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Hölder Continuity of the Integrated Density of States for the Fibonacci Hamiltonian

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Abstract: We obtain the asymptotics of the optimal global Hölder exponent of the integrated density of states of the Fibonacci Hamiltonian for large and small couplings.

1. Introduction

In this paper we are interested in regularity properties of the integrated density of states associated with the Fibonacci Hamiltonian. It follows from known results that it is uniformly Hölder continuous. However, it is clear that the known estimates of the Hölder exponent are far from optimal. Here we employ new methods to obtain estimates for the Hölder exponent in terms of the coupling constant in the regime of large or small coupling, which are shown to be asymptotically optimal in each case.

To motivate our study, let us consider an invertible ergodic transformation T of a probability measure space $(\Omega, d\mu)$ and a bounded measurable function $f : \Omega \to \mathbb{R}$. One associates a family of discrete Schrödinger operators on the line as follows: For $\omega \in \Omega$, the potential $V_\omega : \mathbb{Z} \to \mathbb{R}$ is given by $V_\omega(n) = f(T^n\omega)$ and the operator H_ω in $\ell^2(\mathbb{Z})$ acts as

$$[H_\omega\phi](n) = \phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n).$$

An important quantity associated with such a family of operators, $\{H_\omega\}_{\omega \in \Omega}$, is given by the integrated density of states, which is defined as follows: compare [2,13]. Define the measure dN by

$$\int g(\lambda) dN(\lambda) = \int \langle \delta_0, g(H_\omega)\delta_0 \rangle \, d\mu(\omega). \tag{1}$$

The integrated density of states (IDS), N, is then given by

$$N(E) = \int \chi_{(-\infty, E]}(\lambda) \, dN(\lambda). \tag{2}$$

The terminology is explained by

$$N(E) = \lim_{n \to \infty} \frac{\#\{\text{eigenvalues of } H_{\omega, [1,n]} \le E\}}{n} \quad \text{for } \mu\text{-a.e. } \omega \in \Omega, \tag{3}$$

where $H_{\omega, [1,n]}$ denotes the restriction of H_ω to the interval $[1, n]$ with Dirichlet boundary conditions. It is a basic result that the IDS is always continuous [2]; see [27] for a very short proof that also works in higher dimensions. Craig and Simon, [11] (see [12] for the multi-dimensional case), have shown that more is true. Using the Thouless formula, they proved that N is log-Hölder continuous, that is, there is a constant C such that

$$|N(E_1) - N(E_2)| \le C \left(\log |E_1 - E_2|^{-1} \right)^{-1} \tag{4}$$

for all E_1, E_2 with $|E_1 - E_2| \le 1/2$.

In this general setting, the bound (4) is essentially optimal; see Craig [10] and Gan and Krüger [28]. However, for concrete models, one may hope to improve upon (4). Roughly speaking, one expects stronger regularity properties of the IDS the more random the stochastic process $V_\omega(n)$ is. In the i.i.d. situation, it is always Hölder continuous, that is,

$$|N(E_1) - N(E_2)| \le C|E_1 - E_2|^\gamma \tag{5}$$

for some $C < \infty$ and $\gamma > 0$, as shown by Le Page [40]. If the single-site distribution is nice enough, Simon-Taylor [47] and Campanino-Klein [7] proved that N is C^∞. We refer the reader to the survey article [46] by Simon, which describes the regularity results for the IDS that had been obtained by the mid-1980's.

More recently, there has been renewed interest in the problem of proving regularity better than (4) for some classes of ergodic Schrödinger operators with little randomness. This was initiated by Goldstein and Schlag, who proved Hölder continuity of the IDS, (5), for analytic quasi-periodic potentials in the regime of positive Lyapunov exponents [29]. This paper was followed by [4–6,30] who improved the estimate in some cases or proved regularity results for different models (e.g., with T given by a skew-shift on $\mathbb{T}^2$). Also see Hadj-Amor [31] and Avila-Jitomirskaya [1] for results for analytic quasi-periodic potentials in the regime of zero Lyapunov exponents, and Schlag [45] for results for analytic quasi-periodic models at large coupling in two dimensions.

Due to the typical presence of a dense set of gaps in the spectrum, one does not hope for more than Hölder regularity for quasi-periodic (or, more generally, almost-periodic) models.

Our objective here is to study the regularity properties of the IDS for a prominent quasi-periodic model that is not covered by the Goldstein-Schlag paper and its successors; the Fibonacci Hamiltonian, introduced independently by Kohmoto et al. [37] and Ostlund et al. [41]. It is given by $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $Tx = x + \alpha \mod 1$, where

$$\alpha = \frac{\sqrt{5} - 1}{2}$$

is the inverse of the golden mean, μ is the Lebesgue measure on $\mathbb{T}$, and $f(\omega) = \lambda \chi_{[1-\alpha, 1)}(\omega)$ for some $\lambda > 0$. Thus, the potentials have the form

$$V_\omega(n) = \lambda \chi_{[1-\alpha, 1)}(n\alpha + \omega \mod 1). \tag{6}$$

The associated operators $\{H_\omega\}_{\omega \in \Omega}$ form the family of Fibonacci Hamiltonians. This is the standard model of a one-dimensional quasicrystal. The spectrum of H_ω is easily seen to be independent of ω, and it will henceforth be denoted by Σ_λ. This set is known to be a Cantor set of zero Lebesgue measure [49] (and in fact of Hausdorff dimension strictly between 0 and 1 [8]). Moreover, it is known that H_ω has purely singular continuous spectrum for every λ and ω; see Damanik and Lenz [24], Kotani [38], and Sütő [48,49]. The survey articles [15–17,48] contain information on the results obtained for this model and its generalizations. Here we are interested in the integrated density of states associated with the Fibonacci Hamiltonian, which we will henceforth denote by N_λ since its dependence on the coupling constant λ will be of explicit interest. We mention in passing that dN_λ is the equilibrium measure on Σ_λ in the sense of logarithmic potential theory.

We first note a result that is essentially well-known, but which is stated for the sake of completeness.

Theorem 1. *For every $\lambda > 0$, there are $C_\lambda < \infty$ and $\gamma_\lambda > 0$ such that*

$$|N_\lambda(E_1) - N_\lambda(E_2)| \le C_\lambda |E_1 - E_2|^{\gamma_\lambda}$$

for every E_1, E_2 with $|E_1 - E_2| < 1$.

Proof. It follows from the definition (1)–(2) that the integrated density of states is the distribution function of the μ-average of the spectral measures with respect to H_ω and δ_0. It was shown in [23] that for each λ, these spectral measures are uniformly Hölder continuous with constants $C_\lambda < \infty$ and $\gamma_\lambda > 0$ that are uniform in ω. That is, the μ-average of these measures will also be uniformly Hölder continuous with the same pair of constants.   □

One can infer explicit expressions for C_λ and γ_λ from [23]. However, they are clearly far from optimal and hence we opted not to make them explicit. We also note that [23] had the precursors [14,34], which established uniform Hölder continuity of spectral measure for the single phase $\omega = 0$. Central to all these papers was the work [33] which proved uniform power law upper bounds on transfer matrices for energies from the spectrum.

