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Schoenberg Correspondence on Dual Groups

Michael Schürmann, Stefan Voß

Department of Mathematics and Computer Science, University of Greifswald, Walther-Rathenau-Straße 47, 17487 Greifswald, Germany. E-mail: stefan.voss@uni-greifswald.de

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Dedicated to Wilhelm von Waldenfels

Abstract: As in the classical case of Lévy processes on a group, Lévy processes on a Voiculescu dual group are constructed from conditionally positive functionals. It is essential for this construction that Schoenberg correspondence holds for dual groups: The exponential of a conditionally positive functional is a convolution semigroup of states.

1. Introduction

The original result by Schoenberg [Sch38] says that the Schur exponentials $(\mathrm{e}^{ta_{ij}})_{ij}$, $t \in \mathbb{R}_+$, of a complex $n \times n$-matrix $A = (a_{ij})_{ij}$ are positive semi-definite iff A is conditionally positive semi-definite, i.e. iff $A^* = A$ and

$$\sum_{i,j=1}^{n} \bar{z}_i \, z_j \, a_{ij} \geq 0$$

for all complex numbers $z_1, \ldots, z_n$ with $z_1 + \cdots + z_n = 0$. There are many other examples of Schoenberg correspondence between conditional positivity and positive semigroups. Another elementary example is the correspondence between semigroups $P_t = \mathrm{e}^{tQ}$ of stochastic matrices and their generator Q-matrix which has to have non-negative off-diagonal entries and row sums equal to 0.

If $X_t : \Omega \to G$, $t \in \mathbb{R}_+$, is a Lévy process on a topological group G (that is a G-valued stochastic processes with independent stationary increments) then the distributions μ_t of X_t form a convolution semigroup of probability measures on G, i.e. $\mu_s \star \mu_t = \mu_{s+t}$, where the convolution product is defined by $(\mu_1 \star \mu_2)(f) = \int_G \int_G f(xy) \mu_1(\mathrm{d}x) \mu_2(\mathrm{d}y)$ for probability measures μ, ν and bounded continuous functions f on G. If μ_t is weakly continuous, one can define the generator of the convolution semigroup on an appropriate $*$-algebra of functions on G. This leads to the Lévy–Khintchine formula in the case of $G = \mathbb{R}^d$, or, more generally, to Hunt's formula if G is a Lie group. Again the

generator ψ is conditionally positive, in the sense that ψ is hermitian and $\psi(f) \geq 0$ for functions $f \geq 0$ vanishing at the unit element of G; see [Hun56, vW65]. In the case when G is a locally compact abelian group or a compact group, one can choose as space of functions on G the $*$-algebra formed by the coefficient functions of continuous irreducible representations of G. This coefficient algebra $\mathcal{B}$ is a Hopf $*$-algebra, and convolution semigroups of states on G are precisely given by the conditionally positive linear functionals on the coefficient algebra where conditionally positive now means hermitian and

$$\psi(f) \geq 0 \quad \text{for } f \geq 0,\ f \in \ker \delta.$$

The functional δ is the counit of the coefficient algebra $\mathcal{B}$ and convolution of linear functionals φ_1 and φ_2 on $\mathcal{B}$ is given by

$$\varphi_1 \star \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta, \tag{1.1}$$

where Δ is the comultiplication of $\mathcal{B}$.

We describe the mechanism of constructing Lévy processes. Starting from a conditionally positive linear functional ψ on $\mathcal{B}$, we obtain a convolution semigroup φ_t of linear functionals on $\mathcal{B}$ as the convolution exponentials $\mathrm{e}_\star^{t\psi}$ of ψ. Now it is important that Schoenberg correspondence holds which means that φ_t are positive so that the convolution semigroup consists of states which again are in 1-1-correspondence to probability measures on G. The convolution semigroup defines a projective system of finite-dimensional distributions which by Kolmogorov's theorem allows to construct a Lévy process on G whose convolution semigroup is given by φ_t. This establishes, up to stochastic equivalence of stochastic processes, a 1-1-correspondence between conditionally positive linear functionals on the coefficient algebra of G and Lévy processes on G.

The Hopf $*$-algebras arising from locally compact abelian or compact groups are algebras of functions and as such are commutative. If one generalizes to arbitrary Hopf $*$-algebras (for instance, compact quantum groups), a notion of noncommutative (quantum) Lévy processes has been introduced; see [Sch93]. These quantum Lévy processes are again given by conditionally positive linear functionals, now on the Hopf $*$-algebra $\mathcal{B}$ where conditionally positive now means hermitian and

$$\psi(b^*b) \geq 0 \quad \text{for } b \in \ker \delta. \tag{1.2}$$

The increments of quantum Lévy processes on Hopf $*$-algebras are independent in the sense of *noncommutative tensor independence* where two sub-algebras are called tensor independent if they commute and if expectations factorize.

The tensor product of linear functionals is closely related to the classical notion of stochastic independence. Two random variables $X_1 : \Omega \to G_1, X_2 : \Omega \to G_2$ are independent if their joint distribution is the tensor product of the marginal distributions, that is if

$$\mathbb{P}_{(X_1,X_2)} = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2}. \tag{1.3}$$

Identify the underlying probability measure $\mathbb{P}$ with its expectation $\mathbb{E}(F) = \int F \mathrm{d}P$, $F \in L^\infty(\Omega)$, which is a positive normalized linear functional on the $*$-algebra $L^\infty(\Omega)$. If we think of $\mathbb{P}_{X_1}$ and $\mathbb{P}_{X_2}$ as their expectations $\mathbb{E}_{X_1}$ and $\mathbb{E}_{X_2}$ on the $*$-algebras $L^\infty(G_1)$ and $L^\infty(G_2)$ of bounded measurable functions on G_1 and on G_2 respectively, and of

the joint distribution as the expectation $\mathbb{E}_{(X_1,X_2)}$ on the tensor product $L^\infty(G_1 \times G_2) = L^\infty(G_1) \otimes L^\infty(G_2)$, then (1.3) becomes

$$\mathbb{E}_{(X_1,X_2)} = \mathbb{E}_{X_1} \otimes \mathbb{E}_{X_2}. \tag{1.4}$$

Define the $*$-algebra homomorphisms $j_1 : L^\infty(G_1) \to L^\infty(\Omega)$ and $L^\infty(G_2) \to L^\infty(\Omega)$ by $j_1(f_1) = f_1 \circ X_1$ and $j_2(f_2) = f_2 \circ X_2$ and introduce the $*$-algebra homomorphism

$$j_1 \otimes j_2 : L^\infty(G_1) \otimes L^\infty(G_2) \to L^\infty(\Omega)$$

by

$$(j_1 \otimes j_2)(f_1 \otimes f_2) = j_1(f_1)\, j_2(f_2).$$

Then $\mathbb{E} \circ (j_1 \otimes j_2)$ is the joint distribution of X_1 and X_2, and (1.4) reads

$$\mathbb{E} \circ (j_1 \otimes j_2) = (\mathbb{E} \circ j_1) \otimes (\mathbb{E} \circ j_2). \tag{1.5}$$

It is remarkable that in a noncommutative world there is more than one possibility for a notion of independence. One of these notions, called tensor independence, as was pointed out above is closely related to classical independence and is the noncommutative independence chosen for Lévy processes on Hopf $*$-algebras. In his papers on the broadening of spectral lines [vW73] von Waldenfels used another notion of independence, in some respect the most simple, which now is called *Boolean* independence, because Boolean lattices appear when moments are calculated from their cumulants. Boolean independence also appears as the free product function on the free product of groups in [Boz86].

A central role is played by *free* independence or *freeness* which was introduced by Voiculescu [Voi85]. Muraki showed that under certain natural axioms there are exactly five notions of noncommutative independence ([Mur02, Mur03], see also [Spe97, BGS02]), namely tensor, Boolean, free and monotonic and anti-monotonic independence.

The joint distribution of two classical random variables lives on the tensor product which is the commutative algebra 'freely' generated by the algebras $L^\infty(G_1)$ and $L^\infty(G_2)$. In the noncommutative case the tensor product of algebras has to be replaced by the free product of algebras. The classification result of Muraki classifies the 'natural' products of linear functionals which assign to each pair of algebras $(\mathcal{B}_1, \mathcal{B}_2)$ and linear functionals (φ_1, φ_2), $\varphi_1 : \mathcal{B}_1 \to \mathbb{C}$, $\varphi_2 : \mathcal{B}_2 \to \mathbb{C}$ a linear functional $\varphi_1 \odot \varphi_2 : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to \mathbb{C}$. The product $\odot$ replaces the tensor product of (1.4) of the classical case. If we understand by a noncommutative (quantum) probability space a $*$-algebra $\mathcal{A}$ equipped with a state $\mathbb{E} : \mathcal{A} \to \mathbb{C}$ and by a quantum random variable on a $*$-algebra $\mathcal{B}$ over a quantum probability space $(\mathcal{A}, \mathbb{E})$ a homomorphism $j : \mathcal{B} \to \mathcal{A}$ of $*$-algebras, two random variables $j_1 : \mathcal{B}_1 \to \mathcal{A}$ and $j_2 : \mathcal{B}_2 \to \mathcal{A}$ are called independent (with respect to an independence given by a Muraki natural product $\odot$) if

$$\mathbb{E} \circ (j_1 \sqcup j_2) = (\mathbb{E} \circ j_1) \odot (\mathbb{E} \circ j_2) \tag{1.6}$$

which precisely is the noncommutative version of (1.5).

