Commun. Math. Phys. 325, 105–141 (2014)
Digital Object Identifier (DOI) 10.1007/s00220-013-1854-3

Communications in
**Mathematical
Physics**

Local Existence of Solutions of Self Gravitating Relativistic Perfect Fluids

Uwe Brauer[1,⋆], **Lavi Karp**[2,⋆⋆]

[1] Departamento de Matemática Aplicada, Universidad Complutense Madrid, 28040 Madrid, Spain.
E-mail: oub@mat.ucm.es

[2] Department of Mathematics, ORT Braude College, P.O. Box 78, 21982 Karmiel, Israel.
E-mail: karp@braude.ac.il

Received: 15 December 2011 / Accepted: 28 May 2013
Published online: 17 December 2013 –

Abstract: This paper deals with the evolution of the Einstein gravitational fields which are coupled to a perfect fluid. We consider the Einstein–Euler system in asymptotically flat spacestimes and therefore use the condition that the energy density might vanish or tend to zero at infinity, and that the pressure is a fractional power of the energy density. In this setting we prove local in time existence, uniqueness and well-posedness of classical solutions. The zero order term of our system contains an expression which might not be a C^∞ function and therefore causes an additional technical difficulty. In order to achieve our goals we use a certain type of weighted Sobolev space of fractional order. In Brauer and Karp (J Diff Eqs 251:1428–1446, 2011) we constructed an initial data set for these of systems in the same type of weighted Sobolev spaces.

We obtain the same lower bound for the regularity as Hughes et al. (Arch Ratl Mech Anal 63(3):273–294, 1977) got for the vacuum Einstein equations. However, due to the presence of an equation of state with fractional power, the regularity is bounded from above.

1. Introduction

This paper deals with the Cauchy problem for the Einstein–Euler system describing a relativistic self-gravitating perfect fluid, whose density either has compact support or falls off at infinity in an appropriate manner.

The evolution of the gravitational field is described by the Einstein equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi T_{\alpha\beta}, \tag{1.1}$$

where $g_{\alpha\beta}$ is a semi-Riemannian metric having a signature $(-,+,+,+)$, $R_{\alpha\beta}$ is the Ricci curvature tensor, and R is the scalar curvature. Both tensors are functions of the metric

⋆ Uwe Brauer's research was partially supported by grant MTM2012-31928.

⋆⋆ Lavi Karp's research was supported by ORT Braude College's Research Authority.

$g_{\alpha\beta}$ and its first and second order partial derivatives. The right-hand side of (1.1) consists of the energy–momentum tensor $T_{\alpha\beta}$, which in the case of a perfect fluid takes the form

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}, \tag{1.2}$$

where ϵ is the energy density, p is the pressure and u^α is the four-velocity vector. The vector u^α is a unit timelike vector, which means that it satisfies the normalization condition

$$g_{\alpha\beta}u^\alpha u^\beta = -1. \tag{1.3}$$

The Euler equations describing the evolution of the fluid take the form

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{1.4}$$

where ∇ denotes the covariant derivative associated with the metric $g_{\alpha\beta}$. Equations (1.1) and (1.4) are not sufficient to determinate the structure uniquely, a functional relation between the pressure p and the energy density ϵ (equation of state) is also needed. We choose an equation of state that has been used in astrophysical problems. It is the analogue of the well known polytropic equation of state in the non-relativistic theory, given by

$$p = p(\epsilon) = K\epsilon^\gamma, \qquad K, \gamma \in \mathbb{R}^+, \qquad 1 < \gamma. \tag{1.5}$$

The sound velocity is denoted by

$$\sigma^2 = \frac{dp}{d\epsilon},$$

and the range of the energy density ϵ will be restricted so that the causality condition $\sigma^2 < 1$ will hold.

The unknowns of these equations are the semi-Riemannian metric $g_{\alpha\beta}$, the velocity vector u^α and the energy density ϵ. These are functions of t and x^a, where x^a $(a = 1, 2, 3)$ are the Cartesian coordinates on $\mathbb{R}^3$. The alternative notation $x^0 = t$ will also be used and Greek indices will take the values 0, 1, 2, 3 in the following.

In the present paper we prove the well-posedness of the coupled systems (1.1), (1.2), (1.4) and (1.5) under the harmonic gauge condition in asymptotically flat spacetimes. In order to achieve this, we need to rewrite the above equations as a hyperbolic system.

In astrophysical context the density ϵ is expected to have compact support, or tend to zero at spatial infinity in an appropriate sense. It is well known that the usual symmetrization of the Euler equations is badly behaved in cases where the density tends to zero somewhere. The coefficients of the system degenerate or become unbounded when ϵ approaches zero. It was observed by Makino [19] that this difficulty can be to some extent circumvented in the case of a non-relativistic fluid by using a new matter variable w in place of the mass density. For this reason we introduce the quantity

$$w = M(\epsilon) = \epsilon^{\frac{\gamma-1}{2}}, \tag{1.6}$$

and we call it the Makino variable. A similar device was used by Gamblin [12] and Bezard [2] for the Euler-Poisson equations, and by Rendall [24] and Oliynyk [22] for the Einstein–Euler equations. The common method for solving the Cauchy problem for the Einstein equations consists usually of the following steps.

1. Initial data must satisfy the constraint equations, which are intrinsic to the initial hypersurface. Therefore, the first step is to construct solutions of these constraints.
2. The second step is to use the harmonic coordinate condition and to solve the evolution equations with these initial data.
3. The last step is to prove that the harmonic coordinate condition and the solution of the constraints propagate. That means if they held on an initial hypersurface, they hold for later times.

The last step was treated in detail, for example in Fisher and Marsden[11]. The idea is to work out the condition $\nabla_\alpha G^{\alpha\beta} = 0$. Since our energy–momentum satisfies (1.4), their result can be immediately generalized for our case. But for the sake of brevity we have omitted the details.

However, the presence of the equation of state (1.5) introduces an additional step: the compatibility problem of the initial data for the fluid and the gravitational field [see (2.11)]. There are three types of initial data for the Einstein–Euler system:

- The gravitational data is a triple $(\mathcal{M}, h, K_{ab})$, where $\mathcal{M}$ is a space-like manifold, h is a proper Riemannian metric on $\mathcal{M}$, and K_{ab} is the second fundamental form on $\mathcal{M}$ (extrinsic curvature). The pair (h, K_{ab}) must satisfy the constraint equations (2.10);
- The matter variables, consisting of the energy density z and the momentum density j^a, appear on the right-hand side of the constraints (2.10);
- The initial data for the Makino variable w and the velocity vector u^α of the perfect fluid.

The only type of Sobolev spaces which are known to be useful for existence theorems for the constraint equations in an asymptotically flat manifold, are the weighted Sobolev spaces $H_{k,\delta}$, where $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$. These spaces were introduced by Nirenberg and Walker [21] and Cantor [5], and they are the completion of $C_0^\infty(\mathbb{R}^3)$-functions under the norm

$$\|u\|_{k,\delta}^2 = \sum_{|\alpha|\leq k} \int \left((1 + |x|)^{\delta+|\alpha|} |\partial^\alpha u| \right)^2 dx. \tag{1.7}$$

Due to the presence of the equation of state (1.5) and the Makino variable (1.6), we have to estimate $\|w^{\frac{2}{\gamma-1}}\|_{k,\delta}$. So it is perhaps worth discussing the estimate of Sobolev's norm of u^β in more details for $\beta > 1$. For simplicity we discuss this in the ordinary Sobolev space $H^k = H^k(\mathbb{R}^3)$. The simplest case is when $\beta \in \mathbb{N}$, then $\|u^\beta\|_{H^k} \leq C(\|u\|_{L^\infty})\|u\|_{H^k}$ and there is no restriction on k. When $\beta \notin \mathbb{N}$, then we obtain the same estimate, provided that $k \leq \beta$. This bound on k was improved by Runst and Sickel [25] to $k < \beta + \frac{1}{2}$. Applying this to $\beta = \frac{2}{\gamma-1}$, and taking into account the Sobolev embedding $\|u\|_{L^\infty} \leq C\|u\|_{H^k}$ for $k > \frac{3}{2}$, we get a lower and upper bound for k:

$$\frac{3}{2} < k < \frac{2}{\gamma - 1} + \frac{1}{2}. \tag{1.8}$$

The only exception is the case when $\frac{2}{\gamma-1}$ is an integer. Note that for certain values of γ, inequalities (1.8) possess no integer solution. Hence, for these values of γ it is impossible to obtain a solution to the Einstein–Euler system in Sobolev spaces of integer order. So in order to be able to solve the coupled system for the maximal range of the power γ, and in addition, to improve the regularity of the solutions, we are considering the Cauchy problem in the weighted fractional spaces $H_{s,\delta}$, where s is real number (see Definition 2.1). These spaces were introduced by Triebel [28], and they generalize $H_{k,\delta}$ to a fractional order. In [4] the authors constructed initial data for coupled systems (1.1), (1.2) and (1.4) with the equations of state (1.5). This includes the solution to the constraint equations (2.10), as well as the solution to the compatibility problem between the matter variable (z, j^a) and the fluid variables (w, u^α), (2.11), in the $H_{s,\delta}$-spaces. Here we will establish the well-posedness of Einstein–Euler systems in the weighted fractional Sobolev spaces $H_{s,\delta}$.

The common way to prove well-posedness is to rewrite the evolution equations as a symmetric hyperbolic system. So our first step is to use the Makino variable (1.6) and to reduce the Euler equations (1.4) to a uniformly first order symmetric hyperbolic system. This result was announced in [3] and here we present a detailed proof of it. Our hyperbolic reduction is based on the fluid decomposition; for alternative reductions see [24].

It is well-known that the Einstein equations can be written as a system of quasi-linear wave equations under the harmonic gauge condition [6,7,29]. The proofs of existence and uniqueness either use second order techniques [6,8,13,14,17], or transferring the equations to a first order symmetric hyperbolic system. Fischer and Marsden used the first order techniques and obtained the well-posedness of the reduced vacuum Einstein equations in H^s and for $s > \frac{7}{2}$ [11]. This result was improved by Hughes et al. [14], who obtained $(g_{\alpha\beta}, \partial_t g_{\alpha\beta}) \in H^{s+1} \times H^s$ for $s > \frac{3}{2}$. They used second order theory, and took advantage of the specific form of the quasi-linear system of wave equations, namely, that the coefficients depend only on the semi-metric $g_{\alpha\beta}$, but not on its first order derivatives.

Our aim is to prove existence and uniqueness of the reduced Einstein–Euler system (1.1), (1.2) and (1.4) with the equation of state (1.5). In addition, we would like to achieve the same regularity of the metric as in [14]. But since we have here a coupled system which one of them is a first order, the second order techniques of Hughes, Kato and Marsden in [14] are no longer available for the present problem.

In asymptotically flat spacetimes the initial metric $g_{\alpha\beta}(0)$ differs from the Minkowski metric by a term which is $O(1/r)$ at spatial infinity, and this term does not belong to H^s. It is therefore more appropriate to consider both the constraint and evolution equations in the $H_{s,\delta}$ spaces rather than in the spaces H^s without weights. For the vacuum equations the second author obtained well-posedness of the reduced Einstein equations with $(g_{\alpha\beta}, \partial_t g_{\alpha\beta}) \in H_{s+1,\delta} \times H_{s,\delta+1}$, $s > \frac{3}{2}$ and $\delta > -\frac{3}{2}$, see [16]. But unlike Hughes, Kato and Marsden [14], he treated the quasi-linear system of wave equations as a first order symmetric hyperbolic system. The first order techniques have the advantage that they enable, in a convenient way, the coupling of the gravitational field equations to other matter models, in particular, to perfect fluids. In the Appendix we explain the main idea of [16] which allows us to obtain the regularity index $s > \frac{3}{2}$ by means of first order hyperbolic systems.

A crucial step in the proof of existence and uniqueness of any hyperbolic system is to establish energy estimates for the linearized system. In order to achieve this we define an appropriate inner-product of the $H_{s,\delta}$ spaces, which takes into account the coefficients of the linearized system (see Sect. 5). A similar inner-product was used in [16], and here we rely on these energy estimates.

Once we have obtained the energy estimates for the linearized system, we use Majda's iterative scheme in order to obtain existence and uniqueness of the quasi-linear symmetric hyperbolic system [18]. This procedure uses the fact that solutions to a linear first order symmetric hyperbolic system with C_0^∞ coefficients and initial data are also C_0^∞. But here we encounter a further difficulty, namely, the right-hand side of (1.1) contains the fractional power $w^{\frac{2}{\gamma-1}}$, see (3.16). So even when $w \in C_0^\infty$ and $w \geq 0$, $w^{\frac{2}{\gamma-1}}$ might not be a C^∞ function. We solve that problem by using the fact that $\epsilon = w^{\frac{2}{\gamma-1}}$ satisfies a certain first order linear equation. Gamblin encountered a similar problem for the Euler–Poisson equations [12], but he solved it in a somewhat different way.

Our results improve the existence theory of solutions locally in time of self gravitating relativistic perfect fluids in several aspects. Rendall studied this problem in [24], but he assumed time symmetry, which means that the extrinsic curvature of the initial manifold is zero, and therefore the Einstein constraint equations are reduced to a single scalar equation. In addition, he dealt only with C_0^∞-solutions. In his study of the Newtonian limit of perfect fluids, Oliynyk obtained existence locally in time in the weighted space of integer order $H_{k,\delta}$, for $k \geq 4$ [22]. Both Rendall and Oliynyk assume that the adiabatic exponent of (1.5) satisfies the condition that $\frac{2}{\gamma-1}$ is an integer.

The paper is organized as follows: In the next section we define the weighted Sobolev spaces of fractional order $H_{s,\delta}$ and state the main result. Section 3 has two subsections: the first one deals with the hyperbolic reduction of the Euler equations (1.4); in the second one we spell out the matrices which describe the coupled equations (1.1), (1.2) and (1.4) as a hyperbolic system.

In Sect. 4 we present tools and properties of the $H_{s,\delta}$-spaces which we need in the course of the publication. We also define the corresponding product spaces. The energy estimates for the linearized system are considered in Sect. 5, there we also define the appropriate inner-product. In Sect. 6 we treat the iteration procedure. Parts of the steps are standard and known, but some of them require special attention due to the specific form of the system (3.24) and the product spaces. In this section we will use the fact that the coefficients of the first order derivatives depend only on the semi-metric $g_{\alpha\beta}$. Finally, in Sect. 7 we prove the main result. In the Appendix we give a heuristic idea explaining how the fact that the coefficients of the system of wave equations depend only on the semi-metric $g_{\alpha\beta}$ enables us to obtain the desired regularity by means of symmetric hyperbolic systems.

2. The Main Results

We obtain the well-posdeness in the weighted Sobolev spaces of fractional order. So we first recall their definition.

Let $\{\psi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R}^3)$ be a sequence of cutoff function such that, $\psi_j(x) \geq 0$ for all $j \geq 0$, $\operatorname{supp}(\psi_j) \subset \{x : 2^{j-2} \leq |x| \leq 2^{j+1}\}$, $\psi_j(x) = 1$ on $\{x : 2^{j-1} \leq |x| \leq 2^j\}$ for $j = 1, 2, \ldots,$ $\operatorname{supp}(\psi_0) \subset \{x : |x| \leq 2\}$, $\psi_0(x) = 1$ on $\{x : |x| \leq 1\}$ and

$$|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha| j},$$

where the constant C_α does not depend on j.

We restrict ourselves to the case $p = 2$ and denote the Bessel potential spaces by H^s with the norm given by

$$\|u\|_{H^s}^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u.

Definition 2.1. *For $s, \delta \in \mathbb{R}$,*

$$\left(\|u\|_{H_{s,\delta}}\right)^2 = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|_{H^s}^2, \tag{2.1}$$

where $f_\varepsilon(x) = f(\varepsilon x)$ denotes the scaling by a positive number ϵ. The space $H_{s,\delta}$ is the set of all tempered distributions having a finite norm given by (2.1).

The $H_{s,\delta}$-norm of a distribution u in an open set $\Omega \subset \mathbb{R}^3$ is given by

$$\|u\|_{H_{s,\delta}(\Omega)} = \inf_{f_{|\Omega}=u} \|f\|_{H_{s,\delta}(\mathbb{R}^3)} .$$

Definition 2.2. *Let $\mathcal{M}$ be a 3 dimensional smooth connected manifold and let h be a metric on $\mathcal{M}$ such that $(\mathcal{M}, h)$ is complete. We say that $(\mathcal{M}, h)$ is* ***asymptotically flat*** *of the class $H_{s,\delta}$ if $h \in H^s_{\mathrm{loc}}(\mathcal{M})$ and there is a compact set $S \subset \mathcal{M}$ such that:*

1. *There is a finite collection of charts $\{(U_i, \varphi_i)\}_{i=1}^N$ which covers $\mathcal{M}\backslash S$;*
2. *For each i, $\varphi_i^{-1}(U_i) = E_{r_i} := \{x \in \mathbb{R}^3 : |x| > r_i\}$ for some positive r_i;*
3. *The pull-back $(\varphi_{i*}h)_{ab}$ is uniformly equivalent to the Euclidean metric δ_{ab} on E_{r_i} for each i;*
4. *For each i, $(\varphi_{i*}h)_{ab} - \delta_{ab} \in H_{s,\delta}(E_{r_i})$.*

The $H_{s,\delta}$-norm on the manifold $\mathcal{M}$ is defined as follows. Let $U_0 \subset \mathcal{M}$ be an open set such that $S \subset U_0$ and $\overline{U_0} \Subset \mathcal{M}$. Let $\{\chi_0, \chi_i\}$ be a partition of unity subordinate to $\{U_0, U_i\}$, then

$$\|u\|_{H_{s,\delta}(\mathcal{M})} := \|\chi_0 u\|_{H^s(U_0)} + \sum_{i=1}^{N} \left\|\varphi_i^*(\chi_i u)\right\|_{H_{s,\delta}(\mathbb{R}^3)} \tag{2.2}$$

is the norm of the weighted fractional Sobolev space $H_{s,\delta}(\mathcal{M})$. For the definition of the norm $\|\chi_0 u\|_{H^s(\Omega)}$ on the manifold $\mathcal{M}$, see e.g. [1]. Note that the norm (2.2) depends on the partition of unity, but different partitions of unity result in equivalent norms. In the following we will omit the notation $\mathcal{M}$, that is, we will write $\|u\|_{H_{s,\delta}}$ instead of $\|u\|_{H_{s,\delta}(\mathcal{M})}$.

