

# An Improved Geometric Inequality via Vanishing Moments, with Applications to Singular Liouville Equations

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**Abstract:** We consider a class of singular Liouville equations on compact surfaces motivated by the study of Electroweak and Self-Dual Chern-Simons theories, the Gaussian curvature prescription with conical singularities and Onsager’s description of turbulence. We analyse the problem of existence variationally, and show how the angular distribution of the conformal volume near the singularities may lead to improvements in the Moser-Trudinger inequality, and in turn to lower bounds on the Euler-Lagrange functional. We then discuss existence and non-existence results.

## 1. Introduction

On a compact orientable surface  $(\Sigma, g)$  without boundary and with metric  $g$  we consider the equation

$$-\Delta_g u = \rho \left( \frac{h(x)e^{2u}}{\int_{\Sigma} h(x)e^{2u} dV_g} - a(x) \right) - 2\pi \sum_{j=1}^m \alpha_j \left( \delta_{p_j} - \frac{1}{|\Sigma|} \right), \quad \int_{\Sigma} a(x) dV_g = 1. \quad (1)$$

Here  $\rho$  is a positive parameter,  $a, h : \Sigma \rightarrow \mathbb{R}$  two smooth functions,  $h(x) > 0$  for every  $x \in \Sigma$ ,  $\alpha_j > 0$ ,  $p_j \in \Sigma$  and  $|\Sigma|$  denotes the area of  $\Sigma$ , that is  $|\Sigma| = \int_{\Sigma} dV_g$ .

The analysis of (1) is motivated by the study of vortex type configurations (see for example [4, 11, 26, 62 and 24, 33, 47, 54, 60, 61, 64]) in the Electroweak theory of Glashow-Salam-Weinberg [44], and in Self-Dual Chern-Simons theories [37, 41, 42]. We refer the reader to [25, 68] and to the monographs [67, 72] for further details and an up to date set of references concerning these applications. Other classical problems

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call for the study of (1) such as the prescribed Gaussian curvature problem on surfaces with conical singularities [5, 21, 69, 70] and Onsager's statistical mechanics description of two-dimensional turbulence [15] in the presence of vortex sources [19]. Moreover the study of (1) and of the corresponding Dirichlet problem (see (16) below) on bounded domains  $\Omega \subset \mathbb{R}^2$  has an independent interest related to the description of rotational shear flows [71] and/or Euler flows in the presence of vortex sources [9]. It turns out that the structure of the solutions' set for these elliptic problems is extremely sensitive to the data. For example it is well known (see Proposition 5.7 below) that if  $\Omega$  is the unit ball with just one singularity (i.e.  $m = 1$ ) located at the origin and  $h \equiv 1$ , then we have non existence of solutions for (16) with  $\rho \geq 4\pi(1 + \alpha_1)$ . On the other side, if either  $\Sigma \equiv \mathbb{S}^2$  or  $\Omega$  is simply connected (at least to our knowledge, with few exceptions which will be shortly discussed below), there are no general existence results at all for either (1) or (16) with  $\rho > 4\pi$ . In any case there are still relatively few results at hand concerning existence of a solution for (1) (some of which will be described in more detail below). The reason for this gap consists essentially in the well known issue of non coercivity of the variational functionals (see (6) below) associated to the study of these problems when  $\rho \geq 4\pi$ . It seems in particular that we have a quite unsatisfactory understanding of some of these existence/non existence problems as for example a model case [9] suggests that the Dirichlet problem on simply connected domains should admit at least one solution for each  $\rho \in (0, 4\pi \min_{i=1, \dots, m} \{1 + \alpha_i\}) \setminus 4\pi\mathbb{N}$ . It is one of our motivations to fill this gap here.

In [11] an existence theorem is proved via min-max methods for surfaces with positive genus and for  $\rho \in (4\pi, 8\pi)$ . More recently the latter result has been extended in [4, 5], still for positive genus, for  $\rho$  outside the discrete *critical* set (see (10) and Theorem 2.9 below) found in [11]. In [53] the case of arbitrary genus was treated, but only for  $\alpha_j \leq 1$  for all  $j$  and  $\rho \in (4\pi, 8\pi)$ .

In [24] existence results are deduced by calculating the Leray-Schauder degree when  $\alpha_j \geq 1$  for all  $j$  and  $\rho \in (4\pi, 8\pi)$ . In [25, 26] an on-going project to compute the Leray-Schauder degree of the equation is presented, using refined blow-up analysis and Lyapunov-Schmidt reductions, concerning the case  $\alpha_j \in \mathbb{N}$  for all  $j$ . This approach has been successful for the *regular case*, when all  $\alpha_j$ 's are zero: a formula for the degree of the equation has been derived in [22, 23] building upon previous blow-up analysis and quantization results in [14, 45, 46].

Necessary and sufficient conditions for the existence of a solution for the Dirichlet problem in the critical case  $\rho = 4\pi$  in the presence of singularities on simply connected domains has been recently found in [7] (see also [6]).

We also refer to [29, 38] for some perturbative results providing solutions of multi-bump type (via implicit function theorems) for special values of the parameter  $\rho$  for (1) on bounded two dimensional domains with Dirichlet boundary conditions.

The goal of this paper is to prove a key inequality for treating (1) variationally in general situations, and to present some applications to the existence problem in simple cases. In particular 1) compared to [4, 5 and 11] we remove the assumption on the genus; 2) compared to [24 and 53] we also allow the  $\alpha_j$ 's and  $\rho$  to be arbitrarily large; 3) compared to [25, 26] we do not require  $\alpha_j \in \mathbb{N}$ : we notice that the structure of solutions to (1) might change drastically depending on  $\rho$  when the coefficients  $\alpha_j$  are not integer, see Remark 1.3 (a), and that in some situations one might still have existence when the degree of the equation vanishes (see [34]); 4) compared to [29, 38 and 7], we allow generic values of  $\rho$ . We also expect that our result, combined with those in [17], might allow to treat the case of  $\alpha$ 's with arbitrary sign, more relevant for geometric applications, as well as give

precise homological information on the variational structure (about the latter issue, see more comments after (10)).

