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A Class of Schrödinger Operators with Decaying Oscillatory Potentials

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Abstract: We discuss Schrödinger operators on a half-line with decaying oscillatory potentials, such as products of an almost periodic function and a decaying function. We provide sufficient conditions for preservation of absolutely continuous spectrum and give bounds on the Hausdorff dimension of the singular part of the spectral measure. We also discuss the analogs for orthogonal polynomials on the real line and unit circle.

1. Introduction

In this paper, we investigate half-line Schrödinger operators

$$(Hu)(x) = -u''(x) + V(x)u(x), \tag{1.1}$$

with decaying oscillatory potentials $V : (0, \infty) \to \mathbb{R}$. All operators we consider have 0 as a regular point and are limit point at $+\infty$. Therefore, the expression (1.1), together with a choice of boundary condition $\theta \in [0, \pi)$, defines a Schrödinger operator H on $L^2(0, +\infty)$, with the domain

$$D(H) = \{u \in L^2(0, +\infty) \mid u, u' \in \text{AC}_{\text{loc}}, -u'' + Vu \in L^2, u'(0)\sin\theta = u(0)\cos\theta\}.$$

The operator H is self-adjoint, and for every $z \in \mathbb{C}$ with $\text{Im } z > 0$, there is a nontrivial solution of $-u''_z + Vu_z = zu_z$ which is square-integrable near $+\infty$. This can be used to define the m-function

$$m(z) = \frac{u'_z(0)\cos\theta + u_z(0)\sin\theta}{u_z(0)\cos\theta - u'_z(0)\sin\theta},$$

which, in turn, defines a canonical spectral measure μ by

$$d\mu = \frac{1}{\pi} \underset{\epsilon \downarrow 0}{\text{w-lim}} \, m(x + i\epsilon)dx$$

(the weak limit is with respect to continuous functions of compact support). The importance of μ lies in the fact that the operator H is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu(x))$.

The potentials we consider decay at $+\infty$, so $\sigma_{\text{ess}}(H) = [0, +\infty)$. The purpose of this paper is to characterize the type of spectrum on $[0, +\infty)$. More precisely, for $E > 0$, we study generalized eigenfunctions of H, i.e. solutions of

$$- u''(x) + V(x)u(x) = Eu(x) \tag{1.2}$$

and estimate the Hausdorff dimension of

$$S = \{E > 0 \mid \text{not all solutions of (1.2) are bounded}\} . \tag{1.3}$$

The importance of the set S, from a spectral theorist's point of view, is that by the work of Gilbert–Pearson [9], Behncke [2] and Stolz [28], on $(0, +\infty) \backslash S$, μ is mutually absolutely continuous with the Lebesgue measure.

We denote by $\text{Var}(\gamma, I)$ the variation of the function γ on the interval I,

$$\text{Var}(\gamma, I) = \sup_{k \in \mathbb{N}} \sup_{\substack{x_0, \ldots, x_k \in I \\ x_0 < \cdots < x_k}} \sum_{j=1}^{k} |\gamma(x_j) - \gamma(x_{j-1})|.$$

The following is our main result.

Theorem 1.1. *Let the potential V be given by*

$$V(x) = \sum_{k=1}^{\infty} c_k e^{-i\phi_k x} \gamma_k(x), \tag{1.4}$$

where $c_k \in \mathbb{C}$, $\phi_k \in \mathbb{R}$, and the following conditions hold:

(i) *(uniformly bounded variation) $\gamma_k(x) : (0, \infty) \to \mathbb{C}$ are functions of bounded variation whose variation is bounded uniformly in k, i.e.*

$$\sup_k \text{Var}(\gamma_k, (0, \infty)) < \infty; \tag{1.5}$$

(ii) *(uniform L^p condition) for some $p \in \mathbb{Z}$, $p \geq 2$,*

$$\sup_k \int_0^\infty |\gamma_k(x)|^p dx < \infty; \tag{1.6}$$

(iii) *(decay of coefficients) for some $\alpha \in (0, \frac{1}{p-1})$,*

$$\sum_{k=1}^{\infty} |c_k|^\alpha < \infty. \tag{1.7}$$

Then the set S given by (1.3) has Hausdorff dimension at most $(p-1)\alpha$, and $[0, \infty)$ is the essential support of the absolutely continuous spectrum of H.

Note that conditions (i)–(ii) above imply that $\lim_{x \to \infty} \gamma_k(x) = 0$ for all k.

Bounded variation conditions have been analyzed in spectral theory since Weidmann's theorem [31], but finite sums of the form (1.4) were first analyzed by Wong [32],

in the setting of orthogonal polynomials on the unit circle, in the L^2 case. In the Schrödinger operator literature, Wigner–von Neumann type potentials have attracted attention since Wigner–von Neumann [30] and have been studied by Atkinson [1], Harris–Lutz [12], Reed–Simon [25, Thm. XI.67], Ben-Artzi–Devinatz [3] and Janas–Simonov [13]. Those results are mostly restricted to the L^2 case, with the exception of Janas–Simonov [13] which includes the L^3 case.

Theorem 1.1 continues our earlier work in [21], which was, in turn, the analog of the work [22] on orthogonal polynomials. It is proved in [21] that if the potential V is given by a sum of the form (1.4), with only finitely many non-zero terms and $V \in L^p$, then S is a subset of an explicit finite set which depends only on p and the set of frequencies ϕ_k. The constructions of Krüger [17] and Lukic [21] show that this result is optimal; in particular, when there are finitely many terms, the p-dependence of possible singular spectrum is a real phenomenon, and not just an artifact of the method. This encourages us to conjecture that the p-dependence of possible Hausdorff dimension in Theorem 1.1 is also a real phenomenon; however, no such result is presently known.

In the special case when all the γ_k are equal, the potential becomes the product of an almost periodic function and a decaying function.

Corollary 1.2. *Let $V(x) = \gamma(x)W(x)$, where the following conditions hold:*

(i) *$\gamma(x) : (0, \infty) \to \mathbb{C}$ has bounded variation;*
(ii) *$W(x)$ is an almost periodic function given by*

$$W(x) = \sum_{k=1}^{\infty} c_k e^{-i\phi_k x}, \qquad (1.8)$$

  with $c_k \in \mathbb{C}$, $\phi_k \in \mathbb{R}$, and with (1.7) satisfied for some $\alpha \in (0, \frac{1}{p-1})$;
(iii) *$V \in L^p(0, \infty)$ for some $p \in \mathbb{Z}_+$, $p \geq 2$.*

Then the set S given by (1.3) has Hausdorff dimension at most $(p-1)\alpha$, and $[0, \infty)$ is the essential support of the absolutely continuous spectrum of H.