Since the principal symbol of the field equations (1.1) is characteristic in every direction (see e.g. [10]), it is impossible to solve (1.1) in the present form. We study these equations under the *harmonic gauge condition*

$$F^\mu = g^{\beta\gamma}\Gamma^\mu_{\beta\gamma} = 0, \tag{2.3}$$

where $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}$. Then the field equations (1.1) are equivalent to the *reduced Einstein equations*

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = H_{\alpha\beta}(g, \partial g) - 16\pi T_{\alpha\beta} + 8\pi g^{\mu\nu}T_{\mu\nu}g_{\alpha\beta}, \tag{2.4}$$

where $H_{\alpha\beta}(g, \partial g)$ is a quadratic function of the semi-metric $g_{\alpha\beta}$ and its first order derivatives. Since $g^{\mu\nu}$ has a Lorentzian signature, (2.4) is a system of quasi-linear wave equations. Taking into account the equation of state (1.5), the normalization condition (1.2), and the Makino variable (1.6), then the system of wave equations (2.4) becomes

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = H_{\alpha\beta}(g, \partial g) - 8\pi w^{\frac{2}{\gamma-1}} \left((1 - Kw^2)g_{\alpha\beta} + 2(1 + Kw^2)u_\alpha u_\beta\right). \tag{2.5}$$

So the unknowns of the system (2.5) coupled with the Euler equations (1.4) are the semi-metric $g_{\alpha\beta}$, the velocity vector u^α and the Makino variable w. Note that even if w is a smooth function, $w^{\frac{2}{\gamma-1}}$ might not be smooth in certain regions. The initial data consist of the triple $(\mathcal{M}, h_{ab}, K_{ab})$, where $\mathcal{M}$ is a space-like manifold, h_{ab} is a proper Riemannian metric on $\mathcal{M}$ and K_{ab} is its second fundamental form (extrinsic curvature).

The semi-metric $g_{\alpha\beta}$ takes the following data on M:

$$\begin{cases} g_{00}|_{\mathcal{M}} = -1, \quad g_{0a}|_{\mathcal{M}} = 0, \quad g_{ab}|_{\mathcal{M}} = h_{ab} \\ \qquad -\frac{1}{2}\partial_0 g_{ab}|_{\mathcal{M}} = K_{ab}, \end{cases} \quad a, b = 1, 2, 3. \tag{2.6}$$

The remaining initial data for $\partial_0 g_{\alpha 0_{|\mathcal{M}}}$ are determined through the harmonic condition $F^\mu = 0$. We compute them by inserting the initial data (2.6) in the harmonic gauge condition (2.3). Since $\partial_a g_{00}|_{\mathcal{M}} = \partial_a g_{b0}|_{\mathcal{M}} = 0$, this results in the following expressions for $\partial_0 g_{\alpha 0_{|\mathcal{M}}}$:

$$\begin{cases} \partial_0 g_{00_{|\mathcal{M}}} = 2h^{ab}(K_{ab}) \\ \partial_0 g_{0c_{|\mathcal{M}}} = \frac{1}{2}\left(h^{ab}(\partial_a h_{bc} - \partial_c h_{ab})\right). \end{cases} \tag{2.7}$$

In addition, the initial data of the velocity vector u^α and the Makino variable w are given on $\mathcal{M}$. We denote the Minikowski metric by $\eta_{\alpha\beta}$.

Theorem 2.3 (Main result). *Let $\frac{3}{2} < s < \frac{2}{\gamma-1} + \frac{1}{2}$ and $-\frac{3}{2} \leq \delta$. Assume $\mathcal{M}$ is asymptotically flat of class $H_{s+1,\delta}$, $K_{ab} \in H_{s,\delta+1}$, $\left(u^0 - 1, u^a, w\right)\big|_{\mathcal{M}} \in H_{s+1,\delta+1}$, $w(0) \geq 0$ and $u^\alpha(0)$ is a timelike vector. Then there exists a positive T, a unique semi-metric $g_{\alpha\beta}$, a unit timelike vector u^α and w satisfying the reduced Einstein equations* (2.5) *and the Euler equations* (1.4) *such that*

$$(g_{\alpha\beta}(t) - \eta_{\alpha\beta}) \in C([0,T], H_{s+1,\delta}) \cap C^1([0,T], H_{s,\delta+1}) \tag{2.8}$$

and

$$\left(u^0 - 1, u^a, w\right) \in C([0,T], H_{s+1,\delta+1}) \cap C^1([0,T], H_{s,\delta+2}). \tag{2.9}$$

Remark 2.4 (On the differentiability). Note that we have a lower and an upper bound of the differentiability index s, however, in case $\frac{2}{\gamma-1}$ is an integer, then there is no upper bound.

A necessary and sufficient condition for the equivalence between the reduced Einstein equations (2.5) and the field equations (1.1) is that the geometric data (h, K_{ab}) satisfy the constraint equations

$$\begin{cases} R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 = 16\pi z \\ {}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b(h^{bc}K_{bc}) \ = -8\pi j^a. \end{cases} \tag{2.10}$$

Here $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric h_{ab}. The right-hand-side (z, j^a) consists of the energy density and the momentum density respectively.

Note that the harmonic coordinates F^μ satisfy a homogeneous wave equation. That is why $F^\mu \equiv 0$, if $F^\mu|_{\mathcal{M}} = 0$ and $\partial_0 F^\mu|_{\mathcal{M}} = 0$. The first condition follows from (2.7), and the second holds if the reduced Einstein equations (2.5) and the constraint equations

(2.10) are satisfied [7], [27, §18.8], [29, §10.2]. Although some of these references concern only the vacuum equations, since the proof relies on the Bianchi identities, it is valid for the Einstein-Euler equations, whose energy-momentum tensor $T^{\alpha\beta}$ is divergence free.

Thus solving the constraint equations (2.10) ensures that the solution of (2.5) satisfies the original system (1.1). However, before we solve the constraints, we need to treat the compatibility problem between the matter variables (z, j^a) and the initial data for the velocity u^α and the Makino variable w.

This problem can be described as follows: Let $\bar{u}^\alpha$ denote the projection of the velocity vector u^α on the initial manifold $\mathcal{M}$ and n^α the timelike unit normal vector to $\mathcal{M}$. The energy density z is the double projection of $T_{\alpha\beta}$ on n^α and the momentum density j^a is once the projection of $T_{\alpha\beta}$ on n^α and once on $\mathcal{M}$. Applying these projections to the perfect fluid (1.2) results in

$$\begin{cases} z &= w^{\frac{2}{\gamma-1}}\left(1+(1+Kw^2)\right)h_{ab}\bar{u}^a\bar{u}^b \\ j^a &= w^{\frac{2}{\gamma-1}}\left(1+Kw^2\right)\bar{u}^a\sqrt{1+h_{bc}\bar{u}^b\bar{u}^c} \end{cases}. \tag{2.11}$$

So the compatibility problem consists of solving (2.11) for w, $\bar{u}^a$, when z and j^a are given. This problem combined with a solution to the constraint equations (2.10) was solved in the $H_{s,\delta}$-spaces in [4, Thm. 2.6]. The conditions for this result are that $\frac{3}{2} < s < \frac{2}{\gamma-1} + \frac{1}{2}$, where the metric $h_{ab} - \delta_{ab} \in H_{s+1,\delta}$ with $\delta \in (-\frac{3}{2}, -\frac{1}{2})$, while for the matter variables $(z, j^a) \in H_{s,\delta}$ and δ is just bounded below by $-\frac{3}{2}$. Note that for the hyperbolic equations we need one more degree of regularity, so we need to require that $(z, j^a) \in H_{s+1,\delta+1}$. But then the Makino variable (1.6) causes the upper bound for s to become $\frac{2}{\gamma-1} - \frac{1}{2}$. Given this restriction, we have by Theorem 2.5 of [4] and Proposition 4.9 below that $(w, u^0 - 1, u^a)\big|_{\mathcal{M}} \in H_{s+1,\delta+1}$.

Thus relying on [4], we conclude that there is an initial data set (h_{ab}, K_{ab}) and (w, u^α) belonging to the $H_{s,\delta}$-spaces that satisfies both the constraints (2.10) and the compatibility problem (2.11). The parameter γ of these initial data, however, belongs to the interval $(1, 2)$.

Corollary 2.5. *Under the assumptions of Theorem 2.3 and in addition under the assumption that the initial data (h_{ab}, K_{ab}) and (w, u^α) satisfy the constraint equations (2.10) and the compatibility problem (2.11), there exists a positive T, a semi-metric g, a unit timelike vector u^α and w satisfying the Einstein (1.1) and the Euler equations (1.4) for $t \in [0, T]$. The regularity of g, u^α and w are the same as in Theorem 2.3.*

Remark 2.6 (Existence, uniqueness and regularity). Existence, uniqueness and regularity of solutions of the Einstein equations (1.1) hold relative to the harmonic coordinate condition (2.3). Geometrical uniqueness requires usually one degree more of differentiability [11]. Planchon and Rodnianski [23](see also [9]) gave an argument for the vacuum case to get rid of this additionally regularity. For the Einstein–Euler system, and other matter fields, however, this problem remains still open.

3. Symmetric Hyperbolic Systems

The main result is proved by transforming the coupled system (2.5) and (1.4) into a symmetric hyperbolic system. We therefore recall its definition.

Definition 3.1 (Symmetric hyperbolic system). *A first order quasi-linear $k \times k$ system is a* **symmetric hyperbolic system** *in a region $G \subset \mathbb{R}^k$, if it is of the form*

$$L[U] = A^{\alpha}(U)\partial_{\alpha}U + B(U) = 0, \tag{3.1}$$

where the matrices $A^{\alpha}(U)$ are symmetric and for every arbitrary $U \in G$, there exists a covector ξ such that

$$\xi_{\alpha}A^{\alpha}(U) \tag{3.2}$$

is positive definite. The covectors ξ_{α} for which (3.2) is positive definite, are called **timelike with respect to Eq. (3.1)**.

If ξ can be chosen to be the vector $(1, 0, 0, 0)$, then condition (3.2) implies that the matrix $A^0(U)$ is a positive definite matrix, and we may write system (3.1) in the form

$$A^0(U)\partial_t U = A^a(U)\partial_a U + B(U). \tag{3.3}$$

3.1. The Euler equations written as a symmetric hyperbolic system. It is not obvious that the Euler equations written in the conservative form $\nabla_{\alpha}T^{\alpha\beta} = 0$ are symmetric hyperbolic. In fact these equations have to be transformed in order to be expressed in a symmetric hyperbolic form. Rendall presented such a transformation of these equations in [24]; however, its geometrical meaning is not entirely clear and it might be difficult to generalize it to the non-time symmetric case. Hence we will present a different hyperbolic reduction of the Euler equations and discuss it in some details, for we have not seen it anywhere in the literature.

The basic idea is to perform the standard *fluid decomposition* and then to modify the equation by adding, in an appropriate manner, the normalization condition (1.3) which will be considered as a constraint equation. The fluid decomposition method consists of the projection of equation $\nabla_{\nu}T^{\nu\beta} = 0$ onto u^{α} which leads to $u_{\beta}\nabla_{\nu}T^{\nu\beta} = 0$ and the projection of these equations onto the rest subspace $\mathcal{O}$ orthogonal to u^{α} of a fluid which leads to $P_{\alpha\beta}\nabla_{\nu}T^{\nu\beta} = 0$, where $P_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha}u_{\beta}$. Inserting this decomposition into (1.2) results in a system of the following form:

$$u^{\nu}\nabla_{\nu}\epsilon + (\epsilon + p)\nabla_{\nu}u^{\nu} = 0; \tag{3.4a}$$

$$(\epsilon + p)P_{\alpha\beta}u^{\nu}\nabla_{\nu}u^{\beta} + P^{\nu}{}_{\alpha}\nabla_{\nu}p = 0. \tag{3.4b}$$

Note that we have beside the evolution equations (3.4a) and (3.4b) the following constraint equation: $g_{\alpha\beta}u^{\alpha}u^{\beta} = -1$. We will show in Subsect. 3.1.1 that this constraint equation is conserved under the evolution equation. In order to obtain a symmetric hyperbolic system we have to modify it in the following way. The normalization condition (1.3) gives that $u_{\beta}u^{\nu}\nabla_{\nu}u^{\beta} = 0$, so we add $(\epsilon + p)u_{\beta}u^{\nu}\nabla_{\nu}u^{\beta} = 0$ to Eq. (3.4a) and $u_{\alpha}u_{\beta}u^{\nu}\nabla_{\nu}u^{\beta} = 0$ to (3.4b), which results in

$$u^{\nu}\nabla_{\nu}\epsilon + (\epsilon + p)P^{\nu}{}_{\beta}\nabla_{\nu}u^{\beta} = 0, \tag{3.5a}$$

$$\Gamma_{\alpha\beta}u^{\nu}\nabla_{\nu}u^{\beta} + \frac{\sigma^2}{(\epsilon + p)}P^{\nu}{}_{\alpha}\nabla_{\nu}\epsilon = 0, \tag{3.5b}$$

where $\sigma := \sqrt{\frac{\partial p}{\partial \epsilon}}$ is the speed of sound and

$$\Gamma_{\alpha\beta} = P_{\alpha\beta} + u_{\alpha}u_{\beta} = g_{\alpha\beta} + 2u_{\alpha}u_{\beta}$$

is a reflection with respect to the rest subspace $\mathcal{O}$. As mentioned above, we will introduce a new matter variable which is given by (1.6). The idea behind is the following: The system (3.5a) and (3.5b) is almost of symmetric hyperbolic form, it is symmetric if we multiply the system by appropriate factors, for example, (3.5a) by $\frac{\partial p}{\partial \epsilon} = \sigma^2$ and (3.5b) by $(\epsilon + p)$. However, doing so we will be faced with a system in which the coefficients will either tend to zero or to infinity, as $\epsilon \to 0$. Hence, it is impossible to represent this system in a non-degenerate form using these multiplications.

The central point is now to introduce a new variable $w = M(\epsilon)$ which will regularize the equations even for $\epsilon = 0$. We do this by multiplying Eq. (3.5a) by $\kappa^2 M' = \kappa^2 \frac{\partial M}{\partial \epsilon}$. This results in the following system which we have written in matrix form:

$$\left(\begin{array}{c|c} \kappa^2 u^\nu & \kappa^2(\epsilon + p) M' P^\nu{}_\beta \\ \hline \frac{\sigma^2}{(\epsilon+p)M'} P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array}\right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.6}$$

In order to obtain symmetry we have to demand that

$$M' = \frac{\sigma}{(\epsilon + p)\kappa}, \tag{3.7}$$

where $\kappa \gg 0$ has been introduced in order to simplify the expression for w. If we choose $\kappa = \frac{2}{\gamma - 1} \frac{\sqrt{K\gamma}}{1 + K\epsilon^{\gamma-1}}$, then (3.7) holds and consequently the system (3.6) is transformed into the symmetric system

$$\left(\begin{array}{c|c} \kappa^2 u^\nu & \sigma\kappa P^\nu{}_\beta \\ \hline \kappa\sigma P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array}\right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

The covariant derivative ∇_ν takes in local coordinates the form $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$, where Γ denotes the Christoffel symbols. This expresses the fact that system (3.8) is coupled to system (1.1) for the gravitational field $g_{\alpha\beta}$. In addition, from the definition of the Makino variable (1.6), we see that $\epsilon^{\gamma-1} = w^2$, so $\kappa = \frac{2}{\gamma-1} \frac{\sqrt{K\gamma}}{1+Kw^2}$ and $\sigma = \sqrt{\gamma K} w$. Thus the fractional power of the equation of state (1.5) does not appear in the coefficients of the system (3.8), and these coefficients are C^∞ functions of the scalar w, the four vector u^α and the gravitational field $g_{\alpha\beta}$.

Now we want to show that A^0 of our system (3.8) is indeed positive definite. In order to do that we analyze the principal symbol of this system. For each $\xi_\alpha \in T_x^* V$, the principal symbol is a linear map from $\mathbb{R} \times E_x$ to $\mathbb{R} \times F_x$, where E_x is a fiber in $T_x V$ and F_x is a fiber in the cotangent space $T_x^* V$. Since in local coordinates $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$, the principal symbol of system (3.8) is

$$\xi_\nu A^\nu = \left(\begin{array}{c|c} \kappa^2(u^\nu \xi_\nu) & \sigma\kappa P^\nu{}_\beta \xi_\nu \\ \hline \sigma\kappa P^\nu{}_\alpha \xi_\nu & (u^\nu \xi_\nu)\Gamma_{\alpha\beta} \end{array}\right). \tag{3.9}$$

The characteristics are the set of covectors ξ for which $(\xi_\nu A^\nu)$ is not an isomorphism. Hence the characteristics are the zeros of $Q(\xi) := \det(\xi_\nu A^\nu)$. The geometrical advantages of the fluid decomposition are the following. The operators in the blocks of the matrix (3.9) are the projection on the rest hyperplane $\mathcal{O}$, $P^{\nu}{}_{\alpha}$, and the reflection with respect to the same hyperplane, $\Gamma_{\alpha\beta}$. Therefore, the following relations hold:

$$\Gamma^{\alpha\gamma}\Gamma_{\gamma\beta} = \delta_{\beta}{}^{\alpha}, \qquad \Gamma^{\alpha\gamma}P_{\gamma}{}^{\nu} = P^{\alpha\nu} \qquad \text{and} \qquad P_{\beta}{}^{\alpha}P_{\alpha}{}^{\nu} = P^{\nu}{}_{\beta},$$

which yields

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array}\right)\left(\xi_{\nu}A^{\nu}\right) = \left(\begin{array}{c|c} \kappa^2(u^{\nu}\xi_{\nu}) & \sigma\kappa P^{\nu}{}_{\beta}\xi_{\nu} \\ \hline \sigma\kappa P^{\alpha\nu}\xi_{\nu} & (u^{\nu}\xi_{\nu})\left(\delta^{\alpha}_{\beta}\right) \end{array}\right). \tag{3.10}$$

It is now fairly easy to calculate the determinate of the right hand side of (3.10) and we have

$$\det\left(\begin{array}{c|c} \kappa^2(u^{\nu}\xi_{\nu}) & \sigma\kappa P^{\nu}{}_{\beta}\xi_{\nu} \\ \hline \sigma\kappa P^{\alpha\nu}\xi_{\nu} & (u^{\nu}\xi_{\nu})\left(\delta^{\alpha}_{\beta}\right) \end{array}\right) = \kappa^2(u^{\nu}\xi_{\nu})^3\left((u^{\nu}\xi_{\nu})^2 - \sigma^2P^{\alpha\nu}\xi_{\nu}P^{\nu}_{\alpha}\xi_{\nu}\right).$$

Since P^{α}_{β} is a projection,

$$P^{\alpha\nu}\xi_{\nu}P^{\nu}_{\alpha}\xi_{\nu} = g^{\nu\beta}\xi_{\nu}P^{\alpha}_{\beta}P^{\nu}_{\alpha}\xi_{\nu} = g^{\nu\beta}\xi_{\nu}P^{\nu}{}_{\beta}\xi_{\nu} = P^{\nu}{}_{\beta}\xi_{\nu}\xi^{\beta},$$

and since Γ^{γ}_{β} is a reflection,

$$\det\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array}\right) = \det\left(g^{\alpha\beta}\Gamma^{\gamma}_{\beta}\right) = -\left(\det\left(g_{\alpha\beta}\right)\right)^{-1}.$$

Consequently,

$$Q(\xi) := \det(\xi_{\nu}A^{\nu}) = -\kappa^2\det(g_{\alpha\beta})(u^{\nu}\xi_{\nu})^3\left\{(u^{\nu}\xi_{\nu})^2 - \sigma^2P^{\alpha}{}_{\beta}\xi_{\alpha}\xi^{\beta}\right\}, \tag{3.11}$$

and therefore the characteristic covectors are given by two simple equations:

$$\xi_{\nu}u^{\nu} = 0; \tag{3.12}$$

$$(\xi_{\nu}u^{\nu})^2 - \sigma^2P^{\alpha}{}_{\beta}\xi_{\alpha}\xi^{\beta} = 0. \tag{3.13}$$

Remark 3.2. The characteristics conormal cone is a union of two hypersurfaces in T^*_xV. One of these hypersurfaces is given by the condition (3.12) and it is a three dimensional hyperplane $\mathcal{O}$ with the normal u^{α}. The other hypersurface is given by the condition (3.13) and forms a three dimensional cone, the so-called, *sound cone*.

