Relative Entropy and Squashed Entanglement

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Abstract: We are interested in the properties and relations of entanglement measures. Especially, we focus on the squashed entanglement and relative entropy of entanglement, as well as their analogues and variants.

Our first result is a monogamy-like inequality involving the relative entropy of entanglement and its one-way LOCC variant. The proof is accomplished by exploring the properties of relative entropy in the context of hypothesis testing via one-way LOCC operations, and by making use of an argument resembling that by Piani on the faithfulness of regularized relative entropy of entanglement.

Following this, we obtain a commensurate and faithful lower bound for squashed entanglement, in the form of one-way LOCC relative entropy of entanglement. This gives a strengthening to the strong subadditivity of von Neumann entropy. Our result improves the trace-distance-type bound derived in Brandão et al. (Commun Math Phys, 306:805–830, 2011), where faithfulness of squashed entanglement was first proved. Applying Pinsker's inequality, we are able to recover the trace-distance-type bound, even with slightly better constant factor. However, the main improvement is that our new lower bound can be much larger than the old one and it is almost a genuine entanglement measure.

We evaluate exactly the relative entropy of entanglement under various restricted measurement classes, for maximally entangled states. Then, by proving asymptotic continuity, we extend the exact evaluation to their regularized versions for all pure states. Finally, we consider comparisons and separations between some important entanglement measures and obtain several new results on these, too.

1. Squashed Entanglement and Other Entanglement Measures

As an important concept in quantum mechanics, entanglement plays a central role in quantum information processing. It is the resource responsible for the quantum computational speed-up, quantum communication, quantum cryptography and so on.

Mathematically, quantum entanglement is the most outstanding non-classical feature of compound states that cannot be decomposed as statistical mixtures of product states over subsystems, and has been found to possess a very rich structure. There exist many entanglement measures, defined under various motivations and each characterizing some of its features. The properties and relations of these entanglement measures are very much desirable for our understanding of entanglement. Despite considerable achievements, a lot of issues still remain unclear, even in the bipartite case [1].

Among all the existing entanglement measures, squashed entanglement [2–4] is a particularly interesting one, with many desirable properties. In analogy to the classical intrinsic information [5], squashed entanglement of a bipartite quantum state ρ_{AB} is defined as

$$E_{sq}(\rho_{AB}) := \inf \left\{ \frac{1}{2} I(A; B|E)_\rho : \rho_{ABE} \text{ is an extension of } \rho_{AB} \right\}, \tag{1}$$

where $I(A; B|E)_\rho$ is the quantum conditional mutual information of ρ_{ABE},

$$I(A; B|E)_\rho := S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E) \tag{2}$$

with the von Neumann entropy $S(\rho) := -\operatorname{Tr} \rho \log \rho$. Squashed entanglement satisfies most of the properties that are desired or useful for an entanglement measure. For example, it is monotone under LOCC operations, convex and asymptotically continuous as a function of quantum states, monogamous among one party and other parties, additive on tensor products and superadditive in general [2,6,7]. Moreover, squashed entanglement admits an operational interpretation: it is the minimum rate of qubits transmission at which a quantum state can be redistributed among two parties when arbitrary (quantum) side information is permitted [8–11].

Quantum relative entropy, given by

$$D(\rho \| \sigma) = \begin{cases} \operatorname{Tr}(\rho(\log \rho - \log \sigma)) & \text{if } \operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma), \\ +\infty & \text{otherwise,} \end{cases}$$

measures the distinguishability of two states ρ and σ in the context of asymmetric hypothesis testing [12,13]. Yet it has found important applications in other aspects of quantum information theory: The relative entropy of entanglement [14,15] is another entanglement measure that is of fundamental importance. For the composite system $A \otimes B$, let SEP $(A : B)$ be the set of all separable states, i.e., the states of the form $\sigma_{AB} = \sum_i p_i \sigma_i^A \otimes \sigma_i^B$. Relative entropy of entanglement,

$$E_r(\rho_{AB}) := \min_{\sigma_{AB} \in \text{SEP}} D(\rho \| \sigma), \tag{3}$$

quantifies the amount of entanglement of a state ρ_{AB}, by its relative entropy "distance" to the nearest separable state. Since relative entropy of entanglement is strictly subadditive [16], it is more meaningful in many circumstances to use its regularization,

$$E_r^\infty(\rho_{AB}) := \lim_{n \to \infty} \frac{1}{n} E_r(\rho_{AB}^{\otimes n}).$$

Brandão and Plenio have provided operational interpretations to E_r^∞: it quantifies the optimal rate of transformation between a quantum state and maximally entangled states under non-entangling operations [17,18], and it is also the best error exponent in quantum

hypothesis testing where one of the hypotheses is many copies of the state and the other one is the set of separable states [19].

For each positive operator-valued measurement (POVM) $\{M_i\}_i$, it can be alternatively identified with a measurement operation $\mathcal{M}$, which is a completely positive map from density matrices to probability vectors,

$$\mathcal{M}(\omega) = \sum_i |i\rangle\langle i| \operatorname{Tr}(\omega M_i).$$

On the composite system AB, we define some restricted classes of measurements LO, 1-LOCC, LOCC, SEP and PPT. Here LO, 1-LOCC and LOCC are the sets of measurements that can be implemented by means of local operations, local operations and one-way classical communication, local operations and arbitrary two-way classical communication, respectively; SEP and PPT are the classes of measurements whose POVM elements are separable or positive-partial-transpose, respectively. Without loss of generality, we assume that the one-way classical communication in 1-LOCC is always from A to B.

We see from the definition that squashed entanglement is always non-negative, due to the strong subadditivity of von Neumann entropy, which states that the quantum conditional mutual information can not be negative [20]. However, until very recently proven in [21], the faithfulness of squashed entanglement, meaning that a bipartite quantum state has non-vanishing squashed entanglement if and only if it is entangled, had been a long-standing open question. Note that the infimum in the definition of Eq. (1) cannot be replaced by minimum, because no bound on the dimension of the system E is known. As a result, the evaluation of squashed entanglement becomes very difficult.

The main result of the proof in [21] is the following inequality:

$$E_{sq}(\rho_{AB}) \geq \frac{1}{16 \ln 2} \min_{\sigma_{AB} \in \text{SEP}} \|\rho_{AB} - \sigma_{AB}\|_{1\text{-LOCC}}^2, \tag{4}$$

where

$$\|\rho_{AB} - \sigma_{AB}\|_{1\text{-LOCC}} := \sup_{\mathcal{M} \in 1\text{-LOCC}} \|\mathcal{M}(\rho_{AB}) - \mathcal{M}(\sigma_{AB})\|$$

defines a metric (in fact, a norm) on density operators [22].

The rest of the paper is structured as follows. In Sect. 2 we state our main results. Then, after considering quantum hypothesis testing under one-way LOCC measurements and obtaining a key technical lemma in Sect. 3, we prove these results in Sects. 4, 5 and 6, respectively. In Sect. 7, we deal with the comparisons and separations between entanglement measures and end the paper with a few open questions.

2. Main Results

Before presenting the results, we introduce the variants of relative entropy of entanglement, which will be involved intensively later. Piani defined the relative entropy of entanglement with respect to the set of states G and the restricted class of measurements M [23], as

$$E_{r,\mathsf{M}}^{(\text{G})}(\rho) := \inf_{\sigma \in \text{G}} \sup_{\mathcal{M} \in \mathsf{M}} D\big(\mathcal{M}(\rho) \| \mathcal{M}(\sigma)\big). \tag{5}$$

Using this entanglement measure, he proved that E_r^∞ is faithful, i.e., $E_r^\infty(\rho_{AB}) > 0$ if and only if ρ_{AB} is entangled (same result was derived in [19] independently).

In this paper, G is usually the set of separable states SEP. Therefore, we abbreviate $E_{r,\mathsf{M}}^{(\mathsf{SEP})}$ to $E_{r,\mathsf{M}}$ for simplicity.

Monogamy relation for relative entropy of entanglement. One of the most fundamental properties of entanglement is monogamy: the more a quantum system is entangled with another, then the less it is entangled with the others. For any entanglement measure f, one would expect a quantitative characterization of monogamy of the form

$$f(\rho_{1:23}) \geq f(\rho_{1:2}) + f(\rho_{1:3}).$$

Although this is really the case for squashed entanglement [7], relative entropy of entanglement—along with many other entanglement measures—does not satisfy such a strong relation, with the antisymmetric state being a counterexample [24,25].

Here, we propose and prove a properly weakened monogamy inequality for relative entropy of entanglement, by invoking its one-way LOCC variant.