Corollary 1.2 is an immediate consequence of Theorem 1.1, except for the observation that the L^p condition can be moved from $V(x)$ to $\gamma(x)$, which is proved later. We singled out this special case because it was the main motivation for our work. For various classes of functions $W(x)$, multiplied by a decaying $\gamma(x)$, it has been studied which rate of decay preserves a.c. spectrum. If, instead of being almost periodic, $W(x)$ was sparse (Pearson [23], Kiselev–Last–Simon [15]) or random (Delyon–Simon–Souillard [5], Kotani–Ushiroya [16], Kiselev–Last–Simon [15]), L^2 decay of V would be critical for preservation of a.c. spectrum; however, if $W(x)$ was periodic, any decay would suffice to preserve a.c. spectrum (Golinskii–Nevai [10]). The answer for almost periodic $W(x)$ has been more elusive; Corollary 1.2 gives a partial answer, providing a sufficient condition for preservation of a.c. spectrum.

There have been other results investigating perturbations with bounded variation conditions: Stolz [28,29] investigated higher-order bounded variation conditions, proving purely a.c. spectrum on $[\limsup V, \infty)$. In another direction, Denisov [6,7] and Kaluzhny–Shamis [14] have shown that with a free or periodic background, perturbations of bounded L^2 variation preserve a.c. spectrum. L^2 variation conditions have also found their way into some necessary and sufficient conditions, in the setting of orthogonal polynomials [11,18,19,26,27].

Last [20] has shown that this picture does not carry over in the presence of an arbitrary background potential, by constructing examples of decaying V_0, V such that $\sigma_{ac}(-\Delta + V_0) = [-2, 2]$, but V has bounded variation and $\sigma_{ac}(-\Delta + V_0 + V) = \emptyset$.

The core of the method is summarized by the following technical lemma. To state the lemma, we need to introduce functions h_j of $1 + j$ variables, defined recursively by $h_0(\eta) = 1$ and

$$h_J(\eta; \phi_1, \ldots, \phi_J) = \frac{1}{\eta - \phi_1 - \cdots - \phi_J} \sum_{j=0}^{J-1} h_j(\eta; \phi_1, \ldots, \phi_j)$$

$$\times h_{J-j-1}(\eta; \phi_{j+1}, \ldots, \phi_{J-1}). \tag{1.9}$$

Lemma 1.3. *Let the potential V be given by* (1.4), *with $c_k \in \mathbb{C}$, $\phi_k \in \mathbb{R}$, and let $\eta \in (0, \infty)$ be such that the following conditions hold:*

(i) *(uniformly bounded variation) same as condition (i) of Theorem 1.1;*
(ii) *(uniform L^p condition) same as condition (ii) of Theorem 1.1;*
(iii) *(decay of coefficients)*

$$\sum_{k=1}^{\infty} |c_k| < \infty; \tag{1.10}$$

(iv) *(small divisor conditions) for $j = 1, \ldots, p - 1$,*

$$\sum_{k_1, \ldots, k_j = 1}^{\infty} \left| c_{k_1} \cdots c_{k_j} h_j(\eta; \phi_{k_1}, \ldots, \phi_{k_j}) \right| < \infty. \tag{1.11}$$

Then, for $E = \frac{\eta^2}{4}$, all solutions of (1.2) *are bounded.*

Remark 1.1. The proof of Lemma 1.3 shows that for real solutions $u(x)$, the quantity

$$u'(x)^2 + Eu(x)^2 \tag{1.12}$$

is bounded as $x \to \infty$, and a simple modification (pointed out in the proof) also shows that (1.12) converges as $x \to \infty$. However, the solution $u(x)$ does *not*, except in special cases, obey WKB asymptotics in its usual form. This is because for $p > 2$, there are correction terms in the Prüfer phase which depend on the frequencies ϕ_k, and cannot be expressed directly in terms of $V(x)$.

We also present the analogs of Theorem 1.1 for orthogonal polynomials on the real line and unit circle. Their proofs are largely analogous, so we will only explain the necessary modifications. We first state the result for orthogonal polynomials on the real line (OPRL).

Theorem 1.4. *Let ρ be a nontrivial probability measure on $\mathbb{R}$ with Lebesgue decomposition $d\rho = f(x)dx + d\rho_s$ into an absolutely continuous and a singular part. Let ρ have diagonal Jacobi coefficients $\{b_n\}_{n=1}^{\infty}$ and off-diagonal Jacobi coefficients $\{a_n\}_{n=1}^{\infty}$.*

Assume that there is an integer $p \in \mathbb{Z}$, $p \geq 2$, and a real number $\beta \in (0, \frac{1}{p-1})$, such that each of the sequences $\{a_n^2 - 1\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ can be written in the form

$$\sum_{l=1}^{\infty} c_l e^{-in\phi_l} \gamma_n^{(l)}, \tag{1.13}$$

where $c_l \in \mathbb{C}$, $\phi_l \in \mathbb{R}$, such that the following conditions hold:

(i) *(uniformly bounded variation)*

$$\sup_l \sum_{n=1}^{\infty} |\gamma_{n+1}^{(l)} - \gamma_n^{(l)}| < \infty; \tag{1.14}$$

(ii) *(uniform ℓ^p condition)*

$$\sup_l \sum_{n=1}^{\infty} |\gamma_n^{(l)}|^p < \infty; \tag{1.15}$$

(iii) *(decay of coefficients)*

$$\sum_{l=1}^{\infty} |c_l|^\beta < \infty. \tag{1.16}$$

Then there is a set S of Hausdorff dimension at most $\beta(p-1)$ with $\rho_s((-2,2)\backslash S) = 0$, and $f(x) > 0$ for Lebesgue-a.e. $x \in (-2,2)$.

Remark 1.2. The above theorem assumes that the sequence $\{a_n^2 - 1\}_{n=1}^{\infty}$ is of the form (1.13) and obeys the conditions listed there. The sequence $\{a_n^2 - 1\}_{n=1}^{\infty}$ appears naturally in the proof, but for a spectral theorist, it would be more natural to pose conditions on $\{a_n - 1\}_{n=1}^{\infty}$. However, if $a_n - 1 = (1.13)$, then

$$a_n^2 - 1 = (a_n - 1)^2 + 2(a_n - 1)$$

$$= \sum_{k,l=1}^{\infty} c_k c_l e^{-in(\phi_k + \phi_l)} \gamma_n^{(k)} \gamma_n^{(l)} + 2 \sum_{l=1}^{\infty} c_l e^{-in\phi_l} \gamma_n^{(l)}$$

is of the same form (with the same values of p and β), so there is an immediate corollary where the condition is applied on $\{a_n - 1\}_{n=1}^{\infty}$ instead.

