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The $1/N$ Expansion of Tensor Models Beyond Perturbation Theory

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Abstract: We analyze in full mathematical rigor the most general quartically perturbed invariant probability measure for a random tensor. Using a version of the Loop Vertex Expansion (which we call the mixed expansion) we show that the cumulants write as explicit series in $1/N$ plus bounded rest terms. The mixed expansion recasts the problem of determining the subleading corrections in $1/N$ into a simple combinatorial problem of counting trees decorated by a finite number of loop edges.

As an aside, we use the mixed expansion to show that the (divergent) perturbative expansion of the tensor models is Borel summable and to prove that the cumulants respect an uniform scaling bound. In particular the quartically perturbed measures fall, in the $N \to \infty$ limit, in the universality class of Gaussian tensor models.

1. Introduction

Tensor models [1] generalize matrix models [2,3] to higher dimensions and provide the analytical tool for the study of random geometries in three and more dimensions. Matrix models are probability distributions for random matrices and their moments can be evaluated via a perturbative expansion in ribbon Feynman graphs representing surfaces [3]. A crucial tool in matrix models is the $1/N$ expansion discovered by 't Hooft [4]. The perturbative series of matrix models can be reorganized as a series in $1/N$ (where N is the size of the matrix) indexed by the genus. At leading order only planar graphs [5] contribute and a matrix model undergoes a phase transition to a theory of random continuum surfaces when the coupling constant is tuned to some critical value [6,7].

However many questions concerning the $1/N$ expansion of matrix models remain unanswered. First and foremost the $1/N$ expansion seems inherently a perturbative tool: in order to perform it, one first performs the expansion in the coupling. The perturbative series is not summable hence it is not clear if the subsequent steps are mathematically meaningful. Furthermore, matrix models are defined for a fixed sign of the coupling constant (say positive) for which the perturbation is stable. However, the phase transition to continuum surfaces takes place when tuning the coupling constant to a *negative* critical value. This tuning is meaningful after restricting to the leading order planar series (which is absolutely convergent for both signs of the coupling constant), but what, if any, is the meaning of this tuning to criticality beyond perturbation theory? Can one actually reach this phase transition by an analytic continuation starting from a matrix model? These questions have not yet been satisfactorily answered.

Matrix models have been generalized in higher dimensions to tensor models and group field theories [8–15]. They encode models of random geometries relevant for quantum gravity. However, progress has been slow (beyond model building) in higher dimensions mainly due to the lack of a $1/N$ expansion for tensor models.

This has changed with the advent of the $1/N$ expansion, initially for the colored [16,17] models, and subsequently for all invariant tensor models [18]. Indeed the perturbative series of tensor models supports a $1/N$ expansion [19–23] indexed by the *degree*, a positive integer which plays in higher dimensions the role of the genus.[1] The leading order *melonic* [24,25] graphs triangulate the D-dimensional sphere in any dimension [19–21] and, like their two dimensional counterparts, tensor models undergo a phase transition to a theory of continuous random spaces when tuning to criticality. These results led to significant progress in the understanding of random geometries in higher dimensions [26–38] and the related critical phenomena. Tensor models have been generalized to renormalizable (and generically asymptotically free) tensor field theories, [39–46] leading to the formulation of the "tensor track" approach to quantum gravity [47,48].

Tensor models have been shown to exhibit a powerful universality property [49]: all invariant models respecting a uniform bound on the cumulants become Gaussian in the $N \to \infty$ limit, but the covariance of the limiting Gaussian is a non trivial function of the parameters of the model.

The same questions concerning the status of the $1/N$ for matrix models can also be formulated for tensor models. Can one give a meaning to the $1/N$ expansion beyond perturbation theory? Can one reach the phase transition point by analytic continuation? This paper answers the first question and establishes the $1/N$ expansion in the constructive field theory sense [50]. In order to achieve this result we consider the most general tensor model with a relevant quartic interaction.[2] We generalize to tensors the Loop Vertex Expansion (LVE) [51,52] initially introduced for matrices.[3] and analyze it in detail. The LVE expresses any cumulant as an absolutely convergent series indexed by trees. In this series the coupling constant λ appears in two places: first as an overall factor for each tree and second in the contribution of each tree. This allows, for instance, to prove that, like in matrix models, the perturbative series of tensor models is Borel summable in λ uniformly in N.

However, in the case of tensors the LVE is much more powerful. Although rather involved at first sight, this formulation has two important advantages. First, in the contribution of each tree, the coupling constant always appears rescaled as λ/N^{D-1} where D is the rank of the tensor. Performing a Taylor expansion in λ/N^{D-1} one proves that the rest term is *suppressed in powers of* $|\lambda|/N^{D-2}$. We call this expansion *the mixed expansion*. The mixed expansion provides the $1/N$ expansion of the model beyond perturbation

[1] Unlike the genus, the degree is *not* a topological invariant.

[2] Other quartic interactions can be added but are suppressed in $1/N$.

[3] The LVE has already been used in the literature [53] in a related context.

theory: although shifted from N^{D-1} to N^{D-2}, the scaling with N suppresses the rest term as long as $D \geq 3$. This does not hold for the case of matrices, $D = 2$. Second, at each order in $1/N$, one must sum the series indexed by trees and this can be done explicitly. Somewhat surprisingly, the mixed expansion turns out to be the appropriate computational tool for the study of the $1/N$ series. It reorganizes the corrections in $1/N$ in terms of trees with a finite number of loop edges, and counting such trees is a straightforward combinatorial problem.

The critical behavior of each order in $1/N$ is governed order by order by the divergence of the series indexed by trees. In particular this proves that all terms will diverge for the same critical constant. The mixed expansion (more precisely the $1/N$ expansion derived from it) is an explicit perturbation in $1/N$ around the leading order theory of trees with no loop edges. The leading order (melonic) theory acts effectively as a new "vacuum" around which the subleading terms in $1/N$ act as perturbations.

Beyond providing a tool to analyze order by order the subleading behavior in $1/N$ of the tensor models and an avenue towards establishing their double scaling limits, the present study is a needed step in order to analytically continue the cumulants to the critical constant of the phase transition to the continuum theory.

The situation is more subtle for matrix models. The rest terms in the mixed expansion are not suppressed in powers of $1/N$. In order to obtain a non perturbative definition of the $1/N$ series for matrices one needs to refine further the mixed expansion introduced in the present paper. We believe that an in depth study of the corrections in $1/N$ in the case of tensors will provide the guide towards obtaining the non perturbative $1/N$ expansion for matrices.

We emphasize that the techniques of this paper rely heavily on the LVE formalism: some familiarity with [51,52] would greatly benefit the reader.

This paper is organized as follows. Section 2 is an introductory section. In it we present in some detail the framework of tensor models, we briefly recall the notion of Borel summability and we introduce at length the various notions used in the sequel. In Sect. 3 we introduce the most general model with a relevant quartic perturbation, and we state and comment on our main theorems. Section 4 contains the proofs of our results.

2. Prerequisites

In this section we present the general setting of random tensor models and introduce the various notions and notations we will use in the sequel.

2.1. Generalities on tensor models. We start by a brief overview of the general framework of invariant tensor models presented in detail in [18,49]. This allows us to introduce some relevant notions and notations and to state the universality theorem for random tensors. Later on we will prove that the quartically perturbed measure we deal with in this paper indeed obeys the universality theorem.

We consider rank D covariant tensors $\mathbb{T}_{n^1\dots n^D}$, with $n^1, n^2, \dots n^D \in \{1, \dots N\}$. having **no symmetry** under permutation of their indices. The tensors transform under the *external* tensor product of D fundamental representations of the unitary group $U(N)$ (that is the unitary group acts independently on each index). The complex conjugate tensor, $\bar{\mathbb{T}}_{n^1\dots n^D}$ is a rank D contravariant tensor. A tensor and its complex conjugate can be seen as collections of N^D complex numbers supplemented by the requirement of covariance under base change.

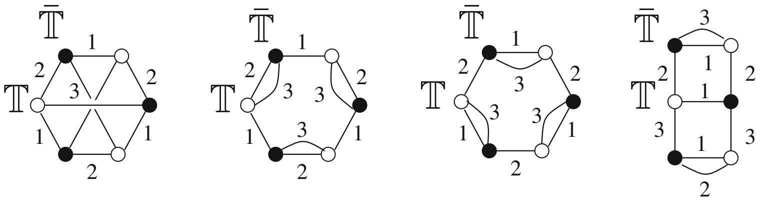

Fig. 1. D-colored graphs representing trace invariants

$$
\begin{aligned}
\mathbb{T}_{a^1\dots a^D} &= \sum_{n^1\dots n^D} U_{a^1n^1}\dots V_{a^Dn^D}\mathbb{T}_{n^1\dots n^D},\\
\bar{\mathbb{T}}_{\bar a^1\dots \bar a^D} &= \sum_{\bar n^1\dots \bar n^D} \bar U_{\bar a^1\bar n^1}\dots \bar V_{\bar a^D\bar n^D}\bar{\mathbb{T}}_{\bar n^1\dots \bar n^D},
\end{aligned}
\tag{1}
$$

where the indices of the complex conjugated tensor are denoted conventionally with a bar. We emphasize that the unitary operators $U, \dots V$ are all *independent*.[4] We denote $\vec n$ the D-tuple of integers $(n^1, \dots n^D)$, and we take $D \ge 3$.

Any invariant polynomial in the tensor entries can be expressed in terms of the **trace invariants** built by contracting in all possible ways pairs of covariant and contravariant indices in a product of tensor entries (see [54,55] for a direct proof relying on averaging over the unitary group). The trace invariants are one to one with closed D-colored graphs.

Definition 1. *A* **bipartite closed** D**-colored graph** *is a graph* $\mathcal{B} = \big(\mathcal{V}(\mathcal{B}), \mathcal{E}(\mathcal{B})\big)$ *with vertex set* $\mathcal{V}(\mathcal{B})$ *and edge set* $\mathcal{E}(\mathcal{B})$ *such that:*

- $\mathcal{V}(\mathcal{B})$ *is bipartite, i.e. the vertex set writes as* $\mathcal{V}(\mathcal{B}) = \mathcal{A}(\mathcal{B}) \cup \bar{\mathcal{A}}(\mathcal{B})$*, such that* $\forall l \in \mathcal{E}(\mathcal{B})$*, then* $l = (v, \bar v)$ *with* $v \in \mathcal{A}(\mathcal{B})$ *and* $\bar v \in \bar{\mathcal{A}}(\mathcal{B})$*. Their cardinalities satisfy* $|\mathcal{V}(\mathcal{B})| = 2|\mathcal{A}(\mathcal{B})| = 2|\bar{\mathcal{A}}(\mathcal{B})|$*. We call* $v \in \mathcal{A}(\mathcal{B})$ *the white vertices and* $\bar v \in \bar{\mathcal{A}}(\mathcal{B})$ *the black vertices of* $\mathcal{B}$*.*
- *The edge set is partitioned into D subsets* $\mathcal{E}(\mathcal{B}) = \bigcup_{i=1}^{D} \mathcal{E}^i(\mathcal{B})$*, where* $\mathcal{E}^i(\mathcal{B}) = \{l^i = (v, \bar v)\}$ *is the subset of edges with color* i*.*
- *All vertices are D-valent with all edges incident to a given vertex having distinct colors.*

The graph associated to an invariant (see Fig. 1 for some examples) is obtained as follows. We represent every $\mathbb{T}_{n^1\dots n^D}$ (respectively $\bar{\mathbb{T}}_{\bar n^1\dots \bar n^D}$) by a white vertex v (respectively a black vertex $\bar v$). The position of an index becomes a *color*: n^1 has color 1, n^2 has color 2 and so on. We represent by an edge the contraction of an index n^i on $\mathbb{T}_{n^1\dots n^D}$ with an index $\bar n^i$ of $\bar{\mathbb{T}}_{\bar n^1\dots \bar n^D}$. The edges $l^i = (v, \bar v) \in \mathcal{E}^i(\mathcal{B})$ inherit the color i of the indices and always connect a white and a black vertex.

The trace invariant associated to $\mathcal{B}$ is

$$
\mathrm{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) = \sum_{n,\bar n} \delta^{\mathcal{B}}_{n\bar n} \prod_{v,\bar v\in\mathcal{V}(\mathcal{B})} \mathbb{T}_{\vec n_v}\bar{\mathbb{T}}_{\vec{\bar n}_{\bar v}}, \qquad \delta^{\mathcal{B}}_{n\bar n} = \prod_{i=1}^{D}\prod_{l^i=(v,\bar v)\in\mathcal{E}^i(\mathcal{B})} \delta_{n^i_v \bar n^i_{\bar v}}. \tag{2}
$$

The trace invariant associated to $\mathcal{B}$ factors over its connected components. We call the invariant *connected* if $\mathcal{B}$ is connected. We denote $k(\mathcal{B})$ the number of white (or black)

[4] One can consider more generally tensors transforming under the external tensor product of fundamental representations of unitary groups of different sizes $U(N_1) \boxtimes U(N_2) \boxtimes \cdots \boxtimes U(N_D)$ with $N_i \ne N_j$.

vertices of the graph $\mathcal{B}$ and $C(\mathcal{B})$ the number of connected components of $\mathcal{B}$, which we label $\mathcal{B}_\rho$. Colored graphs are dual to D-dimensional abstract simplicial pseudo-manifolds [16,17,56,57].

A random tensor is a collection of N^D complex random variables whose joint distribution is encoded in the cumulants (connected moments) of k entries $\mathbb{T}$ and $\bar{k}$ entries $\bar{\mathbb{T}}$, $\kappa_{2k}[\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{\bar{n}}_{\bar{1}}} \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\vec{\bar{n}}_{\bar{k}}}]$. We consider only even distributions, that is the cumulants are nontrivial only if $k = \bar{k}$.

Definition 2. *The probability distribution μ_N of the N^D complex random variables $\mathbb{T}_{\vec{n}}$ is called* **trace invariant** *if its* **cumulants** *are linear combinations of trace invariant operators,*

$$\kappa_{2k}[\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\vec{\bar{n}}_{\bar{1}}} \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\vec{\bar{n}}_{\bar{k}}}] = \sum_{\mathcal{B},\, k(\mathcal{B})=k} \mathfrak{K}(\mathcal{B}, \mu_N) \prod_{\rho=1}^{C(\mathcal{B})} \delta^{\mathcal{B}_\rho}_{n\bar{n}}, \tag{3}$$

where the sum runs over **all** *the D-colored graphs $\mathcal{B}$ with $2k$ vertices.*

There exists a unique D-colored graph with 2 vertices (it has D edges connecting all the two vertices), called the D-dipole and denoted $\mathcal{B}^{(2)}$. We are interested in the large N behavior of a trace invariant probability measure. In order for such a limit to exist, the cumulants must scale with N. We denote the rescaled cumulants

$$\frac{\mathfrak{K}(\mathcal{B}, \mu_N)}{N^{-2(D-1)k(\mathcal{B})+D-C(\mathcal{B})}} \equiv K(\mathcal{B}, N), \tag{4}$$

and we call $K(\mathcal{B}^{(2)}, N)$ the covariance of the distribution.

Definition 3. *We say that the trace invariant probability distribution is* **properly uniformly bounded** *at large N if*

$$\begin{aligned} &\lim_{N\to\infty} K(\mathcal{B}^{(2)}, N) = K(\mathcal{B}^{(2)}) < \infty, \\ &K(\mathcal{B}, N) \le K(\mathcal{B}), \quad \forall \mathcal{B} \neq \mathcal{B}^{(2)} \text{ and } N \text{ large enough}. \end{aligned} \tag{5}$$

The scaling in Eqs. (4) and (5) is *the only* scaling leading to a large N limit. This is a nontrivial statement and the reader should consult [49] for detailed explanations on this point. The simplest example of a probability distribution for random tensors is the normalized Gaussian distribution of covariance σ^2

$$e^{-N^{D-1}\frac{1}{\sigma^2}\sum_{\vec{n},\vec{\bar{n}}} \mathbb{T}_{\vec{n}}\delta_{\vec{n}\vec{\bar{n}}}\bar{\mathbb{T}}_{\vec{\bar{n}}}} \prod_{\vec{n}} \Big(\frac{N^{D-1}}{\sigma^2} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{n}}}{2\pi\imath} \Big), \tag{6}$$

characterized by the expectations of the connected trace invariants

$$\Big\langle \mathrm{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \Big\rangle_{\sigma^2} = \int \Big(\prod_{\vec{n}} \frac{N^{D-1}}{\sigma^2} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{n}}}{2\pi\imath} \Big)\, e^{-N^{D-1}\frac{1}{\sigma^2}\sum_{\vec{n}\vec{\bar{n}}} \mathbb{T}_{\vec{n}}\delta_{\vec{n}\vec{\bar{n}}}\bar{\mathbb{T}}_{\vec{\bar{n}}}}\, \mathrm{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}). \tag{7}$$

The moments of the Gaussian distribution are rather non trivial and have been studied in [49]. For any connected graph $\mathcal{B}$ with $2k(\mathcal{B})$ vertices there exist two non-negative integers, $\Omega(\mathcal{B})$ and $R(\mathcal{B})$ such that

$$\lim_{N\to\infty} N^{-1+\Omega(\mathcal{B})} \Big\langle \mathrm{Tr}_{\mathcal{B}}(\mathbb{T}, \bar{\mathbb{T}}) \Big\rangle_{\sigma^2} = \sigma^{2k(\mathcal{B})} R(\mathcal{B}). \tag{8}$$

The normalization in Eq. (6) is the *only normalization* which ensures that the convergence order is positive and, more importantly, *for all* $\mathcal{B}$, there exists an *infinite* family of invariants (graphs $\mathcal{B}'$) such that $\Omega(\mathcal{B}) = \Omega(\mathcal{B}')$. Again, this is a nontrivial statement and the reader should consult [49] for more details.

Definition 4. *A random tensor* $\mathbb{T}$ *distributed with the probability measure* μ_N **converges in distribution** *to the distributional limit of a Gaussian tensor model of covariance* σ^2 *if, for any connected trace invariant* $\mathcal{B}$,

$$\lim_{N\to\infty} N^{-1+\Omega(\mathcal{B})}\mu_N\Big[\mathrm{Tr}_{\mathcal{B}}(\mathbb{T},\bar{\mathbb{T}})\Big] = \sigma^{2k(\mathcal{B})}R(\mathcal{B}). \tag{9}$$

The universality theorem for tensor models is [49]

Theorem 1 (Universality). *Let* N^D *random variables* $\mathbb{T}_{\vec{n}}$ *whose joint distribution is trace invariant and properly uniformly bounded of covariance* $K(\mathcal{B}^{(2)}, N)$. *Then, in the large N limit, the tensor* $\mathbb{T}_{\vec{n}}$ *converges in distribution to a Gaussian tensor of covariance* $K(\mathcal{B}^{(2)}) = \lim_{N\to\infty} K(\mathcal{B}^{(2)}, N)$.

2.2. Borel summability. The perturbative expansion of tensor models (and of quantum field theories in general) is not summable. The root of the problem is that one usually performs an expansion in some coupling constant λ around $\lambda = 0$. However the interaction is stable for $\lambda > 0$ but unstable for $\lambda < 0$. The partition function and the cumulants are analytic in some domain in the complex plane outside the negative real axis. Hence $\lambda = 0$ belongs to the boundary of the analyticity domain of the cumulants. A Taylor expansion around a point belonging to the boundary of analyticity domain of some function is not absolutely convergent. However, in some cases, such Taylor expansions turn out to be Borel summable.

It is by now a classical result that the perturbation series of the ϕ^4 model in three and four dimensions[5] is indeed Borel summable [58,59]. When dealing with genuine quantum field theories (like in [58,59]) one needs to take into account the renormalization group flow. Although the random tensors we deal with in this paper do not exhibit a flow, establishing Borel summability in our case is rather involved, because tensors have many indices and the large factors of N are hidden in the sums over these indices. It is then necessary to use a constructive expansion in which these sums are organized and controlled appropriately. The techniques of [58,59] must be combined with the ones we introduce below in order to tackle tensor field theories like the ones of [39,45].

To establish Borel summability for random tensors we will use in this paper the following classical result.

