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Determinantal Point Processes and Fermions on Complex Manifolds: Large Deviations and Bosonization

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Abstract: We study determinantal random point processes on a compact complex manifold X associated to a Hermitian metric on a line bundle over X and a probability measure on X. Physically, this setup describes a gas of free fermions on X subject to a $U(1)$-gauge field and when X is the Riemann sphere it specializes to various random matrix ensembles. Our general setup will also include the setting of weighted orthogonal polynomials in $\mathbb{C}^n$, as well as in $\mathbb{R}^n$. It is shown that, in the many particle limit, the empirical random measures on X converge exponentially towards the deterministic pluripotential equilibrium measure, defined in terms of the Monge–Ampère operator of complex pluripotential theory. More precisely, a large deviation principle (LDP) is established with a good rate functional which coincides with the (normalized) pluricomplex energy of a measure recently introduced in Berman et al. (Publ Math de l'IHÉS 117, 179–245, 2013). We also express the LDP in terms of the Ray–Singer analytic torsion. This can be seen as an effective bosonization formula, generalizing the previously known formula in the Riemann surface case to higher dimensions and the paper is concluded with a heuristic quantum field theory interpretation of the resulting effective boson–fermion correspondence.

Contents

1. Introduction

In this paper we study a natural class of determinantal random point processes [39,46] defined on a compact complex manifold X. These processes are induced by the choice of a polarization of X, i.e., an ample line bundle L over X (or more generally a big line bundle) and we will be concerned with their many particle limit. When X is the Riemann sphere this geometric setup contains the extensively studied (Hermitian, unitary and normal) random matrix ensembles. Suitable higher dimensional choices of polarized X give rise to multivariate orthogonal polynomial ensembles, as well as their trigonometric and spherical counterparts (see Sect. 2). On a general complex manifold the point processes represent, from a physical point of view, a gas of (chiral/spin polarized) fermions coupled to a gauge field on X with gauge group $U(1)$ (see Sect. 6).In broad terms the main aim of this paper is to describe the many particle limit in terms of global pluripotential theory and relate it to the notion of bosonization (boson–fermion correspondences) in the physics literature, previously known only in the Riemann surface case [54]. In the companion paper [9] central limit theorems (CLTs) and universality of correlation functions were obtained. Here we will be concerned with the large deviation regime, establishing a large deviation principle (LDP) for the empirical measure of the point process.

Another concrete motivation for the LDP comes from probabilistic methods for locating nearly optimal nodes for interpolating polynomials of large degree on a given set K (which may be realized as a subset of a complex manifold X). Such optimal nodes are commonly defined as configurations of points $(x_1, \ldots, x_N)$ maximizing the density of the probability measure 1.4 below [53] (i.e., as configurations of Fekete points on K [14]).

1.1. An informal introduction to the main results and the relation to Boson–Fermion correspondences. It may be illuminating to first formulate the main results to be obtained in an informal manner, stressing the relations to mathematical physics. Consider a gas of N identical particles on X described by a symmetric probability measure $\mu^{(N)}$ on X^N with density $\rho^{(N)}(x_1, \ldots, x_N)$ wrt a fixed volume form dV on X. The theory of large deviations [25] allows one to give a meaning to the statement that $\mu^{(N)}$ is exponentially concentrated on a deterministic macroscopic measure μ_{eq} (often referred to as the corresponding equilibrium measure) with a rate functional $H(\mu)$. The idea is to think of the large N-limit of the N-particle space X^N of configurations of points $(x_1, \ldots, x_N)$ ("microstates") as being approximated by a space of "macrostates", which is the space $\mathcal{P}(X)$ of all probability measures on X:

$$X^N \sim \mathcal{P}(X),$$

as $N \to \infty$. This "change of variables" from X^N to $\mathcal{P}(X)$ is made precise by introducing the map

$$\delta_N : X^N \to \mathcal{P}(X), \quad \delta_N(x, \ldots, x_N) := \frac{1}{N} \sum_i \delta_{x_i}$$

(which becomes an embedding if we mod out the action of the permutation group S^N on X^N) so that $\mu^{(N)}$ can be pushed forward to define a probability measure $(\delta_N)_* \mu^{(N)}$ on the space $\mathcal{P}(X)$. The exponential concentration referred to above may then be informally written as the following asymptotic relation (as $N \to \infty$):

$$(\delta_N)_* \left(\rho^{(N)}(x_1, \ldots, x_N) dV(x_1) \otimes \cdots \otimes dV(x_N) \right) \sim e^{-a_N H(\mu)} \mathcal{D}\mu, \tag{1.1}$$

where $\mathcal{D}\mu$ denotes a (formal) background measure on the infinite dimensional space $\mathcal{P}(X)$ and where the sequence of numbers a_N is called the *speed* (or rate), which is usually a power of N. Exponential concentration around μ_{eq} appears when $H(\mu) \geq 0$ with $H(\mu) = 0$ precisely when $\mu = \mu_{\text{eq}}$. Thus, in physical terms, the rate functional plays the role of an *effective action*. It should be stressed that defining a suitable measure $\mathcal{D}\mu$ on $\mathcal{P}(X)$ rigorously is notoriously very challenging, but the proper mathematical definition of the exponential concentration in question, i.e., the corresponding LDP, makes no reference to any background probability measure on the space $\mathcal{P}(X)$ (the symbolic expression 1.1 should just be seen as a short hand for saying that a suitable version of the Laplace principle of steepest descent is valid; compare Sect. 4.3).

In the present setting the gas can be represented by spin polarized free fermions on X coupled to a $U(1)$—gauge field A on an ample line bundle $L \rightarrow X$ and we will write $\omega = \frac{i}{2\pi}F_A$ for the corresponding magnetic two-form, normalized so that $[\omega] \in H^2(X, \mathbb{Z})$. Hence $[\omega]$represents the first Chern class $c_1(L)$, normalized so that it becomes an integral class. The probability density

$$\rho^{(N)}(x_1, \ldots x_N) = \frac{1}{\mathcal{Z}_N} \|\det \Psi\|^2 (x_1, \ldots, x_N)$$

is the normalized Slater determinant representing the maximally filled N-particle ground state, i.e., N is the dimension of the space $H^0(X, L)$ of all holomorphic sections on X with values in L. We will consider the limit of increasing field strength, i.e., F_A is replaced by kF_A with $k \rightarrow \infty$ so that $N = N_k = Vk^n + o(k^n) \rightarrow \infty$, where $n = \dim_{\mathbb{C}} X$ and V is the positive number $c_1(L)^n/n!$, since L is assumed ample. It will be shown that an LDP of the form 1.1 holds at a speed Vk^{n+1} with a rate functional $H(\mu)$ that may be decomposed as

$$H(\mu) = E_\omega(\mu) - C, \tag{1.2}$$

where $E_\omega(\mu)$ is the *pluricomplex energy of* μ recently introduced in [15] and the constant $C = C(K, \omega)$ is the *pluricomplex capacity* $C(K, \omega)$ (see Sect. 3 for the precise relation between the notation used here and the notation in [15]). In the case of a Riemann surface, E_ω is nothing but the Dirichlet energy of a unit charge distribution μ subject to the neutralizing exterior charge ω:

$$E_\omega(\mu) = \frac{1}{2}\int_X d\varphi_\mu \wedge d^c\varphi_\mu, \quad dd^c\varphi_\mu = \mu - \omega,$$

where $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$ (see Sect. 1.6). In higher dimensions $E_\omega(\mu)$ may be expressed explicitly in terms of the potential φ_μ of μ obtained by solving the highly non-linear Monge–Ampère equation. We will recall the necessary background from global pluripotential theory in Sect. 3. For the moment we just point out that when μ is a volume form the existence of a smooth potential φ_μ was shown by Yau [60] in his celebrated solution of the Calabi conjecture. The functional E_ω is highly non-linear when $n > 1$—for example, when $n = 2$, one has

$$E_\omega(\mu) = \frac{1}{2}\int_X d\varphi_\mu \wedge d^c\varphi_\mu \wedge \omega + \frac{1}{3}\int_X d\varphi_\mu \wedge d^c\varphi_\mu \wedge \left(dd^c\varphi_\mu\right)$$

and in general it is of degree $n+1$ in the potential φ_μ.

The relation to *bosonization* appears when using a different normalization of the Slater determinant so that it becomes (at least formally) the N-point function $\langle\|\Psi(x_1)\|^2 \ldots$

$\|\Psi(x_N)\|^2\rangle$ of a fermionic quantum field theory on X defined by the corresponding (massless) Dirac action. Mathematically, this amounts to multiplying $\|\det\Psi\|^2(x_1,\dots x_N)$ with the *analytic torsion.* We will then show that this has the effect of canceling the constant C in the expression 1.2 for the rate function. In the final section of the paper we interpret the resulting LDP as an effective bosonization:

$$\left\langle \|\Psi(x_1)\|^2\dots\|\Psi(x_N)\|^2\right\rangle \sim \left\langle e^{i\varphi(x_1)}\dots e^{i\varphi(x_N)}\right\rangle, \tag{1.3}$$

in the large N-limit where the right hand side is expressed in terms of the (formal) quantum field theory for a bosonic field φ with an explicit action, coinciding, up to scaling, with a secondary Bott–Chern class (which in physics terminology is an example of a "higher-derivative action"). In the physics literature bosonization is a well-known phenomenon in 1 + 1 real dimensions (i.e., $n = 1$) [54] and its present higher dimensional incarnation appears to be some what surprising (however, see [23,31,50] for possibly related results). But one explanation, apart from the fact that it only holds effectively/asymptotically, may be the extra condition imposed by the complex/holomorphic structure when $n > 1$: X is a complex manifold and F_A is assumed to be a $(1, 1)$-form, i.e., it determines a holomorphic structure on the underlying line bundle L.

Before turning to the precise formulation of the geometric setup we emphasize that we will be considering a more general setting where the volume form dV on X, used above, is replaced with a suitable measure ν on X supported on a compact set K. One of the main points of considering this more general setting is that it allows one to treat totally *real* situations where X appears as a compactification of the complexification of K. For example, K may be taken to be the real n-sphere S^n or the n-torus T^n.

1.2. The geometric setup. Let $L \to X$ be an ample holomorphic line bundle over a compact complex manifold X of dimension n. We will denote by $H^0(X, L)$ the N-dimensional vector space of all global holomorphic sections of L. Given the geometric data $(\nu, \|\cdot\|)$ consisting of a probability measure ν on X and a continuous Hermitian metric $\|\cdot\|$ on L one obtains an associated probability measure $\mu^{(N)}$ on the N-fold product X^N defined as

$$\mu^{(N)} := \frac{1}{\mathcal{Z}_N}\|\det\Psi\|^2(x_1,\dots x_N)\nu(x_1)\otimes\cdots\otimes\nu(x_N), \tag{1.4}$$

where $\det\Psi$ is any holomorphic section of the pulled-back line bundle $L^{\boxtimes N}$ over X^N representing the complex line $\Lambda^N H^0(X, L)$ and $\mathcal{Z}_N$ is the normalizing constant. Concretely, we may write $\det\Psi$ as a Slater determinant:

$$(\det\Psi)(x_1,\dots,x_N) = \det(\Psi_i(x_j)) \tag{1.5}$$

for a given basis $(\Psi_i)_{=1}^N$ of $H^0(X, L)$. We will denote by ω the real $(1, 1)$-current defined as $\frac{i}{2\pi}$ times the curvature current of the fixed metric $\|\cdot\|$ on L. The normalization has been chosen so that $[\omega]$ defines an integral cohomology class; the first Chern class $c_1(L)$. The *empirical measure* of the ensemble above is the following random measure:

$$(x_1,\dots,x_N)\mapsto\delta_N := \sum_{i=1}^N \delta_{x_i}/N \tag{1.6}$$

which associates to any N-particle configuration $(x_1,\dots,x_N)$ the normalized sum of the delta measures on the corresponding points in X. In probabilistic terms this setting hence defines a *determinantal random point process on X with N particles* [39,42].

If the corresponding L^2-norm on $H^0(X, L)$

$$\|\Psi\|_X^2 = \langle \Psi, \Psi \rangle_X := \int_X \|\Psi(x)\|^2 \, d\nu(x)$$

is non-degenerate (which will always be the case in this paper) then the probability measure $\mu^{(N)}$ on X^N may be expressed as a determinant of the *Bergman kernel* of the Hilbert space $(H^0(X, L), \|\cdot\|_{L^2(X)})$, i.e., the integral kernel of the corresponding orthogonal projection. But one virtue of the definition 1.4 is that it admits a natural generalization to *β-ensembles,* obtained by replacing the power 2 with a general real positive power β (which plays the role of *inverse temperature* from the point of view of statistical mechanics—see Sect. 5 for such generalizations). Replacing L with its k th tensor power, which we will write in additive notation as kL, yields a sequence of point processes on X of an increasing number N_k of particles. We will be concerned with the asymptotic situation when $k \to \infty$. This corresponds to a large N-limit of many particles, since

$$N_k := \dim H^0(X, kL) = V k^n + o(k^n)$$

where the constant V is, by definition, the *volume* of L.

Of course, we can also view point processes above as defined on the support K of the measure ν. There is also a slight variant of the setting above where K may be taken as the non-compact sets $\mathbb{C}^n$ or $\mathbb{R}^n$ (which may be identified with subsets of $X := \mathbb{P}^n$ the complex projective space)—see Sect. 2.3.

1.3. Statement of the main results.

1.3.1. A general large deviation principle. As is well-known, the density of the one-point correlation measure, i.e., of the expectation $\mathbb{E}(\delta_{N_k})$ of the empirical measure, is precisely the normalized point-wise norm of the corresponding Bergman kernel on the diagonal. In the case when the curvature form ω of the fixed Hermitian metric on L is smooth and positive and the measure ν is a volume form on X, it then follows from well-known Bergman kernel asymptotics that

$$\mathbb{E}(\delta_{N_k}) \to \frac{1}{V} \frac{\omega^n}{n!}$$

weakly as $k \to \infty$ where $\mathbb{E}$ denotes the expectation with respect to the determinantal ensemble $(X^N, \mu^{(N)})$ (in fact there is a complete asymptotic expansion in powers of $1/k$ as first shown by Catlin and Zelditch [64]; see the survey and references therein and also [29] for a path integral approach). For a curvature form ω which is smooth, but not necessarily semi-positive, it was shown in [7] that the previous convergence still holds if the right hand side above is replaced by the pluripotential *equilibrium measure* μ_{eq} on X associated to ω, which may be written as

$$\mu_{\text{eq}} = 1_D \frac{1}{V} \frac{\omega^n}{n!},$$

where D is a certain compact subset of X. In the case of a Riemann surface (i.e., $n = 1$) the set D may be obtained by solving a free boundary value problem for the Laplace operator (in the physics literature the set D appears as a limiting Coulomb gas plasma/fluid in the context of the Quantum Hall effect, as well as an eigenvalue droplet in the normal random matrix model—see [61] and references therein). For a general dimension n, the set D is obtained similarly but using the Monge–Ampère operator, which is fully non-linear (see Sect. 3). For general geometric data (ω, ν), with ν supported on a compact subset K, the convergence towards the equilibrium measure μ_{eq} associated to the pair (ω, K) (in the sense of pluripotential theory) was shown to hold very recently in [14] under very weak regularity assumptions on the measure ν and K (more precisely the measure ν was assumed to be Bernstein–Markov (B-M) wrt (K, ω); compare Sect. 4.1).

Our first main result shows that the convergence of the empirical measure is in fact *exponential* in probability with a rate functional which is the (normalized) pluricomplex energy of a measure recently introduced in [15]. More precisely, we have the following LDP for the *laws* of the normalized empirical measures, i.e. for the push forward of the probability measure $\mu^{(N_k)}$ on X^N to the space $\mathcal{P}(K)$ of all probability measures on K, under the map δ_N (we refer to Sect. 3 for the definition of the various notions appearing below).

Theorem 1.1. *(LDP) Let $L \to X$ be an ample line bundle equipped with a fixed continuous Hermitian metric with curvature current ω and ν a probability measure on a compact non-pluripolar set K such that ν is strongly Bernstein–Markov with respect to the set K. Then the laws of the empirical measures of the corresponding determinantal point process 1.4 satisfy a LDP with a* good rate functional *H and speed $Vk^{n+1} (\sim kN_k)$, where*

$$H(\mu) = E_\omega(\mu) - C,$$

where $E_\omega(\mu)$ is the pluricomplex energy (defined wrt ω) of the probability measure μ and C is the constant ensuring that the infimum of H over the space of all probability measures on K vanishes.

It follows from [22] that the pluripotential equilibrium measure μ_{eq} [only depending on (K, ω)] is the unique minimizer of the rate functional H above (anyway this will be reproved here in the course of the proof of the previous theorem). It may also be worth pointing out that the *upper* bound corresponding to the LDP holds without the Bernstein–Markov assumption on the measure ν.

In fact, the LDP above will be shown to hold for any line bundle L that is big, which, by definition, means that N_k is of the order k^n, for k large (the proof of this requires a generalization of the results in Section 5 in [15] to big cohomology classes which should be of independent interest). Moreover, as will be shown in Sects. 4.8 and 5 , respectively, the LDP can be adapted to two other general settings:

- The LDP holds for β-ensembles (Sect. 4.8)
- The LDP holds in a setting where K is replaced by a non-compact subset F

The proof in the non-compact setting proceeds by reducing to the previous case when K is compact. In particular we obtain the following LDP for the *Vandermonde determinant* $\Delta^{(N_k)}$ obtained by taken the basis $(\Psi_i)_{=1}^{N_k}$ in 1.5 to be multinomials in $\mathbb{C}^n$ of total degree at most k. See Sect. 2 for a recap of the relation between the global setup of a line bundle $L \to X$ and the classical setting of orthogonal polynomials in $\mathbb{C}^n$ and Sect. 4.10 (and Corollary 5.4) for a description of the corresponding rate and energy functionals.

Corollary 1.2. *Let $F = \mathbb{R}^n$ (or $F = \mathbb{C}^n$) and let ν be the Euclidean measure on F. Assume that ϕ is a continuous function on F with super logarithmic growth at infinity (see 2.3). Then the push-forward Γ_k of the weighted Vandermonde measure*

$$\tilde{\mu}_{\phi}^{(N_k)} := \left|\Delta^{(N_k)}(z_1, \ldots, z_{N_k})\right|^2 e^{-k\phi(z_1)} \ldots e^{-k\phi(z_{N_k})} \nu^{\otimes N_k}$$

under the map δ_{N_k} (formula 1.6) satisfies an LDP at a speed k^{n+1} and with a good rate functional, which in the case $n = 1$ coincides with the weighted logarithmic energy [49]. More generally, the LDP holds (at a speed $\beta_k k^{n+1}$) when the density of $\tilde{\mu}_{\phi}^{(N_k)}$ is raised to a positive power β_k as long as $\beta_k \leq C$ and $\beta_k k \to \infty$.

As another corollary (see Sect. 5.5) we obtain an LDP for ensembles defined by holomorphic sections vanishing to high order along a given hypersurface in X. Physically, this allows one to consider situations where the fermion ground-state has a filling fraction strictly below 1, as in the fractional Quantum Hall effect, which is well-known in the case when $n = 1$. We also point out some relations to Laplacian growth [38].

It should be stressed that the assumptions in Theorem 1.1 are very weak and they are satisfied in geometrically natural situations. For example, the measure ν may be taken to be defined by integrating against a volume form on a smooth domain K [14]. The measure ν can also be taken as a volume form on K when the latter is either a smooth real hypersurface or a smooth real algebraic variety of dimension n (see Sect. 2).

Given a function φ on X we denote by

$$\epsilon_{N_k,\lambda}(\varphi) := \text{Prob}\left\{\left|\frac{1}{N_k}(\varphi(x_1) + \cdots + \varphi(x_{N_k})) - \int_X \mu_{\text{eq}}\varphi\right| > \lambda\right\} \tag{1.7}$$

the *tail of the linear statistic* $\varphi(x_1) + \cdots + \varphi(x_{N_k})$.