Our main goal is to identify the asymptotic behavior of the Hölder exponent in the regimes of large and small coupling. As mentioned above, the tools used to establish Theorem 1 do not produce optimal results and hence are inadequate to identify the asymptotic behavior precisely. Therefore, different methods are needed in these asymptotic regimes, and we will indeed use different ones in either of these two cases.

In the large coupling regime, we have the following:

Theorem 2. (a) *Suppose $\lambda > 4$. Then, for every*

$$\gamma < \frac{3 \log(\alpha^{-1})}{2 \log(2\lambda + 22)},$$

there is some $\delta > 0$ such that the IDS associated with the family of Fibonacci Hamiltonians satisfies

$$|N_\lambda(E_1) - N_\lambda(E_2)| \le |E_1 - E_2|^\gamma$$

for every E_1, E_2 with $|E_1 - E_2| < \delta$.

(b) *Suppose $\lambda \geq 8$. Then, for every*

$$\tilde{\gamma} > \frac{3\log(\alpha^{-1})}{2\log\left(\frac{1}{2}\left((\lambda-4) + \sqrt{(\lambda-4)^2 - 12}\right)\right)}$$

and every $0 < \delta < 1$, there are E_1, E_2 with $0 < |E_1 - E_2| < \delta$ such that

$$|N_\lambda(E_1) - N_\lambda(E_2)| \geq |E_1 - E_2|^{\tilde{\gamma}}.$$

This shows in particular that the optimal Hölder exponent is asymptotically $\frac{3\log(\alpha^{-1})}{2\log\lambda}$ in the large coupling regime.

Theorem 2 is proved in Subsect. 2.2. The proof is based on the self-similarity of the spectrum of H_ω. In particular, we do not use the Thouless formula and a Hölder continuity result for the Lyapunov exponent, as was the case in many of the works mentioned above. Thus, in a sense, we use a geometric, rather than an analytic, approach. Before turning to the proof, we first recall the canonical periodic approximants, which are obtained by replacing α by its continued fraction approximants. This enables us to describe the self-similarity of the spectrum that is crucial to our proof and it will also establish an explicit way to express $N_\lambda(E)$ in terms of periodic spectra.

In the small coupling regime, we have the following:

Theorem 3. *The integrated density of states $N_\lambda(\cdot)$ is Hölder continuous with Hölder exponent γ_λ, where $\gamma_\lambda \to \frac{1}{2}$ as $\lambda \to 0$, and $\gamma_\lambda < \frac{1}{2}$ for small $\lambda > 0$.*
More precisely,

(a) *For any $\gamma \in (0, \frac{1}{2})$, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there exists $\delta > 0$ such that*

$$|N_\lambda(E_1) - N_\lambda(E_2)| \leq |E_1 - E_2|^{\gamma}$$

for every E_1, E_2 with $|E_1 - E_2| < \delta$.
(b) *For any sufficiently small $\lambda > 0$, there exists $\tilde{\gamma} = \tilde{\gamma}(\lambda) < \frac{1}{2}$ such that for every $\delta > 0$, there are E_1, E_2 with $0 < |E_1 - E_2| < \delta$ and*

$$|N_\lambda(E_1) - N_\lambda(E_2)| \geq |E_1 - E_2|^{\tilde{\gamma}}.$$

Theorem 3 is proved in Subsect. 3.6. The proof uses the trace map formalism and the dynamical properties of the Fibonacci trace map studied previously in [3,8,9,19–21,44]. In particular, a relation between the integrated density of states of the Fibonacci Hamiltonian and the measure of maximal entropy of the trace map was established in [22]. The proof combines this relation with Hölder structures that appear due to hyperbolicity of the trace map in order to get explicit estimates on the Hölder exponent. To show that the obtained asymptotics of the Hölder exponent are optimal, we study the behavior of unstable multipliers of specific periodic points of the trace map.

We conclude this introduction with some general remarks. In [22] we studied the scaling exponents associated with the measures dN_λ. We showed that there exists $0 < \tilde{\lambda}_0 \leq \infty$ such that for $\lambda \in (0, \tilde{\lambda}_0)$, there is $d_\lambda \in (0, 1)$ so that the density of states measure dN_λ is of exact dimension d_λ, that is, for dN_λ-almost every $E \in \mathbb{R}$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{\log N_\lambda(E - \varepsilon, E + \varepsilon)}{\log \varepsilon} = d_\lambda.$$

Moreover,

$$\lim_{\lambda \downarrow 0} d_\lambda = 1.$$

While at first sight this result and the question addressed in Theorem 3 are quite similar, note that in the latter result, one has to establish a uniform estimate in the vicinity of an arbitrarily chosen energy in Σ_λ, whereas in the former result, one may exclude a set of zero dN_λ measure from the consideration. This crucial difference leads to different answers, as the different asymptotics show. Of course the scaling exponent of dN_λ at any energy $E \in \Sigma_\lambda$ bounds the optimal global Hölder exponent of N_λ from above, and hence one gets a one-sided estimate in this way. More specifically, the scaling exponents are worse at gap boundaries of Σ_λ and essentially determine the global Hölder exponent of N_λ, but these points form a countable set and hence a set of measure zero with respect to dN_λ since this measure is always continuous.

It is interesting to compare the large coupling asymptotics of several λ-dependent quantities. As we saw in Theorem 2, the optimal Hölder exponent behaves asymptotically like $1.5 \cdot \frac{\log(\alpha^{-1})}{\log \lambda}$. On the other hand, it was shown in [18] that the Hausdorff dimension of Σ_λ behaves asymptotically like $1.831\ldots \cdot \frac{\log(\alpha^{-1})}{\log \lambda}$, and it was shown in [26] that a certain transport exponent, which measures the rate of wavepacket spreading in the time-dependent Schrödinger equation associated with the Fibonacci Hamiltonian, behaves asymptotically like $2 \cdot \frac{\log(\alpha^{-1})}{\log \lambda}$. Similarly, in the weak coupling regime, we have that the Hausdorff dimension of Σ_λ strictly exceeds the dimension d_λ of dN_λ (a fact that was also proven in [22]), which in turn strictly exceeds the optimal Hölder exponent of N_λ (since they have different asymptotic values). This shows that the strongly coupled and the weakly coupled Fibonacci Hamiltonian serve as a good source of examples demonstrating that certain quantities associated with a discrete Schrödinger operator need not be identical. Moreover, the three different prefactors in the large coupling asymptotics (1.5, 1.831..., and 2) correspond directly, and in a quite beautiful way, to the scaling properties exhibited by Σ_λ (cf. Subsect. 2.1).