Problem (1) admits an equivalent formulation with variational structure: letting  $G_p(x)$  denote the Green’s function of  $-\Delta_g$  on  $\Sigma$  with pole at  $p$ , i.e. the unique solution to

$$-\Delta_g G_p(x) = \delta_p - \frac{1}{|\Sigma|} \quad \text{on } \Sigma, \quad \text{with} \quad \int_{\Sigma} G_p(x) dV_g = 0, \quad (2)$$

by the substitution

$$u \mapsto u + 2\pi \sum_{j=1}^m \alpha_j G_{p_j}(x), \quad h(x) \mapsto \tilde{h}(x) = h(x)e^{-2\pi \sum_{j=1}^m \alpha_j G_{p_j}(x)}, \quad (3)$$

(1) transforms into an equation of the type

$$-\Delta_g u = \rho \left( \frac{\tilde{h}(x)e^{2u}}{\int_{\Sigma} \tilde{h}(x)e^{2u} dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{on } \Sigma. \quad (4)$$

In general the constant  $\frac{1}{|\Sigma|}$  in (4) is replaced by a smooth function  $\tilde{a}(x)$  with  $\int_{\Sigma} \tilde{a}(x)dV_g = 1$ : this term is indeed rather harmless, and we will not comment on this issue any further.

Since  $G_p$  has the asymptotic behavior  $G_{p_j}(x) \simeq \frac{1}{2\pi} \log \frac{1}{d(x,p_j)}$  near  $p_j$ , by (3) the function  $\tilde{h}$  satisfies

$$\tilde{h} > 0 \text{ on } \Sigma \setminus \cup_j \{p_j\}; \quad \tilde{h}(x) \simeq \gamma_j d(x, p_j)^{2\alpha_j} \quad \text{near } p_j \quad (5)$$

for some constant  $\gamma_j > 0$ , where  $d(\cdot, \cdot)$  stands for the distance induced by  $g$ .

Problem (4) is the Euler-Lagrange equation of the functional

$$I_{\rho, \underline{\alpha}}(u) = \int_{\Sigma} |\nabla_g u|^2 dV_g + 2 \frac{\rho}{|\Sigma|} \int_{\Sigma} u dV_g - \rho \log \int_{\Sigma} \tilde{h}(x)e^{2u} dV_g; \quad u \in H^1(\Sigma). \quad (6)$$

One basic tool for treating such kind of functionals is the well known Moser-Trudinger inequality

$$\log \int_{\Sigma} e^{2(u-\bar{u})} dV_g \leq \frac{1}{4\pi} \int_{\Sigma} |\nabla_g u|^2 dV_g + C; \quad u \in H^1(\Sigma), \quad \bar{u} = \int_{\Sigma} u dV_g, \quad (7)$$

see e.g. [39 and 56]. The value  $\frac{1}{4\pi}$  is sharp in (7), as one can insert in the above inequality a test function like

$$\varphi_{\lambda, x}(y) = \log \frac{\lambda}{1 + \lambda^2 \text{dist}(x, y)^2}; \quad \lambda > 0, x \in \Sigma, \quad (8)$$

and check that both sides diverge to infinity at the same rate. This function is usually called a *standard bubble*, since the conformal metric  $\tilde{g} = e^{2\varphi_{\lambda, x}}$  endows  $\Sigma$  with a spherical metric near  $x$ .

In the presence of singularities, namely when a weight  $\tilde{h}$  as in (5) multiplies the exponential term, a modified sharp Moser-Trudinger inequality was derived in [20 and 70] (see also [18], or also [28,40] for extensions to general settings), and takes the form

$$\log \int_{\Sigma} \tilde{h} e^{2(u-\bar{u})} dV_g \leq \frac{1}{4\pi} \frac{1}{\min\{1, \min_j\{1 + \alpha_j\}\}} \int_{\Sigma} |\nabla u|^2 dV_g + C. \tag{9}$$

As one can see, the constant is bigger when one of the coefficients - say  $\alpha_{\bar{j}}$  - is negative, as  $\tilde{h}$  is singular near  $\alpha_{\bar{j}}$ . However, when all of the  $\alpha_j$ 's are positive, as in the case we are considering, the best constant remains  $\frac{1}{4\pi}$ . The fact that  $\tilde{h}$  is equal to zero at all singular points does not give a smaller constant, as one may initially expect: inserting in the inequality a *bubble* at a regular point  $x$  does not pick up any effect of the vanishing of  $\tilde{h}$  near the  $p_j$ 's.

From (9) one has that  $I_{\rho,\underline{\alpha}}$  is bounded from below for  $\rho < 4\pi$ , and hence one can find solutions of (1) by globally minimizing  $I_{\rho,\underline{\alpha}}$ , which is coercive, using the direct methods of the calculus of variations. When  $\rho > 4\pi$  instead the situation becomes more delicate, as  $\inf I_{\rho,\underline{\alpha}} = -\infty$ : one might however try to obtain solutions as saddle points. We describe next some previous results in the literature which rely on this strategy.

Even though  $I_{\rho,\underline{\alpha}}(u)$  is not bounded below on  $H^1(\Sigma)$ , one might hope to find suitable conditions on  $u$  to recover some control. Calling  $\mathfrak{A} \subseteq H^1(\Sigma)$  a set of functions for which this lower bound holds, one can then try to show that  $\mathfrak{A}$  is always intersected along a suitable family of min-max maps.