Let us now consider the timelike vector u_{ν} and insert the covector $-u_{\nu}$ into the principal symbol (3.9), then

$$-u_{\nu}A^{\nu} = \left(\begin{array}{c|c} \kappa^2 & 0 \\ \hline 0 & \Gamma_{\alpha\beta} \end{array}\right)$$

is positive definite. Indeed, $\Gamma_{\alpha\beta}$ is a reflection with respect to a hyperplane having a timelike normal. Hence, $-u_{\nu}$ is a timelike covector with respect to the hydrodynamic equations (3.8). Herewith, we have showed by relatively elegant and elementary methods that the relativistic hydrodynamic equations are symmetric-hyperbolic. We want now to show that the covector $t_\alpha = (1, 0, 0, 0)$ is also timelike with respect to the system (3.8). Since $P^\alpha{}_\beta u_\alpha = 0$, the covector $-u_\nu$ belongs to the sound cone

$$(\xi_\nu u^\nu)^2 - \sigma^2 P^\alpha{}_\beta \xi_\alpha \xi^\beta > 0. \tag{3.14}$$

Inserting $t_\nu = (1, 0, 0, 0)$ the right hand side of (3.14) yields

$$(u^0)^2(1 - \sigma^2) - \sigma^2 g^{00}. \tag{3.15}$$

Since the sound velocity is always less than the light speed, that is $\sigma^2 = \frac{\partial p}{\partial \epsilon} < c^2 = 1$, we conclude from (3.15) that t_ν also belongs to the sound cone (3.14). Hence, the vector $-u_\nu$ can be continuously deformed to t_ν while condition (3.14) holds along the deformation path. Consequently, the determinant of (3.11) remains positive under this process and hence $t_\nu A^\nu = A^0$ is also positive definite. Thus we have proved.

Theorem 3.3. *Let ϵ be a non-negative density function, then the Euler system (1.4) coupled with the equation of state (1.5) can be written as a symmetric hyperbolic system of the form (3.3), and where A^0 is positive definite.*

3.1.1. Conservation of the unit length vector of the fluid.

Proposition 3.4. *The constraint condition $g_{\alpha\beta}u^\alpha u^\beta = -1$ is conserved along the stream lines u^α.*

Proof. Let $k(t)$ be a curve such that $k'(t) = u^\alpha$ and set $Z(t) = (u \circ k)_\beta (u \circ k)^\beta$. In order to establish the conservation of the constraint condition it suffices to establish the following relation:

$$\frac{d}{dt} Z(t) = 2u_\beta \nabla_{k'(t)} u^\beta = 2u^\nu u_\beta \nabla_\nu u^\beta = 0.$$

Multiplying the four last rows of the Euler system (3.8) by u^α and recalling that $P^\nu{}_\alpha$ is the projection on the rest space $\mathcal{O}$ orthogonal to u^α, we have

$$\begin{aligned} 0 &= u^\alpha \left(\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \kappa\sigma P^\nu{}_\alpha \nabla_\nu w\right) \\ &= u^\alpha P_{\alpha\beta} u^\nu \nabla_\nu u^\beta - u^\nu u_\beta \nabla_\nu u^\beta + \kappa\sigma u^\alpha P^\nu{}_\alpha \nabla_\nu w \\ &= -u^\nu u_\beta \nabla_\nu u^\beta. \end{aligned}$$

Therefore, if $g_{\alpha\beta}u^\alpha u^\beta = -1$ on the initial manifold, then it holds along the stream lines u^α. □

3.2. The coupled hyperbolic system. In this section we will transform the coupled system (2.5) and (3.8) into a symmetric hyperbolic system. We will pay attention to the fact that the system will be in a form in which we can apply the energy estimates of [16]. That allows us to obtain the same regularity for the gravitational fields as Hughes, Kato and Marsden [14] got for the Einstein vacuum equations. Note that our system is slightly different from the symmetric hyperbolic system obtained by Fisher and Marsden [11], since our system contains a constant matrix $\mathcal{C}^a$ as given by (3.21).

We consider a spacetime $(V, g_{\alpha\beta})$ of the type $\mathbb{R} \times \mathcal{M}$, where $\mathcal{M}$ is a Riemannian manifold, and we denote local coordinates by (t, x^a). Set

$$h_{\alpha\beta\gamma} = \partial_\gamma g_{\alpha\beta},$$

then the reduced Einstein equations (2.5) take the form

$$\begin{aligned} \partial_t g_{\alpha\beta} &= h_{\alpha\beta 0}, \\ -g^{00}\partial_t h_{\alpha\beta 0} &= \Big\{ 2g^{0a}\partial_a h_{\alpha\beta 0} + g^{ab}\partial_a h_{\alpha\beta b} + H_{\alpha\beta}(g, \partial g) \\ &\quad - 8\pi w^{\frac{2}{\gamma-1}} \Big((1 - Kw^2) g_{\alpha\beta} + 2(1 + Kw^2) u_\alpha u_\beta \Big) \Big\}, \\ g^{ab}\partial_t h_{\alpha\beta a} &= g^{ab}\partial_a h_{\alpha\beta 0}. \end{aligned} \tag{3.16}$$

In order to apply the energy estimates of [16], we need that the coefficients of $\partial_t h_{\alpha\beta 0}$ will be independent of t. This is because of the specific form of the inner-product in $H_{s,\delta}$ spaces which takes into account the matrix A^0 of the system (3.1). In Sect. 5 we will further clarify this issue. Therefore we divide the second row by $-g^{00}$ and in order to preserve the symmetry of the system, we also multiply the third row by $(-g^{00})^{-1}$. Thus the system of wave equations (2.5) is equivalent to the system

$$\begin{aligned} \partial_t g_{\alpha\beta} &= h_{\alpha\beta 0}, \\ \partial_t h_{\alpha\beta 0} &= (-g^{00})^{-1} \Big\{ 2g^{0a}\partial_a h_{\alpha\beta 0} + g^{ab}\partial_a h_{\alpha\beta b} + H_{\alpha\beta}(g, \partial g) \\ &\quad - 8\pi w^{\frac{2}{\gamma-1}} \Big((1 - Kw^2) g_{\alpha\beta} + 2(1 + Kw^2) u_\alpha u_\beta \Big) \Big\}, \\ (-g^{00})^{-1} g^{ab}\partial_t h_{\alpha\beta a} &= (-g^{00})^{-1} g^{ab}\partial_a h_{\alpha\beta 0}. \end{aligned} \tag{3.17}$$

To shorten and simplify the notation, we introduce the auxiliary variables

$$(v, \partial_t v, \partial_x v) = (g_{\alpha\beta} - \eta_{\alpha\beta}, \partial_t g_{\alpha\beta}, \partial_x g_{\alpha\beta}),$$

where $\eta_{\alpha\beta}$ denotes the Minkowski metric and ∂_x denotes the set of all spatial derivatives. We also set $(e_0)^\alpha = (1, 0, 0, 0)$ and $W = (w, u^\alpha - e_0^\alpha)$ represents the Makino and the fluid variables. Finally,

$$U = (v, \partial_t v, \partial_x v, W)$$

represents the unknowns of the coupled system.

We write the matrices in a block form, $A = (\mathbf{a}_{ij})$, the $k \times k$ identity matrix is denoted by $\mathbf{e}_k$ and $\mathbf{0}_{m\times n}$ is the zero matrix.

The coupled system (3.17) and (3.8) can be written in the form of (3.1), where $\mathcal{A}^\alpha$ are 55×55 symmetric matrices which depend only on v and W. We shall describe now the structure of these matrices:

$$\mathcal{A}^0(v, W) = \begin{pmatrix} \mathbf{e}_{10} & \mathbf{0}_{10\times 10} & \mathbf{0}_{10\times 30} & \mathbf{0}_{10\times 5} \\ \mathbf{0}_{10\times 10} & \mathbf{e}_{10} & \mathbf{0}_{10\times 30} & \mathbf{0}_{10\times 5} \\ \mathbf{0}_{30\times 10} & \mathbf{0}_{30\times 10} & \mathbf{a}_{33}^0 & \mathbf{0}_{30\times 5} \\ \mathbf{0}_{5\times 10} & \mathbf{0}_{5\times 10} & \mathbf{0}_{5\times 30} & \mathbf{a}_{44}^0 \end{pmatrix}, \tag{3.18}$$

where

$$\mathbf{a}_{33}^0 = \frac{1}{-g^{00}} \begin{pmatrix} g^{11}\mathbf{e}_{10} & g^{12}\mathbf{e}_{10} & g^{13}\mathbf{e}_{10} \\ g^{21}\mathbf{e}_{10} & g^{22}\mathbf{e}_{10} & g^{23}\mathbf{e}_{10} \\ g^{31}\mathbf{e}_{10} & g^{32}\mathbf{e}_{10} & g^{33}\mathbf{e}_{10} \end{pmatrix},$$

and $\mathbf{a}_{44}^0 = \mathbf{a}_{44}^0(g_{\alpha\beta}, w, u^\alpha)$ is given by (3.8) when $\nu = 0$. From (3.17) we see that the coefficients of $\partial_a U$, $a = 1, 2, 3$, have the form

$$\begin{pmatrix} \mathbf{0}_{10\times 10} & \mathbf{0}_{10\times 40} & \mathbf{0}_{5\times 5} \\ \mathbf{0}_{40\times 10} & \mathbf{a}_{22}^a & \mathbf{0}_{40\times 5} \\ \mathbf{0}_{5\times 10} & \mathbf{0}_{5\times 40} & \mathbf{a}_{33}^a \end{pmatrix},$$

where $\mathbf{a}_{33}^a = \mathbf{a}_{33}^a(g_{\alpha\beta}, w, u^\alpha)$ is from the system (3.8) of the fluid and

$$\mathbf{a}_{22}^a(g_{\alpha\beta}) = \frac{1}{g^{00}} \left(\begin{array}{c|ccc} 2g^{a0}\mathbf{e}_{10} & g^{a1}\mathbf{e}_{10} & g^{a2}\mathbf{e}_{10} & g^{a3}\mathbf{e}_{10} \\ \hline g^{a1}\mathbf{e}_{10} & & & \\ g^{a2}\mathbf{e}_{10} & & \mathbf{0}_{30\times 30} & \\ g^{a3}\mathbf{e}_{10} & & & \end{array}\right). \tag{3.19}$$

It is essential to demand that $\mathbf{a}_{22}^a(g_{\alpha\beta}) \in H_{s,\delta}$, whenever $g_{\alpha\beta} - \eta_{\alpha\beta} \in H_{s,\delta}$. Obviously, this does not hold for the matrix in (3.19). Therefore we need to modify these matrices by a constant matrix

$$\mathbf{c}_{22}^a = \left(\begin{array}{c|ccc} \mathbf{0}_{10\times 10} & \delta^{a1}\mathbf{e}_{10} & \delta^{a2}\mathbf{e}_{10} & \delta^{a3}\mathbf{e}_{10} \\ \hline \delta^{a1}\mathbf{e}_{10} & & & \\ \delta^{a2}\mathbf{e}_{10} & & \mathbf{0}_{30\times 30} & \\ \delta^{a3}\mathbf{e}_{10} & & & \end{array}\right),$$

then $\left(\mathbf{a}_{22}^a - \mathbf{c}_{22}^a\right)(v) \in H_{s,\delta}$ whenever $v \in H_{s,\delta}$. So we set

$$\mathcal{A}^a(v, W) = \begin{pmatrix} \mathbf{0}_{10\times 10} & \mathbf{0}_{10\times 40} & \mathbf{0}_{5\times 5} \\ \mathbf{0}_{40\times 10} & \mathbf{a}_{22}^a - \mathbf{c}_{22}^a & \mathbf{0}_{40\times 5} \\ \mathbf{0}_{5\times 10} & \mathbf{0}_{5\times 40} & \mathbf{a}_{33}^a \end{pmatrix}, \tag{3.20}$$

and a constant matrix

$$\mathcal{C}^a = \begin{pmatrix} \mathbf{0}_{10\times 10} & \mathbf{0}_{10\times 40} & \mathbf{0}_{5\times 5} \\ \mathbf{0}_{40\times 10} & \mathbf{c}_{22}^a & \mathbf{0}_{40\times 5} \\ \mathbf{0}_{5\times 10} & \mathbf{0}_{5\times 40} & \mathbf{0}_{5\times 5} \end{pmatrix}. \tag{3.21}$$

We turn now to the lower order terms. The presence of the fractional power $w^{2/(\gamma-1)}$ in (3.17) causes substantial technical difficulties. We set

$$f(v, W) := -\frac{8\pi w^{\frac{2}{\gamma-1}}}{g^{00}} \left((1 - Kw^2)g_{\alpha\beta} + 2(1 + Kw^2)u_\alpha u_\beta\right), \tag{3.22}$$

then we can write $B(U)$ in the form

$$B(U) = \mathcal{B}(U)(v, \partial_t v, \partial_x v)^T + \mathcal{F}(v, W),$$

where $\mathcal{F}(v, W) = (0, f(v, W), 0, 0)^T$ and

$$\mathcal{B}(U) = \begin{pmatrix} \mathbf{0}_{10\times 10} & \mathbf{e}_{10} & \mathbf{0}_{10\times 10} & \mathbf{0}_{10\times 10} & \mathbf{0}_{10\times 10} \\ \mathbf{0}_{10\times 10} & \mathbf{b}_{22} & \mathbf{b}_{23} & \mathbf{b}_{24} & \mathbf{b}_{25} \\ \mathbf{0}_{30\times 10} & \mathbf{0}_{30\times 10} & \mathbf{0}_{30\times 10} & \mathbf{0}_{30\times 10} & \mathbf{0}_{30\times 10} \\ \mathbf{0}_{5\times 10} & \mathbf{b}_{42} & \mathbf{b}_{43} & \mathbf{b}_{44} & \mathbf{b}_{45} \end{pmatrix}. \tag{3.23}$$

The block $\mathbf{b}_{2j}$, $j = 2, 3, 4, 5$, appears from the quadratic terms in (2.4):

$$H_{\alpha\beta}(g, \partial g) = C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\alpha\beta\gamma\delta\rho\sigma} h_{\epsilon\zeta\eta} h_{\kappa\lambda\mu} g^{\gamma\delta} g^{\rho\sigma},$$

where the term $C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\gamma\delta\alpha\beta\rho\sigma}$ is a combination of Kronecker deltas with integer coefficients. Thus

$$\mathbf{b}_{2j} = (-g^{00})^{-1} C^{\epsilon\zeta\eta\kappa\lambda\mu}_{\alpha\beta\gamma\delta\rho\sigma} h_{\epsilon\zeta\eta} g^{\gamma\delta} g^{\rho\sigma}, \quad \mu = j - 2.$$

The block $\mathbf{b}_{4j}$, $j = 2, 3, 4, 5$, appears from the multiplication of the reflection $\Gamma_{\alpha\beta}$ and u^ν in (3.8) with the Christoffel symbols. So its coefficients consist of multiplications of $g_{\alpha\beta}$, $g^{\alpha\beta}$ and u^ν.

In summary, we can write the coupled systems (3.17) and (3.8) as a symmetric hyperbolic system

$$\mathcal{A}^0(v, W)\partial_t U = \left(\left(\mathcal{A}^a(v, W) + \mathcal{C}^a\right) \partial_a U + \mathcal{B}(U) \begin{pmatrix} v \\ \partial_t v \\ \partial_x v \end{pmatrix} + \mathcal{F}(v, W), \tag{3.24}$$

where $\mathcal{A}^0(U)$ is positive definite in the neighborhood of the initial data (2.6), $\mathcal{A}^0(0) - \mathbf{e}_{55} = \mathcal{A}^a(0) = 0$ and $\mathcal{C}^a$ is a constant symmetric matrix.

4. The $H_{s,\delta}$ Spaces and Their Properties

The definition of the weighted Sobolev spaces of fractional order $H_{s,\delta}$, Definition 2.1, is due to Trieble [28]. Here we quote the propositions and properties which are needed for the proof of the main result. For their proofs see [4,20,28].

We start with some notations.

- Let $\{\psi_j\}$ be the sequence of functions in Definition 2.1. For any positive γ we set

$$\|u\|^2_{H_{s,\delta,\gamma}} = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^\gamma u)_{2^j}\|^2_{H^s}, \tag{4.1}$$

and we will use the convention $\|u\|_{H_{s,\delta,1}} = \|u\|_{H_{s,\delta}}$. The subscripts 2^j mean a scaling by 2^j, that is, $(\psi_j^\gamma u)_{2^j}(x) = (\psi_j^\gamma u)(2^j x)$.