Theorem 2 (Nevanlinna-Sokal, [60]). *A function* $f(\lambda, N)$ *with* $\lambda \in \mathbb{C}$ *and* $N \in \mathbb{R}_+$ *is said to be Borel summable in* λ *uniformly in N if*

- $f(\lambda, N)$ *is analytic in a disk* $\Re\lambda^{-1} > R^{-1}$ *with* $R \in \mathbb{R}_+$ *independent of* N.
- $f(\lambda, N)$ *admits a Taylor expansion at the origin*

$$f(\lambda, N) = \sum_{k=0}^{r-1} f_{N,k}\lambda^k + R_{N,r}(\lambda), \qquad |R_{N,r}(\lambda)| \le K\sigma^r r!|\lambda|^r, \tag{10}$$

for some constants K and σ *independent of N.*

[5] With a fixed UV cutoff for the second case, which is needed in order to control the Landau ghost.

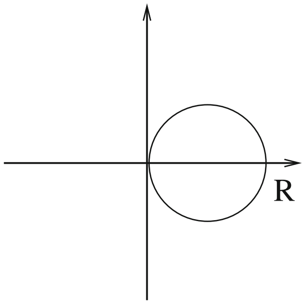

Fig. 2. A Borel disk

If $f(\lambda, N)$ is Borel summable in λ uniformly in N then $B(t, N) = \sum_{k=0}^{\infty} \frac{1}{k!} f_{N,k} t^k$ is an analytic function for $|t| < \sigma^{-1}$ which admits an analytic continuation in the strip $\{z| \; |\Im z| < \sigma^{-1}\}$ such that $|B(t, N)| \le B e^{t/R}$ for some constant B independent of N and $f(\lambda, N)$ is represented by the absolutely convergent integral

$$f(\lambda, N) = \frac{1}{\lambda} \int_0^{\infty} dt \; B(t, N) e^{-\frac{t}{\lambda}}. \tag{11}$$

That is the Taylor expansion of $f(\lambda, N)$ at the origin is Borel summable, and $f(\lambda, N)$ is its Borel sum. The important thing about Borel summability is that it provides a uniqueness criterion: if a divergent series is the Taylor expansion of a Borel summable function $f(\lambda, N)$ at $\lambda = 0$, then $f(\lambda, N)$ is the *unique* Borel summable function whose Taylor series is the original series.

The set $\{\lambda | \Re \lambda^{-1} > R^{-1}, R \in \mathbb{R}_+\}$ is a disk (which we call a Borel disk) in the complex plane centered at $\frac{R}{2}$ and of radius $\frac{R}{2}$ (hence tangent to the imaginary axis, see Fig. 2) as, denoting $\lambda = \frac{R}{2} + a e^{\imath \gamma}$,

$$\Re \lambda^{-1} > R^{-1} \Leftrightarrow \frac{R^2}{4} > a. \tag{12}$$

2.3. Decorated trees. All our results rely on expansions indexed by various kinds of trees which we describe at length in this section. We will denote $\vec{\sigma}$ a D-tuple of permutations over k elements, $\vec{\sigma} = (\sigma_1, \dots \sigma_D)$.

Unrooted plane trees with colored, oriented edges and marked vertices $\mathcal{T}^{\circlearrowright}_{n,\iota}$. An unrooted plane tree is a tree with a cyclic ordering (say clockwise) of the edges at every vertex. We denote the total number of vertices of the tree by n and label them $1, 2, \dots n$. The edges (i, j) of the tree are oriented either from i to j or from j to i and have a color $c \in \{1, 2 \dots D\}$. Plane trees with marked vertices $\iota = \{i_1, \dots i_k\}$ are obtained by selecting a preferred starting point of the cyclic ordering at the vertices $i_1, \dots i_k$. The starting point is represented as a mark (or cilium) on the vertex.[6] We denote such a tree $\mathcal{T}^{\circlearrowright}_{n,\iota}$ with $\iota = \{i_1, \dots i_k\}$ and we denote the abstract tree associated to $\mathcal{T}^{\circlearrowright}_{n,\iota}$ by T_n. Note that several plane trees are associated to the same abstract tree, and a vertex can have at most one cilium. An example is presented in Fig. 3.

All the notions we introduce in the sequel refer to a fixed plane tree $\mathcal{T}^{\circlearrowright}_{n,\iota}$. To simplify notations we consider this dependence implicit.

6 Such trees are sometimes called ciliated plane trees.

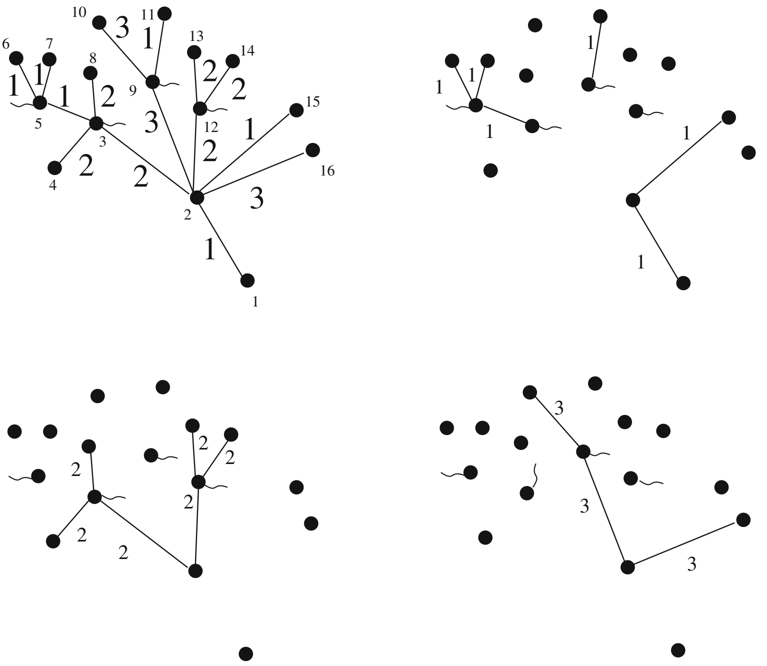

Fig. 3. A ciliated plane tree with colored edges

Every plane tree has an associated contour walk (Harris walk or Dyck path) consisting in the ordered list of vertices encountered when walking clockwise around the tree starting from some position. We identify a tree with its contour walk. Whenever the walk encounters a cilium, the walk steps twice (first a step to the cilium and then a step from the cilium). We signal the presence of a cilium by a semicolon (rather than a comma) separating the two entries in the walk. The contour walk of the tree in Fig. 3 is therefore

$$(1, 2, 3, 4, 3, 5; 5, 6, 5, 7, 5, 3, 8, 3; 3, 2, 9, 10, 9, 11, 9; 9, 2, 12, 13, 12, 14, 12; \\ 12, 2, 15, 2, 16, 2).$$

The contour walk of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ has $2n-2+k$ *steps*, that is entries, denoted $q = 1, 2, \dots 2n-2+k$. The walk in the previous example has 34 steps. We denote the vertex corresponding to the step q by $i(q)$. The contour walk is cyclic.

For every tree $\mathcal{T}_{n,\iota}^{\circlearrowright}$ we consider its connected subgraphs having all the vertices of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ but only the edges of color c of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ (see Fig. 3). Each such subgraph is comprised of several plane trees which we denote f^c. We call them *faces of color c* of $\mathcal{T}_{n,\iota}^{\circlearrowright}$. The walk of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ induces a contour walk for each of its faces f^c obtained by selecting only the (ordered set of) steps $q(f^c)$ at which the contour walk of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ encounters a vertex of f^c. For instance, the face of color 1 with vertices 1, 2, 15 has an induced contour walk $f^1 = (1, 2, 2, 2, 2, 15, 2, 2)$ consisting in the steps $q(f^1) = \{1, 2, 16, 23, 30, 31, 32, 34\}$. Note that the walk of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ can leave and re-intersect several times f^c (in the previous example it leaves f^1 after the step 2 and re-intersects it at the step 16). We partition the faces into two subsets:

- The faces f^c having no cilium. We call them the **internal faces of color** c of $\mathcal{T}^{\circlearrowright}_{n,\iota}$. For the tree in Fig. 3, $f^1 = (1, 2, 2, 2, 2, 15, 2, 2)$ or $f^1 = (4)$ are examples of internal faces of color 1.
- The faces f^c having at least a cilium. We call them the *external faces of color* c of $\mathcal{T}^{\circlearrowright}_{n,\iota}$. Say the vertices $i_{k_1} \dots i_{k_d}$ are the ciliated vertices of f^c properly ordered, that is $f^c = (\dots i_{k_1}; i_{k_1} \dots i_{k_2}; i_{k_2} \dots i_{k_d}; i_{k_d} \dots)$. We can further subdivide f^c into the walks $f^{c;i_{k_1}\to i_{k_2}}$, $f^{c;i_{k_2}\to i_{k_3}}$ up to $f^{c;i_{k_d}\to i_{k_1}}$.[7] We call these walks the **external strands** of $\mathcal{T}^{\circlearrowright}_{n,\iota}$. For the example of 3, the external face of color 1 $f^1 = (3, 3, 5; 5, 6, 5, 7, 5, 3, 3; 3)$ subdivides into the external strands of color 1 $f^{1;3\to 5} = (3, 3, 3, 5)$ and $f^{1;5\to 3} = (5, 6, 5, 7, 5, 3, 3)$. The strands are comprised of the (ordered) entries separated by semicolons in the walks of the faces. Every strand $f^{c;i\to i'}$ has an induced contour walk, made of the steps $q(f^{c;i\to i'})$.

Any vertex belongs to exactly D faces, either internal or external, one for each color. The cilia are identified as the encounters of semicolons (separating the same label) in the contour walk of the tree. We denote $q_1 \dots q_k$ the positions of the cilia in the contour walk of the tree $\mathcal{T}^{\circlearrowright}_{n,\iota}$ (that is $i(q_l)$ precedes a semicolon and $i(q_l) = i(q_l + 1)$). Thus q_1 is the step at which the first cilium is encountered, q_2 the step at which the second cilium is encountered an so on. All the faces f^c to which the ciliated vertices $i(q_l)$ belong are external.

The external faces and strands of color c can be encoded by a permutation ξ_c over $1, \dots k$. The encoding goes as follows. Consider the ciliated vertex $i(q_1)$. It belongs to the external face f^c of color c. Starting from $i(q_1)$, the first ciliated vertex (which is of course unique), we encounter in the contour walk of f^c is of the form $i(q_l)$ for some l (as it is one of the ciliated vertices of $\mathcal{T}^{\circlearrowright}_{n,\iota}$). We set $\xi_c(1) = l$. We repeat the procedure for q_2 and so on and obtain a permutation ξ_c over $1, \dots k$. For the example in Fig. 3 we have $q_1 = 6, q_2 = 14, q_3 = 21, q_4 = 28$ (and $i(q_1) = 5, i(q_2) = 3, i(q_3) = 9, i(q_4) = 12$). The permutations ξ_c write in cycle decomposition as $\xi_1 = (1, 2)(3)(4), \xi_2 = (1)(2, 4)(3)$ and $\xi_3 = (1)(2)(3)(4)$. The external faces of color c correspond to the cycles of ξ_c

$$f^c = \Big(i(q_l) \dots i(q_{\xi_c(l)}); i(q_{\xi_c(l)}) \dots i(q_{\xi_c^2(l)}); \\ i(q_{\xi_c^2(l)}) \dots i(q_{\xi_c^{d-1}(l)}); i(q_{\xi_c^{d-1}(l)}) \dots i(q_l)\Big), \qquad \xi_c^d(l) = l, \tag{13}$$

and the strands write as

$$f^{c;i(q_l)\to i(q_{\xi_c(l)})} = \Big(i(q_l) \dots i(q_{\xi_c(l)})\Big), \\ f^{c;i(q_{\xi_c(l)})\to i(q_{\xi_c^2(l)})} = \Big(i(q_{\xi_c(l)}) \dots i(q_{\xi_c^2(l)})\Big), \dots \tag{14}$$

Consider for instance the example of 3 and $\xi_1 = (1, 2)(3)(4)$. The external faces and strands of color 1 are

$$f^1 = (3, 3, 5; 5, 6, 5, 7, 5, 3, 3; 3) \Rightarrow f^{1;3\to 5} = (3, 3, 3, 5), \\ f^{1;5\to 3} = (5, 6, 5, 7, 5, 3, 3) \\ f^1 = (9, 9, 11, 9; 9) \Rightarrow f^{1;9\to 9} = (9, 9, 9, 11, 9) \\ f^1 = (12, 12, 12; 12) \Rightarrow f^{1;12\to 12} = (12, 12, 12, 12). \tag{15}$$

[7] Where we use the cyclicity of the walk to rearrange the last list in the appropriate order.

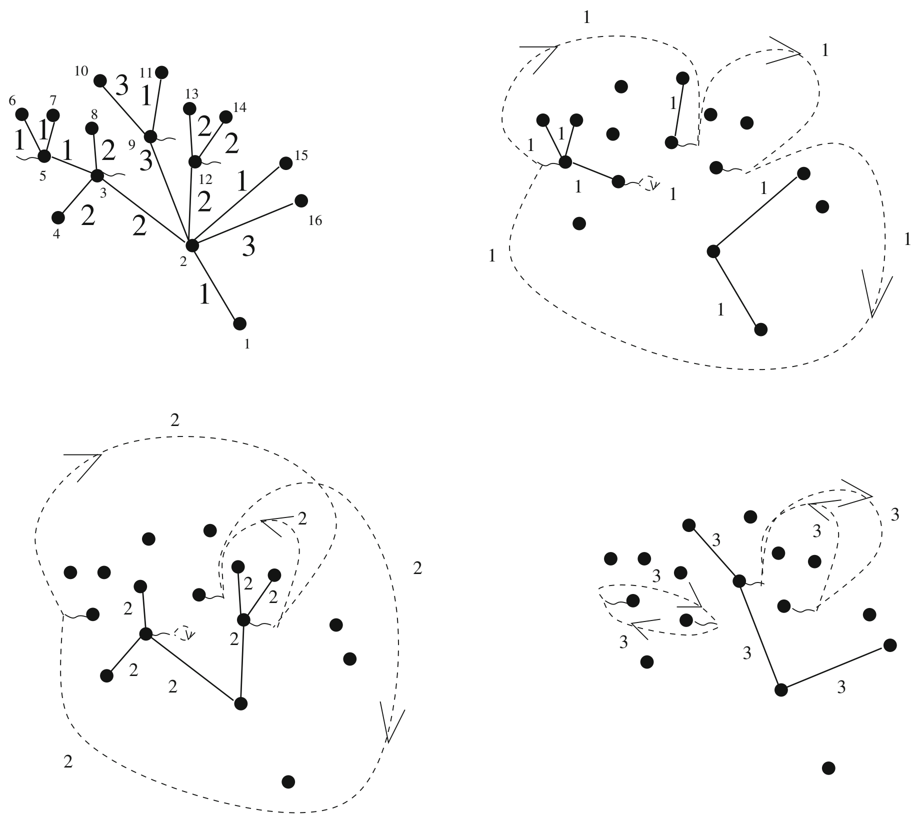

Fig. 4. A ciliated plane tree with external edges

We denote the set of all internal faces of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ by $\mathcal{F}^{\text{int}}(\mathcal{T}_{n,\iota}^{\circlearrowright})$, the set of all external strands of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ by $\mathcal{S}^{\text{ext}}(\mathcal{T}_{n,\iota}^{\circlearrowright})$, and the set of all faces of $\mathcal{T}_{n,\iota}^{\circlearrowright}$ by $\mathcal{F}(\mathcal{T}_{n,\iota}^{\circlearrowright})$.

A tree can be built by adding one by one edges connecting univalent vertices. At every step of this procedure the total number of faces of a tree increases by $D-1$ (every vertex connected by an edge of color c will bring a new face for every color $c' \neq c$), and taking into account that the tree with a unique vertex has D faces, we obtain

$$|\mathcal{F}(\mathcal{T}_{n,\iota}^{\circlearrowright})| = D + (n-1)(D-1) = |\mathcal{F}^{\text{int}}(\mathcal{T}_{n,\iota}^{\circlearrowright})| + \sum_{c=1}^{D} C(\xi_c), \tag{16}$$

where $C(\xi_c)$ denotes the number of cycles of the permutation ξ_c.

Plane trees with external edges. A notion closely related to ciliated plane trees is the one of trees with external edges. For every c, we add to the tree dashed oriented edges of color c connecting pairs of cilia such that every cilium has exactly an incoming and an outgoing dashed edge (we allow an edge to be incoming and outgoing on the same cilium). We call the dashed edges **external edges**. We present in Fig. 4 an example of a tree decorated by external edges (we represented the external edges of color c decorating the faces of color c).

The external edges of color c can be encoded by a permutation τ_c over k elements. The outgoing external edge of color c on the cilium on the vertex $i(q_1)$ is an incoming external edge on some other cilium, say the one on the vertex $i(q_l)$. We set $\tau_c(1) = l$, and repeat the procedure for all the cilia.

The example in Fig. 3 corresponds to the permutations with cycle decomposition $\tau_1 = (2)(1,3,4)$, $\tau_2 = (2)(1,4,3)$ and $\tau_3 = (1,2)(3,4)$ (recall that $q_1 = 6, q_2 = 14, q_3 = 21, q_4 = 28$ and $i(q_1) = 5, i(q_2) = 3, i(q_3) = 9, i(q_4) = 12$).

We denote the tree decorated by external edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}$, with $\vec\tau = (\tau_1, \dots \tau_D)$. The external edges recombine the external strands of $\mathcal{T}^{\circlearrowright}_{n,\iota}$ into **external faces** of $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}$. They are defined as follows.

Starting from the external faces f^c of $\mathcal{T}^{\circlearrowright}_{n,\iota}$,

$$f^c = \Big(i(q_l)\dots i(q_{\xi_c(l)}); i(q_{\xi_c(l)})\dots i(q_{\xi_c^2(l)}); \\ i(q_{\xi_c^2(l)})\dots i(q_{\xi_c^{d-1}(l)}); i(q_{\xi_c^{d-1}(l)})\dots i(q_l)\Big), \qquad \xi_c^d(l) = l, \tag{17}$$

with strands

$$f^{c;i(q_l)\to i(q_{\xi_c(l)})} = \Big(i(q_l)\dots i(q_{\xi_c(l)})\Big), \tag{18}$$

we build the external faces of $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}$ indexed by the cycles of $\tau_c\xi_c$

$$f^c(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}) = \Big(i(q_l)\dots i(q_{\xi_c(l)}); i(q_{\tau_c\xi_c(l)})\dots i(q_{\xi_c\tau_c\xi_c(l)}); \\ i(q_{\tau_c\xi_c\tau_c\xi_c(l)})\dots i(q_{\xi_c(\tau_c\xi_c)^{d-1}(l)})\Big), \qquad (\tau_c\xi_c)^d(l) = l. \tag{19}$$

Consider for instance the example of 4. Recall that we have $\xi_1 = (1,2)(3)(4)$, $\tau_1 = (2)(1,3,4)$ and

$$f^{1;3\to5} = (3,3,3,5), \quad f^{1;5\to3} = (5,6,5,7,5,3,3), \\ f^{1;9\to9} = (9,9,9,11,9), \quad f^{1;12\to12} = (12,12,12,12). \tag{20}$$

We have $\tau_1\xi_1 = (1,2,3,4)$ and the associated external face of $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}$ is

$$f^c(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}) = (5,6,5,7,5,3,3;\,3,3,3,5;\,9,9,9,11,9;\,12,12,12,12) \tag{21}$$

The internal faces of $\mathcal{T}_{n,\iota,\vec\tau}$ are the internal faces of $\mathcal{T}_{n,\iota}$ and we denote $\mathcal{F}(\mathcal{T}_{n,\iota,\vec\tau})$ the set of all the faces, internal and external, of $\mathcal{T}_{n,\iota,\vec\tau}$. We have

$$|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau})| = |\mathcal{F}^{\text{int}}(\mathcal{T}^{\circlearrowright}_{n,\iota})| + \sum_c C(\tau_c\xi_c) \\ = D + (n-1)(D-1) - \sum_{c=1}^{D} C(\xi_c) + \sum_c C(\tau_c\xi_c). \tag{22}$$

Plane trees with external edges and loop edges. The last type of decorated trees we will use is trees with loop edges. We add to the plane tree with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}$ $2s$ new cilia located on the vertices $j_1, j_{1'}, j_2, j_{2'}$ up to $j_s, j_{s'}$. The new cilia are allowed to be located anywhere on the tree, including on one of the vertices in ι. We assign to these

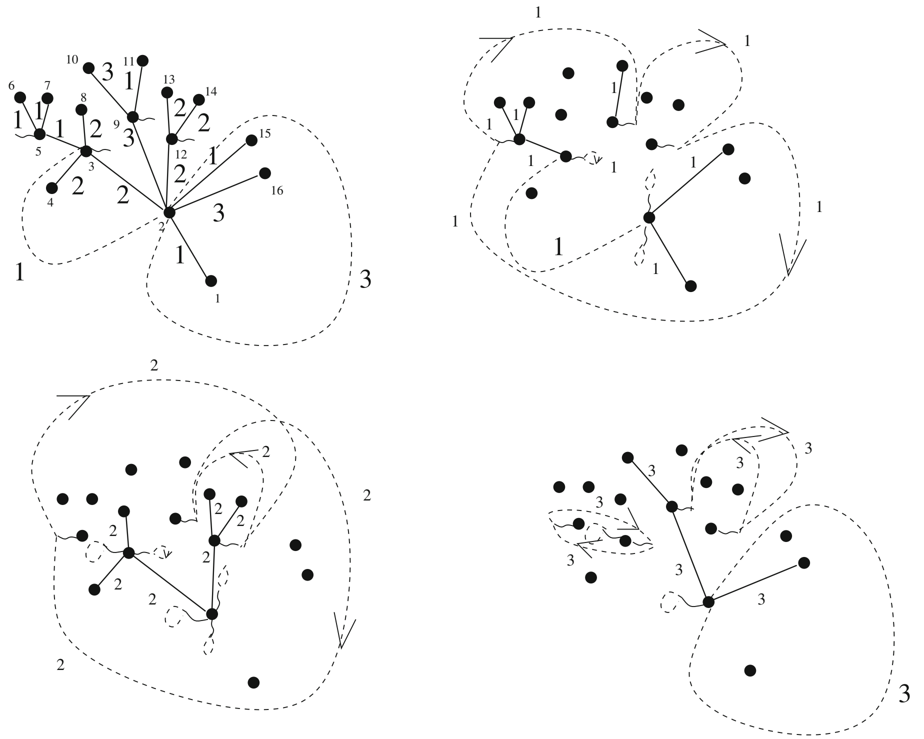

Fig. 5. A ciliated plane tree with external edges and loop edges

new cilia a color such that j_1 and $j_{1'}$ have both color c_1, j_2 and $j_{2'}$ have both color c_2 an do on. We connect the cilia on j_1 and $j_{1'}$ by a dashed edge of color c_1 and so on. We call these new edges **loop edges**. The loop edges are **not** oriented. We denote the graph thus obtained $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}}$ with $\mathcal{L} = \{(j_1, j_{1'}), (j_2, j_{2'}), \dots (j_s, j_{s'})\}$. We present in Fig. 5, in the upper left corner, a plane tree with two loop edges: one of color $c_1 = 3$ and one of color $c_2 = 1$ with $j_1 = 2$, $j_{1'} = 2$, $j_2 = 2$, $j_{2'} = 3$. The loop edges have no orientation.