For the regular case of (1) such a lower bound was obtained in by W.Chen and C.Li in [21] (extending previous results in [1 and 57] for the standard sphere) under the condition that two subsets of  $\Sigma$  with positive mutual distance both contain a finite portion of the total mass. Under such an assumption, one finds that the best constant in (7) can be chosen arbitrarily close to  $\frac{1}{8\pi}$ : as a consequence, when  $\rho < 8\pi$  and when  $I_{\rho,\underline{\alpha}}(u)$  is large negative,  $e^{2u}$  has to concentrate near a single point of  $\Sigma$  (similarly, if  $\rho < 4(k+1)\pi$ ,  $e^{2u}$  concentrates near at most  $k$  points, as shown in [35]). This property was used in [32] to obtain existence on surfaces of positive genus when  $\rho \in (4\pi, 8\pi)$ , and in [34] (relying on an argument in [35] for the  $Q$ -curvature prescription problem) for  $\rho \notin 4\pi\mathbb{N}$  on all surfaces. The restriction  $\rho \notin 4\pi\mathbb{N}$  is a compactness condition which allows to apply the deformation lemma, see [63,49] (see also [23,26] for some results concerning the case  $\rho \in 4\pi\mathbb{N}$ ).

For the singular case, a related approach has been used in [11] where, through a new quantization property (see Theorem 2.9) the result in [32] was extended to the case of positive  $\alpha_j$ 's (and, still, for positive genus). In particular, compactness is obtained provided  $\rho \notin \Lambda_{\underline{\alpha}}$ , where

$$\Lambda_{\underline{\alpha}} = \left\{ 4k\pi + 4\pi \sum_{j \in J} (1 + \alpha_j) \mid k \in \mathbb{N} \cup \{0\}, J \subseteq \{1, \dots, m\} \right\}. \tag{10}$$

The latter existence result was later generalized in [4,5] to the case of arbitrarily large (but positive) values of  $\rho$ .

The last two existence results however do not fully capture the variational features of the presence of the singularities, from three different aspects. They do neither extend

to the case of the sphere or to the case when some negative weights are present (which could be relevant, we recall, for the Gaussian curvature prescription and to the study of turbulent flows interacting with vortex sinks). Finally, the analysis is not sufficient if one wants to fully characterize from the homological point of view the structure of sublevels of  $I_{\rho, \underline{\alpha}}$ , to compute for example the degree of the equation as it has been done in [52] for the regular case.

An improvement of inequality (9), more intrinsically related to the presence of singularities, was derived in [36]: it was shown that for any  $\alpha > -1$  there exists  $C_\alpha > 0$  such that

$$\log \int_B |x|^{2\alpha} e^{2(u-\bar{u})} dV_g \leq \frac{1}{4(1+\alpha)\pi} \int_B |\nabla_g u|^2 dV_g + C_\alpha; \quad u \in H_r^1(B), \quad (11)$$

where  $B$  is the unit ball of  $\mathbb{R}^2$  and  $H_r^1$  the space of radial functions in  $B$  of class  $H^1$ . This result has a previous related counterpart in [55], where the case of curvatures with  $\mathbb{Z}_k$  symmetry and polynomial decay in  $\mathbb{R}^2$  was considered, among others.

In [53] a general improvement (without assuming any symmetry) was found for  $\alpha \in (0, 1]$  and  $\rho \in (4\pi, 8\pi)$ : recall that in this case, by the above discussion, a low energy for  $u$  implies concentration of the volume near at most one point. The novelty in [53] was to derive an extra characterization of this point, which takes into account both the *scale* of concentration of the volume measure and its *center of mass*. More precisely, it was proven that there exists a continuous map  $\beta$  from low sublevels of  $I_{\rho, \underline{\alpha}}$  into  $B$  such that if  $\beta(u)$  hits the singularity then (9) holds with  $\frac{1}{4\pi}$  replaced by  $\frac{1+\varepsilon}{4\pi(1+\alpha)}$ , where  $\varepsilon$  can be chosen arbitrarily small (see Proposition 2.7 for more details). Notice that this condition relaxes the radially in [36] to a two-dimensional constraint, and that it allows an arbitrarily small scale of concentration at a single point (so [21] would not apply). The condition  $\alpha \leq 1$  is indeed sharp in this argument, as one can find counterexamples for  $\alpha > 1$ . We also refer to [54] where a somehow related strategy is used for *Toda systems* (arising from non-abelian theories).

The main goal of this paper is to find a general condition to get an improved inequality for arbitrary  $\alpha$ 's, with no symmetry requirements, and which is flexible enough to be combined with min-max arguments. As we will describe, our approach combines the scaling invariance properties in [53] and the possibility of volume concentration at multiple points (as in [34,35]).

To explain this condition in more detail, suppose we are on the unit ball  $B$  of  $\mathbb{R}^2$  and that we are dealing with only one singularity at the origin with weight  $\alpha$ . Let  $\tilde{f}_u$  denote the probability measure on  $B$ ,

$$\tilde{f}_u = \frac{\tilde{h}(x)e^{2u}}{\int_B \tilde{h}(x)e^{2u} dx}. \quad (12)$$

Roughly speaking, our result can be interpreted as a version of the above concentration property at finitely-many points in a *complete* setting, blowing-up the metric near the singularity as  $\tilde{g} = \frac{1}{|x|^2} (dx)^2$  so that the Euclidean metric becomes cylindrical. To state this property rigorously, assuming that  $\rho \in (4k\pi, 4(k+1)\pi)$ , given  $\delta > 0$  small we define

$$J_{k, \delta}(\tilde{f}_u) = \sup_{x_1, \dots, x_k \neq 0} \int_{\bigcup_{i=1}^k B_{\delta|x_i|}(x_i)} \tilde{f}_u dx. \quad (13)$$

To describe our strategy, we first consider two alternatives which may occur: when  $J_{k,\delta}(\tilde{f}_u)$  is close to 1 and when it is not.

When this quantity is close to 1, we are in a situation similar to Chen and Li’s (but in the cylindrical metric). For the regular case, the argument in [34] (or in [35]) implies that if  $k$  small balls in  $B$  contain most of the volume (as in this first alternative), then it is possible to find a continuous map from these measures into the *formal barycenters of  $B$  of order  $k$* , namely the probability measures of the form  $B_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} : t_i \geq 0, \sum_{i=1}^k t_i = 1, x_i \in B \right\}$ .