Finally, determining the correct Hölder exponent is one important ingredient in a recent study of the spacings of the zeros of a certain class of orthogonal polynomials (or, equivalently, the eigenvalues of $H_{\omega,[1,n]}$ in the notation introduced above) by Krüger and Simon [39] and hence our results feed into their theory.

2. The Large Coupling Regime

2.1. Canonical periodic approximants and scaling properties. In this subsection, we recall some known results for the Fibonacci Hamiltonian, its periodic approximants, and their spectra. The main tools we shall need in the sequel are summarized in Propositions 2.1–2.3 below.

Define the sequence $(F_k)_{k \geq -1}$ of Fibonacci numbers by

$$F_{-1} = 0, \quad F_0 = 1, \quad F_k = F_{k-1} + F_{k-2} \quad \text{for } k \geq 1.$$

For $k \geq 1$, define

$$x_k(E, \lambda) = \frac{1}{2}\text{tr } M_k(E, \lambda),$$

where

$$M_k(E, \lambda) = \begin{pmatrix} E - V_{\omega=0}(F_k) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - V_{\omega=0}(1) & -1 \\ 1 & 0 \end{pmatrix},$$

that is, the transfer matrix for phase $\omega = 0$ and energy E from the origin to the site F_k.

The matrices M_k obey the recursion

$$M_k(E, \lambda) = M_{k-2}(E, \lambda)M_{k-1}(E, \lambda), \tag{7}$$

and as a consequence,

$$x_{k+1}(E, \lambda) = 2x_k(E, \lambda)x_{k-1}(E, \lambda) - x_{k-2}(E, \lambda). \tag{8}$$

With the definitions above, (7) and (8) hold for $k \geq 3$. If we define

$$M_{-1}(E, \lambda) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_0(E, \lambda) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix},$$

these recursions extend to all $k \geq 1$.

The so-called trace map relation, (8), yields the invariant

$$x_{k+1}(E, \lambda)^2 + x_k(E, \lambda)^2 + x_{k-1}(E, \lambda)^2 - 2x_{k+1}(E, \lambda)x_k(E, \lambda)x_{k-1}(E, \lambda) = 1 + \frac{\lambda^2}{4}. \tag{9}$$

The identities (7)–(9) were proved by Sütő in [48].

For fixed λ, define (leaving the dependence on λ implicit)

$$\sigma_k = \{E \in \mathbb{R} : |x_k(E, \lambda)| \leq 1\}.$$

The set σ_k is actually equal to the spectrum of the Schrödinger operator H whose potential V_k results from $V_{\omega=0}$ in (6) by replacing α by F_{k-1}/F_k (see [48]). Hence, V_k is F_k-periodic, $\sigma_k \subset \mathbb{R}$, and it consists of F_k bands (closed intervals).

Next, we recall some results of Damanik and Tcheremchantsev [25], Killip, Kiselev, and Last [36], and Raymond [43]. From now on, we shall always assume $\lambda > 4$ since we will make critical use of the fact that in this case, it follows from the invariant, (9), that three consecutive half-traces cannot simultaneously be bounded in absolute value by 1:

$$\forall \lambda > 4, \ \forall E, k \ : \ \max\{|x_k(E, \lambda)|, |x_{k+1}(E, \lambda)|, |x_{k+2}(E, \lambda)|\} > 1. \tag{10}$$

Following [36], we call a band $I_k \subset \sigma_k$ a type A band if $I_k \subset \sigma_{k-1}$ (and hence $I_k \cap (\sigma_{k+1} \cup \sigma_{k-2}) = \emptyset$). A band $I_k \subset \sigma_k$ is called a type B band if $I_k \subset \sigma_{k-2}$ (and therefore $I_k \cap \sigma_{k-1} = \emptyset$); compare Fig. 1.

By definition of M_{-1} and M_0, $\sigma_{-1} = \mathbb{R}$, $\sigma_0 = [-2, 2]$, and $\sigma_1 = [\lambda - 2, \lambda + 2]$. Hence, σ_0 consists of a single band of type A, and σ_1 consists of a single band of type B.

From (10), one gets the following result on the relative position of bands on successive levels (cf. [36, Lem. 5.3] and [43, Lem. 6.1]):

Proposition 2.1. *For every $\lambda > 4$ and every $k \geq 0$,*

(a) *Every type A band $I_k \subset \sigma_k$ contains exactly one type B band $I_{k+2} \subset \sigma_{k+2}$, and no other bands from σ_{k+1}, σ_{k+2}.*
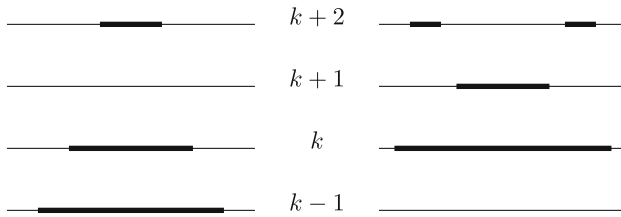
Fig. 1. Left: a type A band in σ_k; Right: a type B band in σ_k

(b) *Every type B band $I_k \subset \sigma_k$ contains exactly one type A band $I_{k+1} \subset \sigma_{k+1}$ and two type B bands from σ_{k+2}, positioned around I_{k+1}.*

The Lebesgue measure of each individual band admits the following geometric (in k) lower bound:

Proposition 2.2. *For $\lambda > 4$, $k \geq 3$, and every band I_k of σ_k, we have*

$$|I_k| \geq \frac{4}{(2\lambda + 22)^{2k/3}},$$

where $|\cdot|$ denotes Lebesgue measure.

Proof. This follows from [25, Lem. 3.5], noting (10) and the fact that on each band I_k of σ_k, we have $\int_{I_k} |x'_k(E)| \, dE = 2$.[1] □

In the large coupling regime, this result is asymptotically optimal in the following sense:

Proposition 2.3. *For $\lambda \geq 8$, $k \geq 4$ and $k \equiv 1 \mod 3$, there exists a type B band I_k of σ_k with*

$$|I_k| \leq \frac{4}{\left(\frac{1}{2} \left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12} \right) \right)^{2k/3}}.$$

Proof. This follows from [36, Lem. 5.5], Proposition 2.1 above (start with the type B band of σ_1 and then cycle periodically through B, A, and "not contained"), and again the fact that on each band I_k of σ_k, we have $\int_{I_k} |x'_k(E)| \, dE = 2$. □

Let us study the scaling properties of the sets σ_k in more detail. In particular, we want to show that, when $n > k$, each band of σ_k contains either F_{n-k} or F_{n-k-2} bands of σ_n.