The presence of the loop edge has several consequences. First, the walk around the tree is modified by the presence of the new cilia. For the example in Fig. 5, in the presence of the loop edge, the walk writes

$$\begin{aligned}(1, 2\boxed{;}2\boxed{;}2, 3, 4, 3\boxed{;}3, 5; 5, 6, 5, 7, 5, 3, 8, 3; 3, 2, 9, 10, 9, 11, 9;\\ 9, 2, 12, 13, 12, 14, 12; 12, 2\boxed{;}2, 15, 2, 16, 2).\end{aligned} \tag{23}$$

where we boxed the semicolons representing the new cilia (a new cilium transforms i, into $i; i$ in the walk). Furthermore the presence of a loop edge modifies the faces. Consider the cilia on j_p and $j_{p'}$. They can either

- belong to two distinct faces of color c_p,

$$f_1^{c_p} = (\dots j_p; j_p \dots) \quad \text{and} \quad f_2^{c_p} = (\dots j_{p'}; j_{p'} \dots). \tag{24}$$

  Then the two faces are merged into the face $f^{c_p} = (\dots j_p; j_{p'} \dots j_{p'}; j_p \dots)$.

- belong to a unique face of color c_p. Then $f^{c_p} = (\dots j_p; j_p \dots j_{p'}; j_{p'} \dots)$ splits into two faces $(j_p \dots j_{p'})$ and $(j_{p'} \dots j_p)$.

More precisely, the tree decorated by loop edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}}$ is the tree with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota',\vec{\tau}'}$ obtained by adding the $2s$ new cilia on $j_1, j_{1'}, j_2, j_{2'}, \dots j_s, j_{s'}$ (hence $\iota' = \iota \cup \{j_1, j_1', j_2, j_2', \dots j_s, j_s'\}$) and connecting by external edges

- j_p with itself (and $j_{p'}$ with itself) by a pair of external edges of opposite orientations for every color $c \neq c_p$.
- j_p with $j_{p'}$ by two external edges (one for each orientation) of color c_p.

The permutation $\vec{\xi}'$ and $\vec{\tau}'$ associated to $\mathcal{T}^{\circlearrowright}_{n,\iota',\vec{\tau}'}$ can easily be identified: for the example of Fig. 5 we have

$$
\begin{aligned}
& q_1 = 2,\ q_2 = 3,\ q_3 = 7,\ q_4 = 9,\ q_5 = 17,\ q_6 = 24,\ q_7 = 31,\ q_8 = 33\\
& \xi_1' = (1,2,8)(3,4,5)(6)(7)\\
& \xi_2' = (1,2,3,5,7,8)(4)(6)\\
& \xi_3' = (1,2,6,8)(3,5)(4)(7)\\
& \tau_1' = (1)(2,3)(4,6,7)(5)(8)\\
& \tau_2' = (1)(2)(3)(4,7,6)(5)(8)\\
& \tau_3' = (1,8)(2)(3)(4,5)(6,7).
\end{aligned}
\tag{25}
$$

However, writing them requires a bit of care as the presence of the new cilia shifts the steps in the contour walk. It is best to present $\mathcal{T}^{\circlearrowright}_{n,\iota',\vec{\tau}'}$ in two stages. The cilia of $\mathcal{T}^{\circlearrowright}_{n,\iota'}$ are either cilia of $\mathcal{T}^{\circlearrowright}_{n,\iota}$ or they are among the $2s$ new cilia. The step q_l corresponding to the l'th cilium encountered in the walk around $\mathcal{T}^{\circlearrowright}_{n,\iota}$ becomes the step $q'_{m(l)}$ corresponding to the $m(l)$'th cilium in the walk around $\mathcal{T}^{\circlearrowright}_{n,\iota'}$. The other steps, denoted $q'_{t(p)}$ and $q'_{t(p')}$ in the walk around $\mathcal{T}^{\circlearrowright}_{n,\iota'}$ corresponding to the new cilia j_p and $j_{p'}$. It follows that the steps q'_r corresponding to cilia in the walk around $\mathcal{T}^{\circlearrowright}_{n,\iota'}$ are partitioned into three categories: $r = m(l)$ for some l, $r = t(p)$ for some p or $r = t(p')$ for some p'.

For the example of Fig. 5 we have

$$
\begin{aligned}
& m(1) = 4,\ m(2) = 5,\ m(3) = 6,\ m(4) = 7,\\
& t(1) = 1,\ t(1') = 8,\ t(2) = 2,\ t(2') = 3.
\end{aligned}
\tag{26}
$$

Step 1. The presence of new cilia modifies the cycle decomposition of ξ_c'. A cycle of ξ_c of the form $(\dots l \xi_c(l) \dots)$ becomes $(\dots m(l) \dots t(p) \dots t(q') \dots \xi_c(m(l)) \dots)$, where $t(p)$ and $t(q')$ denote the (ordered list of) new cilia inserted on the face corresponding to the cycle of ξ_c (for example the cycle $(3,4,5)$ in ξ_1' which comes from the cycle $(1,2)$ in ξ_1). Note that if j_p and $j_{q'}$ are inserted on some internal face of $\mathcal{T}^{\circlearrowright}_{n,\iota}$ then they will form a cycle $\dots t(p) \dots t(q') \dots$ in the permutation ξ_c' having no corespondent in ξ_c (for example the cycle $(1,2,8)$ in ξ_1'). We reconnect the new cilia with trivial external edges

$$
\tilde{\tau}_c(r) = \begin{cases} m(\tau_c(l)) & \text{if } r = m(l) \\ t(p) & \text{if } r = t(p) \\ t(p') & \text{if } r = t(p') \end{cases} .
\tag{27}
$$

We thus obtain an intermediate tree with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota',\vec{\tilde\tau}}$. The permutations $\tilde\tau_c$ acquire extra cycles of length 1 with respect to τ_c. As a function of the cycle structure of ξ'_c, one can at most convert internal faces of color c into external faces of colors c, (if ξ'_c has more cycles than ξ_c) but the total number of faces does not change

$$|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota',\vec{\tilde\tau}})| = |\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})|. \tag{28}$$

Step 2. We convert the permutation $\tilde\tau_c$ into the permutations τ'_c defined as

$$\tau'_c(r) = \begin{cases} m(\tau_c(l)) & \text{if } r = m(l) \text{ for some } l(\tau'_1(4) = 6) \\ t(p') & \text{if } r = t(p) \text{ for some } p \text{ and } c = c_p(\tau'_1(2) = 3) \\ t(p) & \text{if } r = t(p') \text{ for some } p' \text{ and } c = c_p(\tau'_1(3) = 2)\,. \\ t(p) & \text{if } r = t(p) \text{ for some } p \text{ and } c \neq c_p(\tau'_2(2) = 2) \\ t(p') & \text{if } r = t(p') \text{ for some } p' \text{ and } c \neq c_p(\tau'_2(3) = 3) \end{cases} \tag{29}$$

We now obtain the tree with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota',\vec{\tau}'}$ corresponding to the tree with loop edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}}$. For every loop edge of color c, two cycles of length one in $\tilde\tau_c$ are merged into a cycle of length two of τ'_c.

On the other hand, for any permutation ξ, if τ' is obtained from $\tilde\tau$ by merging two cycles of length 1 into a cycle of length two,

$$C(\tau'\xi') \le C(\tilde\tau\xi) + 1, \tag{30}$$

hence

$$\sum_c C(\tau'_c\xi'_c) \le \sum_c C(\tilde\tau_c\xi'_c) + s \Rightarrow |\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})| \le |\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})| + s$$
$$\Rightarrow |\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})| \le D + (n-1)(D-1) - \sum_{c=1}^{D} C(\xi_c) + \sum_c C(\tau_c\xi_c) + s, \tag{31}$$

where $\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})$ is the set of all the faces of $\mathcal{T}_{n,\iota,\vec{\tau},\mathcal{L}}$.

We will associate to each step q in the contour walk of a tree (or of a tree with external edges, or of a tree with external and loop edges) a positive real parameter α_q.

2.4. Interpolated Gaussian measure. Consider an abstract tree T_n with n vertices labeled $1, 2, \dots n$. To every vertex $1, 2, \dots n$ we associate D matrices (one for each color) of size $N \times N$. We denote the matrices associated to the vertex i by $\sigma^{(i)_1}, \sigma^{(i)_2}, \dots \sigma^{(i)_D}$. We associate to every edge of the tree $(i, j) \in T_n$ a positive real variable u^{ij}. To every couple of vertices k and l we associate the function

$$w^{kk}(T_n, u) = 1 \qquad w^{kl}(T_n, u) = \inf_{(i,j)\in P_{k\to l}(T_n)} u^{ij}, \tag{32}$$

with $P_{k\to l}(T_n)$ the unique path in the tree T_n joining the vertices k and l. We denote $\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$ the normalized Gaussian measure of covariance

$$\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)\ \ \sigma^{(k)_c}_{ab}(\sigma^{(l)_{c'}\dagger})_{b'a'}$$
$$= \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)\ \ \sigma^{(k)_c}_{ab}\bar\sigma^{(l)_{c'}}_{a'b'} = w^{kl}(T_n, u)\ \delta_{aa'}\delta_{bb'}\delta^{cc'}. \tag{33}$$

The existence and uniqueness of this measure follows form the positivity of the real symmetric matrix $w^{ij}(T_n, u)$ [61,62]. We include here a short discussion of this point.

We consider a fixed set of parameters u. We denote the edges of the tree by $\ell = (i, j) \in T_n$ and we order them $\ell_1, \dots \ell_{n-1}$ according to the value of the u parameters, $0 \le u^{\ell_1} \le u^{\ell_2} \le \dots \le u^{\ell_{n-1}} \le 1$. We define B^q the partition of the set of vertices into blocks connected by the highest q edges $\ell_{n-1}, \dots \ell_{n-q}$. By convention B^0 is the trivial partition into n sets $\{i\}, \forall i$. The partition B^1 has $n-1$ sets, one is $\{i_{n-1}, j_{n-1}\}$ with $\ell_{n-1} = (i_{n-1}, j_{n-1})$, and the remaining $n-2$ sets are $\{i\},\ i \ne i_{n-1}, j_{n-1}$. The final partition B^{n-1} has a unique set $\{i, \forall i\}$. We label the sets in the partition B^q by $B^{q,(\mu)},\ \mu = 1, \dots n-q$.

For each q we define a $n \times n$ matrix M^q

$$M^q_{ij} = \begin{cases} 1 & i, j \in B^{q,(\mu)} \text{ for some } \mu \\ 0 & i \in B^{q,(\mu_1)}, j \in B^{q,(\mu_2)}\ \mu_1 \ne \mu_2 \end{cases}. \tag{34}$$

Each matrix M^q_{ij} is positive and $w^{ij}(T_n, u)$ is the convex combination with positive weights

$$w^{ij}(T_n, u) = (1 - u^{\ell_{n-1}}) M^0_{ij} + \sum_{q=1}^{n-2} (u^{\ell_{q+1}} - u^{\ell_q}) M^q_{ij} + u^{\ell_1} M^{n-1}_{ij}, \tag{35}$$

hence positive.

A tree T_n decorated by s extra edges $\mathcal{L} = \{(j_p, j_{p'}), p = 1 \dots s\}$ (which we call loop edges) forms a graph G. In the sequel we will encounter the following expression

$$\int_0^1 \Big(\prod_{(i,j) \in T_n} du^{ij} \Big) \prod_{p=1}^{s} w^{j_p j_{p'}}(T_n, u). \tag{36}$$

This integral can be evaluated explicitly [61] and yields the percentage of Hepp sectors of G in which T_n is dominant.

We label the lines of G by fixed labels $l_1, \dots l_{n-1+s}$. A Hepp sector is an ordering σ of the edges of G, $l_{\sigma(1)} \le \dots \le l_{\sigma(n-1+s)}$. The dominant tree in a Hepp sector σ is obtained by collecting one by one the lowest lines in the ordering which do not form loops. That is, defining $F^{(r)}$

$$F^{(1)} = l_{\sigma(1)}, \qquad F^{(r+1)} = F^{(r)} \cup l_{\sigma(k)}, \qquad \sigma(k) = \inf_{\substack{\forall \ell \in F^{(r)}, \ell \le l_{\sigma(k')} \\ F^{(r)} \cup l_{\sigma(k')} \text{ has no cycles}}} \sigma(k'), \tag{37}$$

the dominant tree is $F^{(n-1)}$. The integral in Eq. (36) can be rewritten by introducing a new variable v for every edge in $G \setminus T_n$ as

$$\int_0^1 \Big(\prod_{(i,j) \in T_n} du^{ij} \Big) \int_0^1 \Big(\prod_{(a,b) \in G \setminus T_n} dv^{ab} \Big)\ \chi_{T_n,G}(u, v),$$
$$\chi_{T_n,G}(u, v) = \begin{cases} 1, & \forall l = (a, b) \in G \setminus T_n, \forall \ell \in P_{a \to b}(T_n) \text{ we have } u^\ell \le v^l \\ 0, & \text{otherwise} \end{cases}, \tag{38}$$

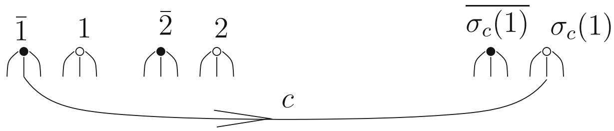

Fig. 6. Graph associated to a D-tuple of permutations

with $P_{a\to b}(T_n)$ the path in T_n from a to b. In a given sector the function $\chi_{T_n,G}$ is either zero or one, hence the integral is zero for the sectors in which T_n is not dominant and $[(n-1+s)!]^{-1}$ in the sectors in which it is.

2.5. The graph of D permutations over k elements. Any D-tuple $\vec{\sigma}$ can be represented as a D colored graph $\mathcal{B}_{\vec{\sigma}}$ with labelled vertices. We draw k black vertices labeled $\bar{1}, \dots \bar{k}$ and k white vertices labeled $1, \dots k$ and we connect the vertex $\bar{l}$ to the vertex $\sigma_c(l)$ by an edge of color c oriented from $\bar{l}$ to $\sigma_c(l)$, see Fig. 6. Conversely, to every colored graph $\mathcal{B}$ with $2k$ vertices labelled $1, \dots k, \bar{1}, \dots \bar{k}$ we associate a unique D-tuple of permutations $\vec{\sigma}(\mathcal{B})$ encoding the connectivity of its edges.

2.6. The Weingarten function. The Weingarten function introduced in [54,55] arises naturally when one considers integrals over the unitary group, namely

$$\int_{U(N)} [dU] \prod_{j=1}^{k} U_{n_j p_j} U^{\dagger}_{p'_j n'_j}$$
$$= \sum_{\sigma,\tau} \delta_{n_1 n'_{\sigma(1)}} \dots \delta_{n_k n'_{\sigma(k)}} \delta_{p_1 p'_{\tau(1)}} \dots \delta_{p_k p'_{\tau(k)}} \mathrm{Wg}(N, \sigma\tau^{-1}), \tag{39}$$

where the sum runs over all the permutations σ and τ of k elements and the Weingarten function is

$$\mathrm{Wg}(N,\tau) = \frac{1}{k!^2} \sum_{\pi} \frac{\chi^{\pi}(1)^2 \chi^{\pi}(\tau)}{s_{\pi,N}(1)}, \tag{40}$$

where the sum runs over the partitions π of N, χ^{π} is the character of the symmetric group corresponding to π and $s_{\pi,N}(x)$ is the Schur function of the unitary group (hence $s_{\pi,N}(1)$ is the dimension of the irreducible representation of $U(N)$ associated with π). We will in particular use the following properties of the Weingarten function [54,55]

$$\mathrm{Wg}(N,(1)) = \frac{1}{N},$$
$$\lim_{N\to\infty} N^{2k-C(\sigma)} \mathrm{Wg}(N,\sigma) = \prod_{s=1}^{C(\sigma)} (-1)^{|C_s(\sigma)|-1} \frac{1}{|C_s(\sigma)|} \binom{2|C_s(\sigma)|-2}{|C_s(\sigma)|-1}, \tag{41}$$

where $C(\sigma)$ denotes the number of cycles of the permutation σ and $|C_s(\sigma)|$ denotes the length of the s'th cycle. It follows that for N large enough we have

$$|\mathrm{Wg}(N,\sigma)| < \frac{1}{N^{2k-C(\sigma)}} 2^{2k}, \tag{42}$$

and we will always assume in this paper that N is large enough such that this bound is respected.

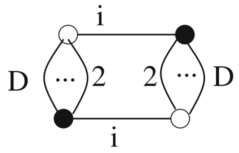

Fig. 7. The graphs of the quartic perturbation terms

3. The Quartically Perturbed Gaussian Measure

Our starting point is the quartically perturbed Gaussian tensor measure

$$d\mu^{(4)} = \frac{1}{Z(\lambda, N)} \Big(\prod_{\vec{n}} N^{D-1} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{n}}}{2\pi\imath} \Big) \, e^{-N^{D-1} S^{(4)}(\mathbb{T}, \bar{\mathbb{T}})},$$
$$S^{(4)}(\mathbb{T}, \bar{\mathbb{T}}) = \sum_{\vec{n}} \mathbb{T}_{\vec{n}} \delta_{\vec{n}\bar{\vec{n}}} \bar{\mathbb{T}}_{\bar{\vec{n}}} + \lambda \sum_{i=1}^{D} \sum_{n\bar{n}} \mathbb{T}_{\vec{n}} \bar{\mathbb{T}}_{\bar{\vec{m}}} \mathbb{T}_{\vec{m}} \bar{\mathbb{T}}_{\bar{\vec{n}}} \; \delta_{n^i \bar{m}^i} \delta_{m^i \bar{n}^i} \prod_{j \neq i} \delta_{n^j \bar{n}^j} \delta_{m^j \bar{m}^j}, \tag{43}$$

with $Z(\lambda, N)$ some normalization constant. The quartic perturbation corresponds to a sum of invariants whose graphs are represented in Fig. 7.