The next result is for orthogonal polynomials on the unit circle (OPUC).

Theorem 1.5. *Let μ be a nontrivial probability measure on $\partial\mathbb{D}$ with Lebesgue decomposition $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$ into an absolutely continuous and a singular part. Let μ have Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ of the form*

$$\alpha_n = \sum_{l=1}^{\infty} c_l e^{-in\phi_l} \gamma_n^{(l)}, \tag{1.17}$$

where $c_l \in \mathbb{C}$, $\phi_l \in \mathbb{R}$, such that conditions (i)–(iii) of Theorem 1.4 hold with some odd integer $p \in \mathbb{Z}$, $p \geq 3$, and some $\beta \in (0, \frac{1}{p-2})$. Then there is a set S of Hausdorff dimension at most $\beta(p-2)$ with $\mu_s(\partial\mathbb{D}\backslash S) = 0$, and $w(\theta) > 0$ for Lebesgue-a.e. θ.

Remark 1.3. Note that in the previous theorem, the only critical values of p for the ℓ^p condition are *odd* integers. The same phenomenon was noticed for the finite frequency case in [22], and is in contrast with orthogonal polynomials on the real line and Schrödinger operators, where the statement changes at every integer value of p. There

is an informal way to understand why this happens. For all systems, the method tells us that critical points are of the form

$$\eta = \phi_{m_1} + \cdots + \phi_{m_k} - (\phi_{n_1} + \cdots + \phi_{n_l})$$

with $k + l < p$. However, only on the unit circle, we can rotate a measure; by rotating the measure by an angle ψ, we shift

$$\eta \mapsto \eta + \psi, \quad \phi_m \mapsto \phi_m + \psi$$

so only the critical points with $k - l = 1$ are preserved. However, increasing p, new points with $k - l = 1$, $k + l < p$ emerge only when p exceeds an odd integer value.

We prove Lemma 1.3 in Sects. 2–3. Sections 4 and 5 contain proofs of Theorem 1.1 and Corollary 1.2, respectively, and Sect. 6 describes the adaptations necessary to carry over the method to prove Theorems 1.4 and 1.5.

2. Preliminaries

To analyze solutions of (1.2), we use Prüfer variables, first introduced by Prüfer [24]. For

$$E = \frac{\eta^2}{4} \tag{2.1}$$

with $\eta > 0$ and for a real-valued nonzero solution $u(x)$ of (1.2), we define modified Prüfer variables $R(x)$, $\theta(x)$ by

$$u'(x) = \tfrac{1}{2}\eta R(x) \cos(\tfrac{1}{2}\eta x + \theta(x)), \tag{2.2}$$

$$u(x) = R(x) \sin(\tfrac{1}{2}\eta x + \theta(x)). \tag{2.3}$$

From (1.2), we obtain a system of first-order differential equations for $\log R$ and θ,

$$\frac{d\theta}{dx} = \frac{V(x)}{\eta}\left(\tfrac{1}{2}e^{i[\eta x + 2\theta(x)]} + \tfrac{1}{2}e^{-i[\eta x + 2\theta(x)]} - 1\right), \tag{2.4}$$

$$\frac{d}{dx}\log R(x) = \mathrm{Im}\left(\frac{V(x)}{\eta}e^{i[\eta x + 2\theta(x)]}\right). \tag{2.5}$$

Note that, by (2.2) and (2.3), boundedness of $R(x)$ implies boundedness of the corresponding solution of (1.2). Thus, the goal becomes to analyze the integral of (2.5),

$$\log R(b) - \log R(a) = \mathrm{Im} \int_a^b \frac{V(x)}{\eta}e^{i[\eta x + 2\theta(x)]}dx. \tag{2.6}$$

Note that we will, indeed, only estimate the imaginary part of the integral in (2.6). The real part does not, in general, converge as $b \to \infty$.

Substituting (1.4) into (2.6), our goal becomes to estimate integrals of the form

$$\int_a^b e^{Ki[\eta x + 2\theta(x)]}e^{-i(\phi_{m_1} + \cdots + \phi_{m_J})x}\gamma_{m_1}(x)\ldots\gamma_{m_J}(x)dx. \tag{2.7}$$

Initially, in (2.6), these integrals appear with $K = J = 1$, but later in the proof they appear with $J \geq 2$ and $0 \leq K \leq J$.

Integrals of the form (2.7) can be estimated by the following lemma, which is just a more quantitative version of Lemma 4.1 from [21]. To avoid placing an absolute continuity assumption on $\gamma_k(x)$, the proof is expressed in terms of Fubini's theorem rather than integration by parts. Remember that by (i), the variations of the γ_k are uniformly bounded,

$$\tau = \sup_k \operatorname{Var}(\gamma_k, (0, \infty)) < \infty. \tag{2.8}$$

Lemma 2.1. *Let $J, K \in \mathbb{Z}$ with $J \geq 1$ and $0 \leq K \leq J$. Let $0 \leq a < b < \infty$, and denote*

$$\Gamma(x) = \gamma_{m_1}(x) \ldots \gamma_{m_J}(x),$$
$$\phi = \phi_{m_1} + \cdots + \phi_{m_J}.$$

Then

$$\left| \int_a^b \left((\phi - K\eta)e^{Ki[\eta x + 2\theta(x)]}e^{-i\phi x}\Gamma(x) - 2Ke^{Ki[\eta x + 2\theta(x)]}e^{-i\phi x}\Gamma(x)\frac{d\theta}{dx} \right) dx \right| \leq 2\tau^J. \tag{2.9}$$

Proof. Without loss of generality assume that γ_k are left continuous. Then there exist finite positive measures ν_k on $\mathbb{R}$ and functions $s_k : \mathbb{R} \to \{-1, 1\}$ such that $\gamma_k(x) = \int_{[x,\infty)} s_k d\nu_k$ and $\nu_k([x, \infty)) = \operatorname{Var}(\gamma_k, [x, \infty)) \leq \tau$ by (2.8). Using Fubini–Tonelli's theorem and then integrating in x, rewrite the integral on the left-hand side of (2.9) as

$$\int_a^b \psi'(x)\Gamma(x)dx$$
$$= \int_{[a,\infty)^J} \int_a^{\min(t_1,\ldots,t_J,b)} \psi'(x)s_{m_1}(t_1) \cdots s_{m_J}(t_J)dx d\nu_{m_1}(t_1) \cdots d\nu_{m_J}(t_J)$$
$$= \int_{[a,\infty)^J} \big(\psi(\min(t_1, \ldots, t_J, b)) - \psi(a) \big) s_{m_1}(t_1) \cdots s_{m_J}(t_J)d\nu_{m_1}(t_1) \cdots d\nu_{m_J}(t_J),$$