Corollary 1.3. *Let φ be a continuous function on X.*

- *If the support K of the measure ν is not pluripolar, then the tail 1.7 satisfies $\epsilon_{k,\epsilon}(\varphi) \leq e^{-Ck^{(n+1)}}$ for some positive constant C depending on φ and ϵ.*
- *In the case when X is a Riemann surface, $K = X$ and the curvature current ω of the metric on L is semi-positive (so that $\mu_{\text{eq}} = \omega$) the following more precise estimate holds:*

$$\epsilon_{N_k,\lambda}(\varphi) \leq 2\exp\left(-N_k^2\left(\frac{2V\lambda^2}{\|d\varphi\|_X^2}(1 + o(1))\right)\right), \tag{1.8}$$

where the error term $o(1)$ denotes a sequence tending to zero as $k \to \infty$ (but depending on φ).

1.3.2. Analytic torsion and effective bosonization. Now fix a smooth Hermitian metric h_X on the tangent bundle TX and take the measure $\nu = dV$ above to be its volume form (in particular, $K = X$). We will also assume that the Hermitian metric on L is smooth—the most interesting case will be when its curvature ω is not semi-positive. The LDP in Theorem 1.1 may then be reformulated in the following way:

Theorem 1.4. *Let $L \to X$ be an ample line bundle over a compact complex manifold X and equip L and TX with smooth metrics as above. Then the push forward of the measures*

$$\prod_{q=1}^{n} \left(\det \Delta_{\bar{\partial},k}^{0,q}\right)^{(-1)^{q+1}q} \|\det \Psi_k\|^2 (x_1, \dots x_{N_k}) dV(x_1) \otimes \dots \otimes dV(x_{N_k}) \tag{1.9}$$

on X^{N_k} under the map δ_N where δ_N is the empirical measure 1.6 satisfy a LDP with the good rate functional $E_\omega(\mu)$ (the pluricomplex energy wrt ω).

Here $\Delta_{\bar{\partial},k}^{0,q}$ denotes the $\bar{\partial}$-Laplacians acting on the space of $(0, q)$-forms with values in kL and their determinants are defined using zeta-function regularization of the product of the positive eigenvalues. We also recall that the corresponding product of determinants appearing in the theorem is, by definition, the *Ray–Singer analytic torsion* associated to the kth tensor power of Hermitian line bundle $(L, \|\cdot\|)$ and the fixed metric on TX [16].

The new input in the previous theorem compared to Theorem 1.1 is the limiting expression for the scaled logarithms of the Ray–Singer analytic torsions which has the effect of canceling the normalization constant appearing in the previous rate functional (see Proposition 4.12). The main reason that we have reformulated the previous LDP in this new form is that it can be seen as an effective (i.e., asymptotic) generalization to higher dimensions of the *bosonization formula* on a Riemann surface, saying that

$$\det \Delta_{\bar{\partial}} \|\det \Psi\|^2 (x_1, \dots x_N) = C_{N,g} \exp\left(\left(\frac{1}{2}\sum_{i \neq j} G(x_i, x_j) + r(x_1, \dots, x_N)\right)\right), \tag{1.10}$$

where G is the Green function of the Laplacian defined wrt the Arakelov metric ω on X (when $g > 0$) [16]. The term r appearing above vanishes for genus $g = 0$. For $g > 0$ it may be expressed in terms of the Riemann theta function on the Jacobian torus of the Riemann surface X. This formula was obtained in [1]; first a heuristic argument was given in op. cit. using a fermion-boson correspondence ansatz (see also [57]) and then the formula was rigorously proved using properties of Quillen metrics and complex algebraic geometry (see the appendix in [1]). Comparing with [1] the present paper thus proceeds in a reversed manner: first we establish the LDP (and in particular Theorem 1.4) rigorously, and then, in the final section of the paper, a heuristic quantum field theory interpretation of the result is given.

Finally, we recall that an explicit expression for the factor $C_{N,g}$ appearing in 6.1 was determined only very recently [59]. Combining Theorem 1.4 with the exact formula 1.10 shows that both $C_{N,g}$ and r are negligible in the large N-limit (in particular it follows that $\log C_{N,g} = o(N^2)$, which is consistent, as it must, with the explicit formula found in [59]).

1.4. Relations to previous results. In the case when $n = 1$ and $K = \mathbb{R}$ the LDP in Corollary 1.2 (for $\beta_k \equiv \beta$) was first obtained by Ben Arous and Guionnet [3] (see also Ben Arous and Zeitouni [4] for the case $K = \mathbb{C}$ and $\phi(z) = |z|^2$). The result in [3] was formulated in terms of the standard Hermitian random matrix ensemble, building on previous work by Voiculescu on free probability theory. In particular, these results imply the convergence of the free energies $k^{-1}N_k^{-1} \log \int \tilde{\mu}_\phi^{(N_k)}$ previously established by Johansson [41] when $K = \mathbb{R}$ using a large deviation type upper bound (see also Hedenmalm and Makarov [38] for the case $K = \mathbb{C}$). An elegant potential theoretic derivation of Johansson's bound for general Bernstein–Markov measures supported on $\mathbb{C}$ was introduced by Bloom and Levenberg [17] (whose global pluripotential version also plays an important role in the present paper). The LDP in Theorem 1.1 in the case when $K \subset \mathbb{P}^1$ for K an arbitary non-polar compact set is contained in the analysis in the recent work [62], where zeroes of random polynomials are considered. In all these works the starting point is the explicit expression for $\log \|\det \Psi\|^2 (x_1, \ldots x_N)$ as a sum of Green functions $G(x_i, x_j)$ of the Laplace operator (which is the simple genus 0 case of the bosonization formula 1.10) and it is shown that the LDP can be expressed in terms of the limiting Green energy of a measure μ. If G were bounded, then the LDP would follow from general asymptotics for Laplace type integrals, but the non-boundedness leads to highly non-trivial analytic issues. The most subtle point in the proof is the lower bound, which is handled using a decomposition argument of the measure μ. The method of proof in the present paper is completely different, as there is no useful analogue of the Green function when $n > 1$, which is a reflection of the fact that the Monge–Ampère operator is fully non-linear.

The present paper is a substantially revised and extended version of the first preprint that appeared on ArXiv (which only contained Theorem 1.1), see [10]. The main new features are that (1) another proof of the LDP is added which uses the general Gärtner–Ellis theorem, (2) the LDP has been extended to a non-compact setting and to β-ensembles, and (3) the relation of the LDP to bosonization is explained and explored. Since the first version of the paper, results equivalent to the LDP in Theorem 1.1 in the case of $\mathbb{P}^n$ have been obtained by Bloom and Levenberg [21]. Their proof of the lower bound in the LDP is different than the ones in the present paper (but it also uses [13]). Moreover, an alternative proof of the LDP in the Riemann surface case is also contained in the arguments in Zelditch's [63] paper, which rely on the explicit bosonization formula 1.10.

1.5. Organization.

- *Section 2*: Here we start by considering concrete examples obtained by specializing the general setting to get ensembles defined by polynomials on $\mathbb{C}^n$ and on complex as well as real algebraic varieties, including spherical polynomials.
- *Section 3*: We recall the global pluripotential theory from [15,22] needed to define and study the rate function of the LDP. In particular, we show that the pluricomplex may be realized as a Legendre transform (even in the general setting of a big class).
- *Section 4*: Two proofs of the LDP in Theorem 1.1 are given. The first one involves the abstract Gärtner–Ellis theorem together with the main results in [13] (which we also give in the general setting of big line bundles). As for second proof it is based on the convergence of Fekete points established in [14]. Technically, one advantage of using the Gärtner–Ellis theorem is that it avoids invoking the variational results on the Monge–Ampère equation in [15]. We also show (Sect. 4.8) how to deduce the LDP in the β-deformed setting and in Sect. 1.3.2 we prove Theorem 1.4, which expresses the LDP in terms of the analytic torsion.
- *Section 5*: The LDP in the non-compact is formulated and proved, essentially by reducing to the previous setting. The section is concluded with applications to ensembles defined by sections vanishing along a given divisor.
- *Section 6*: The relations between the LDP and bosonization in quantum field theory are explored, using some heuristic arguments.

1.6. Notation. Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X which will mainly be assumed to be ample.[1]

1.6.1. Metrics on L. We will fix, once and for all, a continuous Hermitian metric $\|\cdot\|$ on L. Its curvature current times the normalization factor $\frac{i}{2\pi}$ will be denoted by ω. The normalization is made so that $[\omega]$ defines an *integer* cohomology class, i.e., $[\omega] \in H^2(X, \mathbb{Z})$. The local description of $\|\cdot\|$ is as follows: let s be a trivializing local holomorphic section of L, i.e., s is non-vanishing on a given open set U in X. Then we define the local *weight* ϕ of the metric $\|\cdot\|$ by the relation

$$\|s\|^2 = e^{-\phi}.$$

The (normalized) curvature current ω may now be defined by the following expression:

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\phi := dd^c\phi$$

(where we, as usual, have introduced the real operator $d^c := i(-\partial + \overline{\partial})/4\pi$ to absorb the factor $\frac{i}{2\pi}$). The point is that, even though the function ϕ is merely locally well-defined, the form ω is globally well-defined (as any two local weights differ by $\log|g|^2$ for g a non-vanishing holomorphic function). The current ω is said to be *positive* if the weight ϕ is *plurisubharmonic* (psh). If ϕ is smooth this simply means that the Hermitian matrix $\omega_{ij} = (\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j})$ is positive definite (i.e., ω is a Kähler form) and in general it means that, locally, ϕ can be written as a decreasing limit of such smooth functions. Finally, we recall that from the point of view of gauge theory the (non-normalized) curvature form $\partial\overline{\partial}\phi$ is the curvature form F_A of the Chern connection on the complex line bundle L, i.e., the unique connection A on L which is compatible with its given holomorphic structure and Hermitian metric $\|\cdot\|$.

1.6.2. Holomorphic sections of L. We will denote by $H^0(X, L)$ the space of all global holomorphic sections of L. In a local trivialization as above any element Ψ in $H^0(X, L)$ may be represented by a local holomorphic function f, i.e.,

$$\Psi = fs$$

The squared point-wise norm $\|\Psi\|^2(x)$ of Ψ, which is a globally well-defined function on X, may hence be locally written as

$$\|\Psi\|^2(x) = \left(|f|^2 e^{-\phi}\right)(x).$$

It will sometimes be convenient to take the curvature current ω as our geometric data associated to the line bundle L. Strictly speaking, it only determines the metric $\|\cdot\|$ up to a multiplicative constant but, for example, the corresponding random point processes will be independent of this constant.

[1] General references for this section are the books [27,34]. See also [1] for the Riemann surface case.

1.6.3. Metrics and weights vs ω-psh functions. Having fixed a continuous Hermitian metric $\|\cdot\|$ on L with (local) weight ϕ_0 any other metric may be written as

$$\|\cdot\|_{\varphi}^2 := e^{-\varphi} \|\cdot\|^2$$

for a continuous function φ on X, i.e. $\varphi \in C^0(X)$. In other words, the local weight of the metric $\|\cdot\|_{\varphi}$ may be written as $\phi = \varphi + \phi_0$ and hence its curvature current may be written as

$$dd^c\phi = \omega + dd^c\varphi := \omega_{\varphi}$$

This means that we have a correspondence between the space of all (singular) metrics on L with positive curvature current and the space $\mathrm{PSH}(X, \omega)$ of all upper-semi continuous functions φ on X such that $\varphi + \phi_0$ is locally psh. Note for example, that if $\Psi \in H^0(X, L)$ then $\log \|\Psi\|^2 \in \mathrm{PSH}(X, \omega)$.

1.6.4. Norms on $H^0(X, L)$. Given a compact subset K of X the fixed metric $\|\cdot\|$ on L induces, in the usual way, an L^{∞}-norm on $H^0(X, L)$:

$$\|\Psi\|_{L^{\infty}(K)} := \sup_{x \in K} \|\Psi(x)\| .$$

This is a non-degenerate norm if K is not contained in a analytic subvariety of X and in particular if K is *non-pluripolar,* i.e., K is not locally contained in the $-\infty$ set of a plurisubharmonic function. We will fix a compact non-pluripolar subset K once and for all.

Similarly, any given finite measure ν on a compact set K induces an L^2-norm on $H^0(X, L)$ (which will always be non-degenerate in the present paper):

$$\|\Psi\|_{L^2(K,\nu)}^2 := \int_X \|\Psi\|^2 \, d\nu.$$

Sometimes we will also use the notation

$$\|\Psi\|_{L^2(\nu, e^{-\varphi})}^2 := \int_X \|\Psi\|_{\varphi}^2 \, d\nu = \int_X \|\Psi\|^2 e^{-\varphi} d\nu$$

if $\varphi \in C^0(X)$ and similarly for L^{∞}-norms. In practice, the measures ν that we will be concerned with enjoy regularity properties of Bernstein–Markov type (see Sect. 4.1).

1.6.5. Scaling by k. The Hermitian line bundle (L, ϕ) over X induces, in a functorial way, Hermitian line bundles over all products of X (and its conjugate $\overline{X}$) and we will usually keep the notation ϕ for the corresponding weights. When studying asymptotics we will replace L by its kth tensor power, written as kL in additive notation. The induced weights on kL may then be represented as $k\phi$.

1.6.6. Functionals. In Sect. 3 we will introduce primitive of the Monge–Ampère operator, denoted by $\mathcal{E}(\varphi)$, which defines a functional on the space $\mathrm{PSH}(X, \omega)$. Following [22] this functional is then used, by a duality construction, to define the pluricomplex energy $E_{\omega}(\mu)$ of a measure μ, which thus defines a functional on the space $\mathcal{P}(X)$ of all probability measures on X. It should however be pointed out that our notation differs from the notation in [22], where the functional $\mathcal{E}$ is denoted by E and the functional E_{ω} by $\mathcal{E}^*$.

2. Examples: Orthogonal Polynomial Ensembles

2.1. Multivariate polynomial ensembles. Let ν be a measure supported on a compact subset K of $\mathbb{C}^n$, which up to a trivial scaling, may be assumed to be contained in the open unit-ball. We next briefly recall how to fit this situation into the previous global setup of an ample line bundle $L \to X$ over a compact complex manifold X. First we can identify $\mathbb{C}^n$ with an affine piece U of the projective space $\mathbb{P}^n(:= X)$ as follows. By definition, $\mathbb{P}^n$ is the complex quotient $\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ denotes the standard action $((Z_0, \ldots, Z_n), \lambda) \mapsto (\lambda Z_0, \ldots, \lambda Z_n)$ and we write, as usual, $(Z_0, \ldots, Z_n) \mapsto [Z_0 : \ldots : Z_n]$ for the corresponding projection map from $\mathbb{C}^{n+1} - \{0\}$ to $\mathbb{P}^n$. The fibers of the latter maps are complex lines in $\mathbb{C}^{n+1}$, which thus define a line bundle L^* over $\mathbb{P}^n$ and we denote the dual line bundle L by $\mathcal{O}(1)$, which is usually referred to as the *hyperplane line bundle* over $\mathbb{P}^n$. By construction the "homogeneous coordinates" Z_i define holomorphic sections of the latter line bundle. In this notation the embedding of $\mathbb{C}^n$ into $\mathbb{P}^n$ is given by $z \mapsto [1, z]$, so that the image of $\mathbb{C}^n$ is given by $U := \{Z_0 \neq 0\}$. Accordingly, the section $s := Z_0$ of $\mathcal{O}(1)$ defines a trivialization over $\mathbb{C}^n \approxeq U$, so that the restriction to U of a metric on $\mathcal{O}(1) \to U$ gets identified with a weight ϕ defined on $\mathbb{C}^n$ (as explained in Sect. 1.6). In this way one obtains a correspondence between locally bounded metrics on $\mathcal{O}(1) \to \mathbb{P}^n$ and functions $\phi(z)$ on $\mathbb{C}^n$ such that

$$\phi(z) = \phi_0(z) + O(1) := \log^+ |z|^2 + O(1) \tag{2.1}$$

where $\phi_0(z) := \log^+ |z|^2$, i.e it is equal to $\log |z|^2$ for $|z| > 1$ and 0 otherwise. Equivalently, fixing $\omega_0 := dd^c \log^+ |z|^2$, which extends to a form on $\mathbb{P}^n$ with locally continuous potentials, and letting $\varphi = \phi - \log^+ |z|^2$ hence yields a bijection between all *bounded* $\varphi \in \mathrm{PSH}(\mathbb{P}^n, \omega_0)$ and all ϕ as above which are psh on $\mathbb{C}^n$ (compare [35]). Also, since we have assumed that the compact set K is contained in the open unit-ball we have that $\phi = \varphi$ on a neighborhood of K and hence

$$\omega_\varphi := \omega + dd^c\varphi = dd^c\phi$$

on a neighborhood of K. Moreover, using the trivialization above, the space $H^0(\mathbb{P}^n, k\mathcal{O}(1))$ of all homogenous polynomials Ψ_k of total degree k gets identified with the space of all polynomials $p_k(z)$ in $z_1, \ldots, z_n$ on $\mathbb{C}^n$ of total degree at most k and the point-wise norm induced by ϕ is given by

$$\|\Psi_k\|^2 (z) := |p_k(z)|^2 e^{-k\phi(z)}$$

Hence, also fixing a suitable measure ν on K the corresponding L^2-norm of $\|\Psi_k\|^2 (z)$ coincides with the weighted L^2-norm of the polynomial p_k appearing in the theory of orthogonal polynomials (see for example the appendix in [49]). Note that since K is compact the classical unweighted theory in $\mathbb{C}^n$ may be obtained by taking $\phi = 0$ on K and then extending ϕ so that 2.1 holds (for example $\phi = \log^+ |z|^2$, will do).

Now, as explained in the introduction of the paper, the pair $(\nu, \|\cdot\|)$ defines, for any k, a determinantal point process on K concretely obtained by taking $\Psi_i^{(k)}$ to be a basis for the space of all polynomials in $z_1, \ldots, z_n$ on $\mathbb{C}^n$ of degree at most k. If the basis consists of multinomials then the corresponding Slater determinant is known as the (multivariate) *Vandermonde determinant*:

$$\det \Psi_k = \Delta^{(N_k)}(z_1, \ldots, z_{N_k}) \tag{2.2}$$

The conditions in Theorem 1.1 are satisfied if, for example, K is compact domain in $\mathbb{C}^n$ with smooth boundary or its boundary and ν is the measure defined by a volume form on K (see Sect. 4.1).

More generally, we can replace $\mathbb{C}^n$ with an affine algebraic subvariety $X_0 := \{p_1(z) = \ldots p_m(z) = 0\}$ cut out by polynomials p_i on $\mathbb{C}^n$ and ν with a measure supported on a compact subset of X_0. Then we let X be the associated projective variety obtained by taking the Zariski-closure of X_0 in $\mathbb{P}^n$ and let L be the restriction $\mathcal{O}_X(1)$ (strictly speaking we have to assume that X is non-singular, but otherwise one could pass to a resolution of X and use that the results to be proved hold in the general setting of a big line bundle L). The Slater determinant is then defined in terms of a given basis in $H^0(X, k\mathcal{O}_X(1))$, which for k large may be identified with the vector space spanned by the restriction to X of all polynomials of degree at most k in $\mathbb{C}^n$. Again the conditions in Theorem 1.1 are satisfied if K is a bounded domain in X or its boundary and ν is a measure as above.

2.2. Real examples. It is interesting to apply the previous setup to a completely "real" setting. For example, we can take the measure ν to be supported on a compact subset K of $\mathbb{R}^n$. Embedding $\mathbb{R}^n$ in $\mathbb{C}^n$ and taking a basis of polynomials defined over $\mathbb{R}$, i.e. with real coefficients, then induces a determinantal point-process on K to which Theorem 1.1 applies if, for example, K is a smooth domain in $\mathbb{R}^n$ and ν is taken as the usual Euclidean (Lebesgue) measure on K (the assumptions in the theorem are indeed satisfied as follows from the results in [18]). When $n = 1$ the corresponding ensemble may be realized by random Hermitian matrices with eigenvalues conditioned to lie in K (a finite union of intervals) [24].