In this paper, general quantum Lévy processes are considered, the independence of increments coming from one of the five notions of independence of Muraki's classification. To treat the general independence case, Hopf $*$-algebras have to be replaced by

their 'free' counterparts where tensor products of algebras are replaced by free products of algebras. Such objects appeared already in work of Voiculescu [Voi87] and had been called 'dual groups' . We will call 'dual semigroup' a $*$-algebra $\mathcal{B}$ equipped with a comultiplication Δ which is a $*$-algebra homomorphism from $\mathcal{B}$ to the free product $\mathcal{B} \sqcup \mathcal{B}$ of $\mathcal{B}$ with itself such that coassociativity and the counit property hold. If there is also an antipode we will speak of (algebraic) dual groups.

Suppose that $(\mathcal{B}, \Delta)$ is a dual semigroup. In addition let there be given a (fixed) natural product $\odot$ with its associated notion of noncommutative independence; see [Mur03, BGS02]. We define the convolution product of two linear functionals φ_1, φ_2 on $\mathcal{B}$ by

$$\varphi_1 \star \varphi_2 = (\varphi_1 \odot \varphi_2) \circ \Delta \tag{1.7}$$

in complete analogy to the tensor case (1.1). Quantum Lévy processes on the dual semigroup $\mathcal{B}$ are again determined by convolution semigroups of states on $\mathcal{B}$. Convolution exponentials $\mathrm{e}_\star^\psi$ of linear functionals ψ on $\mathcal{B}$ can be defined as before (see Sect. 3). Schoenberg correspondence (Theorem 3.1) says that the (point-wise) continuous convolution semigroups φ_t of states on $\mathcal{B}$ are precisely given by $\varphi_t = \mathrm{e}_\star^{t\psi}$ with ψ the (point-wise) derivative of φ_t and ψ conditionally positive.

In Sect. 4 we start from a conditionally positive linear functional ψ on a dual semigroup $\mathcal{B}$. Using Schoenberg correspondence, we associate with it a convolution semigroup of states on $\mathcal{B}$. By an inductive limit procedure we then construct a quantum Lévy process on $\mathcal{B}$ with convolution semigroup given by $\varphi_t = \mathrm{e}_\star^{t\psi}$.

2. Preliminaries

Algebras will be over the complex numbers and will assumed to be associative. A $*$-algebra is an algebra $\mathcal{A}$ with an involution $*$, i.e. an anti-linear map $a \mapsto a^*$ on $\mathcal{A}$ such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$. A unital algebra is an algebra such that there exists an element $\mathbf{1}$ (called the unit element) in $\mathcal{A}$ with $a\,\mathbf{1} = a = \mathbf{1}\,a$. A (counital) coalgebra is a triplet $(\mathcal{C}, \Delta, \delta)$ consisting of a (complex) vector space $\mathcal{C}$ and linear mappings $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\delta : \mathcal{C} \to \mathbb{C}$ such that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and $(\delta \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \delta) \circ \Delta$, where for vector spaces $\mathcal{V}$ and $\mathcal{W}$ we write $\mathcal{V} \otimes \mathcal{W}$ for the vector space tensor product. Either $\mathcal{C}$ is the trivial vector space or there exists an element $e \in \mathcal{C}$ with $\delta e = 1$. In the latter case $\mathcal{C} = \mathbb{C}e \oplus \mathcal{C}_0$ with $\mathcal{C}_0 = \ker \delta$, and for $c \in \mathcal{C}$, $c_0 = c - \delta(c)e$, we have $\Delta c - (\delta(c)e \otimes e + e \otimes c_0 + c_0 \otimes e) \in \mathcal{C}_0 \otimes \mathcal{C}_0$. In particular, $\Delta e = e \otimes e + B$, $B \in \mathcal{C}_0 \otimes \mathcal{C}_0$, and $\Delta c = e \otimes c + c \otimes e + B$, $B \in \mathcal{C}_0 \otimes \mathcal{C}_0$, for $c \in \mathcal{C}_0$.

A bialgebra is a coalgebra $(\widetilde{\mathcal{B}}, \widetilde{\Delta}, \delta)$, where $\widetilde{\mathcal{B}}$ is a unital algebra such that $\widetilde{\Delta}, \delta$ are algebra homomorphisms. If we put $\mathcal{B} = \ker \delta$, then $\widetilde{\Delta}\mathcal{B} \subset \mathcal{B} \oplus \mathcal{B} \oplus (\mathcal{B} \otimes \mathcal{B}) =: \mathcal{B} \otimes_0 \mathcal{B}$, and the pair $(\mathcal{B}, \Delta)$, $\Delta = \widetilde{\Delta}\restriction \mathcal{B} : \mathcal{B} \to \mathcal{B} \otimes_0 \mathcal{B}$ consists of an algebra $\mathcal{B}$ and a 'comultiplication' Δ such that

$$(\Delta \otimes_0 \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes_0 \Delta) \circ \Delta \tag{2.1}$$

and

$$(0 \otimes_0 \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes_0 0) \circ \Delta\,. \tag{2.2}$$

This shows that a bialgebra equivalently can be defined to be a pair $(\mathcal{B}, \Delta)$ consisting of an algebra $\mathcal{B}$ and an algebra homomorphism $\Delta : \mathcal{B} \to \mathcal{B} \otimes_0 \mathcal{B}$ such that (2.1) and (2.2) hold.

For an index set $I \neq \emptyset$, we put

$$\mathbb{A}(I) = \{(\varepsilon_1, \ldots, \varepsilon_n) \,|\, n \in \mathbb{N}, \varepsilon_l \in I, l = 1, \ldots, n, \varepsilon_l \neq \varepsilon_{l+1}, l = 1, \ldots, n-1\}.$$

For a family $(\mathcal{A}_i)_{i\in I}$ of algebras and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{A}(I)$ we denote by $\mathcal{A}_\varepsilon$ the algebraic tensor product $\mathcal{A}_\varepsilon = \mathcal{A}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{A}_{\varepsilon_n}$ of the algebras $\mathcal{A}_{\varepsilon_1}, \ldots, \mathcal{A}_{\varepsilon_n}$. Define the free product $\bigsqcup_{i\in I} \mathcal{A}_i$ of the family $(\mathcal{A}_i)_{i\in I}$ as the vector space direct sum

$$\bigsqcup_{i\in I} \mathcal{A}_i = \bigoplus_{\varepsilon\in\mathbb{A}(I)} \mathcal{A}_\varepsilon$$

with multiplication given by

$$(a_1 \otimes \ldots \otimes a_n)\,(b_1 \otimes \ldots \otimes b_m) = \begin{cases} a_1 \otimes \ldots \otimes a_n \otimes b_1 \otimes \ldots \otimes b_m & \text{if } \varepsilon_n \neq \gamma_1 \\ a_1 \otimes \ldots \otimes a_{n-1} \otimes (a_n b_1) \otimes b_2 \otimes \ldots \otimes b_m & \text{if } \varepsilon_n = \gamma_1 \end{cases}$$

for $n, m \in \mathbb{N}$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{A}(I)$, $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{A}(I)$ and $a_1 \otimes \ldots \otimes a_n \in \mathcal{A}_\varepsilon$, $b_1 \otimes \ldots \otimes b_m \in \mathcal{A}_\gamma$. For example, if $I = \{1, 2\}$,

$$\mathcal{A}_1 \sqcup \mathcal{A}_2 = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus (\mathcal{A}_1 \otimes \mathcal{A}_2) \oplus (\mathcal{A}_2 \otimes \mathcal{A}_1) \oplus \ldots.$$

The free product is the co-product in the category of algebras, i.e. given two algebra homomorphisms $j_1 : \mathcal{B}_1 \to \mathcal{A}$ and $j_2 : \mathcal{B}_2 \to \mathcal{A}$ with the same target $\mathcal{A}$, there is a unique algebra homomorphism $j_1 \sqcup j_2 : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to \mathcal{A}$ such that $j_{1/2} = (j_1 \sqcup j_2) \circ i_{1/2}$, where i_1, i_2 denote the natural embeddings of $\mathcal{B}_1, \mathcal{B}_2$ into $\mathcal{B}_1 \sqcup \mathcal{B}_2$. We frequently write $j_1 \sqcup\!\!\sqcup j_2$ for $(i_1 \circ j_1) \sqcup (i_2 \circ j_2) : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ for algebra homomorphisms $j_{1/2} : \mathcal{B}_{1/2} \to \mathcal{A}_{1/2}$. The free product $\mathcal{A}_1 \sqcup_{\mathbf{1}} \mathcal{A}_2$ of unital algebras $\mathcal{A}_1, \mathcal{A}_2$ is obtained from $\mathcal{A}_1 \sqcup \mathcal{A}_2$ by dividing by the ideal generated by $\mathbf{1}_{\mathcal{A}_1} - \mathbf{1}_{\mathcal{A}_2}$. Then $\sqcup_{\mathbf{1}}$ is the co-product in the category of unital algebras.