where $\psi(x) = ie^{i(K\eta - \phi)x}e^{2iK\theta(x)}$. Since $|\psi(x)| = 1$, this implies

$$\left| \int_a^b \psi'(x)\Gamma(x)dx \right| \leq 2 \int_{[a,\infty)^J} |s_{m_1}(t_1) \cdots s_{m_J}(t_J)|d\nu_{m_1}(t_1) \cdots d\nu_{m_J}(t_J),$$

and integrating in $t_1, \ldots, t_J$ implies (2.9). $\quad \square$

We must keep track of integrals of the form (2.7) and the multiplicative constants with which they will appear in the method. We need to introduce quite a bit of notation, whose importance will become clear in Sect. 3 (or see [21] for more motivation). For instance, the integral (2.7) will appear multiplied by $f_{J,K}(\eta; \phi_{m_1}, \ldots, \phi_{m_J})$, with a function $f_{J,K}$ which we are about to define.

The functions $f_{J,K}$ and $g_{J,K}$ are introduced in [21], for $J, K \in \mathbb{Z}$ with $J \geq 1$ and $0 \leq K \leq J$. For other pairs $(J, K) \in \mathbb{Z}^2$, we take those functions to be zero by convention. They are functions of $1 + J$ variables, defined recursively by

$$f_{1,0}(\eta; \phi_1) = -\frac{1}{\eta}, \qquad f_{1,1}(\eta; \phi_1) = \frac{1}{\eta}, \tag{2.10}$$

and

$$g_{J,K}(\eta; \{\phi_j\}_{j=1}^J) = -\frac{2K}{K\eta - \sum_{j=1}^J \phi_j} f_{J,K}(\eta; \{\phi_j\}_{j=1}^J), \tag{2.11}$$

$$f_{J,K}(\eta; \{\phi_j\}_{j=1}^J) = \frac{1}{\eta} \sum_{k=K-1}^{K+1} \sum_{\sigma \in S_J} \frac{1}{J!} \omega_{K-k} g_{J-1,k}(\eta; \{\phi_{\sigma(j)}\}_{j=1}^{J-1}), \quad J \geq 2, \tag{2.12}$$

where S_J denotes the symmetric group in J elements and

$$\omega_a = \begin{cases} -1 & a = 0 \\ \frac{1}{2} & a = \pm 1 \\ 0 & |a| \geq 2 \end{cases} \tag{2.13}$$

are constants which come from an alternative way of writing (2.4) as

$$\frac{d\theta}{dx} = \frac{V(x)}{\eta} \sum_{a=-1}^{1} \omega_a e^{ia[\eta x + 2\theta(x)]}. \tag{2.14}$$

Notation can be simplified by the following symmetric product.

Definition 2.1. *For a function p_I of $1 + I$ variables and a function q_J of $1 + J$ variables, their symmetric product is a function $p_I \odot q_J$ of $1 + (I + J)$ variables defined by*

$$(p_I \odot q_J)(\eta; \{\phi_i\}_{i=1}^{I+J}) = \frac{1}{(I+J)!} \sum_{\sigma \in S_{I+J}} p_I(\eta; \{\phi_{\sigma(i)}\}_{i=1}^I) q_J(\eta; \{\phi_{\sigma(i)}\}_{i=I+1}^{I+J}).$$

Further, it will be convenient to think of ω_a, with $a \in \mathbb{Z}$, as a function of $1 + 1$ variables, with values given by (2.13), and to introduce $\xi_{J,K}$, for $0 \leq K \leq J$, as a function of $1 + J$ variables,

$$\xi_{J,K}(\eta; \{\phi_j\}_{j=1}^J) = \begin{cases} \frac{(-1)^{K-1}}{\eta} & J = 1 \\ 0 & J \geq 2. \end{cases} \tag{2.15}$$

We can now rewrite (2.10), (2.12) as

$$f_{J,K} = \xi_{J,K} + \frac{1}{\eta} \sum_{a=-1}^{1} \omega_a \odot g_{J-1,K+a}. \tag{2.16}$$

It will also be useful to have notation for the corresponding functions with flipped signs of all but the first parameter,

$$\check{f}_{J,K}(\eta; \{\phi_j\}_{j=1}^J) = f_{J,K}(\eta; \{-\phi_j\}_{j=1}^J), \tag{2.17}$$

$$\check{g}_{J,K}(\eta; \{\phi_j\}_{j=1}^J) = g_{J,K}(\eta; \{-\phi_j\}_{j=1}^J), \tag{2.18}$$

and for

$$\mathcal{G}_{J,0} = \sum_{j=1}^{J-1} \sum_{k=1}^{\min\{j, J-j\}} \frac{1}{4k} g_{j,k} \odot \check{g}_{J-j,k}. \tag{2.19}$$

We now point out some identities among the functions just defined. The importance of these identities is mostly in locating singularities of those functions, rather than in the precise form of the identities. For instance, (2.11) seems to indicate that $g_{J,K}$ has a singularity when $K\eta = \sum_{j=1}^{J} \phi_j$, but (2.21) below implies that many of those singularities are removable and that all non-removable singularities stem from $g_{j,1}$ for some $j \le J$, with $\eta = \sum_{i=1}^{j} \phi_{m_i}$.

Lemma 2.2. (i) *For $0 \le K \le J$ and $0 < k < K$,*

$$f_{J,K} = \tfrac{1}{2} \sum_{j=0}^{J} f_{j,k} \odot g_{J-j,K-k}, \tag{2.20}$$

$$g_{J,K} = \tfrac{1}{2} \sum_{j=0}^{J} g_{j,k} \odot g_{J-j,K-k}. \tag{2.21}$$

(ii) *For $J \ge 2$,*

$$f_{J,0} - \check{f}_{J,0} = (\phi_1 + \cdots + \phi_J)\mathcal{G}_{J,0}, \tag{2.22}$$

assuming the parameters $\eta; \phi_1, \ldots, \phi_J$ for both sides of the identity.