Theorem 1. *For every tripartite quantum state ρ_{ABE}, we have*

$$E_r(\rho_{B:AE}) \geq E_{r,\mathsf{1\text{-}LOCC}}(\rho_{AB}) + E_r^\infty(\rho_{BE}), \tag{6}$$

and

$$E_r^\infty(\rho_{B:AE}) \geq E_{r,\mathsf{1\text{-}LOCC}}^\infty(\rho_{AB}) + E_r^\infty(\rho_{BE}). \tag{7}$$

Equation (7) is obtained from Eq. (6) by regularizing both sides, and it becomes stronger due to the subadditivity of E_r and superadditivity of $E_{r,\mathsf{1\text{-}LOCC}}$ [16,23].

It is worth mentioning that Eq. (6) and Eq. (7) are in the form similar to Piani's superadditivity-like relation

$$E_r(\rho_{A_1A_2:B_1B_2}) \geq E_{r,\mathsf{M}}(\rho_{A_1B_1}) + E_r(\rho_{A_2B_2}),$$

with M either LOCC or SEP. The difference is that in our result, there is only one single system B on the left side, while it appears twice on the right side. As a result, the price we have to pay is degrading the measurement class to 1-LOCC and imposing a regularization in the second term of the right side, respectively (see Eq. (6)). One the other hand, our proof needs a new technique (Lemma 5 in the next section), which is derived in the context of quantum hypothesis testing under the restricted measurement class 1-LOCC.

Commensurate lower bound for squashed entanglement. We provide in this paper a commensurate and faithful lower bound for squashed entanglement. Instead of the one-way LOCC trace distance as in Eq. (4), our result is in the form of one-way LOCC relative entropy of entanglement, which is more natural and stronger.

Theorem 2. *For any quantum state ρ_{AB}, we have*

$$E_{sq}(\rho_{AB}) \geq \frac{1}{2} E_{r,\mathsf{1\text{-}LOCC}}^\infty(\rho_{AB}) \geq \frac{1}{2} E_{r,\mathsf{1\text{-}LOCC}}(\rho_{AB}). \tag{8}$$

The core inequality for von Neumann entropy, strong subadditivity, states that for any tripartite state ρ_{ABE},

$$I(A; B|E)_\rho \geq 0.$$

Recalling the definition of squashed entanglement, Theorem 2 implies

$$I(A; B|E)_\rho \geq E_{r,\text{1-LOCC}}(\rho_{AB}),$$

and hence strengthens the strong subadditivity inequality by relating it to a distance-like entanglement measure on two of the subsystems.

To see how our result of Theorem 2 improves the lower bound proven in [21], we explain in more detail as follows. On the one hand, applying Pinsker's inequality [26], we are able to recover the trace-distance bound of Eq. (4), even with a slightly better constant factor:

$$E_{sq}(\rho_{AB}) \geq \frac{1}{4\ln 2} \min_{\sigma_{AB}\in\text{SEP}} \|\rho_{AB} - \sigma_{AB}\|^2_{\text{1-LOCC}}.$$

On the other hand, while the trace-distance bound can be at most $O(1)$, our new bound (8) can be very large. Indeed, $E_{r,\text{1-LOCC}}$ is asymptotically normalized, in the sense of Proposition 4.

Asymptotic continuity. To quantify the resources in quantum protocols in a physically robust way, entanglement measures are expected to be asymptotically continuous. Piani's paper [23] contains the proofs of several properties of $E_{r,\text{M}}^{(\text{G})}$ for a certain combination of G and M. Now we also show asymptotic continuity under very general conditions.

We say that a set S is star-shaped with respect to some $x_0 \in$ S, if $px + (1 - p)x_0 \in$ S for all $x \in$ S and $0 \leq p \leq 1$.

Proposition 3. *Let* M *be any set of measurements, and* G *be a set of states on a quantum system with Hilbert space dimension k, containing the maximally mixed state τ and such that in fact* G *is star-shaped with respect to τ. Let ρ, ρ' be two states of the quantum system with $\|\rho - \rho'\|_\text{M} \leq \epsilon \leq \frac{1}{e}$. Then,*

$$\left| E_{r,\text{M}}^{(\text{G})}(\rho) - E_{r,\text{M}}^{(\text{G})}(\rho') \right| \leq 2\epsilon \log \frac{6k}{\epsilon}.$$

Evaluation on maximally entangled states and pure states. The entanglement measure $E_{r,\text{M}}$ is difficult to calculate due to the two optimizations in its definition. Here we conduct the first exact evaluation on maximally entangled states, with M any of {LO, 1-LOCC, LOCC, SEP, PPT}. The basic idea is to make use of the symmetry of $\frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$, namely, invariance under unitary operation $U \otimes \overline{U}$. Then, with the help of asymptotic continuity of Proposition 3, we further obtain their regularized versions on general pure states.

At first glance, the restricted class of measurements M may make $E_{r,\text{M}}$ much smaller than the normal relative entropy of entanglement. However, in our case we find that they are almost the same when the local dimension is very large.

Proposition 4. *For the rank-d maximally entangled state Φ_d,*

$$E_{r,\text{LO}}(\Phi_d) = E_{r,\text{1-LOCC}}(\Phi_d) = E_{r,\text{LOCC}}(\Phi_d) = E_{r,\text{SEP}}(\Phi_d)$$
$$= E_{r,\text{PPT}}(\Phi_d) = \log(d + 1) - 1. \tag{9}$$

As a corollary, this implies that for the pure state ψ_{AB}, the regularized versions are equal to the entropic pure state entanglement:

$$E_{r,\mathsf{LO}}^{\infty}(\psi_{AB}) = E_{r,\mathsf{1\text{-}LOCC}}^{\infty}(\psi_{AB}) = E_{r,\mathsf{LOCC}}^{\infty}(\psi_{AB})$$
$$= E_{r,\mathsf{SEP}}^{\infty}(\psi_{AB}) = E_{r,\mathsf{PPT}}^{\infty}(\psi_{AB}) = S(\mathrm{Tr}_B\,\psi). \tag{10}$$

3. Quantum Hypothesis Testing Under One-way LOCC Operations with Limited Disturbance

In quantum hypothesis testing, we are given many copies of an information source, which is statistically described by state ρ (the null hypothesis) or σ (the alternative hypothesis). The task is to decide which state the source is really in. This is achieved by doing a two outcome measurement $\{L_n, \mathbb{1} - L_n\}$ on n realizations of the source. We define two types of errors. Type I error is the probability that we falsely conclude that the state is σ while it is actually ρ, given by $\alpha_n(L_n) := \mathrm{Tr}\,\rho^{\otimes n}(\mathbb{1} - L_n)$; type II error instead is the probability that we mistake σ for ρ, given by $\beta_n(L_n) := \mathrm{Tr}\,\sigma^{\otimes n} L_n$. In an asymmetric situation, we want to minimize the type II error while only simply requiring that the type I error converges to 0. The quantum Stein's lemma states that the maximal error exponent of type II is the relative entropy $D(\rho\|\sigma)$ [12,13]: On the one hand, there exists a test $\{L_n, \mathbb{1} - L_n\}$ satisfying

$$\alpha_n(L_n) \to 0 \quad \text{and} \quad -\frac{1}{n}\log\beta_n(L_n) \to D(\rho\|\sigma).$$

On the other hand, if a test $\{L_n, \mathbb{1} - L_n\}$ is such that

$$\liminf_{n\to\infty} -\frac{1}{n}\log\beta_n(L_n) > D(\rho\|\sigma),$$

then $\alpha_n(L_n) \to 1$. This also applies to the classical setting, if we replace quantum states ρ and σ by classical probability distributions and the quantum measurement by a classical decision function [27].

When ρ and σ are compound quantum states, it is natural to put locality constraints on the measurements $\{L_n, \mathbb{1} - L_n\}$. In this case, the problem of quantum hypothesis testing becomes much more difficult, and solutions are known only in some specific situations [28,29]. Here, we focus on the family of measurements which are implementable by means of local operations and one-way classical communication (one-way LOCC). Our goal is not to derive a single-letter formula for the optimal error exponent; instead, we are interested in how the disturbance on the quantum states induced by the measurement is limited, when a certain error exponent of type II is achieved.