One can in principle consider more general quartic perturbations (in the associated graphs the vertices share q and $D - q$ lines respectively with $q > 1$) however such perturbations are suppressed in powers of $1/N$. The generating function of the moments of $\mu^{(4)}$ is

$$Z(J, \bar{J}; \lambda, N) = \int \Big(\prod_{\vec{n}} N^{D-1} \frac{d\mathbb{T}_{\vec{n}} d\bar{\mathbb{T}}_{\vec{n}}}{2\pi\imath} \Big) \, e^{-N^{D-1} S^{(4)}(\mathbb{T}, \bar{\mathbb{T}}) + \sum_{\bar{\vec{n}}} \bar{\mathbb{T}}_{\bar{\vec{n}}} J_{\bar{\vec{n}}} + \sum_{\vec{n}} \mathbb{T}_n \bar{J}_n}, \tag{44}$$

and the generating function of the cumulants is $W(J, \bar{J}; \lambda, N) = \ln Z(J, \bar{J}; \lambda, N)$,

$$\kappa\big(\mathbb{T}_{\vec{n}_1}, \bar{\mathbb{T}}_{\bar{\vec{n}}_1}, \dots \mathbb{T}_{\vec{n}_k}, \bar{\mathbb{T}}_{\bar{\vec{n}}_k}\big) = \frac{\partial^{(2k)}}{\partial \bar{J}_{\vec{n}_1} \partial J_{\bar{\vec{n}}_1} \dots \partial \bar{J}_{\vec{n}_k} \partial J_{\bar{\vec{n}}_k}} W(J, \bar{J}; \lambda, N) \Big|_{J=\bar{J}=0}. \tag{45}$$

Our first result concerns the constructive expansion of $W(J, \bar{J}; \lambda, N)$.

Theorem 3 (Constructive Expansion 1). *The generating function of the cumulants of* $\mu^{(4)}$ *in Eq.* (43) *writes as a sum over plane trees* $\mathcal{T}^{\circlearrowright}_{n,\imath}$

$$W(J, \bar{J}; \lambda, N) = \sum_{n \geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{N^{(k+n-1)(D-1)}} \sum_{\substack{i_1, i_2 \dots, i_k = 1 \\ i_d \neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\imath}} \mathfrak{T}(\mathcal{T}^{\circlearrowright}_{n,\imath}), \tag{46}$$

where the contribution of a tree is

$$\mathfrak{T}(\mathcal{T}_{n,\iota}^{\circlearrowright}) = \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij} \Big) \int d\mu_{w^{ij}(T_n,u) 1^{\otimes D}}(\sigma) \int \Big(\prod_{q=1}^{2n-2+k} d\alpha_q \Big) \, e^{-\sum_{q=1}^{2n-2+k} \alpha_q}$$
$$\times \prod_{f^c \in \mathcal{F}^{\text{int}}(\mathcal{T}_{n,\iota}^{\circlearrowright})} \text{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big] \sum_{\{p_l^c, n_l^c\}} \prod_{l=1}^{k} \bar{J}_{p_l^1,\dots p_l^D} J_{n_l^1,\dots n_l^D}$$
$$\times \prod_{f^{c;i(q_l)\to i(q_{\xi_c(l)})} \in \mathcal{S}^{\text{ext}}(\mathcal{T}_{n,\iota}^{\circlearrowright})} \Big[\overrightarrow{\prod_{q\in q(f^{c;i(q_l)\to i(q_{\xi_c(l)})})}} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big]_{p_l^c n_{\xi_c(l)}^c},$$

where we used the notations of Sect. 2.3 and $\rightarrow$ *means that the products are ordered.*

This expansion is the generalization to tensor models of the constructive Loop Vertex Expansion (LVE) introduced in [51] for matrix models. Note that our expansion looks somewhat different from the LVE of [51], notably the step parameters α_p have no equivalent in the initial formulation. The usual LVE is recovered from 3 by restricting to trees having no ciliated vertices. In this case the integrals over α_p can be computed explicitly and one recovers the formulation in terms of resolvents of [51].

As a consequence of the (LVE) we can derive an expansion for the cumulants of our measure.

Theorem 4 (Constructive Expansion 2). *The cumulants of the measure* $\mu^{(4)}$ *in Eq.* (43) *are trace invariants*

$$\kappa\big(\mathbb{T}_{\vec{p}_1}, \bar{\mathbb{T}}_{\vec{n}_1}, \dots \mathbb{T}_{\vec{p}_k}, \bar{\mathbb{T}}_{\vec{n}_k}\big) = \sum_{\mathcal{B},\; k(\mathcal{B})=k} \mathfrak{K}(\mathcal{B}, \mu_N^{(4)}) \prod_{\rho=1}^{C(\mathcal{B})} \delta_{n\bar{n}}^{\mathcal{B}_\rho}, \tag{47}$$

where the sum runs over D-colored graphs with $2k$ *vertices labelled* $1, \dots k, \bar{1}, \dots \bar{k}$ *and admit an expansion as a sum over plane trees with external edges* $\mathcal{T}_{n,\iota,\vec{\tau}}$

$$\mathfrak{K}(\mathcal{B}, \mu_N^{(4)}) = \sum_{n\geq k} \frac{1}{n!} (-\lambda)^{n-1} \frac{1}{N^{(k+n-1)(D-1)}} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d \neq i_{d'}}}^{n} \sum_{\mathcal{T}_{n,\iota}^{\circlearrowright}} \sum_{\vec{\tau}} \mathfrak{T}^{\mathfrak{E}}(\mathcal{T}_{n,\iota,\vec{\tau}}^{\circlearrowright}), \tag{48}$$

where the contribution of a tree with external edges is

$$\mathfrak{T}^{\mathfrak{E}}(\mathcal{T}_{n,\iota,\vec{\tau}}^{\circlearrowright}) = k! \Big(\prod_{c=1}^{D} \text{Wg}\big(N, \tau_c \sigma_c^{-1}(\mathcal{B})\big) \Big) \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij} \Big) \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$$
$$\times \int \Big(\prod_{q=1}^{2n-2+k} d\alpha_q \Big) e^{-\sum_{q=1}^{2n-2+k} \alpha_q} \prod_{f^c \in \mathcal{F}(\mathcal{T}_{n,\iota,\vec{\tau}}^{\circlearrowright})} \text{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big].$$

The main advantage of the constructive expansion of the cumulants is that, unlike the perturbative expansion in λ, it leads to a series which is absolutely convergent uniformly in N.

Theorem 5 (Absolute Convergence). *The series in Eq.* (48) *is absolutely convergent for* $\lambda \in \mathbb{R}$, $\lambda \in [0, 2^{-3}D^{-1})$. *Moreover, the cumulants are bounded by*

$$|\mathfrak{K}(\mathcal{B}, \mu_N^{(4)})| \leq N^{D-2k(D-1)-C(\mathcal{B})}|\lambda|^{k-1}K(\mathcal{B}), \tag{49}$$

for some constant $K(\mathcal{B})$ *independent of* N *(and independent of* λ *for* $|\lambda|$ *small enough).*

The scaling bound of Eq. (49) coincides with the proper uniform bound of definition 3. However we can not yet conclude that $\mu^{(4)}$ is properly uniformly bounded. Indeed, in order to conclude this, one still needs to prove that the second cumulant converges when $N \to \infty$. We will show this later in this paper.

The domain of convergence $0 \leq \lambda < 2^{-3}D^{-1}$ is optimal. We will see below that the leading order in $1/N$, the "melonic" sector yields a series whose radius of convergence is exactly $2^{-3}D^{-1}$. It is not surprising that the full non perturbative expression diverges when one reaches the radius of convergence of the leading order.

The series (48) computes the cumulants for real, positive (and small) coupling constant. The cumulants can be analytically continued to some domain in the complex plane.

Corollary 1. *The cumulants* $\mathfrak{K}(\mathcal{B}, \mu_N^{(4)})$ *can be analytically continued for* $\lambda = |\lambda|e^{\imath\varphi}$ *with* $\varphi \in (-\pi, \pi)$ *and* $|\lambda| < \big(\cos\frac{\varphi}{2}\big)^2 2^{-3}D^{-1}$. *In this domain they are represented by the absolutely convergent series*

$$\mathfrak{K}(\mathcal{B}, \mu_N^{(4)}) = \sum_{n\geq k}\frac{1}{n!}(-\lambda)^{n-1}\frac{1}{N^{(k+n-1)(D-1)}}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}}\sum_{\vec{\tau}}\mathfrak{T}^{\mathfrak{E}}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}),$$

$$\mathfrak{T}^{\mathfrak{E}}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = k!\Big(\prod_{c=1}^{D}\mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$$
$$\times\int\Big(\prod_{q=1}^{2n-2+k}e^{-\imath\frac{\varphi}{2}}d\alpha_q\Big)e^{-\sum_{q=1}^{2n-2+k}e^{-\imath\frac{\varphi}{2}}\alpha_q}$$
$$\times\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})}\mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow}e^{-\alpha_q\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]. \tag{50}$$

We have thus a well defined expression for the cumulants in a heart-shaped domain (see Fig. 8 below where we represented by a dashed circle the circle of radius $2^{-3}D^{-1}$). As expected, $\lambda = 0$ is a point belonging to the boundary of this analyticity domain.

The interplay between the non perturbative LVE expansions presented so far and the $1/N$ expansion is captured by the following theorem. Consider the rescaled cumulants, which according to Eq. (48) write as

$$K(\mathcal{B}, N) = N^{-D+2k(D-1)+C(\mathcal{B})}\mathfrak{K}(\mathcal{B}, \mu_N^{(4)})$$
$$= \sum_{n\geq k}\frac{1}{n!}(-\lambda)^{n-1}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}}\sum_{\vec{\tau}}T^{E}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}), \tag{51}$$

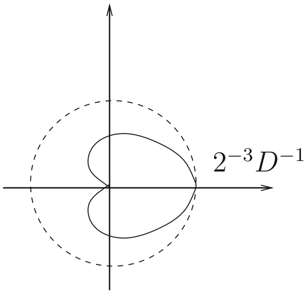

Fig. 8. Domain of convergence

where the rescaled contribution of each tree with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}$ is

$$T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)}\; k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)$$
$$\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$$
$$\times \int \Big(\prod_{q=1}^{2n-2+k} e^{-\iota\frac{\varphi}{2}} d\alpha_q\Big)\, e^{-\sum_{q=1}^{2n-2+k} e^{-\iota\frac{\varphi}{2}}\alpha_q}$$
$$\times \prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})} \mathrm{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]. \tag{52}$$

Theorem 6 (The Mixed Expansion). *The contribution of a tree with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota}$ admits an expansion in terms of trees with external edges and loop edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}}$*

$$T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = \sum_{q=0}^{s-1} T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) + R^{(s)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}),$$
$$T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = \sum_{\mathcal{L},|\mathcal{L}|=q}\ \sum_{c_1\dots c_q} T^{EL,(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}}) \tag{53}$$

where $\mathcal{L}$ runs over all possible ways to decorate $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}$ with q loop edges $(j_1, j_{1'})$, $(j_2, j_{2'})$ up to $(j_q, j_{q'})$ and $c_1, \dots c_q$ run over the possible colorings of the loop edges and the contribution of a tree with external edges and loop edges is

$$T^{EL,(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}}) = \frac{1}{q!}\Big(-\frac{\lambda}{N^{D-1}}\Big)^q k!\Big(\prod_{c=1}^{D}\mathrm{Wg}\big(N,\tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)$$
$$\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \prod_{p=1}^{s} w^{j_p j_{p'}}(T_n, u)$$
$$\times N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)+|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})|}, \tag{54}$$

while the rest term is

$$R^{(s)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = \int_0^1 dt (1-t)^{s-1} \Bigg[N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)} \\ \times \frac{1}{(s-1)!}\Big(-\frac{\lambda}{N^{D-1}}\Big)^s k! \Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c \sigma_c^{-1}(\mathcal{B})\big)\Big) \sum_{\mathcal{L},|\mathcal{L}|=s} \sum_{c_1 \dots c_s} \\ \times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \prod_{p=1}^{s} w^{j_p j'_p}(T_n, u) \int d\mu_{w^{ij}(T_n,u) 1^{\otimes D}}(\sigma) \\ \times \int_0^\infty \Big(\prod_{q=1}^{2n-2+k+2s} e^{-\iota \frac{\varphi}{2}} d\alpha_q \Big) e^{-\sum_{q=1}^{2n-2+k+2s} e^{-\iota\frac{\varphi}{2}}\alpha_q} \\ \times \prod_{f^c \in \mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})} \mathrm{Tr}\Big[\overrightarrow{\prod_{q \in q(f^c)}} e^{-\alpha_q \sqrt{t} \sqrt{\frac{|\lambda|}{N^{D-1}}} (\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big] \Bigg]. \tag{55}$$

Furthermore the terms in the mixed expansion admit the bounds

$$\big| T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) \big| \le \frac{|\lambda|^q}{N^{q(D-2)}} (k! 2^{2Dk} D^q) \frac{(2n+2q+k-3)!}{q!(2n+k-3)!} \\ |R^{(s)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})| \le \frac{1}{\big(\cos\frac{\varphi}{2}\big)^{2n+2s+k-2}} \frac{|\lambda|^s}{N^{s(D-2)}} (k! 2^{2Dk} D^s) \frac{(2n+2s+k-3)!}{(s-1)!(2n+k-3)!}. \tag{56}$$

We call this expansion the mixed expansion because it is at the same time an expansion in λ *and* an expansion in $1/N$. More precisely, being an expansion in $\frac{\lambda}{N^{D-2}}$, one can use it to establish the Borel summability of the cumulants or, alternatively, one can use it to establish the $\frac{1}{N}$ expansion of the cumulants at all orders.

Theorem 7 (Borel summability). *The rescaled cumulants*

$$K(\mathcal{B}, N) \equiv N^{-D+2k(D-1)+C(\mathcal{B})} \mathfrak{K}(\mathcal{B}, \mu_N^{(4)}), \tag{57}$$

are Borel summable in λ uniformly in N.

A crucial point is that, as we are interested in the large N regime, both the convergence of the constructive expansion in its analyticity domain and the Borel summability around $\lambda = 0$ are uniform in N.

Theorem 8 (The $1/N$ expansion of the cumulants). *Using the mixed expansion, the rescaled cumulants of $\mu^{(4)}$ write as*

$$K(\mathcal{B}, N) = \sum_{n \ge k} \frac{1}{n!} (-\lambda)^{n-1} \sum_{\substack{i_1, i_2 \dots, i_k = 1 \\ i_d \ne i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} \sum_{\vec{\tau}} \sum_{q=0}^{s-1} \sum_{\mathcal{L},|\mathcal{L}|=q} \sum_{c_1 \dots c_q} T^{EL,(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}}) \\ + R_N^{(s)}(\mathcal{B}, \lambda), \tag{58}$$

and for $|\lambda| < 5^{-2}2^{-1}D^{-1}\big(\cos\frac{\varphi}{2}\big)^2$ *the rest term admits the bound*

$$|R_N^{(s)}(\mathcal{B},\lambda)| \le K\sigma^s s! \frac{1}{N^{s(D-2)}} \frac{|\lambda|^{s+k-1}}{\big(\cos\frac{\varphi}{2}\big)^{2s+3k-1}}. \tag{59}$$

for some constants K *and* σ.

One can now use the $1/N$ expansion as follows: the terms up to order $N^{-s(D-2)}$ are indexed by trees with at most s loop edges. Such corrections can be evaluated order by order. In particular all corrections at fixed order in $1/N$ will reach criticality when the sum over n becomes critical, i.e., all terms in the $1/N$ expansion will diverge for *the same* critical constant.

The factorial bound in Eq. (59) suggests that the cumulants are Borel summable in $1/N$. This is most likely the case, however the attentive reader will notice that we did not yet establish analyticity of the cumulants in $1/N$. This is difficult, because, besides the explicit occurrences of N, we also must take into account that N is the size of the matrices $\sigma_c^{(i)}$. In order to establish analyticity in $1/N$ one needs to find a better representation of the cumulants in which N appears exclusively as a parameter. We postpone this to future work.

Before we conclude this paper we present as an example the leading order behaviour of the two point cumulant. A number of simplifications arise in this case : $k = 1$, $\mathrm{Wg}\big(N,(1)\big) = \frac{1}{N}$, $C(\mathcal{B}_{(\vec{1})}) = 1$. Using the $1/N$ expansion up to order $s = 1$ we get

$$\begin{aligned} K(\mathcal{B}^{(2)},N) &= \sum_{n\ge k}\frac{1}{n!}(-\lambda)^{n-1}\sum_{i_1=1}^{n}\sum_{\mathcal{T}^{\circlearrowright}_{n,\{i_1\}}} T^{EL,(0)}(\mathcal{T}^{\circlearrowright}_{n,\{i_1\},(\vec{1}),\emptyset}) + R_N^{(1)}(\mathcal{B}^{(2)},\lambda),\\ T^{EL,(0)}(\mathcal{T}^{\circlearrowright}_{n,\{i_1\},(\vec{1}),\emptyset}) &= \frac{1}{N^D}N^{-D+2(D-1)+1-(1+n-1)(D-1)+|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,(\vec{1}),\emptyset})|},\\ R_N^{(1)}(\mathcal{B}^{(2)},\lambda) &\le \frac{1}{N^{D-2}}K\frac{|\lambda|}{\big(\cos\frac{\varphi}{2}\big)^4}. \end{aligned} \tag{60}$$

The leading order behavior can be resummed, as

$$|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,(\vec{1}),\emptyset})| = D + (n-1)(D-1) \Rightarrow T^{EL,(0)}(\mathcal{T}^{\circlearrowright}_{n,\{i_1\},(\vec{1}),\emptyset}) = 1, \tag{61}$$

and the sum over trees can be computed explicitly (we do this for arbitrary trees in Eq. (78) below),

$$K(\mathcal{B}^{(2)},N) = \sum_{n\ge k}(-2D\lambda)^{n-1}\frac{(2n-2)!}{(n-1)!n!} + \frac{1}{N^{D-2}}K\frac{|\lambda|}{\big(\cos\frac{\varphi}{2}\big)^4}. \tag{62}$$

Corollary 2 (The large N limit). *For* $\frac{|\lambda|}{\big(\cos\frac{\varphi}{2}\big)^2}$ *small enough*

$$\lim_{N\to\infty} K(\mathcal{B}^{(2)},N) = \frac{-1+\sqrt{1+8D\lambda}}{4D\lambda}. \tag{63}$$

In particular this coupled with the uniform bound in Theorem 5 proves that the measure $\mu^{(4)}$ is properly uniformly bounded, hence according to Theorem 1, becomes Gaussian in the large N limit.

4. Proofs

Before going to the core of the proofs of the various theorems in the text we establish a number of results we will use recurrently in the sequel.

4.1. Technical prerequisites. **The main inequality on permutations.** Recall that $C(\sigma)$ denotes the number of cycles of the permutation σ and $C(\mathcal{B}_{\vec{\sigma}})$ denotes the number of connected components of the graphs associated to the D-tuple of permutations $\vec{\sigma}$.

Lemma 1. *Let $\vec{\xi}$ and $\vec{\sigma}$ and $\vec{\tau}$ be three D-tuples of permutations over k elements such that $\vec{\xi}$ are permutations encoding the external faces of a tree. We have the bound*

$$\sum_c C(\tau_c\sigma_c^{-1}) - \sum_c C(\xi_c) + \sum_c C(\tau_c\xi_c) \leq (D+1)k - C(\mathcal{B}_{\vec{\sigma}}). \tag{64}$$

This bound is a trivial consequence of the following two propositions.

Proposition 1. *Let ξ and σ be any two fixed permutations over k elements, and let τ be any permutation over k elements. Then*

$$C(\tau\sigma^{-1}) + C(\tau\xi) \leq C(\xi^{-1}\sigma^{-1}) + k. \tag{65}$$

Proof. The bound is saturated if $\tau\xi$ is the identity permutation.

Consider then the sum $C(\tau\sigma^{-1}) + C(\tau\xi)$ and suppose that $\tau\xi$ is not the identity permutation. The cycle structure of the permutation $\tau\xi$ can be easily read of by drawing the graph of the two permutations ξ^{-1}, τ (where we reverse the orientation of the edges representing the permutation ξ^{-1}), see Fig. 9: a moment of reflection reveals that the number of cycles of $\tau\xi$ is the number of connected components of the graph as the $\tau\xi$ jumps from a white vertex to the next white vertex following the arrows.