(iii) *The functions $g_{J,1}$ are just rescaled and symmetrized h_J, namely,*

$$g_{J,1}(\eta; \{\phi_j\}_{j=1}^{J}) = -\frac{2}{\eta^J} \frac{1}{J!} \sum_{\sigma \in S_J} h_J(\eta; \{\phi_{\sigma(j)}\}_{j=1}^{J}). \tag{2.23}$$

Proof. (i) is a rescaled version of [21, Lem. 5.1(i)].
   (ii) Start from (2.11) to note

$$\frac{1}{2k}(\phi_1 + \cdots + \phi_J)g_{j,k} \odot \check{g}_{J-j,k} = -f_{j,k} \odot \check{g}_{J-j,k} + g_{j,k} \odot \check{f}_{J-j,k}.$$

Summing in j and k and using (2.16), we have

$$2(\phi_1 + \cdots + \phi_J)\mathcal{G}_{J,0} = -\xi_{1,1} \odot \check{g}_{J-1,1} - \frac{1}{\eta} \sum_{j=1}^{J-1} \sum_{k=1}^{\min\{j,J-j\}} \sum_{a=-1}^{1} \omega_a \odot g_{j-1,k+a} \odot \check{g}_{J-j,k}$$

$$+ \xi_{1,1} \odot g_{J-1,1} + \frac{1}{\eta} \sum_{j=1}^{J-1} \sum_{k=1}^{\min\{j,J-j\}} \sum_{a=-1}^{1} g_{j,k} \odot \omega_a \odot \check{g}_{J-j-1,k+a},$$

which implies (2.22) since the triple sums are equal (after a relabeling of indices) and $f_{J,0} = \frac{1}{\eta}\omega_1 \odot g_{J-1,1} = \frac{1}{2}\xi_{1,1} \odot g_{J-1,1}$.
   (iii) We prove (2.23) by induction on J. Start by verifying

$$g_{1,1}(\eta; \phi_1) = -\frac{2}{\eta} \frac{1}{\eta - \phi_1} = -\frac{2}{\eta} h_1(\eta; \phi_1).$$

For $J \geq 2$, by (2.11), (2.16) and (2.21), we have

$$g_{J,1}(\eta; \{\phi_j\}_{j=1}^J) = -\frac{2}{\eta - \sum_{j=1}^J \phi_j} f_{J,1}$$

$$= -\frac{2}{\eta - \sum_{j=1}^J \phi_j} \frac{1}{\eta} \left(\omega_0 \odot g_{J-1,1} + \tfrac{1}{2}\omega_1 \odot \sum_{j=1}^{J-2} g_{j,1} \odot g_{J-j-1,1} \right).$$

By the inductive hypothesis, this implies

$$g_{J,1}(\eta; \{\phi_j\}_{j=1}^J) = -\frac{2}{\eta - \sum_{j=1}^J \phi_j} \frac{1}{\eta^J} \left(-2\omega_0 \odot h_{J-1} + 2\omega_1 \odot \sum_{j=1}^{J-2} h_j \odot h_{J-j-1} \right).$$

Using (2.13) and $h_0 = 1$, the inductive step is completed. □

3. Proof of Lemma 1.3

In this section, we freely use all assumptions of Lemma 1.3. We break up its proof into several lemmas. Let us start by denoting

$$\sigma = \sup_k \|\gamma_k\|_p, \tag{3.1}$$

which is finite by assumption (ii).

Denoting

$$S_{J,K}(x) = \sum_{m_1,\ldots,m_J=1}^\infty f_{J,K}(\eta; \phi_{m_1}, \ldots, \phi_{m_J})\beta_{m_1}(x) \ldots \beta_{m_J}(x)e^{iK[\eta x + 2\theta(x)]}, \tag{3.2}$$

where

$$\beta_k(x) = c_k e^{-i\phi_k x} \gamma_k(x), \tag{3.3}$$

(2.6) becomes

$$\log R(b) - \log R(a) = \text{Im} \int_a^b S_{1,1}(x)dx. \tag{3.4}$$

The idea of the proof is to iteratively replace $S_{1,1}$ by a sum of $S_{J,K}$'s with ever higher values of J. We will have to keep track of the errors, so denote

$$E_{J,K} = \sum_{m_1,\ldots,m_J=1}^\infty \left| c_{m_1} \ldots c_{m_J} g_{J,K}(\eta; \phi_{m_1}, \ldots, \phi_{m_J}) \right| \tag{3.5}$$

(note that $E_{J,K}$ is trivially zero unless $1 \leq K \leq J$, since the same is true of $g_{J,K}$) and

$$\mathcal{E}_{J,0} = \sum_{m_1,\ldots,m_J=1}^\infty \left| c_{m_1} \ldots c_{m_J} \mathcal{G}_{J,0}(\eta; \phi_{m_1}, \ldots, \phi_{m_J}) \right| \tag{3.6}$$

for $K = 0$.

Lemma 3.1. $E_{J,K}$ *is finite when* $1 \leq K \leq J \leq p-1$ *and* $\mathcal{E}_{J,0}$ *is finite for* $2 \leq J \leq p$.

Proof. By (2.23), since the condition (1.11) holds for $J = 1, \ldots, p-1$, $E_{J,1}$ is finite for the same values of J. Now note that (2.21) implies

$$E_{J,K} \leq \tfrac{1}{2} \sum_{j=0}^{J} E_{j,k} E_{J-j,K-k}, \tag{3.7}$$

and (2.22) implies

$$\mathcal{E}_{J,0} \leq \sum_{j=1}^{J-1} \sum_{k=1}^{\min\{j,J-j\}} \tfrac{1}{4k} E_{j,k} E_{J-j,k}, \tag{3.8}$$

and the lemma follows from these two identities. $\square$

Lemma 3.2. *The sum* $\mathcal{S}_{J,K}(x)$ *is absolutely convergent when* $0 \leq K \leq J \leq p$, *and if in addition* $J \geq 2$, *then*

$$\sum_{m_1,\ldots,m_J=1}^{\infty} \left| f_{J,K}(\eta; \phi_{m_1}, \ldots, \phi_{m_J}) \beta_{m_1}(x) \ldots \beta_{m_J}(x) \right|$$

$$\leq \tfrac{1}{\eta} \sum_{a=-1}^{1} |\omega_a| E_{J-1,K+a} \sum_{l=1}^{\infty} |c_l| \tau^J. \tag{3.9}$$

If $J = p$, *then also*

$$\int_0^{\infty} \sum_{m_1,\ldots,m_J=1}^{\infty} \left| f_{J,K}(\eta; \phi_{m_1}, \ldots, \phi_{m_J}) \beta_{m_1}(x) \ldots \beta_{m_J}(x) \right| dx$$

$$\leq \tfrac{1}{\eta} \sum_{a=-1}^{1} |\omega_a| E_{p-1,K+a} \sum_{l=1}^{\infty} |c_l| \sigma^p. \tag{3.10}$$

Proof. Equation (2.16) implies

$$|f_{J,K}| \leq |\xi_{J,K}| + \tfrac{1}{\eta} \sum_{a=-1}^{1} |\omega_a| \odot |g_{J-1,K+a}|. \tag{3.11}$$