Let the null hypothesis and alternative hypothesis be $\rho_{ABE}^{\otimes n}$ and $\sigma_{ABE}^{\otimes n}$, respectively. Let the allowed operations be restricted to one-way LOCC which is performed on systems A^n and B^n, with classical communication from Alice's side (A^n) to Bob's side (B^n). On the one hand, it is easy to see that, for any one-way LOCC measurement $\mathcal{M}^{AB\to X}$, $D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma))$ is an achievable error exponent of type II. This is because, after doing the measurement $\mathcal{M}$ on each copy of the quantum states, the two states $\rho^{\otimes n}$ and $\sigma^{\otimes n}$ are replaced by classical probability distributions $(\mathcal{M}(\rho))^{\otimes n}$ and $(\mathcal{M}(\sigma))^{\otimes n}$. Then applying Stein's lemma in the classical setting, we know that there exists a classical decision rule which can achieve the above-mentioned error figure. Hence, the corresponding quantum measurement $\{L_n, \mathbb{1} - L_n\}$ can be constructed from $\mathcal{M}^{\otimes n}$ and this decision rule. On

the other hand, when the two kinds of errors are sufficiently small, the one-way LOCC test $\{L_n, \mathbb{1} - L_n\}$ can be performed in such a way that the reduced states on system $B^n E^n$, $\rho_{BE}^{\otimes n}$ and $\sigma_{BE}^{\otimes n}$, are kept almost undisturbed. This is a consequence of the "gentle measurement lemma" [30]. Note that, generally speaking, the full states $\rho_{ABE}^{\otimes n}$ and $\sigma_{ABE}^{\otimes n}$ will be inevitably disturbed significantly by the measurement, because in the one-way LOCC procedure, Bob's choice of measurement is based on the outcome of Alice's measurement, and the extracting of such classical information generically has to damage the states at Alice's side.

Lemma 5. *For any two states ρ_{ABE} and σ_{ABE}, and any one-way LOCC measurement $\mathcal{M}^{AB \to X}$ acting on system AB, with classical communication from A to B, there exists a sequence of quantum instruments $\mathcal{T}_n^{A^n B^n \to X B^n}$, which are implementable via local operations and one-way classical communication from A^n to B^n, such that*

$$\lim_{n \to \infty} \frac{1}{n} D\big(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})\big) = D\big(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})\big), \tag{11}$$

$$\lim_{n \to \infty} \left\| \mathcal{T}_n^q \otimes \mathrm{id}^{E^n}(\rho_{ABE}^{\otimes n}) - \rho_{BE}^{\otimes n} \right\|_1 = 0, \tag{12}$$

where $\mathcal{T}_n^c := \mathrm{Tr}_{B^n} \circ \mathcal{T}_n^{A^n B^n \to X B^n}$, and $\mathcal{T}_n^q := \mathrm{Tr}_X \circ \mathcal{T}_n^{A^n B^n \to X B^n}$.

Proof. Let the POVM elements of the measurement $\mathcal{M}$ be $\{R_k^A \otimes S_{k,\ell}^B\}_{k,\ell}$, with $\sum_k R_k = \mathbb{1}^A$ and $\sum_\ell S_{k,\ell} = \mathbb{1}^B$ for all k. Operationally, this means that Alice does a measurement $\{R_k\}_k$ on system A, then she tells Bob the outcome k, and according to what he receives, Bob does a measurement $\{S_{k,\ell}\}_\ell$ on the system B. For $\mathcal{M}^{\otimes n}$ acting on $A^n B^n$, we denote the measurement outcomes $(k_1 k_2 \ldots k_n, \ell_1 \ell_2 \ldots \ell_n) =: (k^n, \ell^n)$, and the corresponding measurement elements $\bigotimes_{i=1}^n (R_{k_i}^{A_i} \otimes S_{k_i,\ell_i}^{B_i}) =: R_{k^n} \otimes S_{k^n,\ell^n}$.

For the problem of quantum hypothesis testing with the null hypothesis $\rho_{ABE}^{\otimes n}$ and the alternative hypothesis $\sigma_{ABE}^{\otimes n}$, and the permitted operations are one-way LOCC on parties A^n and B^n, we consider the protocol as follows. First, we apply the measurement $\mathcal{M}$ to each copy of the states ρ and σ, resulting in classical probability distributions $\mathcal{M}^{\otimes n}(\rho^{\otimes n})$ and $\mathcal{M}^{\otimes n}(\sigma^{\otimes n})$. Then, we partition the set $\{(k^n, \ell^n)\}$ of all measurement outcomes into two disjoint subsets $\mathcal{O}_{n,\mathrm{Null}}$ and $\mathcal{O}_{n,\mathrm{Alt}}$, and make a classical decision: if the measurement outcome is in $\mathcal{O}_{n,\mathrm{Null}}$, we infer that the state is $\rho^{\otimes n}$ (null hypothesis); otherwise, it belongs to $\mathcal{O}_{n,\mathrm{Alt}}$ and we conclude that the state is $\sigma^{\otimes n}$ (alternative hypothesis). In such a protocol, the two types of errors are

$$\alpha_n = \sum_{(k^n,\ell^n) \in \mathcal{O}_{n,\mathrm{Alt}}} \mathrm{Tr}\, \rho_{AB}^{\otimes n}(R_{k^n} \otimes S_{k^n,\ell^n}), \tag{13}$$

$$\beta_n = \sum_{(k^n,\ell^n) \in \mathcal{O}_{n,\mathrm{Null}}} \mathrm{Tr}\, \sigma_{AB}^{\otimes n}(R_{k^n} \otimes S_{k^n,\ell^n}). \tag{14}$$

By the classical Stein's lemma [27], there exists a partition $\{(k^n, \ell^n)\} = \mathcal{O}_{n,\mathrm{Null}} \dot{\cup} \mathcal{O}_{n,\mathrm{Alt}}$ such that

$$\lim_{n \to \infty} \alpha_n = 0, \tag{15}$$

$$\lim_{n \to \infty} -\frac{1}{n} \log \beta_n = D(\mathcal{M}(\rho) \| \mathcal{M}(\sigma)), \tag{16}$$

which leads to

$$\lim_{n\to\infty}\frac{1}{n}D(\{1-\alpha_n,\alpha_n\}\|\{\beta_n,1-\beta_n\})=D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)). \qquad (17)$$

From now on, we fix such a partition of $\{(k^n,\ell^n)\}$ into $\mathcal{O}_{n,\text{Null}}$ and $\mathcal{O}_{n,\text{Alt}}$. Let

$$Q_{k^n,x}:=\sqrt{\sum_{\ell^n:(k^n,\ell^n)\in\mathcal{O}_{n,x}}S_{k^n,\ell^n}}, \qquad (18)$$

where the index x can be "Null" or "Alt". It is obvious that $\{Q_{k^n,\text{Null}},Q_{k^n,\text{Alt}}\}$ forms a complete set of Kraus operators, i.e. $Q_{k^n,\text{Null}}^{\dagger}Q_{k^n,\text{Null}}+Q_{k^n,\text{Alt}}^{\dagger}Q_{k^n,\text{Alt}}=\mathbb{1}^{B^n}$. We are now ready for the definition of quantum instrument $\mathcal{T}_n^{A^nB^n\to XB^n}$:

$$\mathcal{T}_n(\omega_{A^nB^n}):=\sum_{x=\text{Null},\text{Alt}}|x\rangle\langle x|^X\otimes\sum_{k^n}\text{Tr}_{A^n}\big(\sqrt{R_{k^n}}\otimes Q_{k^n,x}\big)\omega_{A^nB^n}\big(\sqrt{R_{k^n}}\otimes Q_{k^n,x}\big).$$

$$(19)$$

To complete the proof, we will demonstrate that $\mathcal{T}_n$ satisfies all the requirements as advertised. First, it is obvious that $\mathcal{T}_n$ can be realized by means of one-way LOCC. Alice does a measurement $\{R_{k^n}\}$ on the system A^n, then she communicates the outcome k^n to Bob; upon receiving k^n, Bob does a two-outcome measurement with Kraus operators $\{Q_{k^n,\text{Null}},Q_{k^n,\text{Alt}}\}$ on the system B^n, at the same time he stores the measurement results "Null" or "Alt" in the classical register X.