In our case, it is natural to incorporate the dilation invariance of the problem (corresponding to a translation along the axis of the cylinder), and to project onto the barycenters of order  $k$  of  $S^1$ , which coincides with the cylinder factoring out the translations.

For doing this, we define the probability measure on the circle

$$\mu_u(A) = \int_{\tilde{A}} \tilde{f}_u dx; \quad A \subseteq S^1, \quad \tilde{A} = \cup_{t \in (0,1]} tA. \tag{14}$$

When  $J_{k,\delta}(f)$  is close to 1 then  $\mu_u \simeq \sum_{i=1}^k t_i \delta_{\theta_i}$  (in the distributional sense) for some  $t_i \geq 0$  and some  $\theta_i \in S^1$ . The  $k$ -barycenters of  $S^1$ ,  $(S^1)_k$ , are known to be homotopically equivalent to  $S^{2k-1}$ , see Theorem 1.1 and Corollary 1.5 in [43]. It is however convenient for us to understand this set in more detail, proving that it is indeed *homeomorphic* to a (piecewise smooth) immersed sphere  $\mathcal{S}_k$  in  $\mathbb{C}^k$  with interior  $\mathcal{U}_k$  being a neighborhood of the origin, see Sect. 3. This is useful in order to construct a continuous projection of a small neighborhood  $\mathcal{N}_k$  of  $\mathcal{S}_k$  onto  $\mathcal{S}_k$  itself.

More precisely, let

$$F_k(\tilde{f}_u) = \left( \int_{S^1} z d\mu_u, \int_{S^1} z^2 d\mu_u, \dots, \int_{S^1} z^k d\mu_u \right) \tag{15}$$

map the probability measures on  $S^1$  into  $\mathbb{C}^k$ . Using this map, we define  $\mathcal{S}_k$  to be  $F_k((S^1)_k)$ , which can be seen to be homeomorphic to  $\mathcal{S}_k$ : we check that in the first alternative ( $J_{k,\delta}(\tilde{f}_u)$  close to 1)  $F_k(\tilde{f}_u)$  lies in  $\mathcal{N}_k$ , so we can project continuously onto  $\mathcal{S}_k \simeq (S^1)_k$ .

We consider next the second of the two alternatives, namely when  $J_{k,\delta}(\tilde{f}_u)$  is bounded away from 1. One thing to be immediately noticed is that while in a compact situation one always obtains weak convergence of a sequence of probability measures to a probability measure, in the complete case some part of the *mass* (or the whole one) might *vanish* (borrowing the terminology of concentration-compactness theory).

We show by a covering argument that if  $J_{k,\delta}(\tilde{f}_u)$  is bounded away from 1 then either some volume concentrates near at least  $k + 1$  well separated points with respect to the cylindrical metric, or that some part of the measure spreads on the cylinder (giving rise to a vanishing), see Lemma 4.2. In either case, using harmonic liftings and some argument in [53] (see Proposition 2.7) we show that the constant in (9) improves by a factor  $\min\{1+k, 1+\alpha\}$ , see Proposition 4.1 (recall that  $\alpha$  stands for the weight of the singularity at the origin). One condition, easy to verify, which guarantees this improvement is the vanishing of the moments of the measure  $\mu_u$  up to order  $k$ , see Corollary 4.6. Qualitatively, this is quite similar to the requirement on  $\beta(u)$  in [53], see the comments after (11). Furthermore, it would apply to a symmetric case as in [55], but it only imposes finitely-many integral constraints on  $u$ .

We employ the previously described inequality to find new existence results for (1) and its analogous Dirichlet problem on bonded domains  $\Omega \subset \mathbb{R}^2$ , that is

$$\begin{cases} -\Delta u = \rho \frac{\tilde{h}e^{2u}}{\int_B \tilde{h}e^{2u} dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where,  $\tilde{h}(x) = h(x)e^{-2\pi \sum_{j=1}^m \alpha_j G_{p_j,0}(x)}$  for some strictly positive and smooth  $h$  on  $\Omega$  and, for  $p \in \Omega$ ,  $G_{p,0}(x)$  denotes the Green's function uniquely defined by

$$-\Delta G_{p,0}(x) = \delta_p \quad \text{on } \Omega, \quad G_{p,0}(x) = 0 \quad \text{on } \partial\Omega. \tag{17}$$

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^2$  be an open and bounded domain, and let  $m \neq 0$ . Then problem (16) admits a solution for every  $\rho \in (0, 4\pi \min_{i=1, \dots, m} \{1 + \alpha_i\}) \setminus 4\pi\mathbb{N}$ .*

**Theorem 1.2.** *Suppose  $\Sigma$  is a topological sphere, and let  $m \geq 2$ . Then (1) has a solution provided  $\rho \in (0, 4\pi \min_{i=1, \dots, m} \{1 + \alpha_i\}) \setminus 4\pi\mathbb{N}$ .*

- Remark 1.3.* (a) The upper bounds on  $\rho$  in Theorems 1.1 and 1.2 are sharp: in Sect. 5 we complement our results with Propositions 5.7 and 5.8, giving non existence for larger values of  $\rho$  (for  $m = 1$  in the unit ball or  $\mathbb{R}^2$  and for  $m = 2$  on  $S^2$ ).
- (b) We will prove in detail Theorem 1.1 only for simply connected domains and for  $m = 1$ , since this case is the simplest one requiring our new estimates. We sketch the argument for the other cases in Remark 5.6, since the proof adapts quite easily. The counterpart of Theorem 1.2 for surfaces with positive genus (actually a more general version of it, without upper bounds on  $\rho$ ) was proved in [5]: in Remark 5.6 we will briefly discuss how our method can also be adapted to these surfaces when  $\rho < 4\pi \min_{i=1, \dots, m} \{1 + \alpha_i\}$ .

To prove Theorem 1.1 we employ a min-max scheme which uses the formal barycenters of  $S^1$ . More precisely, let  $k$  be the unique integer for which  $\rho \in (4k\pi, 4(k + 1)\pi)$ . Then, since  $J_{k,\delta}(\tilde{f}_u)$  separate from 1 leads to a lower bound for  $I_{\rho,\alpha}$ , the above discussion suggests that if  $u$  has low energy, then the measure  $\mu_u$  on  $S^1$  (see (14)) should be close to some element of  $(S^1)_k$  in the distributional sense.