Similarly, if the polynomials $p_1, \ldots, p_m$ defining the affine algebraic variety X_0 above have real coefficients and the corresponding real algebraic variety $K := X_0 \cap \mathbb{R}^n$ is compact then we can take ν to be the measure on X defined by any given volume form on $K \subset X$. The following two particular cases have been extensively studied in the literature, in particular in the context of approximation theory, and as we will explain the Bernstein–Markov properties can be proved directly in these cases.

2.2.1. Spherical polynomials. Let $K = S^n$ be the unit n-sphere in $\mathbb{R}^{n+1}$ and ν the usual $O(n, \mathbb{R})$-invariant probability measure on S^n. We can then let the algebraic variety X_0 above be the complex quadric defined by $\{p(z) = z_1^2 + \cdots z_{n+1}^2 - 1 = 0\}$ so that its real points are precisely S^n. Then $H^0(X, \mathcal{O}_X(1))$ with the Hermitian product induced by $(\nu, 0)$ gets identified with the complexification of the space $H_k(S^n)$ of all spherical polynomials on the n-sphere S^n of degree at most k equipped with its usual $O(n, \mathbb{R})$-invariant scalar product. The corresponding Slater determinant naturally appears in numerical problems as the determinant of the interpolation matrix (see for example [53] and references therein). From the physics point of view the process represents the ground state of a gas of free spin-polarized fermions on S^n (which can be seen as a compact version of the $\mathbb{R}^n$-case studied in [51,52] in connection to sphere packings etc).

It may be illuminating to give the following direct proof of the Bernstein–Markov property (Sect. 4.1) in this case. From the $O(n+1, \mathbb{R})$-invariance it follows immediately that $\mathbb{E}(\delta_N) = \nu$, which equivalently means that the Bergman function [14] is identically equal to the dimension N_k of $H_k(S^n)$, i.e.

$$\sup_{p_k \in H_k(S^n)} |p_k(z)|^2 / \int_{S^n} |p_k(z)|^2 d\nu = N_k$$

Since N_k is of the order k^n and hence of sub-exponential growth in k we conclude that $(\nu, 0)$ has the BM-property. But then it follows from general principles that it in fact has the BM-property wrt *any* continuous function on $K := S^n$. This was shown by Bloom in the case when $K \subset \mathbb{R}^n$ [18] (Thm. 3.2), but the same arguments work in the present case. Indeed, writing

$$|p_k(z)|^2 e^{-k\varphi} = \left| p_k(z) \left(e^{-\varphi/2}\right)^k \right|^2$$

for a continuous function φ on S^n we may extend $-\varphi$ continuously to all of $\mathbb{R}^n$ and then, using the Stone–Weierstrass theorem approximate it uniformly by polynomials on $\mathbb{R}^n$. Truncating the Taylor expansion of $e^{-\varphi}$ then allows us to approximate $e^{-\varphi}$ with polynomials p_ϵ such that

$$1 - 2\epsilon \leq p_\epsilon / e^{-\varphi} \leq 1 + 2\epsilon$$

Applying the BM-property for $(\nu, 0)$ to the polynomial $p_k(z)p_\epsilon(z)$ and rescaling then proves that (ν, φ) also has the BM-property, as desired.

2.2.2. Trigonometric polynomials. We let $K := T^n$ be the unit n-torus in $\mathbb{R}^{2n}_{x,y}$ and set $p_i(z, w) = z_i^2 + w_i^2 - 1$ for $i = 1, \ldots n$ in $\mathbb{C}^{2n}_{z,w}$ so that $K \subset X \subset \mathbb{P}^{2n}$. Then $H^0(X, \mathcal{O}_X(1))$ with the Hermitian product induced by $(\nu, 0)$ gets identified with the complexification of the space $H_k(T^n)$ of all trigonometric polynomials on T^n of total degree at most k (i.e., the corresponding frequencies lie in k times the unit simplex) equipped with its usual $O(2, \mathbb{R})^{\otimes n}$-invariant scalar product. Similarly, replacing $\mathbb{P}^{2n}$ with $(\mathbb{P}^2)^n$ gives trigonometric polynomials with frequencies in $[0, k]^n$.

2.3. The case of non-compact subsets of $\mathbb{C}^n$. There is a non-compact variant of the previous setting when K is assumed to be a merely *closed* subset of $\mathbb{C}^n$, but where the continuous weight ϕ on $\mathbb{C}^n$ has super logarithmic growth:

$$\phi(z) \geq (1 + \epsilon) \ln |z|^2 , \quad \text{when } |z| \gg 1 \tag{2.3}$$

In particular ϕ is *not* the restriction of a locally bounded metric on $\mathcal{O}(1) \to \mathbb{P}^n$. In the case when $n = 1$ this is the setting of weighted potential theory considered in the book [49] (see also the appendix in [49] for the case when $n > 1$). In particular, we may then take K as all of $\mathbb{C}^n$ (or $\mathbb{R}^n$) and $\nu = d\lambda$ as the corresponding Lebesgue measure. For example, in the case of $n = 1$ the corresponding ensembles may be realized by random normal (or Hermitian) random matrices whose eigenvalues are subject to the “confining potential” ϕ [24,61]. As shown in Sect. 2.3 the LDP in Theorem 1.1 is still valid in this non-compact setting.

3. The Monge–Ampere Operator, Energy and Rate Functionals

In this section we will recall the global complex pluripotential theory that is needed to define and study the rate functional for the LDP. Almost all the material on the complex Monge–Ampere equation is contained in [15,22] (apart from the results in Sects. 3.4 , 3.6).

3.1. The Monge–Ampère operator and the functional $\mathcal{E}_\omega(u)$. Let (X, ω) be a compact complex manifold equipped with a fixed $(1,1)$-current ω with continuous local potentials and assume that the class $[\omega] \in H^2(X, \mathbb{R})$ is Kähler. In other words, ω is assumed to be cohomologous to a Kähler form, i.e. a $(1,1)$-form which is smooth and strictly positive (see Sect. 3.6 for the more general setting of a big class). To simplify the presentation it will sometimes be convenient to assume that the local potentials of ω are smooth, but note that we do not assume that ω is semi-positive.

Let us start by recalling the definition of the Monge–Ampère measure $\mathrm{MA}_\omega(\varphi)$ in the case when φ is smooth. It is defined by

$$\mathrm{MA}_\omega(\varphi) := \frac{(\omega + dd^c\varphi)^n}{Vn!} =: \frac{(\omega_\varphi)^n}{Vn!}$$

where the normalization constant V ensures that $\mathrm{MA}_\omega(\varphi)$ has total unit charge. For simplicity we will omit the subscript ω in the definition of MA_ω.

The Monge–Ampère MA operator may be naturally identified with a one-form on the vector space $C^\infty(X)$ by letting

$$\left\langle \mathrm{MA}_{|\varphi}, v \right\rangle := \int_X MA(\varphi)v$$

for $\varphi, v \in C^\infty(X)$. As observed by Mabuchi, in the context of Kähler–Einstein geometry, the one-form MA is closed and hence it has a primitive $\mathcal{E}$ (defined up to an additive constant) on the space $C^\infty(X)$, i.e.

$$d\mathcal{E}_{|\varphi} = \mathrm{MA}(\varphi) \tag{3.1}$$

We fix the additive constant by requiring $\mathcal{E}(0) = 0$. Sometimes we will use a sub-script ω to indicate the dependence of $\mathcal{E}$ on ω. Integrating $\mathcal{E}_\omega$ along line segments one arrives at the following well-known formula

$$\mathcal{E}_\omega(\varphi) := \frac{1}{(n+1)!V} \sum_{j=0}^{n} \int_X \varphi \omega_\varphi^j \wedge (\omega)^{n-j} \tag{3.2}$$

Conversely, one can simply take this latter formula as the definition of $\mathcal{E}_\omega$ and observe that the following proposition holds (compare [13,15,22] for a more general singular setting).

Proposition 3.1. *The following holds*

- *The differential of the functional $\mathcal{E}_\omega$ at a smooth function φ is represented by the measure* $\mathrm{MA}(\varphi)$, *i.e.*
$$\frac{d}{dt}_{t=0}(\mathcal{E}_\omega(\varphi + tv)) = \int_X \mathrm{MA}(\varphi)v \tag{3.3}$$
- *$\mathcal{E}_\omega$ is increasing on the space of all smooth ω-psh functions*
- *$\mathcal{E}_\omega$ is concave on the space of all smooth smooth ω-psh functions [when $n = 1$ it is concave on all of $C^\infty(X)$].*

Note that the first point implies the second one, since the differential of $\mathcal{E}_\omega$ is represented by a (positive) measure.

Finally, we recall that the functional $\mathcal{E}_\omega(\varphi)$ may also be expressed as secondary Bott–Chern class. Indeed, comparing formula 3.2 with the notation in [56] gives $\mathcal{E}_\omega(\varphi) = \tilde{\mathrm{ch}}(h_0 e^{-\varphi}, h_0)$ where h_0 is a fixed metric whose curvature form is equal to ω and $\tilde{\mathrm{ch}}(h_1, h_0)$ is, up to normalization, the secondary Bott–Chern class attached to the first Chern class of L.

3.1.1. The singular setting and the space $\mathcal{E}^1(X,\omega)$. The subspace $\mathcal{E}^1(X,\omega)$ of $\mathrm{PSH}(X,\omega)$ consisting of all ω-psh functions of *finite energy* may be defined as follows (generalizing the classical Dirichlet spaces on Riemann surfaces). First we extend the functional $\mathcal{E}_\omega$ (formula 3.2) to all ω-psh functions by demanding that it still be increasing and usc, i.e. we define

$$\mathcal{E}_\omega(\varphi) := \inf_{\psi \geq \boldsymbol{\varphi}} \mathcal{E}_\omega(\psi) \in [-\infty, \infty[$$

where ψ ranges over all smooth ω-psh functions such that $\psi \geq \varphi$. Next, we let

$$\mathcal{E}^1(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) : \mathcal{E}_\omega(\varphi) > -\infty\},$$

which is a convex subspace, since $\mathcal{E}_\omega$ is concave. As a consequence of the monotonicity of $\mathcal{E}_\omega(u)$ and Bedford–Taylor's fundamental local continuity result for mixed Monge–Ampère operators one obtains the following proposition (cf. [22], Prop 2.10; note that $\mathcal{E}_\omega = -E_\chi$ for $\chi(t) = t$ in the notation in op. cit.)

Proposition 3.2. *The functional $\mathcal{E}_\omega$ is upper semi-continuous on* $\mathrm{PSH}(X,\omega)$, *concave and non-decreasing. Moreover, it is continuous wrt decreasing sequences in* $\mathrm{PSH}(X,\omega)$.

For any $\varphi \in \mathcal{E}^1(X,\omega)$ the (non-pluripolar) Monge–Ampère measure $\mathrm{MA}(\varphi)$ is well-defined and does not charge any pluripolar sets [22]. We collect the continuity properties that we will use in the following [22]

Proposition 3.3. *Let $(\varphi^{(i)}) \subset \mathcal{E}^1(X,\omega)$ be a sequence decreasing to $\varphi \in \mathcal{E}^1(X,\omega)$. Then, as $i \to \infty$,*

$$\mathrm{MA}(\varphi_i) \to \mathrm{MA}(\varphi), \quad \varphi_i \mathrm{MA}(\varphi_i) \to \varphi \mathrm{MA}(\varphi)$$

in the weak topology of measures and $\mathcal{E}_\omega(\varphi_j) \to \mathcal{E}_\omega(\varphi)$.

Remark 3.4. Here we have, for concreteness, chosen to phrase the definition of the functional $\mathcal{E}_\omega$ on $\mathrm{PSH}(X,\omega)$ [and the corresponding finite energy space $\mathcal{E}^1(X,\omega)$] in terms of its restriction to *smooth* ω-psh functions (implicitly assuming that ω has smooth potentials), where the classical formula 3.2 makes sense. However, this procedure is limited to the case when the class $[\omega]$ is Kähler, where Demailly's regularization result [26] may be applied. Anyway, following [15,22], a direct definition of $\mathcal{E}_\omega$ can be given in the general setting of a big class $[\omega]$ (assuming that ω has continuous local potentials), by replacing smooth ω-psh functions with ω-psh functions with minimal singularities and classical wedge products with the Bedford–Taylor product of positive currents with locally bounded potentials (computed on the Kähler locus of X). An equivalent direct definition can also be given using the non-pluripolar product of positive currents introduced in [22] to make sense of the formula 3.2. One then defines $\mathcal{E}^1(X,\omega)$ as the subspace of all ω-psh functions φ with full Monge–Ampère mass (which with our normalizations means hat $\mathrm{MA}(\varphi)$ is a probability measure) such that $\mathcal{E}_\omega(\varphi)$ is finite.

3.2. The pluricomplex energy $E_\omega(\mu)$ and potentials. Following [15] we define, for any given probability measure μ on X its (pluricomplex) energy by

$$E_\omega(\mu) := \sup_{\varphi \in PSH(X,\omega)} \mathcal{E}_\omega(\varphi) - \langle \varphi, \mu \rangle, \tag{3.4}$$

where the sup is taken over all $\varphi \in \mathrm{PSH}(X, \omega)$ (sometimes we will omit the subscript ω and simply write $E_\omega = E$). We will denote the subspace of all finite energy probability measures by

$$E_1(X) := \{\mu : E_\omega(\mu) < \infty\}$$

(which only depends on the *class* $[\omega]$ and not on the representative ω).

By Propositions 3.3 and Demailly's approximation theorem it is enough to take the sup over all Kähler potentials. But one point of working with less regular functions is that the sup can be attained. Indeed, as recalled in the following theorem

$$E_\omega(\mu) := \mathcal{E}_\omega(\varphi_\mu) - \langle \varphi_\mu, \mu \rangle \tag{3.5}$$

for a unique potential $\varphi_\mu \in \mathcal{E}^1(X, \omega)/\mathbb{R}$ of the measure μ if $E_\omega(\mu) < \infty$ where

$$\mathrm{MA}(\varphi_\mu) = \mu \tag{3.6}$$

Theorem 3.5. [15] *The following is equivalent for a probability measure μ on X:*

- $E_\omega(\mu) < \infty$
- $\langle \varphi, \mu \rangle < \infty$ for all $\varphi \in \mathcal{E}^1(X, \omega)$
- *μ has a potential $\varphi_\mu \in \mathcal{E}^1(X, \omega)$, i.e. Eq. 3.6 holds*

Moreover, φ_μ is uniquely determined mod $\mathbb{R}$, i.e. up to an additive constant and can be characterized as the function maximizing the functional whose sup defines $E_\omega(\mu)$ (formula 3.4). Even more generally: if φ_j is an asymptotically maximizing sequence sequence (normalized so that $\sup_X \varphi_j = 0$), i.e.

$$\liminf_{j\to\infty} \mathcal{E}(\varphi_j) - \langle \varphi_j, \mu \rangle = E_\omega(\mu)$$

then $\varphi_j \to \varphi_\mu$ in $L^1(X, \mu)$ and $\mathcal{E}(\varphi_j) \to \mathcal{E}(\varphi_j)$.

Remark 3.6. In the proof of the LDP in Sect. 4 we will only use the existence of a potential φ_μ for a given measure μ of finite energy (and not the uniqueness). As for the maximization property of φ_μ it will follow from the proof of the LDP, but it is also a simple consequence of the concavity of the functional $\mathcal{E}_\omega$ on the space $\mathcal{E}^1(X, \omega)$.

The previous theorem was proved in [15] using the variational approach in the more general setting of a big class $[\omega]$—one crucial ingredient in the proof is the differentiability Theorem 3.7 below. In the case when μ is a volume form Yau's [60] seminal theorem furnishes a *smooth* potential φ_μ, i.e a Kähler potential (using the continuity method for PDEs and delicate a priori estimates).

3.3. The psh projection P and the equilibrium measure. Given a compact non-pluripolar set K and an upper semi-continuos function φ we first define

$$(\Pi_{(K,\omega)}\varphi)(x) := (\sup\{\psi(x) : \psi \in \mathrm{PSH}(X, \omega),\ \psi \leq \varphi \text{ on } K\}) \tag{3.7}$$

We then define $P_K\phi$ as the upper-semi continuous regularization of the function $\Pi_{(K,\omega)}\varphi$. If φ is continuous, then $P_{(K,\omega)}\varphi$ is a bounded ω-psh function, which follows from the assumption that K be non-pluripolar [35]. We also recall that by a classical result of

Beford-Taylor concerning negligible sets $P_{(K,\omega)}\varphi = \Pi_{(K,\omega)}\varphi$ quasi-everywhere, i.e. away from a pluripolar subset of X.

Now the pluripotential equilibrium measure of a weighted non-pluripolar set (K,ω) may be defined as the following measure

$$\mu_{\text{eq}} := \text{MA}(P_{(K,\omega)}0)$$

supported on X. This is the global version of the original definition given by Siciak in the context of approximation theory in $\mathbb{C}^n$ (see [35] and references there in). An alternative *variational* characterization of the equilibrium measure was given very recently in [15], which will play a prominent role in this paper (see below).

When $K = X$ we will simply write

$$P_{(X,\omega)} = P_\omega (= P)$$

Note that if φ is usc then $\Pi_{(X,\omega)}\varphi = P_{(X,\omega)}\varphi$, using that the function $P_{(X,\omega)}\varphi$ is a contender for the sup defining $\Pi_{(K,\omega)}\varphi$. Moreover, if φ is continuous then so is $P_\omega\varphi$. Indeed, the lower semi-continuity of $P_\omega\varphi$ follows from Demailly's approximation result which allows us to write $P_\omega\varphi$ as an upper envelope of continuous functions (see [7] for further regularity results).

One of the main results in [13] is the following differentiability result which will play a crucial role in the present paper.

Theorem 3.7 (Berman and Boucksom [13]). *Let K be a compact non-pluripolar subset of X. Then the functional $\mathcal{E}_\omega \circ P_{(K,\omega)}$ is concave and Gateaux differentiable on $C^0(X)$. More precisely,*

$$d(\mathcal{E}_\omega \circ P_{(K,\omega)})_{|\varphi} = \text{MA}(P_{(K,\omega)}\varphi)$$

It should be emphasized that the differentiability result above is in a sense very surprising (even when φ is smooth and $K = X$) . Indeed, the projection operator $P_{(K,\omega)}$ is certainly not differentiable. Moreover, the functional $\mathcal{E}_\omega \circ P_{(K,\omega)}$ is in general not *two* times differentiable. From a statistical mechanical point of view the one time differentiability corresponds to an absence of a first order phase transition (see [9]). An important ingredient in the proof of the previous theorem is the following orthogonality relation:

$$\left\langle \text{MA}(P_{(K,\omega)}\varphi)), \varphi - P_{(K,\omega)}\varphi \right\rangle = 0 \tag{3.8}$$

saying that $P_{(K,\omega)}\varphi = \varphi$ a.e. wrt the measure $\text{MA}(P_{(K,\omega)}\varphi)$

3.4. Further properties of the energy $E_\omega(\mu)$. By the first equality in formula 3.9, appearing in the following proposition, the pluricomplex energy E_ω on can be seen as the *Legendre–Fenchel transform* of the functional $u \mapsto -(\mathcal{E}_\omega \circ P_{(K,\omega)})(-u)$ (compare formula 4.9):

Proposition 3.8. *The following properties of the energy $E(\mu)$ (formula 3.4) hold:*

- *Assume that $E(\mu) < \infty$. Then the sup defining the energy $E(\mu)$ may be taken over the subset of all* continuous *ω-psh functions. More generally, if μ is supported on a non-pluripolar compact set K in X, then*

$$E(\mu) := \sup_{\varphi \in C^0(X)} \left(\mathcal{E}(P_K\varphi) - \int_X \varphi\mu \right) = \sup_{\varphi \in C^0(X) \cap PSH(X,\omega)} \left(\mathcal{E}(P_K\varphi) - \int_X \varphi\mu \right) \tag{3.9}$$

- *The functional E is lower semi-continuous (lsc) on $\mathcal{P}(X)$.*

Proof. (compare the proof of Theorem 5.3 in [15]). Given the set K and $\varphi \in C^0(X) \cap \mathrm{PSH}(X,\omega)$ we note that. by definition, $P_K\varphi \geq \varphi$ on X (and $P_K\varphi = \varphi$ on K). Hence, $\mathcal{E}(P_K\varphi) - \int_X \varphi\mu \geq \mathcal{E}(\varphi) - \int_X \varphi\mu$ and since $P_K\varphi$ is a candidate for the sup defining $E(\mu)$ this proves the statement when φ ranges over continuous ω-psh functions. Next, if φ is merely continuous then we decompose

$$\mathcal{E}(P_K\varphi) - \int_X \varphi\mu = \left(\mathcal{E}(P_K\varphi) - \int_X P_K\varphi\mu\right) + \int_X (P_K\varphi - \varphi)\mu$$

Note that $P_K\varphi \leq \varphi$ away from a pluripolar set (since $P_K\varphi = \Pi_K\varphi$ quasi-everywhere). But since $E(\mu) < \infty$ the measure μ does not charge pluripolar sets [15] and hence setting $\psi := P_K\varphi$ gives $\mathcal{E}(\psi) - \int_X \psi\mu \leq E(\mu)$. Finally, writing ψ as a decreasing limit of elements $\psi_j \in C^0(X) \cap PSH(X,\omega)$ and using the previous case for $\varphi = \psi_j$ finishes the proof of the first point. As for the lower semi-continuity of E it follows immediately from the fact that E is defined as a sup of continuous functionals. □

In Sect. 3.6 we will give a different proof of the Legendre transform relation referred to above, which has the virtue of applying to a general big class. In our second proof of Theorem 1.1 we will also have use for the following approximation lemma of independent interest.