Secondly, we verify Eq. (11). Clearly, we can write

$$\mathcal{T}_n\otimes\text{id}^{E^n}(\rho_{ABE}^{\otimes n})=\sum_x|x\rangle\langle x|^X\otimes\tilde{\rho}_{B^nE^n}^x, \qquad (20)$$

with

$$\tilde{\rho}_{B^nE^n}^x=\sum_{k^n}\text{Tr}_{A^n}\big(\sqrt{R_{k^n}}\otimes Q_{k^n,x}\otimes\mathbb{1}^{E^n}\big)\rho_{ABE}^{\otimes n}\big(\sqrt{R_{k^n}}\otimes Q_{k^n,x}\otimes\mathbb{1}^{E^n}\big). \qquad (21)$$

Equations (13), (18) and (21) together guarantee that

$$\text{Tr}\,\tilde{\rho}_{B^nE^n}^{\text{Alt}}=\alpha_n\quad\text{and}\quad\text{Tr}\,\tilde{\rho}_{B^nE^n}^{\text{Null}}=1-\alpha_n, \qquad (22)$$

which together with Eq. (20) results in

$$\mathcal{T}_n^c(\rho_{AB}^{\otimes n})=(1-\alpha_n)|\text{Null}\rangle\langle\text{Null}|^X+\alpha_n|\text{Alt}\rangle\langle\text{Alt}|^X. \qquad (23)$$

Similarly, from Eqs. (14), (18) and (19), we derive that

$$\mathcal{T}_n^c(\sigma_{AB}^{\otimes n})=\beta_n|\text{Null}\rangle\langle\text{Null}|^X+(1-\beta_n)|\text{Alt}\rangle\langle\text{Alt}|^X. \qquad (24)$$

So, Eqs. (17), (23) and (24) imply

$$\lim_{n\to\infty}\frac{1}{n}D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n})\|\mathcal{T}_n^c(\sigma_{AB}^{\otimes n}))=D(\mathcal{M}(\rho_{AB})\|\mathcal{M}(\sigma_{AB})),$$

which is exactly Eq. (11).

Finally, we prove that $\mathcal{T}_n$ satisfies Eq. (12). Making use of Eqs. (20) and (22), we have

$$
\begin{aligned}
\left\| \mathcal{T}_n^q \otimes \mathrm{id}^{E^n}(\rho_{ABE}^{\otimes n}) - \rho_{BE}^{\otimes n} \right\|_1 &= \left\| \tilde{\rho}_{B^n E^n}^{\mathrm{Alt}} + \tilde{\rho}_{B^n E^n}^{\mathrm{Null}} - \rho_{BE}^{\otimes n} \right\|_1 \\
&\leq \alpha_n + \left\| \tilde{\rho}_{B^n E^n}^{\mathrm{Null}} - \rho_{BE}^{\otimes n} \right\|_1 .
\end{aligned}
\tag{25}
$$

Paying attention to the definition of $\tilde{\rho}_{B^n E^n}^{\mathrm{Null}}$, namely Eq. (21), we easily check that

$$
\tilde{\rho}_{B^n E^n}^{\mathrm{Null}} = \mathrm{Tr}_{K^n} \sqrt{\Lambda} \, \tilde{\rho}_{K^n B^n E^n} \sqrt{\Lambda},
\tag{26}
$$

where $\tilde{\rho}_{K^n B^n E^n} := \mathrm{Tr}_{A^n} \sum_{k^n} |k^n\rangle\langle k^n|^{K^n} \otimes (\sqrt{R_{k^n}} \rho_{ABE}^{\otimes n} \sqrt{R_{k^n}})$ is a normalized quantum state, and $\Lambda := \sum_{k^n} |k^n\rangle\langle k^n|^{K^n} \otimes Q_{k^n, \mathrm{Null}}^2$ is a POVM element satisfying $0 \leq \Lambda \leq \mathbb{1}$. As a result,

$$
\begin{aligned}
\left\| \tilde{\rho}_{B^n E^n}^{\mathrm{Null}} - \rho_{BE}^{\otimes n} \right\|_1 &= \left\| \mathrm{Tr}_{K^n} \sqrt{\Lambda} \, \tilde{\rho}_{K^n B^n E^n} \sqrt{\Lambda} - \mathrm{Tr}_{K^n} \tilde{\rho}_{K^n B^n E^n} \right\|_1 \\
&\leq \left\| \sqrt{\Lambda} \, \tilde{\rho}_{K^n B^n E^n} \sqrt{\Lambda} - \tilde{\rho}_{K^n B^n E^n} \right\|_1 \\
&\leq 2\sqrt{1 - \mathrm{Tr} \, \tilde{\rho}_{K^n B^n E^n} \Lambda} \\
&= 2\sqrt{\alpha_n},
\end{aligned}
\tag{27}
$$

where the first line is by Eq. (26) and the fact that $\mathrm{Tr}_{K^n} \tilde{\rho}_{K^n B^n E^n} = \rho_{BE}^{\otimes n}$, the second line is because of the monotonicity of trace distance under partial trace, the third line makes use of the gentle measurement lemma [30], and the last line follows from Eqs. (22) and (26). Eventually, inserting Eq. (27) into Eq. (25), and invoking Eq. (15), we arrive at

$$
\left\| \mathcal{T}_n^q \otimes \mathrm{id}^{E^n}(\rho_{ABE}^{\otimes n}) - \rho_{BE}^{\otimes n} \right\|_1 \leq \alpha_n + 2\sqrt{\alpha_n} \to 0,
$$

which is precisely Eq. (12). $\quad\square$

4. Entanglement Monogamy Relation and Commensurate Lower Bound for Squashed Entanglement

Proof of Theorem 1. As discussed in Sect. 2, it suffices to prove Eq. (6). Let $\sigma_{B:AE}$ be the nearest separable state to $\rho_{B:AE}$ with respect to the measure of relative entropy. That is to say,

$$
E_r(\rho_{B:AE}) = D(\rho_{ABE} \| \sigma_{ABE}) = \frac{1}{n} D(\rho_{ABE}^{\otimes n} \| \sigma_{ABE}^{\otimes n}).
\tag{28}
$$

Let $\mathcal{M}^{AB \to X}$ be an arbitrary one-way LOCC measurement. Applying Lemma 5 to ρ_{ABE}, σ_{ABE} and $\mathcal{M}^{AB \to X}$, we know that there exists a sequence of quantum instruments $\mathcal{T}_n^{A^n B^n \to X B^n}$, which are implementable via local operations and classical communication from A^n to B^n, such that

$$
\lim_{n \to \infty} \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) = D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})),
\tag{29}
$$

$$
\lim_{n \to \infty} \left\| \mathcal{T}_n^q \otimes \mathrm{id}^{E^n}(\rho_{ABE}^{\otimes n}) - \rho_{BE}^{\otimes n} \right\|_1 = 0,
\tag{30}
$$

where $\mathcal{T}_n^c := \text{Tr}_{B^n} \circ \mathcal{T}_n^{A^n B^n \to X B^n}$, and $\mathcal{T}_n^q := \text{Tr}_X \circ \mathcal{T}_n^{A^n B^n \to X B^n}$. Write $\mathcal{T}_n \otimes \text{id}^{E^n}(\rho_{ABE}^{\otimes n})$ $= \sum_{i_n} p_{i_n} |i_n\rangle\langle i_n|^X \otimes \rho_{B^n E^n}^{i_n}$ and $\mathcal{T}_n \otimes \text{id}^{E^n}(\sigma_{ABE}^{\otimes n}) = \sum_{i_n} q_{i_n} |i_n\rangle\langle i_n|^X \otimes \sigma_{B^n E^n}^{i_n}$. It is easy to check that

$$
\begin{aligned}
&D(\mathcal{T}_n \otimes \text{id}^{E^n}(\rho_{ABE}^{\otimes n}) \| \mathcal{T}_n \otimes \text{id}^{E^n}(\sigma_{ABE}^{\otimes n})) \\
&= D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \sum_{i_n} p_{i_n} D(\rho_{B^n E^n}^{i_n} \| \sigma_{B^n E^n}^{i_n}) \\
&\geq D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + D(\mathcal{T}_n^q \otimes \text{id}^{E^n}(\rho_{ABE}^{\otimes n}) \| \sum_{i_n} p_{i_n} \sigma_{B^n E^n}^{i_n}) \\
&\geq D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + E_r\big(\mathcal{T}_n^q \otimes \text{id}^{E^n}(\rho_{ABE}^{\otimes n})\big),
\end{aligned}
\tag{31}
$$

where the first line is by direct calculation, the second line follows from convexity of quantum relative entropy, and for the last line, note that the state $\sum_{i_n} p_{i_n} \sigma_{B^n:E^n}^{i_n}$ is still separable because of the LOCC feature of $\mathcal{T}_n$. By the Lindblad–Uhlmann theorem [31,32], quantum relative entropy is monotonic under cptp quantum operations. So, combining Eqs. (28) and (31) results in

$$
E_r(\rho_{B:AE}) \geq \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \frac{1}{n} E_r\big(\mathcal{T}_n^q \otimes \text{id}^{E^n}(\rho_{ABE}^{\otimes n})\big).
\tag{32}
$$