We build a min-max scheme based on this idea: starting from  $\sigma \in (S^1)_k$ , we define a test function (see (63)) for which the associated conformal volume resembles  $\sigma$ , and on which  $I_{\rho,\alpha}$  attains large negative values, see Lemma 5.1. Since  $(S^1)_k \simeq S^{2k-1}$  (see the comments before (15)), the family of these test functions forms a  $2k - 1$ -dimensional sphere in  $H_0^1(B)$ , on which the supremum of  $I_{\rho,\alpha}$  is very low.

We then complete this family with a map from a topological ball in  $\mathbb{C}^k$  into  $H_0^1(B)$ , and we consider the min-max value associated to this construction, see Proposition 5.2. The improved inequality in Proposition 4.1 is used to show that the min-max value is strictly larger than the maximal value at the boundary, otherwise by Proposition 3.1 we would be able to find (naively) a retraction from the unit ball of  $\mathbb{C}^k$  onto its boundary, which is a contradiction. Details are given in Sect. 5.

The compactness issue due to a lack of knowledge about the Palais-Smale condition can be tackled via by now standard means: varying the parameter  $\rho$  and reasoning as in [49,63] it is possible to find  $\rho_n \rightarrow \rho$  for which  $I_{\rho_n,\alpha}$  has a bounded Palais-Smale sequence and hence a solution. Convergence can then be proved using Theorem 2.9 and Corollary 2.10.

The proof of Theorem 1.2 can be handled using minor modifications: the main point is that concentration of conformal volume may occur either near the singularity  $p_1$  or near  $p_2$ . The corresponding improved inequality is given in Proposition 5.3, where one can see that both weights  $\alpha_1, \alpha_2$  play a role. The proof of the non existence results in Propositions 5.7 and 5.8 are shown using well-known Pohožaev type identities (see also the recent paper [16] for non existence results on surfaces with positive genus).

The paper is organized as follows. In Sect. 2 we list some preliminary results on elementary inequalities, the Moser-Trudinger inequality and some of its improvements, together with the compactness result from [11]. In Sect. 3 we show how to embed continuously the barycenters  $(S^1)_k$  into  $\mathbb{C}^k$  using moments of measures on the unit circle, and how to project continuously onto this image the family of functions for which  $J_{k,\delta}$  is close to 1. In Sect. 4 we then analyse the complementary situation, proving a dichotomy result in Proposition 4.2 and then the improved inequality in both alternatives. In Sect. 5 we then prove both existence and non existence results.

### 2. Notation and preliminaries

This section contains some useful preliminary material, including elementary inequalities, some variants of the Moser-Trudinger inequality from [21,34,35,53], and a compactness results from [11].

We will deal with either compact Riemannian surfaces  $(\Sigma, g)$ , with or without boundary, or with the unit ball  $B$  of  $\mathbb{R}^2$ . We let  $d(x, y)$  be the distance of two points  $x, y$ , while  $B_r(p)$  will stand for the open metric ball of radius  $r$  and center  $p$ . We also set, for convenience

$$B_r = \{x \in B \mid d_g(x, 0) < r\}, \quad \text{and} \quad A(s, t) = \{x \in B \mid s < d_g(x, 0) < t\}.$$

The symbol  $\int_{\Omega} u \, dV_g$  denotes the average integral  $\frac{1}{|\Omega|} \int_{\Omega} u \, dV_g$ . For  $\alpha > 0$  we set

$$k_{\alpha} = \min \{k \in \mathbb{N} \mid k \geq \alpha\}. \tag{18}$$

If  $u \in H^1(\Sigma)$  or  $u \in H_0^1(B)$ , and if  $\Omega$  has smooth boundary and is compactly contained in the domain of  $u$ , we denote by  $\mathcal{H}_{\Omega}(u)$  the harmonic extension of  $u$  inside  $\Omega$ , namely we set

$$\mathcal{H}_{\Omega}(u) = \begin{cases} v & \text{in } \Omega; \\ u & \text{in } \Sigma \setminus \Omega \quad (\text{resp. } B \setminus \Omega), \end{cases} \tag{19}$$

where  $v$  is the solution of

$$\begin{cases} \Delta_g v = 0 & \text{in } \Omega; \\ v = u & \text{on } \partial\Omega. \end{cases}$$

For two probability measures  $\mu_1, \mu_2$  defined on  $B$  the *Kantorovich-Rubinstein* distance is defined as

$$d_{KR}(\mu_1, \mu_2) = \sup_{\|f\|_{Lip} \leq 1} \left| \int_B f \, d\mu_1 - \int_B f \, d\mu_2 \right|. \tag{20}$$



If  $\tilde{h}$  is as in (5), for  $u$  of class  $H^1$  we will set

$$f_u = \tilde{h}(x)e^{2u}; \quad \tilde{f}_u = \frac{\tilde{h}(x)e^{2u}}{\int_B \tilde{h}(x)e^{2u} dx}. \tag{21}$$

We will use similar notations for functions which are defined on a compact surface  $\Sigma$ .

Generic large positive constants are always denoted by  $C, \tilde{C}$ , etc.: even though we allow constants to vary, we will often stress their dependence on other constants or parameters, as we need sometimes to be careful in the ordering of their choices.

*2.1. Some elementary inequalities.* We begin with two elementary lemmas: the first can be proved e.g. using Fourier analysis, while the second follows from Poincaré’s inequality.