- For a non-negative integer m and $\beta \in \mathbb{R}$, the space C^m_β is the set of all functions having continuous partial derivatives up to order m and such that the norm
$$\|u\|_{C^m_\beta} = \sum_{|\alpha|\leq m} \sup_{\mathbb{R}^3} \Big((1+|x|)^{\beta+|\alpha|}|\partial^\alpha u(x)|\Big) \tag{4.2}$$
is finite.
- We will use the notation $A \lesssim B$ to denote an inequality $A \leq CB$ where the positive constant C does not depend on the parameters in question.

We recall that $\{\psi_j\}$ are cutoff functions, hence $\psi_j^\beta \in C_0^\infty(\mathbb{R}^3)$ for any positive β. Furthermore, for a given α, there are two constants $C_1(\beta,\alpha)$ and $C_2(\beta,\alpha)$ such that
$$C_1(\beta,\alpha)|\partial^\alpha \psi_j(x)| \leq |\partial^\alpha \psi_j^\beta(x)| \leq C_2(\beta,\alpha)|\partial^\alpha \psi_j(x)|,$$
and these inequalities are independent of j. Therefore Proposition 4.1 below is a consequence of [28, Thm. 1].

Proposition 4.1. *For any positive γ, there are two positive constants $c_0(\gamma)$ and $c_1(\gamma)$ such that*
$$c_0(\gamma)\|u\|_{H_{s,\delta}} \leq \|u\|_{H_{s,\delta,\gamma}} \leq c_1(\gamma)\|u\|_{H_{s,\delta}}.$$

Proposition 4.2. *For any nonnegative integer m, positive γ and δ,*
$$\|u\|^2_{H_{m,\delta,\gamma}} \lesssim \|u\|^2_{m,\delta} \lesssim \|u\|^2_{H_{m,\delta,\gamma}} \textit{holds},$$
where $\|u\|_{m,\delta}$ is defined by (1.7).

Proposition 4.3. *If $s_1 \leq s_2$ and $\delta_1 \leq \delta_2$, then*
$$\|u\|_{H_{s_1,\delta_1}} \leq \|u\|_{H_{s_2,\delta_2}}.$$

Proposition 4.4. *If $u \in H_{s,\delta}$, then*
$$\|\partial_i u\|_{H_{s-1,\delta+1}} \leq \|u\|_{H_{s,\delta}}.$$

Proposition 4.5. *Let $s_1, s_2 \geq s$, $s_1+s_2 > s+\frac{3}{2}$, $s_1+s_2 \geq 0$ and $\delta_1+\delta_2 \geq \delta-\frac{3}{2}$. If $u \in H_{s_1,\delta_1}$ and $v \in H_{s_2,\delta_2}$, then*
$$\|uv\|_{H_{s,\delta}} \lesssim \|u\|_{H_{s_1,\delta_1}} \|v\|_{H_{s_2,\delta_2}}. \tag{4.3}$$

Remark 4.6. If for a fixed constant c_0, $u-c_0 \in H_{s_1,\delta_1}$ and $v \in H_{s_2,\delta_2}$, then we can apply the multiplication property (4.3) to $(u-c_0)v$ and obtain
$$\|uv\|_{H_{s,\delta}} \lesssim \Big(\|u-c_0\|_{H_{s_1,\delta_1}} + |c_0|\Big) \|v\|_{H_{s_2,\delta_2}}.$$

Proposition 4.7. *Let $u \in H_{s,\delta} \cap L^\infty$, $1<\beta$, $0<s<\beta+\frac{1}{2}$ and $\delta \in \mathbb{R}$, then*
$$\||u|^\beta\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty})\|u\|_{H_{s,\delta}}.$$

Proposition 4.8. *If $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \geq \beta$, then*

$$\|u\|_{C^m_\beta} \lesssim \|u\|_{H_{s,\delta}}, \tag{4.4}$$

where $\|u\|_{C^m_\beta}$ is given by (4.2).

Proposition 4.9. *Let $F : \mathbb{R}^m \to \mathbb{R}^l$ be a C^{N+1}-function such that $F(0) = 0$ and where $N \geq [s] + 1$. Then there is a constant C such that for any $u \in H_{s,\delta}$,*

$$\|F(u)\|_{H_{s,\delta}} \leq C\|F\|_{C^{N+1}} \left(1 + \|u\|^N_{L^\infty}\right) \|u\|_{H_{s,\delta}}. \tag{4.5}$$

Proposition 4.10. (a) *The class $C^\infty_0(\mathbb{R}^3)$ is dense in $H_{s,\delta}$.*
(b) *Given $u \in H_{s,\delta}$, $s' > s \geq 0$ and $\delta' \geq \delta$. Then for $\rho > 0$ there is $u_\rho \in C^\infty_0(\mathbb{R}^3)$ and a positive constant $C(\rho)$ such that*

$$\|u_\rho - u\|_{H_{s,\delta}} \leq \rho \quad \text{and} \quad \|u_\rho\|_{H_{s',\delta'}} \leq C(\rho)\|u\|_{H_{s,\delta}}.$$

4.1. Product spaces. The unknown of system (3.24) is a vector valued function

$$U = (v, \partial_t v, \partial_x v, W),$$

where $v = g_{\alpha\beta} - \eta_{\alpha\beta}$ stands for the field variables and $W = (w, u^\alpha - e^\alpha_0)$ stands for the fluid variables. We consider them in the space

$$X_{s,\delta} := H_{s,\delta} \times H_{s,\delta+1} \times H_{s,\delta+1} \times H_{s+1,\delta+1}, \tag{4.6}$$

with the norm [see (4.1)]

$$\|U\|^2_{X_{s,\delta}} = \|v\|^2_{H_{s,\delta,2}} + \|\partial_t v\|^2_{H_{s,\delta+1,2}} + \|\partial_x v\|^2_{H_{s,\delta+1,2}} + \|W\|^2_{H_{s+1,\delta+1,2}}. \tag{4.7}$$

Remark 4.11. Note that if $U = (v, \partial_t v, \partial_x v, W) \in X_{s,\delta}$, then $v \in H_{s+1,\delta}$. Because $U \in X_{s,\delta}$ implies that $v \in H_{s,\delta}$ and $\partial_x v \in H_{s,\delta+1}$, so by the integral representation of the norm $H_{s,\delta}$ (see [4, §2)]), we obtain that

$$\|v\|_{H_{s+1,\delta}} \lesssim \left(\|v\|_{H_{s,\delta}} + \|\partial_x v\|_{H_{s,\delta+1}}\right).$$

5. Energy Estimates

In this section we will derive the energy estimates for a linear symmetric hyperbolic system, a system which we have obtained by linearising (3.24). So we consider

$$\mathcal{A}^0 \partial_t U = \left(\mathcal{A}^a + \mathcal{C}^a\right) \partial_a U + \mathcal{B} \begin{pmatrix} v \\ \partial_t v \\ \partial_x v \end{pmatrix} + \mathcal{F} + \mathcal{D}, \tag{5.1}$$

where $U = (v, \partial_t v, \partial_x v, W)$, the matrices $\mathcal{A}^0$, $\mathcal{A}^a$, $\mathcal{B}$ and $\mathcal{C}^a$ have the same structural form as the corresponding matrices in (3.24), $\mathcal{C}^a$ is a constant matrix, and the vectors $\mathcal{F}$ and $\mathcal{D}$ have the form $(0, f, 0, 0)$ and $(0, d_2, d_3, d_4)$ respectively.

Assumption 5.1. All the matrices have the same block structure as (3.18), (3.20) and (3.23) and satisfy:

$$\left(\mathcal{A}^0(t,\cdot)-\mathbf{e}_{55}\right),\ \mathcal{A}^a(t,\cdot)\in H_{s+1,\delta}; \tag{5.2a}$$

$$\exists\, c_0\geq 1 \text{ such that } {c_0}^{-1}V^TV\leq V^T\mathcal{A}^0V\leq c_0V^TV,\quad \forall V\in\mathbb{R}^{55}; \tag{5.2b}$$

$$\partial_t\mathcal{A}^0(t,\cdot)\in L^\infty; \tag{5.2c}$$

$$\mathbf{b}_{2j}(t,\cdot),\mathbf{b}_{4j}(t,\cdot)\in H_{s,\delta+1},\quad j=2,3,4,5; \tag{5.2d}$$

$$\mathcal{F}(t,\cdot),\mathcal{D}(t,\cdot)\in H_{s,\delta+1}. \tag{5.2e}$$

5.1. $X_{s,\delta}$-energy estimates. We turn now to the definition of an inner-product of the space $X_{s,\delta}$ which takes into account the structure of the matrix $\mathcal{A}^0$ of the system (5.1).

Let $F(u)$ denote the Fourier transform of a distribution u, then we set

$$\Lambda^s(u)=(1-\Delta)^{\frac{s}{2}}(u)=F^{-1}\left(\left(1+|\xi|^2\right)^{\frac{s}{2}}F\right)(u).$$

The standard inner-product of the Bessel-potential spaces H^s is

$$\langle u_1,u_2\rangle_s=\langle\Lambda^s(u_1),\Lambda^s(u_2)\rangle_{L^2}.$$

Taking into account the term-wise definition of the norm (2.1), we define the inner-product of $H_{s,\delta}$ as follows:

$$\langle u_1,u_2\rangle_{s,\delta}:=\sum_{j=0}^{\infty}2^{\left(\delta+\frac{3}{2}\right)2j}\left\langle\Lambda^s\left(\psi_j^2u_1\right)_{2^j},\Lambda^s\left(\psi_j^2u_2\right)_{2^j}\right\rangle_{L^2}, \tag{5.3}$$

where $(u)_{2^j}$ denotes scaling by 2^j. By Proposition 4.1, $\langle u,u\rangle_{s,\delta}=\|u\|^2_{H_{s,\delta,2}}\simeq\|u\|^2_{H_{s,\delta}}$. To each component of the space

$$X_{s,\delta}:=H_{s,\delta}\times H_{s,\delta+1}\times H_{s,\delta+1}\times H_{s+1,\delta+1}$$

we assign its own inner-product. Since $\mathcal{A}^0=(\mathbf{a}^0_{ij})$, where $\mathbf{a}^0_{ij}$ is the zero matrix for $i\neq j$, $\mathbf{a}^0_{ii}$ is the identity for $i=1,2$, we assign to the first two components the inner-product (5.3), while for the other terms we insert $\mathcal{A}^0$ termwise.

Definition 5.2 (Inner-product in $X_{s,\delta}$). *Let $U_i=(v_i,\partial_tv_i,\partial_xv_i,W_i)\in X_{s,\delta},\ i=1,2$ and assume that the matrix $\mathcal{A}^0$ satisfies Assumption 5.1, then we denote the inner-product of $X_{s,\delta}$ by*

$$\begin{aligned}\langle U_1,U_2\rangle_{X_{s,\delta,\mathcal{A}^0}}:=&\ \langle v_1,v_2\rangle_{s,\delta}+\langle\partial_tv_1,\partial_tv_2\rangle_{s,\delta+1}\\&+\langle\partial_xv_1,\partial_xv_2\rangle_{s,\delta+1,\mathbf{a}^0_{33}}+\langle W_1,W_2\rangle_{s+1,\delta+2,\mathbf{a}^0_{44}},\end{aligned} \tag{5.4}$$

where the terms are defined in the following way:

- *An inner-product of $H_{s,\delta}$ of the form: $\langle v_1,v_2\rangle_{s,\delta}$, where the inner-product $\langle\cdot,\cdot\rangle_{s,\delta}$ is defined by (5.3);*
- *An inner-product of $H_{s,\delta+1}$ of the form: $\langle\partial_tv_1,\partial_tv_2\rangle_{s,\delta+1}$, where $\langle\cdot,\cdot\rangle_{s,\delta+1}$ is defined by (5.3) with $\delta+1$;*

- *An inner-product on $H_{s,\delta+1}$ of the form:*

$$\langle \partial_x v_1, \partial_x v_2 \rangle_{s,\delta+1,\mathbf{a}^0_{33}} := \sum_{j=0}^{\infty} 2^{\left(\delta+1+\frac{3}{2}\right)2j} \left\langle \Lambda^s \left(\psi_j^2 \partial_x v_1\right)_{2^j}, \left(\mathbf{a}^0_{33}\right)_{2^j} \Lambda^s \left(\psi_j^2 \partial_x v_2\right)_{2^j} \right\rangle_{L^2}. \quad (5.5)$$

- *An inner-product of $H_{s+1,\delta+1}$ of the form:*

$$\langle W_1, W_2 \rangle_{s+1,\delta+2,\mathbf{a}^0_{44}} := \sum_{j=0}^{\infty} 2^{\left(\delta+1+\frac{3}{2}\right)2j} \left\langle \Lambda^{s+1} \left(\psi_j^2 W_1\right)_{2^j}, \left(\mathbf{a}^0_{44}\right)_{2^j} \Lambda^{s+1} \left(\psi_j^2 W_2\right)_{2^j} \right\rangle_{L^2}. \quad (5.6)$$

We denote by $\|U\|_{X_{s,\delta,\mathcal{A}^0}}$ the norm associated with the inner-product (5.4). Since the matrix $\mathcal{A}^0$ satisfies (5.2b), the following equivalence:

$$\|U\|_{X_{s,\delta}} \lesssim \|U\|_{X_{s,\delta,\mathcal{A}^0}} \lesssim \|U\|_{X_{s,\delta}} \quad (5.7)$$

holds. In order to simplify the notation we set $U(t) = U(t, x^1, x^2, x^3)$.

Lemma 5.3. *Let $s > \frac{3}{2}$, $\delta \geq -\frac{3}{2}$ and assume the coefficients of (5.1) satisfy Assumptions 5.1. If $U(t) \in C_0^\infty(\mathbb{R}^3)$ is a solution of (5.1), then*

$$\frac{d}{dt} \langle U(t), U(t) \rangle_{X_{s,\delta,\mathcal{A}^0}} \leq C c_0 \left(\langle U(t), U(t) \rangle_{X_{s,\delta,\mathcal{A}^0}} + 1 \right), \quad (5.8)$$

where the constant C depends on the corresponding norms of the coefficients, s and δ.

The corresponding energy estimates for the vacuum Einstein equations were obtained in [16]. The same techniques can be applied here with some obvious modifications. We therefore give only a short sketch of the proof.

Sketch of the proof. From the inner-product (5.4) we see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \langle U, U \rangle_{X_{s,\delta,\mathcal{A}^0}} &= \langle v, \partial_t v \rangle_{s,\delta} + \langle \partial_t v, \partial_t^2 v \rangle_{s,\delta+1} + \langle \partial_x v, \partial_x \partial_t v \rangle_{s,\delta+1,\mathbf{a}^0_{33}} \\ &+ \langle W, \partial_t W \rangle_{s+1,\delta+2,\mathbf{a}^0_{44}} \\ &+ \sum_{j=0}^{\infty} 2^{\left(\delta+1+\frac{3}{2}\right)2j} \left\langle \Lambda^s \left(\psi_j^2 \partial_x v\right)_{2^j}, \partial_t \left(\mathbf{a}^0_{33}\right)_{2^j} \Lambda^s \left(\psi_j^2 \partial_x v\right)_{2^j} \right\rangle_{L^2} \\ &+ \sum_{j=0}^{\infty} 2^{\left(\delta+2+\frac{3}{2}\right)2j} \left\langle \Lambda^{s+1} \left(\psi_j^2 W\right)_{2^j}, \partial_t \left(\mathbf{a}^0_{44}\right)_{2^j} \Lambda^{s+1} \left(\psi_j^2 W\right)_{2^j} \right\rangle_{L^2}. \end{aligned}$$

By the Cauchy Schwarz inequality, we obtain

$$|\langle v, \partial_t v \rangle_{s,\delta}| \leq \|v\|_{H_{s,\delta,2}} \|\partial_t v\|_{H_{s,\delta,2}} \leq \frac{1}{2} \left(\|v\|^2_{H_{s,\delta,2}} + \|\partial_t v\|^2_{H_{s,\delta+1,2}} \right),$$

and by Assumption (5.2c), the first infinite sum is less than

$$C\left\|\partial_t \mathbf{a}_{33}^0\right\|_{L^\infty}\sum_{j=0}^{\infty}2^{\left(\delta+1+\frac{3}{2}\right)2j}\left\langle\Lambda^s\left(\psi_j^2\partial_x v\right)_{2^j},\Lambda^s\left(\psi_j^2\partial_x v\right)_{2^j}\right\rangle_{L^2}$$
$$=C\left\|\partial_t \mathbf{a}_{33}^0\right\|_{L^\infty}\|\partial_x v\|^2_{H_{s,\delta+1,2}}.$$

A similar estimate holds for the second infinite sum. The most difficult part is the estimate

$$\left|\langle\partial_t v,\partial_t^2 v\rangle_{s,\delta+1}+\langle\partial_x v,\partial_x\partial_t v\rangle_{s,\delta+1,\mathbf{a}_{33}^0}\right|\lesssim\left(\|U\|^2_{X_{s,\delta}}+1\right), \tag{5.9}$$

and here it is essential to use the assumption that $\mathcal{A}^\alpha\in H_{s+1,\delta}$ and $s>\frac{3}{2}$. We present here the main ideas of this estimate and for a detailed proof we refer to [16, §4].

Let

$$E_{\partial_t}(j)=\left\langle\Lambda^s\left(\psi_j^2\partial_t v\right)_{2^j},\Lambda^s\left(\psi_j^2(\partial_t^2 v)\right)_{2^j}\right\rangle_{L^2} \tag{5.10}$$

and

$$E_{\partial_x}(j)=\left\langle\Lambda^s\left(\psi_j^2\partial_x v\right)_{2^j},\left(\mathbf{a}_{33}^0\right)_{2^j}\Lambda^s\left(\psi_j^2\partial_t(\partial_x v)\right)_{2^j}\right\rangle_{L^2}. \tag{5.11}$$

In order to use Eq. (5.1) we need to commute $\left(\mathbf{a}_{33}^0\right)_{2^j}$ both with the operator Λ^s and ψ_j^2. An essential ingredient is the Kato–Ponce commutator estimate (8.3) (see the Appendix). Usually this estimate is used in similar situations with the operator Λ^s, we, however, apply it to the pseudodifferential operator $P=\Lambda^s\partial_x$. This enables us to use estimate (8.3) with the index $s+1$ and to exploit the assumption that $\mathcal{A}^\alpha\in H^{s+1}_{\mathrm{loc}}$.