Lemma 3.9. *Assume that μ is a probability measure supported on a compact set K such that $E(\mu)$ is finite. Let φ_μ be a potential of μ and take a sequence φ_j in $C^0(X) \cap$ $\mathrm{PSH}(X,\omega)$ such that φ_j decreases to φ_μ. Then $\mu_j := \mathrm{MA}(P_{(K,\omega)}\varphi_j) \to \mu$ and $E(\mu_j) \to \mu$.*

Proof. First observe that $P_{(K,\omega)}\varphi_j$ decreases to $P_{(K,\omega)}\varphi_\mu$, using that $P_{(K,\omega)}$ is decreasing (compare Sect. 3.6). Moreover, by Lemma 3.13 below $P_{(K,\omega)}\varphi_\mu = \varphi_\mu$ and hence $P_{(K,\omega)}\varphi_j$ decreases to φ_μ. But then it follows from Proposition 3.3 that $\mu_j :=$ $\mathrm{MA}(P_{(K,\omega)}\varphi_j) \to \mu$ and $E(\mu_j) \to \mu$. □

Remark 3.10. The Lemma remains true if φ_j is merely assumed to be continuous, which is useful when generalizing Theorem 1.1 to the case when the line bundle L is merely big (compare Sect. 3.6).

We also recall that a direct computation yields the following explicit expression for the energy of a measure in terms of the potential φ_μ:

$$E_\omega(\mu) = \frac{1}{V}\sum_{j=0}^{n-1} \frac{1}{j+2}\int d\varphi_\mu \wedge d^c\varphi_\mu \wedge \frac{\left(dd^c\varphi_\mu\right)^j}{j!} \wedge \frac{\omega^{n-1-j}}{(n-1-j)!}, \tag{3.10}$$

(where the right hand side may be written as $(I_\omega - J_\omega)(\phi_\mu)$, in terms of Aubin's functionals I_ω and J_ω and where the wedge products should, in general, be interpreted as non-pluripolar products; compare [15] and references therein). Even though we will not use this formula in the proofs it will appear in the discussion in Sect. 6. Note when $n = 1 E_\omega$ is hence a multiple of the classical Dirichlet energy and may also be expressed as

$$E_\omega(\mu) = -\frac{1}{2}\int G_\omega(x,y)\mu(x) \otimes \mu(y) \tag{3.11}$$

(where we have assumed $V = 1$ for simplicity), where $G_\omega(x,y)$ is the Green function defined by $d_x d^c_x G_\omega(x,y) = \delta_y(x) - \omega(x)$ and the normalization condition $\int G(x,y)\omega(y)$ $= 0$.

3.5. The rate functional and electrostatic capacity. Given a (weighted) non-pluripolar compact set K we define the *rate functional* $H_{(K,\omega)}$ as the normalized energy functional:

$$H_{(K,\omega)}(\mu) := E_{\omega}(\mu) - C(K,\omega) \tag{3.12}$$

where $C(K,\omega)$ is the following constant

$$C(K,\omega) := \inf_{\mu \in \mathcal{M}_1(K)} E_{\omega}(\mu) \tag{3.13}$$

The constant $e^{-\frac{n}{n+1}C(K,\omega)}$ was called the *pluricomplex electrostatic capacity* in [15]; it generalizes the logarithmic capacity in $\mathbb{C}$ and Leja's transfinite diameter in $\mathbb{C}^n$. Moreover, as shown in [15]

$$C(K,\omega) := \mathcal{E}_{\omega}(P_{(K,\omega)}0) < \infty \tag{3.14}$$

and hence the rate functional $H_{(K,\omega)}$, defined above, may also be expressed as

$$H_{(K,\omega)} = E_{\omega}(\mu) - \mathcal{E}_{\omega}(P_{(K,\omega)}0) \tag{3.15}$$

It is in this latter form that the rate functional will appear in our second proof of the LDP and as a byproduct of the LDP we will then rederive formula 3.13.

Proposition 3.11. *Let K be a non-pluripolar compact subset of X. The functional $H_{(K,\omega)} : \mathcal{P}(K) \to [0,\infty]$ is a* good rate functional*, i.e. it is lower semi-continuous and proper. It has a unique minimizer which coincides with μ_{eq}, the equilibrium measure of (K,ω), defined in Sect. 3.3.*

Proof. First observe that, by definition, the functional $H_{(K,\omega)}$ is lsc iff the functional E is lsc, which holds by Proposition 3.8. To prove that $H_{(K,\omega)}$ is proper we must prove that the sublevel sets $\{H_{(K,\omega)} \leq C\}$ are compact for any constant C. Since $H_{(K,\omega)}$ is lsc these sets are closed in $\mathcal{P}(K)$. But by the compactness of $\mathcal{P}(K)$ any closed set is compact. As for the uniqueness it follows from the differentiability Theorem 3.7 combined with standard convexity arguments (see [15])—alternatively it will follow from the LDP in Theorem 1.1). □

3.6. Complements: the pluricomplex energy as a Legendre transform in the big case. In this section we will show that the pluricomplex energy E_{ω} may be realized as a Legendre–Fenchel transform when the class $[\omega] \in H^2(X,\mathbb{R})$ is merely assumed *big,* which by definition means that the class contains some Kähler current T, i.e. a positive current T such that $T \geq \omega_0$ for some smooth and strictly positive form ω_0 on X. This will be needed in order to extend Theorem 1.1 to big (but not necessarily ample) line bundles; compare Sect. 4.4.1. We recall that the setting of a big class $[\omega]$ is the most general setting for the global pluripotential theory outlined above and we refer to [15,22] for further references (compare Remark 3.4).

The new difficulty that appears in the case of a general big cohomology class is that we can no longer approximate a given element in $\text{PSH}(X,\omega)$ with elements in $\text{PSH}(X,\omega) \cap C^0(X)$, as the latter space will in general be empty (in fact, typically, no element will even be bounded from below!). Anyway, we will show how to bypass this problem by taking the approximating sequence to consist of functions which are merely continuous and use the orthogonality relation 3.8.

Proposition 3.12. *Let X be a compact complex manifold X with a a big class and $[\omega]$ and K a non-pluripolar compact subset of X. Then*

$$E_\omega(\mu) = \sup_{\varphi \in C^0(X)} \left(\mathcal{E}_\omega \circ P_{(K,\omega)}(\varphi) - \int_X \varphi d\mu \right)$$

for any probability measure μ supported on K. In other words, $E_\omega(\mu)$ coincides with the Legendre–Fenchel transform at μ of the functional $u \mapsto -\mathcal{E}_\omega \circ P_{(K,\omega)}(-u)$ on $C^0(X)$ (compare formula 4.9).

Proof. First we note that the *lower* bound is proved exactly as before: since $P_{(K,\omega)}(\varphi)$ is a contender for the sup defining $E_\omega(\mu)$ we have

$$\begin{aligned} E_\omega(\mu) &\geq \sup_{\varphi \in C^0(X)} \left(\mathcal{E}_\omega(P_{(K,\omega)}(\varphi)) - \int_X P_{(K,\omega)}(\varphi) d\mu \right) \\ &\geq \sup_{\varphi \in C^0(X)} \left(\mathcal{E}_\omega(P_{(K,\omega)}(\varphi)) - \int_X \varphi d\mu \right), \end{aligned}$$

using, in the last step, that

$$P_{(K,\omega)}(\varphi) \leq \varphi \quad \mu - \text{a.e.} \tag{3.16}$$

Indeed, the inequality holds quasi-everywhere on K and since μ is supported on K and does not charge pluripolar sets (since $E_\omega(\mu) < \infty$) the inequality 3.16 follows.

Next we turn to the proof of the *upper* bound. To fix ideas we start with the case when $X = K$ and write P for the corresponding projection operator $P_{(X,\omega)}$. We will denote by ψ a potential of the measure μ, i.e. the ω-psh function which is uniquely determined mod $\mathbb{R}$ by the property that $\mathrm{MA}(\psi) = \mu$ [15]. Since ψ is ω-psh and in particular usc there exists a sequence φ_j in $C^0(X)$ decreasing to ψ. Setting $\psi_j := P(\varphi_j)$ then gives a decreasing sequence of ω-psh function converging to ψ, as follows immediately from the fact that P is decreasing (indeed, $\varphi_j \geq \psi$ gives $P(\varphi_j) \geq P(\psi) = \psi$ and $\tilde{\psi} := \lim_{j\to\infty} P(\varphi_j) \leq \lim_{j\to\infty} \varphi_j = \psi$, forcing $\tilde{\psi} = \psi$ as desired). In particular,

$$E_\omega(\mu) = \mathcal{E}_\omega(\psi) - \int_X \psi \mathrm{MA}(\psi) = \lim_{j\to\infty} \mathcal{E}_\omega(\psi_j) - \int_X \psi_j \mathrm{MA}(\psi)$$

To prove the upper bound in the proposition it will thus be enough to show the following

$$\text{Claim: } \liminf_{j\to\infty} \int_X (\psi_j - \varphi_j)\mathrm{MA}(\psi) \geq 0$$

By the orthogonality relation 3.8 it is equivalent to show that

$$\liminf_{j\to\infty} \int_X (\psi_j - \varphi_j) \left(\mathrm{MA}(\psi) - \mathrm{MA}(\psi_j)\right) \geq 0$$

To this end we rewrite the previous integral as a sum of three terms

$$\begin{aligned} A_j + B_j + C_j = &\int (\psi_j - \psi) \left(\mathrm{MA}(\psi) - \mathrm{MA}(\psi_j)\right)) \\ &+ \int (\psi - \varphi_j)\mathrm{MA}(\psi) + \int (\varphi_j - \psi)\mathrm{MA}(\psi_j) \end{aligned}$$

First note that $A_j \to 0$. Indeed, since ψ_j decreases to ψ in $\mathcal{E}^1(X,\omega)$ this follows immediately from Proposition 3.3 (which also holds in the big case [22]). Next, by the usual monotone convergence theorem of integration theory we also have that $B_j \to 0$.

Finally since $\varphi_j \geq \psi$ we have $C_j \geq 0$ for any j, which concludes the proof of the claim above.

Finally, to we turn to the case of a general compact subset K. First, a simple modification of the previous argument gives that $\psi_j := P_{(K,\omega)}\varphi_j$ decreases to $P_{(K,\omega)}\psi$. Then the previous argument can be repeated word for word once we have established that $P_{(K,\omega)}\psi = \psi$, which is the content of the following Lemma. □

Lemma 3.13. *Assume that $\psi \in \mathcal{E}^1(X,\omega)$ and that $\mu := MA(\psi)$ is supported on a compact subset K of X. Then $P_{(K,\omega)}\psi = \psi$.*

Proof. First observe that $P_{(K,\omega)}\psi \geq \psi$ (indeed, $P_{(K,\omega)}\psi \geq \Pi_{(K,\omega)}\psi \geq \psi$ since ψ is a contender for the sup defining $\Pi_{(K,\omega)}\psi$). Moreover, as explained above $P_{(K,\omega)}\psi \leq \psi$ a.e. wrt $\mu(:= MA(\psi))$ and hence

$$\mathcal{E}_\omega(\psi) - \int_X \psi d\mu := E(\mu) \leq \mathcal{E}_\omega(P_{(K,\omega)}(\psi)) - \int_X P_{(K,\omega)}(\psi) d\mu,$$

but then it follows from the uniqueness of maximizer in Theorem 3.5 (which also holds in the big case [15]) that $P_{(K,\omega)}(\psi) - \psi = c \in \mathbb{R}$. Finally, integrating against μ and using that $P_{(K,\omega)}\psi = \psi$ a.e. wrt μ concludes the proof. □

Remark 3.14. The property that $P_K\varphi_\mu = \varphi_\mu$ for any measure μ of finite energy supported on a compact set K was used without any explicit proof in the previous version [10] of the present paper (thanks to Norm Levenberg for pointing this out). When φ_μ is continuous this property is an immediate consequence of the standard domination principle saying that $\psi \leq \varphi MA(\varphi)-$ a.e. implies that $\psi \leq \varphi$ everywhere, if ψ and φ are ω-psh and φ is continuous. A proof of the more general domination principle where φ is only assumed to have finite energy, was supplied by Dinew in [21] in the setting of a Kähler class. It may be worth pointing out that the variational argument used in the proof of the previous lemma also gives a simple proof of the latter domination principle in the general setting of a big class (just set $\mu := \mathrm{MA}(\varphi)$ and apply the variational argument to $\max\{\varphi, \psi\}$).

4. The Large Deviation Principle in the Compact Case

In the following we will assume given an ample line bundle $L \to X$ over a compact complex manifold X and a continuous Hermitian metric $\|\cdot\|$ on L, whose normalized curvature current we will denote by ω. We will also assume given a (finite) measure ν on X satisfying certain (very weak) regularity properties. These are of Bernstein–Markov type and they are formulated with respect to a compact subset K. In practice the latter set often appears as the support of ν, but this is not necessary.

4.1. Bernstein–Markov measures. Following [14], we will say that a measure ν is *Bernstein–Markov wrt* (K,ω) if, given any positive number ϵ, there exist C_ϵ such that

$$\sup_{x\in K} \|s_k\|^2(x) \leq C_\epsilon e^{k\epsilon} \int_X \|s_k\|^2 d\nu \tag{4.1}$$

for any element s_k of $H^0(X, kL)$, where the norms are taken wrt the metric whose curvature current is ω (in particular, if K is non-pluripolar, then any such measure ν defines a non-degenerate L^2-norm on the spaces $H^0(X, kL)$). Strictly speaking, it may be appear more logical to say that ν is *Bernstein–Markov wrt* $(K, \|\cdot\|)$ since it is the metric $\|\cdot\|$ which appears in the previous equality. But the point is that since $\|\cdot\|$ is determined by ω up to a multiplicative number it does not matter which metric with curvature form ω we take.

More generally, we will say that the measure ν is Bernstein–Markov wrt (K, ω, φ) (or simply *B-M wrt* (K, φ) since ω has been fixed) if

$$\sup_{x\in K}(\|s_k\|^2 e^{-k\varphi}(x)) \leq C_\epsilon e^{k\epsilon} \int_X \|s_k\|^2 e^{-k\varphi} d\nu \tag{4.2}$$

for a given $\varphi \in C^0(K)$ (where, of course, the constant C_ϵ depends on φ). The measure ν is *strongly Bernstein–Markov wrt the set* K if the previous inequality holds for *any* continuous φ. Note that this definition is independent of the choice of a ω or equivalently on the choice of a fixed continuous metric $\|\cdot\|$ on L, since the logarithm of the quotient of any two such metrics is a continuous function.

There is also a stronger variant of the latter Bernstein–Markov property defined as follows: the measure ν is said to be *strongly Bernstein–Markov wrt quasi-psh functions* on K if for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\sup_{x\in K} e^{p\psi} \leq C_\epsilon e^{p\epsilon} \int_X e^{p\psi} d\nu \tag{4.3}$$

for all $p > 0$ and any function ψ on X which is ω-psh wrt some positive current $\omega \in c_1(L)$ with continuous potentials. This notion thus only depends on the pair (ν, K), once we have fixed the line bundle L. Taking $\psi(x) := \frac{1}{k} \log \|s_k\|^2 (x) - \varphi(x)$ and $p = k$ shows that the latter notion of BM-property indeed implies the previous one.

It should be stressed that, in practice, almost all measures ν appearing in geometrically reasonable situations satisfy the last and strongest notion of Bernstein–Markov property above. But we will not go further into concrete examples, referring the reader instead to [14] and references therein.

Remark 4.1. Since we are using the language of ω-psh functions, as opposed to the reference [14], where psh weights on line bundles are used, it may be helpful to compare our notation with the notion in op. cit. In fact, a measure ν is strongly B-M wrt quasi-psh functions on K precisely when it is *B-M with respect to the weighted set* (K, ϕ) for any continuous weight ϕ on the line bundle L, in the sense of [14]. One virtue of the definition used here is that it also applies without change to the case when $c_1(L)$ is replaced with a fixed pseudoeffective class in $H^{1,1}(X)$. It may also be illuminating to compare with the classical terminology in pluripotential theory in $\mathbb{C}^n$ (see [18]) where ν is said to have the B-M-property wrt a compact set K in $\mathbb{C}^n$ if ν has the B-M-property wrt the weighted set $(K, 0)$ (i.e. $\omega = 0$ on K) in our terminology. The point is that for a general complex manifolds X there is no canonical choice of form ω for a given set K which is the reason for the terminology used here.

4.2. Definition of the determinantal probability measure. We start by recalling the definition of the determinantal probability measure given in the introduction of the paper. Given an ample line bundle $L \to X$ with a continuous Hermitian metric $\|\cdot\|$ (whose normalized curvature current is denoted by ω) and a measure ν on X one obtains a sequence of probability measures $\mu^{(N_k)}$ on X^{N_k} defined as follows. First we set $k = 1$ and recall that N denotes the dimension of the vector space $H^0(X, L)$. Hence, the top exterior power $\Lambda^N H^0(X, L)$ is one-dimensional and we fix a a non-zero element

$\det \Psi \in \Lambda^N H^0(X, L)$. We may identify $\det \Psi$ with a holomorphic section of $L^{\boxtimes N}$ over the N-fold product X^N, using the natural embedding

$$\Lambda^N H^0(X, L) \hookrightarrow H^0(X, L)^{\otimes N} \simeq H^0(X^N L^{\boxtimes N})$$

Now we may define the probability measure $\mu^{(N)}$ on X^N by

$$\mu^{(N)} := \frac{\|\det \Psi\|^2}{\mathcal{Z}} \nu^{\otimes N}$$

where the point-wise norm is computed the fixed Hermitian metric on L and where the normalizing constant is the L^2-norm of $\det \Psi$ induced by the pair $(\nu, \|\cdot\|)$:

$$\mathcal{Z} := \|\det \Psi\|^2_{L^2(X^N, \nu^{\otimes N})}$$

By homogeneity $\mu^{(N)}$ is invariant under scaling of $\|\cdot\|$ and hence it only depends on the data (ν, ω). Now the whole sequence $\mu^{(N_k)}$ (and the corresponding normalization constants $\mathcal{Z}_k$) is defined by replacing L with its kth tensor power kL and using the induced norms. The constant $\mathcal{Z}_k$ depends multiplicatively on the choice of generator $\det \Psi_k \in \Lambda^{N_k} H^0(X, kL)$ but, by homogeneity, the corresponding probability measure $\mu^{(N_k)}$ does not.