Multiplying by

$$|\beta_{m_1}(x) \ldots \beta_{m_J}(x)| \leq |c_{m_1} \ldots c_{m_J}| \tau^J$$

(which follows from (2.8)) and summing in $m_1, \ldots, m_J$ proves (3.9). For $J = p$, multiplying (3.11) instead by

$$\int_0^{\infty} |\beta_{m_1}(x) \ldots \beta_{m_p}(x)| dx \leq |c_{m_1} \ldots c_{m_p}| \sigma^p$$

(which follows from (3.1)) and summing in $m_1, \ldots, m_p$ proves (3.10). $\square$

Lemma 3.3. *For $J = 1, \ldots, p-1$ and $0 \le a < b < \infty$,*

$$\left| \int_a^b \left(\sum_{K=1}^J \mathcal{S}_{J,K} - \sum_{K=0}^{J+1} \mathcal{S}_{J+1,K} \right) dx \right| \le \sum_{K=1}^J \frac{1}{K} E_{J,K} \tau^J. \tag{3.12}$$

Proof. For $K \ge 1$, use Lemma 2.1 and multiply (2.9) by $\frac{1}{2K} g_{J,K}(\eta; \phi_{m_1}, \ldots, \phi_{m_J})$ to conclude

$$\left| \int_a^b \left(f_{J,K} e^{Ki[\eta x + 2\theta(x)]} e^{-i\phi x} \Gamma(x) - g_{J,K} e^{Ki[\eta x + 2\theta(x)]} e^{-i\phi x} \Gamma(x) \frac{d\theta}{dx} \right) dx \right| \le \frac{1}{K} |g_{J,K}| \tau^J,$$

where we have used (2.11) and the notation in Lemma 2.1. Multiply by $c_{m_1} \ldots c_{m_J}$, sum in $m_1, \ldots, m_J$ from 1 to ∞, and sum in K from 1 to J to conclude (3.12). The sum containing the $g_{J,K}$ turns into the sum of $\mathcal{S}_{J+1,K}$ by using (1.4), (2.14) and (2.12).

The infinite summation is justified by Fubini's Theorem, by Lemmas 3.1 and 3.2. $\quad\square$

Lemma 3.4. *For $J = 2, \ldots, p$ and $0 \le a < b < \infty$,*

$$\left| \text{Im} \int_a^b \mathcal{S}_{J,0}(x) dx \right| \le \mathcal{E}_{J,0} \tau^J. \tag{3.13}$$

Proof. Without loss of generality, we can assume that for each term (3.3) in the sum (1.4), the sum also contains a term $\bar{c}_k e^{i\phi_k x} \bar{\gamma}_k(x)$; we can fulfill this assumption by taking the representation (1.4) and averaging it with its complex conjugate, since $V(x)$ is real-valued. Then, note that for every term

$$f_{J,0}(\eta; \phi_{m_1}, \ldots, \phi_{m_J}) \beta_{m_1}(x) \ldots \beta_{m_J}(x)$$

in $\mathcal{S}_{J,0}$, there is another term with opposite signs of the ϕ_{m_j},

$$f_{J,0}(\eta; -\phi_{m_1}, \ldots, -\phi_{m_J}) \bar{\beta}_{m_1}(x) \ldots \bar{\beta}_{m_J}(x).$$

Averaging those two terms and using Lemma 2.2(ii) and Lemma 2.1, we can estimate

$$\frac{1}{2} \left| \text{Im} \int_a^b \left(f_{J,0} \beta_{m_1}(x) \ldots \beta_{m_J}(x) + \check{f}_{J,0} \bar{\beta}_{m_1}(x) \ldots \bar{\beta}_{m_J}(x) \right) dx \right|$$

$$= \frac{1}{2} \left| \text{Im} \int_a^b \left((f_{J,0} - \check{f}_{J,0}) \beta_{m_1}(x) \ldots \beta_{m_J}(x) \right) dx \right|$$

$$\le |\mathcal{G}_{J,0}(\eta; \phi_{m_1}, \ldots, \phi_{m_J})| \tau^J.$$

Summing in $m_1, \ldots, m_J$ implies (3.13). $\quad\square$

Proof of Lemma 1.3. Summing (3.12) in $J = 1, \ldots, p-1$, we obtain

$$\left| \int_a^b \left(\mathcal{S}_{1,1}(x) - \sum_{K=1}^p \mathcal{S}_{p,K}(x) - \sum_{j=2}^p \mathcal{S}_{j,0}(x) \right) dx \right| \le \sum_{j=1}^{p-1} \sum_{k=1}^j \frac{1}{k} E_{j,k} \tau^j. \tag{3.14}$$

Meanwhile, using Lemma 3.2 for $J = p$, using the triangle inequality on (3.10) and summing in K,

$$\left| \sum_{K=1}^p \int_a^b \mathcal{S}_{p,K}(x) dx \right| \le \frac{2}{\eta} \sum_{k=0}^{p-1} E_{p-1,k} \sum_{l=1}^{\infty} |c_l| \sigma^p. \tag{3.15}$$

Taking the imaginary part of (3.14) and using Lemma 3.4 and (3.15), we conclude (remembering (3.4)) that

$$
|\log R(b) - \log R(a)| \le \sum_{j=1}^{p-1} \sum_{k=1}^{j} \frac{1}{k} E_{j,k} \tau^j + \sum_{J=2}^{p} \mathcal{E}_{J,0} \tau^J + \frac{2}{\eta} \sum_{k=0}^{p} E_{p-1,k} \sum_{l=1}^{\infty} |c_l| \sigma^p.
$$

(3.16)

All we need from this inequality is that it is an estimate independent of b. Thus, $\log R(b)$ is a bounded function as $b \to \infty$, which shows that $u(x)$ is a bounded function.

We presented the proof in this way for (relative) clarity. If we had, instead of using τ and σ, written the estimates in terms of the γ_k and used that throughout the method, (3.16) would be an inequality with a right-hand side that converges to 0 as $a \to \infty$. This would automatically imply that $\log R(x)$ is Cauchy as $x \to \infty$, and that a finite limit

$$
\lim_{x \to \infty} \log R(x)
$$

exists.   □

4. Proof of Theorem 1.1

To prove Theorem 1.1 starting from Lemma 1.3, we need to estimate the Hausdorff dimension of the set of η which the small divisor condition (1.11) fails for some $j < p$. For instance, for $p = 2$ we need to estimate the dimension of the set of η where

$$
\sum_{l=1}^{\infty} \left| \frac{c_l}{\eta - \phi_l} \right| = \infty;
$$

for $p = 3$ we also need to estimate the dimension of the set where

$$
\sum_{k,l=1}^{\infty} \left| \frac{c_k c_l}{(\eta - \phi_k)(\eta - \phi_k - \phi_l)} \right| = \infty;
$$

etc. We will use measures with the following property: for $\beta \in [0, 1]$, a Borel measure ν on $\mathbb{R}$ is uniformly β-Hölder continuous (or UβH) if there exists $\tilde{C} < \infty$ such that for every interval $I \subset \mathbb{R}$ with $|I| < 1$,

$$
\nu(I) \le \tilde{C}|I|^{\beta},
$$

(4.1)

where $|\cdot|$ denotes Lebesgue measure. If ν is finite, the condition $|I| < 1$ can be removed (while possibly changing $\tilde{C}$). The condition (4.1) enters the proof through the following lemma.