It was proven in [33] that the relative entropy of entanglement satisfies a strong continuity condition: for two states ρ_1 and ρ_2 on system AB with $\|\rho_1 - \rho_2\|_1 \leq \frac{1}{e}$, we have

$$
|E_r(\rho_1) - E_r(\rho_2)| \leq 2(2 + \log|A| + \log|B|)\|\rho_1 - \rho_2\|_1 + 2\eta(\|\rho_1 - \rho_2\|_1),
\tag{33}
$$

where $\eta(x) = -x \log x$. Now, letting $n \to \infty$ in Eq. (32), and then making use of Eqs. (29), (30) and (33), we obtain

$$
\begin{aligned}
E_r(\rho_{B:AE}) &\geq \lim_{n\to\infty} \frac{1}{n} D(\mathcal{T}_n^c(\rho_{AB}^{\otimes n}) \| \mathcal{T}_n^c(\sigma_{AB}^{\otimes n})) + \lim_{n\to\infty} \frac{1}{n} E_r\big(\mathcal{T}_n^q \otimes \text{id}^{E^n}(\rho_{ABE}^{\otimes n})\big) \\
&= D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})) + E_r^\infty(\rho_{BE}).
\end{aligned}
\tag{34}
$$

Since $\mathcal{M}$ is arbitrary, it follows from Eq. (34) that

$$
\begin{aligned}
E_r(\rho_{B:AE}) &\geq \sup_{\mathcal{M}\in 1\text{-LOCC}} D(\mathcal{M}(\rho_{AB}) \| \mathcal{M}(\sigma_{AB})) + E_r^\infty(\rho_{BE}) \\
&\geq E_{r,1\text{-LOCC}}(\rho_{AB}) + E_r^\infty(\rho_{BE}),
\end{aligned}
\tag{35}
$$

where the second inequality is by the definition of $E_{r,1\text{-LOCC}}$, and we are done. $\square$

Proof of Theorem 2. It is shown in [21, Lem. 1] that

$$
I(A; B|E)_\rho \geq E_r^\infty(\rho_{B:AE}) - E_r^\infty(\rho_{BE}).
\tag{36}
$$

Equation (7) in Theorem 1, together with Eq. (36), gives us

$$
I(A; B|E)_\rho \geq E_{r,1\text{-LOCC}}^\infty(\rho_{AB}).
\tag{37}
$$

Then, recalling the definition of squashed entanglement and by the superadditivity of $E_{r,1\text{-LOCC}}$ [23], we arrive at

$$
E_{sq}(\rho_{AB}) \geq \frac{1}{2} E_{r,1\text{-LOCC}}^\infty(\rho_{AB}) \geq \frac{1}{2} E_{r,1\text{-LOCC}}(\rho_{AB}),
$$

which concludes the proof. $\square$

5. Asymptotic Continuity

Proof of Proposition 3. For $0 \le x \le 1$, let $\mathsf{G}_x := x\mathsf{G} + (1 - x)\tau$, so that $\mathsf{G}_1 = \mathsf{G}$ and $\mathsf{G}_0 = \tau$. We follow very closely [33], and start by the observation that because of $\mathsf{G}_x \subset \mathsf{G}$ and the operator monotonicity of the log function,

$$E_{r,\mathsf{M}}^{(\mathsf{G})} \le E_{r,\mathsf{M}}^{(\mathsf{G}_x)} \le E_{r,\mathsf{M}}^{(\mathsf{G})} - \log x. \tag{38}$$

We will later see that $x = 1 - \epsilon$ is a good choice. However, it is clear already that if it is close to 1, then we reduce our problem to proving asymptotic continuity for G_x, which has the property that all of its elements are of full rank. In fact, the smallest eigenvalue of a $\sigma \in \mathsf{G}_x$ is $\ge \frac{1-x}{k}$.

Now fix $\sigma \in \mathsf{G}_x$ and $\mathcal{M} \in \mathsf{M}$, and consider

$$E_{r,\{\mathcal{M}\}}^{(\sigma)}(\rho) = D\big(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)\big) = \sum_i \operatorname{Tr} \rho M_i \log \frac{\operatorname{Tr} \rho M_i}{\operatorname{Tr} \sigma M_i}.$$

Since $0 \le M_i \le \mathbb{1}$, we can write $M_i = 3k\lambda_i Q_i$ with operators $Q_i \ge 0$ s.t. $\operatorname{Tr} Q_i = \frac{1}{3}$, and $\lambda_i \ge 0$, $\sum_i \lambda_i = 1$. Then, $\frac{1}{3} \ge \operatorname{Tr} \sigma Q_i \ge \frac{1-x}{3k}$ for all i. We can also rewrite the above quantity as

$$E_{r,\{\mathcal{M}\}}^{(\sigma)}(\rho) = 3k \sum_i \lambda_i \operatorname{Tr} \rho Q_i \log \frac{\operatorname{Tr} \rho Q_i}{\operatorname{Tr} \sigma Q_i}$$

$$= -\sum_i \operatorname{Tr} \rho M_i \log \operatorname{Tr} \sigma Q_i + 3k \sum_i \lambda_i \operatorname{Tr} \rho Q_i \log \operatorname{Tr} \rho Q_i,$$

and we will treat the two latter sums separately; call them $\mathrm{I}(\rho)$ and $\mathrm{II}(\rho)$, respectively. For the first one,

$$\big|\mathrm{I}(\rho) - \mathrm{I}(\rho')\big| = \left| \sum_i \operatorname{Tr}(\rho - \rho') M_i \log \operatorname{Tr} \sigma Q_i \right|$$

$$\le \sum_i \log \frac{3k}{1-x} \big|\operatorname{Tr} \rho M_i - \operatorname{Tr} \rho' M_i\big|$$

$$= \log \frac{3k}{1-x} \|\rho - \rho'\|_{\{\mathcal{M}\}} \le \epsilon \log \frac{3k}{\epsilon}.$$

For the second term, we use the function $\eta(t) = -t \log t$, which is concave, non-negative on the unit interval and has the elementary property that for all $s, t \ge 0$, $\eta(s + t) \le \eta(s) + \eta(t)$. Furthermore, for $0 \le t \le \frac{1}{e}$ it is monotonically increasing. Now, $\mathrm{II}(\rho) = -3k \sum_i \lambda_i \eta(\operatorname{Tr} \rho Q_i)$, and so

$$\big|\mathrm{II}(\rho) - \mathrm{II}(\rho')\big| \le 3k \sum_i \lambda_i \big|\eta(\operatorname{Tr} \rho Q_i) - \eta(\operatorname{Tr} \rho' Q_i)\big|$$

$$\le 3k \sum_i \lambda_i \eta\big(|\operatorname{Tr}(\rho - \rho') Q_i|\big)$$

$$\le 3k\eta \left(\sum_i \lambda_i |\operatorname{Tr}(\rho - \rho') Q_i| \right)$$

$$= 3k\eta \left(\frac{1}{3k} \|\rho - \rho'\|_{\{\mathcal{M}\}} \right)$$

$$\leq 3k\eta \left(\frac{\epsilon}{3k} \right) = \epsilon \log \frac{3k}{\epsilon},$$

where in the third line we have used the concavity of η.

Putting these two observations together, we obtain (recall $\sigma \in G_x$, $x = 1 - \epsilon$)

$$\left| E_{r,\{\mathcal{M}\}}^{(\sigma)}(\rho) - E_{r,\{\mathcal{M}\}}^{(\sigma)}(\rho') \right| \leq 2\epsilon \log \frac{3k}{\epsilon}. \tag{39}$$

From this, the rest of the argument is pretty standard, all we need to implement is the maximization over $\mathcal{M} \in \mathsf{M}$ and the minimization over $\sigma \in G_x$. First, fix $\sigma \in G_x$; then,

$$\left| E_{r,\mathsf{M}}^{(\sigma)}(\rho) - E_{r,\mathsf{M}}^{(\sigma)}(\rho') \right| = \left| \sup_{\mathcal{M}} E_{r,\{\mathcal{M}\}}^{(\sigma)}(\rho) - \sup_{\mathcal{M}'} E_{r,\{\mathcal{M}'\}}^{(\sigma)}(\rho') \right|$$

$$\leq \sup_{\mathcal{M} \in \mathsf{M}} \left| E_{r,\{\mathcal{M}\}}^{(\sigma)}(\rho) - E_{r,\{\mathcal{M}\}}^{(\sigma)}(\rho') \right|$$

$$\leq 2\epsilon \log \frac{3k}{\epsilon},$$

invoking Eq. (39). Similarly,

$$\left| E_{r,\mathsf{M}}^{(G_x)}(\rho) - E_{r,\mathsf{M}}^{(G_x)}(\rho') \right| = \left| \inf_{\sigma} E_{r,\mathsf{M}}^{(\sigma)}(\rho) - \inf_{\sigma'} E_{r,\mathsf{M}}^{(\sigma')}(\rho') \right|$$

$$\leq \sup_{\sigma \in G_x} \left| E_{r,\mathsf{M}}^{(\sigma)}(\rho) - E_{r,\mathsf{M}}^{(\sigma)}(\rho') \right|$$

$$\leq 2\epsilon \log \frac{3k}{\epsilon},$$

using the relation for fixed σ. From this and Eq. (38), using $-\log x = -\log(1-\epsilon) \leq 2\epsilon$, the proposition follows. □

6. Evaluation on Maximally Entangled States and Pure States

Proof of Proposition 4. We show separately $E_{r,\mathsf{LO}}(\Phi_d) \geq \log(d+1) - 1$ and $E_{r,\mathsf{PPT}}(\Phi_d) \leq \log(d+1) - 1$, which together complete the proof, since by definition, $E_{r,\mathsf{LO}} \leq E_{r,\mathsf{1\text{-}LOCC}} \leq E_{r,\mathsf{LOCC}} \leq E_{r,\mathsf{SEP}} \leq E_{r,\mathsf{PPT}}$.