We follow [Fra06] and define (stochastic) independence in the language of category theory. Let $(\mathbf{C}, \square, i)$ be a *tensor category with injections*, i.e. $(\mathbf{C}, \square)$ is a tensor category such that for each pair C_1, C_2 of objects there exists a pair i_{C_1}, i_{C_2} of morphisms $i_{C_1} : C_1 \to C_1 \square C_2$, $i_{C_2} : C_2 \to C_1 \square C_2$, such that for any pair $j_1 : C_1 \to D_1$, $j_2 : C_2 \to D_2$ of morphisms we have

$$(j_1 \square j_2) \circ i_{C_1} = i_{D_1} \circ j_1,$$
$$(j_1 \square j_2) \circ i_{C_2} = i_{D_2} \circ j_2.$$

Then two morphisms $j_1 : C_1 \to C$, $j_2 : C_2 \to C$ with the same target are called *independent* if there exists a morphism $j : C_1 \square C_2 \to C$ such that $j_1 = j \circ i_{C_1}$ and $j_2 = j \circ i_{C_2}$. For example, consider the category formed by 'dual probability spaces' that is by pairs (C, φ) with C a commutative von-Neumann algebra and φ a normal state on C. This is a tensor category with injections if we choose the von-Neumann algebra tensor product with the tensor product of states and the natural injections. The morphisms from (C, φ) to (D, θ) are the von-Neumann algebra homomorphisms j with $\varphi = \theta \circ j$, that is they are precisely the random variables. Two random variables are stochastically independent in the classical sense iff they are independent in the above sense of categories with injections.

Since we will stay in an algebraic framework, we consider tensor products in the category formed by pairs $(\mathcal{B}, \varphi)$, where $\mathcal{B}$ is an algebra and $\varphi : \mathcal{B} \to \mathbb{C}$ is a linear

functional on $\mathcal{B}$. There is a type of tensor product with injections in this category given by

$$(\mathcal{B}_1, \varphi_1)\square(\mathcal{B}_2, \varphi_2) = (\mathcal{B}_1 \sqcup \mathcal{B}_2, \varphi_1 \odot \varphi_2),$$

where $\varphi_1 \odot \varphi_2$ is a linear functional on the free product $\mathcal{B}_1 \sqcup \mathcal{B}_2$ of algebras such that the product $\odot$ satisfies the axioms

$$(\varphi_1 \odot \varphi_2) \circ i_{1/2} = \varphi_{1/2}, \tag{A1}$$

$$(\varphi_1 \odot \varphi_2) \odot \varphi_3 = \varphi_1 \odot (\varphi_2 \odot \varphi_3), \tag{A2}$$

$$(\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2) = (\varphi_1 \odot \varphi_2) \circ (j_1 \sqcup j_2). \tag{A3}$$

Consider the additional axioms

$$(\varphi_1 \odot \varphi_2)(b_1 b_2) = (\varphi_1 \odot \varphi_2)(b_2 b_1) = \varphi_1(b_1)\varphi_2(b_2) \tag{A4}$$

for all $b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2$, and

$$\varphi_1 \odot \varphi_2 = \varphi_2 \odot \varphi_1. \tag{A5}$$

Muraki [Mur02,Mur03] showed that there are exactly five products satisfying (A1)–(A4), the tensor product, the free product [Voi85], the Boolean product [vW73], and the monotonic and anti-monotonic products [Mur97,Mur01,Lu97]. It was shown in [Spe97,BGS02] that the tensor, the free and the Boolean products are the only three products satisfying (A1)–(A5). An independence coming from a product with (A1)–(A3) will be called a $\odot$-*independence*.

A linear functional φ on a $*$-algebra $\mathcal{A}$ is called hermitian if $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in \mathcal{A}$. We call φ *conditionally positive* if φ is hermitian and if $\varphi(a^* a) \geq 0$ for all $a \in \mathcal{A}$. In this paper, we call φ a *state* if we have $\widetilde{\varphi}(a^* a) \geq 0$ for all $a \in \widetilde{\mathcal{A}}$, where $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to \mathbb{C}$ is the normalized linear extension of φ to $\widetilde{\mathcal{A}} = \mathbb{C}\mathbf{1} \oplus \mathcal{A}$. A state is conditionally positive whereas the converse is not always true. For example, on the $*$-algebra of complex polynomials in one self-adjoint indeterminate x, with constant part equal to 0 the linear functional ψ with $\psi(x^n) = \delta_{2,n}$ is conditionally positive but not a state. The point-wise limit of states is a state. If $\mathcal{I}$ is a two-sided $*$-ideal of the $*$-algebra $\mathcal{A}$, then for a state φ on $\mathcal{A}$ which vanishes on $\mathcal{I}$ we have that $\hat{\varphi}, \hat{\varphi}(a + \mathcal{I}) = \varphi(a)$, is a state on $\mathcal{A}/\mathcal{I}$.

We say that a $\odot$-independence is *positive* if for two $*$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ and states φ_1 and φ_2 on $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, the product $\varphi_1 \odot \varphi_2$ is a state. Another notion of states is the following. Call φ a strong state if $\min\{\lambda \in \mathbb{C} \,|\, \varphi_\lambda(a^* a) \geq 0 \; \forall a \in \widetilde{\mathcal{A}}\} = 1$, where φ_λ is the extension of φ to $\mathbb{C}\mathbf{1} \oplus \mathcal{A}$ with $\varphi_\lambda(\mathbf{1}) = \lambda$. Then a strong state is a state, and the converse is false in general. We say that a $\odot$-independence is strongly positive if the product of two strong states is a strong state. Then each positive $\odot$-independence is strongly positive. This holds because φ_1 and φ_2 are the restrictions of $\varphi_1 \odot \varphi_2$ to $\mathcal{A}_1$ and $\mathcal{A}_2$. It is well-known that Muraki's five notions of independence are positive. In fact,

Proposition 2.1. *The only positive $\odot$-independences are Muraki's five. In particular, a $\odot$-independence is strongly positive iff it is positive.*

Proof. It follows from [BGS02], Lem. 1, that there are 'universal' complex constants q_1, q_2 such that $(\varphi_1 \odot \varphi_2)(c_1 c_2) = q_1\, \varphi_1(c_1)\varphi_2(c_2)$ and $(\varphi_1 \odot \varphi_2)(c_2 c_1) = q_2\, \varphi_1(c_1)\varphi_2(c_2)$ for $c_1 \in \mathcal{B}_1, c_2 \in \mathcal{B}_2$.

We show that a strongly positive $\odot$-independence must be one of Muraki's five. Take $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{C}[x]$ and $\varphi(x^n) = 1, n \in \mathbb{N}$. Then φ is a strong state. If $\odot$ is strongly positive $\varphi \odot \varphi$ must be a state. With $\mathbb{C}[x] \sqcup \mathbb{C}[x] = \mathbb{C}\langle x, y\rangle$, we must have $(\varphi \odot \varphi)(xy) = q_1$ and $\varphi \odot \varphi(yx) = q_2$. We have

$$0 \leq (\varphi \odot \varphi)\big((\lambda\mathbf{1} + \alpha x + \beta y)^*(\lambda\mathbf{1} + \alpha x + \beta y)\big)$$

for all $\lambda, \alpha, \beta \in \mathbb{C}$ and the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & q_1 \\ 1 & q_2 & 1 \end{pmatrix}$$

must be positive semi-definite which forces $q_1 = q_2 = 1$. It follows that a strongly positive $\odot$-independence satisfies (A4) so that Muraki's result can be applied. □