Let $\Psi_k=\left(\sum_{j=0}^\infty\psi_j\right)^{-1}\psi_k$, where $\{\psi_j\}$ is the dyadic resolution of the norm (2.1). Since $\operatorname{supp}(\Psi_k\psi_j)=\operatorname{supp}(\Psi_k)\cap\operatorname{supp}(\psi_j)$, $\Psi_k(x)\psi_j(x)\neq 0$ only when $k=j-3$, $\ldots, j+3$, and hence we have that

$$E_{\partial_x}(j)=\sum_{k=j-3}^{j+3}\left\langle\Lambda^s\left(\psi_j^2\partial_x v\right)_{2^j},\left(\mathbf{a}_{33}^0\right)_{2^j}\Lambda^s\left(\Psi_k\psi_j^2\partial_t(\partial_x v)\right)_{2^j}\right\rangle_{L^2}$$
$$=:\sum_{k=j-3}^{j+3}E_{\partial_x}(j,k), \tag{5.12}$$

and similarly

$$E_{\partial_t}(j)=\sum_{k=j-3}^{j+3}\left\langle\Lambda^s\left(\psi_j^2\partial_t v\right)_{2^j},\Lambda^s\left(\Psi_k\psi_j^2\partial_t(\partial_t v)\right)_{2^j}\right\rangle_{L^2}=:\sum_{k=j-3}^{j+3}E_{\partial_t}(j,k). \tag{5.13}$$

Writing

$$\left(\Psi_k\psi_j^2\partial_t(\partial_x v)\right)_{2^j}=2^{-j}\partial_x\left(\Psi_k\psi_j^2\partial_t v\right)_{2^j}-\partial_x\left(\Psi_k\psi_j^2\right)_{2^j}(\partial_t v)_{2j}, \tag{5.14}$$

and

$$\Lambda^s\left(\partial_x\left(\Psi_k\psi_j^2\partial_t v\right)_{2^j}\right)=\left(\Lambda^s\partial_x\right)\left(\Psi_k\psi_j^2\partial_t v\right)_{2^j}-(\Psi_k)_{2^j}\left(\Lambda^s\partial_x\right)\left(\left(\psi_j^2\partial_t v\right)_{2^j}\right)$$
$$+(\Psi_k)_{2^j}\left(\Lambda^s\partial_x\right)\left(\left(\psi_j^2\partial_t v\right)_{2^j}\right), \tag{5.15}$$

we have that

$$E_{\partial_x}(j,k) = 2^{-j}\left\langle\Lambda^s\left(\psi_j^2\partial_x v\right)_{2^j},\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\left(\Lambda^s\partial_x\right)\left(\psi_j^2\partial_t v\right)_{2^j}\right\rangle+R(a,j,k)$$
$$=: E_{\partial_x}(a,j,k)+R(a,j,k), \tag{5.16}$$

where

$$R(a,j,k)$$
$$=\left\langle\Lambda^s\left(\psi_j^2\partial_x v\right)_{2^j},\left(\mathbf{a}_{33}^0\right)_{2^j}\left[\left(\Lambda^s\partial_x\right)\left(\Psi_k\psi_j^2\partial_t v\right)_{2^j}-(\Psi_k)_{2^j}\left(\Lambda^s\partial_x\right)\left(\psi_j^2\partial_t v\right)_{2^j}\right]\right\rangle_{L^2}$$
$$+\left\langle\Lambda^s\left(\psi_j^2\partial_x v\right)_{2^j},\left(\mathbf{a}_{33}^0\right)_{2^j}\Lambda^s\left[\partial_x\left(\Psi_k\psi_j^2\right)_{2^j}(\partial_t v)_{2j}\right]\right\rangle_{L^2}.$$

Using the Kato–Ponce estimate with the operator $(\Lambda^s\partial_x)$, we conclude that

$$\left\|\left[\left(\Lambda^s\partial_x\right)\left(\Psi_k\psi_j^2\partial_t v\right)_{2^j}-(\Psi_k)_{2^j}\left(\Lambda^s\partial_x\right)\left(\psi_j^2\partial_t v\right)_{2^j}\right]\right\|_{L^2}$$
$$\lesssim\left\{\left\|\nabla(\Psi_k)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}+\left\|(\Psi_k)_{2^j}\right\|_{H^s}\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{L^\infty}\right\}.$$

Since $k=j-3,\ldots,j+3$, the norms involving Ψ_k are bounded by a constant independent of j. So by the Cauchy–Schwarz inequality and the Sobolev embedding theorem we obtain that

$$|R(a,j,k)|\lesssim\left\|\left(\mathbf{a}_{33}^0\right)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2\partial_x v\right)_{2^j}\right\|_{H^s}\left(\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}+\left\|(\psi_j\partial_t v)_{2^j}\right\|_{H^s}\right). \tag{5.17}$$

Next, we commute $\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}$ with $\Lambda^s\partial_x$, that is, we write,

$$\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\left(\Lambda^s\partial_x\right)\left(\psi_j^2\partial_t v\right)_{2^j}$$
$$=\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\left(\Lambda^s\partial_x\right)\left(\psi_j^2\partial_t v\right)_{2^j}-\left(\Lambda^s\partial_x\right)\left(\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\left(\psi_j^2\partial_t v\right)_{2^j}\right)$$
$$+\Lambda^s\left(\left(\partial_x\left(\Psi_k\mathbf{a}_{33}^0\psi_j^2\right)\right)_{2^j}(\partial_t v)_{2^j}\right)+2^j\Lambda^s\left(\left(\Psi_k\mathbf{a}_{33}^0\psi_j^2\right)_{2^j}(\partial_t\partial_x v)_{2^j}\right),$$

then

$$E_{\partial_x}(a,j,k) = \left\langle\Lambda^s\left(\psi_j^2\partial_x v\right),\Lambda^s\left(\Psi_k\mathbf{a}_{33}^0\psi_j^2\partial_t\partial_x v\right)_{2^j}\right\rangle_{L^2}+R(b,j,k)$$
$$:= E_{\partial_x}(b,j,k)+R(b,j,k). \tag{5.18}$$

Since $\mathbf{a}_{33}^0\in H^{s+1}_{\rm loc}$, the Kato–Ponce commutator estimate (8.3) implies that

$$\left\|\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\left(\Lambda^s\partial_x\right)\left(\psi_j^2\partial_t v\right)_{2^j}-\left(\Lambda^s\partial_x\right)\left(\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\left(\psi_j^2\partial_t v\right)_{2^j}\right)\right\|_{L^2}$$
$$\lesssim\left\|\nabla\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\right\|_{L^\infty}\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}+\left\|\left(\Psi_k\mathbf{a}_{33}^0\right)_{2^j}\right\|_{H^{s+1}}\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{L^\infty}. \tag{5.19}$$

Taking into account that $\left\|\left(\Psi_k \mathbf{a}_{33}^0\right)_{2^j}\right\|_{H^{s+1}} \simeq \left\|\left(\psi_k \mathbf{a}_{33}^0\right)_{2^k}\right\|_{H^{s+1}}$, we obtain in a similar manner, as in the estimate of the previous remainder term, the following

$$|R(b,j,k)| \lesssim \left(\left\|\left(\psi_k\left(\mathbf{a}_{33}^0-\mathbf{e}_{33}\right)\right)_{2^k}\right\|_{H^{s+1}}+1\right)\left\|\left(\psi_j^2\partial_x v\right)_{2^j}\right\|_{H^s} \times\left(\left\|\left(\psi_j\partial_t v\right)_{2^j}\right\|_{H^s}+\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}\right). \tag{5.20}$$

Adding $E_{\partial_t}(j,k)$ to $E_{\partial_x}(b,j,k)$ enables us now to use Eq. (5.1), that is,

$$\begin{aligned} &E_{\partial_t}(j,k)+E_{\partial_x}(b,j,k)\\ &=\left\langle\Lambda^s\left(\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j},\Lambda^s\left(\Psi_k\psi_j^2\begin{pmatrix}\mathbf{e}_{10} & \mathbf{0}_{10\times 30}\\ \mathbf{0}_{30\times 10} & \mathbf{a}_{33}^0\end{pmatrix}\partial_t\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}\right\rangle_{L^2}\\ &=\sum_{a=1}^{3}\left\langle\Lambda^s\left(\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j},\Lambda^s\left(\Psi_k\psi_j^2\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\partial_a\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}\right\rangle_{L^2}\\ &\quad+\left\langle\Lambda^s\left(\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j},\Lambda^s\left(\Psi_k\psi_j^2\mathbf{B}\right)_{2^j}\right\rangle_{L^2}, \end{aligned}$$

where the matrices $\mathbf{a}_{22}^a$ and $\mathbf{c}_{22}^a$ are defined in Subsect. 3.2 and $\mathbf{c}_{22}^a$ are constant matrices. In the last expression $\mathbf{B}$ contains the zero and first order derivatives of v. It is straightforward to estimate this term since it does not contain second order derivatives.

In order to estimate the second order terms, we write

$$\begin{aligned}\left(\Psi_k\psi_j^2\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\partial_a\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j} &= 2^{-j}\partial_a\left(\Psi_k\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}\\ &\quad-\left(\partial_a\left(\Psi_k\psi_j^2\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\right)\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}, \end{aligned}\tag{5.21}$$

and then

$$\begin{aligned} &\left(\Lambda^s\partial_a\right)\left(\Psi_k\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}\\ &=\left(\Lambda^s\partial_a\right)\left(\Psi_k\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}-\left(\Psi_k\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\left(\Lambda^s\partial_a\right)\left(\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)\right)_{2^j}\\ &\quad+\left(\Psi_k\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\left(\Lambda^s\partial_a\right)\left(\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)\right)_{2^j}. \end{aligned}\tag{5.22}$$

The first term of the right-hand side of (5.21) is being estimated by the Kato–Ponce commutator estimate (8.3) and with the operator $(\Lambda^s\partial_a)$, which is of order $s+1$. For the second term we use the common method of integration by parts. As to the second term of (5.21), we have that

$$\begin{aligned}&\left(\partial_a\left(\Psi_k\psi_j^2\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\right)\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)\\ &=\left(\partial_a\left(\Psi_k\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\right)\psi_j+2\Psi_k\left(\mathbf{a}_{22}^a+\mathbf{c}_{22}^a\right)\partial_a\psi_i\right)\psi_j\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}.\end{aligned}$$

So by the Cauchy–Schwarz inequality we obtain that

$$\left|\left\langle \Lambda^s\left(\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}, \Lambda^s\left(\partial_a\left(\Psi_k\left(\mathbf{a}_{22}^a\right)\right)\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}\right\rangle_{L^2}\right|$$
$$\lesssim\left(\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}+\left\|\left(\psi_j^2\partial_x v\right)_{2^j}\right\|_{H^s}\right)\left\|\left(\partial_a\left(\Psi_k\left(\mathbf{a}_{22}^a\right)\right)\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}\right\|_{H^s}, \tag{5.23}$$

and since $\mathcal{A}^a\in H_{s+1,\delta}$, $s>\frac{3}{2}$, we have by the multiplicity property of the H^s spaces that

$$\left\|\left(\partial_a\left(\Psi_k\left(\mathbf{a}_{22}^a\right)\right)\psi_j^2\begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right)_{2^j}\right\|_{H^s}$$
$$\lesssim\left\|\left(\psi_k\mathbf{a}_{22}^a\right)_{2^j}\right\|_{H^{s+1}}\left(\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}+\left\|\left(\psi_j^2\partial_x v\right)_{2^j}\right\|_{H^s}\right). \tag{5.24}$$

The other term can be estimated by similar arguments. From inequalities (5.17), (5.20), (5.23) and (5.24), we conclude that

$$\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+3}2^{(\delta+1+\frac{3}{2})2j}\left|E_{\partial_t}(j,k)+E_{\partial_x}(j,k)\right|$$
$$\lesssim\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+3}2^{(\delta+1+\frac{3}{2})2j}\left\|\left(\psi_k\mathbf{a}_{22}^a\right)_{2^k}\right\|_{H^{s+1}}\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j\partial_x v\right)_{2^j}\right\|_{H^s}+\cdots, \tag{5.25}$$

where $\cdots$ represent terms that are sums of similar structure. For example ψ_j is replaced by ψ_j^2, $\mathbf{a}_{22}^a$ is replaced by $(\mathbf{a}_{33}^0-\mathbf{e}_{33})$, or the H^{s+1}-norm is replaced by the L^∞-norm.

Applying the Hölder inequality and using the equivalence of norms (Propositions 4.1), we have that

$$\sum_{j=0}^{\infty}\sum_{k=j-3}^{j+3}2^{(\delta+1+\frac{3}{2})2j}\left\|\left(\psi_k\mathbf{a}_{22}^a\right)_{2^k}\right\|_{H^{s+1}}\left\|\left(\psi_j^2\partial_t v\right)_{2^j}\right\|_{H^s}\left\|\left(\psi_j\partial_x v\right)_{2^j}\right\|_{H^s}$$
$$\lesssim\left\|\mathbf{a}_{22}^a\right\|_{H_{s+1,\delta}}\|\partial_t v\|_{H_{s+1,\delta}}\|\partial_x v\|_{H_{s+1,\delta}}\le\left\|\mathbf{a}_{22}^a\right\|_{H_{s+1,\delta}}\left(\|\partial_t v\|^2_{H_{s+1,\delta}}+\|\partial_x v\|^2_{H_{s+1,\delta}}\right). \tag{5.26}$$

So recalling the properties of the inner-products (5.3), (5.5), (5.4), (5.10), (5.11), (5.12) and (5.13), we obtain (5.9). The estimate of $\langle W,\partial_t W\rangle_{s+1,\delta+2,\mathbf{a}_{44}^0}$ relies on similar ideas to those of (5.9), but it is simpler, since $W\in H^{s+1}_{\mathrm{loc}}$. Having collected the estimates of all the terms, we have

$$\frac{1}{2}\frac{d}{dt}\langle U,U\rangle_{X_{s,\delta,\mathcal{A}^0}}\lesssim\left(\|U\|^2_{X_{s,\delta}}+1\right)$$

and by the equivalence (5.7) we obtain (5.8). □

Remark 5.4. Note that if the coefficients of $\partial_t v$ in the matrix $\mathcal{A}^0$ were dependent on t, then we would have to reiterate the commutation (5.15), but with the operator $\Lambda^s\partial_t$ instead of $\Lambda^s\partial_x$. However, $\Lambda^s\partial_t$ is a pseudodifferential operator of order s, and hence we would not get the desired regularity. This is the reason for dividing Eqs. (3.16) by $-g^{00}$.

5.2. L^2_δ-energy estimates. The next section deals with the existence of solutions to the nonlinear symmetric hyperbolic systems by means of an iteration scheme. The $X_{s,\delta}$-energy estimates are used to obtain boundedness of the sequence, while L^2_δ-energy estimates are needed in order to establish the contraction.

Let

$$\langle u_1, u_2\rangle_\delta = \int_{\mathbb{R}^3} (1+|x|)^{2\delta} u_1^T(x) u_2(x) dx$$

denote a weighted L^2 inner-product, where $u_1^T u_2$ denote the scalar product between two vectors. The $L^2_\delta(\mathbb{R}^3)$-space is the closure of all continuous functions under the norm $\|u\|^2_{L^2_\delta} = \langle u, u\rangle_\delta$, and this norm is equivalent to the norm $\|u\|_{H_{0,\delta}}$ (see [28]). Similar to (4.6), we set

$$Y_\delta = L^2_\delta \times L^2_{\delta+1} \times L^2_{\delta+1} \times L^2_{\delta+1},$$

and $\|U\|^2_{Y_\delta} = \|v\|^2_{L^2_\delta} + \|\partial_t v\|^2_{L^2_{\delta+1}} + \|\partial_x v\|^2_{L^2_{\delta+1}} + \|W\|^2_{L^2_{\delta+1}}$. We also define the inner-product of Y_δ in accordance with the system (5.1):

$$\langle U_1, U_2\rangle_{Y_\delta,\mathcal{A}^0} = \langle v_1, v_2\rangle_\delta + \langle \partial_t v_1, \partial_t v_2\rangle_{\delta+1} + \left\langle \partial_x v_1, \mathbf{a}^0_{33}\partial_x v_2\right\rangle_{\delta+1} + \left\langle W_1, \mathbf{a}^0_{44} W_2\right\rangle_{\delta+1},$$

and the associated norm $\|U\|^2_{Y_\delta,\mathcal{A}^0} = \langle U, U\rangle_{Y_\delta,\mathcal{A}^0}$. By assumption (5.2b), $\|U\|_{Y_\delta,\mathcal{A}^0} \simeq \|U\|_{Y_\delta}$.

Lemma 5.5. *Assume the coefficients of* (5.1) *satisfy Assumptions 5.1. If $U(t) \in X_{1,\delta}$ is a solution of* (5.1)*, then*

$$\frac{d}{dt}\langle U(t), U(t)\rangle_{Y_\delta,\mathcal{A}^0} \le C c_0 \left(\langle U(t), U(t)\rangle_{Y_\delta,\mathcal{A}^0} + \|\mathcal{F}\|^2_{L^2_{\delta+1}} + \|\mathcal{D}\|^2_{L^2_{\delta+1}}\right), \tag{5.27}$$

where the constant C depends upon the L^∞-norms of $\mathcal{A}^\alpha$, $\partial_\alpha \mathcal{A}^\alpha$ and $\mathcal{B}$.