For the former, we need to show that for each separable state there exists an LO measurement such that the relative entropy of the measurement outcomes is at least $\log \frac{d+1}{2}$. In fact, it suffices to employ the $U \otimes \overline{U}$-twirl followed by local measurements in the computational basis. Although this requires shared randomness, it is easy to see that derandomization can be done due to the joint convexity of relative entropy. The twirl leaves Φ_d invariant and transforms the separable state into a separable *isotropic* state

$$\sigma = p\Phi_d + (1 - p)\frac{1}{d^2}\mathbb{1},$$

where the separability is equivalent to $p \leq \frac{1}{d+1}$ [34]. Now, the measurement of the maximally entangled state and of σ yield distributions

$$P(xy|\Phi_d) = \frac{1}{d}\delta_{xy},$$

$$Q(xy|\sigma) = \frac{p}{d}\delta_{xy} + \frac{1-p}{d^2}.$$

From this it is straightforward to calculate the relative entropy

$$D(P\|Q) = \sum_x \frac{1}{d}\log\frac{1/d}{p/d + (1-p)/d^2}$$

$$= -\log\left(p + \frac{1-p}{d}\right) \geq \log\frac{d+1}{2},$$

and we are done.

For the second (upper) bound, we need to show that there is no better measurement once we choose an appropriate separable state, which predictably we set

$$\sigma = \frac{1}{d+1}\Phi_d + \frac{d}{d+1}\frac{1}{d^2}\mathbb{1} = \frac{1}{d}\Phi_d + \frac{d-1}{d}\frac{1}{d^2-1}(\mathbb{1} - \Phi_d).$$

Now our entangled state and the separable candidate are isotropic This means that whatever PPT measurement we have, i.e. with POVM elements M_i such that $M_i^\Gamma \geq 0$, the covariant POVM $\left(dU\,(U\otimes\overline{U})M_i\,(U\otimes\overline{U})^\dagger\right)_{i,U}$ will achieve the same relative entropy. Note however that the probabilities $\mathrm{Tr}\,\rho(U\otimes\overline{U})M_i(U\otimes\overline{U})^\dagger$ are independent of the unitary U for isotropic $\rho \in \{\Phi_d, \sigma\}$, so we get the same relative entropy for the twirled POVM with operators

$$\widehat{M}_i = \int dU\,(U\otimes\overline{U})M_i\,(U\otimes\overline{U})^\dagger,$$

which are all isotropic: $\widehat{M}_i = \alpha_i\Phi_d + \beta_i(\mathbb{1} - \Phi_d)$, with $\alpha_i, \beta_i \geq 0$ and separately adding up to 1. The PPT condition is $\beta_i \geq \frac{1}{d+1}\alpha_i$ for all i. Next, the maximum of the relative entropy will be attained on an extremal measurement from this class, which restricts (w.l.o.g.) the number of outcomes to two. The only nontrivial POVM with these properties is composed of the two operators

$$\widehat{M}_0 = \Phi_d + \frac{1}{d+1}(\mathbb{1} - \Phi_d),$$

$$\widehat{M}_1 = \frac{d}{d+1}(\mathbb{1} - \Phi_d).$$

For this measurement, the probabilities observed on Φ_d are 1 and 0, respectively; for the above σ they are $\frac{2}{d+1}$ and $\frac{d-1}{d+1}$, yielding indeed a relative entropy of $\log\frac{d+1}{2}$.

Now, we can conclude that $E_{r,\mathsf{M}}^\infty$, with M be any of {LO, 1-LOCC, LOCC, SEP, PPT} coincides with the entropic entanglement measure on pure states. This follows now easily from the asymptotic theory of pure state entanglement and the asymptotic continuity. To be precise, let ψ be a pure state on $A \otimes B$; then there is a sequence of $\epsilon_n \to 0$ and of LO protocols(!) to convert $\psi^{\otimes n}$ into $\rho^{(n)}$ with $\left\|\Phi_2^{\otimes n(E(\psi)-\epsilon_n)} - \rho^{(n)}\right\|_1 \leq \epsilon_n$. By the monotonicity of $E_{r,\mathsf{M}}$ under local operations and for large enough n,

$$E_{r,\mathsf{M}}(\psi^{\otimes n}) \geq E_{r,\mathsf{M}}\left(\rho^{(n)}\right)$$

$$\geq E_{r,\mathsf{M}}\left(\Phi_2^{\otimes n(E(\psi)-\epsilon_n)}\right) - 2\epsilon_n \log \frac{6 \cdot 2^{2nE(\psi)}}{\epsilon_n}$$

$$\geq nE(\psi) - O(n)\epsilon_n - O(1).$$

Conversely, there are one-way LOCC protocols to convert $\Phi_2^{\otimes n(E(\psi)+\epsilon_n)}$ into $\omega^{(n)}$ with $\left\|\psi^{\otimes n} - \omega^{(n)}\right\|_1 \leq \epsilon_n$. Hence, for large enough n,

$$E_{r,\mathsf{M}}(\psi^{\otimes n}) \leq E_{r,\mathsf{M}}\left(\omega^{(n)}\right) + 2\epsilon_n \log \frac{6|A|^n|B|^n}{\epsilon_n}$$

$$\leq E_{r,\mathsf{M}}\left(\Phi_2^{\otimes n(E(\psi)+\epsilon_n)}\right) + 2\epsilon_n \log \frac{6|A|^n|B|^n}{\epsilon_n}$$

$$\leq nE(\psi) + O(n)\epsilon_n + O(1).$$

Together, we obtain, for $\mathsf{M} \in \{\text{1-LOCC, LOCC, SEP, PPT}\}$,

$$\left|\frac{1}{n}E_{r,\mathsf{M}}(\psi^{\otimes n}) - E(\psi)\right| \leq O(\epsilon_n) + O\left(\frac{1}{n}\right) \to 0,$$

as $n \to \infty$. For LO the above reasoning does not apply because we need one-way LOCC operations in the converse (dilution) part. However, as $E_{r,\mathsf{LO}} \leq E_{r,\text{1-LOCC}}$, the lower bound for the former and the upper bound for the latter suffice. $\square$

7. Comparisons Between Entanglement Measures

In this section, we consider the relations between entanglement measures. Especially, we are interested in two classes of them. The first class consists of squashed-like measures. This includes the squashed entanglement E_{sq} itself, the conditional entanglement of mutual information $E_I(\rho_{AB}) := \frac{1}{2}\inf\{I(AA'; BB')_\rho - I(A'; B')_\rho\}$ with $\rho_{AA'BB'}$ being an extension of ρ_{AB} [35], and the c-squashed entanglement $E_{sq,c}(\rho_{AB}) := \frac{1}{2}\inf\{I(A; B|E)_\rho\}$, where the infimum is taken over all the extension state ρ_{ABE} of the form $\sum_i p_i \rho_{AB}^i \otimes |i\rangle\langle i|_E$ [36]. It is known that these entanglement measures satisfy the chain of inequalities [37,38]

$$E_d \leq K_d \leq E_{sq} \leq E_I \leq E_{sq,c}^\infty \leq E_c,$$

where E_d is the distillable entanglement, K_d is the distillable key and E_c the entanglement cost.