4.2.1. The free energy functional $\mathcal{F}_k$ and reference data. A leading role in the proof of Theorem 1.1 will be played by the following (free energy type) functional on $C^0(X)$:

$$\mathcal{F}_k[\varphi] := -\log \int_{X^{N_k}} \|\det \Psi_k\|^2 e^{-k\varphi} (d\nu)^{\otimes N_k} \tag{4.4}$$

Strictly speaking this functional depends on the choice of generator $\det \Psi_k$ of $\Lambda^{N_k} H^0$ (X, kL), but the point is that *differences* $\mathcal{F}_k[\varphi] - \mathcal{F}_k[\psi]$ are independent of the choice of generator. Let us explain this in more detail. First, fixing a basis $\Psi_1^{(k)}, \ldots, \Psi_N^{(k)}$ in $H^0(X, kL)$ induces a generator $\det \Psi_k$ which may be expressed as follows, when evaluated $(x_1, \ldots, x_{N_k})$:

$$(\det \Psi_k)(x_1, \ldots, x_{N_k}) = \det \left(\Psi_i^{(k)}(x_j) \right) \in kL_{x_1} \otimes \cdots \otimes kL_{x_{N_k}} \tag{4.5}$$

Now changing the basis to $\Phi_1^{(k)}, \ldots, \Phi_N^{(k)}$ we note that $\det \Phi_k = A \det \Phi_k$ where A is the corresponding change of base matrix, which is thus constant over X^{N_k}. In particular,

$$\|\det \Phi_k\|^2 (x_1, \ldots, x_{N_k}) = \det G \|\det \Psi_k\|^2 ((x_1, \ldots, x_{N_k}) \quad G_{ij} = \left\langle \Phi_i^{(k)}, \Phi_j^{(k)} \right\rangle_{L^2} \tag{4.6}$$

where the L^2-norm on $H^0(X, kL)$ is the one for which $\Psi_i^{(k)}$ is an orthonormal basis. To fix a specific generator it will (in particular in Sect. 4.5) be convenient to to fix some auxiliary "reference data" consisting of a measure ν_0, a continuous function φ_0 and compact set K_0 such that ν_0 is Bernstein–Markov wrt (K_0, φ_0). We can then take the base $\Psi_{k,1}, \ldots, \Psi_{k,N}$ above to be orthonormal wrt the inner product on $H^0(X, kL)$ induced by (ν_0, φ_0) (i.e. by $(\nu_0, \|\cdot\| e^{-\varphi_0})$. The point is that the corresponding functional $\mathcal{F}_k[\varphi]$ then has, after rescaling, a limit as $k \to \infty$ (if ν has the Bernstein–Markov wrt (K, φ); see Theorem 4.6 below).

Remark 4.2. In order to compare with the concrete cases studied in Sect. 2, it may be illuminating to express $\mathcal{Z}_k$ in terms of the local notation introduced in Sect. 1.6. First we note that, given a local holomorphic trivialization of L, the section $\det \Psi_k$ above may be locally represented by a local holomorphic function f_k on X^{N_k}, hence, $\mathcal{Z}_k$ may, using local notation, be written as

$$\mathcal{Z}_k = \int_{X^{N_k}} |(f_k(x_1, \ldots, x_k)|^2 e^{-k(\phi(x_1)+\cdots+\phi(x_{N_k})}(d\nu)^{\otimes N_k}$$

where ϕ is the local weight of the fixed metric $\|\cdot\|$ on L (in particular, locally $\omega = dd^c\phi$). In the setting of Sect. 2 the function f_k thus appears as the corresponding Vandermonde determinant.

4.3. Definition of a large deviation principle (LDP). Let us recall the general definition of a LDP due to Donsker and Varadhan (see for example the book [25]):

Definition 4.3. *Let $\mathcal{P}$ be a Polish space, i.e. a complete separable metric space.*

(i) *A function $H : \mathcal{P} \to]-\infty, \infty]$ is a* rate function *iff it is lower semi-continuous. It is a* good rate function *if it is also proper.*
(ii) *A sequence Γ_k of measures on $\mathcal{P}$ satisfies a* LDP *with* speed r_k *and* rate function H *iff*

$$\limsup_{k\to\infty} \frac{1}{r_k} \log \Gamma_k(\mathcal{F}) \leq - \inf_{\mu\in\mathcal{F}} H(\mu)$$

for any closed subset $\mathcal{F}$ of $\mathcal{P}$ and

$$\liminf_{k\to\infty} \frac{1}{r_k} \log \Gamma_k(\mathcal{G}) \geq - \inf_{\mu\in G} H(\mu)$$

for any open subset $\mathcal{G}$ of $\mathcal{P}$.

Note that when the measures Γ_k are probability measures the LDP implies that $\inf_{\mu\in\mathcal{P}} H(\mu) = 0$ (just take $\mathcal{F} = \mathcal{G} = \mathcal{P}$).

Let now $\mathcal{P} = \mathcal{P}(K)$ be the space of all probability measures on X which is a Polish space, where the topology corresponds to the weak convergence of measures.

Given a set $\mathcal{F}$ in $\mathcal{P}(K)$ we will, somewhat abusively, write

$$K^N \cap \mathcal{F} := K^N \cap (\delta_N)^{-1}(\mathcal{F})$$

where δ_N denotes the map defined by the empirical measure 1.6.

4.4. Proof of Theorem 1.1 using the Gärtner–Ellis theorem.. We start by recalling one of the main results in [13] (see Thm A in opus. cit) which may be formulated as follows:

Theorem 4.4 (Berman and Boucksom [13]). *Assume that the measure ν has the Bernstein–Markov property wrt (K, φ) and (K, ψ). Then the following asymptotics for the functional $\mathcal{F}_k$* (formula 4.4) hold:

$$\lim_{k\to\infty} \frac{1}{kN_k}(\mathcal{F}_k[\varphi] - (\mathcal{F}_k[\psi]) = \mathcal{F}_K(\varphi) - \mathcal{F}_K(\psi),$$

where $\mathcal{F}_K(\varphi) := \mathcal{E} \circ P_{(K,\omega)}$.

Moreover, by Theorem 3.7 (which is Theorem B in [13]) the functional $\mathcal{F}_K$ is Gateau differentiable on $C^0(K)$. As will be next explained Theorem 1.1 now follows from an application of a suitable version of the Gärtner–Ellis theorem. The version we will need may be formulated as follows (see [25] (Cor. 4.6.14, p. 148) and references therein):

Theorem 4.5 (Abstract Gärtner–Ellis theorem). *Let $\mathcal{M}$ be a locally convex Hausdorff topological vector space and Γ_k a sequence of Borel measures on $\mathcal{M}$ which is exponentially tight. Assume that there is a sequence of positive numbers r_k such that the Laplace transforms $\widehat{\Gamma}_k$ (see formula 4.7) seen as functionals on the dual $\mathcal{M}^*$, satisfy*

$$\frac{1}{r_k}\log \widehat{\Gamma}_k[r_k u] \to \Lambda[u]$$

for any u in $\mathcal{M}^*$ where the functional Λ is Gateau differentiable on $\mathcal{M}^*$. Then Γ_k satisfies a LDP with speed r_k and with a rate functional $H := \Lambda^*$ on $\mathcal{M}$, i.e. H is the Legendre–Fenchel transform of Λ (see formula 4.9)

To apply this theorem to the present setting we let $\mathcal{M}(= \mathcal{M}(K))$ be the space of all signed finite Borel measures on K with its usual weak topology, i.e. $\mu_j \to \mu$ iff

$$\langle u, \mu_j \rangle := \int_X u\mu_j \to \int_X u\mu$$

for all $u \in C^0(K)$. As is well-known $\mathcal{M}$ is a locally convex Hausdorff topological vector space and it is the topological dual of the vector space $C^0(K)$. We let Γ_k be the laws of the empirical measure of the determinant process defined above:

$$\Gamma_k := \delta_{N_k}\left(\mu^{(N_k)}\right)$$

which are thus supported on the convex subspace $\mathcal{P}(K)$ in $\mathcal{M}(K)$ consisting of all probability measures. In particular, since K is compact so is $\mathcal{P}(K)$ and hence the tightness condition on Γ_k is automatically satisfied. The Laplace transform of Γ_k is, by definition (with our sign convention), the following functional $\widehat{\Gamma}_k$ on the dual $\mathcal{M}^* = C^0(K)$:

$$\widehat{\Gamma}_k[u] := \int_{\mathcal{M}} d\Gamma_k(\mu)e^{\langle u,\mu\rangle} \tag{4.7}$$

Hence, pulling back the integral above to K^N allows us to write

$$-\log \widehat{\Gamma}_k[-kN_k\varphi] = \mathcal{F}_k[\varphi] - \mathcal{F}_k[0],$$

and thus, by Theorem 4.4,

$$\frac{1}{kN_k}\log \widehat{\Gamma}_k[kN_k u] \to \Lambda[u] := -\left((\mathcal{E}\circ P_{(K,\omega)})(-u) - \mathcal{E}_\omega(P_{(K,\omega)}0)\right) \tag{4.8}$$

if ν has the Bernstein–Markov property wrt (K, φ). In other words, the convergence holds for *all* u if ν has the *strong* Bernstein–Markov property wrt K, which we have indeed assumed. All in all this means that we can apply the Gärtner–Ellis theorem above and deduce that an LDP holds on the space $\mathcal{M}(K)$ with rate functional

$$H(\mu) = \Lambda^*(\mu) := \sup_{u\in C^0(K)} (\Lambda(u) - \langle u, \mu\rangle) \tag{4.9}$$

In particular, restricting to the subspace $\mathcal{P}(K)$, setting $\varphi = -u$ and using Proposition 3.8 concludes the proof of Theorem 1.1.

4.4.1. The case of a big line bundle. Let us briefly point out that Theorem 1.1 remains true for a general *big* line bundle L, i.e. a holomorphic line bundle $L \to X$ over a complex manifold such that $V := \limsup_{k\to\infty}(\dim H^0(X, kL)/k^n)$ is strictly positive. Equivalently, this means that the class $c_1(L)$ is big in the sense of Sect. 3.6 (compare [22] and references therein). To see this first recall that the results in [13] (i.e. Theorem A and B) invoked in the previous proof where already shown in [13] to hold for any big line bundle. Hence, the Gärtner–Ellis theorem provides, just as before, a LDP with rate functional $\Lambda[u]$ as in formula 4.8. Finally, as shown in Sect. 3.6 the Legendre transform of the latter functional still coincides with the pluricomplex energy $E_\omega(\mu)$ in the general setting of a big class.

4.5. A direct proof of Theorem 1.1 using Fekete points. In this section we will give a different proof of 1.1, which uses the convergence of Fekete points shown in [14] as a replacement for the Gärtner–Ellis theorem. The argument will also reveal that the upper bound in the LDP holds without any Bernstein–Markov assumption on the measure ν.

4.5.1. Preliminaries on asymptotics. In this section we will fix reference data (ν_0, K, φ_0) as in Sect. 4.2.1 and a generator $\det \Psi_k$ compatible with this data. We then set

$$\tilde{\mu}^{(N_k)} := \|\det \Psi_k\|^2 \, \nu^{\otimes N} \tag{4.10}$$

for the corresponding *non-normalized* measure on X^{N_k}. Given a continuous function φ on X it will also be convenient to use the notation $\mu^{(N_k)}_{k\varphi}$ for the determinantal probability measure on X^{N_k} obtained by replacing the fixed point-wise norms $\|\cdot\|^2$ by $\|\cdot\|^2 \, e^{-k\varphi(\cdot)}$. Equivalently, $\mu^{(N_k)}_{k\varphi}$ can be written as the "tilted" probability measure

$$\mu^{(N_k)}_{k\varphi} = \frac{1}{\mathcal{Z}_{k\varphi}} \mu^{(N_k)} e^{-k\varphi} \tag{4.11}$$

where φ is the corresponding *linear statistic* $\varphi(x_1) + \cdots + \varphi(x_{N_k})$ on X^{N_k} and

$$\mathcal{Z}_{k\varphi} = \int_{X^{N_k}} \mu^{(N_k)} e^{-k\varphi}$$

Recall that ν is assumed to be strongly Bernstein–Markov with respect to a compact set K.

First we collect the following results proved in [13,14].

Theorem 4.6. *Let K be non-pluripolar subset of X.*

- *[13] The following convergence holds:*

$$k^{-(n+1)} \log \|\det \Psi_k\|^2_{L^\infty(k\varphi, K^{N_k})} \to -\mathcal{E}(P_K\varphi) + \mathcal{E}(P_{K_0}\varphi_0) \tag{4.12}$$

(where the norms are computed wrt the metric on L whose curvature current is ω) and if the measure ν has the Bernstein–Markov property wrt (K, φ), then we also have

$$k^{-(n+1)} \log \|\det \Psi_k\|^2_{L^2(k\varphi, K^{N_k}, \nu)} \to -\mathcal{E}(P_K\varphi) + \mathcal{E}(P_{K_0}\varphi_0) \tag{4.13}$$

- *[14] Let* $(\mathbf{x}_k)$ *be a sequence of configurations in* K *(i.e.* $\mathbf{x}_k \in K^{N_k}$*) which are asymptotically Fekete, i.e. such that*

$$\liminf_{k\to\infty} k^{-(n+1)} \log \left(\mu_{k\varphi}^{(N_k)}(\mathbf{x}_k)/\nu^{\otimes N_k}\right) \geq 0$$

Then $\mu_k := \delta_{N_k}(\mathbf{x}_k)$ *converges weakly to the equilibrium measure* $\mathrm{MA}(P_{(K,\omega)}\varphi)$.
- *[14] If the measure* ν *has the Bernstein–Markov property wrt* (K, φ),

$$\left(\mathbb{E}_{k\varphi}(\delta_{N_k}) :=\right) \int_{K^{N-1}} \mu_{k\varphi}^{(N_k)} \to \mathrm{MA}(P_{(K,\omega)}\varphi)$$

weakly.

In Sect. 4.8 we will repeat the simple argument used in [13] to deduce 4.13 from 4.12 in the previous theorem.

Before continuing, it may be illuminating to compare the previous theorem with the results in the previous section. First, the convergence in 4.13 is a refinement of Theorem 4.4, using the transformation property 4.6 (in fact, this is the way Theorem 4.4 is proved in [13]). Next, we also recall that the second point in the previous theorem is obtained from the first point together with the differentiability Theorem 3.7, using a completely elementary convex analysis argument.

We will also need a localized version of the last two points in the previous theorem. To this end, define the following set:

$$A_{k\varphi} := \left\{\mathbf{x}_k \in K^{N_k} : k^{-(n+1)} \log \left(\mu_{k\varphi}^{(N_k)}(\mathbf{x}_k)/\nu^{\otimes N_k}\right) \geq -1/k\right\} \tag{4.14}$$

Lemma 4.7. *Let* φ *be a continuous function on* X. *Then*

$$\liminf_{k\to\infty} k^{-(n+1)} \log\left(\int_{A_{k\varphi}} \|\det \Psi_k\|^2 \, d\nu^{N_k}\right) \geq \mathcal{E}(P_{K_0}\varphi_0) - \mathcal{E}(P_K\varphi) + \int \varphi \mathrm{MA}(P_K\varphi)$$

Proof. Decomposing the point-wise norm $\|\det \Psi_k\|^2 = \|\det \Psi_k\|_{k\varphi}^2 \, e^{k\varphi}$ and using Jensen's inequality applied to the convex function e^t gives

$$\left(\int_{A_k} \|\det \Psi_k\|_{k\varphi}^2 \, e^{k\varphi} d\nu^{N_k}\right) \geq \mathcal{Z}'_{k\varphi} \exp\left(\int_{A_{k\varphi}} \frac{\|\det \Psi_k\|_{k\varphi}^2}{\mathcal{Z}'_{k\varphi}} k\varphi d\nu^{N_k}\right)$$

where

$$\mathcal{Z}'_{k\varphi} := \left(\int_{A_{k\varphi}} \|\det \Psi_k\|_{k\varphi}^2 \, d\nu^{N_k}\right)$$

Hence, the sequence in the r.h.s in the statement of the lemma is bounded from below by

$$k^{-(n+1)} \log \mathcal{Z}_{k\varphi} + k^{-n}\left(\int_{A_{k\varphi}} \frac{\|\det \Psi_k\|_{k\varphi}^2}{\mathcal{Z}_{k\varphi}} \varphi d\nu^{N_k}\right)\left(\mathcal{Z}_{k\varphi}/\mathcal{Z}'_{k\varphi}\right)$$
$$+ k^{-(n+1)} \log \left(\mathcal{Z}'_{k\varphi}/\mathcal{Z}_{k\varphi}\right)$$

But by the "exponential" decay of the probability measure $\mu_{k\varphi}^{(N_k)}$ on the complement of $A_{k\varphi}$:

$$\mathcal{Z}_{k\varphi}/\mathcal{Z}'_{k\varphi} \to 1 \tag{4.15}$$

Indeed,

$$\mathcal{Z}'_k/\mathcal{Z}_k = \int_{A_k} \mu_{k\varphi}^{(N_k)} = 1 - \int_{K^{N_k}-A_k} \mu_{k\varphi}^{(N_k)}$$

and on $K^{N_k} - A_k$ we have, by definition, $\mu_{k\varphi}^{(N_k)} < e^{-k^n} d\nu^{N_k}$ proving the convergence 4.15. Finally, using (ii) and (iii) in Theorem 4.6 combined with the exponentially decay of $\mu_{k\varphi}^{(N_k)} (= \frac{\|\det \Psi_k\|^2_{k\varphi}}{\mathcal{Z}_{k\varphi}})$ on the complement of $A_{k\varphi}$ finishes the proof of the lemma. □

4.6. Proofs of upper and lower bounds in Theorem 1.1. To simplify the notation we assume that $V = 1$ (the general case is obtained by a trivial rescaling). First we will prove the upper bound of the theorem (without the normalization factor). It does not use the Bernstein–Markov property of the measure ν. It will be convenient to first establish the LDP for the non-normalized measures $\tilde{\mu}^{(N_k)}$ (formula 4.10).