A *dual semigroup* is a pair $(\mathcal{B}, \Delta)$ consisting of a $*$-algebra $\mathcal{B}$ and a $*$-algebra homomorphism $\Delta : \mathcal{B} \to \mathcal{B} \sqcup \mathcal{B}$ such that

$$(\Delta \sqcup \mathrm{id}) \circ \Delta = (\mathrm{id} \sqcup \Delta) \circ \Delta$$

and

$$(0 \sqcup \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \sqcup 0) \circ \Delta;$$

cf. [Zha91, Voi87] and [BGS05]. Put $\widetilde{\mathcal{B}} = \mathbb{C}\mathbf{1} \oplus \mathcal{B}$, $\widetilde{\Delta} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}} \sqcup_{\mathbf{1}} \widetilde{\mathcal{B}} = \mathbb{C}\mathbf{1} \oplus \mathcal{B} \sqcup \mathcal{B}$, $\widetilde{\Delta}\lceil\mathcal{B} = \Delta$, $\widetilde{\Delta}\mathbf{1} = \mathbf{1}$, $\delta : \widetilde{\mathcal{B}} \to \mathbb{C}$, $\delta\lceil\mathcal{B} = 0$, $\delta\mathbf{1} = 1$. Then the triplet $(\widetilde{\mathcal{B}}, \widetilde{\Delta}, \delta)$ satisfies

$$(\widetilde{\Delta} \sqcup_{\mathbf{1}} \mathrm{id}) \circ \widetilde{\Delta} = (\mathrm{id} \sqcup_{\mathbf{1}} \widetilde{\Delta}) \circ \widetilde{\Delta}, \tag{2.3}$$

$$(\delta \sqcup_{\mathbf{1}} \mathrm{id}) \circ \widetilde{\Delta} = \mathrm{id} = (\mathrm{id} \sqcup_{\mathbf{1}} \delta) \circ \widetilde{\Delta}. \tag{2.4}$$

Conversely, given a triplet $(\widetilde{\mathcal{B}}, \widetilde{\Delta}, \delta)$ such that $\widetilde{\mathcal{B}}$ is a unital $*$-algebra and $\widetilde{\Delta} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}} \sqcup_{\mathbf{1}} \widetilde{\mathcal{B}}$, $\delta : \widetilde{\mathcal{B}} \to \mathbb{C}$ are unital $*$-algebra homomorphisms with (2.3) and (2.4), it can be shown that the pair $(\ker\delta, \widetilde{\Delta}\lceil\ker\delta)$ is a dual semigroup; see [BGS05]. A dual semigroup is called a *dual group* if there is an endomorphism S on $\mathcal{B}$ such that $(S \sqcup \mathrm{id}) \circ \Delta = 0 = (\mathrm{id} \sqcup S) \circ \Delta$.

Let $\mathbf{C}$ and $\mathbf{D}$ be two categories and let $\mathbf{F}$ be a functor from $\mathbf{C}$ to $\mathbf{D}$. For an object D in $\mathbf{D}$ a *universal pair* or arrow [ML98] from D to $\mathbf{F}$ is a pair (C, i) with C an object in $\mathbf{C}$ and i a morphism $i : D \to \mathbf{F}(C)$ such that the following universal property is fulfilled. For each object A in $\mathbf{C}$ and morphism $k : D \to \mathbf{F}(A)$ there is a unique morphism $j : C \to A$ such that $k = \mathbf{F}(j) \circ i$. In the case when $\mathbf{C}$ is the category of algebras, $\mathbf{D}$ is the category of vector spaces, and $\mathbf{F}$ is the forgetful functor, a universal pair from a vector space $\mathcal{V}$ to $\mathbf{F}$ can be realized as the tensor algebra over $\mathcal{V}$ which is the vector space direct sum of the vector space tensor powers $\mathcal{V}^{\otimes n}$ of $\mathcal{V}$. This is an algebra with multiplication $(v_1 \otimes \ldots v_n)(w_1 \otimes \ldots \otimes w_m) = v_1 \otimes \ldots \otimes v_n \otimes w_1 \otimes \ldots \otimes w_m, v_1, \ldots, v_n, w_1, \ldots, w_m \in \mathcal{V}$. The morphism $i_{\mathcal{V}}$ is given by the natural embedding. The unique morphism j associated with a morphism k is denoted by $T(k)$. Similarly, the tensor $*$-algebra $T(\mathcal{V})$ over a $*$-vector space $\mathcal{V}$, that is a vector space $\mathcal{V}$ equipped with an anti-linear self-inverse map $v \mapsto v^*$, is the universal pair from $\mathcal{V}$ to the forgetful functor from the category of $*$-algebras to the category of $*$-vector spaces. The involution of $T(\mathcal{V})$ is given by $(v_1 \otimes \ldots \otimes v_n)^* = v_n^* \otimes \ldots \otimes v_1^*$.

Now let **C** be the category of *commutative* algebras and let **D** again be the category of vector spaces with the forgetful functor from **C** to **D**. The universal pair is denoted by $(S(\mathcal{V}), i_{\mathcal{V}})$, and a realization of $S(\mathcal{V})$ is the symmetric tensor algebra over $\mathcal{V}$ which is the quotient of $T(\mathcal{V})$ by the ideal generated by $v \otimes w - w \otimes v$, $v, w \in \mathcal{V}$. We frequently write $v_1 \otimes_s \ldots \otimes_s v_n$ for the equivalence class of $v_1 \otimes \ldots \otimes v_n$. Of course, there is also the symmetric tensor $*$-algebra over a $*$-vector space $\mathcal{V}$. If $\{v_i \mid i \in I\}$ is a (self-adjoint) vector space basis of $\mathcal{V}$, then $S(\mathcal{V})$ can be identified with the polynomial algebra $\mathbb{C}[x_i; i \in I]$.

It follows from [BGS02], Lem. 2.1, that if $\odot$ satisfies (A3) there are linear maps

$$\sigma_{\mathcal{B}_1,\mathcal{B}_2} : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to S(\mathcal{B}_1) \otimes S(\mathcal{B}_2) \cong S(\mathcal{B}_1 \oplus \mathcal{B}_2)$$

such that

$$\varphi_1 \odot \varphi_2 = (S(\varphi_1) \otimes S(\varphi_2)) \circ \sigma_{\mathcal{B}_1,\mathcal{B}_2}.$$

By Theorem 3.4 of [BGS05], for a fixed $\odot$-independence and a dual semigroup $(\mathcal{B}, \Delta)$, we can form the commutative $*$-bialgebra $(S(\mathcal{B}), S(\sigma \circ \Delta))$, where we put $\sigma = \sigma_{\mathcal{B},\mathcal{B}}$. Thus a $\odot$-independence gives rise to a functor from the category of dual semigroups to the category of commutative $*$-bialgebras. Now put, for a fixed $\odot$-independence,

$$\varphi_1 \star \varphi_2 = (\varphi_1 \odot \varphi_2) \circ \Delta \tag{2.5}$$

for linear functionals φ_1 and φ_2 on a dual semigroup $\mathcal{B}$. Then (see [BGS02])

$$S(\varphi_1 \star \varphi_2) = S(\varphi_1) \star S(\varphi_2), \tag{2.6}$$

where the second convolution product is with respect to the comultiplication $S(\sigma \circ \Delta)$.

3. Schoenberg Correspondence

The intersection of two coalgebras is again a coalgebra so that the sub-coalgebra generated by a subset of a coalgebra is well-defined. The fundamental theorem of coalgebras (see e.g. [DNR01]) says that the sub-coalgebra generated by a single element (and thus by a finite number of elements) is finite-dimensional. It follows that a coalgebra is the inductive limit of its finite-dimensional sub-coalgebras. For a linear functional ψ on a coalgebra $\mathcal{C}$ define the linear map T_ψ on $\mathcal{C}$ by $T_\psi = (\mathrm{id} \otimes \psi) \circ \Delta$. Then T is a unital algebra homomorphism from the convolution algebra formed by linear functionals on $\mathcal{C}$ to the algebra of linear operators on $\mathcal{C}$ and $L \mapsto \delta \circ L$ is the left inverse of T. Moreover, T_ψ leaves invariant all sub-coalgebras of $\mathcal{C}$. Denote by e^{T_ψ} the inductive limit of the (matrix) exponentials of the restrictions of T_ψ to finite-dimensional sub-colagebras. Put $\exp_\star \psi := \delta \circ \mathrm{e}^{T_\psi}$. It follows that the series

$$\sum_{n=0}^{\infty} \frac{\psi^{\star n}}{n!}(c) \tag{3.1}$$

converges for all $c \in \mathcal{C}$ and that this limit equals $\exp_\star \psi$. We have

$$\exp_\star(\psi_1 + \psi_2) = (\exp_\star \psi_1) \star (\exp_\star \psi_2)$$

if $\psi_1 \star \psi_2 = \psi_2 \star \psi_1$ and

$$(\exp_\star \psi)(c) = \lim_{n\to\infty} (\delta + \frac{\psi}{n})^{\star n}(c).$$

More generally, [SV08,SSV10].