The other class contains the relative entropy of entanglement E_r and its relatives $E_{r,\leftrightarrow}(\rho_{AB}) := E_{r,\text{LOCC}}(\rho_{AB})$ and $E_{r,\to}(\rho_{AB}) := \sup\{E_{r,\text{1-LOCC}}(\Lambda(\rho_{AB})) : \Lambda \text{ being}$ LOCC$\}$. Here $E_{r,\to}$ is an "update" of $E_{r,\text{1-LOCC}}$ such that it is LOCC monotone. Note that in the definition of $E_{r,\to}$, the supremum is taken over all LOCC operations, in contrast to the smaller set of LOCC measurements. It is known that [39]

$$E_d \leq K_d \leq E_r^\infty \leq E_c.$$

On the other hand, it is obvious from the definitions that $E_{r,\to}^\infty \leq E_{r,\leftrightarrow}^\infty \leq E_r^\infty$, and we will show that $E_d \leq E_{r,\to}^\infty$ later in Proposition 7. Hence, we have also

$$E_d \leq E_{r,\to}^\infty \leq E_{r,\leftrightarrow}^\infty \leq E_r^\infty \leq E_c.$$

Although these two classes of entanglement measures are defined in different ways, we are able to make comparisons between them, and obtain the relations in Proposition 6.

Proposition 6. *The following universal relations between entanglement measures hold:*

$$
\begin{aligned}
1. &\quad 2E_{sq,c}^{\infty} \geq E_r^{\infty}, \\
2. &\quad 2E_I \geq E_{r,\leftrightarrow}^{\infty}, \\
3. &\quad 2E_{sq} \geq E_{r,\rightarrow}^{\infty}.
\end{aligned}
$$

These relations also hold true if we replace the regularized entanglement measures by their corresponding non-regularized versions.

Proof. The first inequality is easy. For any classical extension $\rho_{ABE} = \sum_i p_i \rho_{AB}^i \otimes |i\rangle\langle i|_E$ of a state ρ_{AB}, we have

$$
\begin{aligned}
I(A; B|E)_\rho &= \sum_i p_i D(\rho_{AB}^i \| \rho_A^i \otimes \rho_B^i) \\
&\geq D\left(\rho_{AB} \| \sum_i p_i \rho_A^i \otimes \rho_B^i\right) \\
&\geq E_r(\rho_{AB}),
\end{aligned}
$$

using the joint convexity of the relative entropy. This, together with the definition of $E_{sq,c}$, implies that $2E_{sq,c} \geq E_r$. Regularizing both sides, we get the regularized version as desired.

For the second inequality, we employ the idea for the proof of [21, Lem. 1], and apply it to the partial state merging protocol [35]. Let $\rho_{AA'BB'E}$ be a pure state, where the AA' system is with Alice, BB' is with Bob, and E is at Eve's hand. Alice and Bob are to transmit their systems A and B to Eve, by sending as few qubits as possible to her, provided that unlimited entanglement is available between Alice (Bob) and Eve. In the i.i.d. case, this task can be expressed as the transformation $\rho_{AA':BB':E}^{\otimes n} \longrightarrow \rho_{A':B':EAB}^{\otimes n}$. Asymptotically, it requires a minimal sum-rate $\frac{1}{2}\{I(AA'; BB') - I(A'; B')\}$ of quantum communication [35]. On the other hand, because the relative entropy of entanglement E_r is unlockable [40], the decrease of entanglement between Alice and Bob in this protocol, measured by E_r, is no larger than 2 times the qubits transmitted. This means

$$
I(AA'; BB') - I(A'; B') \geq E_r^{\infty}(\rho_{AA':BB'}) - E_r^{\infty}(\rho_{A':B'}). \tag{40}
$$

The right side of Eq. (40) satisfies [23]

$$
E_r^{\infty}(\rho_{AA':BB'}) - E_r^{\infty}(\rho_{A':B'}) \geq E_{r,\leftrightarrow}^{\infty}(\rho_{AB}) \geq E_{r,\leftrightarrow}(\rho_{AB}). \tag{41}
$$

Equations (40) and (41), together with the definition of E_I, lead to the second inequality and its non-regularized version as advertised.

The last inequality and its non-regularized version is essentially due to Theorem 2, since squashed entanglement is non-increasing under any LOCC operations. □

Proposition 7. *We have $E_{r,\rightarrow}^{\infty} \geq E_d$.*

Proof. Let Λ_n be a LOCC operation that satisfies

$$
\|\Lambda_n(\rho_{AB}^{\otimes n}) - \Phi_{d_n}\|_1 \leq \epsilon
$$

with $\epsilon \leq \frac{1}{e}$. We have

$$
E_{r,\rightarrow}(\rho_{AB}^{\otimes n}) \geq E_{r,\text{1-LOCC}}(\Lambda_n(\rho_{AB}^{\otimes n})) \geq \log \frac{d_n + 1}{2} - 2\epsilon \log \frac{6d_n^2}{\epsilon}, \tag{42}
$$

where the first inequality is by definition of $E_{r,\to}$, and the second one makes use of Proposition 3 and Proposition 4. Recall that the distillable entanglement can be written as

$$E_d(\rho_{AB}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{\Lambda_n \in \text{LOCC}} \left\{ \frac{\log d_n}{n} : \|\Lambda_n(\rho^{\otimes n}) - \Phi_{d_n}\|_1 \leq \epsilon \right\}. \qquad (43)$$

Equation (42) and Eq. (43) together imply

$$E_d(\rho_{AB}) \leq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{E_{r,\to}(\rho_{AB}^{\otimes n}) + 1 - 2\epsilon \log \epsilon + 2\epsilon \log 6}{n(1 - 4\epsilon)}$$
$$= E_{r,\to}^{\infty}(\rho_{AB})$$

and we are done. □

We summarize the relations between these entanglement measures in Fig. 1. Since we are mainly interested in the regularized versions, some relations between the non-regularized entanglement measures are not reflected here. These include $E_{sq,c} \leq E_f$, $E_r \leq E_f$, $E_r \leq 2E_{sq,c}$ and $E_{r,\to} \leq E_{r,\leftrightarrow}$ (E_f is the entanglement of formation). Some pairs of these entanglement measures are incomparable, meaning that—depending on the state—they can be larger than each other. This is really the case for E_{sq} and E_r
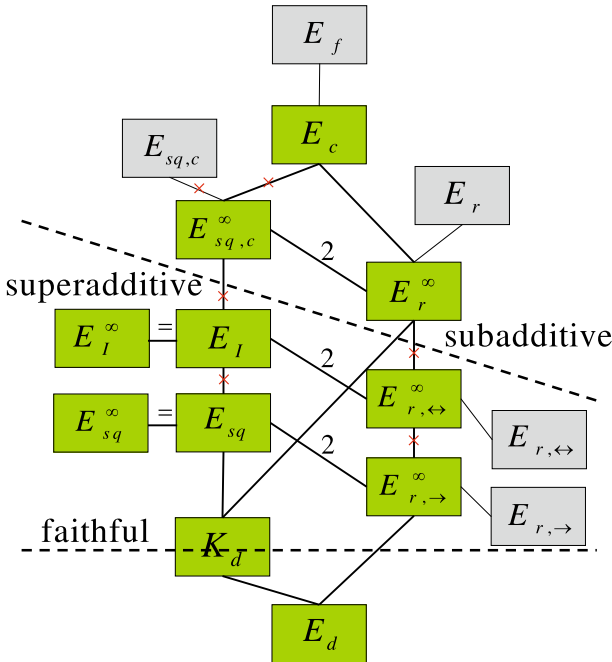
Fig. 1. Relations between some entanglement measures. When two quantities are connected by a line with a constant above (constant 1 is omitted), it means that the higher one multiplied by the constant is no smaller than the lower one. For those entanglement measures of which the separation is still unknown, we mark a *cross* (*red* in the online version) on the line that connects them. The *upper dashed line* divides these entanglement measures into two groups: the upper ones are subadditive and the lower ones are superadditive. Entanglement measures above the *lower dashed line* are faithful, while the only one below this line, E_d, is not faithful [42]. Whether the distillable key, K_d, is faithful or not, is still an open question, hence we put the line on it

(E_r^∞). $E_{sq} \gg E_r$ is known for certain "flower states", due to the lockability of E_{sq} and non-locking of E_r [40,41]; the other direction $E_r^\infty \gg E_{sq}$ holds for $d \times d$ antisymmetric states [24,25]. We conjecture that the same situation occurs between E_I and E_r (E_r^∞), E_{sq} and $E_{r,\leftrightarrow}^\infty$ ($E_{r,\leftrightarrow}$), K_d and $E_{r,\leftrightarrow}^\infty$, K_d and $E_{r,\rightarrow}^\infty$, which are left as open questions. Note that the possibility of $E_I > E_r$ and $E_{sq} > E_{r,\leftrightarrow}^\infty$ for certain states, are known from the relations in Fig. 1 and that E_{sq} can be larger than E_r.