Proposition 4.8. *Assume that ν is a probability measure supported on the compact set K in X and let $\mathcal{F}$ be a closed set in $\mathcal{P}(K)$. Then*

$$\limsup_{k\to\infty} k^{-(n+1)} \log\left(\|\det \Psi_k\|^2_{L^2(\nu, K^{N_k}\cap\mathcal{F})} \leq - \inf_{\mu\in\mathcal{F}} E_\omega(\mu) + \mathcal{E}_\omega(P_{(K_0,\omega)}\varphi_0 \right)$$

Proof. We may assume that K is not pluri-polar; otherwise the right hand side is infinite [15]. Since ν is a probability measure supported on K we have

$$(kN_k)^{-1} \log\left(\|\det \Psi_k\|^2_{L^2(\nu, K^{N_k}\cap\mathcal{F})}\right) \leq (kN_k)^{-1} \log\left(\|\det \Psi_k\|^2_{L^\infty(K^{N_k}\cap\mathcal{F})}\right). \tag{4.16}$$

Given $\mathbf{x}_k \in X^N$ we will write $\mu_{\mathbf{x}_k} := \delta_N(\mathbf{x}_k)$ for the corresponding normalized sum of Dirac masses. Let $\mathbf{x}_k \in K^{N_k} \cap \mathcal{F}$ be a configuration realizing the sup in the r.h.s. above and fix a continuous ω-psh function φ on X. Then the r.h.s above may be written as

$$\begin{aligned}(kN_k)^{-1} \log\left(\|\det \Psi_k\|^2 (\mathbf{x}_k)\right) &= (kN_k)^{-1} \log\left(\|\det \Psi_k\|^2_{k\varphi} (\mathbf{x}_k)\right) + \langle \varphi, \mu_{\mathbf{x}_k}\rangle \\ &\leq (kN_k)^{-1} \log\left(\|\det \Psi_k\|^2_{L^\infty(k\varphi, K^{N_k})}\right) + \langle \varphi, \mu_{\mathbf{x}_k}\rangle \end{aligned} \tag{4.17}$$

After passing to a subsequence we may assume, by compactness, that

$$\mu_{\mathbf{x}_k} \to \mu \in \mathcal{F}$$

weakly, since $\mathcal{F}$ is closed. In particular, since φ is a continuous function on X it follows that

$$(kN_k)^{-1} \log\left(\|\det \Psi_k\|^2_{L^\infty(k\varphi, K^{N_k})}\right) + \langle \varphi, \mu_{\mathbf{x}_k}\rangle \to \mathcal{E}_\omega(P_{K_0}\varphi_0) - \mathcal{E}_\omega(P_K\varphi) + \int_X \varphi\mu, \tag{4.18}$$

using 4.12. Since this holds for any such φ combining 4.17 and 4.18 gives, also using 3.9 in Proposition 3.8,

$$\limsup_{k\to\infty}(kN_k)^{-1}\log\left(\|\det\Psi_k\|^2(\mathbf{x}_k)\right)\leq -E_\omega(\mu)+\mathcal{E}_\omega(P_{K_0}\varphi_0)) \tag{4.19}$$

for the chosen subsequence of configurations. Hence, by 4.16

$$\limsup_{k\to\infty}(kN_k)^{-1}\log\left(\|\det\Psi_k\|^2_{L^2(\nu,K^{N_k}\cap\mathcal{F})}\right)\leq\sup_{\mu\in\mathcal{F}}(-E_\omega(\mu))+\mathcal{E}_\omega(P_{K_0}\varphi_0))$$

which finishes the proof of the proposition. □

Finally, we will prove the following lower bound:

Proposition 4.9. *Suppose that the measure ν has the Bernstein–Markov property wrt the set K in X. Then for any open set $\mathcal{G}$ in $\mathcal{P}(K)$*

$$\liminf_{k\to\infty}k^{-(n+1)}\log\|\det\Psi_k\|^2_{L^2(\nu,K^{N_k}\cap\mathcal{G})}\geq-\inf_{\mu\in\mathcal{G}}E_\omega(\mu)+\mathcal{E}_\omega(P_{(K_0,\omega)}\varphi_0)$$

Proof. For a given $\mu\in\mathcal{G}$ we have to prove

$$\liminf_{k\to\infty}k^{-(n+1)}\log\|\det\Psi_k\|^2_{L^2(\nu,K^{N_k}\cap\mathcal{G})}\geq-E_\omega(\mu)+\mathcal{E}_\omega(P_{(K_0,\omega)}\varphi_0) \tag{4.20}$$

We may assume that $E_\omega(\mu)<\infty$ (otherwise the statement is trivially true). But then Theorem 3.5 gives that there exists $\varphi_\mu\in\mathcal{E}^1(X,\omega)$ such that $\mathrm{MA}(\varphi_\mu)=\mu$. To fix ideas we first assume that φ_μ is continuous. Now,

$$\liminf_{k\to\infty}k^{-(n+1)}\log\|\det\Psi_k\|^2_{L^2(\nu,K^{N_k}\cap\mathcal{G})}\geq\liminf_{k\to\infty}k^{-(n+1)}\log\|\det\Psi_k\|^2_{L^2(\nu,A_{k\varphi_\mu})}$$
$$\geq\left(-\mathcal{E}(P_K\varphi_\mu)+\int\varphi_\mu\mathrm{MA}(P_K\varphi_\mu)\right)+\mathcal{E}_\omega(P_{K_0}\varphi_0)$$

using lemma 4.7 in the last step. Since, φ_μ is assumed continuous $P_K\varphi_\mu=\varphi_\mu$ almost everywhere wrt $\mathrm{MA}(P_K\varphi_\mu)$ (by the orthogonality relation 3.8) and hence the first terms above equals $-E(\mu)$, proving the desired bound. Finally, in the general case when φ_μ is a general potential of finite energy we take a sequence φ_j of continuous ω-psh functions decreasing to φ_μ. By Lemma 3.9

$$\mu_j:=\mathrm{MA}(P_K\varphi_j)\to\mathrm{MA}(\varphi_\mu)=\mu$$

in $\mathcal{P}(K)$. In particular, since $\mathcal{G}$ is assumed open in $\mathcal{P}(K)$,

$$\mu_j\in\mathcal{G} \tag{4.21}$$

for $j\gg 1$. Next, fix a large index j and consider the set $A_{k\varphi_j}$, defined as in formula 4.14. Then for $k\gg 1$

$$A_{k\varphi_j}\subset(\delta_{N_k})^{-1}\mathcal{G} \tag{4.22}$$

Indeed, assume for a contradiction that the previous statement is false. Then there is a sequence $(\mathbf{x}_{k_i})$ of configurations $\mathbf{x}_{k_i}\in K^{N_k}-(\delta_{N_k})^{-1}\mathcal{G}$ such that

$$\liminf_{k_i}k_i^{-(n+1)}\inf_{K^{N_{k_i}}}\left(\log\left(\mu_{k_i\varphi_j}(\mathbf{x}_{k_i})/\nu^{\otimes N}\right)\right)\geq 0$$

But then Theorem 4.6 gives that

$$\mu_{\mathbf{x}_{k_i}} \to \mu_j$$

in $\mathcal{P}(K)$, forcing $\mu_j \in \mathcal{P}(K) - \mathcal{G}$, which contradicts 4.21. We may now repeat the previous argument with φ_μ replaced by φ_j for $j \gg 1$ and instead get the lower bound

$$\left(-\mathcal{E}(P_K\varphi_j) + \int \varphi_j \mathrm{MA}(P_K\varphi_j)\right) + \mathcal{E}_\omega(P_{K_0}\varphi_0) = -E(\mu_j) + \mathcal{E}_\omega(P_{K_0}\varphi_0)$$

Finally, letting j tend to infinity and using the convergence in Lemma 3.9 concludes the proof in the general case. □

Finally, note that to obtain the rate functional $H_{(K,\omega)}$ for the LDP wrt the normalized measures $\mu^{(N_k)}$ we just have to normalize by dividing by $\mathcal{Z}_k$ which, by the first point in Theorem 4.6 gives the rate functional

$$H_{(K,\omega)}(\mu) := (E_\omega(\mu)) + \mathcal{E}_\omega(P_{K_0}\varphi_0) - \mathcal{E}_\omega(P_K 0) - \mathcal{E}_\omega(P_{K_0}\varphi_0) = (E_\omega(\mu)) - \mathcal{E}_\omega(P_K 0)$$

which coincides with the definition in formula 3.15 of $H_{(K,\omega)}$. This completes the proof of the theorem.

Remark 4.10. Applying the LDP established above (for the sequence of probability measures) to $\mathcal{F} = \mathcal{G} = \mathcal{P}(K)$ gives

$$\log(1) = 0 = \inf_{\mu\in\mathcal{P}(K)} (E_\omega(\mu)) + \mathcal{E}_\omega(P_K 0) - \mathcal{E}_\omega(P_{K_0}\varphi_0),$$

which proves the formula 3.13.

4.7. Proof of Corollary 1.3. Given $\lambda > 0$ and $u \in C^0(X)$ we let F_λ be the set of all probability measures μ on K such that $\int_X \varphi(\mu - \mu_{\mathrm{eq}}) \geq \lambda$. Since this is a compact set, not containing μ_{eq}, and since the rate functional $H(= H_{(K,\omega)})$ is good it follows immediately that $\inf_{\mu\in F_\lambda} H = C_\lambda > 0$ and hence the corollary is a consequence of the upper bound contained in Theorem 1.1. Next, assuming that $n = 1$, let us show that when $\mu_{\mathrm{eq}} = \omega$ we have

$$\inf_{\mu\in F_\lambda} E_\omega(\mu) = \frac{2\lambda^2}{\|d\varphi\|_X^2}$$

(to simplify the notation we will assume that $V = 1$). To this end we first note that

$$E(\mu) = -\frac{1}{2}\int_{:X} u_\mu dd^c u_\mu = \frac{1}{2}\int_{:X} du_\mu \wedge d^c u_\mu := \frac{1}{2}(u_\mu, u_\mu)$$

where u_μ is satisfies $dd^c u_\mu = \mu - \omega$. Since, $(\cdot,\cdot)$ defines a positive definite inner product on the Dirichlet space $H := \{v : dv \in L^2(X)\}/\mathbb{R}$ and since $F_\lambda = \{\mu = \omega + dd^c v : (\varphi, v) \geq \lambda\}$ it follows immediately from the Cauchy–Schwartz inequality that $E(\mu) = \frac{2\lambda^2}{\|d\varphi\|_X^2}$. The proof is concluded by noting that (in any dimension) we have when $\omega \geq 0$ that $H_{(X,\omega)} = E_\omega$. Indeed, $C(X,\omega) := \inf_{\mu\in\mathcal{P}(X)} E_\omega(\mu) = \mathcal{E}_\omega(P_{(X,\omega)}0) = \mathcal{E}_\omega(0) = 0$.

4.8. A Generelized LDP for β-ensembles. Given a weighted measure (ν, ω) and a sequence $(\beta_k) \subset \mathbb{R}_+$ we obtain a sequence of random point process on X, defined by the following probability measure on X^{N_k}:

$$\mu_{\beta_k}^{(N_k)} := \frac{1}{\mathcal{Z}_{N_k,\beta_k}} \|\det \Psi_k\|^{\beta_k} (\nu(x_1) \otimes \cdots \otimes \nu(x_N))$$

Theorem 4.11. *Suppose that $\beta_k \leq C$ and $\beta_k k \to \infty$ (in particular, $\beta_k \equiv \beta$ is allowed) and that the measure ν satisfies the* strong *B-M-property wrt quasi-psh functions on the non-pluripolar set K. Then the random point processes above satisfy a LDP with the same rate functional as in Theorem 1.1, but with the speed $\beta_k k N_k$.*

Proof. We will first show that

$$\mathcal{F}_{N_k,\beta_k}[\varphi] := \frac{1}{kN_k} \log \left\| \det(\Psi_k) e^{-k\varphi} \right\|_{L^{\beta_k}(X^{N_k}, \nu^{\otimes N_k})} \to -\mathcal{E}_\omega(P_{(K,\omega)}\varphi) \tag{4.23}$$

as $k \to \infty$, where we have taken Ψ_k to be defined wrt a basis $(\Psi_{i,k})$ which is orthonormal wrt the Hermitian metric determined by $(\nu, \|\cdot\|)$. To this end first note that for any $\epsilon > 0$ there is a constant C_ϵ such that

$$\sup_{K^{N_k}} \left\| \det(\Psi_k) e^{-k\varphi} \right\| \leq C_\epsilon^{N_k} e^{\epsilon \beta_k k N_k} \left\| \det(\Psi_k) e^{-k\varphi} \right\|_{L^{\beta_k}(X^{N_k}, \nu^{\otimes N_k})}$$

Indeed, this follows immediately from applying the inequality 4.3 N_k times, one time for each variable of $\psi(x_1, \ldots, x_{N_k}) := \frac{1}{k} \log \left\| \det(\Psi_k) e^{-k\varphi} \right\|$ and with $p = \beta_k k$. But then the first point in Theorem 4.6 gives that

$$\mathcal{F}_{N_k,\beta_k}[\varphi] \leq 0 - \mathcal{E}_\omega(P_{(K,\omega)}\varphi) + o(1) \leq \mathcal{F}_{N_k,\beta_k}[\varphi] + \epsilon\beta_k + \frac{1}{\beta_k k} \log C_\epsilon$$

which finishes the proof of 4.23. Finally, since

$$\frac{1}{\beta_k k N_k} \log \mathbb{E}(e^{-\beta_k k(\varphi(x_1)+\cdots+)}) = \mathcal{F}_{N_k,\beta_k}[\varphi] - \mathcal{F}_{N_k,\beta_k}[0] \to -\mathcal{E}_\omega(P_{(K,\omega)}\varphi) + \mathcal{E}_\omega(P_{(K,\omega)}0)$$

the abstract Gärtner–Ellis theorem gives the desired LDP (alternatively, the direct proof in Sect. 4.5 can also be used to conclude the proof of the LDP, just as in the case $\beta = 2$). □

4.9. The LDP in terms of analytic torsion: proof of Theorem 1.4. In this section we will explain how to deduce Theorem 1.4 from Theorem 1.1. We thus fix a smooth Hermitian metric on L with curvature form ω (where we recall that the most interesting case will be when ω is non-negative) and a Hermitian metric h_X on TX. It will be convenient to write

$$\tau_k := \frac{1}{k^{n+1}} \sum_{q=1}^{n} q(-1)^q \log \det \left(\Delta_{\bar{\partial},k}^{0,q} \right)$$

for the corresponding scaled analytic torsions. In fact, since the rate function appearing in Theorem 1.1 is of the form $E_\omega(\mu) - C$ where C is a constant, Theorem 1.4 is an immediate consequence of Theorem 1.1 combined with the following

Proposition 4.12. *Let $L \to X$ be an ample line bundle equipped with a smooth metric with curvature form ω and fix a Hermitian metric on TX. Then the scaled limit of the corresponding analytic torsions for the k th tensor power of L may be expressed as follows:*

$$\lim_{k\to\infty} \tau_k = \inf_{\mu\in\mathcal{P}(X)} E_\omega(\mu)$$

Proof. As shown in [13] in the case when the metric h_X on TX is Kähler and in [11] in the general case

$$\lim_{k\to\infty} \tau_k = Vn!(\mathcal{E}_\omega(P_\omega 0) - \mathcal{E}_\omega(0)) = Vn!(\mathcal{E}_\omega(P_\omega 0)$$

(where the factors $Vn!$ come from the conventions for the functional $\mathcal{E}$ in the present paper). Hence the desired convergence follows immediately from the identities 3.13 and 3.14 (with $K = X$). Note that even if X is always Kähler here (i.e. it admits some Kähler metric) the reason that we need to use [11] is that the Hermitian metric h_X is not assumed to be Kähler. □

4.10. Remarks on normalizations of rate functionals, energy and Vandermonde determinants. As shown above the LDP wrt the non-normalized measures $\|\det \Psi_k\|^2 \nu^{\otimes N}$ has a rate functional

$$\tilde{E}_\omega(\mu) := E_\omega(\mu) - \mathcal{E}_\omega(P_{(K_0,\omega)}\varphi_0) \tag{4.24}$$

where the constant $\mathcal{E}_\omega(P_{(K_0,\omega)}\varphi_0)$ is independent (as it must) of the support K of ν. It follows immediately from the LDP that

$$\tilde{E}_{\omega+dd^c\varphi}(\mu) := \tilde{E}_\omega(\mu) + \int_X \varphi\mu, \tag{4.25}$$

which of course could also be proved directly using the explicit expression 4.24. Let us illustrate this in the case of multivariate polynomials ensembles (Sect. 2). We fix ω_0 such that $\omega_0 = dd^c\phi_0$ for ϕ_0 with logarithmic growth at infinity in $\mathbb{C}^n$ and such that ϕ_0 vanishes on a large ball in $\mathbb{C}^n$ (more precisely: large enough to contain the compact subsets of $\mathbb{C}^n$ appearing below). We take the reference measure ν_0 to be the invariant probability measure on the unit-torus in $\mathbb{C}^n$ and set $\varphi_0 = 0$ (compare the setup in Sect. 4.2.1). Then the basis $(\Psi_{k,i})$ can be taken as monomials and $\det \Psi_k = \Delta^{(N_k)}(z_1, \ldots, z_{N_k})$ becomes the multivariate *Vandermonde determinant* (as in Sect. 2). The LDP in the corresponding *weighted* setting on a compact subset K in $\mathbb{C}^n$ can now be symbolically written as follows (where μ denotes a probability measure on K):

$$\left|\Delta^{(N_k)}(z_1, \ldots, z_{N_k})\right|^2 e^{-k(\phi(z_1)+\cdots)} \sim e^{-\frac{1}{n!}k^{n+1}(\tilde{E}_0(\mu)+\int \phi\mu)}, \tag{4.26}$$

where the "normalized energy" (or "non-weighted energy") $\tilde{E}_0(\mu)$ is independent of ϕ and K (for example, by the transformation property 4.25). In the classical case when $n = 1$ it is not hard to check that $\tilde{E}_0(\mu)$ is the classical logarithmic energy of a measure μ:

$$\tilde{E}_0(\mu) = -\int_{\mathbb{C}} \log|z - w|\mu(z) \otimes \mu(w) \tag{4.27}$$

Indeed, taking ω to vanish on a neighborhood of the support of μ and using the Green function expression 3.11 shows that $\tilde{E}_0(\mu) = -\int_{\mathbb{C}} \log|z-w|\mu(z)\otimes\mu(w) + C$. To see that $C = 0$ we take $\mu = \omega_0$ to be the invariant measure on S^1, i.e. $\mu = dd^c\phi_0$ for $\phi_0 = \log^+|z|^2$ so that $\tilde{E}_0(\mu) = 0 + C$. Since $\phi_0 = 0$ on the support of $\mu = \omega_0$ we can then use the formula 4.24 with $\omega = dd^c\phi_0$ which gives $\tilde{E}_0(\mu) = \tilde{E}_{\omega_0}(\mu) = E_{\omega_0}(\omega_0) - \mathcal{E}_{\omega_0}(P_{(S^1,\omega_0)}0) = 0-0$ using that $P_{(S^1,\omega_0)}0 = 0$ (by the maximum principle). All in all this forces $C = 0$ showing that 4.27 holds. Hence the rate functional in 3.15 is, when $n = 1$, precisely the *weighted logarithmic energy* of μ (which is the subject of the book [49]). In physical terms ϕ hence acts as an exterior potential. In fact, the weighted energy appearing in the rate functional in 4.26 can be extended to the setting when K is replaced by a closed *non-compact* set as explained in the following section.

5. The Large Deviation Principle in the Non-Compact Case

In this section we will obtain a variant of the LDP which applies to non-compact sets F and in particular to $F = \mathbb{R}^n$ or $F = \mathbb{C}^n$.

5.1. Functional analytic setup. In order to formulate the LDP in the case when the random point process is defined on a non-compact topological space F we need to specify the topology that we put on the space $\mathcal{M}(F)$ of all signed measures. We will take the standard choice, namely the weak topology generated by the space $C_b(F)$ of all bounded continuous functions on F. Then the topological dual $\mathcal{M}(F)^*$ of $\mathcal{M}(F)$, i.e. with the space of all linear continuous functions on $\mathcal{M}(F)$, may be identified with $C_b(F)$, using the integration pairing as before [25]. We will apply the Gärtner–Ellis theorem (Theorem 4.5) in this setting and we hence recall the notion of exponential tightness (which is vacuous when F is compact): a sequence Γ_k of probability measure on a space $\mathcal{P}$ is *exponentially tight* (wrt the speed r_k) if $\mathcal{P}$ may be exhausted by compact subsets $\mathcal{F}_\alpha$ for $\alpha > 0$ such that $\limsup_{k\to\infty} \log(\Gamma_k(\mathcal{P}-\mathcal{F}_\alpha)/r_k) < -\alpha$.

5.2. The setting and statement of the LDP. We will consider the following general setting. Start with an open set U of X such that $X - U$ is locally pluripolar (in the applications that we have in mind $X - U$ will even be an analytic subvariety). We will say that a pair (ν, φ) of a measure ν on U and a continuous function φ is *admissible* if

- $\varphi \to \infty$ at infinity in U
- $\int e^{-k\varphi} d\nu < \infty$ for $k \geq k_0$.

We will denote the support of ν in U by F, which is a closed set in U (but possibly non-compact in X!). In the following we fix a continuous metric $\|\cdot\|$ on $L \to X$ with normalized curvature form ω_0. Then

$$\mu_{k\varphi}^{(N_k)} := \|\det \Psi_k\|^2 e^{-k\varphi} \nu^{\otimes N} / \mathcal{Z}_k$$

is a well-defined probability measure on F^{N_k} for $k \geq k_0$.

The next theorem gives a LDP which is a variant of Theorem 1.1.