Proof. We find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle U(t), U(t)\rangle_{Y_\delta,\mathcal{A}^0} &= \langle v, \partial_t v\rangle_\delta + \left\langle \partial_t v, \partial_t^2 v\right\rangle_{\delta+1} + \left\langle \partial_x v, \mathbf{a}^0_{33}\partial_t\partial_x v\right\rangle_{\delta+1} \\ &+ \left\langle W, \mathbf{a}^0_{44}\partial_t W\right\rangle_{\delta+1} + \left\langle \partial_x v, \partial_t(\mathbf{a}^0_{33})\partial_x v\right\rangle_{\delta+1} + \left\langle W, \partial_t(\mathbf{a}^0_{44}) W\right\rangle_{\delta+1}.\end{aligned}$$

By the Cauchy–Schwarz inequality $|\langle v, \partial_t v\rangle_\delta| \le (\|v\|^2_{L^2_\delta} + \|\partial_t v\|^2_{L^2_\delta})$, $|\langle \partial_x v, \partial_t(\mathbf{a}^0_{33})\partial_x v\rangle_{\delta+1}| \lesssim \|\partial_t(\mathbf{a}^0_{33})\|_{L^\infty}\|\partial_x v\|^2_{L^2_{\delta+1}}$ and $|\langle W, \partial_t(\mathbf{a}^0_{44}) W\rangle_{\delta+1}| \lesssim \|\partial_t(\mathbf{a}^0_{44})\|_{L^\infty} \|W\|^2_{L^2_{\delta+1}}$. Since the system (5.1) is semi-decoupled, we may estimate the expressions with $\partial_t v$ and $\partial_x v$ first, and later the term W. Using Eq. (5.1) and recalling the structure matrices (3.18), (3.20) and (3.23), we have

$$\begin{aligned}\left\langle \partial_t v, \partial_t^2 v\right\rangle_{\delta+1} + \left\langle \partial_x v, \mathbf{a}^0_{33}\partial_t\partial_x v\right\rangle_{\delta+1} &= \sum_{a=1}^{3}\left\langle \begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}, \left(\mathbf{a}^a_{22} + \mathbf{c}^a_{22}\right)\partial_a \begin{pmatrix}\partial_t v\\ \partial_x v\end{pmatrix}\right\rangle_{\delta+1} \\ &+ \langle \partial_t v, \mathbf{b}_{22}\partial_t v + (\mathbf{b}_{23} + \mathbf{b}_{24} + \mathbf{b}_{25})\partial_x v\rangle_{\delta+1} + \langle \partial_t v, f + d_2\rangle_{\delta+1} + \langle \partial_x v, d_3\rangle_{\delta+1}.\end{aligned}$$

Exploring the symmetry of the matrices and using integration by parts, we have that

$$2\left\langle \begin{pmatrix} \partial_t v \\ \partial_x v \end{pmatrix}, \left(\mathbf{a}_{22}^a + \mathbf{c}_{22}^a\right) \partial_a \begin{pmatrix} \partial_t v \\ \partial_x v \end{pmatrix} \right\rangle_{\delta+1} = -\left\langle \begin{pmatrix} \partial_t v \\ \partial_x v \end{pmatrix}, \partial_a \left(\mathbf{a}_{22}^a\right) \begin{pmatrix} \partial_t v \\ \partial_x v \end{pmatrix} \right\rangle_{\delta+1}$$
$$-2(\delta+1)\left\langle \begin{pmatrix} \partial_t v \\ \partial_x v \end{pmatrix}, \frac{x^a}{|x|} \left(\mathbf{a}_{22}^a + \mathbf{c}_{22}^a\right) \begin{pmatrix} \partial_t v \\ \partial_x v \end{pmatrix} \right\rangle_{\delta+\frac{1}{2}}.$$

Thus

$$\left| \left\langle \partial_t v, \partial_t^2 v \right\rangle_{\delta+1} + \left\langle \partial_x v, \mathbf{a}_{33}^0 \partial_t \partial_x v \right\rangle_{\delta+1} \right|$$
$$\lesssim \left(\sum_{a=1}^{3} \left(\|\partial_a(\mathbf{a}_{22}^a)\|_{L^\infty} + \|\mathbf{a}_{22}^a\|_{L^\infty} \right) + \|\mathcal{B}\|_{L^\infty} \right) \left(\|\partial_t v\|_{L^2_{\delta+1}}^2 + \|\partial_x v\|_{L^2_{\delta+1}}^2 \right)$$
$$+ \|\partial_t v\|_{L^2_{\delta+1}}^2 + \|\partial_x v\|_{L^2_{\delta+1}}^2 + \|f\|_{L^2_{\delta+1}}^2 + \|d_2\|_{L^2_{\delta+1}}^2 + \|d_3\|_{L^2_{\delta+1}}^2.$$

And for W we have,

$$\left\langle W, \mathbf{a}_{44}^0 \partial_t W \right\rangle_{\delta+1} = \sum_{a=1}^{3} \langle W, \mathbf{a}_{33}^a \partial_a W \rangle_{\delta+1} + \langle W, d_4 \rangle_{\delta+1},$$

so similar arguments give that

$$\left| \left\langle W, \mathbf{a}_{44}^0 \partial_t W \right\rangle_{\delta+1} \right| \lesssim \left(\sum_{a=1}^{3} \left(\|\partial_a(\mathbf{a}_{33}^a)\|_{L^\infty} + \|\mathbf{a}_{33}^a\|_{L^\infty} \right) + 1 \right) \|W\|_{L^2_{\delta+1}}^2 + \|d_4\|_{L^2_{\delta+1}}^2.$$

Finally, using the equivalence of the norms we obtain (5.27). □

6. The Iteration Process

In this section we adopt Majda's iterative scheme[18] in order to prove the well-posdeness of the coupled hyperbolic system (3.24) in the $H_{s,\delta}$-spaces. A similar approach was carried out in [16] for the vacuum Einstein equations, but in the presence of a perfect fluid there are additional difficulties. The zero order term B is not necessarily a C^∞-function since it contains the fractional power of the Makino variable $w^{2/(\gamma-1)}$. Hence we could not apply the standard iteration scheme in order to prove the existence theorem for symmetric hyperbolic systems. We denote the initial data by $U_0 = (\phi, \varphi, \partial_x \phi, W_0)$, where $W_0 = (w_0, (u_0)^\alpha - e_0^\alpha)$ represents the initial data for the Makino variable w and the four velocity vector $u^\alpha - e_0^\alpha$.

Theorem 6.1. *Let* $\frac{3}{2} < s < \frac{2}{\gamma-1} + \frac{1}{2}$, $-\frac{3}{2} \leq \delta$. *Assume* $U_0 \in X_{s,\delta}$, $w_0 \geq 0$ *and that there exists a positive constant* μ *such that*

$$\frac{1}{\mu} V^T V \leq V^T \mathcal{A}^0(\phi, W_0) V \leq \mu V^T V \quad \text{for all } V \in \mathbb{R}^{55}. \tag{6.1}$$

Then there exists a positive constant T *and a unique solution* $U(t) = (v(t), \partial_t v(t), \partial_x v(t), W(t))$ *to the system* (3.24) *such that* $U(0, x) = U_0(x)$,

$$U \in C\left([0, T], X_{s,\delta}\right) \quad \text{and} \quad W \in C^1\left([0, T], H_{s,\delta+2}\right). \tag{6.2}$$

The size of the time interval depends only on the norms of the initial data.

The proof of Theorem 6.1 will be carried out in several steps:

1. Setting up the iterative scheme;
2. Proving that the fractional power $(w^k)^{2/(\gamma-1)}$ is a C_0^∞-function;
3. Boundedness of the iteration sequence in the $X_{s,\delta}$-norm;
4. Contraction in a lower norm;
5. Convergence;
6. Uniqueness;
7. Continuity in the $X_{s,\delta}$-norm.

A part of the above proofs are standard, but some of them require special attention due to the specific form of the system (3.24) and use of the space $X_{s,\delta}$. Moreover, the fact that the matrices $\mathcal{A}^\alpha = \mathcal{A}^\alpha(v, W)$ are not dependent on the derivative of the semi-metric plays an essential role here.

Step 1. From condition (6.1) and the embedding into the continuous, Proposition 4.8, we see that there is a bounded domain $G \subset \mathbb{R}^{55}$ containing the initial value U_0 and a constant $c_0 \geq 1$ such that

$$\frac{1}{c_0} V^T V \leq V^T \mathcal{A}^0(v, W) V \leq c_0 V^T V \tag{6.3}$$

for all $U = (v, \partial_t v, \partial_x v, W) \in G$ and $V \in \mathbb{R}^{55}$. By means of the density properties of $H_{s,\delta}$, Proposition 4.10, there exist positive constants C and R, and a sequence

$$\left\{U_0^k\right\}_{k=0}^\infty = \left\{(\phi^k, \varphi^k, \partial_x \phi^k, w_0^k, (u_0^\alpha)^k - e_0^\alpha)\right\}_{k=0}^\infty \subset C_0^\infty(\mathbb{R}^3), \tag{6.4}$$

such that

$$\left\|U_0^0\right\|_{X_{s+1,\delta}} \leq C \left\|U_0\right\|_{X_{s,\delta}}, \tag{6.5}$$

$$\left\|U - U_0^0\right\|_{X_{s,\delta}} \leq R \Rightarrow U \in G, \tag{6.6}$$

and

$$\left\|U_0^k - U_0\right\|_{X_{s,\delta}} \leq \frac{R 2^{-k}}{4 c_0}. \tag{6.7}$$

The iterative scheme is defined as follows: let $U^0(t,x) = U_0^0(x)$ and

$$U^{k+1}(t,x) = (v^{k+1}(t,x), \partial_t v^{k+1}(t,x), \partial_x v^{k+1}(t,x), W^{k+1}(t,x))$$

be a solution of the linear Cauchy problem

$$\begin{cases} \mathcal{A}^0(v^k, W^k)\partial_t U^{k+1} = \left(\mathcal{A}^a(v^k, W^k) + \mathcal{C}^a\right) \partial_a U^{k+1} \\ \qquad\qquad\qquad + \mathcal{B}(U^k) \begin{pmatrix} v^{k+1} \\ \partial_t v^{k+1} \\ \partial_x v^{k+1} \end{pmatrix} + \mathcal{F}(v^k, W^k), \\ U^{k+1}(0,x) = U_0^{k+1}(x), \end{cases} \tag{6.8}$$

where $\mathcal{F}(v^k, W^k) = (0, f(v^k, W^k), 0, 0)$, $f(v^k, W^k)$ is given by (3.22) and $W^k = (w^k, (u^\alpha)^k - e_0^\alpha)$.

Step 2. The iterative method relies on the fact that solutions of linear symmetric hyperbolic systems with C_0^∞ coefficients and initial data, are also C_0^∞. However, even if $w^k \geq 0$ and $w^k \in C_0^\infty$, it does not guarantee that $(w^k)^{2/(\gamma-1)}$ is a C_0^∞-function. Since the function $f(v^k, W^k)$ contains the term $(w^k)^{2/(\gamma-1)}$, we must assure that it is a C_0^∞-function.

Proposition 6.2. *Let $u \in H_{s,\delta}$ be non-negative and $\beta > 0$. Then there is a sequence $\{u^k\} \subset C_0^\infty$ such that $u^k \to u$ in the $H_{s,\delta}$-norm and $(u^k)^\beta \in C_0^\infty$.*

Proof. Let $\varepsilon > 0$, then by Proposition 4.10 there is $u^\varepsilon \in C_0^\infty$ with $\|u - u^\varepsilon\|_{H_{s,\delta}} < \varepsilon$. Take now a positive number M so that $\operatorname{supp}(u^\varepsilon) \subset \{|x| \leq M\}$, and let χ_M be the cut-off function satisfying $\chi_M(x) = 1$ for $|x| \leq M$ and $\chi_M(x) = 0$ for $|x| \geq M + 1$. For any positive number ϱ, we set

$$u^{\varepsilon,\varrho}(x) = \chi_M(x)\left(u^\varepsilon(x) + \varrho\right).$$

Then $(u^{\varepsilon,\varrho})^\beta \in C_0^\infty$, since $(u^\varepsilon + \varrho) > 0$ and χ_M is a cut-off function. Moreover, $u^{\varepsilon,\varrho} - u^\varepsilon = \chi_M \varrho$, hence $u^{\varepsilon,\varrho} \to u^\varepsilon$ in the $H_{s,\delta}$-norm as $\varrho \to 0$. □

Thus we may assume that $\{w_0^k\}_{k=0}^\infty$, the C_0^∞ approximation of the initial data of the Makino variable w_0 in (6.4), satisfies $(w_0^k)^{2/(\gamma-1)} \in C_0^\infty$. We turn now to showing that for $t \geq 0$,

$$\epsilon^k(t,x) = \left(w^k\right)^{\frac{2}{\gamma-1}}(t,x)$$

is also a C_0^∞-function.

Proposition 6.3. *For each integer $k \geq 0$, $\epsilon^k(t,\cdot) \in C_0^\infty(\mathbb{R}^3)$.*

Proof. We conduct the proof by induction. Obviously the statement holds when $k = 0$. Now the 51st row of the system (6.8) is equivalent to (3.5a), so the linearization of it results in

$$\begin{aligned}(u^0)^k \partial_t \epsilon^{k+1} &+ (u^a)^k \partial_a \epsilon^{k+1} + \epsilon^k (1 + K(w^k)^2) P_\alpha^\nu \left(g_{\alpha\beta}^k, (u^\beta)^k\right) \partial_\nu (u^\alpha)^{k+1} \\ &+ \epsilon^k (1 + K(w^k)^2) P_\alpha^\nu \left(g_{\alpha\beta}^k, (u^\beta)^k\right) \left(\Gamma_{\nu\mu}^\alpha\right)^k (u^\mu)^k = 0, \end{aligned} \tag{6.9}$$

where the $P_\alpha^\nu(\cdot,\cdot)$ is the projection of Eq. (3.5a) and $\left(\Gamma_{\nu\mu}^\alpha\right)^k$ are the Christoffel symbols with respect to the semi-metric $g_{\alpha\beta}^k$. It follows from [4, Thm. 2.6], that $u^0(0,x) \geq 1$, hence $(u^0)^k(0,x) \geq 1$ and therefore we can divide (6.9) by $(u^0)^k$ and we conclude that ϵ^{k+1} satisfies a first order linear equation of the form

$$\begin{cases} \partial_t \epsilon^{k+1} + b_a(t,x)\partial_a \epsilon^{k+1} + c(t,x) = 0 \\ \epsilon^{k+1}(0,x) = (w_0^{k+1}(x))^{\frac{2}{\gamma-1}} \end{cases}. \tag{6.10}$$

Note that $c(t,x)$ contains the term ϵ^k, but this term is C_0^∞ by the induction hypothesis. Hence all the coefficients of (6.10) are C_0^∞-functions. We solve (6.10) by means of the characteristic method. So let $\Phi(s,y)$ be the solution of the system

$$\frac{dt}{ds} = 1, \quad \frac{dx_a}{ds} = b_a(t, x), \quad t(0) = 0, \quad x(0) = y.$$

Then $\epsilon^{k+1}(t, x) = Z(\Phi^{-1}(t, x))$, where $Z(s, y)$ is the solution of the initial value problem

$$\frac{dZ}{ds} = -c(s, x), \qquad Z(0, y) = (w_0^{k+1}(y))^{\frac{2}{\gamma-1}}.$$

Obviously

$$Z(s, y) = (w_0^{k+1}(y))^{\frac{2}{\gamma-1}} - \int_0^s c(\tau, x(\tau))d\tau.$$

Since $c(\tau, \cdot) \in C_0^\infty$ and $(w_0^{k+1})^{\frac{2}{\gamma-1}} \in C_0^\infty$ by Proposition 6.2, $Z(s, \cdot)$ also belongs to C_0^∞, and hence also $Z(\Phi^{-1}(t, \cdot)) = \epsilon^{k+1}(t, \cdot) = \left(w^{k+1}\right)^{\frac{2}{\gamma-1}}(t, \cdot) \in C_0^\infty$. □

Step 3. We conclude from Proposition 6.2, and the theory of linear symmetric hyperbolic systems (cf. [15]), that for each k there is a solution $U^k(t, x)$ of the linear system (6.8) such that $U^k(t, \cdot) \in C_0^\infty(\mathbb{R}^3)$. Therefore by (6.5), (6.6) and (6.7), for each k we have

$$T_k = \sup\left\{T : \sup_{0\le t\le T}\left\|U^k(t) - U_0^0\right\|_{X_{s,\delta}} \le R\right\} > 0.$$

Proposition 6.4. *There are positive constants T^* and L such that*

$$\sup\left\{T : \sup_{0\le t\le T}\left\|U^k(t) - U_0^0\right\|_{X_{s,\delta}} \le R\right\} \ge T^* \quad \textit{for all } k, \tag{6.11}$$

and

$$\sup_{0\le t\le T^*}\left\|\partial_t W^k\right\|_{H_{s,\delta+2}} \le L \quad \textit{for all } k. \tag{6.12}$$

Proof. We prove it by induction. Set $V^{k+1} = U^{k+1} - U_0^0$, then V^k satisfies the linear initial value problem

$$\begin{cases} \mathcal{A}^0(v^k, W^k)\partial_t V^{k+1} = \left(\mathcal{A}^a(v^k, W^k) + \mathcal{C}^a\right)\partial_a V^{k+1} \\ +\mathcal{B}(U^k)\begin{pmatrix} v^{k+1} - \phi_0^0 \\ \partial_t v^{k+1} - \varphi_0^0 \\ \partial_x\left(v^{k+1} - \phi_0^0\right)\end{pmatrix} + \mathcal{F}(v^k, W^k) + \mathcal{D}^k \\ V^{k+1}(0, x) = U_0^{k+1}(x) - U_0^0(x), \end{cases} \tag{6.13}$$

where

$$\mathcal{D}^k = \left(\mathcal{A}^a(v^k, W^k) + \mathcal{C}^a\right)\partial_a U_0^0 + \mathcal{B}(U^k)\begin{pmatrix}\phi_0^0 \\ \varphi_0^0 \\ \partial_x\phi_0^0\end{pmatrix}. \tag{6.14}$$

In order to apply the energy estimate, Lemma 5.3, to (6.13) we have to check that Assumptions 5.1 are satisfied and the corresponding norms are independent of k. Note that all the matrices have the same structure, and from (6.3) we see that condition 5.2b holds. By the induction hypothesis,

$$\left\| v^k(t) - \phi_0^0 \right\|_{H_{s,\delta,2}}^2 + \left\| \partial_x v^k(t) - \partial_x \phi_0^0 \right\|_{H_{s,\delta+1,2}}^2 \leq \left\| U^k(t) - U_0^0 \right\|_{X_{s,\delta}}^2 \leq R^2,$$

therefore by Remark 4.11, $\|v^k(t) - \phi_0^0\|_{H_{s+1,\delta}} \lesssim R$. Applying the Moser type estimate, Proposition 4.9, we have

$$\begin{aligned} \left\| \mathcal{A}^0(v^k, W^k) - \mathbf{e}_{55} \right\|_{H_{s+1,\delta}}^2 &\lesssim \left(\left\| v^k \right\|_{H_{s+1,\delta}}^2 + \left\| W^k \right\|_{H_{s+1,\delta}}^2 \right) \\ &\lesssim \left\| U^k - U_0^0 \right\|_{X_{s,\delta}}^2 + \left\| U_0^0 \right\|_{X_{s,\delta}}^2 \lesssim \left(R^2 + \left\| U_0^0 \right\|_{X_{s,\delta}}^2 \right). \end{aligned}$$

In a similar way we get that $\left\| \mathcal{A}^a(v^k, W^k) \right\|_{H_{s+1,\delta}}$ is bounded by a constant depending on R. By Propositions 4.3, 4.5 ,4.9, Remark 4.6 and the structure of the matrix $\mathcal{B}$ in (3.23), we obtain that $\left\| \mathbf{b}_{2j}(U^k) \right\|_{H_{s,\delta+1}} \lesssim \|U^k\|_{X_{s,\delta}} \lesssim \left(R + \|U_0^0\|_{X_{s,\delta}} \right)$ and a similar estimate holds for $\left\| \mathbf{b}_{4j}(U^k) \right\|_{H_{s,\delta+1}}$, $j = 2, 3, 4, 5$. We recall that $\mathcal{F}(v^k, W^k) = (0, f(v^k, W^k), 0, 0)$, where $f(v^k, W^k)$ is given by (3.22). Applying again Propositions 4.5, 4.9 and Remark 4.6, we obtain that

$$\left\| f(v^k, W^k) \right\|_{H_{s,\delta+1}} \lesssim \left\| (w^k)^{\frac{2}{\gamma-1}} \right\|_{H_{s,\delta+1}} \left(\left\| v^k \right\|_{H_{s,\delta+1}} + \left\| (u^\alpha)^k - e_0^\alpha \right\|_{H_{s,\delta+1}} + 1 \right). \tag{6.15}$$

Now, for $\frac{3}{2} < s < \frac{2}{\gamma-1} + \frac{1}{2}$, we apply the estimate for the fractional power, Proposition 4.7, and obtain that

$$\left\| (w^k)^{\frac{2}{\gamma-1}} \right\|_{H_{s,\delta+1}} \lesssim \left\| w^k \right\|_{H_{s,\delta+1}} \leq \left\| w^k \right\|_{H_{s+1,\delta+1}}. \tag{6.16}$$

Since $\left\| v^k \right\|_{H_{s,\delta+1}}$ and $\left\| W^k \right\|_{H_{s,\delta+1}}$ are bounded by a constant independent of k, it follows from (6.15) and (6.16) that also $\left\| f(v^k, W^k) \right\|_{H_{s,\delta+1}}$ is bounded by a constant independent of k.