The separation between entanglement measures is another interesting topic. Proposition 4 provides us with the strict inequalities $E_{r,\leftrightarrow} < E_{r,\leftrightarrow}^\infty$ and $E_{r,\rightarrow} < E_{r,\rightarrow}^\infty$ for maximally entangled states. The fact that $2E_{sq,c}^\infty \geq E_r^\infty$ (cf. Proposition 6) and E_r^∞ can be much larger than E_{sq} implies the separation between $E_{sq,c}^\infty$ and E_{sq}, disproving the conjecture that $E_{sq,c}^\infty$ and E_{sq} may be the same [37]. Similarly, the relations shown in Fig. 1, together with the fact that E_{sq} and E_r^∞ can be much larger than the other, lead to separations for the pairs (E_c, E_r^∞), $(E_{sq,c}^\infty, E_r^\infty)$, $(E_r^\infty, E_{r,\rightarrow}^\infty)$, (E_r^∞, K_d), $(E_I, E_{r,\leftrightarrow}^\infty)$, $(E_{sq}, E_{r,\rightarrow}^\infty)$ and (E_{sq}, K_d). Separation between $E_{r,\rightarrow}^\infty$ and E_d is witnessed by the bound entangled states, since the former is faithful. A separation between E_d and K_d [39] had been discovered previously, that between E_c and E_f is by Hastings [43,44], that between E_r and E_r^∞ due to Vollbrecht and Werner [16].

At last, separations between pairs of entanglement measures that are still unknown, are marked in Fig. 1, and we leave them as open questions.

References

1. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865 (2009)
2. Christandl, M., Winter, A.: 'Squashed entanglement'—an additive entanglement measure. J. Math. Phys. **45**, 829 (2004)
3. Tucci, R.R.: Quantum entanglement and conditional information transmission. http://arXiv.org/abs/quant-ph/9909041v1, 1999
4. Tucci, R.R.: Entanglement of distillation and conditional mutual information. http://arXiv.org/abs/quant-ph/0202144v2, 2002
5. Maurer, U.M., Wolf, S.: Unconditionally secure key agreement and the intrinsic conditional information. IEEE Trans. Inf. Theory **45**, 499 (1999)
6. Alicki, R., Fannes, M.: Continuity of quantum conditional information. J. Phys. A: Math. Gen. **37**, L55 (2004)
7. Koashi, M., Winter, A.: Monogamy of entanglement and other correlations. Phys. Rev. A **69**, 022309 (2004)
8. Devetak, I., Yard, J.: Exact cost of redistributing quantum states. Phys. Rev. Lett. **100**, 230501 (2008)
9. Yard, J., Devetak, I.: Optimal quantum source coding with quantum information at the encoder and decoder. IEEE Trans. Inf. Theory **55**, 5339 (2009)
10. Ye, M.-Y., Bai, Y.-K., Wang, Z.D.: Quantum state redistribution based on a generalized decoupling. Phys. Rev. A **78**, 030302(R) (2008)
11. Oppenheim, J.: A paradigm for entanglement theory based on quantum communication. http://arXiv.org/abs/0801.0458v1 [quant-ph], 2008
12. Hiai, F., Petz, D.: The proper formula for relative entropy and its asymptotics in quantum probability. Commun. Math. Phys. **143**, 99 (1991)
13. Ogawa, T., Nagaoka, H.: Strong converse and Stein's lemma in quantum hypothesis testing. IEEE. Tran. Inf. Theory **46**, 2428 (2000)
14. Vedral, V., Plenio, M.B., Rippin, M.A., Knight, P.L.: Quantifying entanglement. Phys. Rev. Lett. **78**, 2275 (1997)

15. Vedral, V., Plenio, M.B.: Entanglement measures and purification procedures. Phys. Rev. A **57**, 1619 (1998)
16. Vollbrecht, K.G.H., Werner, R.F.: Entanglement measures under symmetry. Phys. Rev. A **64**, 062307 (2001)
17. Brandão, F.G.S.L., Plenio, M.B.: Entanglement theory and the second law of thermodynamics. Nat. Phys. **4**, 873 (2008)
18. Brandão, F.G.S.L., Plenio, M.B.: A reversible theory of entanglement and its relation to the second law. Commun. Math. Phys. **295**, 829 (2010)
19. Brandão, F.G.S.L., Plenio, M.B.: A generalization of quantum Stein's lemma. Commun. Math. Phys. **295**, 791 (2010)
20. Lieb, E.H., Ruskai, M.B.: Proof of the strong subadditivity of quantum-mechanical entropy. J. Math. Phys. **14**, 1938 (1973)
21. Brandão, F.G.S.L., Christandl, M., Yard, J.: Faithful squashed entanglement. Commun. Math. Phys. **306**, 805–830 (2011); Erratum to: Faithful Squashed Entanglement. Commun. Math. Phys. **316**, 287–288 (2012)
22. Matthews, W., Wehner, S., Winter, A.: Distinguishability of quantum states under restricted families of measurements with an application to quantum data hiding. Commun. Math. Phys. **291**, 813 (2009)
23. Piani, M.: Relative entropy of entanglement and restricted measurements. Phys. Rev. Lett. **103**, 160504 (2009)
24. Christandl, M., Schuch, N., Winter, A.: Entanglement of the antisymmetric state. Commun. Math. Phys. **311**, 397 (2012)
25. Christandl, M., Schuch, N., Winter, A.: Highly entangled states with almost no secrecy. Phys. Rev. Lett. **104**, 240405 (2010)
26. Fuchs, C.A., van de Graaf, J.: Cryptographic distinguishability measures for quantum-mechanical states. IEEE. Tran. Inf. Theory **45**, 1216 (1999)
27. Cover, T.M., Thomas, J.A.: *Elements of Information Theory*. New York: Wiley, 1991
28. Matthews, W., Winter, A.: On the Chernoff distance for asymptotic LOCC discrimination of bipartite quantum states. Commun. Math. Phys. **285**, 161 (2008)
29. Owari, M., Hayashi, M.: Asymptotic local hypothesis testing between a pure bipartite state and the completely mixed state. http://arXiv.org/abs/1105.3789v1 [quant-ph], 2011
30. Winter, A.: Coding theorem and strong converse for quantum channels. IEEE. Tran. Inf. Theory **45**, 2481 (1999)
31. Lindblad, G.: Completely positive maps and entropy inequalities. Commun. Math. Phys. **40**, 147 (1975)
32. Uhlmann, A.: Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory. Commun. Math. Phys. **54**, 21 (1977)
33. Donald, M., Horodecki, M.: Continuity of relative entropy of entanglement. Phys. Lett. A **264**, 257 (1999)
34. Horodecki, M., Horodecki, P.: Reduction criterion of separability and limits for a class of distillation protocols. Phys. Rev. A **59**, 4206 (1999)
35. Yang, D., Horodecki, M., Wang, Z.: An additive and operational entanglement measure: conditional entanglement of mutual information. Phys. Rev. Lett. **101**, 140501 (2008)
36. Yang, D., Horodecki, K., Horodecki, M., Horodecki, P., Oppenheim, J., Song, W.: Squashed entanglement for multipartite states and entanglement measures based on the mixed convex roof. IEEE Trans. Inf. Theory **55**(7), 3375 (2009)
37. Yang, D., Horodecki, M., Wang, Z.: Conditional entanglement. http://arXiv.org/abs/quant-ph/0701149v3, 2007
38. Christandl, M.: The structure of bipartite quantum states—insights from group theory and cryptography. Ph. D. thesis, University of Cambridge. http://arXiv.org/abs/quant-ph/0604183v1, 2006
39. Horodecki, K., Horodecki, M., Horodecki, P., Oppenheim, J.: Secure key from bound entanglement. Phys. Rev. Lett. **94**, 160502 (2005)
40. Horodecki, K., Horodecki, M., Horodecki, P., Oppenheim, J.: Locking entanglement measures with a single qubit. Phys. Rev. Lett. **94**, 200501 (2005)
41. Christandl, M., Winter, A.: Uncertainty, monogamy, and locking of quantum correlations. IEEE Trans. Inf. Theory **51**(9), 3159 (2005)
42. Horodecki, M., Horodecki, P., Horodecki, R.: Mixed-state entanglement and distillation: Is there a 'bound' entanglement in nature? Phys. Rev. Lett. **80**, 5239 (1998)
43. Hastings, M.B.: Superadditivity of communication capacity using entangled inputs. Nat. Phys. **5**, 255 (2009)
44. Shor, P.W.: Equivalence of additivity questions in quantum information theory. Commun. Math. Phys. **246**, 453 (2003)

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