Lemma 3.1. *Let ψ be a linear functional on a coalgebra $\mathcal{C}$. Suppose that $R_n, n \in \mathbb{N}$, are linear functionals on $\mathcal{C}$ such that for each $b \in \mathcal{C}$ there is a constant $C_b \in \mathbb{R}_+$ with*

$$|R_n(b)| \leq \frac{1}{n^2} C_b \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Then

$$(\delta + \frac{\psi}{n} + R_n)^{\star n}$$

converges to $\exp_\star \psi$ *point-wise.*

Proof. By the fundamental theorem of coalgebras we can assume that $\mathcal{C}$ is finite-dimensional. Then with some norm $||\ ||$ on $\mathcal{C}$,

$$\begin{aligned} ||T_{R_n}|| &= ||(\mathrm{id} \otimes R_n) \circ \Delta|| \\ &\leq ||\mathrm{id} \otimes R_n||\, ||\Delta|| \\ &= ||R_n||\, ||\Delta||. \end{aligned}$$

Choose a vector space basis $\{b_1, \ldots, b_k\}$ of $\mathcal{C}$. Then for $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$,

$$\begin{aligned} &|R_n(\alpha_1 b_1 + \cdots + \alpha_k b_k)| \\ &\quad \leq \max_{1 \leq j \leq k} |R_n(b_j)|\, (|\alpha_1| + \cdots + |\alpha_k|) \\ &\quad \leq \frac{1}{n^2} (\max_{1 \leq j \leq k} C_{b_j})(|\alpha_1| + \cdots + |\alpha_k|), \end{aligned}$$

which implies $||T_{R_n}|| \leq \frac{1}{n^2} C$ for some constant C. Now

$$(\mathrm{id} + \frac{T_\psi}{n} + T_{R_n})^n \to \mathrm{e}^{T_\psi},$$

and thus

$$(\delta + \frac{\psi}{n} + R_n)^{\star n} \to \mathrm{e}^{\psi}.$$

□

A family $(\varphi_t)_{t \in \mathbb{R}_+}$ of linear functionals on a coalgebra $\mathcal{C}$ is called a *continuous convolution semigroup* (CCSG) if $\varphi_{s+t} = \varphi_s \star \varphi_t, \varphi_0 = \delta$, and $\varphi_t \to \delta$ point-wise for $t \to 0+$. For a CCSG the operators T_{φ_t} form a semigroup of linear operators on $\mathcal{C}$. Using the fundamental theorem of coalgebras and a well-known result for continuous semigroups of matrices, we obtain that $\lim_{t \to 0+} \frac{1}{t}(\varphi_t - \delta)$ exists point-wise and that we have $\exp_\star(t\psi) = \varphi_t$ for the limiting functional ψ. It follows that the CCSGs are exactly given by the convolution exponentials $\exp_\star(t\psi)$.

Now let ψ be a linear functional on a dual semigroup $\mathcal{B}$. Moreover, fix a $\odot$-independence, and put $D(\psi)(B) = \frac{\mathrm{d}}{\mathrm{d}t} S(t\,\psi)(B)|_{t=0}$ for $B \in S(\mathcal{B})$. We define $\exp_\star \psi$ point-wise by $(\exp_\star D(\psi)) \circ i_{\mathcal{B}}$. Then $\exp_\star(t\psi)$, $t \geq 0$, form a CCSG of linear functionals on $\mathcal{B}$ with the convolution product now given by (2.5); see [BGS02]. We have [SV08]

Lemma 3.2. *Let ψ be a linear functional on a dual semigroup $\mathcal{B}$. Suppose that R_n, $n \in \mathbb{N}$, are linear functionals on $\mathcal{B}$ such that for each $b \in \mathcal{B}$ there is a constant $C_b \in \mathbb{R}_+$ with*

$$|R_n(b)| \le \frac{1}{n^2} C_b \quad \forall n \in \mathbb{N}.$$

Then

$$(\frac{\psi}{n} + R_n)^{\star n}$$

converges to $\exp_\star \psi$ *point-wise.*

Proof. By (2.6) we have

$$S((\frac{\psi}{n} + R_n)^{\star n}) = S(\frac{\psi}{n} + R_n)^{\star n}$$

and

$$\begin{aligned}(\frac{\psi}{n} + R_n)^{\star n} &= S((\frac{\psi}{n} + R_n)^{\star n})\lceil\mathcal{B} \\ &= S(\frac{\psi}{n} + R_n)^{\star n}\lceil\mathcal{B}.\end{aligned}$$

Moreover, $\exp_\star \psi = \exp_\star D(\psi)\lceil\mathcal{B}$. We will prove that

$$S(\frac{\psi}{n} + R_n)^{\star n} \to \exp_\star D(\psi).$$

For $b_1, \dots, b_k \in \mathcal{B}$, $k \ge 1$,

$$\begin{aligned}&S(\frac{\psi}{n} + R_n)(b_1 \otimes_s \dots \otimes_s b_k) \\ &= (\frac{\psi}{n} + R_n)(b_1) \dots (\frac{\psi}{n} + R_n)(b_k) \\ &= \sum_{A \subset \{1,\dots,k\}} \frac{1}{n^{\#A}} \prod_{j \in A} \psi(b_j) \prod_{j \notin A} R_n(b_j) \\ &= R_n(b_1) \dots R_n(b_k) + \frac{1}{n}\big(\psi(b_1) R_n(b_2) \dots R_n(b_k) \\ &\quad + R_n(b_1)\psi(b_2) R_n(b_3) \dots R_n(b_k) + \dots + R_n(b_1) \dots R_n(b_{k-1})\psi(b_k)\big) \\ &\quad + \frac{1}{n^2} T_n(b_1 \otimes_s \dots \otimes_s b_k)\end{aligned}$$

with $|T_n(b_1 \otimes_s \dots \otimes_s b_k)| \le D_1$ for all $n \in \mathbb{N}$ for some constant $D_1 \in \mathbb{R}_+$. Also $|R_n(b_1) \dots R_n(b_k)| \le \frac{1}{n^2} D_2$ for all $n \in \mathbb{N}$ for some $D_2 \in \mathbb{R}_+$, and for a suitable constant D_3,

$$\begin{aligned}&|\psi(b_1) R_n(b_2) \dots R_n(b_k) + R_n(b_1)\psi(b_2) R_n(b_3) \dots R_n(b_k) + \dots \\ &\quad + R_n(b_1) \dots R_n(b_{k-1})\psi(b_k)| \le \frac{1}{n^2} D_3\end{aligned}$$

if $k \geq 2$ so that

$$S(\frac{\psi}{n} + R_n)(b_1 \otimes_s \ldots \otimes_s b_k) \leq \frac{1}{n^2} D$$

if $k \geq 2$ for some $D \in \mathbb{R}_+$. For $k = 1$ we have $S(\frac{\psi}{n} + R_n)(b) = \frac{\psi}{n}(b) + R_n(b)$. Moreover, $S(\frac{\psi}{n} + R_n)(\mathbf{1}) = 1$. It follows that

$$S(\frac{\psi}{n} + R_n) = S(0) + \frac{1}{n} D(\psi) + \widetilde{R}_n$$

with $\widetilde{R}_n : S(\mathcal{B}) \to \mathbb{C}$ linear, $\widetilde{R}_n(\mathbf{1}) = 0$, and $|\widetilde{R}_n(x)| \leq \frac{1}{n^2} \widetilde{C}_x$ for all $n \in \mathbb{N}$ for a suitable constant $\widetilde{C}_x$. By Lemma 3.1,

$$S(\frac{\psi}{n} + R_n) = \big(S(0) + \frac{1}{n} D(\psi) + \widetilde{R}_n\big)^{\star n} \longrightarrow \exp_\star D(\psi)$$

point-wise. □

We say that *Schoenberg correspondence holds* on a dual semigroup $\mathcal{B}$ if the convolution exponential $\exp_\star \psi$ is a state for each conditionally positive linear functional ψ on $\mathcal{B}$ and for each positive $\odot$-independence. Then Schoenberg correspondence holds on $\mathcal{B}$ iff the CCSG of states (with respect to a positive $\odot$-independence) on $\mathcal{B}$ are exactly given by $\exp_\star(t\psi)$ with ψ conditionally positive. We will show that Schoenberg correspondence holds on *all* dual semigroups (Theorem 3.1).

Let $(\mathcal{B}, \Delta)$ and $(\mathcal{C}, \Lambda)$ be two dual semigroups and let $\kappa : \mathcal{C} \to \mathcal{B}$ be a $*$-algebra homomorphism. For a linear functional ψ on $\mathcal{B}$ we put

$$\gamma_t = \exp_\star(t\psi)$$
$$\varphi_t = \exp_\star(t(\psi \circ \kappa))$$

for $t \in \mathbb{R}_+$.

Proposition 3.1.

$$(\gamma_{t/n} \circ \kappa)^{\star n} \to \varphi_t$$

point-wise for all $t \in \mathbb{R}_+$.

Proof. We can assume that $t = 1$. We will show that there are linear functionals $R_n : \mathcal{C} \to \mathcal{B}$ and constants $C_b \in \mathbb{R}_+$, $b \in \mathcal{C}$, such that

$$(\gamma_{1/n} \circ \kappa)(b) = \frac{1}{n}(\psi \circ \kappa)(b) + R_n(b)$$

with

$$|R_n(b)| \leq \frac{1}{n^2} C_b$$

for all $n \in \mathbb{N}$. An application of Lemma 3.2 will then prove the proposition.