The required estimate for $\mathcal{D}^k$ defined in (6.14) follows from the multiplicity property (4.3) and the estimates which we have already obtained for $\left\| \mathcal{A}^\alpha(v^k, W^k) \right\|_{H_{s+1,\delta}}$ and $\left\| \mathcal{B}(U^k) \right\|_{H_{s,\delta+1}}$.

It remains to verify (5.2c); note that by the induction hypothesis (6.11), condition (6.6) and the embedding (4.4), we have

$$\begin{aligned} \left\| \partial_t \mathcal{A}^0(v^k, W^k) \right\|_{L^\infty} &\leq \sup_G \left| \frac{\partial \mathcal{A}^0}{\partial v}(v, W) \right| \left\| \partial_t v^k \right\|_{L^\infty} \\ &+ \sup_G \left| \frac{\partial \mathcal{A}^0}{\partial W}(v, W) \right| \left\| \partial_t W^k \right\|_{L^\infty} \lesssim \left(\left\| \partial_t v^k \right\|_{H_{s,\delta+1}} + \left\| \partial_t W^k \right\|_{H_{s,\delta+2}} \right). \end{aligned}$$

Since $\left\|\partial_t W^k\right\|_{H_{s,\delta+2}}$ is bounded by hypothesis (6.12), we see that $\left\|\partial_t \mathcal{A}^0(v^k, W^k)\right\|_{L^\infty}$ is also bounded by a constant depending on R but not on k. We now apply Lemma 5.3, and with the combination of Gronwall's inequality, condition (6.7) and the equivalence (5.7), we conclude that there is a constant C depending on R such that

$$\sup_{0\leq t\leq T}\left\|V^{k+1}(t)\right\|^2_{X_{s,\delta}} \leq e^{Cc_0T}\left(\frac{R^2}{8}+Cc_0^2T\right). \tag{6.17}$$

We turn now to show (6.12). It follows from the structure of the matrices $\mathcal{A}^0$, $\mathcal{A}^a$ and $\mathcal{B}$ (see (3.18), (3.20) and (3.23)) that

$$\partial_t W^{k+1} = \left(\mathbf{a}^0_{44}(v^k, W^k)\right)^{-1}\left[\sum_{a=1}^3 \mathbf{a}^a_{33}(v^k, W^k)\partial_a W^{k+1}\right.$$
$$\left.+\mathbf{b}_{42}(U^k)\partial_t v^{k+1} + \sum_{a=1}^3 \mathbf{b}_{4(a+2)}(U^k)\partial_a v^{k+1}\right]. \tag{6.18}$$

Note that $\|v^k(t)\|_{H_{s+1,\delta}}$, $\|W^k(t)\|_{H_{s+1,\delta+1}}$ and $\|U^k(t)\|_{X_{s,\delta}}$ are bounded by a constant independent of k, while $\|\partial_t v^{k+1}(t)\|_{H_{s,\delta+1}}$, $\|\partial_a v^{k+1}(t)\|_{H_{s,\delta+1}}$ and $\|\partial_a W^{k+1}(t)\|_{H_{s,\delta+2}}$ are bounded by (6.17). Hence applying the calculus of the $H_{s,\delta}$-spaces to (6.18), we obtain that $\left\|\partial_t W^{k+1}\right\|_{H_{s,\delta+2}}$ is also bounded by a constant independent of k. Choosing T^* so that

$$e^{Cc_0T^*}\left(\frac{R^2}{8}+Cc_0^2T^*\right) < R^2$$

completes the proof of the proposition. □

Step 4. Here we show contraction in the weighted L^2-norm. Our method relies on the L^2_δ-energy estimates, Lemma 5.5.

Proposition 6.5. *There exist positive constants $T^{**} \leq T^*$ and $\Lambda < 1$, and a positive sequence $\{\beta_k\}$ with $\sum \beta_k < \infty$ such that*

$$\sup_{0\leq t\leq T^{**}}\left\|U^{k+1}(t)-U^k(t)\right\|_{Y_\delta,\mathcal{A}^0} \leq \Lambda \sup_{0\leq t\leq T^{**}}\left\|U^k(t)-U^{k-1}(t)\right\|_{Y_\delta,\mathcal{A}^0} + \beta_k. \tag{6.19}$$

Proof. The function $(U^{k+1}-U^k)$ satisfies the linear system

$$\mathcal{A}^0(v^k, W^k)\partial_t(U^{k+1}-U^k) = \mathcal{A}^a(v^k, W^k)\partial_a(U^{k+1}-U^k) + \mathcal{B}(U^k)\begin{pmatrix} v^{k+1}-v^k \\ \partial_t(v^{k+1}-v^k) \\ \partial_x(v^{k+1}-v^k)\end{pmatrix}$$
$$+\mathcal{F}(v^k, W^k) - \mathcal{F}(v^{k-1}, W^{k-1}) + \mathcal{D}^k, \tag{6.20}$$

where

$$\mathcal{D}^k = -\left(\mathcal{A}^0(v^k, W^k) - \mathcal{A}^0(v^{k-1}, W^{k-1})\right)\partial_t U^k$$
$$+\left(\mathcal{A}^a(v^k, W^k) - \mathcal{A}^a(v^{k-1}, W^{k-1})\right)\partial_a U^k + \left(\mathcal{B}(U^k)-\mathcal{B}(U^{k-1})\right)\begin{pmatrix} v^k \\ \partial_t v^k \\ \partial_x v^k\end{pmatrix}. \tag{6.21}$$

By (6.11), (6.12) and Proposition 4.8, $\|\mathcal{A}^\alpha(v^k, W^k)\|_{L^\infty}$, $\|\partial_\alpha \mathcal{A}^\alpha(v^k, W^k)\|_{L^\infty}$ and $\left\|\mathcal{B}(U^k)\right\|_{L^\infty}$ are bounded by a constant independent of k, so we may apply Lemma 5.5 and get that

$$\frac{d}{dt}\left\|U^{k+1}(t) - U^k(t)\right\|^2_{Y,\mathcal{A}^0} \leq C_1\left(\left\|U^{k+1}(t) - U^k(t)\right\|^2_{Y,\mathcal{A}^0}\right.$$
$$\left. + \left\|\mathcal{F}(v^k, W^k) - \mathcal{F}(v^{k-1}, W^{k-1})\right\|^2_{L_{\delta+1}} + \left\|\mathcal{D}^k\right\|^2_{L_{\delta+1}}\right). \tag{6.22}$$

Thus our main task is to estimate the two last terms of the above inequality by means of the difference $\|U^k - U^{k-1}\|^2_{Y_\delta}$. Recall that $\mathcal{F}(v, W) = (0, f(v, w), 0, 0)^T$, where f is a scalar function given in (3.22), so

$$\begin{aligned}
&f(v^k, W^k) - f(v^{k-1}, W^{k-1})\\
&= \left(\frac{\partial f}{\partial v}\left(\tau v^k + (1-\tau)\, v^{k-1}, \tau W^k + (1-\tau)\, W^{k-1}\right)\right)\left(v^k - v^{k-1}\right)\\
&\quad + \left(\frac{\partial f}{\partial W}\left(\tau v^k + (1-\tau)\, v^{k-1}, \tau W^k + (1-\tau)\, W^{k-1}\right)\right)\left(W^k - W^{k-1}\right)
\end{aligned}$$

for some $\tau \in [0, 1]$. Note that $(v^k - v^{k-1})$ belong to L^2_δ, while we need to estimate the $L^2_{\delta+1}$-norm of the above expressions. However, since the Makino $w \in H_{s+1,\delta+1} \subset H_{s,\delta+1}$, we get from (3.22), and Propositions 4.7, 4.5 and 4.9 that $\frac{\partial f}{\partial v} \in H_{s,\delta+1}$. Therefore, by the equivalence $\|u\|_{L^2_\delta} \simeq \|u\|_{H_{0,\delta}}$ (Proposition 4.2) and the multiplicity property (4.3), we obtain

$$\left\|\frac{\partial f}{\partial v}\left(v^k - v^{k-1}\right)\right\|_{H_{0,\delta+1}} \lesssim \left\|\frac{\partial f}{\partial v}\right\|_{H_{s,\delta+1}}\left\|v^k - v^{k-1}\right\|_{H_{0,\delta}} \lesssim \left\|\frac{\partial f}{\partial v}\right\|_{H_{s,\delta+1}}\left\|v^k - v^{k-1}\right\|_{L^2_\delta}.$$

The other term is somewhat easier to treat, since $(W^k - W^{k-1}) \in L^2_{\delta+1}$, and hence

$$\left\|\frac{\partial f}{\partial W}\left(W^k - W^{k-1}\right)\right\|_{L^2_{\delta+1}} \leq \left\|\frac{\partial f}{\partial W}\right\|_{L^\infty}\left\|W^k - W^{k-1}\right\|_{L^2_{\delta+2}}.$$

Now $\left\|\frac{\partial f}{\partial W}\right\|_{L^\infty}$ is bounded by a constant depending on $\left\|U^k\right\|_{X_{s,\delta}}$ and $\left\|U^{k-1}\right\|_{X_{s,\delta}}$, and these are independent of k. Thus

$$\left\|\mathcal{F}(v^k, W^k) - \mathcal{F}(v^{k-1}, W^{k-1})\right\|_{L^2_{\delta+1}} \leq C_2\left\|U^k - U^{k-1}\right\|_{Y,\delta}, \tag{6.23}$$

where the constant C_2 is independent of k. We shall now estimate the first term of $\mathcal{D}^k$ in (6.21). From the structure of $\mathcal{A}^0(v, W)$ we see that

$$\begin{aligned}
&\left(\mathcal{A}^0(v^k, W^k) - \mathcal{A}^0(v^{k-1}, W^{k-1})\right)\partial_t U^k\\
&= \left(\mathbf{a}^0_{33}(v^k) - \mathbf{a}^0_{33}(v^{k-1})\right)\partial_t\partial_x v^k + \left(\mathbf{a}^0_{44}(v^k, W^k) - \mathbf{a}^0_{44}(v^{k-1}, W^{k-1})\right)\partial_t W^k.
\end{aligned}$$

Now

$$\mathbf{a}^0_{33}(v^k) - \mathbf{a}^0_{33}(v^{k-1}) = \frac{\partial \mathbf{a}^0_{33}}{\partial v}\left(\tau v^k + (1-\tau)v^{k-1}\right)\left(v^k - v^{k-1}\right)$$

for some $\tau \in [0, 1]$. But since $(v^k - v^{k-1}) \notin L^2_{\delta+1}$, we cannot use the $L^\infty - L^2$ estimates. We therefore apply the algebra (or multiplication) property, once with $s_1 = 1,\ s_2 = s-1$ and $s = 0$, and once with $s_1 = s,\ s_2 = 1$ and $s = 1$, which results in the following inequality:

$$\begin{aligned} &\left\| \left(\mathbf{a}^0_{33}(v^k) - \mathbf{a}^0_{33}(v^{k-1})\right) \partial_t \partial_x v^{k-1} \right\|_{H_{0,\delta+1}} \\ &\lesssim \left\| \frac{\partial \mathbf{a}^0_{33}}{\partial v} \left(\tau v^k + (1-\tau) v^{k-1}\right) \left(v^k - v^{k-1}\right) \right\|_{H_{1,\delta}} \left\| \partial_t \partial_x v^{k-1} \right\|_{H_{s-1,\delta+2}} \\ &\lesssim \left\| \frac{\partial \mathbf{a}^0_{33}}{\partial v} \left(\tau v^k + (1-\tau) v^{k-1}\right) \right\|_{H_{s,\delta}} \left\| v^k - v^{k-1} \right\|_{H_{1,\delta}} \left\| \partial_t v^{k-1} \right\|_{H_{s,\delta+1}}. \end{aligned} \tag{6.24}$$

We use now the third Moser inequality (4.5) and $\|v^k - v^{k-1}\|_{L^\infty} \lesssim \|v^k - v^{k-1}\|_{H_{s+1,\delta}}$ in order to obtain

$$\left\| \frac{\partial \mathbf{a}^0_{33}}{\partial v} \left(\tau v^k + (1-\tau) v^{k-1}\right) \right\|_{H_{s+1,\delta}} \leq C\left(\left\| v^k \right\|_{H_{s+1,\delta}}, \left\| v^{k-1} \right\|_{H_{s+1,\delta}} \right), \tag{6.25}$$

where

$$C\left(\left\| v^k \right\|_{H_{s+1,\delta}}, \left\| v^{k-1} \right\|_{H_{s+1,\delta}} \right)$$

denotes that the constant depends in some way on the terms $\|v^k\|_{H_{s+1,\delta}}$ and $\left\|v^{k-1}\right\|_{H_{s+1,\delta}}$.

In a similar way we obtain

$$\begin{aligned} &\left\| \left(\mathbf{a}^0_{44}(v^k, W^k) - \mathbf{a}^0_{44}(v^{k-1}, W^{k-1})\right) \partial_t W^k \right\|_{L^2_{\delta+1}} \\ &\lesssim \left(\left\| \frac{\partial \mathbf{a}^0_{44}}{\partial v} \left(\tau v^k + (1-\tau) v^{k-1}\right) \right\|_{H_{s,\delta+1}} \left\| v^k - v^{k-1} \right\|_{H_{1,\delta}} \right. \\ &\qquad \left. + \left\| \frac{\partial \mathbf{a}^0_{44}}{\partial W} \right\|_{L^\infty} \left\| W^k - W^{k-1} \right\|_{L^2_{\delta+1}} \right) \left\| \partial_t W^k \right\|_{H_{s,\delta+2}}. \end{aligned} \tag{6.26}$$

We recall that $\left\|\partial_t W^k\right\|_{H_{s,\delta+2}}$ is bounded by (6.12) and $\left\|v^k - v^{k-1}\right\|^2_{H_{1,\delta}} \simeq \left\|v^k - v^{k-1}\right\|^2_{L^2_\delta} + \left\|\partial_x v^k - \partial_x v^{k-1}\right\|^2_{L^2_{\delta+1}}$, therefore from (6.24), (6.25) and (6.26) we obtain that

$$\left\| \left(\mathcal{A}^0(v^k, W^k) - \mathcal{A}^0(v^{k-1}, W^{k-1})\right) \partial_t U^k \right\|_{L^2_{\delta+1}} \leq C_3 \left\| U^k - U^{k-1} \right\|_{Y_\delta}. \tag{6.27}$$

In a similar manner we can estimate the difference involving the $\mathcal{A}^a$ matrices. The estimate of $\mathcal{B}(U^k) - \mathcal{B}(U^{k-1})$ is simpler, since its first column of the matrix $\mathcal{B}$ is zero and therefore this expression does not contain the element $(v^k - v^{k-1})$. Thus we conclude from inequalities (6.22), (6.23) and (6.27) that

$$\frac{d}{dt} \left\| U^{k+1}(t) - U^k(t) \right\|^2_{Y,\mathcal{A}^0} \leq C_4 \left(\left\| U^{k+1}(t) - U^k(t) \right\|^2_{Y,\mathcal{A}^0} + \left\| U^k(t) - U^{k-1}(t) \right\|^2_{Y,\mathcal{A}^0} \right),$$

where C_4 is independent of k. Therefore by Gronwall's inequality, we obtain that for any $T^{**} \leq T^*$,

$$\sup_{0\leq t\leq T^{**}} \left\| U^{k+1}(t) - U^k(t) \right\|^2_{Y,\mathcal{A}^0} \leq e^{C_4 T^{**}} \Bigg(\left\| U_0^{k+1} - U_0^k \right\|^2_{Y_\delta,\mathcal{A}^0} + T^{**} C_4 \sup_{0\leq t\leq T^{**}} \left\| U^k(t) - U^{k-1}(t) \right\|^2_{Y,\mathcal{A}^0} \Bigg),$$

and hence inequality (6.19) holds if we choose T^{**} so that $\Lambda := \sqrt{2C_4 e^{C_4 T^{**}} T^{**}} < 1$, and set $\beta_k := \sqrt{2e^{C_4 T^{**}}} \left\| U_0^{k+1} - U_0^k \right\|_{Y_\delta,\mathcal{A}^0}$. □

Step 5. We discuss here the convergence. It follows from Proposition 6.5 that

$$\sum_k \left\| U^{k+1}(t) - U^k(t) \right\|_{Y_\delta} < \infty,$$

hence $\{U^k(t)\}$ is a Cauchy sequence in $L^\infty([0, T^{**}], Y_\delta)$. Applying the Gagliardo-Nirenberg-Moser estimate $\|u\|_{H^{s'}} \leq \|u\|_{H^s}^{\frac{s'}{s}} \|u\|_{L^2}^{1-\frac{s'}{s}}$ term-wise to the norm (2.1), we get that

$$\|u\|_{H_{s',\delta}} \leq \|u\|_{H_{s,\delta}}^{\frac{s'}{s}} \|u\|_{L^2_\delta}^{1-\frac{s'}{s}} \quad \text{for} \quad 0 < s' < s \text{ and } \delta \in \mathbb{R}. \tag{6.28}$$

Hence $\{U^k(t)\}$ is a Cauchy sequence in $L^\infty\left([0, T^{**}], X_{s',\delta}\right)$ and therefore $U^k(t) \to U(t)$ in the $X_{s',\delta}$-norm for any $0 < s' < s$ and $\delta \geq -\frac{3}{2}$. Furthermore, by Remark 4.6, $v^k(t) \to v(t)$ in $H_{s'+1,\delta}$-norm. Thus if we choose $\frac{3}{2} < s' < s$, then by the embedding (4.4),

$$v^k(t) \to v(t), \; W^k(t) \to W(t) \quad \text{in} \quad C^1$$

and

$$\partial_t v^k(t) \to \partial_t v(t), \; \partial_t W^k(t) \to \partial_t W(t) \quad \text{in} \quad C^0.$$

Thus, $U(t) = (v(t), \partial_t v(t), \partial_x v(t), W(t))$ is a solution of the system (3.24).

