t)\Big| \le Ct^{-1}\|S\|^2_{H^1} + Ct^{2N-2\nu}. \tag{3.10}$$

3.1.2. Control of the L^2 norm. Consider $J_0(t) = \int_{\mathbb{R}^2} dy|S|^2$. We have

$$t^{1+2\nu}\frac{d}{dt}J_0(t) = \mathcal{E}_5+\mathcal{E}_6+\mathcal{E}_7,$$
$$\mathcal{E}_5 = 2\int dy(U^{(N)}\times\Delta S, S),$$
$$\mathcal{E}_6 = -2(1+2\nu)t^{2\nu}J_0(t),$$
$$\mathcal{E}_7 = 2\int dy(R^{(N)}, S).$$

Consider $\mathcal{E}_5$. Decomposing $U^{(N)}$ and S in the basis f_1, f_2, Q:

$$U^{(N)}(y,t) = e^{\theta R}((1+z_3^{(N)}(\rho,t))Q(\rho)+z_1^{(N)}(\rho,t)f_1(\rho)+z_2^{(N)}(\rho,t)f_2(\rho)),$$
$$S(y,t) = e^{\theta R}(\zeta_1(\rho,t)f_1(\rho)+\zeta_2(\rho,t)f_2(\rho)+\zeta_3(\rho,t)Q(\rho)),$$

one can rewrite $\mathcal{E}_5$ as follows.

$$\mathcal{E}_5 = \mathcal{E}_8+\mathcal{E}_9+\mathcal{E}_{10},$$
$$\mathcal{E}_8 = -4\int_{\mathbb{R}_+} d\rho\rho\frac{h_1}{\rho}\zeta_2\partial_\rho\zeta_3,$$
$$\mathcal{E}_9 = -2\int_{\mathbb{R}_+} d\rho\rho(\partial_\rho z^{(N)}\times\partial_\rho\zeta, \zeta), \quad z^{(N)} = (z_1^{(N)}, z_2^{(N)}, z_3^{(N)}),\ \zeta = (\zeta_1,\zeta_2,\zeta_3),$$
$$\mathcal{E}_{10} = 2\int_{\mathbb{R}_+} d\rho\rho(z^{(N)}\times l, \zeta),$$

where

$$l = (-\frac{1}{\rho^2}\zeta_1 - \frac{2h_1}{\rho}\partial_\rho\zeta_3, -\frac{1}{\rho^2}\zeta_2, \kappa(\rho)\zeta_3+\frac{2h_1}{\rho}\partial_\rho\zeta_1 - \frac{2h_1h_3}{\rho^2}\partial_\rho\zeta_1).$$

Clearly,

$$|l| \leq C\rho^{-2}(|\zeta| + |\partial_\rho \zeta|).$$

Therefore,

$$|\mathcal{E}_{10}| \leq Ct^{2\nu}\|S\|_{H^1}^2. \tag{3.11}$$

Consider $\mathcal{E}_8$. It follows from

$$2(\zeta, \mathbf{k} + z^{(N)}) + |\zeta|^2 = 0, \tag{3.12}$$

that

$$|\partial_\rho \zeta_3| \leq C(|\partial_\rho z^{(N)}||\zeta| + |z^{(N)}||\partial_\rho \zeta| + |\partial_\rho \zeta||\zeta|).$$

As a consequence,

$$|\mathcal{E}_8| \leq C\big[t^{2\nu}\|S\|_{H^1}^2 + \|\nabla S\|_{L^2}^3\big]. \tag{3.13}$$

Consider $\mathcal{E}_9$. Denote $e_0 = \mathbf{k} + z^{(N)}$ and write $\zeta = \zeta^\perp + \mu e_0$, $\mu = (\zeta, e_0)$. It follows from (3.12) that

$$|\mu| \leq C|\zeta|^2,$$
$$|\mu_\rho| \leq C|\zeta||\partial_\rho \zeta|.$$

Therefore, $\mathcal{E}_9$ can be written as

$$\mathcal{E}_9 = -2\int_{\mathbb{R}_+} d\rho\rho(\partial_\rho \zeta^\perp \times \zeta^\perp, \partial_\rho e_0) + O(\|S\|_{H^1}^2\|\nabla S\|_{L^2}). \tag{3.14}$$