If $\tau\xi$ is not the identity, then there exists p such that $\xi(p) \neq \tau^{-1}(p)$ (and $\tau(\xi(p)) \neq p$). We compare $C(\tau\sigma^{-1}) + C(\tau\xi)$ with $C(\tau'\sigma^{-1}) + C(\tau'\xi)$, with τ' defined as

$$\tau'(q) = \begin{cases} \tau(q) & \forall q \neq \tau^{-1}(p), \xi(p) \\ \tau'\big(\tau^{-1}(p)\big) = \tau\big(\xi(p)\big) & \\ \tau'\big(\xi(p)\big) = p & \end{cases}. \tag{66}$$

The graph of the permutations ξ^{-1}, τ' is represented in Fig. 10. By substituting τ with τ' we created a new connected component, $C(\tau'\xi) = C(\tau\xi) + 1$. Representing now the graph of the permutations σ, τ and σ, τ', we see that replacing τ by τ' amounts

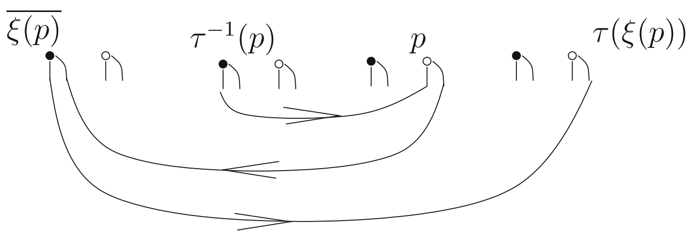

Fig. 9. Graph associated to ξ^{-1}, τ

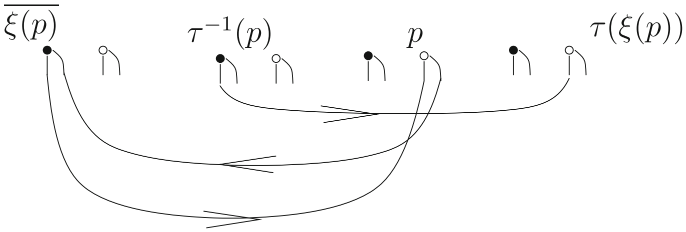

Fig. 10. Graph associated to ξ^{-1}, τ'

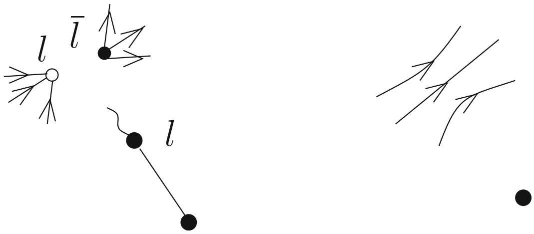

Fig. 11. Ciliated vertex and the associated vertices of $\mathcal{B}_{\vec{\sigma}}$

to permuting the endpoints of the edges representing the τ permutation originating in $\tau^{-1}(p)$ and $\xi(p)$, hence the number of cycles of $\tau\sigma^{-1}$ can not decrease by more than 1. Thus

$$C(\tau\sigma^{-1}) + C(\tau\xi) \leq C(\tau'\sigma^{-1}) + C(\tau'\xi). \tag{67}$$

Iterating, we find that the maximum is achieved for τ such that $\tau\big(\xi(p)\big) = p$ for all p. □

Proposition 2. *Let $\vec{\xi}$ and $\vec{\sigma}$ be two D-tuples of permutations (with $D \geq 2$) such that $\vec{\xi}$ are permutations encoding the external faces of a tree. Then*

$$\sum_c C(\sigma_c \xi_c) - \sum_c C(\xi_c) + C(\mathcal{B}_\sigma) \leq k. \tag{68}$$

Proof. For every cilium l we draw a black and white vertex and represent the graph $\mathcal{B}_{\vec{\sigma}}$ associated to the permutation $\vec{\sigma}$, see Fig. 11.

Consider the tree with associated permutations $\vec{\xi}$. As the univalent vertices with no cilia of the tree have no bearing over the permutations $\vec{\xi}$ we can eliminate them. Consider a ciliated univalent vertex l in the tree, and say that the line touching it has color c. It follows that we are in one of the two cases

$$\begin{cases} \text{case I} & \xi_c(l) \neq l, \qquad \xi_{c_1}(l) = l \quad \forall c_1 \neq c, \\ \text{case II} & \xi_c(l) = l, \qquad \xi_{c_1}(l) = l \quad \forall c_1 \neq c. \end{cases} \tag{69}$$

The permutations $\vec{\sigma}$ fall in one of the following three categories

$$\begin{cases} \text{case 1} & \sigma_c(l) \neq l, \quad \forall c, \\ \text{case 2} & \sigma_{c_i}(l) = l, i = 1 \dots q < D, \quad \sigma_{c_j}(l) \neq l \quad \{c_i\} \cup \{c_j\} = \{1, \dots D\}, \\ \text{case 3} & \sigma_c(l) = l, \quad \forall c. \end{cases} \tag{70}$$

We build the graph obtained by eliminating the ciliated vertex l in the tree (by cutting the tree line touching it) and deleting the vertices l and $\bar{l}$ in $\mathcal{B}_{\vec{\sigma}}$ and reconnecting the σ lines respecting the colors (see Fig. 11 on the right). The new graph is characterized by permutations $\vec{\xi}'$ and $\vec{\sigma}'$ over $\{1, \dots k\}\backslash\{l\}$

$$\begin{cases} \text{case I} & \xi'_c(p) = \xi_c(p), \forall p \neq \xi_c^{-1}(l), \qquad \xi'_c\big(\xi_c^{-1}(l)\big) = \xi_c(l) \\ & \xi'_{c_1}(p) = \xi_{c_1}(p), \forall p \neq l \\ \text{case II} & \xi_c(p) = \xi_c(p), \forall p \neq l \\ & \xi'_{c_1}(p) = \xi_{c_1}(p), \forall p \neq l \end{cases},$$

$$\begin{cases} \text{case 1} & \sigma'_c(p) = \sigma_c(p), \forall p \neq \sigma_c^{-1}(l), \qquad \sigma'_c\big(\sigma_c^{-1}(l)\big) = \sigma_c(l) \\ \text{case 2} & \sigma'_{c_j}(p) = \sigma_{c_j}(p), \forall p \neq \sigma_{c_j}^{-1}(l), \qquad \sigma'_{c_j}\big(\sigma_{c_j}^{-1}(l)\big) = \sigma_{c_j}(l) \\ & \sigma'_{c_i}(p) = \sigma_{c_i}(p), \forall p \neq l \\ \text{case 3} & \sigma'_c(p) = \sigma_c(p), \forall p \neq l \end{cases}. \tag{71}$$

Consider now the graph of two permutations ξ_c^{-1}, σ_c. The permutations $(\xi'_c)^{-1}, \sigma'_c$ correspond to the graph whose vertices l and $\bar{l}$ have been deleted and whose edges have been reconnected coherently. The number of cycles changes as

- if $\sigma_c(l) \neq l$, and $\xi_c(l) \neq l$ then $C(\sigma_c\xi_c) \leq C(\sigma'_c\xi'_c) + 1$,
- if $\sigma_c(l) \neq l$, and $\xi_c(l) = l$ then $C(\sigma_c\xi_c) = C(\sigma'_c\xi'_c)$,
- if $\sigma_c(l) = l$, and $\xi_c(l) \neq l$ then $C(\sigma_c\xi_c) = C(\sigma'_c\xi'_c)$,
- if $\sigma_c(l) = l$ and $\xi_c(l) = l$ then $C(\sigma_c\xi_c) = C(\sigma'_c\xi'_c) + 1$.

Finally, the number of connected components of $\mathcal{B}_{\vec{\sigma}}$ changes as

- case 1, $\mathcal{B}_{\vec{\sigma}} \leq \mathcal{B}_{\vec{\sigma}'} + 1$,
- case 2, $\mathcal{B}_{\vec{\sigma}} = \mathcal{B}_{\vec{\sigma}'}$,
- case 3, $\mathcal{B}_{\vec{\sigma}} = \mathcal{B}_{\vec{\sigma}'} + 1$.

As a function of the case we are into, we then have

$$I1: \sum_c C(\sigma_c\xi_c) \leq \sum_c C(\sigma'_c\xi'_c) + 1,$$
$$\sum_c C(\xi_c) = \sum_c C(\xi'_c) + D - 1, \quad C(\mathcal{B}_{\vec{\sigma}}) \leq C(\mathcal{B}_{\vec{\sigma}'}) + 1,$$
$$II1: \sum_c C(\sigma_c\xi_c) = \sum_c C(\sigma'_c\xi'_c),$$
$$\sum_c C(\xi_c) = \sum_c C(\xi'_c) + D, \quad C(\mathcal{B}_{\vec{\sigma}}) \leq C(\mathcal{B}_{\vec{\sigma}'}) + 1,$$
$$I2: \sum_c C(\sigma_c\xi_c) \leq \sum_c C(\sigma'_c\xi'_c) + q + 1,$$
$$\sum_c C(\xi_c) = \sum_c C(\xi'_c) + D - 1, \quad C(\mathcal{B}_{\vec{\sigma}}) = C(\mathcal{B}_{\vec{\sigma}'}),$$
$$II2: \sum_c C(\sigma_c\xi_c) = \sum_c C(\sigma'_c\xi'_c) + q,$$
$$\sum_c C(\xi_c) = \sum_c C(\xi'_c) + D, \quad C(\mathcal{B}_{\vec{\sigma}}) = C(\mathcal{B}_{\vec{\sigma}'}),$$

$$
\begin{aligned}
I3: \ & \sum_c C(\sigma_c \xi_c) = \sum_c C(\sigma'_c \xi'_c)) + D - 1, \\
& \sum_c C(\xi_c) = \sum_c C(\xi'_c) + D - 1, \quad C(\mathcal{B}_{\vec{\sigma}}) = C(\mathcal{B}_{\vec{\sigma}'}) + 1, \\
II3: \ & \sum_c C(\sigma_c \xi_c) = \sum_c C(\sigma'_c \xi'_c) + D, \\
& \sum_c C(\xi_c) = \sum_c C(\xi'_c) + D, \quad C(\mathcal{B}_{\vec{\sigma}}) = C(\mathcal{B}_{\vec{\sigma}'}) + 1.
\end{aligned} \tag{72}
$$

In all cases

$$
\sum_c C(\xi_c \sigma_c^{-1}) - \sum_c C(\xi_c) + C(\mathcal{B}_\sigma) \leq \sum_c C(\xi'_c (\sigma'_c)^{-1}) - \sum_c C(\xi'_c) + C(\mathcal{B}_{\vec{\sigma}'}) + 1, \tag{73}
$$

and the bound can be attained only for $I3$, $II3$, $I2$ with $q = D - 1$, or $I1$ if $D = 2$. We iterate the procedure, taking into account that after eliminating the univalent vertex l of the tree one might need to eliminate some new univalent vertices with no cilia (as the latter have no bearing on the permutations $\vec{\xi}'$). Iterating up to permutations of 1 element $\xi_c = \sigma_c = (1)\ \forall c$, we get

$$
\sum_c C(\xi_c \sigma_c^{-1}) - \sum_c C(\xi_c) + C(\mathcal{B}_\sigma) \leq (k-1) + 1. \tag{74}
$$

□

Evaluating derivatives. We will repeatedly use in the sequel the following result.

Lemma 2. *For any $N \times N$ matrices F and G and any function $H(\alpha_1, \alpha_2)$ we have*

$$
\begin{aligned}
& \int_0^\infty d\alpha_1 d\alpha_2 \ H(\alpha_1, \alpha_2) \sum_{p_1 n_1 p_2 n_2} F_{n_1 p_1} G_{n_2 p_2} \\
& \quad \times \sum_{ab} \frac{\partial}{\partial \sigma_{ab}^{(i)_c}} \Big[e^{-\alpha_1 \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(i)_c} - \sigma^{(i)_c \dagger})} \Big]_{p_1 n_1} \\
& \quad \times \frac{\partial}{\partial \sigma_{ba}^{(j)_c \dagger}} \Big[e^{-\alpha_2 \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(j)_c} - \sigma^{(j)_c \dagger})} \Big]_{p_2 n_2} \\
& = -\frac{\lambda}{N^{D-1}} \int_0^\infty d\beta_1 d\gamma_1 d\beta_2 d\gamma_2 \ H(\beta_1 + \gamma_1, \beta_2 + \gamma_2) \\
& \quad \times \mathrm{Tr} \Big[F e^{-\beta_1 \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(i)_c} - \sigma^{(i)_c \dagger})} e^{-\gamma_2 \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(j)_c} - \sigma^{(j)_c \dagger})} \\
& \qquad G e^{-\beta_2 \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(j)_c} - \sigma^{(j)_c \dagger})} e^{-\gamma_1 \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(i)_c} - \sigma^{(i)_c \dagger})} \Big].
\end{aligned} \tag{75}
$$

Proof. Expanding the exponentials and evaluating the derivatives the integral writes

$$
\begin{aligned}
& = \int_0^\infty d\alpha_1 d\alpha_2 \ H(\alpha_1, \alpha_2) \sum_{p_1 n_1 p_2 n_2} F_{n_1 p_1} G_{n_2 p_2} \\
& \quad \sum_{ab} \Big[\sum_{u_1, v_1 \geq 0} \frac{(-\alpha_1)^{u_1 + v_1 + 1}}{(u_1 + v_1 + 1)!} \sqrt{\frac{\lambda}{N^{D-1}}}^{u_1 + v_1 + 1}
\end{aligned}
$$

$$\Big[(\sigma^{(i)_c} - \sigma^{(i)_c\dagger})^{u_1}\Big]_{p_1 a}\Big[(\sigma^{(i)_c} - \sigma^{(i)_c\dagger})^{v_1}\Big]_{bn_1}\Big]$$

$$\Big[\sum_{u_2,v_2\geq 0}\frac{(-\alpha_2)^{u_2+v_2+1}}{(u_2+v_2+1)!}(-1)\sqrt{\frac{\lambda}{N^{D-1}}}^{u_2+v_2+1}$$

$$\Big[(\sigma^{(j)_c} - \sigma^{(j)_c\dagger})^{u_2}\Big]_{p_2 b}\Big[(\sigma^{(j)_c} - \sigma^{(j)_c\dagger})^{v_2}\Big]_{an_2}\Big]. \tag{76}$$

Using $\frac{1}{(u_1+v_1+1)!} = \int_0^1 dx_1 \frac{(1-x_1)^{u_1}}{u_1!}\frac{x_1^{v_1}}{v_1!}$ this rewrites as

$$-\frac{\lambda}{N^{D-1}}\int_0^\infty d\alpha_1 d\alpha_2\ H(\alpha_1,\alpha_2)\int_0^1 \alpha_1 dx_1 \alpha_2 dx_2 \sum_{p_1 n_1 p_2 n_2} F_{n_1 p_1} G_{n_2 p_2}$$

$$\sum_{u_1,v_1,u_2,v_2\geq 0}\frac{[-\alpha_1(1-x_1)]^{u_1}}{u_1!}\frac{[-\alpha_1 x_1]^{v_1}}{v_1!}\frac{[-\alpha_2(1-x_2)]^{u_2}}{u_2!}\frac{[-\alpha_2 x_2]^{v_2}}{v_2!}$$

$$\sqrt{\frac{\lambda}{N^{D-1}}}^{u_1+v_1+u_2+v_2}\Big[(\sigma^{(i)_c} - \sigma^{(i)_c\dagger})^{u_1}(\sigma^{(j)_c} - \sigma^{(j)_c\dagger})^{v_2}\Big]_{p_1 n_2}$$

$$\Big[(\sigma^{(j)_c} - \sigma^{(j)_c\dagger})^{u_2}(\sigma^{(i)_c} - \sigma^{(i)_c\dagger})^{v_1}\Big]_{p_2 n_1}, \tag{77}$$

and changing variables to $\beta_1 = \alpha_1(1-x_1)$, $\gamma_1 = \alpha_1 x_1$ and similarly for 2 and summing over u_1, v_1, u_2, v_2 the lemma follows □

Combinatorial countings. We count the number of plane trees with n vertices, k ciliated vertices $i_1, \dots i_k$ and colored oriented edges. Every combinatorial tree T_n with degrees of the vertices $d_1, \dots d_n$, has $(2D)^{n-1} d_{i_1}! \dots d_{i_k}! \prod_{i\neq i_k}(d_i - 1)!$ associated plane trees with colored oriented edges and marked vertices $\mathcal{T}^{\circlearrowright}_{n,\iota}$, corresponding to the two possible orientations of the edges, the D possible colorings of every edge and the permutations of all but one of the halfedges touching each vertex (plus a choice d_{i_r} of where to place the cilium on the marked vertices). The number of combinatorial trees with assigned degrees $d_1, \dots d_n$ is $\frac{(n-2)!}{(d_1-1)!\dots(d_n-1)!}$ and we get

$$\Big(\sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} 1\Big)_{n,\iota\ \text{fixed}}$$

$$= (2D)^{n-1}\sum_{\substack{d_1,\dots d_n=1\\ \sum d_i = 2n-2}}^{n}\frac{(n-2)!}{(d_1-1)!\dots(d_n-1)!} d_{i_1}!\dots d_{i_k}!\prod_{i\neq i_k}(d_{i_k}-1)!$$

$$= (2D)^{n-1}(n-2)!\sum_{\substack{d_1,\dots d_n=1\\ \sum d_i = 2n-2}}^{n} d_{i_1}\dots d_{i_k} = (2D)^{n-1}\frac{(2n+k-3)!}{(n+k-1)!}, \tag{78}$$

as the sums over d_i yield the coefficient of the term of degree x^{2n-2} in the expansion of

$$\Big[x\big(\frac{1}{1-x}\big)'\Big]^k \frac{x^{n-k}}{(1-x)^{n-k}} = \frac{x^n}{(1-x)^{n+k}} = x^n\sum_p \binom{n+k+p-1}{p} x^p. \tag{79}$$

For every tree $\mathcal{T}^{\circlearrowright}_{n,\iota}$ one has $k!^D$ trees with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}$,

$$\Big(\sum_{\vec{\tau}} 1\Big)\Big|_{\mathcal{T}^{\circlearrowright}_{n,\iota}\ \text{fixed}} = k!^D. \tag{80}$$

The number of plane trees decorated by loop edges is counted as follows. The walk around $\mathcal{T}^{\circlearrowright}_{n,\iota}$ has $2n-2+k$ steps. The insertion of s loop edges consists in the choice of $2s$ positions to insert new cilia. One can insert a new cilium at every step of the walk. However, with every insertion of a cilium the walk acquires a new step, hence one has

$$[2(n-1)+k][2(n-1)+k+1]\dots[2(n-1)+k+2s-1] = \frac{(2n+2s+k-3)!}{(2n+k-3)!}, \tag{81}$$

ways to connect the $2s$ loop edges on $\mathcal{T}^{\circlearrowright}_{n,\iota}$. Furthermore for each choice, one has D^s possible colorings of the loop edges, hence

$$\Big(\sum_{\mathcal{L},|\mathcal{L}|=s}\sum_{c_1,\dots c_s} 1\Big)\Big|_{\mathcal{T}_{n,\iota,\vec{\tau}}\ \text{fixed}} = D^s\frac{(2n+2s+k-3)!}{(2n+k-3)!}. \tag{82}$$

Finally,

$$\frac{1}{n!}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} 1 = \frac{1}{(n-k)!}. \tag{83}$$

Computing the logarithm of a Gaussian integral. The logarithm of a Gaussian integral can be computed using the universal Brydges–Kennedy–Abdesselam–Rivasseau forest formula [62] and a replica trick.