We have

$$\begin{aligned}(\gamma_{1/n}\circ\kappa)(b) &= \gamma_{1/n}(\kappa(b))\\ &= \big(\exp_\star \frac{1}{n}D(\psi)\big)(\kappa(b))\\ &= \frac{1}{n}D(\psi)(\kappa(b)) + \frac{1}{2n^2}D(\psi)^{\star 2}(\kappa(b))\\ &\quad + \frac{1}{3!n^3}D(\psi)^{\star 3}(\kappa(b)) + \cdots\\ &= \frac{1}{n}\psi(\kappa(b)) + \frac{1}{n^2}R_n(b)\end{aligned}$$

with $|R_n(b)| \le C_b$ for all $n \in \mathbb{N}$ for a suitable constant C_b. □

Proposition 3.2. *Suppose that Schoenberg correspondence holds on $\mathcal{B}$. Then for a conditionally positive linear functional ψ on $\mathcal{B}$, we have that, given a positive $\odot$-independence, $\varphi_t = \exp_\star(t(\psi\circ\kappa))$ is a CCSG of states on $\mathcal{C}$.*

Proof. We fix a positive $\odot$-independence. If Schoenberg correspondence holds on $\mathcal{B}$, then $\gamma_{t/n}$ are states on $\mathcal{B}$. This implies that $\gamma_{t/n}\circ\kappa$ are states on $\mathcal{C}$ which by the positivity of the independence gives that $(\gamma_{t/n}\circ\kappa)^{\star n}$ are states on $\mathcal{C}$. By Proposition 3.1 φ_t is the point-wise limit of $(\gamma_{t/n}\circ\kappa)^{\star n}$. Since the point-wise limit of states is a state, the proposition follows. □

Let ψ be a conditionally positive linear functional on a $*$-algebra $\mathcal{A}$. We form the left ideal $\mathcal{N}_\psi = \{a \in \mathcal{A} \mid \psi(ba) = 0\ \forall b \in \mathcal{A}\}$. The quotient space $D = \mathcal{A}/\mathcal{N}_\psi$ is an inner product space with inner product $\langle\eta(a),\eta(b)\rangle = \psi(a^*b)$, where $\eta : \mathcal{A} \to D$ denotes the canonical map. Moreover, $\rho(a)\eta(b) = \eta(ab)$ defines a $*$-representation of $\mathcal{A}$ on D, i.e. a $*$-algebra homomorphism from $\mathcal{A}$ to the $*$-algebra $L(D)$ of adjointable linear operators on D; see [Fra06,Sch93]. We have

Proposition 3.3. *If $(a_i)_{i\in I}$ is a set of generators of the algebra $\mathcal{A}$, the maps ρ, η and ψ are determined by their values on the a_i.*

Since a $*$-vector space $\mathcal{V}$ generates the tensor $*$-algebra $T(\mathcal{V})$, we have that conditionally positive linear functionals are given by an inner product space D, a $*$-map $\rho : \mathcal{V} \to L(D)$, a linear map $\eta : \mathcal{V} \to D$, and a hermitian linear functional $\psi : \mathcal{V} \to \mathbb{C}$; cf. [Sch91,Fra06].

Clearly, a linear functional on a $*$-algebra $\mathcal{A}$ is a state if it is the expectation of a quantum random variable. Consider the case of tensor independence. It is well known that the Gelfand–Naimark–Segal (GNS) representation π of $\varphi = \exp_\star\psi$, ψ a conditionally positive linear functional on $T(\mathcal{V})$, is given by

$$\pi(v) = A^*(\eta(v)) + \Lambda(\rho(v)) + A(\eta(v^*)) + \psi(v)\mathrm{id},$$

$v \in \mathcal{V}$, where A^*, Λ, A are the creation, preservation and annihilation operators on Bose Fock space over the completion H of the inner product space D, i.e.

$$\varphi(v_1 \otimes \ldots \otimes v_n) = \mathbb{E}(\pi(v_1)\ldots\pi(v_n)),$$

where expectation is taken in the vacuum state of the Fock space, see e.g. [Sch91]. It follows that φ is a state. The analogous result holds in the free case if the, Bose Fock

space is replaced by the full Fock space and A^*, Λ, A are the free creation, preservation and annihilation operators; see for example [GSS92]. In the remaining three cases of independences we can apply the reduction theory of Franz [Fra06] to realize the GNS of φ on a Bose Fock space. We have

Proposition 3.4. *Schoenberg correspondence holds on tensor $*$-algebras.*

We will apply Proposition 3.2 to the following situation. The tensor $*$-algebra $T(\mathcal{B})$ of a dual semigroup $\mathcal{B}$ (viewed as a $*$-vector space) carries another dual group structure than that given by the primitive comultiplication $b \mapsto i_1(b) + i_2(b)$. The second dual semigroup structure is given by extending the map

$$(i_{\mathcal{B}} \sqcup i_{\mathcal{B}}) \circ \Delta : \mathcal{B} \to T(\mathcal{B}) \sqcup T(\mathcal{B}),$$

with $i_{\mathcal{B}} : \mathcal{B} \to T(\mathcal{B})$ the natural embedding, to a homomorphism

$$T(\Delta) : T(\mathcal{B}) \to T(\mathcal{B}) \sqcup T(\mathcal{B}).$$

We denote by $M : T(\mathcal{B}) \to \mathcal{B}$ the multiplication map $T(\mathrm{id})$.

Proposition 3.5. *We have for linear functionals $\varphi_1, \varphi_2, \psi$ on a dual semigroup $\mathcal{B}$*
(a)

$$(\varphi_1 \circ M) \star_{T(\Delta)} (\varphi_2 \circ M) = (\varphi_1 \star_{\Delta} \varphi_2) \circ M,$$

(b)

$$\exp_{\star T(\Delta)}(\psi \circ M) = (\exp_{\star_{\Delta}} \psi) \circ M,$$

(c) *the linear functional* $\exp_{\star T(\Delta)}(\psi \circ M)$ *vanishes on the two-sided $*$-ideal* $\ker M$.

Proof. From (A1)

$$(\varphi_1 \circ M) \odot (\varphi_2 \circ M) = (\varphi_1 \odot \varphi_2) \circ (M \sqcup M).$$

Since $(M \sqcup M) \circ T(\Delta) = \Delta \circ M$, part (a) follows. (b) is a consequence of (a) and 3.2 for $R_n = 0$. (c) follows from (b). □

Theorem 3.1. *Schoenberg correspondence holds for all dual semigroups.*

Proof. For a positive $\odot$-independence and a conditionally positive linear functional ψ on a dual semigroup $(\mathcal{E}, \Delta)$ we have that $\psi \circ M$ is conditionally positive on the tensor $*$-algebra $T(\mathcal{E})$. Now apply Proposition 3.2 to $\mathcal{C} = (T(\mathcal{E}), T(\Delta))$, $\mathcal{B} = T(\mathcal{E})$ with the primitive comultiplication, and $\kappa = \mathrm{id}$. By Proposition 3.4, Schoenberg correspondence holds on $\mathcal{B}$. By Proposition 3.2 this means that Schoenberg correspondence holds on $\mathcal{C}$. Thus $\exp_{\star}(\psi \circ M)$ is a state on $\mathcal{C}$. But, using (c) of Proposition 3.5 and the fact that $T(\mathcal{E})/\ker M = \mathcal{E}$, it follows that $\exp_{\star}\psi$ is a state on $\mathcal{E}$. □

4. Quantum Lévy Processes

A *quantum Lévy process* (QLP) on a dual semigroup with respect to a positive $\odot$-independence (over a quantum probability space $(\mathcal{A}, \mathbb{E})$) is a family of quantum random variables $j_{st} : \mathcal{B} \to \mathcal{A}$, $0 \le s \le t$, such that

$$(j_{rs} \sqcup j_{st}) \circ \Delta = j_{rt} \quad \text{for all } 0 \le r \le s \le t, \tag{4.1}$$

$$j_{t_1,t_2}, \ldots, j_{t_n,t_{n+1}} \text{ are independent for all } n \in \mathbb{N}, 0 \le t_1 \le \cdots \le t_{n+1}, \tag{4.2}$$

$$\mathbb{E} \circ j_{st} \text{ only depends on } t - s, \tag{4.3}$$

$$\lim_{t \to 0+} (\mathbb{E} \circ j_{0t})(b) = 0 \quad \text{for all } b \in \mathcal{B}. \tag{4.4}$$

Property (4.1) is the increment property, (4.2) expresses the independence of increments with respect to the underlying $\odot$-independence, (4.3) reflects the stationarity of increments, and (4.4) is a condition of weak continuity.