Proposition 6.6. *For any* $\Phi \in X_{s,\delta}$,

$$\lim_k \left\langle U^k(t), \Phi \right\rangle_{X_{s,\delta}} = \langle U(t), \Phi \rangle_{X_{s,\delta}}, \tag{6.29}$$

uniformly for $0 \leq t \leq T^{**}$*, and where* $\langle \cdot, \cdot \rangle_{X_{s,\delta}}$ *denote the inner-product* (5.4) *with* $\mathcal{A}^0$ *being the identity matrix.*

As a consequence of the weak convergence (6.29), we have that

$$\|U(t)\|_{X_{s,\delta}} \leq \liminf_k \left\| U^k(t) \right\|_{X_{s,\delta}}.$$

Thus the solution $U(t)$ belongs to $X_{s,\delta}$. For the proof of Proposition 6.6 see [16, §5].

Step 6. Here we shall prove uniqueness.

Proposition 6.7. *Suppose* $U_1(t), U_2(t) \in X_{s,\delta}$ *are two solutions of* (3.24) *with the same initial data, then* $U_1(t) \equiv U_2(t)$.

Proof. Let $V(t) = U_1(t) - U_2(t)$, then it satisfies the same type of a linear system as (6.20), therefore by similar estimates as in Step 4, we obtain that

$$\frac{d}{dt}\|V(t)\|^2_{Y_{\delta,\mathcal{A}^0}} \lesssim \|V(t)\|^2_{Y_{\delta,\mathcal{A}^0}},$$

and since $V(0) \equiv 0$, Gronwall's inequality implies that $V(t) \equiv 0$. □

Step 7. Since $X_{s,\delta}$ is a Hilbert space it suffices to show that

$$\limsup_{t\to 0^+} \|U(t)\|_{X_{s,\delta,\mathcal{A}^0}} \leq \|U(0)\|_{X_{s,\delta,\mathcal{A}^0}} \tag{6.30}$$

in order to establish the continuity in the norm. Here $\mathcal{A}^0$ depends on the initial data ϕ and W_0, that is, $\mathcal{A}^0 = \mathcal{A}^0(\phi, W_0)$. The proof of (6.30) relies on the same arguments as in [18] and therefore we leave it out. This completes the proof of Theorem 6.1.

7. Proof of the Main Result

The proof of the main result, Theorem 2.3, actually follows from Theorem 6.1, we just have to check whether the initial data of the gravitational fields and of the fluid satisfy the assumptions of Theorem 6.1. We recall that $v(t) = g_{\alpha\beta}(t) - \eta_{\alpha\beta}$, so setting $\phi = v(0)$, we have by the assumptions of Theorem 2.3 that $\phi \in H_{s+1,\delta}$. The initial data for the time derivative φ are given by $\partial_t g_{\alpha\beta}(0)$, where $\partial_t g_{ab}(0) = -2K_{ab}$ $(a, b = 1, 2, 3)$, and $\partial_t g_{\alpha 0}(0)$ is given by expression (2.7). By the assumption of Theorem 2.3, $K_{ab} \in H_{s,\delta+1}$ and therefore by Propositions 4.5 and 4.9, $\partial_t g_{\alpha 0}(0)$ also belongs to $H_{s,\delta+1}$. Thus $\varphi = \partial_t g_{\alpha\beta}(0)$ satisfies the initial condition of Theorem 6.1. Note that $\mathbf{a}^0_{33}(0) = h^{ab}$, where h_{ab} is a proper Riemannian metric. Since $w(0) \geq 0$ and $u^\alpha(0)$ is a unit timelike vector, $\mathbf{a}^0_{44}(0)$ is a positive definite matrix by Theorem 3.3. Hence $\mathcal{A}^0(\phi, W_0)$ satisfy condition (6.1) and we conclude that $U(t) = \big(g_{\alpha\beta}(t) - \eta_{\alpha\beta}, \partial_t g_{\alpha\beta}(t), \partial_x g_{\alpha\beta}(t), W(t)\big) \in C([0, T], X_{s,\delta})$. Hence (2.8) follows from Remark 4.11, and (2.9) from (6.2). That completes the proof.

8. Appendix

The classical paper of Hughes, Kato and Marsden [14] established the short time existence of the vacuum Einstein equations by solving a second order quasi-linear hyperbolic system whose solutions $(g_{\alpha\beta}, \partial_t g_{\alpha\beta})$ belong to $H^{s+1} \times H^s$ for $s > \frac{3}{2}$.

On the other hand, Fisher and Marsden treated the Einstein vacuum equation by means of the theory of symmetric hyperbolic systems. However, they only obtained the regularity of $s > \frac{7}{2}$. In [16] we generalized the result of [14] to the $H_{s,\delta}$ spaces, treating however, the Einstein equations as a symmetric hyperbolic system. Since the techniques of [16], and in particular the energy estimates, play an essential role in the present paper, we outline its main idea that enables us to obtain the same regularity as in [14].

We present a heuristic argument explaining the essential idea. First, if a function v satisfies a wave equation, then the vector $V = (v, \partial_t v, \partial_x v)$ satisfies a symmetric hyperbolic system. The general condition for existence and uniqueness in the $H^s(\mathbb{R}^3)$ spaces is $s > \frac{5}{2}$. Hence, we have by this method that $\partial_t v, \partial_x v \in H^s$ for $s > \frac{5}{2}$.

However, in our case we improve this regularity to $(v, \partial_t v, \partial_x v) \in H^{s+1} \times H^s \times H^s$ for $s > \frac{3}{2}$. This is because we do not consider a general quasi-linear symmetric hyperbolic system where the matrices $A^a(V)$ depend on V, but a system in which the matrices $A^a(v)$ only depend on v but **not** on its derivatives.

In order to see how this fact allows us to improve the regularity of the solution we will derive energy estimates for the linearized symmetric hyperbolic system. For the sake of clarity we consider a simple hyperbolic system

$$\partial_t V = A^a(v)\partial_a V,$$

then its linearized form is

$$\partial_t V = \tilde{A}^a \partial_a V. \tag{8.1}$$

Note that in each iteration we solve the linear system (8.1) with $\widetilde{A}^a = \widetilde{A}^a(v^k)$, and since $V^k = (v^k, \partial_t v^k, \partial_x v^k) \in H^s$, $v^k \in H^{s+1}$, and hence $\widetilde{A}^a = \widetilde{A}^a(v^k) \in H^{s+1}$ by Moser type estimates. The crucial step is to derive the energy estimate

$$\frac{d}{dt}\left(\frac{1}{2}\|V\|_{H^s}^2\right) \leq C\|V\|_{H^s}^2 \tag{8.2}$$

for $s > \frac{3}{2}$ and whenever V satisfies the linear system (8.1). We recall that $\|V\|_{H^s} = \|\Lambda^s V\|_{L^2}$, where Λ^s is the pseudodifferential operator $(1-\Delta)^{\frac{s}{2}}$.

One of the basic tools for obtaining (8.2) are the commutator estimates. Here we shall use the following Pseudodifferential operator version of the Kato–Ponce estimate [26, §3.6]: Let P be a differential operator in the class $OPS^s_{1,0}$, then

$$\|P(fg) - fP(g)\|_{L^2} \leq C\left\{\|\nabla f\|_{L^\infty}\|g\|_{H^{s-1}} + \|f\|_{H^s}\|g\|_{L^\infty}\right\}, \tag{8.3}$$

for any $f \in H^s \cap C^1$ and $g \in H^{s-1} \cap L^\infty$.

The standard way to obtain (8.2) is to differentiate $\|V\|_{H^s}^2$ with respect to time, to insert the differential equation (8.1) and then apply a suitable commutator which leads to

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|V\|_{H^s}^2 &= \left\langle \Lambda^s(V), \Lambda^s(\partial_t V)\right\rangle_{L^2} = \left\langle \Lambda^s(V), \Lambda^s\left(\widetilde{A}^a\partial_a V\right)\right\rangle_{L^2} \\
&= \left\langle \Lambda^s(V), \widetilde{A}^a\left(\Lambda^s(\partial_a V)\right)\right\rangle_{L^2} \\
&\quad + \left\langle \Lambda^s(V), \left[\Lambda^s\left(\widetilde{A}^a\partial_a V\right) - \widetilde{A}^a\left(\Lambda^s(\partial_a V)\right)\right]\right\rangle_{L^2},
\end{aligned}$$

and then the first term is taken care of by integration by parts and the second one is by applying the above Kato–Ponce estimate to the operator Λ^s. But this procedure results in a term of the form $\|\partial_a V\|_{L^\infty}$ which contains $\|\partial_a\partial_x v\|_{L^\infty}$. In order to estimate it by $\|\partial_a\partial_x v\|_{H^{s-1}} \lesssim \|\partial_x v\|_{H^s}$ we need to require that $s - 1 > \frac{3}{2}$, and hence we do not get the desired result.

We circumvent this difficulty by writing

$$\widetilde{A}^a\partial_a V = \partial_a\left(\widetilde{A}^a V\right) - \left(\partial_a\widetilde{A}^a\right)V,$$

and making the commutation

$$\Lambda^s\left(\partial_a\left(\widetilde{A}^a V\right)\right) = \left[\left(\Lambda^s\partial_a\right)\left(\widetilde{A}^a V\right) - \widetilde{A}^a\left(\Lambda^s\partial_a\right)(V)\right] + \widetilde{A}^a\left(\Lambda^s\partial_a\right)(V),$$

which we insert into the first row of Eq. (8.4). Then we have to estimate three terms:

$$\begin{aligned}
I &= \left\langle \Lambda^s(V), \left[\left(\Lambda^s\partial_a\right)\left(\widetilde{A}^a V\right) - \widetilde{A}^a\left(\Lambda^s\partial_a\right)(V)\right]\right\rangle_{L^2}, \\
II &= \left\langle \Lambda^s(V), \widetilde{A}^a\left(\Lambda^s\partial_a\right)(V)\right\rangle_{L^2},
\end{aligned}$$

and

$$III = \left\langle \Lambda^s(V), \Lambda^s\left((\partial_a \widetilde{A}^a)V\right)\right\rangle_{L^2}.$$

For the first term we apply the Kato-Ponce commutator (8.3). However, this time we do it for the operator $(\Lambda^s \partial_a)$ which has order $s+1$, and hence

$$\begin{aligned}|I| &\leq \|V\|_{H^s} \left\|\left(\Lambda^s \partial_a\right)\left(\widetilde{A}^a V\right) - \widetilde{A}^a \left(\Lambda^s \partial_a\right)(V)\right\|_{L^2} \\ &\lesssim \|V\|_{H^s} \left\{\|\nabla \widetilde{A}^a\|_{L^\infty} \|V\|_{H^s} + \|\widetilde{A}^a\|_{H^{s+1}} \|V\|_{L^\infty}\right\}.\end{aligned}$$

So by the Sobolev embedding theorem, we see that $|I| \lesssim \|\widetilde{A}^a\|_{H^{s+1}} \|V\|^2_{H^s}$. Likewise, since H^s is an algebra for $s > \frac{3}{2}$,

$$|III| \lesssim \|V\|_{H^s} \|(\partial_a \widetilde{A}^a)V\|_{H^s} \lesssim \|V\|^2_{H^s} \|(\partial_a \widetilde{A}^a)\|_{H^s} \lesssim \|\widetilde{A}^a\|_{H^{s+1}} \|V\|^2_{H^s}.$$

Since $\Lambda^s \partial_a = \partial_a \Lambda^s$ and $\widetilde{A}^a$ is symmetric, we obtain a similar estimate for II by using integration by parts. Hence we conclude that the energy estimate (8.2) holds. Note that in the estimate of all three terms above we have used the fact that $\widetilde{A}^a \in H^{s+1}$.

For the general case where $A^0 \neq I$, one has to define an appropriated inner-product which takes into account the matrix A^0. Details for the vacuum equations in the weighted spaces $H_{s,\delta}$ and a positive definite A^0 can be found in [16, §4] and only slight modifications are needed in order to extend the energy estimates of [16] to the coupled system (3.24).

References

1. Aubin, T.: *Some Nonlinear Problems in Riemannian Geometry*. Berlin-Heidelberg-New York: Springer-Verlag, 1998
2. Bezard, M.: Existence locale de solutions pour les equations d'Euler–Poisson. Japan J. Ind. Appl. Math. **10**, 431–450 (1993)
3. Brauer, U., Karp, L.: Local existence of classical solutions of the Einstein–Euler system using weighted Sobolev spaces of fractional order. Les Comptes Rendus de l'Académie Des Sciences /Série. Mathematique **345**(1), 49–54 (2007)
4. Brauer, U., Karp, L.: Well-posedness of the Einstein–Euler system in asymptotically flat spacetimes: the constraint equations. J. Diff. Eqs. **251**, 1428–1446 (2011)
5. Cantor, M.: Spaces of functions with asymptotic conditions on $\mathbb{R}^n$. Indiana Univ. Math. J. **24**(9), 897–902 (1975)
6. Choquet-Bruhat, Y.: Théorème d'existence pour certains systèmes d'équations aux dérivées partelles non linéaires. Acta Math. **88**, 141–225 (1952)
7. Choquet-Bruhat, Y., York, J.W.: *The Cauchy problem*. In: General relativity and Gravitation, A. Held, ed., International Society of General Relativity, New York: Prenum Press, 1980, pp. 99–172
8. Christodoulou, D.: The boost problem for weakly coupled quasilinear hyperbolic systems of the second order. J. Math. Pures Appl. **60**(3), 99–130 (1981)
9. Chruściel, P., Galloway, G., Pollack, D.: Mathematical general relativity: a sampler. Bull. Am. Math. Soc. **47**, 567–638 (2010)
10. DeTurck, D.M.: The Cauchy problem for Lorentz metrics with prescribed Ricci curvature. Compositio Math. **48**(3), 157–162 (1983)
11. Fischer, A.E., Marsden, J.E.: The Einstein evolution equations as a first–order quasi–linear symmetric hyperbolic system. Commun. Math. Phys. **28**, 1–38 (1972)
12. Gamblin, P.: Solution réguliére á temps petit por l'équation d'Euler–Poisson. Comm. Part. Diff. Eqs. **18**, 731–745 (1993)
13. Hawking, S.W., Ellis, G.R.W.: *The Large Scale Structure of Space–time*. Cambridge: Cambridge University Press, 1973
14. Hughes, T.J.R., Kato, T., Marsden, J.E.: Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. Arch. Ratl. Mech. Anal. **63**(3), 273–294 (1977)

15. John, F.: *Partial differential equations*. Berlin-Heidelberg-New York: Springer, 1986
16. Karp, L.: On the well-posedness of the vacuum Einstein's equations. J. Evol. Eqs. **11**, 641–673 (2011)
17. Klainerman, S., Rodnianski, I.: Rough solutions of the Einstein-vacuum equations. Ann. Math. **161**(3), 1143–1193 (2005)
18. Majda, A.: *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*. New York: Springer, 1984
19. Makino, T.: *On a local existence theorem for the evolution equation of gaseous stars*. In: Patterns and Waves, T. Nishida, M. Mimura, H. Fujii, eds., Amsterdam: North–Holland, 1986, pp. 459–479
20. Maxwell, D.: Rough solutions of the Einstein constraint equations. J. Reine Angew. Math. **590**, 1–29 (2006)
21. Nirenberg, L., Walker, H.: The null spaces of elliptic differential operators in $\mathbb{R}^n$. J. Math. Anal. Appl. **42**, 271–301 (1973)
22. Oliynyk, T.A.: The Newtonian limit for perfect fluids. Commun. Math. Phys. **276**(1), 131–188 (2007)
23. Planchon, F., Rodnianski, I.: *On Uniqueness for the Cauchy Problem in General Relativity*, unpublished
24. Rendall, A.D.: The initial value problem for a class of general relativistic fluid bodies. J. Math. Phys. **33**(2), 1047–1053 (1992)
25. Runst, T., Sickel, W.: *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*. Berlin: Walter de Gruyter, 1996
26. Taylor, M.E.: *Pseudodifferential Operators and Nonlinear PDE*, Progress in Mathematics, Vol. **100**, Boston-Basel-Berlin: Birkhäuser, 1991, Second printing, 1993
27. Taylor, M.E.: *Partial Differential Equations, Vol. iii, Nonlinear Equations*. Berlin-Heidelberg-New York: Springer, 1997
28. Triebel, H.: Spaces of Kudrjavcev type I. Interpolation, embedding, and structure. J. Math. Anal. Appl. **56**(2), 253–277 (1976)
29. Wald, R.M.: *General Relativity*. Chicago, IL: The University Press of Chicago, 1984

Communicated by P. T. Chruściel