Recall that σ_0 consists of a single band of type A, and σ_1 consists of a single band of type B. A repeated application of Proposition 2.1 yields the entries in the following table:

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[1] Here we caution the reader that the literature on the Fibonacci Hamiltonian is not consistent in the sense that some papers use half-traces, as we do here, and other papers use traces instead. The papers [25,36,43], on which much of the present subsection is based, belong to the second group and hence their results need to be slightly reformulated when given here.

k	# of bands in σ_k of type A	# of bands in σ_k of type B
0	1	0
1	0	1
2	1	1
3	1	2
4	2	3
5	3	5
6	5	8
$\vdots$	$\vdots$	$\vdots$

We observe the following:

Lemma 2.4. *For $\lambda > 4$ and $k \geq 2$, we have*

$$\text{\# of type A bands in } \sigma_k = F_{k-2},$$
$$\text{\# of type B bands in } \sigma_k = F_{k-1}.$$

In particular, for $k \geq 0$, every band of σ_k is either of type A or of type B.

Proof. This is a straightforward induction. $\square$

A similar straightforward application of Proposition 2.1 gives the following result.

Lemma 2.5. *Suppose $\lambda > 4$, $k \geq 0$, and $n > k$.*

(a) *Every type A band of σ_k contains F_{n-k-2} bands of σ_n.*
(b) *Every type B band of σ_k contains F_{n-k} bands of σ_n.*

2.2. Proof of Theorem 2. In this subsection we prove Theorem 2. Our main tools will be Proposition 2.2, Lemmas 2.4 and 2.5, and a formula, given in (11) below, connecting the IDS of the Fibonacci Hamiltonian and the band structure of the periodic spectra introduced in the previous subsection.

Let us first note that in the Fibonacci case, as a consequence of results of Hof for uniquely ergodic models, [32], the convergence in (3) takes place for every, rather than almost every, $\omega \in \Omega$. Moreover, Dirichlet boundary conditions can be replaced by other boundary conditions, such as Neumann or periodic boundary conditions. If we choose periodic boundary conditions, and consider $\omega = 0$ and convergence only along the subsequence $(F_n)_{n \geq 1}$, we obtain the following formula (which had already been noted by Raymond [43]),

$$N_\lambda(E) = \lim_{n \to \infty} \frac{\#\{\text{bands of } \sigma_n \leq E\}}{F_n}, \tag{11}$$

since each band of σ_n contains exactly one eigenvalue of the operator $H_{\omega=0}$, restricted to $[1, F_n]$ with periodic boundary conditions.

Proof of Theorem 2. (a) Given $E_1 < E_2$ (with $E_2 - E_1$ small; less than $4/(2\lambda + 22)^2$, say), we want to estimate $N_\lambda(E_2) - N_\lambda(E_1)$ from above. It follows from (11) that

$$N_\lambda(E_2) - N_\lambda(E_1) = \lim_{n \to \infty} \frac{\#\{\text{bands of } \sigma_n \text{ contained in } [E_1, E_2]\}}{F_n}. \tag{12}$$

Thus, we need to estimate from above the number of bands of σ_n that are contained in $[E_1, E_2]$.

Let $k \geq 3$ be the integer with

$$\frac{4}{(2\lambda + 22)^{2(k+1)/3}} \leq E_2 - E_1 < \frac{4}{(2\lambda + 22)^{2k/3}}.$$

Consider, for $n > k$, the bands of σ_n that are contained in $[E_1, E_2]$. By the results of the previous section, each of these bands is contained in a band of σ_k or in a band of σ_{k-1}. By the definition of k and Proposition 2.2, at most two bands from $\sigma_k \cup \sigma_{k-1}$ can occur as associated bands. Thus, Lemmas 2.4 and 2.5 imply that at most $2F_{n-(k-1)}$ bands of σ_n can be contained in $[E_1, E_2]$.

Define γ_k by

$$\gamma_k = \frac{(k-3)\log(\alpha^{-1})}{\frac{2}{3}(k+1)\log(2\lambda + 22) - \log 4}.$$

Then, we infer from (12) that

$$N_\lambda(E_2) - N_\lambda(E_1) = \lim_{n\to\infty} \frac{\#\{\text{bands of } \sigma_n \text{ contained in } [E_1, E_2]\}}{F_n}$$

$$\leq \lim_{n\to\infty} \frac{2F_{n-(k-1)}}{F_n}$$

$$\leq \lim_{n\to\infty} \frac{F_{n-(k-3)}}{F_n}$$

$$= \alpha^{k-3}$$

$$= \left(\frac{4}{(2\lambda + 22)^{2(k+1)/3}}\right)^{\gamma_k}$$

$$\leq (E_2 - E_1)^{\gamma_k}.$$

Now, given

$$0 < \gamma < \frac{3\log(\alpha^{-1})}{2\log(2\lambda + 22)},$$

choose $k_0 \geq 3$ such that $\gamma_k \geq \gamma$ for every $k \geq k_0$. Let

$$\delta = \frac{4}{(2\lambda + 22)^{2k_0/3}} \in (0, 1).$$

We obtain that for every E_1, E_2 with $|E_1 - E_2| < \delta$,

$$|N_\lambda(E_1) - N_\lambda(E_2)| \leq |E_1 - E_2|^{\gamma_k} \leq |E_1 - E_2|^{\gamma},$$

where $k \geq k_0$ is the integer associated with $|E_1 - E_2|$.

(b) Suppose $\lambda \geq 8$ and let

$$\tilde{\gamma} \in \left(\frac{3\log(\alpha^{-1})}{2\log\left(\frac{1}{2}\left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12}\right)\right)}, 1\right)$$

and $0 < \delta < 1$ be given.

Choose k_0 such that

$$\frac{4}{\left(\frac{1}{2}\left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12}\right)\right)^{2k_0/3}} < \delta$$

and such that

$$\tilde{\gamma}_k := \frac{(k+1)\log\alpha^{-1}}{\frac{2}{3}k\,\log\left(\frac{1}{2}\left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12}\right)\right) - \log 4} < \tilde{\gamma}$$

for every $k \geq k_0$.

Next, choose $k \geq \max\{k_0, 4\}$ with $k \equiv 1 \mod 3$. By Proposition 2.3, there exists a type B band I_k of σ_k with

$$|I_k| \leq \frac{4}{\left(\frac{1}{2}\left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12}\right)\right)^{2k/3}}.$$

Denote the endpoints of I_k by E_1, E_2, that is, $I_k = [E_1, E_2]$.