**Lemma 2.1.** *Let  $p \in B$ , let  $s > 0$ , and suppose  $B_{2s}(p) \subseteq B$ . For  $u \in H^1(B_{2s}(p))$ , let  $\mathcal{H}_{B_s(p)}(u)$  be as in (19). Then there exists a universal constant  $C_0$ , independent of  $u, p$  and  $s$ , such that*

$$\int_{B_s(p)} |\nabla \mathcal{H}_{B_s(p)}(u)|^2 dx \leq C_0 \int_{B_{2s}(p) \setminus B_s(p)} |\nabla u|^2 dx.$$

*Remark 2.2.* A similar result holds when  $u \in H^1_0(B)$ ,  $B_s(p) \cap \partial B \neq \emptyset$  and  $d(p, \partial B) < \frac{s}{2}$ , with  $s \leq \frac{1}{4}$ : this condition controls from below the angle formed by  $\partial B_s(p)$  and  $\partial B$  at their intersection points. Analogous statements can be also proved for small metric balls on a given compact surface.

**Lemma 2.3.** *Let  $p \in B, s > 0$ , and suppose  $B_s(p) \subseteq B$ . Let  $\bar{C}_0$  be a fixed constant, and suppose that  $B_r(q) \subseteq B_s(p)$  with  $r \geq \bar{C}_0^{-1}s$ , and  $d(B_r(q), \partial B_s(p)) \geq \bar{C}_0^{-1}s$ . Let  $u \in H^1(B_s(p))$ : then there exists another constant  $\tilde{C}_0$ , depending on  $\bar{C}_0$  but independent of  $s, q, r$  and  $u$ , such that*

$$\left| \int_{\partial B_r(q)} u d\sigma - \int_{\partial B_s(p)} u d\sigma \right| \leq \tilde{C}_0 \|\nabla u\|_{L^2(B_s(p))}.$$

*2.2. Improved Moser-Trudinger inequalities.* We start by recalling the well known Moser-Trudinger inequality for surfaces with or without boundary (see e.g. [18, 39, 56]).

**Proposition 2.4.** *Let  $\Sigma$  be a compact surface. Then*

a) *If  $\Sigma$  has no boundary,*

$$\log \int_{\Sigma} e^{2u} dV_g \leq \frac{1}{2\pi} \int_{\Sigma} |\nabla_g u|^2 dV_g + 2 \int_{\Sigma} u dV_g + C \quad \text{for every } u \in H^1(\Sigma). \tag{22}$$

b) *If  $\Sigma$  has boundary,*

$$\log \int_{\Sigma} e^{2u} dV_g \leq \frac{1}{4\pi} \int_{\Sigma} |\nabla_g u|^2 dV_g + C \quad \text{for every } u \in H^1_0(\Sigma). \tag{23}$$

As we remarked in the Introduction, the constant  $\frac{1}{4\pi}$  in (22) is sharp, as one can see plugging in the inequality test functions as in (8).

The next result, proven in [21] for  $\ell = 1$  and in [34,35] for general  $\ell$ , gives a criterion for getting a smaller multiplicative constant in the Moser-Trudinger inequality. Basically, it asserts that the more  $e^{2u}$  is spread, the smaller constant can be chosen in (22). The proof, not reported here, relies on localizing the Moser-Trudinger inequality (through suitable cut-off functions) near the sets, called  $\Omega_i$ 's, on which there is concentration of volume.

**Proposition 2.5.** *Let  $\Sigma$  be a compact surface with no boundary,  $\tilde{h} : \Sigma \rightarrow \mathbb{R}$  with  $0 \leq \tilde{h}(x) \leq C_0$ . Let  $\Omega_1, \dots, \Omega_{\ell+1}$  be subsets of  $\Sigma$  with  $\text{dist}(\Omega_i, \Omega_j) \geq \delta_0$  for some  $\delta_0 > 0$  if  $i \neq j$ , and fix  $\gamma_0 \in (0, \frac{1}{\ell+1})$ . Then, for any  $\varepsilon > 0$  there exists a constant  $C = C(C_0, \varepsilon, \delta_0, \gamma_0, \ell)$  such that*

$$\log \int_{\Sigma} f_u dV_g \leq C + \frac{1 + \varepsilon}{4(\ell + 1)\pi} \int_{\Sigma} |\nabla_g u|^2 dV_g + 2 \int_{\Sigma} u dV_g$$

for all functions  $u \in H^1(\Sigma)$  satisfying

$$\int_{\Omega_i} \tilde{f}_u dV_g \geq \gamma_0, \quad i = 1, \dots, \ell + 1. \tag{24}$$

A similar result holds, without the average of  $u$  on the right-hand side, if  $\Sigma$  has boundary and  $u \in H_0^1(\Sigma)$ .

The case of surfaces with boundary is not explicitly written in [21] but their proof can be adapted with minor changes to cover this situation as well. A useful corollary of Proposition 2.5 is the following, which describes the set of functions for which the Euler-Lagrange functional is large negative. For the proof, which uses Proposition 2.5 and a covering argument, see [34 and 35] (see also [21 or 32] for  $k = 1$ ).

**Corollary 2.6.** *Suppose  $\rho < 4(k + 1)\pi$ . Then, given any  $\varepsilon, r > 0$  there exists  $L = L(\varepsilon, r) > 0$  such that*

$$I_{\rho, \underline{\alpha}}(u) \leq -L \quad \Rightarrow \quad \int_{\cup_{j=1}^k B_r(x_j)} \tilde{f}_u dV_g > 1 - \varepsilon \quad \text{for some } x_1, \dots, x_k \in \Sigma.$$

The next improved Moser-Trudinger inequality is established in [53], and exploits the role played by the singularities in a more subtle way. While Proposition 2.5 is based on the separation of concentration regions, Proposition 2.7 involves a separation in the scales of concentration.