Let $\mathcal{I} \subset \mathbb{R}_+$ be a compact interval or equal to $\mathbb{R}_+$. Denote by M the set

$$M = \{\sigma \subset \mathcal{I} \,|\, 1 < \#\sigma < \infty\}$$

of finite subsets of $\mathcal{I}$ with the inclusion of sets as partial ordering. We write $\sigma = \{t_1 < \cdots < t_{n+1}\}$ for a set $\sigma = \{t_1, \ldots, t_{n+1}\} \in M, t_1 < \cdots < t_{n+1}$. Define the n^{th} comultiplication $\Delta : \mathcal{B} \to \mathcal{B}^{\sqcup n}$, $n \in \mathbb{N}_0$, of a dual semigroup $(\mathcal{B}, \Delta)$ recursively by

$$\Delta_0 = 0; \;\; \Delta_{n+1} = (\Delta_n \sqcup \mathrm{id}) \circ \Delta.$$

For $\{s < t\} \in M$ we denote by $\mathcal{B}_{st}$ a copy of $\mathcal{B}$ and by $\iota_{st} : \mathcal{B} \to \mathcal{B}_{st}$ the identification map. Put $\mathcal{B}_\sigma = \bigsqcup_{l=1}^n \mathcal{B}_{t_l,t_{l+1}}$ and let

$$f_{\{t_1,t_{n+1}\},\sigma} : \mathcal{B}_{t_1,t_{n+1}} \to \mathcal{B}_\sigma$$

be the mapping

$$(\iota_{t_1,t_2} \sqcup \ldots \sqcup \iota_{t_n,t_{n+1}}) \circ \Delta_n \circ \iota_{t_1,t_{n+1}}^{-1}.$$

Moreover, for $\sigma = \{t_1 < \cdots < t_{n+1}\}$ and $\tau \supset \sigma$,

$$\begin{aligned}\tau = \{&t_1 = t_{11} < \cdots < t_{1,m_1} < t_{1,m_1+1} = t_2 = t_{21} < \cdots < \cdots t_{2,m_2} < t_{2,m_2+1} \\ &= t_3 < \cdots < t_{n-1,m_{n-1}+1} = t_n = t_{n1} < \cdots < t_{n,m_n} < t_{n,m_n+1} = t_{n+1}\},\end{aligned}$$

we define $f_{\sigma\tau} : \mathcal{B}_\sigma \to \mathcal{B}_\tau$ by

$$f_{\sigma\tau} = f_{\{t_1,t_2\},\{t_{11},\ldots,t_{1,m_1+1}\}} \sqcup \ldots \sqcup f_{\{t_n,t_{n+1}\},\{t_{n1},\ldots,t_{n,m_n+1}\}}.$$

For $\tau = \{t_1 < \cdots < t_{n+1}\}$ and $\sigma = \{t_k < \cdots < t_l\}, k, l \in \{1, \ldots, n+1\}$ we put $f_{\sigma\tau} : \mathcal{B}_\sigma \to \mathcal{B}_\tau$ equal to the natural embedding. For the general case $\tau \supset \sigma$ when $\sigma = \{s_1 < \cdots < s_{m+1}\}$ with $s_1 = t_k, s_{m+1} = t_l$ for $k, l \in \{1, \ldots, n+1\}$ we put

$$f_{\sigma\tau} = f_{\{t_k<\cdots<t_l\},\tau} \circ f_{\sigma,\{t_k<\cdots<t_l\}}.$$

Let ψ be conditionally positive on $\mathcal{B}$. By Schoenberg correspondence, $\varphi_t = \mathrm{e}_\star^{t\psi}$ form a convolution semigroup of states on $\mathcal{B}$. For $\sigma = \{t_1 < \cdots < t_{n+1}\} \in M$ we put

$$\varphi_\sigma = (\varphi_{t_2-t_1} \odot \ldots \odot \varphi_{t_{n+1}-t_n}) \circ (\iota_{t_1,t_2}^{-1} \sqcup \ldots \sqcup \iota_{t_n,t_{n+1}}^{-1})$$

to obtain a state on $\mathcal{B}_\sigma$. The family $(\mathcal{B}_\sigma, \varphi_{gs}, f_{\sigma\tau})$ is an inductive system in the category formed by pairs $(\mathcal{B}, \varphi)$ where, $\mathcal{B}$ is a $*$-algebra and φ a state on $\mathcal{B}$. It is not difficult to see that inductive limits exist in this category. Let $(\mathcal{A}, \mathbb{E}, f_\sigma)$ be the inductive limit of the above inductive system. Then $\mathcal{A}$ is a $*$-algebra, $\mathbb{E}$ a state on $\mathcal{A}$ and $f_\sigma : \mathcal{B}_\sigma \to \mathcal{A}$ are $*$-algebra homomorphisms such that $\mathbb{E} \circ f_\sigma = \varphi_\sigma$ and $f_\tau \circ f_{\sigma\tau} = f_\sigma$.

Theorem 4.1.

$$j_{st} = f_{\{s,t\}} \circ \iota_{st} : \mathcal{B} \to \mathcal{A},\ s < t$$

defines a QLP on $\mathcal{B}$ whose convolution semigroup of states is given by $\varphi_t = \mathrm{e}_\star^{t\psi}$.

Proof. We have

$$\begin{aligned} j_{rs} \star j_{st} &= (j_{rs} \sqcup j_{st}) \circ \Delta \\ &= f_{\{r,s,t\}} \circ (f_{\{r,s\},\{r,s,t\}} \amalg f_{\{r,t\},\{r,s,t\}}) \circ \Delta \\ &= f_{\{r,s,t\}} \circ f_{\{r,t\},\{r,s,t\}} \\ &= j_{rt}. \end{aligned}$$

Next

$$\begin{aligned} \mathbb{E}(j_{t_1,t_2} \sqcup \ldots \sqcup j_{t_n,t_{n+1}}) &= \mathbb{E} \circ j_{\{t_1,\ldots,t_{n+1}\}} \\ &= \varphi_{\{t_1,\ldots,t_{n+1}\}} \\ &= \varphi_{t_2-t_1} \odot \ldots \odot \varphi_{t_{n+1}-t_n} \\ &= (\mathbb{E} \circ j_{t_1,t_2}) \odot \ldots \odot (\mathbb{E} \circ j_{t_n,t_{n+1}}), \end{aligned}$$

which gives the independence of increments. Since $\mathbb{E} \circ j_{st} = \varphi_{t-s}$, stationarity of increments holds, too. Weak continuity follows from the continuity of φ_t. We have that j_{st} is a QLP. Its convolution semigroup clearly is given by φ_t. □

Example 1. A classical time-indexed stochastic process on the group $\mathcal{U}_d$ of unitary $d \times d$-matrices is a process U_t, $t \geq 0$, of unitary operators on the Hilbert space $\mathbb{C}^d \otimes L^2(\Omega)$. The process U_t is a Lévy process if the $*$-algebras $\mathcal{A}_{st} \subset L^\infty(\Omega) \subset \mathcal{B}(L^2(\Omega))$, $0 \leq s \leq t$, generated by the entries $(U_{st})_{kl}$, $k, l = 1, \ldots, d$, are independent for disjoint intervals, where we write U_{st} for the increment $U_s^{-1} U_t \in \mathcal{B}(\mathbb{C}^d \otimes L^2(\Omega)) \cong \mathrm{M}_d(\mathcal{B}(L^2(\Omega))$. This means

$$\mathbb{E}(a_1 \cdots a_n) = \mathbb{E}(a_1) \cdots \mathbb{E}(a_n)$$

for $a_i \in \mathcal{A}_{t_i,t_{i+1}}$, $i = 1, \ldots, n$, $t_1 < \cdots < t_{n+1}$. Here $\mathbb{E}(a) = \int_\Omega a \, \mathrm{d}\mathbb{P} = \langle \xi, a\, \xi \rangle$, ξ the constant function equal to 1. Moreover, the expectation restricted to $\mathcal{A}_{st}$ only depends on $t - s$, and U_t converges to the identity in the sense that

$$\langle \xi, (U_{st}^{(*)})_{k_1,l_1} \cdots (U_{st}^{(*)})_{k_n,l_n} \xi \rangle \to \delta_{k_1,l_1} \cdots \delta_{k_n,l_n} \tag{4.5}$$

for $t \downarrow s$. Since $\mathcal{A}_{st} \subset L^\infty(\Omega)$ for all $0 \leq s \leq t$, the algebras $\mathcal{A}_{t_i,t_{i+1}}$ commute. We pass to the noncommutative case (cf. [vW84]) by considering processes U_t of unitary, on a Hilbert space $\mathbb{C}^d \otimes \mathcal{H}$, H a Hilbert space, with expectation given by a unit vector ξ in $\mathcal{H}$. The algebras $\mathcal{A}_{st}$ are defined as before, independence is in the state given by ξ in the sense of a fixed noncommutative independence. Stationarity of increments is still defined by (4.5).