Lemma 3. *Let X be a complex vector of components $X_1,\dots X_N$ and let I be the Gaussian integral of covariance C defined as*

$$I = \int d\mu_C(X)\ e^{V(\bar{X},X)}, \qquad \int d\mu_C(X)\ X_a\bar{X}_{\bar{b}} = C_{a\bar{b}}. \tag{84}$$

Then

$$\begin{aligned}\ln I = &\sum_{n\geq 0}\frac{1}{n!}\sum_{T_n}\int_0^1\Big(\prod_{(i,j)\in T_n} du^{ij}\Big); \int d\mu_{w^{ij}(T_n,u)C}(X^{(i)})\\ &\times\Big[\prod_{(i,j)\in T_n}\Big(\sum_{a\bar{b}}\frac{\partial}{\partial X^{(i)}_a}C_{a\bar{b}}\frac{\partial}{\partial \bar{X}^{(j)}_{\bar{b}}} + \sum_{a\bar{b}}\frac{\partial}{\partial X^{(j)}_a}C_{a\bar{b}}\frac{\partial}{\partial \bar{X}^{(i)}_{\bar{b}}}\Big)\Big]\\ &\times\prod_{i=1}^{n} V(\bar{X}^{(i)},X^{(i)}),\end{aligned} \tag{85}$$

where T_n runs over all the combinatorial trees with n vertices labelled 1, 2 *up to n, (i,j) denotes the tree edge connecting the vertices i and j, the parameters $w^{ij}(T_n,u)$ are defined as*

$$w^{ii}(T_n,u) = 1, \qquad w^{ij}(T_n,u) = \inf_{(k,l)\in P_{i\to j}(T_n)} u^{kl}, \tag{86}$$

where $P_{i\to j}(T_n)$ denotes the unique path in the tree T_n joining i and j and the interpolated Gaussian measure $d\mu_{w^{ij}(T_n,u)C}(X^{(i)})$ is

$$\int d\mu_{w^{ij}(T_n,u)C}(X^{(i)})\ X_a^{(i)}\bar{X}_{\bar{b}}^{(j)} = w^{ij}(T_n,u)C_{a\bar{b}}. \tag{87}$$

Proof. We Taylor expand in $V(X,\bar{X})$ to get

$$I = \int d\mu_C(X)\ e^{V(\bar{X},X)} = \int d\mu_C(X)\ \sum_{n\geq 0}\frac{1}{n!}\ V(\bar{X},X)^n. \tag{88}$$

The term of degree n can be rewritten as a Gaussian integral over n replicas $X^{(1)}$, $X^{(2)}$ up to $X^{(n)}$ with degenerate covariance between the replicas $C_{a\bar{b}}^{(i,j)} = C_{a\bar{b}}$,

$$I = \sum_{n\geq 0}\frac{1}{n!}\int d\mu_{C_{a\bar{b}}^{(i,j)}}(X^{(i)})\ \ \prod_{i=1}^{n} V(\bar{X}^{(i)},X^{(i)}). \tag{89}$$

Each term in this expansion is a function of parameters $x^{ij} = x^{ji}$, evaluated for $x^{ij} = 1$, corresponding to a Gaussian measure with covariance

$$C_{a\bar{b}}^{(i,i)} = C_{a\bar{b}}, \qquad C_{a\bar{b}}^{(i,j)} = x^{ij}C_{a\bar{b}},\ i\neq j. \tag{90}$$

Consider n vertices labeled $1, 2\dots n$ and a function f depending on $\frac{n(n-1)}{2}$ edge variables x^{ij} with $i\neq j$. The universal Brydges–Kennedy–Abdesselam–Rivasseau forest formula [62] states that

$$f(1,\dots 1) = \sum_{F_n}\int_0^1\Big(\prod_{(i,j)\in F_n} du^{ij}\Big)\left(\frac{\partial^{|\mathcal{E}(F_n)|}f}{\prod_{(i,j)\in F_n}\partial x^{ij}}\right)\Bigg|_{x^{kl}=w^{kl}(F_n,u)},$$
$$w^{kl}\big(F_n,u\big) = \inf_{(i,j)\in\mathcal{P}_{k\to l}(\mathcal{F}_n)} u^{ij}, \tag{91}$$

where F_n runs over all the forests built over the n sites, $|\mathcal{E}(F_n)|$ denotes the number of edges in the forest, $\mathcal{P}_{k\to l}(F_n)$ is the unique path in F_n joining the vertices k and l, and the infimum is set to zero if k and l do not belong to the same tree in the forest. In order to apply the BKAR formula to the term of degree n we evaluate

$$\frac{\partial^{|\mathcal{E}(F_n)|}}{\prod_{(i,j)\in\mathcal{F}_n}\partial x^{ij}}\Big[\int d\mu_{x^{ij}C_{a\bar{b}}}(X^{(i)})\ \ \prod_{i=1}^{n}V(\bar{X}^{(i)},X^{(i)})\Big]$$
$$= \frac{\partial^{|\mathcal{E}(F_n)|}}{\prod_{(i,j)\in\mathcal{F}_n}\partial x^{ij}}\Big[e^{\sum_{a,b,i}\frac{\partial}{\partial X_a^{(i)}}C_{a\bar{b}}\frac{\partial}{\partial\bar{X}_{\bar{b}}^{(i)}}+\sum_{a,b,i\neq j}x^{ij}\frac{\partial}{\partial X_a^{(i)}}C_{a\bar{b}}\frac{\partial}{\partial\bar{X}_{\bar{b}}^{(j)}}}$$
$$\prod_{i=1}^{n}V(\bar{X}^{(i)},X^{(i)})\Big]\Big|_{X^{(i)}=\bar{X}^{(i)}=0}$$
$$= e^{\sum_{a,b,i}\frac{\partial}{\partial X_a^{(i)}}C_{a\bar{b}}\frac{\partial}{\partial\bar{X}_{\bar{b}}^{(i)}}+\sum_{a,b,i\neq j}x^{ij}\frac{\partial}{\partial X_a^{(i)}}C_{a\bar{b}}\frac{\partial}{\partial\bar{X}_{\bar{b}}^{(j)}}}$$

$$
\times\Big[\prod_{(i,j)\in F_n}\Big(\sum_{a\bar b}\frac{\partial}{\partial X_a^{(i)}}C_{a\bar b}\frac{\partial}{\partial \bar X_{\bar b}^{(j)}}+\sum_{a\bar b}\frac{\partial}{\partial X_a^{(j)}}C_{a\bar b}\frac{\partial}{\partial \bar X_{\bar b}^{(i)}}\Big)\Big]
\prod_{i=1}^n V(\bar X^{(i)},X^{(i)})\Big|_{X^{(i)},\bar X^{(i)}=0}
$$
$$
=\int d\mu_{x^{ij}C_{a\bar b}}(X^{(i)})\Big[\prod_{(i,j)\in \mathcal{F}_n}\Big(\sum_{a\bar b}\frac{\partial}{\partial X_a^{(i)}}C_{a\bar b}\frac{\partial}{\partial \bar X_{\bar b}^{(j)}}+\sum_{a\bar b}\frac{\partial}{\partial X_a^{(j)}}C_{a\bar b}\frac{\partial}{\partial \bar X_{\bar b}^{(i)}}\Big)\Big]
\prod_{i=1}^n V(\bar X^{(i)},X^{(i)}), \tag{92}
$$

where we have taken into account that $x^{ij}=x^{ji}$. Thus

$$
I=\sum_{n\geq 0}\frac{1}{n!}\sum_{F_n}\int_0^1\Big(\prod_{(i,j)\in F_n}du^{ij}\Big)\int d\mu_{w^{ij}(F_n,u)C_{a\bar b}}(X^{(i)})
\times\Big[\prod_{(i,j)\in F_n}\Big(\sum_{a\bar b}\frac{\partial}{\partial X_a^{(i)}}C_{a\bar b}\frac{\partial}{\partial \bar X_{\bar b}^{(j)}}+\sum_{a\bar b}\frac{\partial}{\partial X_a^{(j)}}C_{a\bar b}\frac{\partial}{\partial \bar X_{\bar b}^{(i)}}\Big)\Big]
\prod_{i=1}^n V(\bar X^{(i)},X^{(i)}). \tag{93}
$$

The lemma follows by noticing that the Gaussian integral factors over the trees in the forest and recalling that the logarithm of a function which is a sum over forests of contributions factored over the trees is the sum over trees of the tree contribution. □

The most important feature of the BKAR formula is that the matrix $w^{ij}(T_n,u)$ is *positive* [62]. The Gaussian measure is thus well defined and the expectation of any function of $X,\bar X$ is bounded by its supremum.

4.2. Proofs of the theorems. In the reminder of this section we present the proofs of the theorems enunciated in the text.

4.2.1. Proof of the first constructive expansion Theorem 3. The Loop Vertex Expansion of $W(J,\bar J;\lambda,N)$ in Eq. (46) is obtained by combining the Hubbard Stratonovich intermediate field representation and the BKAR formula. We will sometimes drop the bar over the indices of $\bar{\mathbb{T}}$.

Step 1: Hubbard Stratonovich intermediate field representation. For any complex numbers Z_1 and Z_2, $e^{-Z_1Z_2}$ can be represented as a Gaussian integral

$$
\int\frac{d\bar z dz}{2\imath\pi}e^{-z\bar z-zZ_1+\bar z Z_2}\;\underset{\bar z=x-\imath y}{\overset{z=x+\imath y}{=\!=}}\;\int\frac{dxdy}{\pi}e^{-x^2-y^2-x(Z_1-Z_2)-\imath y(Z_1+Z_2)}
=e^{\frac{(Z_1-Z_2)^2}{4}-\frac{(Z_1+Z_2)^2}{4}}=e^{-Z_1Z_2}. \tag{94}
$$

It follows that a term in the quartic perturbation of the measure in Eq. (43) can be represented using $N \times N$ integration variables σ^c_{ab} as

$$\begin{aligned}
&e^{-N^{D-1}\lambda \sum \mathbb{T}_{\vec n}\bar{\mathbb{T}}_{\vec{\bar m}}\mathbb{T}_{\vec m}\bar{\mathbb{T}}_{\vec{\bar n}}\ \delta_{n^c\bar m^c}\delta_{m^c\bar n^c}\prod_{c'\neq c}\delta_{n^{c'}\bar n^{c'}}\delta_{m^{c'}\bar m^{c'}}} \\
&= e^{-N^{D-1}\lambda \sum \delta_{n^c\bar m^c}\delta_{m^c\bar n^c}\left(\sum \mathbb{T}_{\vec n}\bar{\mathbb{T}}_{\vec{\bar n}}\prod_{c'\neq c}\delta_{n^{c'}\bar n^{c'}}\right)\left(\sum \bar{\mathbb{T}}_{\vec{\bar m}}\mathbb{T}_{\vec m}\prod_{c'\neq c}\delta_{m^{c'}\bar m^{c'}}\right)} \\
&= \int \Big(\prod_{ab}\frac{d\sigma^c_{ab}d\bar\sigma^c_{ab}}{2\pi\imath}\Big)e^{-\sum_{ab}\sigma^c_{ab}\bar\sigma^c_{ab}} \\
&e^{-\sqrt{\lambda}N^{\frac{D-1}{2}}\sum\left(\mathbb{T}_{\vec n}\bar{\mathbb{T}}_{\vec{\bar n}}\prod_{c'\neq c}\delta_{n^{c'}\bar n^{c'}}\right)\sigma^c_{n^c\bar n^c}+\sqrt{\lambda}N^{\frac{D-1}{2}}\sum\left(\bar{\mathbb{T}}_{\vec{\bar m}}\mathbb{T}_{\vec m}\prod_{c'\neq c}\delta_{m^{c'}\bar m^{c'}}\right)\bar\sigma^c_{\bar m^c m^c}}.
\end{aligned} \tag{95}$$

The new integration variables σ^c_{ab} form D matrices of size $N\times N$, known as intermediate fields. Denoting 1 the identity matrix of size $N \times N$ we write more compactly

$$\begin{aligned}
&e^{-N^{D-1}\lambda\sum_{c=1}^D\sum_{n\bar n}\mathbb{T}_{\vec n}\bar{\mathbb{T}}_{\vec{\bar m}}\mathbb{T}_{\vec m}\bar{\mathbb{T}}_{\vec{\bar n}}\ \delta_{n^c\bar m^c}\delta_{m^c\bar n^c}\prod_{c'\neq c}\delta_{n^{c'}\bar n^{c'}}\delta_{m^{c'}\bar m^{c'}}} \\
&= \int\Big(\prod_{cab}\frac{d\sigma^c_{ab}d\bar\sigma^c_{ab}}{2\pi\imath}\Big)\ e^{-\mathrm{tr}(\sigma^c\sigma^{c\dagger})-N^{D-1}\sqrt{\frac{\lambda}{N^{D-1}}}\sum\mathbb{T}_{\vec n}\left(\sum_c 1^{\otimes c-1}\otimes(\sigma^c-\sigma^{c\dagger})\otimes 1^{\otimes D-c}\right)\bar{\mathbb{T}}_{\vec{\bar n}}},
\end{aligned} \tag{96}$$

thus $Z(J,\bar J;\lambda,N)$ becomes

$$\begin{aligned}
Z(J,\bar J;\lambda,N) &= \int\Big(\prod N^{D-1}\frac{d\mathbb{T}_{\vec n}d\bar{\mathbb{T}}_{\vec n}}{2\pi\imath}\Big)\Big(\prod\frac{d\sigma^c_{ab}d\bar\sigma^c_{ab}}{2\pi\imath}\Big)\ e^{-\sum_c\mathrm{tr}(\sigma^c\sigma^{c\dagger})} \\
&\times e^{-N^{D-1}\sum\mathbb{T}_{\vec n}\left(1^{\otimes D}+\sqrt{\frac{\lambda}{N^{D-1}}}\sum_c 1^{\otimes c-1}\otimes(\sigma^c-\sigma^{c\dagger})\otimes 1^{\otimes D-c}\right)\bar{\mathbb{T}}_{\vec{\bar n}}+\sum\bar{\mathbb{T}}_{\vec{\bar n}}J_{\vec{\bar n}}+\sum\mathbb{T}_{\vec n}\bar J_{\vec n}}.
\end{aligned} \tag{97}$$

As $\sigma-\sigma^\dagger$ is anti hermitian the operator $R(\sigma)=\Big[1^{\otimes D}+\sqrt{\frac{\lambda}{N^{D-1}}}\sum_c 1^{\otimes c-1}\otimes(\sigma^c-\sigma^{c\dagger})\otimes 1^{\otimes D-c}\Big]^{-1}$, which we call the resolvent, is well defined. The intermediate field representation renders the integration over $\mathbb{T}\bar{\mathbb{T}}$ Gaussian, thus

$$Z(J,\bar J;\lambda,N)=\int\Big(\prod_{c;ab}\frac{d\sigma^c_{ab}d\bar\sigma^c_{ab}}{2\pi\imath}\Big)e^{-\sum_c\mathrm{tr}\sigma^c\sigma^{c\dagger}+\mathrm{tr}\ln\left(R(\sigma)\right)+\frac{1}{N^{D-1}}\langle\bar J|R(\sigma)|J\rangle}, \tag{98}$$

where $\langle\bar J|R(\sigma)|J\rangle=\sum_{\vec n,\vec m}\bar J_{\vec n}R(\sigma)_{\vec n\vec m}J_{\vec m}$.

Step 2: Extracting the logarithm. Using lemma 3 the generating function of the connected moments $W(J,\bar J;\lambda,N)$ is

$$\begin{aligned}
W(J,\bar J;\lambda,N) &= \sum_{n\geq 1}\frac{1}{n!}\sum_{T_n}\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma) \\
&\times\Big[\prod_{(i,j)\in T_n}\sum_{c=1}^D\Big(\sum_{ab}\frac{\partial}{\partial\sigma^{(i)c}_{ab}}\frac{\partial}{\partial\sigma^{(j)c\dagger}_{ba}}+\sum_{ab}\frac{\partial}{\partial\sigma^{(j)c}_{ab}}\frac{\partial}{\partial\sigma^{(i)c\dagger}_{ba}}\Big)\Big] \\
&\times\prod_{i=1}^n\Big\{\mathrm{tr}\ln\big[R(\sigma^{(i)})\big]+\frac{1}{N^{D-1}}\langle\bar J|R(\sigma^{(i)})|J\rangle\Big\},
\end{aligned} \tag{99}$$

where the interpolated Gaussian measure writes formally as

$$d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma) = e^{\sum_{c=1}^{D}\left(\sum_i \sum_{ab}\left[\frac{\partial}{\partial\sigma_{ab}^{(i)c}}\frac{\partial}{\partial\sigma_{ba}^{(i)c\dagger}}+\frac{\partial}{\partial\sigma_{ab}^{(i)c}}\frac{\partial}{\partial\sigma_{ba}^{(i)c\dagger}}\right]+\sum_{i<j}x^{ij}\sum_{ab}\left[\frac{\partial}{\partial\sigma_{ab}^{(i)c}}\frac{\partial}{\partial\sigma_{ba}^{(j)c\dagger}}+\frac{\partial}{\partial\sigma_{ab}^{(j)c}}\frac{\partial}{\partial\sigma_{ba}^{(i)c\dagger}}\right]\right)}. \tag{100}$$

Expanding the product over i, we get

$$\begin{aligned} W(J,\bar{J};\lambda,N) &= \sum_{n\geq 1}\frac{1}{n!}\sum_{T_n}\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma^{(i)_c}) \\ &\times\Big[\prod_{(i,j)\in T_n}\sum_{c=1}^{D}\Big(\sum_{ab}\frac{\partial}{\partial\sigma_{ab}^{(i)_c}}\frac{\partial}{\partial\sigma_{ba}^{(j)_c\dagger}}+\sum_{ab}\frac{\partial}{\partial\sigma_{ab}^{(j)_c}}\frac{\partial}{\partial\sigma_{ba}^{(i)_c\dagger}}\Big)\Big] \\ &\times\sum_{k=0}^{n}\frac{1}{k!}\frac{1}{N^{k(D-1)}}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\langle\bar{J}|R(\sigma^{(i_1)})|J\rangle\dots\langle\bar{J}|R(\sigma^{(i_k)})|J\rangle\prod_{\substack{i=1\\ i\neq i_1,\dots i_k}}^{n}\operatorname{tr}\ln\big[R(\sigma^{(i)})\big]. \end{aligned} \tag{101}$$

We need to evaluate the action of the derivative operators on the product resolvents. This is done in Lemma 4 below. For each combinatorial tree T_n one obtains a sum over all the plane trees $\mathcal{T}_{n,\iota}^{\circlearrowright}$ with colored oriented edges compatible with it, and indexing the sum by these plane trees we get

$$\begin{aligned} W(J,\bar{J};\lambda,N) &= \sum_{n\geq 1}\frac{1}{n!}(-\lambda)^{n-1}\sum_{k=0}^{n}\frac{1}{k!}\frac{1}{N^{(k+n-1)(D-1)}}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\sum_{\mathcal{T}_{n,\iota}^{\circlearrowright}} \\ &\times\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)\int\Big(\prod_{q=1}^{2n-2+k}d\alpha_q\Big)\,e^{-\sum_{q=1}^{2n-2+k}\alpha_q} \\ &\prod_{f^c\in\mathcal{F}^{\text{int}}(\mathcal{T}_{n,\iota}^{\circlearrowright})}\operatorname{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}}e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)_c}-\sigma^{i(q)_c\dagger})}\Big]\sum_{pn}\prod_{l=1}^{k}\bar{J}_{p_l^1,\dots p_l^D}J_{n_l^1,\dots n_l^D} \\ &\prod_{f^{c;i(q_l)\to i(q_{\xi_c(l)})}\in\mathcal{S}^{\text{ext}}(\mathcal{T}_{n,\iota}^{\circlearrowright})}\Big[\overrightarrow{\prod_{q\in q(f^{c;i(q_l)\to i(q_{\xi_c(l)})})}}e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)_c}-\sigma^{i(q)_c\dagger})}\Big]_{p_l^c n_{\xi_c(l)}^c}, \end{aligned} \tag{102}$$

which proves the theorem. □

Lemma 4. *The contribution of a tree T_n with marked vertices $i_1, \dots i_k$ is*

$$\Big[\prod_{(i,j)\in T_n} \sum_{c=1}^{D} \Big(\sum_{ab} \frac{\partial}{\partial \sigma^{(i)_c}_{ab}} \frac{\partial}{\partial \sigma^{(j)_c \dagger}_{ba}} + \sum_{ab} \frac{\partial}{\partial \sigma^{(j)_c}_{ab}} \frac{\partial}{\partial \sigma^{(i)_c \dagger}_{ba}} \Big) \Big]$$

$$\times \langle \bar J | R(\sigma^{(i_1)}) | J \rangle \dots \langle \bar J | R(\sigma^{(i_k)}) | J \rangle \prod_{\substack{i=1 \\ i\neq i_1,\dots i_k}}^{n} \operatorname{tr} \ln \big[R(\sigma^{(i)}) \big]$$

$$= \frac{(-\lambda)^{n-1}}{N^{(n-1)(D-1)}} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} \int \Big(\prod_{q=1}^{2n-2+k} d\alpha_q \Big) e^{-\sum_{q=1}^{2n-2+k} \alpha_q}$$

$$\prod_{f^c \in \mathcal{F}^{\rm int}(\mathcal{T}^{\circlearrowright}_{n,\iota})} \operatorname{Tr} \Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big] \sum_{pn} \prod_{l=1}^{k} \bar J_{p^1_l,\dots p^D_l} J_{n^1_l,\dots n^D_l}$$

$$\prod_{f^{c;i(q_l)\to i(q_{\xi_c(l)})} \in \mathcal{S}^{\rm ext}(\mathcal{T}^{\circlearrowright}_{n,\iota})} \Big[\prod_{q\in q(f^{c;i(q_l)\to i(q_{\xi_c(l)})})}^{\rightarrow} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big]_{p^c_l n^c_{\xi_c(l)}}, \qquad (103)$$

where the sum runs over all the plane trees with colored oriented edges and ciliated vertices $i_1, \dots i_k$ which reduce to the combinatorial tree T_n.