By (12), Lemma 2.5, the definition of $\tilde{\gamma}_k$, the choice of $I_k = [E_1, E_2]$, the choice of k, and the fact that $0 < E_2 - E_1 < \delta < 1$, we find that

$$\begin{aligned} N_\lambda(E_2) - N_\lambda(E_1) &= \lim_{n\to\infty} \frac{\#\{\text{bands of } \sigma_n \text{ contained in } [E_1, E_2]\}}{F_n} \\ &= \lim_{n\to\infty} \frac{F_{n-k-1}}{F_n} \\ &= \alpha^{k+1} \\ &= \left(\frac{4}{\left(\frac{1}{2}\left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12}\right)\right)^{2k/3}}\right)^{\tilde{\gamma}_k} \\ &\geq (E_2 - E_1)^{\tilde{\gamma}_k} \\ &\geq (E_2 - E_1)^{\tilde{\gamma}}. \end{aligned}$$

This completes the proof. □

3. The Small Coupling Regime

Here we will use the relation between the IDS for the Fibonacci Hamiltonian and the measure of maximal entropy for the so called Trace Map associated with the discrete Schrödinger operator with Fibonacci potential.

3.1. IDS for the free laplacian. It is well known that $\Sigma_0 = [-2, 2]$ and

$$N_0(E) = \begin{cases} 0 & E \leq -2 \\ \frac{1}{\pi}\arccos\left(-\frac{E}{2}\right) & -2 < E < 2 \\ 1 & E \geq 2. \end{cases} \tag{13}$$

In particular, N_0 is Hölder continuous with Hölder exponent $1/2$. We will need a somewhat more detailed description of the continuity properties of N_0 (which also follows directly from the explicit form (13)).

Lemma 3.1. *For any $\varepsilon > 0$, there exists $C > 0$ such that the following hold:*

(a) *If $E_1, E_2 \in [-2 + \varepsilon, 2 - \varepsilon]$, then*

$$|N_0(E_2) - N_0(E_1)| \leq C|E_2 - E_1|.$$

(b) *If $E_1, E_2 \in [-2, 2]$, then*

$$|N_0(E_2) - N_0(E_1)| \leq C|E_2 - E_1|^{1/2}.$$

(c) *If $E_1, E_2 \in [-2, -2 + \varepsilon)$ and $-2 < E_1 < E_2$, then*

$$|N_0(E_2) - N_0(E_1)| \leq \frac{C}{|2 + E_1|^{1/2}}|E_2 - E_1|.$$

Similarly, if $E_1, E_2 \in (2 - \varepsilon, 2]$ and $E_1 < E_2 < 2$, then

$$|N_0(E_2) - N_0(E_1)| \leq \frac{C}{|2 - E_2|^{1/2}}|E_2 - E_1|.$$

(d) *There exists $C_0 > 0$ such that for any $E \in (-2, 2)$, one has*

$$|N_0(-2) - N_0(E)| \geq C_0|E + 2|^{1/2} \quad and \quad |N_0(2) - N_0(E)| \geq C_0|E - 2|^{1/2}.$$

3.2. The trace map. The recursion (8) gives rise to a fundamental connection between the spectral properties of the Fibonacci Hamiltonian and the dynamics of the *trace map*

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \ T(x, y, z) = (2xy - z, x, y).$$

The function $G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$ is invariant[2] under the action of T (cf. (9)), and hence T preserves the family of cubic surfaces[3]

$$S_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2xyz = 1 + \frac{\lambda^2}{4} \right\}.$$

It is therefore natural to consider the restriction T_λ of the trace map T to the invariant surface S_λ. That is, $T_\lambda : S_\lambda \to S_\lambda$, $T_\lambda = T|_{S_\lambda}$. We denote by Λ_λ the set of points in S_λ whose full orbits under T_λ are bounded (it is known that Λ_λ is equal to the non-wandering set of T_λ).

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[2] It is usually called the Fricke-Vogt invariant.
[3] The surface S_0 is known as Cayley cubic.

3.3. Hyperbolicity of the trace map. Recall that an invariant closed set Λ of a diffeomorphism $f : M \to M$ is *hyperbolic* if there exists a splitting of the tangent space $T_x M = E_x^u \oplus E_x^u$ at every point $x \in \Lambda$ such that this splitting is invariant under Df, the differential Df exponentially contracts vectors from the stable subspaces $\{E_x^s\}$, and the differential of the inverse, Df^{-1}, exponentially contracts vectors from the unstable subspaces $\{E_x^u\}$. A hyperbolic set Λ of a diffeomorphism $f : M \to M$ is *locally maximal* if there exists a neighborhood U of Λ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

It is known that for $\lambda > 0$, Λ_λ is a locally maximal hyperbolic set of $T_\lambda : S_\lambda \to S_\lambda$; see [8,9,19].

3.4. Properties of the trace map for $\lambda = 0$. The surface

$$\mathbb{S} = S_0 \cap \{(x, y, z) \in \mathbb{R}^3 : |x| \le 1, |y| \le 1, |z| \le 1\}$$

is homeomorphic to S^2, invariant under T, smooth everywhere except at the four points $P_1 = (1, 1, 1)$, $P_2 = (-1, -1, 1)$, $P_3 = (1, -1, -1)$, and $P_4 = (-1, 1, -1)$, where $\mathbb{S}$ has conic singularities, and the trace map T restricted to $\mathbb{S}$ is a factor of the hyperbolic automorphism of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by

$$\mathcal{A}(\theta, \varphi) = (\theta + \varphi, \theta) \,(\text{mod } 1).$$

The semi-conjugacy is given by the map

$$F : (\theta, \varphi) \mapsto (\cos 2\pi(\theta + \varphi), \cos 2\pi\theta, \cos 2\pi\varphi). \tag{14}$$

The map $\mathcal{A}$ is hyperbolic, and is given by the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which has eigenvalues

$$\mu = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad -\mu^{-1} = \frac{1 - \sqrt{5}}{2}.$$

A Markov partition for the map $\mathcal{A} : \mathbb{T}^2 \to \mathbb{T}^2$ is shown in Fig. 2. Its image under the map $F : \mathbb{T}^2 \to \mathbb{S}$ is a Markov partition for the pseudo-Anosov map $T : \mathbb{S} \to \mathbb{S}$.

3.5. Spectrum and trace map. Denote by ℓ_λ the line

$$\ell_\lambda = \left\{ \left(\frac{E - \lambda}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}.$$

It is easy to check that $\ell_\lambda \subset S_\lambda$. An energy $E \in \mathbb{R}$ belongs to the spectrum Σ_λ of the Fibonacci Hamiltonian if and only if the positive semiorbit of the point $(\frac{E-\lambda}{2}, \frac{E}{2}, 1)$ under iterates of the trace map T is bounded; see [48]. Moreover, the stable manifolds of points in Λ_λ intersect the line ℓ_λ transversally if $\lambda > 0$ is sufficiently small [19] or if $\lambda \ge 16$ [9].