**Proposition 2.7.** *Consider the case of one singularity at the origin in  $B$  (in our previous notation,  $m = 1$  and  $p_1 = 0$ ), and let  $\alpha = \alpha_1$ . Let  $\eta$  be a small positive constant, and fix  $\tau > 0$ . Let  $u \in H_0^1(B)$ , and suppose there exists  $s \in (0, \frac{1}{4})$  such that*

$$\int_{s < |x| < 4s} |\nabla u|^2 dx \leq \eta \int_B |\nabla u|^2 dx,$$

and such that

$$\int_{|x| < s} \tilde{f}_u dx \geq \tau; \quad \int_{|x| > 4s} \tilde{f}_u dx \geq \tau. \tag{25}$$

Then, there exists a universal constant  $C_0 > 0$  and  $\tilde{C} = \tilde{C}(\eta, \tau, \alpha)$  (independent of  $s$ ) such that one has the inequality

$$(1+\alpha) \log \int_B f_u dx \leq \frac{1}{4\pi} \left( \alpha \int_{|x|<2s} |\nabla u|^2 dx + \int_{|x|>2s} |\nabla u|^2 dx + C_0 \eta \int_B |\nabla u|^2 dx \right) + \tilde{C}.$$

*Proof.* The details of the proof can be found in Proposition 4.1 of [53]: for the reader's convenience, since we will also need some modified version of this result (see Remark 2.8) we will sketch here the main arguments.

First of all, we modify  $u$  in  $B_{4s} \setminus B_s$  so it becomes constant in  $B_{3s} \setminus B_{2s}$ : precisely, if we let

$$\chi_s(r) = \min \left\{ \frac{1}{s}(r - s), 1, \frac{1}{s}(4s - r) \right\}; \quad \hat{u}(s) = \int_{B_{4s} \setminus B_s} u \, dx,$$

and define

$$\tilde{u}(x) = \begin{cases} \chi_s(|x|)\hat{u}_s + (1 - \chi_s(|x|)) u(x) & \text{for } x \in B_{4s} \setminus B_s; \\ u(x) & \text{for } x \in B \setminus (B_{4s} \setminus B_s). \end{cases}$$

By Poincaré's inequality and our assumptions we have that (choosing possibly a larger universal  $C_0$ )

$$\int_{B_{4s} \setminus B_s} |\nabla \tilde{u}|^2 dx \leq C_0 \eta \int_B |\nabla u|^2 dx. \tag{26}$$

Hence, by the first inequality in (25), the asymptotics of  $\tilde{h}$ , a change of variables (a dilation bringing  $B_{2s}$  into  $B$ ), by (23) (used on  $\tilde{u} - \hat{u}(s)$ ), and (26) one finds

$$\log \int_B f_u dx \leq \frac{1}{4\pi} \left( \int_{B_s} |\nabla u|^2 dx + C_0 \eta \int_B |\nabla u|^2 dx \right) + 2\hat{u}(s) + 2(1 + \alpha) \log s + \tilde{C}. \tag{27}$$

Moreover one has

$$\int_B f_u dx \leq \frac{1}{\tau} \int_{B \setminus B_{2s}} f_{\tilde{u}} dx = \int_{B \setminus B_{2s}} \frac{\tilde{h}}{|x|^{4\alpha}} |x|^{4\alpha} e^{2\tilde{u}} dx \leq \frac{C}{s^{2\alpha}} \int_{B \setminus B_{2s}} e^{2v} dx, \tag{28}$$

with  $v(x) = \tilde{u}(x) + 2\alpha w(x)$ , where

$$w(x) = \begin{cases} \log(2s) & x \in B_{2s}, \\ \log |x| & x \in B \setminus B_{2s}. \end{cases}$$

Notice that

$$\begin{aligned} \int_{B \setminus B_{2s}} |\nabla v|^2 dx &= \int_{B \setminus B_{2s}} |\nabla \tilde{u}|^2 dx + 4\alpha^2 \int_{B \setminus B_{2s}} \frac{1}{|x|^2} dx \\ &\quad + 4\alpha \int_{B \setminus B_{2s}} \langle \nabla \tilde{u}, \nabla(\log |x|) \rangle dx. \end{aligned}$$

We integrate by parts to obtain

$$\int_{B \setminus B_{2s}} |\nabla v|^2 dx \leq \int_{B \setminus B_{2s}} |\nabla \tilde{u}|^2 dx + 8\pi\alpha^2 \log \frac{1}{s} - 8\pi\alpha \int_{\partial B_{2s}} \tilde{u} \, d\sigma + \tilde{C}. \tag{29}$$

Next, by using the second inequality in (25), (28), (23) for  $v$  and the fact that  $\tilde{u} \equiv \hat{u}(s)$  on  $\partial B_{2s}$ , we get

$$\begin{aligned} \log \int_B f_u dx &\leq 2\alpha \log \frac{1}{s} + \frac{1}{4\pi} \int_{B \setminus B_{2s}} |\nabla \tilde{u}|^2 dx + 2\alpha^2 \log \frac{1}{s} \\ &\quad - 2\alpha \hat{u}(s) + C_0 \eta \int_B |\nabla u|^2 dx + \tilde{C}. \end{aligned} \tag{30}$$

By using (27) together with (30) we finally deduce

$$(1 + \alpha) \log \int_B f_u dx \leq \frac{\alpha}{4\pi} \int_{B_s} |\nabla u|^2 dx + \frac{1}{4\pi} \int_{B \setminus B_{3s}} |\nabla u|^2 dx + C_0 \eta \int_B |\nabla u|^2 dx + \tilde{C},$$

which is the desired conclusion.  $\square$

*Remark 2.8.* (a) The above proposition also works when the center of the ball is shifted by an amount of order of the radius. More precisely, if we have the same assumptions of Proposition 2.7 replacing  $B_s$  (resp.  $B_{4s}$ ) by  $B_s(p)$  (resp.  $B_{4s}(p)$ ) with  $|p| \leq \bar{C}s$ , the same result will hold provided  $B_{4s}(p) \subseteq B$ , and allowing the dependence of  $\tilde{C}$  also on  $\bar{C}$ . The proof follows the same lines as before (combined in particular with Lemma 2.3) and requires minor adaptations from [53], where this case is treated on compact surfaces.

(b) The same assertion as in the previous part of this remark holds if the condition  $B_{4s}(p) \subseteq B$  is replaced by  $d(p, \partial B) \leq \frac{s}{2}$  (see Remark 2.2). To see this, one can simply use a localization argument for (23) as in the proof in [21], see the comments before Proposition 2.5.