Proof. We orient the edge (i, j) of the tree from i to j for the term $\partial_{\sigma^{(i)}} \partial_{\sigma^{(j)\dagger}}$. Taking into account the sum over c we obtain the sum over trees with colored, oriented edges. In order to compute the contribution of each such tree we need to evaluate the action of the derivative operators on the product of traces.

We set $M(\sigma^{(i)}) = [R(\sigma^{(i)})]^{-1} = 1^{\otimes D} + \sqrt{\frac{\lambda}{N^{D-1}}} \sum_{c=1}^{D} 1^{\otimes c-1} \otimes (\sigma^{(i)_c} - \sigma^{(i)_c\dagger}) \otimes 1^{\otimes D-c}$ and we represent the resolvents with the help of a new parameter

$$R(\sigma^{(i)}) = \int_0^\infty d\alpha \quad e^{-\alpha} \bigotimes_{c=1}^{D} e^{-\alpha \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(i)_c} - \sigma^{(i)_c\dagger})}. \qquad (104)$$

Note that, denoting $M_{\hat p \hat q}$ the minor of M with the line p and column q deleted, we have $\frac{\partial}{\partial m_{pq}} \operatorname{tr} \ln M = \frac{\partial}{\partial m_{pq}} \ln \det M = \frac{1}{\det M} \frac{\partial}{\partial m_{pq}} \det M = \frac{1}{\det M} (-1)^{p+q} M_{\hat p \hat q} = [M^{-1}]_{qp}$, hence the derivatives of the vertices are

$$\frac{\partial}{\partial \sigma^{(i)_c}_{ab}} \operatorname{tr} \ln \big[R(\sigma^{(i)}) \big] = - \frac{\partial}{\partial \sigma^{(i)_c}_{ab}} \operatorname{tr} \ln M(\sigma^{(i)}) = - \sum_{\vec n \vec p} \frac{\partial m_{\vec n \vec p}}{\partial \sigma^{(i)_c}_{ab}} \frac{\partial}{\partial m_{\vec n \vec p}} \operatorname{tr} \ln M(\sigma^{(i)})$$

$$= - \sqrt{\frac{\lambda}{N^{D-1}}} \sum_{\vec n \vec p} \Big(\delta_{n^c a} \delta_{p^c b} \prod_{c'\neq c} \delta_{n^{c'} p^{c'}} \Big) R(\sigma^{(i)})_{\vec p \vec n}$$

$$= - \sqrt{\frac{\lambda}{N^{D-1}}} \int_0^\infty d\alpha \quad e^{-\alpha} \Big[e^{-\alpha \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(i)_c} - \sigma^{(i)_c\dagger})} \Big]_{ab}$$

$$\times \prod_{c'\neq c} \operatorname{Tr} \Big[e^{-\alpha \sqrt{\frac{\lambda}{N^{D-1}}} (\sigma^{(i)_{c'}} - \sigma^{(i)_{c'}\dagger})} \Big],$$

$$\frac{\partial}{\partial\sigma_{ba}^{(i)_c\dagger}}\operatorname{tr}\ln\left[R(\sigma^{(i)})\right]$$
$$=\sqrt{\frac{\lambda}{N^{D-1}}}\int_0^\infty d\alpha\quad e^{-\alpha}\left[e^{-\alpha\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{(i)_c}-\sigma^{(i)_c\dagger})}\right]_{ba}$$
$$\times\prod_{c'\neq c}\operatorname{Tr}\left[e^{-\alpha\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{(i)_{c'}}-\sigma^{(i)_{c'}\dagger})}\right]. \tag{105}$$

It follows that a tree made of one line of color c connecting two non ciliated vertices i and j will yield a contribution

$$\frac{(-\lambda)}{N^{D-1}}\int_0^\infty d\alpha_1 d\alpha_2\quad e^{-\alpha_1-\alpha_2}\operatorname{Tr}\left[e^{-\alpha_1\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{(i)_c}-\sigma^{(i)_c\dagger})}e^{-\alpha_2\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{(j)_c}-\sigma^{(j)_c\dagger})}\right]$$
$$\times\prod_{c'\neq c}\operatorname{Tr}\left[e^{-\alpha_1\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{(i)_{c'}}-\sigma^{(i)_{c'}\dagger})}\right]\prod_{c'\neq c}\operatorname{Tr}\left[e^{-\alpha_2\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{(j)_{c'}}-\sigma^{(j)_{c'}\dagger})}\right], \tag{106}$$

which reproduces the Eq. (103) for the tree with two vertices labelled i and j connected by one line. Note that this tree has $2D-2$ internal faces of colors $c'\neq c$ and one internal face of color c. On the other hand, a ciliated vertex writes as

$$\langle\bar{J}|R(\sigma^{(i_1)})|J\rangle=\int_0^\infty d\alpha\;\; e^{-\alpha}\sum_{pn}\prod_{c=1}^{D}\left[e^{-\alpha\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{(i_1)_c}-\sigma^{(i_1)_c\dagger})}\right]_{p^c n^c}\quad\bar{J}_{\vec{p}}J_{\vec{n}}, \tag{107}$$

reproducing Eq. (103) for the tree with a unique ciliated vertex i_1.

The proof proceeds by induction on the number of vertices. Note that any tree can be obtained by adding one by one its edges. At each step two trees are joined by the new edges. As the lemma holds for the initial trees (as they have less vertices), when evaluating the derivative with respect to $\sigma^{(i)}$ and $\sigma^{(j)\dagger}$ one obtains a sum over terms, one for each occurrence of $\sigma^{(i)}$ and $\sigma^{(j)\dagger}$, i.e. a sum over all possible ways to join the two plane trees together into a plane tree with n vertices. Lemma 2 shows that the right hand side of Eq. (103) is reproduced. □

4.2.2. Proof of the second constructive expansion Theorem 4. We add D fictitious integral over the unitary group $U(N)$, i.e. we write

$$W(J,\bar{J};\lambda,N)=\int_{U(N)}[dU^1]\dots\int[dU^D]\;W(J,\bar{J};\lambda,N), \tag{108}$$

which of course holds as $\int_{U(N)}[dU]=1$. Now, for all fixed U^c, we perform the change of variables of Jacobian 1, $\sigma^{(i)_c}\to U^{c\dagger}\sigma^{(i)_c}U^c$. in Eq. (102). The Gaussian measure is invariant under this change of variables, hence

$$W(J,\bar{J};\lambda,N)=\sum_{n\geq 1}\frac{1}{n!}(-\lambda)^{n-1}\sum_{k=0}^{n}\frac{1}{k!}\frac{1}{N^{(k+n-1)(D-1)}}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\sum_{\mathcal{T}_{n,\iota}^{\circlearrowright}}$$
$$\times\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)\int\Big(\prod_{q=1}^{2n-2+k}d\alpha_q\Big)\,e^{-\sum_{q=1}^{2n-2+k}\alpha_q}$$

$$
\times \prod_{f^c \in \mathcal{F}^{\text{int}}(\mathcal{T}^{\circlearrowright}_{n,\iota})} \text{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big]
$$
$$
\times \sum_{pn,uv} \prod_{l=1}^{k} \Big(\bar{J}_{p_l^1,\dots p_l^D} J_{n_l^1,\dots n_l^D} \prod_{c=1}^{D} U^{c\dagger}_{p_l^c u^c(q_l)} U^{c}_{v^c(q_{\xi_c(l)}) n^c_{\xi_c(l)}} \Big)
$$
$$
\times \prod_{f^{c;i(q_l)\to i(q_{\xi_c(l)})} \in \mathcal{S}^{\text{ext}}(\mathcal{T}^{\circlearrowright}_{n,\iota})} \Big[\overrightarrow{\prod_{q \in q(f^{c;i(q_l)\to i(q_{\xi_c(l)})})}} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big]_{u^c(q_l) v^c(q_{\xi_c(l)})}, \tag{109}
$$

where the indices $u^c(q_l)$ and $v^c(q_{\xi_c(l)})$ are summed. The integral over the unitary group of a product of matrix elements is, according to Eq. (39) (see [54,55] for details)

$$
\int_{U(N)} [dU^c] \prod_{l=1}^{k} U^c_{v^c(q_{\xi_c(l)}) n^c_{\xi_c(l)}} U^{c\dagger}_{p_l^c u^c(q_l)}
$$
$$
= \sum_{\sigma_c,\tau_c} \text{Wg}(N, \tau_c \sigma_c^{-1}) \prod_{l=1}^{k} \delta_{v^c(q_{\xi_c(l)}) u^c(q_{\tau_c(l)})} \delta_{n^c_{\xi_c(l)} p^c_{\sigma_c(l)}}
$$
$$
= \sum_{\sigma_c,\tau_c} \text{Wg}(N, \tau_c \sigma_c^{-1}) \prod_{l=1}^{k} \delta_{v^c(q_l) u^c(q_{\tau_c(l)})} \delta_{n^c_l p^c_{\sigma_c(l)}}, \tag{110}
$$

where σ_c and τ_c run over all the permutations of k elements (and in the second line we shifted both σ_c and τ_c by the permutation ξ_c^{-1}). We obtain

$$
W(J, \bar{J}; \lambda, N) = \sum_{n\geq 1} \frac{1}{n!} (-\lambda)^{n-1} \sum_{k=0}^{n} \frac{1}{k!} \frac{1}{N^{(k+n-1)(D-1)}} \sum_{\substack{i_1, i_2 \dots, i_k = 1 \\ i_d \neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}}
$$
$$
\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij} \Big) \int d\mu_{w^{ij}(T_n,u) 1^{\otimes D}}(\sigma) \int \Big(\prod_{q=1}^{2n-2+k} d\alpha_q \Big) e^{-\sum_{q=1}^{2n-2+k} \alpha_q}
$$
$$
\times \sum_{\vec{\sigma},\vec{\tau}} \Big(\prod_{c=1}^{D} \text{Wg}(N, \tau_c \sigma_c^{-1}) \Big)
$$
$$
\times \prod_{f^c \in \mathcal{F}^{\text{int}}(\mathcal{T}^{\circlearrowright}_{n,\iota})} \text{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big]
$$
$$
\times \sum_{pn,uv} \Big(\prod_{l=1}^{k} \prod_{c=1}^{D} \delta_{v^c(q_l) u^c(q_{\tau_c(l)})} \Big) \prod_{l=1}^{k} \Big(\bar{J}_{p_l^1,\dots p_l^D} J_{n_l^1,\dots n_l^D} \prod_{c=1}^{D} \delta_{n^c_l p^c_{\sigma_c(l)}} \Big)
$$
$$
\times \prod_{f^{c;i(q_l)\to i(q_{\xi_c(l)})} \in \mathcal{S}^{\text{ext}}(\mathcal{T}^{\circlearrowright}_{n,\iota})} \Big[\overrightarrow{\prod_{q \in q(f^{c;i(q_l)\to i(q_{\xi_c(l)})})}} e^{-\alpha_q \sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)c} - \sigma^{i(q)c\dagger})} \Big]_{u^c(q_l) v^c(q_{\xi_c(l)})}. \tag{111}
$$

The external sources group into trace invariants: each permutation $\vec{\sigma}$ is one to one with a D-colored graph $\mathcal{B}_{\vec{\sigma}}$ with $2k$ labelled vertices. The external strands recompose along the faces of the plane tree with external edges $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}$ to yield

$$
\begin{aligned}
W(J,\bar{J};\lambda,N) &= \sum_{n\geq 1}\frac{1}{n!}(-\lambda)^{n-1}\sum_{k=0}^{n}\frac{1}{k!}\frac{1}{N^{(k+n-1)(D-1)}}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}}\\
&\times\sum_{\vec{\sigma},\vec{\tau}}\Big(\prod_{c=1}^{D}\mathrm{Wg}(N,\tau_c\sigma_c^{-1})\Big)\ \mathrm{Tr}_{\mathcal{B}_{\vec{\sigma}}}(J,\bar{J})\\
&\times\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)\int\Big(\prod_{q=1}^{2n-2+k}d\alpha_q\Big)\,e^{-\sum_{q=1}^{2n-2+k}\alpha_q}\\
&\times\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})}\mathrm{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}}e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)_c}-\sigma^{i(q)_c\dagger})}\Big].
\end{aligned}
\tag{112}
$$

It follows that the cumulants of the measure $\mu^{(4)}$ are sums over graphs $\mathcal{B}$

$$
\begin{aligned}
\kappa\big(\mathbb{T}_{\vec{p}_1},\bar{\mathbb{T}}_{\vec{n}_1},\dots\mathbb{T}_{\vec{p}_k},\bar{\mathbb{T}}_{\vec{n}_k}\big) &= \frac{\partial^{(2k)}}{\partial\bar{J}_{\vec{p}_1}\partial J_{\vec{n}_1}\dots\partial\bar{J}_{\vec{p}_k}\partial J_{\vec{n}_k}}W(J,\bar{J};\lambda,N)\Big|_{J=\bar{J}=0}\\
&= \sum_{\mathcal{B},\ k(\mathcal{B})=k}\mathfrak{K}(\mathcal{B},\mu_N^{(4)})\prod_{\rho=1}^{C(\mathcal{B})}\delta^{\mathcal{B}_\rho}_{n\bar{n}},
\end{aligned}
\tag{113}
$$

where, denoting the (unique) permutation $\vec{\sigma}$ associated to the D-colored graph $\mathcal{B}$ with labelled vertices by $\vec{\sigma}(\mathcal{B})$, we have

$$
\begin{aligned}
\mathfrak{K}(\mathcal{B},\mu_N^{(4)}) &= \sum_{n\geq k}\frac{1}{n!}(-\lambda)^{n-1}\frac{1}{N^{(k+n-1)(D-1)}}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}}\sum_{\vec{\tau}}\\
&\times k!\Big(\prod_{c=1}^{D}\mathrm{Wg}(N,\tau_c\sigma_c^{-1}(\mathcal{B}))\Big)\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)\\
&\times\int\Big(\prod_{q=1}^{2n-2+k}d\alpha_q\Big)\,e^{-\sum_{q=1}^{2n-2+k}\alpha_q}\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})}\mathrm{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}}e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)_c}-\sigma^{i(q)_c\dagger})}\Big],
\end{aligned}
\tag{114}
$$

and Theorem 4 holds. □

4.2.3. Proof of the absolute convergence Theorem 5. The operator $\sigma^{i(q)_c}-\sigma^{i(q)_c\dagger}$ is an anti-Hermitian operator hence, denoting $||\cdot||$ the operator norm, $\Big|\Big|e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)_c}-\sigma^{i(q)_c\dagger})}\Big|\Big|\leq 1$ and

$$\Big|\mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]\Big| \le N\Big\|\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big\|$$
$$\le N\prod_{q\in q(f^c)}^{\rightarrow}\Big\|e^{-\alpha_q\sqrt{\frac{\lambda}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big\| \le N. \tag{115}$$

The integrals over α_q and u^{ij} are bounded by 1 as well as the Gaussian integral (as $\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$ is normalized and positive). The Weingarten function is bounded, from Eq. (41), by $\mathrm{Wg}(N,\tau_c\sigma_c^{-1}(\mathcal{B}))\le\frac{2^{2k}}{N^{2k-C(\tau_c\sigma_c^{-1}(\mathcal{B}))}}$ and we get a bound

$$|\mathfrak{K}(\mathcal{B},\mu_N^{(4)})| \le \sum_{n\ge k}\frac{1}{n!}|\lambda|^{n-1}\frac{1}{N^{(k+n-1)(D-1)}}\sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n}\sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}}\sum_{\vec\tau}$$
$$\times k!2^{2Dk}N^{-2Dk+\sum_c C(\tau_c\sigma_c^{-1}(\mathcal{B}))+\left|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau})\right|} \tag{116}$$

The total scaling with N is therefore, using Eq. (22),

$$-(k+n-1)(D-1)-2Dk+\sum_c C(\tau_c\sigma_c^{-1}(\mathcal{B}))$$
$$+D+(n-1)(D-1)-\sum_c C(\xi_c)+\sum_c C(\tau_c\xi_c)$$
$$= D-2Dk-k(D-1)+\sum_c C(\tau_c\sigma_c^{-1}(\mathcal{B}))-\sum_c C(\xi_c)+\sum_c C(\tau_c\xi_c), \tag{117}$$

which is bounded from Lemma 1 by

$$D-2Dk-k(D-1)+(D+1)k-C(\mathcal{B}_{\vec\sigma}) = D-2(D-1)k-C(\mathcal{B}_{\vec\sigma}). \tag{118}$$

Using the bounds in Eqs. (78) and (83) we get

$$|\mathfrak{K}(\mathcal{B},\mu_N^{(4)})| \le N^{D-2(D-1)k-C(\mathcal{B}_{\vec\sigma})}k!^{D+1}2^{2Dk}$$
$$\sum_{n\ge k}|\lambda|^{n-1}(2D)^{n-1}\frac{(2n+k-3)!}{(n-k)!(n+k-1)!}. \tag{119}$$

A (not tight) bound on the combinatorial factor is

$$\frac{(2n+k-3)!}{(n-k)!(n+k-1)!} \le (2n+k-3)^{k-2}\frac{(2n-1)!}{(n-k)!(n+k-1)!} \le (3n)^{k-2}2^{2n}, \tag{120}$$

and we get

$$|\mathfrak{K}(\mathcal{B},\mu_N^{(4)})| \le N^{D-2k(D-1)-C(\mathcal{B})}k!^{D+1}2^{2Dk}3^{k-2}\sum_{n\ge k}n^{k-2}|8D\lambda|^{n-1}. \tag{121}$$

The series is absolutely convergent for $|\lambda| < 2^{-3}D^{-1}$ and the cumulants are bounded by

$$|\mathfrak{K}(\mathcal{B},\mu_N^{(4)})| \le N^{D-2k(D-1)-C(\mathcal{B})}|\lambda|^{k-1}K(\mathcal{B}), \tag{122}$$

for some constant $K(\mathcal{B})$ independent of N (and independent of λ for $|\lambda|$ small enough). □