Let us denote $L_\lambda : \mathbb{R} \to \ell_\lambda$, $L_\lambda(E) = (\frac{E-\lambda}{2}, \frac{E}{2}, 1)$. It is affine and contracts distances by the multiplicative factor $\frac{1}{\sqrt{2}}$.
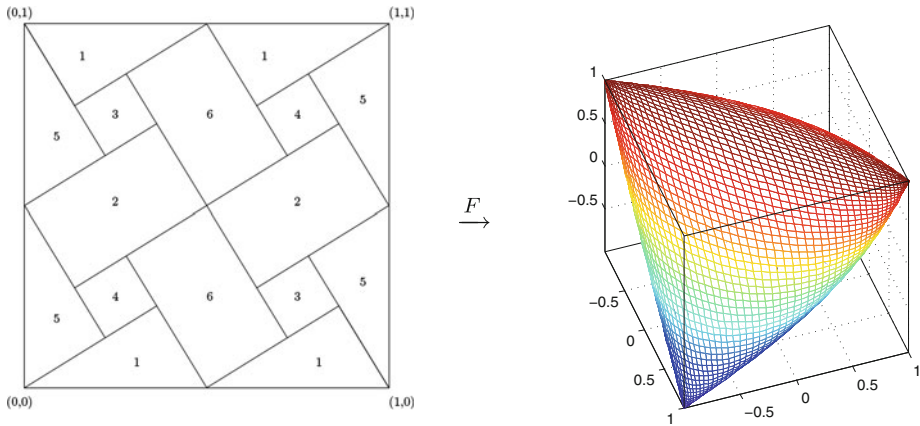
Fig. 2. The semi-conjugacy F between the linear map $\mathcal{A}$ and the trace map T on the central part $\mathbb{S}$ of the Cayley cubic

Define the map $\Psi_\lambda : \Sigma_\lambda \to [0, 2]$ by

$$\Psi_\lambda(x) = y \quad \Leftrightarrow \quad N_\lambda(x) = N_0(y). \tag{15}$$

It turns out that there is a direct relation between the map Ψ_λ and dynamical structures of the trace map T_λ. The following statement is implicitly contained in [22, Claim 3.2].

Proposition 3.2. *There exists $\lambda_0 > 0$ such that the following holds. Take any $\lambda \in [0, \lambda_0)$ and $x_1, x_2 \in \Omega_\lambda$. Consider the stable manifolds $W^s(x_1)$ and $W^s(x_2)$, and take some points $p_1 = W^s(x_1) \cap \ell_\lambda$ and $p_2 = W^s(x_2) \cap \ell_\lambda$. When λ changes, there are unique continuations of the points $x_1, x_2 \in \Omega_\lambda$, denote them by $x_1(\lambda), x_2(\lambda)$. The continuations of the intersections $p_i(\lambda) = W^s(x_i(\lambda)) \cap \ell_\lambda$, $i = 1, 2$, are also well defined, and the value of the difference $N_\lambda(L_\lambda^{-1}(p_2(\lambda))) - N_\lambda(L_\lambda^{-1}(p_1(\lambda)))$ is independent of $\lambda \in [0, \lambda_0)$.*

Notice that Proposition 3.2 gives a dynamical description of the map Ψ_λ. In [22] this description was used to establish a relation between the IDS N_λ and the measure of maximal entropy for $T|_{\Lambda_\lambda}$.

3.6. Hölder continuity of the IDS in the small coupling regime.

Here we prove Theorem 3.

Choose sufficiently small neighborhoods $U(P_i)$ of the singularities $\{P_1, P_2, P_3, P_4\}$ of the Cayley cubic. Let $U^*(P_i) \subset U(P_i)$ be an essentially smaller neighborhood of the singularity P_i. Set $U = \bigcup_{i=1,2,3,4} U(P_i)$ and $U^* = \bigcup_{i=1,2,3,4} U^*(P_i)$. Take any $E_1, E_2 \in \Sigma_\lambda$, and denote by b the interval on ℓ_λ between the points $L_\lambda(E_1)$ and $L_\lambda(E_2)$.

Notice that

$$|N_\lambda(E_1) - N_\lambda(E_2)| = |N_0(\Psi_\lambda(E_1)) - N_0(\Psi_\lambda(E_2))|.$$

Let a be the interval on ℓ_0 between the points $L_0(\Psi_\lambda(E_1))$ and $L_0(\Psi_\lambda(E_2))$. We will consider separately three cases:

(i) a and b are away from the neighborhoods $U^*(P_i)$ of the singularities.

(ii) a and b are in a neighborhood $U(P_i)$ of a singularity (that could be either $P_1 = (1, 1, 1)$ or $P_2 = (-1, -1, 1)$), and one of the edges of a is on a local strong stable manifold of the singularity (which implies that one of the edges of b is on the local strong stable manifold of a periodic orbit of period 2 or 6). This is equivalent to the case when E_1 or E_2 is equal to $\min \Sigma_\lambda$ or $\max \Sigma_\lambda$.

(iii) a and b are in a neighborhood $U(P_i)$ of a singularity (that could be either $P_1 = (1, 1, 1)$ or $P_2 = (-1, -1, 1)$), and none of the edges of a is on a local strong stable manifold of the singularity.

Note that if $|E_1 - E_2|$ is sufficiently small (depending on the choice of U and U^*), then exactly one of the cases (i)–(iii) applies.

Consider the case (i). Let us iterate a and b until they grow up to the length of order one. To simplify the estimates, let us introduce the following notation:

$$x \sim_C y \quad \Leftrightarrow \quad C^{-1}|y| \le |x| \le C|y|.$$

There exists a constant $C > 0$ that is independent of a, b such that for some $M \in \mathbb{N}$, we have $|T^M(a)| \sim_C |T^M(b)| \sim_C 1$. Let us split the sequence of iterates $\{a, T(a), T^2(a), \ldots, T^M(a)\}$ into finite intervals $\{a, T(a), \ldots, T^{k_1}(a)\}$, $\{T^{k_1+1}(a), \ldots, T^{k_2}(a)\}, \ldots$ (and similarly for $\{b, T(b), T^2(b), \ldots, T^M(b)\}$) in such a way that for each k_i, one of the following cases holds:

(a) $\{T^{k_{i-1}+1}(a), \ldots, T^{k_i}(a)\}$ as well as $\{T^{k_{i-1}+1}(b), \ldots, T^{k_i}(b)\}$ are away from U^*, and moreover, $\frac{|T^{k_i}(a)|}{|T^{k_{i-1}}(a)|} > 2$ and $\frac{|T^{k_i}(b)|}{|T^{k_{i-1}}(b)|} > 2$,

(b) $\{T^{k_{i-1}+1}(a), \ldots, T^{k_i}(a)\}$ as well as $\{T^{k_{i-1}+1}(b), \ldots, T^{k_i}(b)\}$ are inside of U.