Let $e_1,\ e_2$ be a smooth orthonormal basis of the tangent space $T_{e_0}S^2$ that verifies $e_2 = e_0 \times e_1$. Then the expression $(\partial_\rho \zeta^\perp \times \zeta^\perp, \partial_\rho e_0)$ can be written as follows:

$$(\partial_\rho \zeta^\perp \times \zeta^\perp, \partial_\rho e_0) = (\zeta^\perp, \partial_\rho e_0)\left[(\zeta^\perp, e_2)(\partial_\rho e_0, e_1) - (\zeta^\perp, e_1)(\partial_\rho e_0, e_2)\right],$$

which leads to the estimate

$$\left|\int_{\mathbb{R}_+} d\rho\rho(\partial_\rho \zeta^\perp \times \zeta^\perp, \partial_\rho e_0)\right| \leq C\|\partial_\rho z^{(N)}\|_{L^\infty}^2 J_0(t) \leq Ct^{2\nu} J_0(t). \tag{3.15}$$

Combining (3.11), (3.13), (3.14), (3.15) we obtain

$$\left|\frac{d}{dt}J_0(t)\right| \leq C\left[t^{-1}\|S\|_{H^1}^2 + t^{-1-2\nu}\|S\|_{H^1}^2\|\nabla S\|_{L^2} + t^{2N-2\nu}\right]. \tag{3.16}$$

3.1.3. Control of the weighted L^2 norm. Using (3.4) to compute the derivative $\frac{d}{dt}\|yS(t)\|^2_{L^2}$, we obtain

$$t^{1+2\nu}\frac{d}{dt}\||y|S(t)\|^2_{L^2} = -4\int dy y_i(U^{(N)}\times\partial_i S, S)$$
$$-2\int dy|y|^2(\partial_i U^{(N)}\times\partial_i S, S)$$
$$-2(1+2\nu)t^{2\nu}\||y|S(t)\|^2_{L^2}+2\int dy|y|^2(R^{(N)}, S).$$

Here and below ∂_j stands for ∂_{y_j}, the summation over the repeated indexes being assumed.

As a consequence, we get

$$\left|\frac{d}{dt}\||y|S(t)\|^2_{L^2}\right| \le \frac{C}{t}\left[\||y|S(t)\|^2_{L^2}+t^{-4\nu}\|S\|^2_{H^1}+t^{2N-4\nu}\right]. \tag{3.17}$$

3.1.4. Control of the higher regularity. In addition to (3.5), assume that

$$\|S(t)\|_{H^3}+\||y|S(t)\|_{L^2}\le t^{2N/5}. \tag{3.18}$$

We will control $\dot H^3$ norm of the solution by means of $\|\nabla S_t\|_{L^2}$. More precisely, consider the functional

$$J_3(t)=t^{2+4\nu}\int dx|\nabla s_t(x,t)|^2+t^{1+2\nu}\int dx\kappa(t^{-1/2-\nu}x)|s_t(x,t)|^2,$$

where $s(x,t)$ is defined by

$$s(x,t)=e^{\alpha(t)R}S(\lambda(t)x,t).$$

Write $s_t(x,t)=e^{\alpha(t)R}\lambda^2(t)g(\lambda(t)x,t)$. In terms of g, J_3 can be written as $J_3(t)=\int dy|\nabla g(y,t)|^2+\int dy\kappa(\rho)|g(y,t)|^2$. Let us compute the derivative $\frac{d}{dt}J_3(t)$. Clearly, $g(y,t)$ solves

$$\begin{aligned}&t^{1+2\nu}g_t+\alpha_0 t^{2\nu}Rg-(\nu+\frac{1}{2})t^{2\nu}(2+y\cdot\nabla)g\\&=(S+U^{(N)})\times\Delta g+g\times(\Delta U^{(N)}+\Delta S)\\&\quad+(U^{(N)}\times\Delta U^{(N)}-R^{(N)})\times\Delta S\\&\quad+S\times\Delta(U^{(N)}\times\Delta U^{(N)}-R^{(N)})+t^{2+4\nu}r_t^{(N)}.\end{aligned} \tag{3.19}$$

Therefore, we get

$$\begin{aligned}t^{1+2\nu}\frac{d}{dt}J_3(t)=&(2+4\nu)t^{2\nu}\|\nabla g\|^2_{L^2}+(\frac{1}{2}+\nu)t^{2\nu}\int(2\kappa-\rho\kappa')|g|^2dy\\&+E_1+E_2+E_3+E_4+E_5,\end{aligned} \tag{3.20}$$

where

$$E_1 = -2\int dy(g\times\Delta\chi^{(N)},\Delta g) + 2\int dy\kappa(\chi^{(N)}\times\Delta g, g),$$