4.2.4. Proof of Corollary 1. Consider first the simpler problem of proving the function of a complex variable $z = re^{\imath\varphi}$ defined as

$$f(z) = \int_0^\infty e^{-\imath\frac{\varphi}{2}} d\alpha \; e^{-\alpha e^{-\imath\frac{\varphi}{2}} - \imath\alpha r^{1/2} x} \tag{123}$$

is analytic for $\varphi \in (-\pi, \pi)$. First it is easy to see that f is bounded as $|f(z)| < \frac{1}{\cos\frac{\varphi}{2}} < \infty$. Second, f respects the Cauchy–Riemann equations

$$\begin{aligned} r\frac{\partial}{\partial r} f(z) &= \int_0^\infty d\alpha \; e^{-\imath\frac{\varphi}{2}} e^{-\alpha e^{-\imath\frac{\varphi}{2}}} \frac{1}{2}\alpha \frac{d}{d\alpha} e^{-\imath\alpha r^{1/2} x} \\ &= \int_0^\infty d\alpha \; e^{-\imath\frac{\varphi}{2}} \frac{1}{2}\frac{d}{d\alpha}\Big(\alpha e^{-\alpha e^{-\imath\frac{\varphi}{2}}}\Big) e^{-\imath\alpha r^{1/2} x} \\ &= \int_0^\infty d\alpha \; e^{-\imath\frac{\varphi}{2}} \frac{1}{2}\Big(e^{-\alpha e^{-\imath\frac{\varphi}{2}}} - \alpha e^{-\imath\frac{\varphi}{2}} e^{-\alpha e^{-\imath\frac{\varphi}{2}}}\Big) e^{-\imath\alpha r^{1/2} x} = \imath\frac{\partial}{\partial\varphi} f, \end{aligned} \tag{124}$$

hence it is analytic. For our case similar partial integration with respect to α shows that each function

$$\begin{aligned} \mathfrak{T}^{\mathfrak{E}}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}) = k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big) \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma) \\ \times \int \Big(\prod_{q=1}^{2n-2+k} e^{-\imath\frac{\varphi}{2}} d\alpha_q\Big) e^{-\sum_{q=1}^{2n-2+k} e^{-\imath\frac{\varphi}{2}}\alpha_q} \\ \times \prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau})} \mathrm{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c} - \sigma^{i(q)c\dagger})}\Big], \end{aligned} \tag{125}$$

respects the Cauchy–Riemann equations and, being bounded, is analytic in λ. It follows that the full cumulant

$$\mathfrak{K}(\mathcal{B}, \mu_N^{(4)}) = \sum_{n\geq k}\frac{1}{n!}(-\lambda)^{n-1}\frac{1}{N^{(k+n-1)(D-1)}} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}}\sum_{\vec\tau} \mathfrak{T}^{\mathfrak{E}}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}), \tag{126}$$

which is a sum of products of analytic functions, is analytic whenever the sum over n converges. Reproducing step by step the proof of Theorem 5 but taking into account that the integrals over α are bounded by $\frac{1}{\cos\frac{\varphi}{2}}$ instead of 1, we obtain that the series in Eq. (126) is absolutely convergent for

$$|\lambda| < \big(\cos\frac{\varphi}{2}\big)^2 2^{-3} D^{-1}. \tag{127}$$

□

4.2.5. Proof of the mixed expansion Theorem 6. Consider the contribution of a tree with external edges

$$T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)}\; k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N,\tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)$$
$$\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$$
$$\times \int \Big(\prod_{q=1}^{2n-2+k} e^{-\iota\frac{\varphi}{2}} d\alpha_q\Big)\; e^{-\sum_{q=1}^{2n-2+k} e^{-\iota\frac{\varphi}{2}}\alpha_q}$$
$$\times \prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]. \tag{128}$$

We use a Taylor expansion

$$f(\sqrt{|\lambda|}e^{\iota\frac{\varphi}{2}}) = \sum_{q=0}^{s-1}\frac{1}{q!}\Big[\frac{d^q}{dt^q} f(\sqrt{t|\lambda|}e^{\iota\frac{\varphi}{2}})\Big]_{t=0}$$
$$+\frac{1}{(s-1)!}\int_0^1 (1-t)^{s-1}\frac{d^s}{dt^s}\Big(f(\sqrt{t|\lambda|}e^{\iota\frac{\varphi}{2}})\Big)dt. \tag{129}$$

We must evaluate the derivative with respect to t acting on $T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})$. In each operator $\sqrt{|\lambda|}$ multiplies a difference $\sigma-\sigma^\dagger$ and the derivative with respect to t acting on an exponential computes to

$$\frac{d}{dt}\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{t}\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]$$
$$=\sum_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})}\;\prod_{f^{c'}\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}),\; f^{c'}\neq f^c} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{t}\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]$$
$$\times\sum_{q\in q(f^c)}\frac{1}{2t}\sum_{ab}\Big(\sigma^{i(q)_c}_{ab}\frac{\partial}{\partial\sigma^{i(q)_c}_{ab}}+\sigma^{i(q)_c\dagger}_{ba}\frac{\partial}{\partial\sigma^{i(q)_c\dagger}_{ba}}\Big)$$
$$\times\mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{t}\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]$$
$$=\sum_{i=1}^{n}\sum_{c=1}^{D}\frac{1}{2t}\sum_{ab}\Big(\sigma^{(i)_c}_{ab}\frac{\partial}{\partial\sigma^{(i)_c}_{ab}}+\sigma^{(i)_c\dagger}_{ba}\frac{\partial}{\partial\sigma^{(i)_c\dagger}_{ba}}\Big)$$
$$\times\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{t}\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big], \tag{130}$$

Integrating by parts the Gaussian integral we get

$$\frac{d}{dt}\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma) \prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau})} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q t\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]$$

$$= \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$$

$$\times\Big[\sum_{i,j=1}^{n} w^{ij}(T_n,u)\sum_{c=1}^{D}\frac{1}{2t}\sum_{ab}\Big(\frac{\partial}{\partial\sigma^{(j)c\dagger}_{ba}}\frac{\partial}{\partial\sigma^{(i)c}_{ab}}+\frac{\partial}{\partial\sigma^{(j)c}_{ab}}\frac{\partial}{\partial\sigma^{(i)c\dagger}_{ba}}\Big)\Big]$$

$$\times\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau})} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q t\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]. \tag{131}$$

The action of the derivatives on a product of traces has been evaluated in Lemma 4. The sums over i, j and c yields a sum over all the possible ways to add a colored loop edge to the tree $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}$. The two new auxiliary parameters β_1 and γ_1 (which we relabel as two supplementary α parameters) don't have any $e^{-\iota\frac{\varphi}{2}}$ in the measure, hence we need to explicitly add them. Thus we get

$$\sum_{j_1j'_1,c}\frac{1}{2t}\Big(-\frac{t|\lambda|e^{\iota\varphi}}{N^{D-1}}\Big)2w^{j_1j'_1}(T_n,u)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$$

$$\times\int\Big(\prod_{q=1}^{2n-2+k+2}e^{-\iota\frac{\varphi}{2}}d\alpha_q\Big)\,e^{-\sum_{q=1}^{2n-2+k+2}e^{-\iota\frac{\varphi}{2}}\alpha_q}$$

$$\times\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau,\{j_1j'_1\}})} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]. \tag{132}$$

Note that both derivative terms lead to the same contribution (the loop edge does not have any orientation) hence

$$\frac{d}{dt}T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau}) = N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)}\,k!\Big(\prod_{c=1}^{D}\mathrm{Wg}\big(N,\tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)$$

$$\times\int_0^1\Big(\prod_{(i,j)\in T_n}du^{ij}\Big)\int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)\sum_{\mathcal{L}=\{(j_1j'_1)\};c_1}\Big(-\frac{\lambda}{N^{D-1}}\Big)w^{j_1j'_1}(T_n,u)$$

$$\times\int\Big(\prod_{q=1}^{2n-2+k+2}e^{-\iota\frac{\varphi}{2}}d\alpha_q\Big)\,e^{-\sum_{q=1}^{2n-2+k+2}e^{-\iota\frac{\varphi}{2}}\alpha_q}$$

$$\times\prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec\tau,\mathcal{L}})} \mathrm{Tr}\Big[\prod_{q\in q(f^c)}^{\rightarrow} e^{-\alpha_q\sqrt{t}\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big], \tag{133}$$

and the derivative of order s is

$$\frac{d^s}{dt^s} T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)}\; k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)$$
$$\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma) \sum_{\mathcal{L};c_1\dots c_s} \Big(-\frac{\lambda}{N^{D-1}}\Big)^s \prod_{p=1}^{s} w^{j_p j'_p}(T_n,u)$$
$$\times \int_0^\infty \Big(\prod_{q=1}^{2n-2+k+2s} e^{-\iota\frac{\varphi}{2}} d\alpha_q\Big)\, e^{-\sum_{q=1}^{2n-2+k+2s} e^{-\iota\frac{\varphi}{2}}\alpha_q}$$
$$\times \prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})} \mathrm{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q\sqrt{t}\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big], \tag{134}$$

where $\mathcal{L}\equiv\{(j_1,j'_1),\dots(j_s,j'_s)\}$ runs over all the possible ways to decorate $\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}$ with unoriented loop edges $\mathcal{L}$ and $c_1,\dots c_s$ run over the possible colorings of the $2s$ loop edges. Taking into account that the Gaussian measures are normalized we evaluate

$$\Big[\frac{d^q}{dt^q} T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})\Big]_{t=0} = N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)}\; k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)$$
$$\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big)\Big(-\frac{\lambda}{N^{D-1}}\Big)^q \sum_{\mathcal{L};c_1\dots c_q} \prod_{p=1}^{s} w^{j_p j'_p}(T_n,u) N^{|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})|}$$
$$= \Big(-\frac{\lambda}{N^{D-1}}\Big)^q k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big) \sum_{\mathcal{L},c_1\dots c_q} \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \prod_{p=1}^{s} w^{j_p j'_p}(T_n,u)$$
$$N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)+|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})|}, \tag{135}$$

which yields the terms $T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})$ and $T^{EL,(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})$ in Theorem 6. The rest term is

$$R^{(s)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = \frac{1}{(s-1)!}\int_0^1 dt(1-t)^{s-1}\Big[N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)}$$
$$\times k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big) \sum_{\mathcal{L};c_1\dots c_s} \Big(-\frac{\lambda}{N^{D-1}}\Big)^s$$
$$\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \prod_{p=1}^{s} w^{j_p j'_p}(T_n,u) \int d\mu_{w^{ij}(T_n,u)1^{\otimes D}}(\sigma)$$
$$\times \int_0^\infty \Big(\prod_{q=1}^{2n-2+k+2s} e^{-\iota\frac{\varphi}{2}} d\alpha_q\Big)\, e^{-\sum_{q=1}^{2n-2+k+2s} e^{-\iota\frac{\varphi}{2}}\alpha_q}$$
$$\times \prod_{f^c\in\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})} \mathrm{Tr}\Big[\overrightarrow{\prod_{q\in q(f^c)}} e^{-\alpha_q\sqrt{t}\sqrt{\frac{|\lambda|}{N^{D-1}}}(\sigma^{i(q)c}-\sigma^{i(q)c\dagger})}\Big]\Big], \tag{136}$$

which establishes the mixed expansion. Concerning the bounds, we first bound the term $T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})$

$$T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) = \sum_{\mathcal{L},c_1\dots c_q} \frac{1}{q!}\Big[\Big(-\frac{\lambda}{N^{D-1}}\Big)^q k!\Big(\prod_{c=1}^{D} \mathrm{Wg}\big(N, \tau_c\sigma_c^{-1}(\mathcal{B})\big)\Big)$$
$$\times \int_0^1 \Big(\prod_{(i,j)\in T_n} du^{ij}\Big) \prod_{p=1}^{s} w^{j_p j'_p}(T_n, u) N^{-D+2k(D-1)+C(\mathcal{B})-(k+n-1)(D-1)+|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})|}\Big]. \tag{137}$$

We bound the product of Weingarten functions by $2^{2Dk}N^{-2Dk+C(\tau_c\sigma_c^{-1}(\mathcal{B}))}$. The integrals over u^{ij} are bounded by 1, hence

$$\big|T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})\big| \le \frac{1}{q!}\frac{|\lambda|^q}{N^{(D-1)q}} k! 2^{2Dk} \sum_{\mathcal{L},c_1\dots c_q}$$
$$\times N^{-D+2k(D-1)+C(\mathcal{B})-2Dk+C(\tau_c\sigma_c^{-1}(\mathcal{B}))-(k+n-1)(D-1)+|\mathcal{F}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau},\mathcal{L}})|}. \tag{138}$$

By Eq. (31) we find that the scaling with N is bounded by

$$-D+2k(D-1)+C(\mathcal{B}) - 2Dk+C(\tau_c\sigma_c^{-1}(\mathcal{B}))-(k+n-1)(D-1)$$
$$+D+(n-1)(D-1)-\sum_c C(\xi_c)+\sum_c C(\tau_c\xi_c)+q-q(D-1)\le -q(D-2), \tag{139}$$

where we used Lemma 1. Thus

$$\big|T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})\big| \le \frac{|\lambda|^q}{N^{q(D-2)}} k! 2^{2Dk} D^q \frac{(2n+2q+k-3)!}{q!(2n+k-3)!}. \tag{140}$$

For the rest term, $R^{(s)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})$ we use similar bounds and taking into account that the integrals over α are bounded by $\frac{1}{\cos\frac{\varphi}{2}}$ and the integral over t is bounded by 1 we find

$$|R^{(s)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})| \le \frac{1}{\big(\cos\frac{\varphi}{2}\big)^{2n+2s+k-2}} \frac{|\lambda|^s}{N^{s(D-2)}} k! 2^{2Dk} D^s \frac{(2n+2s+k-3)!}{(s-1)!(2n+k-3)!}. \tag{141}$$

□

4.2.6. Proof of the Borel summability Theorem 7. We now show that the rescaled cumulants

$$K(\mathcal{B}, N) = N^{-D+2k(D-1)+C(\mathcal{B})}\mathfrak{K}(\mathcal{B}, \mu_N^{(4)})$$
$$= \sum_{n\ge k} \frac{1}{n!}(-\lambda)^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} \sum_{\vec{\tau}} T^E(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}), \tag{142}$$

are Borel summable in λ uniformly in N. First, the Corollary 1 ensures that the series (51) is absolutely convergent for $|\lambda| < \big(\cos\frac{\varphi}{2}\big)^2 2^{-3} D^{-1}$, hence it certainly is absolutely convergent in a Borel disk of fixed radius.

Second, using the mixed expansion theorem we write

$$K(\mathcal{B},N) = \sum_{n\geq k} \frac{1}{n!}(-\lambda)^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} \sum_{\vec{\tau}} \Big[\sum_{q=1}^{s-1} T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}) + R^{(s)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})\Big]. \tag{143}$$

We perform a Taylor expansion of $K(\mathcal{B},N)$ in λ up to order $r > k$. All the terms corresponding to trees with $n \geq r+1$ are in the reminder, hence for them we use the mixed expansion for $s=0$. For the terms corresponding to trees with $n < r+1$ we use the mixed expansion up to order $s = r-(n-1)$. The explicit terms $T^{(q)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})$ yield a series in λ which is nothing but the Taylor expansion of the rescaled cumulant. The reminder term of the Taylor expansion of $K(\mathcal{B},N)$ writes as

$$R_{N,r}(\mathcal{B},\lambda) = \sum_{n=k}^{r} \frac{1}{n!}(-\lambda)^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} \sum_{\vec{\tau}} R^{(r+1-n)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})$$
$$+ \sum_{n\geq r+1} \frac{1}{n!}(-\lambda)^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} \sum_{\vec{\tau}} R^{(0)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}}). \tag{144}$$

The terms with $n \geq r+1$ then admit a bound

$$\Big|\sum_{n\geq r+1} \frac{1}{n!}(-\lambda)^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} \sum_{\vec{\tau}} R^{(0)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})\Big|$$
$$\leq \sum_{n\geq r+1} |\lambda|^{n-1}(2D)^{n-1} \frac{(2n+k-3)!}{(n-k)!(n+k-1)!} \frac{1}{\big(\cos\frac{\varphi}{2}\big)^{2n-2+k}} k!^{D+1} 2^{2Dk}$$
$$\leq k!^{D+1}2^{2Dk} \sum_{n\geq r+1} |\lambda|^{n-1}(2D)^{n-1} \frac{1}{\big(\cos\frac{\varphi}{2}\big)^{2n-2+k}} (3n)^{k-2}2^{2n}$$
$$\leq \frac{K}{\big(\cos\frac{\varphi}{2}\big)^{k}} \Big(\frac{|\lambda|}{\big(\cos\frac{\varphi}{2}\big)^{2}}\Big)^{r}, \tag{145}$$

for some constant K and $|\lambda| < 2^{-3}D^{-1}\big(\cos\frac{\varphi}{2}\big)^2$. The terms with $r \leq n$ admit a bound

$$\Big|\sum_{n=k}^{r} \frac{1}{n!}(-\lambda)^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowright}_{n,\iota}} R^{(r+1-n)}(\mathcal{T}^{\circlearrowright}_{n,\iota,\vec{\tau}})\Big|$$
$$\leq \sum_{n=k}^{r} |\lambda|^{n-1}(2D)^{n-1} \frac{(2n+k-3)!}{(n-k)!(n+k-1)!}$$
$$\frac{1}{\big(\cos\frac{\varphi}{2}\big)^{2n-2+k+2(r+1-n)}} |\lambda|^{r+1-n} N^{-(r+1-n)(D-2)} k!^{D+1} 2^{2Dk}$$
$$\times D^{r+1-n} \frac{[2n+k+2(r+1-n)-3]!}{(r-n)![2n+k-3]!} \tag{146}$$

Taking into account that

$$\begin{aligned}\frac{(2n+k-3)!}{(n-k)!(n+k-1)!} &< 3^{2n+k-1}k! < 3^{3r}k! \\ \frac{(2r+k-1)!}{(r-n)![2n+k-3]!} &< 3^{2r+k-1}(r-n+2)! < 3^{3r}(r+1)!\end{aligned} \tag{147}$$

these terms are bounded by

$$\frac{|\lambda|^r}{\left(\cos\frac{\varphi}{2}\right)^{2r+k}} k!^{D+2} 2^{2Dk} D^r 3^{6r}(r+1)!2^r \tag{148}$$

Overall we thus derive a bound

$$R_{N,r}(\mathcal{B},\lambda) \le K \frac{|\lambda|^r}{\left(\cos\frac{\varphi}{2}\right)^{2r+k}} \sigma^r r!, \tag{149}$$

for some constants K and σ which proves the Theorem 7. □

4.2.7. Proof of the 1/N expansion Theorem 8. Using the mixed expansion up to order s for every $\mathcal{T}^{\circlearrowleft}_{n,\iota,\vec{\tau}}$ leads to the rest term

$$R_N^{(s)}(\mathcal{B},\lambda) = \sum_{n\ge k} \frac{1}{n!}(-\lambda)^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowleft}_{n,\iota}} \sum_{\vec{\tau}} R^{(s)}(\mathcal{T}^{\circlearrowleft}_{n,\iota,\vec{\tau}}), \tag{150}$$

hence using the bound on $R^{(s)}(\mathcal{T}^{\circlearrowleft}_{n,\iota,\vec{\tau}})$ we get

$$\begin{aligned}|R_N^{(s)}(\mathcal{B},\lambda)| \le{}& \sum_{n\ge k} \frac{1}{n!}|\lambda|^{n-1} \sum_{\substack{i_1,i_2\dots,i_k=1\\ i_d\neq i_{d'}}}^{n} \sum_{\mathcal{T}^{\circlearrowleft}_{n,\iota}} \sum_{\vec{\tau}} \\ &\times \frac{1}{\left(\cos\frac{\varphi}{2}\right)^{2n+2s+k-2}} \frac{|\lambda|^s}{N^{s(D-2)}} (k!2^{2Dk}D^s) \frac{(2n+2s+k-3)!}{(s-1)!(2n+k-3)!}.\end{aligned} \tag{151}$$

By the combinatorial countings Eq. (78) we get

$$\begin{aligned}|R_N^{(s)}(\mathcal{B},\lambda)| \le{}& \frac{1}{N^{s(D-2)}}(k!^{D+1}2^{2Dk}D^s) \frac{1}{\left(\cos\frac{\varphi}{2}\right)^k} \sum_{n\ge k} \left(\frac{|\lambda|}{\left(\cos\frac{\varphi}{2}\right)^2}\right)^{s+n-1} \\ &\times (2D)^{n-1} \frac{(2n+k-3)!}{(n-k)!(n+k-1)!} \frac{(2n+2s+k-3)!}{(s-1)!(2n+k-3)!},\end{aligned} \tag{152}$$

that is, denoting K and σ two constants depending only on k,

$$\begin{aligned}|R_N^{(s)}(\mathcal{B},\lambda)| \le{}& \frac{1}{N^{s(D-2)}} K\sigma^s \frac{|\lambda|^{s+k-1}}{\left(\cos\frac{\varphi}{2}\right)^{2s+3k-1}} \\ &\times \sum_{q\ge 0} \left(\frac{|2D\lambda|}{\left(\cos\frac{\varphi}{2}\right)^2}\right)^q \frac{(2q+2s+3k-3)!}{(s-1)!q!(q+2k-1)!}.\end{aligned} \tag{153}$$

A non optimal bound on the combinatorial factor is

$$\frac{(2q+2s+3k-3)!}{(s-1)!q!(q+2k-1)!} \leq (s+1)!k!5^{2q+2s+3k-3}, \tag{154}$$

and, as the sum over q converges absolutely for $|\lambda| < 5^{-2}2^{-1}D^{-1}\left(\cos\frac{\varphi}{2}\right)^2$, we get

$$|R_N^{(s)}(\mathcal{B},\lambda)| \leq K\sigma^s s! \frac{1}{N^{s(D-2)}} \frac{|\lambda|^{s+k-1}}{\left(\cos\frac{\varphi}{2}\right)^{2s+3k-1}}. \tag{155}$$

for some constants K and σ. □

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