In the case (a), since for small $\lambda > 0$, the maps $T_0|_{\mathbb{S}\setminus U^*}$ and $T_\lambda|_{S_\lambda\setminus U^*}$ are C^1-close, for some $\alpha_\lambda < 1$ with $\alpha_\lambda \to 1$ as $\lambda \to 0$, we have $\frac{|T^{k_i}(a)|}{|T^{k_{i-1}}(a)|} \ge \left(\frac{|T^{k_i}(b)|}{|T^{k_{i-1}}(b)|}\right)^{\alpha_\lambda}$. In the case (b), by [21, Prop. 3.15], we have $\frac{|T^{k_i}(a)|}{|T^{k_{i-1}}(a)|} \ge C\mu_0^{\frac{k_i-k_{i-1}}{2}}$, where μ_0 is an unstable multiplier at a singularity, and at the same time we have $\frac{|T^{k_i}(b)|}{|T^{k_{i-1}}(b)|} \le \tilde{C}\mu_\lambda^{k_i-k_{i-1}}$, where μ_λ is an unstable multiplier of a periodic orbit P_λ, and which obeys $\mu_\lambda \to \mu_0$ as $\lambda \to 0$. Choose $\gamma_\lambda < \frac{1}{2}$, $\gamma_\lambda \to \frac{1}{2}$ as $\lambda \to 0$, in such a way that $\mu_\lambda^{2\gamma_\lambda} < \mu_0$. Then, (since in this case, the smallness of U^* relative to U guarantees that $k_i - k_{i-1}$ is large enough) we have $\frac{|T^{k_i}(a)|}{|T^{k_{i-1}}(a)|} \ge \left(\frac{|T^{k_i}(b)|}{|T^{k_{i-1}}(b)|}\right)^{\gamma_\lambda}$.

Now we have

$$1 \sim_C |T^M(a)| = |a| \cdot \frac{|T^{k_1}(a)|}{|a|} \cdot \frac{|T^{k_2}(a)|}{|T^{k_1}(a)|} \cdot \ldots \cdot \frac{|T^{k_m}(a)|}{|T^{k_{m-1}}(a)|}$$

and

$$1 \sim_C |T^M(b)| = |b| \cdot \frac{|T^{k_1}(b)|}{|b|} \cdot \frac{|T^{k_2}(b)|}{|T^{k_1}(b)|} \cdot \ldots \cdot \frac{|T^{k_m}(b)|}{|T^{k_{m-1}}(b)|}.$$

Therefore,

$$|a| = \frac{|T^M(a)|}{\prod_{j=1}^{m} \frac{|T^{k_j}(a)|}{|T^{k_{j-1}}(a)|}} \tag{16}$$

$$\leq \frac{C}{\left(\prod_{j=1}^{m} \frac{|T^{k_j}(b)|}{|T^{k_{j-1}}(b)|}\right)^{\gamma_\lambda}} = \frac{C}{\left(\frac{|T^M(b)|}{|b|}\right)^{\gamma_\lambda}} \leq C^{1+\gamma_\lambda}|b|^{\gamma_\lambda} = C'|b|^{\gamma_\lambda}. \tag{17}$$

Since $|b| = \frac{1}{\sqrt{2}}|E_1 - E_2|$ and $|a| = \frac{1}{\sqrt{2}}|\Psi_\lambda(E_1) - \Psi_\lambda(E_2)|$, using Lemma 3.1.(a), we find

$$|N_\lambda(E_1) - N_\lambda(E_2)| = |N_0(\Psi_\lambda(E_1)) - N_0(\Psi_\lambda(E_2))|$$
$$\leq C|\Psi_\lambda(E_1) - \Psi_\lambda(E_2)| \leq C\sqrt{2}|a| \leq CC'\sqrt{2}|b|^{\gamma_\lambda} \leq C''|E_1 - E_2|^{\gamma_\lambda}.$$

Now let us consider the case (ii). In this case, one of the edges of a and b will never leave the neighborhood U^*. Let us iterate a and b sufficiently many times to have

$$|T^M(a)| \sim_C |T^M(b)| \sim_C 1. \tag{18}$$

Consider the coordinate system in a neighborhood of the singularity that rectifies all the invariant manifolds (see [19, Sect. 4] or [21, Subsect. 3.2]). The lines ℓ_0 and ℓ_λ are transversal to the central-stable manifolds, hence we can apply [21, Prop. 3.15] with some bounded (independent of a and b) number k^* from [21, Lem. 3.16]. Since $\mu_\lambda^{2\gamma_\lambda} < \mu_0$, we have

$$|a| \leq C\mu_0^{-M} \leq C\mu_\lambda^{-2\gamma_\lambda M} \leq C'|b|^{2\gamma_\lambda}.$$

Using Lemma 3.1.(b), we find

$$|N_\lambda(E_1) - N_\lambda(E_2)| = |N_0(\Psi_\lambda(E_1)) - N_0(\Psi_\lambda(E_2))|$$
$$\leq C|\Psi_\lambda(E_1) - \Psi_\lambda(E_2)|^{1/2} \leq C'|E_1 - E_2|^{\gamma_\lambda}.$$

Finally, let us consider the case (iii). Suppose that $E_1, E_2 \in \Sigma_\lambda$ are such that $E_1 < E_2 < \max \Sigma_\lambda$ and $\Psi_\lambda(E_1)$ is close to 2 (the case when $\Psi_\lambda(E_1)$ is close to -2 is similar). Then, $a \subset \ell_0, b \subset \ell_\lambda$ are in the neighborhood $U(P_1)$ of $P_1 = (1, 1, 1)$, and the distance from a to the central-stable manifold is of order μ_0^{-s}, where s is a number of iterates needed for a to leave $U(P_1)$; see [21, Prop. 3.14]. This implies that

$$|2 - \Psi_\lambda(E_2)| \sim_C \mu_0^{-s}.$$

Consider the arcs $\tilde{a} = T^s(a)$ and $\tilde{b} = T^s(b)$. These arcs are away from U, and therefore from (16) we deduce that $|\tilde{a}| \leq C|\tilde{b}|^{\gamma_\lambda}$. On the other hand, we have

$$|\tilde{a}| \sim_C \mu_0^s|a|, \quad |\tilde{b}| \sim_C \mu_\lambda^s|b|,$$

and therefore

$$|a| \leq C\mu_0^{-s}|\tilde{a}| \leq C^2\mu_0^{-s}|\tilde{b}|^{\gamma_\lambda} \leq C^{2+\gamma_\lambda}\mu_0^{-s}\mu_\lambda^{s\gamma_\lambda}|b|^{\gamma_\lambda} = C^{2+\gamma_\lambda}\mu_0^{-\frac{s}{2}}\left(\frac{\mu_\lambda^{\gamma_\lambda}}{\mu_0^{1/2}}\right)^s|b|^{\gamma_\lambda}$$

$$\leq C'\mu_0^{-\frac{s}{2}}|b|^{\gamma_\lambda}.$$

Now, using Lemma 3.1.(c), we find

$$|N_\lambda(E_1) - N_\lambda(E_2)| = |N_0(\Psi_\lambda(E_1)) - N_0(\Psi_\lambda(E_2))|$$

$$\leq \frac{C}{|2 - \Psi_\lambda(E_2)|^{\frac{1}{2}}} |\Psi_\lambda(E_1) - \Psi_\lambda(E_2)| \leq \frac{C^2 C'}{\mu_0^{-\frac{s}{2}}} \mu_0^{-\frac{s}{2}} |b|^{\gamma_\lambda} \leq C''|E_1 - E_2|^{\gamma_\lambda}.$$

Finally, notice that in all cases (i), (ii), (iii), the constant in the inequality can be set to 1 if one takes a slightly smaller Hölder exponent and sufficiently small $|E_1 - E_2|$. This finishes the proof of part (a) of Theorem 3.