Lemma 4.1. *Let ν be a finite UβH measure on $\mathbb{R}$.*

(i) *If $\alpha \in (0, \beta)$, then for all $\psi \in \mathbb{R}$,*

$$
\int \frac{1}{|\eta - \psi|^{\alpha}} d\nu(\eta) \le D_{\alpha},
$$

(4.2)

where D_{α} is a finite constant which depends only on α and not on ψ.

(ii) *For $J \geq 1$ and $\alpha \in (0, \frac{\beta}{J})$,*

$$\int |h_J(\eta; \phi_1, \ldots, \phi_J)|^\alpha \, d\nu(\eta) \leq C_J D_{J\alpha}, \tag{4.3}$$

where $C_J = \frac{1}{J+1} \binom{2J}{J}$ are Catalan numbers.

Proof. (i) By Fubini's theorem, and picking an arbitrary $\epsilon \in (0, \infty)$,

$$\int \frac{1}{|\eta - \psi|^\alpha} d\nu(\eta) = \int_0^\infty \nu\left(\left\{\eta : \frac{1}{|\eta - \psi|^\alpha} > t\right\}\right) dt$$

$$\leq \epsilon \nu(\mathbb{R}) + \int_\epsilon^\infty \tilde{C}(2t^{-1/\alpha})^\beta dt$$

$$\leq \epsilon \nu(\mathbb{R}) + \tilde{C} \frac{2^\beta}{\frac{\beta}{\alpha} - 1} \epsilon^{1 - \beta/\alpha},$$

which is a bound independent on ψ, concluding the proof.

(ii) The proof proceeds by induction. For $J = 0$ the statement is trivial.

Assume the statement is true for all $j < J$. Integrating one term of the sum on the right-hand side of (1.9) and using Hölder's inequality and the inductive hypothesis, we get

$$\int \left|\frac{1}{\eta - \phi_1 - \cdots - \phi_J} h_j h_{J-j-1}\right|^\alpha d\nu(\eta) \leq D_{J\alpha}^{1/J} (C_j D_{J\alpha})^{j/J} (C_{J-j-1} D_{J\alpha})^{(J-j-1)/J}$$

$$\leq C_j C_{J-j-1} D_{J\alpha}.$$

Summing in j, using (1.9), and remembering that Catalan numbers obey the recursion relation

$$C_J = \sum_{j=0}^{J-1} C_j C_{J-j-1},$$

we complete the inductive step. □

Lemma 4.2. *Assume that (1.7) holds. Then, for a positive integer j, the set of η for which the condition (1.11) fails has Hausdorff dimension at most $j\alpha$.*

Proof. Denote by T the set of η where the condition (1.11) fails. If the Hausdorff dimension of T was greater than $j\alpha$, then for some $\beta > j\alpha$ we would have $h^\beta(T) = \infty$. Thus, there would exist a subset $T' \subset T$ such that $\nu = \chi_{T'} h^\beta$ is a finite UβH measure with $\nu(T) > 0$ (see, e.g.,[8, Thm. 5.6]).

Then Lemma 4.1(ii) implies

$$\int \sum_{k_1, \ldots, k_j=1}^\infty |c_{k_1} \cdots c_{k_j} h_j(\eta; \phi_{k_1}, \ldots, \phi_{k_j})|^\alpha \, d\nu(\eta) \leq C_j D_{j\alpha} \left(\sum_{k=1}^\infty |c_k|^\alpha\right)^j.$$

Since the integral is finite, the integrand must be ν-a.e. finite. However, for $\alpha \in (0, 1]$ and a sequence x_n of nonnegative numbers,

$$\sum_{n=1}^\infty x_n^\alpha < \infty \implies \sum_{n=1}^\infty x_n < \infty;$$

thus, (1.11) holds for ν-a.e. η, contradicting $\nu(T) > 0$. □

Proof of Theorem 1.1. Conditions (i)–(iii) of Lemma 1.3 are trivially satisfied. By Lemma 4.2, the condition (1.7) holds for all $j = 1, \ldots, p - 1$ away from a set of Hausdorff dimension at most $(p - 1)\alpha$. Thus, by Lemma 1.3, the Hausdorff dimension of the set S is at most $(p - 1)\alpha$ (the map $\eta \mapsto \frac{\eta^2}{4}$ obviously preserves Hausdorff dimension).

By the results of Gilbert–Pearson [9], Behncke [2] and Stolz [28], boundedness of solutions for $E \in (0, \infty) \backslash S$ implies that the canonical spectral measure $d\mu$ and Lebesgue measure are mutually absolutely continuous on $(0, \infty) \backslash S$, which completes the proof. $\square$

5. Proof of Corollary 1.2

Corollary 1.2 is a special case of Theorem 1.1, with all the $\gamma_k(x)$ taken to be equal to the same function $\gamma(x)$; by the following lemma, $V(x) \in L^p$ then implies $\gamma(x) \in L^p$, and the corollary is immediate.

Lemma 5.1. *Let $W(x)$ be (uniformly) almost periodic and not identically zero, and let $\gamma : (0, \infty) \to \mathbb{R}$ have bounded variation. Let $p \in [1, \infty)$. Then $W\gamma \in L^p(0, \infty)$ implies $\gamma \in L^p(0, \infty)$.*

Proof. If W is almost periodic, then so is $|W|^p$, since the map $t \mapsto |t|^p$ is uniformly continuous on compacts. If γ has bounded variation, then so does $|\gamma|^p$, since (by the mean value theorem for $t \mapsto t^p$)

$$\big| |\gamma(x)|^p - |\gamma(y)|^p \big| \leq p \|\gamma\|_\infty^{p-1} |\gamma(x) - \gamma(y)|.$$

Thus, it suffices to prove the lemma for $p = 1$.