Theorem 5.1. *Let (ν, φ) be an admissible pair such that $(\nu, \varphi + u)$ satisfies the BM-property 4.2 for any $u \in C_b(F)$ and such that that the support F of ν is non-pluripolar.*

Then the laws on $\mathcal{P}(F)$ of the probability measure $\mu_{k\varphi}^{(N_k)}$ on F^{N_k} satisfy a LDP with a good rate functional $H(= H_{(F,\omega_0,\varphi)})$ *and speed Vk^{n+1}. On the space $\mathcal{P}(F)$ the rate functional $H(\mu)$ is minimized (and vanishes) precisely on the pluripotential equilibrium measure $\mu_{\text{eq}}(= \text{MA}(P_{(K,\omega_0)}\varphi))$. Moreover, the rate functional may be decomposed as*

$$H(\mu) = E_{\omega_0}(\mu) + \int \varphi\mu - C \tag{5.1}$$

where C is the normalizing constant.

Remark 5.2. The reason that we have now denoted the given curvature form by ω_0 (and not ω) is that, in the case F is compact, the role of ω in Theorem 1.1 is in Theorem 5.1 played by $\omega_0 + dd^c\varphi$. The point is that in the present compact setting $\|\cdot\| e^{-\varphi}$ does not, in general, extend to a continuous metric (or even bounded) metric on L over X.

Before starting the proof it will be convenient to make the additional (very weak) assumption that F be regular in the sense that $\Pi_{(F,\omega_0)}(\varphi+u) \in C^0(X)$ if $u \in C_b^0(F)$ and hence the operator $\Pi_{(F,\omega_0)}$ will in the following coincide with its regularization $P_{(F,\omega_0)}$. This assumption may be removed by approximation just as in the proof of Theorem A in [13] (and anyway it is automatically satisfied in the main cases $F = \mathbb{R}^n$ or $F = \mathbb{C}^n$ considered below). To simplify the notation we will often omit the subscript ω_0 in $P_{(F,\omega_0)}$ and $\mathcal{E}_{\omega_0}$ and simply write P_F and $\mathcal{E}$, respectively.

5.3. Start of the proof: localization to a "ball" B_R. Let ρ be a given exhaustion function of U and write $B_R := \{\rho \leq R\}$ so that B_R is sequence of increasing compact sets covering U. We first note that the support of the equilibrium measure $\mu_{\text{eq}} := \text{MA}(P_F\varphi)$ is compact in U. Indeed, in general it is contained in the closed set $D := \{P_F\varphi \geq \varphi\}$ (this is a well-known property of "free envelopes" and follows for example from Prop 1.10 in [13] or rather its proof) and since $\varphi \to \infty$ at infinity in U and $P_F\varphi \leq C$ (using $P_F\varphi \leq P_{F\cap B_R}\varphi$) it follows that D is compact in U. Let us fix a "ball" B_R in U containing the support of μ_{eq}.

Lemma 5.3. *We have that $P_{F\cap B_R}\varphi = P_F\varphi$. Moreover, for any $\Psi_k \in H^0(X, kL)$*

$$\sup_F \|\Psi_k\|^2 e^{-k\varphi} = \sup_F \|\Psi_k\|^2 e^{-kP_F\varphi} \tag{5.2}$$

and for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\left(\|\Psi_k\|^2 e^{-k\varphi}\right)(x) \leq C_\epsilon e^{k(P_F\varphi-\varphi)} e^{\epsilon k} \int \|\Psi_k\|^2 e^{-k\varphi} d\nu \tag{5.3}$$

Proof. (compare Lemma 2.2 in the appendix of [49]). By definition $P_{F\cap B_R}\varphi \geq P_F\varphi$ and $P_{F\cap B_R}\varphi \leq \varphi$ on the support of $\text{MA}(P_F\varphi)$. Since this latter set is contained in D (see above) this means that $P_{F\cap B_R}\varphi \leq P_F\varphi$ a.e. wrt $\text{MA}(P_F\varphi)$ and hence the inequality holds everywhere accord to the domination principle (see [22] for a very general version of this principle). This shows that $P_{F\cap B_R}\varphi = P_F\varphi$. Next, note that 5.2 follows directly from the definition of P_F (just as in [13]). To prove 5.3 we set $\psi := \frac{1}{k}(\log(\|\Psi_k\|^2)(x)/C_\epsilon e^{\epsilon k} \int \|\Psi_k\|^2 e^{-k\varphi} d\nu)$ where C_ϵ is chosen so that the BM-inequality holds wrt K, i.e. so that $\psi \leq \varphi$ on K. Since ψ is a candidate for the sup defining $P_F\varphi$ it follows that $\psi \leq P_F\varphi$ on X which finishes the proof. □

Let us first prove that the analogue of the first point in Theorem 4.6 holds and in particular:

$$k^{-(n+1)}\log\|\det\Psi_k\|^2_{L^2(k\varphi,\nu^{\otimes N_k})}\to-\mathcal{E}_{\omega_0}(P_{(K,\omega_0)}(\varphi))+\mathcal{E}_{\omega_0}(P_{(K_0,\omega_0)}\varphi_0),\tag{5.4}$$

To this end we first apply the previous lemma N_k times to $\Psi_k^{(j)}(x)\in H^0(X,kL)$ defined as $\det\Psi_k$ evaluated at $(x_1,x_2,x_{j-1}x,x_j,\dots,x_{N_k})$ for $j=1,\dots,N_k$. This gives

$$\begin{aligned}&\left(\|\det\Psi_k\|^2e^{-k\varphi}\right)(x_1,\dots,x_N)\\&\leq C_\epsilon^{N_k}e^{\epsilon N_kk}e^{-k((\varphi-P_F\varphi)(x_1,\dots,x_N)}\int_{X^{N_k}}\|\det\Psi_k\|^2e^{-k\varphi}\nu^{\otimes N_k}\end{aligned}\tag{5.5}$$

Moreover, decomposing $k\varphi=(k-k_0)\varphi+k_0\varphi$ and using $P_Fv\leq v$ on the support F of ν gives

$$\begin{aligned}&\int_{X^{N_k}}\|\det\Psi_k\|^2e^{-k\varphi}\nu^{\otimes N_k}\\&\leq C_\epsilon^{N_k}e^{\epsilon N_kk}\sup_F\left(\|\det\Psi_k\|^2e^{-kP_F\left(\left(1-\frac{k_0}{k}\right)\varphi\right)}\right)\left(\int_Xe^{-k_0\varphi}\nu\right)^{N_k}\end{aligned}$$

Using that P_F is concave we get $P_F((1-\frac{k_0}{k})\varphi)\geq((1-\frac{k_0}{k})P_F\varphi+(\frac{k_0}{k})P_F0$ and since $P_F\varphi$ and P_F0 are both bounded on X it follows that

$$\int_{X^{N_k}}\|\det\Psi_k\|^2e^{-k\varphi}\nu^{\otimes N_k}\leq C\sup_F\left(\|\det\Psi_k\|^2e^{-kP_F\varphi}\right)C'^{N_k}e^{\epsilon N_kk}\tag{5.6}$$

Since, by Lemma 5.3 $\sup_F\|\det\Psi_k\|^2e^{-k\varphi}=\sup_F(\|\det\Psi_k\|^2e^{-kP_F\varphi})$ combining 5.5 and 5.6 (and using that $\varphi\geq P_F\varphi$) gives

$$k^{-(n+1)}\log\sup_F\left(\|\det\Psi_k\|^2e^{-kP_F\varphi}\right)=k^{-(n+1)}\log\|\det\Psi_k\|^2_{L^2(k\varphi,\nu)}+o(1)\tag{5.7}$$

Next, by Lemma 5.3 we have $P_F\varphi=P_{F\cap B_R}\varphi$ and hence we can apply the second point in Theorem 4.6 to the function $P_{F\cap B_R}\varphi$ on X and deduce that 5.4 indeed holds (where we again used that $P_F\varphi=P_{F\cap B_R}\varphi$, but now in the rhs in 5.4).

Next, we note that the analogue of Theorem 3.7 holds: the functional $u\mapsto\mathcal{E}_{\omega_0}(P_{(F,\omega_0)}(\varphi+u))$ is Gateaux differentiable on $C_b(F)$ with differential $\mathrm{MA}(P_{(F,\omega_0)}(\varphi+u))$. In other words,

$$\left.\frac{\mathcal{E}_{\omega_0}(P_{(F,\omega_0)}(\varphi+tu))}{dt}\right|_{t=0}=\int_X\mathrm{MA}(P_{(F,\omega_0)}\varphi)u$$

Indeed, since t stays in a bounded set and u is bounded we may, as explained above, assume the support of $\mathrm{MA}(P_{(F,\omega_0)}(\varphi+tu))$ is contained in B_R giving, just as before, that $P_{(F,\omega_0)}(\varphi+tu)=P_{(F\cap B_R,\omega_0)}(\varphi+tu)$ and hence the differentiability follows from Theorem 3.7.

5.4. Exponential tightness and application of the Gärtner–Ellis theorem. Given the previous estimates the proof could be obtained by repeating the arguments in Sect. 4.5. But for simplicity we will instead apply the Gärtner–Ellis theorem to this non-compact setting and we will start by verifying the exponential tightness. We let $\mathcal{F}_\alpha$ to be the set of all measures on $\mathcal{P}(F)$ such that $\int(\varphi - P_F\varphi)\mu \leq 3\alpha$. Since $\varphi - P_F\varphi \to \infty$ at infinity in U, the set $\mathcal{F}_\alpha$ is indeed compact. By definition

$$\Gamma_k(\mathcal{P}(F) - \mathcal{F}_\alpha) = \int_{\{\varphi - P_F\varphi > N_F 3\alpha\}} \frac{(\|\det \Psi_k\|^2 \, e^{-k\varphi})(x_1, \ldots, x_N)}{\|\det \Psi_k\|^2_{L^2(k\varphi, \nu^{\otimes N_k})}} \nu^{\otimes N_k}$$

Now, by 5.5 the density in the previous integral may be estimated from above by $C_\epsilon^{N_k} e^{\epsilon N_k k} e^{-k(\varphi - P_F\varphi)}$ for some fixed small $\epsilon > 0$ (taken so that $\epsilon < \alpha/2$). Hence, decomposing

$$e^{-k(\varphi - P_F\varphi)} = e^{-\frac{1}{2}k(\varphi - P_F\varphi)} e^{-\frac{1}{2}k(\varphi - P_F\varphi)} \leq e^{-\frac{1}{2}kN_F 3\alpha} e^{-\frac{1}{2}k\varphi} C^k$$

and integrating wrt $\nu^{\otimes N_k}$ (and using that φ is admissible) finishes the proof of the exponential tightness.

All in all this means that we may apply the Gärtner–Ellis Theorem 4.5 as before and obtain an LDP with a rate functional expressed as a Legendre transform

$$H(\mu) := \left(\sup_{C_b(F)} \mathcal{E}_{\omega_0}(P_{(F,\omega_0)}(\varphi + u)) - \int u\mu \right) - C \tag{5.8}$$

Properties of the rate functional. The existence, uniqueness and form of the minimizer of the rate functional H follows exactly as in the proof of Proposition 3.11. The fact that H is good is a well-known consequence of the LDP and the exponential tightness obtained above (see Lemma 1.2.18 in [25]). But it could also be proved directly from the decomposition 5.9, to whose proof we now turn. We first rewrite the first bracket in 5.8 as

$$\left(\sup_{\varphi' \in \{\varphi\} + C_b(F)} (\mathcal{E}_{\omega_0}(P_{(F,\omega_0)}(\varphi') - \int \varphi'\mu) \right) + \int \varphi\mu$$

Next we have, just as in the proof of Proposition 3.1 that

$$\mathcal{E}_{\omega_0}(P_{(F,\omega_0)}(\varphi')) - \int \varphi'\mu \leq \mathcal{E}_{\omega_0}(P_{(F,\omega_0)}(\varphi') - \int P_{(F,\omega_0)}(\varphi')\mu \leq E_{\omega_0}(\mu)$$

as $P_{(F,\omega_0)}\varphi'$ is a candidate for the sup defining $E_{\omega_0}(\mu)$. As for the lower bound we take φ'_j smooth functions on X increasing to the unbounded function φ'. Then $P_{(F,\omega_0)}(\varphi'_j) \leq P_{(F,\omega_0)}(\varphi')$ and hence

$$\mathcal{E}_{\omega_0}\left(P_{(F,\omega_0)}(\varphi'_j)\right) - \int \varphi'_j\mu \leq \mathcal{E}_{\omega_0}\left(P_{(F,\omega_0)}(\varphi')\right) - \int P_{(F,\omega_0)}(\varphi')\mu + \epsilon_j$$

where $\epsilon_j \to 0$ by the monotone convergence theorem of integration theory. Noting that $P_{(F,\omega_0)}(\varphi'_j) = P_{(\bar{F},\omega_0)}(\varphi'_j)$ (since by assumption $\bar{F} - F$ is locally pluripolar in X; see [35]) we can apply Proposition 3.1 to deduce that

$$E_{\omega_0}(\mu) = \sup_{\varphi' \in \{\varphi\} + C_b(F)} \left(\mathcal{E}_{\omega_0}(P_{(F,\omega_0)}(\varphi')) - \int \varphi'\mu \right)$$

finishing the proof of the decomposition formula 5.1. Note that since φ is bounded from below it then follows that $H(\mu) < \infty$ iff $E_{\omega_0}(\mu) < \infty$ and $\int \varphi\mu < \infty$.

Corollary 5.4. *Let $F = \mathbb{R}^n$ (or $F = \mathbb{C}^n$) and let ν be the Euclidean measure on F. Assume that ϕ is a function on F with super logarithmic growth at infinity. Then the push-forward Γ_k of the Vandermonde measure*

$$\tilde{\mu}^{(N_k)} := \left|\Delta^{(N_k)}\right|^2 e^{-k\phi}\nu^{\otimes N_k}$$

under the map δ_{N_k} (formula 1.6) satisfies a LDP with a good rate functional

$$\tilde{E}_\phi(\mu) = \tilde{E}_0(\mu) + \int \phi\mu \tag{5.9}$$

with $\tilde{E}_0$ independent of ϕ and F. The rate functional has a unique minimizer, coinciding with the pluripotential equilibrium measure $\mu_{\mathrm{eq}} := \mathrm{MA}(P_F\phi)$.

Proof. Let $(X, L, \omega_0) := (\mathbb{P}^n, \mathcal{O}(1), \omega_0)$ as in the beginning of Sect. 2 and write $\varphi = \phi - \log^+ |z|^2 := \phi - \phi_0$ on $U := \mathbb{C}^n$. Then we have $\|\cdot\|^2 e^{-k\varphi} = |\cdot|^2 e^{-k\phi}$. To see that the BM-property is satisfied we first note that replacing F with $F \cap B_R$ for an ordinary ball of radius R the pair (ν, φ) has the BM-property wrt $F \cap B_R$ for *any* $\varphi \in C^0(F)$ (and hence for $\varphi + u$ as above), as is well-known [18]. Next we will apply the arguments in the proof of Lemma 5.3: fixing φ and defining ψ as in that lemma hence gives $\psi \leq \varphi$ on B_R and in particular $\psi \leq \varphi$ on the support of $\mathrm{MA}(P_F\varphi)$ if $R \gg 1$. But then it follows from the domination principle that $\psi \leq P_F\varphi$ on all of X and hence the inequality 5.3 holds (even when restricting ν to B_R). Since, $P_F\varphi \leq \varphi$ on all of X this finishes the proof of the BM-property wrt F. Hence we may apply the previous theorem to obtain an LDP with a rate functional of the form 5.1 for some constant C. Finally, we may simply define $\tilde{E}_0(\mu) := E_{\omega_0}(\mu) - \int \phi_0\mu - C$ so that formula 5.9 holds, as desired. □

When $n = 1$ the functional $\tilde{E}_0(\mu)$ in 5.9 can still be written in the classical form 4.27, even if μ does not have compact support in $\mathbb{C}(= U)$. This follows for example from the latter case by using the following lemma valid in any dimension:

Lemma 5.5. *Let μ be a measure such that $E(\mu) < \infty$ and write $\mu_R := 1_{B_R}\mu / \int_X \mu_R$, where B_R is a sequence of sets such that $1_{B_R}\mu \to \mu$. Then $E_{\omega_0}(\mu_R) \to E_{\omega_0}(\mu)$ as $R \to \infty$.*

Proof. Let φ_{μ_R} be the potential of the probability measure μ_R normalized so that $\sup_X \varphi_{\mu_R} = 0$. Now by the variational property of φ_{μ_R} we have

$$\mathcal{E}(\varphi_{\mu_R}) - \int \varphi_{\mu_R}\mu \geq \left(\mathcal{E}(\varphi_\mu) - \int \varphi_\mu\mu\right) + \delta_R \int \varphi_{\mu_R}\mu$$

where $\delta_R \to 0$. Moreover, by Prop 3.4 in [15] the following estimate holds (using that μ has finite energy): $|\int \varphi_{\mu_R}\mu| \leq C|\mathcal{E}(\varphi_{\mu_R})|^{1/2}$. Hence, the inequality above forces $-\mathcal{E}(\varphi_{\mu_R}) \leq C$ and as consequence φ_{μ_R} is an asymptotic maximizer in the sense of Theorem 3.5. Hence, the latter theorem gives $\mathcal{E}(\varphi_{\mu_R}) \to \mathcal{E}(\varphi_\mu)$ and $\int \varphi_{\mu_R}\mu \to \int \varphi_\mu\mu$. Using that $\mu_R = 1_{B_R}\mu(1 + \delta_R) \leq C\mu$ and dominated convergence hence finally gives $E(\mu_R) \to E(\mu)$. □

5.5. Applications to sections vanishing along a given hypersurface and Laplacian growth. Let Z by a smooth hypersurface in X. Let H_{kZ} be the subspace of $H^0(X, kL)$ consisting of all sections vanishing to order k along Z. Then any continuous Hermitian metric $\|\cdot\|$ (with curvature form ω) and a volume form ν on X induce by restriction, an inner product on the subspace H_{kZ} (that will be non-degenerate under the assumptions below). Hence, we can associate a sequence of determinantal point-processes to the corresponding sequence of Hilbert spaces. We will assume that the line bundle $L - \mathcal{O}(Z)$ is ample, where $(\mathcal{O}(Z), s_Z)$ is the line bundle with a holomorphic section s_Z cutting out Z, i.e. $Z = \{s_z = 0\}$ (using, as before, additive notation for tensor products of line bundles).