$$\begin{aligned}E_2 = &-2\int dy((U^{(N)}\times\Delta U^{(N)} - R^{(N)})\times\Delta S,\Delta g)\\ &+2\int dy(\Delta(U^{(N)}\times\Delta U^{(N)} - R^{(N)})\times S,\Delta g)\\ &+2\int dy\kappa((U^{(N)}\times\Delta U^{(N)} - R^{(N)})\times\Delta S, g)\\ &-2\int dy\kappa(\Delta(U^{(N)}\times\Delta U^{(N)} - R^{(N)})\times S, g),\end{aligned}$$

$$\begin{aligned}E_3 &= -2\int dy(g\times\Delta S,\Delta g),\\ E_4 &= 2\int dy\kappa(S\times\Delta g, g),\\ E_5 &= -2t^{2+4\nu}\int dy(r_t,\Delta g) + 2t^{2+4\nu}\int dy\kappa(r_t, g).\end{aligned}$$

The terms E_j, $j = 1, 4, 5$ can be estimated as follows.

$$\begin{aligned}|E_1| &\le Ct^{2\nu}\|g\|_{H^1}^2,\\ |E_4| &\le C\|g\|_{H^1}^2\|S\|_{H^3} \le Ct^{2\nu}\|g\|_{H^1}^2,\\ |E_5| &\le C(t^{2\nu}\|g\|_{H^1}^2 + t^{2N+3+4\nu}),\end{aligned} \tag{3.21}$$

provided N is sufficiently large and $t \le t_0$ with some $t_0 = t_0(N) > 0$.

For E_2 we have

$$\begin{aligned}|E_2| \le\, &C(\|\Delta\chi^{(N)}\|_{W^{2,\infty}} + \|R^{(N)}\|_{H^3})\|g\|_{H^1}\|S\|_{H^3}\\ &+ C\|\langle y\rangle^{-1}\nabla\Delta^2\chi^{(N)}\|_{L^\infty}\|\nabla g\|_{L^2}\|\langle y\rangle S\|_{L^2}.\end{aligned}$$

As a consequence,

$$|E_2| \le Ct^{2\nu}(\|g\|_{H^1}\|S\|_{H^3} + \|\nabla g\|_{L^2}\|\langle y\rangle S\|_{L^2}). \tag{3.22}$$

Note that since

$$g = (U^{(N)} + S)\times\Delta S + S\times\Delta U^{(N)} + R^{(N)}, \tag{3.23}$$

the bootstrap assumption (3.18) implies

$$\begin{aligned}\|g\|_{L^2} &\le C(\|S\|_{H^2} + \|R^{(N)}\|_{L^2}),\\ \|\nabla g\|_{L^2} &\le C(\|S\|_{H^3} + \|\nabla R^{(N)}\|_{L^2}).\end{aligned} \tag{3.24}$$

Therefore, (3.21), (3.22) can be rewritten as

$$\begin{aligned}|E_1| + |E_2| + |E_4| + |E_5| \le\, &Ct^{2\nu}[\|S\|_{H^3}^2 + (\|S\|_{H^3} + t^{N+1+2\nu})\|\langle y\rangle S\|_{L^2}]\\ &+Ct^{2N+1+4\nu}.\end{aligned} \tag{3.25}$$

Consider E_3. One has

$$\begin{aligned} g \times \Delta S &= (U^{(N)} + S, \Delta S)\Delta S - |\Delta S|^2 (U^{(N)} + S) \\ &\quad + (S \times \Delta U^{(N)} + R^{(N)}) \times \Delta S, \\ \Delta g &= (U^{(N)} + S) \times \Delta^2 S + Y, \\ Y &= 2(\partial_j U^{(N)} + \partial_j S) \times \Delta \partial_j S + S \times \Delta^2 U^{(N)} \\ &\quad + 2\partial_j S \times \Delta \partial_j U^{(N)} + \Delta R^{(N)}. \end{aligned} \tag{3.26}$$

Therefore, one can write E_3 as $E_3 = E_6 + E_7 + E_8$, where

$$\begin{aligned} E_6 &= -2\int dy (U^{(N)} + S, \Delta S)(\Delta S, \Delta g), \\ E_7 &= 2\int dy |\Delta S|^2 (U^{(N)} + S, \Delta g) = 2\int dy |\Delta S|^2 (U^{(N)} + S, Y), \\ E_8 &= -2\int dy ((S \times \Delta U^{(N)} + R^{(N)}) \times \Delta S, \Delta g). \end{aligned}$$