We may pick $T > 0$ for which there exist $\delta, \Delta \in (0, \infty)$ such that for all $a \geq 0$,

$$\delta \leq \int_a^{a+T} |W(x)| dx \leq \Delta. \tag{5.1}$$

The upper bound is trivial with $\Delta = T \|W\|_\infty$, whereas existence of the lower bound for large enough T is a standard fact for non-zero almost periodic functions (see, e.g., [4, p. 20]).

For $x, y \in [a, a + T]$, by the triangle inequality,

$$|\gamma(y)| \leq |\gamma(x)| + |\gamma(x) - \gamma(y)| \leq |\gamma(x)| + \mathrm{Var}(\gamma, [a, a + T]).$$

Integrating in y from a to $a + T$, we conclude

$$\frac{1}{T} \int_a^{a+T} |\gamma(y)| dy \leq |\gamma(x)| + \mathrm{Var}(\gamma, [a, a + T]).$$

Multiplying by $|W(x)|$, integrating in x from a to $a + T$, and using (5.1), we obtain

$$\frac{\delta}{T} \int_a^{a+T} |\gamma(y)| dy \leq \int_a^{a+T} |W(x)\gamma(x)| dx + \mathrm{Var}(\gamma, [a, a + T]) \Delta.$$

Specialize to $a = nT$ and sum in n to obtain

$$\frac{\delta}{T} \int_0^\infty |\gamma(y)| dy \leq \int_0^\infty |W(x)\gamma(x)| dx + \mathrm{Var}(\gamma, [0, \infty)) \Delta,$$

which completes the proof since the right-hand side is finite. $\square$

6. An Outline of the Proofs of Theorems 1.4 and 1.5

The proofs of Theorems 1.4 and 1.5 follow the same ideas, adapted to the discrete case. We present a discussion of the necessary adaptations, omitting the computational details.

In [22], we have developed an iterative scheme for proving theorems similar to Theorems 1.4 and 1.5, but where (1.13) and (1.17) are finite sums. For both OPRL and OPUC, the essential spectrum can be parametrized by $\eta \in [0, 2\pi]$, by $\eta \mapsto 2\cos(\eta/2)$ or by $\eta \mapsto e^{i\eta}$, respectively. The suitable analog of Prüfer variables can be presented in a unified way for both OPRL and OPUC (see [22, Sect. 4]), as sequences r_n, θ_n obeying the recursion relations

$$\frac{r_{n+1}}{r_n} = \frac{|1 - \alpha_n e^{i[(n+1)\eta + 2\theta_n]} - c\bar{\alpha}_n|}{\sqrt{(1 - c\alpha_n)(1 - c\bar{\alpha}_n) - \alpha_n\bar{\alpha}_n}}, \tag{6.1}$$

$$e^{2i(\theta_{n+1} - \theta_n)} = \frac{1 - \bar{\alpha}_n e^{-i[(n+1)\eta + 2\theta_n]} - c\alpha_n}{1 - \alpha_n e^{i[(n+1)\eta + 2\theta_n]} - c\bar{\alpha}_n}, \tag{6.2}$$

where

$$c = \begin{cases} 0 & \text{for OPUC,} \\ 1 & \text{for OPRL.} \end{cases}$$

Here, for OPUC, α_n are just Verblunsky coefficients, whereas for OPRL,

$$\alpha_n = \frac{a_n^2 - 1 + e^{i\eta/2}b_{n+1}}{e^{i\eta} - 1}.$$

Thus, in either case, the sequence α_n is of the form (1.13). To discuss boundedness of the sequence r_n, we estimate partial sums of (6.1),

$$\sum_{n=M}^{N} e^{-in\phi} \Gamma_n e^{ik[(n+1)\eta + 2\theta_n]}, \tag{6.3}$$

where

$$\Gamma_n = \gamma_n^{(k_1)} \cdots \gamma_n^{(k_s)} \bar{\gamma}_n^{(l_1)} \cdots \bar{\gamma}_n^{(l_t)}, \tag{6.4}$$

$$\phi = \phi_{k_1} + \cdots + \phi_{k_s} - \phi_{l_1} - \cdots - \phi_{l_t}, \tag{6.5}$$

and $s + t < K$. The analog of Lemma 2.1 becomes (compare with [22, Lem. 6.1])

Lemma 6.1. *With notation as above,*

$$\sum_{n=M}^{N} \Big((e^{-i(k\eta - \phi)} - 1)e^{-in\phi} \Gamma_n e^{ik[(n+1)\eta + 2\theta_n]}$$

$$- e^{-in\phi} \Gamma_n e^{ik[(n+1)\eta + 2\theta_n]} \big(e^{2ik(\theta_{n+1} - \theta_n)} - 1 \big) \Big) \leq 2\tau^{s+t},$$

where τ is a uniform bound on the variation of the $\gamma^{(l)}$,

$$\tau = \sup_l \sum_{n=M}^{\infty} |\gamma_{n+1}^{(l)} - \gamma_n^{(l)}|.$$

This lemma drives an iterative procedure, and it is proved that the Prüfer amplitude r_n is bounded in n if certain small divisor conditions are met. The singularities involved are of the form

$$\frac{1}{e^{-i(\eta-\phi)} - 1},$$

with ϕ as in (6.5), and since these are first order singularities at $\eta \in \phi + 2\pi\mathbb{Z}$, they can be handled as in Sect. 4. For instance, in the ℓ^2 case, the Prüfer amplitude is bounded if

$$\sum_{l=1}^{\infty} \left| \frac{c_l}{e^{-i(\eta-\phi_l)} - 1} \right| < \infty.$$

In the general case, the algebra is more complicated than for Schrödinger operators, and one needs to work with functions $f_{I,J,K,L}$, $g_{I,J,K,L}$ parametrized by four indices I, J, K, L, as defined in [22, Sect. 8]. The proof needs identities analogous to those in Lemma 2.2. For some of those identities, [22] avoided finding them explicitly, and instead proved by contradiction that the functions obey desired properties. This indirect proof is easily adapted to the current needs; for instance, if in Sect. 2 we hadn't known that (2.22) held, but we knew that $f_{J,0} - \check{f}_{J,0} = 0$ whenever $\phi_1 + \cdots + \phi_J = 0$, that and the fact that $f_{J,0} - \check{f}_{J,0}$ is a rational function would suffice to conclude existence of a rational function $\mathcal{G}_{J,0}$ such that (2.22) holds.

A closer look at the algebra shows that for OPRL, we obtain small divisor conditions for integers j with $j < p$, whereas for OPUC, we only obtain small divisor conditions for odd integers j with $j < p$. This explains why Theorems 1.4 and 1.5 give different estimates on the Hausdorff dimension, as was already motivated in Remark 1.3.

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