Corollary 5.6. *The laws of the normalized empirical measure of the determinantal point process associated to H_{kZ} satisfy a LDP on $X - Z$ with a good rate functional whose unique minimizer is compactly supported on $X - Z$.*

Proof. The map $\Psi_k \mapsto \Psi_k / s_Z$ establishes a unitary isomorphism between the Hilbert space H_{kZ} above and the vector space $(H^0(X, k(L - \mathcal{O}(Z)))$ with the inner product induced by the volume form ν and the (non-continuous) metric $\|\dot\| \, e^{\log |s_Z|}$ on the line bundle $L - \mathcal{O}(Z)$ over X (the metric blows up along Z). Since, ν satisfies the BM-property wrt any domain in X (as is proved just as in the proof of the previous corollary) the LDP follows from the previous theorem applied to the line bundle $L - \mathcal{O}(Z)$ and the set U. □

In fact, the previous LDP holds for much more general measures ν. For example we can take $\nu = f dV_X$ where f is continuous and positive on $X - Z$. Then we just have to assume that the corresponding inner products on H_k are finite for $k \gg 1$. Moreover, the assumption that $L - \mathcal{O}(Z)$ be ample can be relaxed to assuming that $L - \mathcal{O}(Z)$ is *big* as in Sect. 4.4.1. But even in the ample case the point is really that the previous corollary can be adapted to the setting where Z is replaced with by an effective $\mathbb{R}$-divisor with simple normal crossings:

$$Z := t_1 Z_1 + \cdots t_m Z_m$$

where $t_i \geq 0$ if we let H_k be defined by taking the vanishing order on Z_i to be the round-down of kt_i. In particular, if L is ample then so is $L - \mathcal{O}(Z)$ for t_i sufficiently small (and rational, so that the ampleness makes sense). For example, when $Z = t_1 Z_1$ and s_{Z_1} is a section of L then we can take $t_1 \in [0, 1[$ so that $t_1 = 0$ corresponds to the situation in the previous sections where H_{kZ} is the full Hilbert space $H^0(X, kL)$ and as $t_1 \to 1$ the scaled dimension of H_{kZ} tends to zero. Hence, physically t_1 plays the role of the "filling fraction" familiar from the Quantum Hall effect when $n = 1$ (where X is the Riemann sphere $\mathbb{P}^1$ and Z_1 is the point at infinity). This latter case has also been studied extensively from the point of view of Laplacian growth (for example in connection to the Hele-Shaw flow); see [38,61] and references therein. Here we just briefly point out the relation to Laplacian growth on a Riemann surface of arbitrary genus:

Proposition 5.7. *Fix a point Z on a compact Riemann surface (X, ω) equipped with a real two-form ω such that $\int_X \omega = 1$. For $t \in]0, 1[$ the equilibrium measure associated to (X, ω, tZ) (i.e. the minimizer appearing in 5.6) may be written as*

$$\mu_t := 1_{D_t} \omega, \tag{5.10}$$

for a closed set D_t. The right derivative of μ_t exists and its value at $t = t_0$ coincides with the "non-weighted" equilibrium measure of the compact subset D_{t_0} of $X - Z$ (see *the proof for the precise meaning of this term). In particular, if D_{t_0} is a domain with piecewise smooth boundary, then this latter measure is supported on ∂D_{t_0} and its density is the normal derivative of the Green function of D_{t_0} with a pole at Z.*

Proof. The regularity statement can be deduced from the general $\mathcal{C}^{1,1}$-regularity result of envelopes in [7]. Next, it will be convenient to switch to the weight notation, i.e., we let ω be the curvature form for a weight ϕ on a line bundle of degree one (see Sect. 1.6) and we write ϕ_t for the upper envelope of all psh weights χ_t on L such that $\chi_t \leq \phi$ on X and $\chi_t \leq t \log |s|^2 + C_{\chi_t}$ for some constant C_{χ_t}. We let D_t be the closed subset where $\chi_t = \phi$. Then μ_t is the curvature current of ϕ_t, i.e., $\mu_t = dd^c \phi_t$. Moreover, it follows immediately from the definition that $\phi_{t+h} \leq \phi_t$ if $h > 0$ and that $t \mapsto \phi_t$ is concave in t. Accordingly, $D_{t+h} \subset D_t$ and by concavity the right derivative $\dot{\phi}_t$ is thus a decreasing limit of psh weights on L and hence it exists and is itself a psh weight on L. Moreover, by definition, the right derivative of μ_t is equal to $dd^c \dot{\phi}_t$. To see the relation to Green functions we note that on $U := X - Z$ we get a well-defined function by setting $u_t := \phi_t - \log |s|^2$ and we denote by g_t its right derivative. It follows from the previous discussion that g_t is a psh function on U such that $g_t = 0$ on D_t, $dd^c g_t = 0$ in the exterior and g_t has a logarithmic pole at infinity in U in the sense that in a fixed local holomorphic coordinate z centered at Z we have that $g_t - \log |s|^2$ is bounded (since by definition $\log |s|^2 - C \leq \dot{\phi}_t \leq \log |s|^2 + C$). By construction $d\mu_t/dt = 1_U dd^c g_t$ which we call the "non-weighted" equilibrium measure of the compact subset D_{t_0} of U. Finally, in the regular case we can use Stokes theorem to get $d\mu_t/dt_{|t=t_0} = dd^c g_{t_0} = [\partial D_{t_0}] \wedge d^c g_{t_0}$ which gives the required normal derivative on the boundary. □

In other words the proposition says (in the regular case) that the one-parameter family of domains defined by the supports of the equilibrium measures μ_t are decreasing in t and their evolution is driven by the normal derivative of the corresponding Green functions, which is the definition of Laplacian growth. It would be interesting to know if an analogous theory of "Monge–Ampère growth" can be developed in the higher dimensional case where Z is a submanifold cut out by a holomorphic section s of a line bundle L. For example, it seems natural to conjecture that in the general higher dimensional case the right derivative of the corresponding weighted equilibrium measures μ_t still exists (at least under suitable regularity assumptions) and is equal to the corresponding non-weighted equilibrium measure of the set D_t. The problem with extending the previous proof to higher dimensions is that it is no longer a priori clear that $\dot{\phi}_t$ is a psh weight. Interestingly, in another direction it was recently shown in [48] that, in any dimension, the partial Legendre transform in t of the corresponding equilibrium potentials defines a weak geodesic ray in the (closure) of the space of all Kähler potentials on L, equipped with the Mabuchi metric.

6. Relation to Bosonization and Effective Field Theories

We will conclude with a heuristic discussion about some relations to the notion of bosonization (or fermion–boson correspondence) in the physics literature (see [54]). We will compare the present setting with the one in the paper [1] which concerns the case of a Riemann surface X (but see also [57]). Another useful reference on field theory linking mathematical and physical terminology is [32]. As established in [1] (see also [57]) there is a correspondence between a certain theory of fermions Ψ on one hand and bosons φ on the other, on a Riemann surface X. In the present context Ψ is a complex spinor coupled to the line bundle L and φ is a smooth function on X. The main ingredient in the correspondence is the equality

$$\left\langle \|\Psi(x_1)\|^2 \dots \|\Psi(x_N)\|^2 \right\rangle = \left\langle e^{i\varphi(x_1)} \dots e^{i\varphi(x_N)} \right\rangle, \tag{6.1}$$

(see formula 4.4 in [1]) where the brackets denote integration against (formal) functional integral measures of the form $\mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{-S_{\mathrm{Ferm}}(\Psi,\bar{\Psi})}$ and $\mathcal{D}\varphi e^{-S_{\mathrm{bose}}(\varphi)}$, respectively. The fermionic action S_{Ferm} has a standard form (see below) and the problem is to find a bosonic action S_{bose} so that the ansatz 6.1 above holds.

Comparing with the geometric setup in the present paper we will, in this section, consider the case when the compact set K is all of X and equip L with a smooth metric with (normalized) curvature form ω, that will however not be assumed to be positive. We will also fix an Hermitian metric on X with volume form dV. In the previous terminology we will hence consider the determinantal point process associated to the weighted measure (dV, ω) on X.

Let us now explain how the LDP in Theorem 1.1 (and in particular its variant in Theorem 1.4) can be interpreted as an asymptotic/effective version of the boson–fermion correspondence on a complex manifold of arbitrary dimension n with the choice

$$S_{\mathrm{bose}}(\varphi) = -\frac{1}{(-i)^{n-1}}\mathcal{E}_{-i\omega}(\varphi), \tag{6.2}$$

where it will, in this section, be convenient to remove the normalization factor V from the definition 3.2 of the functional $\mathcal{E}_\omega$ and simply set

$$\mathcal{E}_\omega(\varphi) = \frac{1}{(n+1)!}\sum_{j=0}^{n}\int_X \varphi\omega_\varphi^j \wedge (\omega)^{n-j},$$

where we recall that $\omega_\varphi := \omega + dd^c\varphi (= \omega + \frac{i}{2\pi}\partial\bar{\partial}\varphi)$. With this normalization the following $n+1$-homogeneity holds:

$$\mathcal{E}_{c\omega}(c\psi) = c^{n+1}\mathcal{E}_\omega(\psi). \tag{6.3}$$

In particular, when $n = 1$ we get

$$S_{\mathrm{bose}}(\varphi) = -\mathcal{E}_{-i\omega}(\varphi) = \frac{1}{2}\int_X d\varphi \wedge d^c\varphi + i\int_X \varphi\omega$$

which in the notation of [1] corresponds precisely to the decomposition

$$S_{\mathrm{bose}}(\varphi) = S_1 + S_2$$

given in [1], in the case when X has genus zero. In the higher genus case soliton type terms are added in [1] to the action, which describe the topological sector of the bosonic theory. This is related to the fact that the field φ should really be assumed to be circle valued (i.e. its values are only well-defined up to integer periods). But as explained in [1] one may assume that φ is single-valued when dealing with the non-topological sector—compare the discussion in connection to formula 3.19 in [1] (anyway the topological sector can be shown to give a lower order contribution in the large k limit studied below). See also the discussion in Sect. 2.2.1 of [12] for a more precise comparison with the normalization conventions used in [1].

To make the connection with the LDP we will, as before, consider the limit when L is replaced by the large tensor power kL. Since, $N_k \sim k^n$ this is also the limit of many particles and the asymptotic boson-correspondence will hence give a bosonic field theory description of a collective theory of fermions. As explained in the introduction of the paper we will consider "clouds" of points $(x_1, \ldots, x_N)$ that can be described by a "macro state", i.e. a limiting continuous distribution, or more precisely a measure $\mu = \rho dV$ on X:

$$\frac{1}{N_k}\sum_{i=1}^{N_k}\delta_{x_i} \approx \mu \tag{6.4}$$

in a suitable smeared out sense.

6.1. The fermionic side. First recall the representation of the Slater determinant 4.5 (expressed in an orthonormal basis wrt to (dV, ω)) as a functional integral over Grassman (anti-commuting) fields (compare formula 4.5 in [1]):

$$\|(\det\Psi)(x_1, \ldots, x_N)\|^2 = C_N \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{-S_{\text{ferm}}(\Psi,\bar{\Psi})}\,\|\Psi(x_1)\|^2 \ldots \|\Psi(x_1)\|^2, \tag{6.5}$$

integrating of over all complex (Dirac) spinors Ψ, i.e. over all smooth sections of the exterior algebra $\Lambda^{0,*}(T^*X) \otimes L$. Here $S_{\text{ferm}}(\Psi, \bar{\Psi})$ is the fermionic action

$$S_{\text{ferm}}(\Psi, \bar{\Psi}) = \int_X \langle \mathbb{D}_A\Psi, \Psi\rangle\, dV,$$

expressed in terms of the Dirac operator $\mathbb{D}_A$ on $\Lambda^{0,*}(T^*X) \otimes L$ induced by the complex structure on X and L and coupled to the gauge field/connection A whose curvature is $-i2\pi\omega$ and to the Hermitian metric on X with volume form dV. Concretely, we have $\mathbb{D}_A = \overline{\partial} + \overline{\partial}^*$ expressed in terms of the adjoint of the $\overline{\partial}$-operator. For the Riemann surface case see [1]. This latter case, i.e. when $n = 1$, is special since $S_{\text{ferm}}(\Psi, \bar{\Psi})$ is then independent of the choice of Hermitian metric on X (i.e. the action is conformally invariant). Indeed, decomposing $\Psi = \Psi_0 + \Psi_1$ gives

$$S_{\text{ferm}}(\Psi, \bar{\Psi}) = i\int_X \overline{\partial}\Psi_0 \wedge \overline{\Psi_1} + \Psi_0 \wedge \overline{\partial\Psi_1},$$

where the integrand is naturally a $(1, 1)$-form and may hence be integrated over X (to simplify the formula we have assumed that L is trivial above—in general one also has to use the metric on L or equivalently couple $\overline{\partial}$ to the gauge field A). The integer N appearing in 6.5 is the dimension of the space of zero-modes of $\mathbb{D}_A$ on $\Lambda^{0,*}(T^*X) \otimes L$ which coincides with $H^0(X, L)$ since we have assumed that L is ample (more precisely, this will be true once we replace L with kL for k sufficiently large due to Kodaira vanishing in positive degrees). Moreover, the constant C_N in 6.5 is the inverse of the Ray–Singer analytic torsion of the complex $\Lambda^{0,*}(T^*X) \otimes L$ (compare Sect. 1.3.2), if we use zeta function regularization of the corresponding formal determinants.

Now applying the LDP in Theorem 1.4 for the Slater determinants multiplied by the analytic torsion and using 6.5 and 6.4 hence gives

$$\left\langle \|\Psi(x_1)\|^2 \ldots \|\Psi(x_N)\|^2 \right\rangle \sim e^{-kN_k E_\omega(\mu)}.$$

6.2. The bosonic side. We will use an argument involving analytic continuation in a real parameter t (which will be set to $-i$ in the end). To this end we consider the path integral

$$\int \mathcal{D}\varphi e^{-S_t(\varphi)} e^{t\varphi(x_1)} \dots e^{t\varphi(x_N)}, \quad -S_t(\varphi) = t^{-(n-1)} \mathcal{E}_{t\omega}(\varphi).$$

Next, we will show that, in the limit when L gets replaced with kL and ω with $k\omega$ we have

$$\int \mathcal{D}\varphi e^{-S_t(\varphi)} e^{t\varphi(x_1)} \dots e^{t\varphi(x_N)} \sim e^{t^2 k^{n+1} E_\omega(\mu)}. \tag{6.6}$$

Accepting this for the moment we see that the effective bosonization explained above follows by invoking analytic continuation and setting $t = -i$.

To see how the asymptotics above come about first note that we get the following exponent in the integral, after setting $\varphi = k\psi$:

$$t^{-(n-1)} \mathcal{E}_{tk\omega}(k\psi) - tk(\psi(x_1) + \cdots \psi(x_N)).$$

Using the $n + 1$-homogeneity 6.3 the previous expression may be written as

$$k^{(n+1)} \left(t^{-(n-1)} \mathcal{E}_{t\omega}(\psi) - t \frac{1}{k^n} (\psi(x_1) + \cdots \psi(x_N)) \right)$$

and hence, using 6.4, it can be approximated by

$$k^{(n+1)} \left(t^{-(n-1)} \mathcal{E}_{t\omega}(\psi) - t \int_X \psi \mu \right).$$

Now we get

$$\int \mathcal{D}\varphi e^{-S_t(\varphi)} e^{t\varphi(x_1)} \dots e^{t\varphi(x_N)} \sim e^{k^{n+1} \sup_\psi \left(t^{-(n-1)} \mathcal{E}_{t\omega}(\psi) - t \int_X \psi \mu \right)}.$$

Denote by ψ_t a function where the sup is above is attained. Since it is a stationary point we get the equation

$$t^{-(n-1)} \frac{1}{n!} \left(t\omega + dd^c \psi_t \right)^{n+1} = t\mu \Leftrightarrow \frac{1}{n!} \left(t\omega + dd^c \psi_t \right)^{n+1} = t^n \mu$$

(see the variational properties in Proposition 3.1). Hence a solution is obtained by setting $\psi_t = t\psi_\mu$ where ψ_μ is a potential for μ (solving the equation for $t = 1$). Finally, since

$$\left(t^{-(n-1)} \mathcal{E}_{t\omega}(t\psi_\mu) - t \int_X (t\psi_\mu)\mu \right) = t^2 (\mathcal{E}_\omega(\psi_\mu) - \int_X \psi_\mu \mu) = t^2 E_\omega(\mu)$$

(using the homogeneity 6.3 for $c = t$ in the first step and the formula 3.5 for $E_\omega(\mu)$. This proves the asymptotics 6.6 up to a subtle point that was neglected in the previous argument: when $n > 1$ the function ψ_t may not be a maximizer even though it is a stationary point. Equivalently, this means that the potential φ_μ of μ may not maximize the functional

$$\varphi \mapsto \mathcal{E}_\omega(\varphi) - \int_X \varphi \mu$$

over the *whole* space $C^\infty(X)$. By Theorem 3.5 it is a maximizer on the subspace $\mathcal{H}_\omega$ of $C^\infty(X)$ where $\omega_\varphi \geq 0$, but it is known that the functional above is not even bounded from above on the whole space $C^\infty(X)$. Coming back to the original variable φ this problem could have been circumvented by restricting the integration to all φ such that $dd^c\varphi \geq -kt\omega$. This can be seen as a regularization similar to frequency cut-offs usually used in quantum field theory. Alternatively, one could expect that there is a choice of action

$$S_{\text{bos}} = -\frac{1}{(-i)^{n-1}}\mathcal{E}_{-i\omega} + S',$$

where the term S' has the effect of localizing the integral to the subspace $\mathcal{H}_{\omega/k}$ in the large k limit and that S' will be lower order in k and hence negligible on the space $\mathcal{H}_{\omega/k}$. It would be very interesting to find a model (when $n > 1$) where such a mechanism could be analyzed.

6.3. Consistency with the central limit theorem. Instead of going further into the subtle points raised above we will just observe that the choice of bosonic action 6.2 is consistent with the CLT proved in [9]. We will consider the case when the curvature form ω is strictly positive. Then the CLT referred to above says, in physical terms, that the (suitably scaled) fluctuations of the empirical measure 1.6 converge to a random measure/charge-distribution for a statistical Coulomb gas ensemble described by the Boltzmann weight

$$e^{-\tilde{E}(\nu)}\mathcal{D}\nu,$$

where $\nu(= \rho dV)$ is a signed measure (i.e. a difference of two positive measures) such that $\int_X \nu = 0$ and

$$\tilde{E}(\nu) = -\frac{1}{2}\int_X (\Delta u_\nu)u_\nu dV \left(= \frac{1}{2}\|du_\nu\|_X^2\right), \quad \nu = \Delta u_\nu,$$

where Δ is the Laplacian taken wrt a Riemannian metric g on X and dV is the volume form of g. Equivalently, the Green potential u_ν is a *massless boson field* on X (i.e., a *Gaussian free field* in mathematical terms). Using Fourier transforms (see [9] and references therein) the precise mathematical meaning of the convergence is

$$\int_{X^N} dV(x_1)\ldots dV(x_N)\rho^{(N)}(x_1,\ldots,x_N)e^{\frac{1}{a_k}i(u(x_1)+\cdots+u(x_N))} \to e^{-\frac{1}{2}\|du\|_X^2}, \tag{6.7}$$

where $\rho^{(N)}$ is the Slater determinant appearing above and $a_k = k^{(n-1)/2}$. To obtain the latter convergence from the previous bosonization ansatz we approximate the lhs above by

$$C_{N_k}\int \mathcal{D}\mu e^{\frac{k^n}{a_k}i\int u\left(\mu-\frac{\omega^n}{n!}\right)} D\varphi e^{S_{\text{bose},k}(\varphi)+k^n i\int\varphi\mu}. \tag{6.8}$$

Note that when $k = 1$ we may expand

$$S_{\text{bose},1}(\varphi) = -\frac{1}{(-i)^{n-1}}\mathcal{E}_{-i\omega}(\varphi) = -i\int_X \varphi\frac{\omega^n}{n!} - \frac{1}{2}\int_X d\varphi\wedge d^c\varphi\wedge\frac{\omega^{n-1}}{(n-1)!} + \cdots,$$

where the dots indicate a sum of $n-1$ terms of the form

$$\frac{1}{2}\int_X d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^j \wedge \omega^{n-j}, \quad j \geq 1$$

which hence is of order ≥ 3 in φ (some of the coefficients will be imaginary). In the limit when we are changing L by kL and ω by $k\omega$ we write $\varphi = k\psi$ as before, so that

$$S_{\text{bose},k}(\varphi) = k^{n+1} S_{\text{bose},1}(\psi)$$

and hence the exponent in the φ integral in 6.8 may be written as

$$k^{n+1}\left(i\int \psi \left(\mu - \frac{\omega^n}{n!}\right) - \frac{1}{2}\left\|d\psi\right\|_X^2 + \cdots\right).$$

Les us now make the following change of variables:

$$\mu = \frac{\omega^n}{n!} + \frac{1}{k^{(n+1)/2}}\nu, \quad \psi = \frac{1}{k^{(n+1)/2}}v.$$

Then the previous expression may be written as

$$i\int v\nu - \frac{1}{2}\left\|dv\right\|_X^2 + O(k^{-1})$$

and hence the integral over $D\varphi$ may be approximated by a Gaussian integral over v which, as usual, may be evaluated as

$$e^{-\frac{1}{2}\|du_\nu\|_X^2}$$

All in all this means that the integral 6.8 may be approximated by

$$\int \mathcal{D}\nu e^{i\int u\nu} e^{-\frac{1}{2}\|dv_\nu\|_X^2}$$

and performing the Gaussian integration again finally gives the end result $e^{-\frac{1}{2}\|du\|_X^2}$, thus confirming 6.7.

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