For E_6 we have:

$$\begin{aligned} E_6 &= 2\int dy [(\Delta U^{(N)}, S) + 2(\partial_j U^{(N)}, \partial_j S) + (\partial_j S, \partial_j S)](\Delta S, \Delta g) \\ &= -2\int dy [(\Delta U^{(N)}, S) + 2(\partial_j U^{(N)}, \partial_j S) + (\partial_j S, \partial_j S)](\Delta \partial_k S, \partial_k g) \\ &\quad - 2\int dy (\Delta S, \partial_k g)\partial_k [(\Delta U^{(N)}, S) + 2(\partial_j U^{(N)}, \partial_j S) + (\partial_j S, \partial_j S)]. \end{aligned}$$

As a consequence, one obtains:

$$|E_6| \le C\|S\|_{H^3}^2 \|g\|_{H^1} \le Ct^{2\nu}\|S\|_{H^3}^2. \tag{3.27}$$

Consider E_7. From (3.26) we have

$$\|Y\|_{L^2} \le C(\|S\|_{H^3} + t^N).$$

Therefore, we obtain:

$$|E_7| \le Ct^{2\nu}\|S\|_{H^3}^2. \tag{3.28}$$

Finally, E_8 can be estimated as follows

$$|E_8| \le C\|g\|_{H^1}(\|S\|_{H^3}^2 + t^N\|S\|_{H^3}) \le Ct^{2\nu}\|S\|_{H^3}^2 + Ct^{3N}. \tag{3.29}$$

Combining (3.27), (3.29), (3.28) we get

$$|E_3| \le C(t^{2\nu}\|S\|_{H^3}^2 + t^{3N}), \tag{3.30}$$

which together with (3.25) gives

$$\left|\frac{d}{dt} J_3(t)\right| \le \frac{C}{t}\left[\|S\|_{H^3}^2 + (\|S\|_{H^3} + t^{N+1+2\nu})\||y|S\|_{L^2})\right] + Ct^{2N+2\nu}. \tag{3.31}$$

3.1.5. Proof of Proposition 3.1. To prove the proposition it is sufficient to show that (3.5), (3.18) implies (3.2), (3.3).

Under the bootstrap assumption (3.18), (3.10), (3.16) become

$$\Big|\frac{d}{dt}J_1(t)\Big| + \Big|\frac{d}{dt}J_0(t)\Big| \le Ct^{-1}\|S\|_{H^1}^2 + Ct^{2N-2\nu}, \quad \forall t \le t_0, \tag{3.32}$$

provided N is sufficiently large, t_0 sufficiently small.

Note that for $c_0 > 0$ sufficiently large one has $\|S\|_{H^1}^2 \le J_1 + c_0 J_0$. Therefore, denoting $J(t) = J_1(t) + c_0 J_0(t)$ one can rewrite (3.32) as

$$\Big|\frac{d}{dt}J(t)\Big| \le Ct^{-1}J(t) + Ct^{2N-2\nu}. \tag{3.33}$$

Integrating this inequality with zero initial condition at t_1 one gets

$$J(t) \le \frac{C}{N}t^{2N+1-2\nu}, \quad \forall t \in [t_1, t_0], \tag{3.34}$$

provided N is sufficiently large. As a consequence, we obtain

$$\|S\|_{H^1}^2 \le \frac{C}{N}t^{2N+1-2\nu}, \quad \forall t \in [t_1, t_0]. \tag{3.35}$$

Consider $\||y|S(t)\|_{L^2}$. From (3.17), (3.35) we have

$$\left|\frac{d}{dt}\||y|S(t)\|_{L^2}^2\right| \le \frac{C}{t}\left[\||y|S(t)\|_{L^2}^2 + t^{2N+1-6\nu}\right]. \tag{3.36}$$

Integrating this inequality and assuming that N is sufficiently large, we get

$$\||y|S(t)\|_{L^2}^2 \le \frac{C}{N}t^{2N+1-6\nu}, \quad \forall t \in [t_1, t_0], \tag{3.37}$$

which gives in particular,

$$\||x|s(t)\|_{L^2}^2 \le t^{N/2}, \quad \forall t \in [t_1, t_0]. \tag{3.38}$$

We next consider $\|\nabla\Delta s(t)\|_{L^2(\mathbb{R}^2)}$. It follows from (3.23), (3.18) that for any $j = 1, 2$

$$\|\partial_j g - (U^{(N)} + S) \times \Delta\partial_j S\|_{L^2} \le C(\|S\|_{H^2(\mathbb{R}^2)} + t^{N+1+2\nu}). \tag{3.39}$$