Let $\mathcal{K}[d]$ be the Kreîn dual of the compact group $\mathcal{U}_d$ that is the Hopf $*$-algebra generated by commuting indeterminates x_{kl} and x_{kl}^* with relations $x^* x = \mathbf{1}$ in matrix form where $x = (x_{kl})_{kl}$ and $x^* = (x_{kl}^*)_{kl}$. The comultiplication is given by $\Delta x = x \otimes x$, the counit by $\delta x = 1$. Lévy processes on $\mathcal{U}_d$ and QLPs on $\mathcal{K}[d]$ are in 1-1-correspondence via $j_{st}(x_{kl}) = (U_{st})_{kl}$. Denote by $\mathcal{K}\langle d \rangle$ the dual group generated by non-commuting indeterminates, again given by the entries of x and x^* with relations $x^* x = \mathbf{1} = x x^*$ and

with comultipication defined in the same manner as in the commutative case. Notice that now Δ is a map from $\mathcal{K}\langle d\rangle$ to $\mathcal{K}\langle d\rangle \sqcup\!\sqcup_{\mathbf{1}} \mathcal{K}\langle d\rangle$ that is we consider $\mathcal{K}\langle d\rangle$ as a unital dual group. For example, $\Delta x_{kl} = \sum_{n=1}^{d} \iota_1(x_{kn})\iota_2(x_{nl})$. The antipode is the $*$-algebra automorphism given by $Sx_{kl} = x_{lk}^*$. Noncommutative unitary Lévy processes as described above and QLPs on $\mathcal{K}\langle d\rangle$ are the same objects, again via $j_{st}(x_{kl}) = (U_{st})_{kl}$.

Let ψ be a conditionally positive hermitian linear functional on $\mathcal{K}\langle d\rangle$ and let (D, ρ, η) be as in Sect. 3. Denote by H the completion of D. Since the x_{kl} generate $\mathcal{K}\langle d\rangle$ as a $*$-algebra, ρ is determined by the unitary operator $W = \rho(x_{kl})_{kl}$ on $\mathbb{C}^d \otimes H$. Moreover,

$$0 = \eta\Big(\sum_{n=1}^{d} x_{nk}^* x_{nl}\Big) = \sum_{n=1}^{d} (W_{nk}^* \eta(x_{nl}) + \eta(x_{nk})\delta_{nl})$$

and $\eta(x_{kl}^*) = -\sum_{n=1}^{d} W_{nl}^* \eta(x_{nk})$. It follows from Proposition 3.3 that the QLP with generator ψ is determined by W, the matrix $L \in \mathrm{M}_d(H)$, $L_{kl} = \eta(x_{kl})$, and the matrix $G \in \mathrm{M}_d(\mathbb{C})$, $G_{kl} = \psi(x_{kl})$. Conversely, each such triplet (W, L, G) defines a conditionally positive hermitian linear functional ψ on $\mathcal{K}\langle d\rangle$ by Proposition 3.3. We described the QLPs on $\mathcal{K}\langle d\rangle$ by triplets (W, L, G) consisting of a unitary W on $\mathbb{C}^d \otimes H$, a $d \times d$-matrix L with entries in H, and a scalar $d \times d$-matrix G.

Example 2. Let $\mathbb{F}_n$ denote the free group with $n \in \mathbb{N}$ generators $g_1, \ldots, g_n$. The group algebra $\mathbb{CF}_n$ of $\mathbb{F}_n$ is a dual group with $g^* = g^{-1}$, $\Delta : \mathbb{CF}_n \to \mathbb{CF}_n \sqcup\!\sqcup_{\mathbf{1}} \mathbb{CF}_n$, $\Delta g_i = \iota_1(g_i)\iota_2(g_i)$, $\delta g_i = 1$. A QLP on $\mathbb{F}_n$ (that is on $\mathbb{CF}_n$) is given by a vector $(U_t^{(1)}, \ldots\ldots, U_t^{(n)})$ of unitary operators $U_t^{(i)}$, $i = 1, \ldots, n$, $t \in \mathbb{R}_+$, on a Hilbert space $\mathcal{H}$ such that the $*$-algebras $\mathcal{A}_{st} \subset \mathcal{B}(\mathcal{H})$ generated by the operators $(U_s^{(i)})^{-1} U_t^{(i)}$ are independent for disjoint intervals in a state on $\mathcal{B}(\mathcal{H})$ given by a unit vector in $\mathcal{H}$. Stationarity of increments and continuity are defined as in the case of $\mathcal{K}\langle d\rangle$. We find that conditionally positive hermitian linear functionals, and thus QLPs, on $\mathbb{CF}_n$ are given by triplets (W, L, G) consisting of a vector $W = (W^{(1)}, \ldots, W^{(n)})$ of unitary operators on a Hilbert space H, a vector $L = (L^{(1)}, \ldots, L^{(n)})$ of elements of H and $G \in \mathbb{C}^n$.

References

[BGS02] Ben Ghorbal, A., Schürmann, M.: Non-commutative notions of stochastic independence. Math. Proc. Camb. Philos. Soc. **133**(3), 531–561 (2002)

[BGS05] Ben Ghorbal, A., Schürmann, M.: Quantum lévy processes on dual groups. Math. Z. **251**(1), 147–165 (2005)

[Boz86] Bożejko, M.: Positive definite functions on the free group and the noncommutative riesz product. Boll. Un. Mat. Ital. A (6) **5**(1), 13–21 (1986)

[DNR01] Dăscălescu, S., Năstăsescu, C., Raianu, Ş.: *Hopf algebras*. In: Monographs and textbooks in pure and applied mathematics, Vol. **235**, New York: Marcel Dekker Inc., 2001

[Fra06] Franz, U.: *Lévy processes on quantum groups and dual groups*. In: Quantum independent increment processes II. Lecture Notes in Mathematics, Vol. **1866**, Berlin: Springer, 2006, pp. 161–257

[GSS92] Glockner, P., Schürmann, M., Speicher, R.: Realization of free white noises. Arch. Math. (Basel) **58**(4), 407–416 (1992)

[Hun56] Hunt, G.A.: Semi-groups of measures on lie groups. Trans. Am. Math. Soc. **81**, 264–293 (1956)

[Lu97] Lu, Y.G.: An interacting free fock space and the arcsine law. Prob. Math. Stat. **17**(1, Acta Univ. Wratislav. No. 1928), 149–166 (1997)

[ML98] Mac Lane, S.: *Categories for the working mathematician.* In: Graduate texts in mathematics, Vol. **5**, 2nd edn., New York: Springer, 1998

[Mur97] Muraki, N.: Noncommutative brownian motion in monotone fock space. Commun. Math. Phys. **183**(3), 557–570 (1997)

[Mur01] Muraki, N.: Monotonic independence, monotonic central limit theorem and monotonic law of small numbers. Inf. Dime. Anal. Quan. Prob. Rel. Top. **4**, 39–58 (2001)
[Mur02] Muraki, N.: The five independences as quasi-universal products. Inf. Dime. Anal. Quan. Prob. Rel. Top. **5**(1), 113–134 (2002)
[Mur03] Muraki, N.: The five independences as natural products. Inf. Dime. Anal. Quan. Prob. Rel. Top. **6**(3), 337–371 (2003)
[Sch38] Schoenberg, I.J.: Metric spaces and positive definite functions. Trans. Am. Math. Soc. **44**(3), 522–536 (1938)
[Sch91] Schürmann, M.: Quantum stochastic processes with independent additive increments. J. Multivar. Anal. **38**(1), 15–35 (1991)
[Sch93] Schürmann, M.: *White noise on bialgebras*. Lecture Notes in Mathematics, Vol. **1544**, Berlin: Springer, 1993
[Spe97] Speicher, R.: *On universal products*. In: Free probability theory (Waterloo, ON, 1995). *Fields Inst. Commun.*, Vol. **12**, Providence: Am. Math. Soc., 1997, pp. 257–266
[SSV10] Schürmann, M., Skeide, M., Volkwardt, S.: Transformation of quantum lévy processes on hopf algebras. Commun. Stoch. Anal. **4**(4), 553–577 (2010)
[SV08] Schürmann, M., Voß, S.: *Positivity of free convolution semigroups*. In: Quantum probability and related topics. QP–PQ: Quantum Probab. White Noise Anal., Vol. **23**, Hackensack: World Sci. Publ., 2008, pp. 225–235
[Voi85] Voiculescu, D.: *Symmetries of some reduced free product C^*-algebras*. In: Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983). Lecture Notes in Mathematics, Vol. **1132**, Berlin: Springer, 1985, pp. 556–588
[Voi87] Voiculescu, D.: Dual algebraic structures on operator algebras related to free products. J. Oper. Theory **17**, 85–98 (1987)
[vW65] von Waldenfels, W.: Fast positive operatoren. Z. Wahr. Verw. Geb. **4**, 159–174 (1965)
[vW73] von Waldenfels, W.: *An approach to the theory of pressure broadening of spectral lines*. In: Behara, M., Krickeberg, K., Wolfowitz, J. (eds.) Probability and information theory II. Lecture Notes in Mathematics, Vol. **296**, Berlin: Springer, 1973
[vW84] von Waldenfels, W.: *Itô solution of the linear quantum stochastic differential equation describing light emission and absorption*. In: Quantum probability and applications to the quantum theory of irreversible processes (Villa Mondragone, 1982). Lecture Notes in Mathematics, Vol. **1055**, Berlin: Springer, 1984, pp. 384–411
[Zha91] Zhang, J.J.: *H*-algebras. Adv. Math. **89**(2), 144–191 (1991)

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