In order to show part (b) of Theorem 3, consider the periodic points of period 2 that are born out of the singularity $(1, 1, 1)$. The strong stable manifolds of these points correspond to the boundaries of the gaps in the spectrum. These periodic points form a curve

$$Per_2 = \left\{(x, y, z) : x \in \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right), y = \frac{x}{2x - 1}, z = x, \right\}.$$

For the map T^2, these points are fixed points, and $DT^2(1, 1, 1) = \begin{pmatrix} 6 & 3 & -2 \\ 2 & 2 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, with

eigenvalues 1 and $\frac{7 \pm 3\sqrt{5}}{2}$.

Lemma 3.3. *For $\lambda > 0$ sufficiently small, the largest eigenvalue of DT^2 at the periodic point of period 2 near the singularity $(1, 1, 1)$ is strictly larger than the largest eigenvalue of $DT^2(1, 1, 1)$.*

Proof. Take a periodic point $\left(x, \frac{x}{2x-1}, x\right) \in Per_2$. We have

$$DT^2\left(x, \frac{x}{2x-1}, x\right) = \begin{pmatrix} \frac{2x(4x-1)}{2x-1} & \frac{4x-1}{(2x-1)^2} & \frac{2x}{1-2x} \\ 2x & \frac{2x}{2x-1} & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

and the largest eigenvalue is

$$\lambda^u(x) = \frac{1 - 2x + 8x^2 + \sqrt{-3 + 12x + 4x^2 - 32x^3 + 64x^4}}{2(2x - 1)}.$$

Now we have

$$\frac{d}{dx}\lambda^u(x)|_{x=1} = 0, \quad \frac{d^2}{dx^2}\lambda^u(x)|_{x=1}$$

$$= \frac{8(-3(1+3\sqrt{5})+2(9+18\sqrt{5}+2(-27+3\sqrt{5}+2(55 - 12\sqrt{5}+4(-1+6\sqrt{5})))))}{135\sqrt{5}}$$

$$= 16.247987... > 0.$$

$\square$

In other words, Lemma 3.3 claims that $\mu_\lambda > \mu_0$ if $\lambda > 0$ is small enough. Fix a small $\lambda > 0$ and take any $\tilde{\gamma} \in (0, \frac{1}{2})$ such that $\mu_\lambda^{2\tilde{\gamma}} > \mu_0$. We claim that if $E_2 = \max \Sigma_\lambda$, $E_1 \in \Sigma_\lambda$, and $|E_1 - E_2|$ is sufficiently small, then

$$|N_\lambda(E_1) - N_\lambda(E_2)| \geq |E_1 - E_2|^{\tilde{\gamma}}.$$

Indeed, $\Psi(E_2) = 2$, and the interval $a \subset \ell_0$ between the points $L_0(\Psi_\lambda(E_2))$ and $L_0(\Psi_\lambda(E_1))$ has one of its end points at the singularity $P_1 = (1, 1, 1)$. Consider also the interval $b \subset \ell_\lambda$ between $L_\lambda(E_2)$ and $L_\lambda(E_1)$. As in the case (ii) above, consider M iterates of a and b, where M is such that (18) holds. Then, due to [21, Prop. 3.14], we have

$$|a| \sim_C \mu_0^{-M}, \quad |b| \sim_C \mu_\lambda^{-M}.$$

Using Lemma 3.1.(d), we find

$$|N_\lambda(E_1) - N_\lambda(E_2)| = |N_0(\Psi_\lambda(E_1)) - N_0(\Psi_\lambda(E_2))|$$

$$\geq C_0 |\Psi_\lambda(E_1) - \Psi_\lambda(E_2)|^{1/2} = 2^{1/4} C_0 |a|^{1/2} \geq C' \mu_0^{-\frac{M}{2}} = C' \left(\frac{\mu_\lambda^{M\tilde{\gamma}}}{\mu_0^{\frac{M}{2}}} \right) \mu_\lambda^{-M\tilde{\gamma}}$$

$$\geq C'' \left(\frac{\mu_\lambda^{2\tilde{\gamma}}}{\mu_0} \right)^{\frac{M}{2}} |b|^{\tilde{\gamma}} \geq C''' \left(\frac{\mu_\lambda^{2\tilde{\gamma}}}{\mu_0} \right)^{\frac{M}{2}} |E_2 - E_1|^{\tilde{\gamma}} \geq |E_2 - E_1|^{\tilde{\gamma}},$$

provided M is large enough (i.e., if $|a|$ and $|b|$ are small enough, or, equivalently, if $|E_2 - E_1|$ is small enough). This completes the proof of Theorem 3. □

Notice that as a byproduct of this proof, we also get the following statement:

Proposition 3.4. *The map* $\Psi_\lambda : \Sigma_\lambda \to [0, 2]$ *given by* (15) *is Hölder continuous with a Hölder exponent* γ_λ *that obeys* $\gamma_\lambda \to \frac{1}{2}$ *as* $\lambda \to 0$.

Remark 3.5. In terms of the dynamics of the trace map, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, the semiconjugacy $\Phi_\lambda : \Lambda_\lambda \to \mathbb{S}$, $\Phi_\lambda \circ T_\lambda|_{\Lambda_\lambda} = T_0|_\mathbb{S} \circ \Phi_\lambda$, is Hölder continuous with a Hölder exponent γ_λ such that $\gamma_\lambda \to \frac{1}{2}$ as $\lambda \to 0$. Since we are not using this statement here, we do not elaborate on it.

This is related to the following classical fact from hyperbolic dynamics. Suppose Λ_f is a compact locally maximal hyperbolic set of a surface diffeomorphism $f : M^2 \to M^2$, g is C^1-close to f, and Λ_g is a continuation of Λ_f. Then there is a continuous conjugacy $h : \Lambda_f \to \Lambda_g$, $h \circ f = g \circ h$, and h has to be Hölder continuous with Hölder exponent close to one (see [35,42]). In our case, the Hölder exponent of the conjugacy Φ_λ is close to 1/2, not to one, due to essentially different behavior of $T_0|_\mathbb{S}$ and $T_\lambda|_{\Lambda_\lambda}$ near the singularities of the Cayley cubic.

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