Note also that since $|U^{(N)} + S| = 1$, we have

$$\begin{aligned}
|(U^{(N)} + S) \times \Delta\partial_j S|^2 &= |\Delta\partial_j S|^2 - (U^{(N)} + S, \Delta\partial_j S)^2,\\
(U^{(N)} + S, \Delta\partial_j S) &= -(\Delta U^{(N)} + \Delta S, \partial_j S) - \Delta(\partial_j U^{(N)}, S)\\
&\quad - 2(\partial_k U^{(N)} + \partial_k S, \partial_{jk}^2 S),
\end{aligned}$$

which together with (3.18) gives

$$\|\Delta\partial_j S\|_{L^2}^2 - \|(U^{(N)} + S) \times \Delta\partial_j S\|_{L^2}^2 \le C\|S\|_{H^2}^2. \tag{3.40}$$

Consider the functional $\tilde{J}_3(t) = J_3(t) + c_1 J_0(t)$. It follows from (3.24), (3.39), (3.40) that for $c_1 > 0$ sufficiently large we have

$$c_2\|S\|_{H^3}^2 - Ct^{2N+1+2\nu} \le \tilde{J}_3(t) \le C(\|S\|_{H^3}^2 + t^{2N+1+2\nu}), \tag{3.41}$$

with some $c_2 > 0$.

From (3.31), (3.32), (3.37) one gets

$$\begin{aligned} \left|\frac{d}{dt}\tilde{J}_3(t)\right| &\le C\left[t^{-1}(\|S\|_{H^3(\mathbb{R}^2)}^2 + \||y|S\|_{L^2(\mathbb{R}^2)}^2) + t^{2N-2\nu}\right] \\ &\le Ct^{-1}\tilde{J}_3(t) + Ct^{2N-6\nu}. \end{aligned} \tag{3.42}$$

Integrating this inequality between t_1 and t and observing that $\tilde{J}_3(t_1) = t_1^{2+4\nu}\int dx|\nabla r^{(N)}(x,t_1)|^2 + t_1^{1+2\nu}\int dx\kappa(t^{-1/2+\nu}x)|r^{(N)}(x,t_1)|^2$, and therefore, $|\tilde{J}_3(t_1)| \le Ct_1^{2N+1+2\nu}$, we obtain

$$\tilde{J}_3(t) \le Ct^{2N+1-6\nu}, \quad \forall t \in [t_1, t_0].$$

Combining this inequality with (3.41), one gets

$$\|S\|_{H^3(\mathbb{R}^2)}^2 \le Ct^{2N+1-6\nu}, \quad \forall t \in [t_1, t_0],$$

which implies that

$$\|s\|_{H^3(\mathbb{R}^2)} \le t^{N/2}, \quad \forall t \in [t_1, t_0].$$

This concludes the proof of Proposition 3.1.

3.2. Proof of the theorem. The proof of the theorem is now straightforward. Fix N such that Proposition 3.1 holds. Take a sequence (t^j), $0 < t^j < t_0$, $t^j \to 0$ as $j \to \infty$. Let $u_j(x,t)$ be the solution of

$$\begin{aligned} \partial_t u_j &= u_j \times \Delta u_j, \quad t \ge t^j, \\ u_j|_{t=t^j} &= u^{(N)}(t^j). \end{aligned} \tag{3.43}$$

By Proposition 3.1, for any j, $u_j - u^{(N)} \in C([t^j, t_0], H^3)$ and satisfies

$$\|u_j(t) - u^{(N)}(t)\|_{H^3} + \|\langle x\rangle(u_j(t) - u^{(N)}(t))\|_{L^2} \le 2t^{N/2}, \quad \forall t \in [t^j, t_0]. \tag{3.44}$$

This implies in particular, that the sequence $u_j(t_0) - u^{(N)}(t_0)$ is compact in H^2 and therefore after passing to a subsequence we can assume that $u_j(t_0) - u^{(N)}(t_0)$ converges in H^2 to some 1-equivariant function $w \in H^3$, with $\|w\|_{H^3} \le \delta^{2\nu}$, $|u^{(N)}(t_0) + w| = 1$.

Consider the Cauchy problem

$$\begin{aligned} u_t &= u \times \Delta u, \quad t \le t_0, \\ u|_{t=t_0} &= u^{(N)}(t_0) + w. \end{aligned} \tag{3.45}$$

By the local well-posedness, (3.45) admits a unique solution $u \in C((t^*, t_0], \dot{H}^1 \cap \dot{H}^3)$ with some $0 \le t^* < t_0$. By H^1 continuity of the flow (see [10]), $u_j \to u$ in $C((t^*, t_0], \dot{H}^1)$, which together with (3.44) gives

$$\|u(t) - u^{(N)}(t)\|_{H^3} \le 2t^{N/2}, \quad \forall t \in (t^*, t_0]. \tag{3.46}$$

This implies that $t^* = 0$ and combined with Proposition 2.1 gives the result stated in Theorem 1.1.

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