**2.3. Compactness of solutions.** Concerning (4), we have the following result, proved in [11] via blow-up analysis, extending previous theorems in [13, 14 and 46] for the regular case (see also [10] for the case of negative  $\alpha_i$ 's).

**Theorem 2.9.** *Let  $\Sigma$  be a compact surface, and let  $u_i$  solve (4) with  $\tilde{h}$  as in (5),  $\rho = \rho_i$ ,  $\rho_i \rightarrow \bar{\rho}$ , with  $\alpha_j > 0$  and  $p_j \in \Sigma$ . Suppose that  $\int_{\Sigma} f_{u_i} dV_g \leq \bar{C}$  for some fixed  $\bar{C} > 0$ . Then along a subsequence  $u_{i_k}$  one of the following alternatives holds:*

- (i)  $u_{i_k}$  is uniformly bounded from above on  $\Sigma$ ;
- (ii)  $\max_{\Sigma} \left( 2u_{i_k} - \log \int_{\Sigma} f_{u_{i_k}} dV_g \right) \rightarrow +\infty$  and there exists a finite blow-up set  $S = \{q_1, \dots, q_l\} \in \Sigma$  such that
  - (a) for any  $s \in \{1, \dots, l\}$  there exist  $x_k^s \rightarrow q_s$  such that  $u_{i_k}(x_k^s) \rightarrow +\infty$  and  $u_{i_k} \rightarrow -\infty$  uniformly on the compact sets of  $\Sigma \setminus S$ ,
  - (b)  $\rho_{i_k} \tilde{f}_{u_{i_k}} \rightharpoonup \sum_{s=1}^l \beta_s \delta_{q_s}$  in the sense of measures, with  $\beta_s = 4\pi$  for  $q_s \neq \{p_1, \dots, p_m\}$ , or  $\beta_s = 4\pi(1 + \alpha_j)$  if  $q_s = p_j$  for some  $j = \{1, \dots, m\}$ . In particular one has that

$$\bar{\rho} = 4\pi n + 4\pi \sum_{j \in J} (1 + \alpha_j),$$

for some  $n \in \mathbb{N} \cup 0$  and  $J \subseteq \{1, \dots, m\}$  (possibly empty) satisfying  $n + |J| > 0$ , where  $|J|$  is the cardinality of the set  $J$ .

A similar result holds on bounded domains, for Dirichlet boundary data.

From the above result we obtain immediately the following corollary, which will be useful to prove our existence results. Recall the definition of  $k_\alpha$  in (18) and of  $\Lambda_\alpha$  in (10).

**Corollary 2.10.** *Consider problem (4) in  $B$ , with Dirichlet boundary data, and suppose  $m = 1$ . Let  $\rho \in (4\pi, 4(k_\alpha + 1)\pi)$ . Then the set of solutions is uniformly bounded in  $C^2(\overline{B})$  provided*

$$\rho \neq 4\pi(1 + \alpha) \quad \text{and} \quad \rho \neq 4k\pi, \quad k = 1, \dots, k_\alpha.$$

### 3. Momenta of Probability Measures and the Set of Formal Barycenters of $S^1$

For fixed  $k \in \mathbb{N}$  we denote by  $\underline{z}_k \in \mathbb{C}^k$  the vector  $\underline{z}_k = (z_1, \dots, z_k)$ , by  $\mathcal{D}_k$  the following subset of  $\mathbb{C}^k$ :

$$\mathcal{D}_k = \{\underline{z}_k \in \mathbb{C}^k \mid |z_1| = |z_2| = \dots = |z_k| = 1\},$$

and for  $R > 0$ ,

$$B_R^{(k)} = \{\underline{z}_k \in \mathbb{C}^k \mid |z_1|^2 + |z_2|^2 + \dots + |z_k|^2 < R^2\}.$$

We also set

$$\mathbb{R}_k^{(+)} := \{\underline{t}_k \in \mathbb{R}^k : t_i > 0, \forall i = 1, \dots, k\},$$

and

$$S_k := \left\{ \underline{t}_k \in [0, 1]^k : \sum_{i=1}^k t_i = 1 \right\}, \quad \mathring{S}_k := \left\{ \underline{t}_k \in (0, 1)^k : \sum_{i=1}^k t_i = 1 \right\}.$$

For  $\underline{t}_k \in S_k$  we denote by  $\sigma_k$  an element in the space of  $k$ -barycenters of  $S^1$ , that is

$$(S^1)_k \ni \sigma_k = \sum_{i=1}^k t_i \delta_{\theta_i}, \quad \theta_i \in [0, 2\pi), \forall i = 1, \dots, k,$$

and finally define  $F_k : (S^1)_k \mapsto \mathbb{C}^k$  to be the following map,

$$F_k(\sigma_k) = \left( \int_{S^1} z \, d\sigma_k, \int_{S^1} z^2 \, d\sigma_k, \dots, \int_{S^1} z^k \, d\sigma_k \right).$$

*Remark 3.1.* In Propositions 3.1 and its proof, the convergence  $f_n \rightarrow \sigma \in (S^1)_k$  with respect to the Kantorovich-Rubinstein metric is an abuse of notations (since the K-R distance was defined for measures supported in  $B$ ). The problem is solved just considering the  $f_n$  to be defined in an open neighborhood of  $B$  and considering the convergence of the measures induced by the  $f_n$  as specified in (14).

**Proposition 3.1.** *There exist small constants  $\delta_k, \tau_k > 0$  with the following property. If  $\delta \leq \delta_k$  there exists a continuous (with respect to the Kantorovich-Rubinstein distance) map  $\Xi_k$  from the set of functions  $f \in L^1, \int_B f dx = 1$  satisfying*

$$J_{k,\delta}(f) > 1 - \tau_k, \tag{31}$$

*into  $\mathcal{S}_k$ . Moreover, if  $f_n \rightarrow \sigma \in (S^1)_k$  in the Kantorovich-Rubinstein metric, then  $\Xi_k(f_n) \rightarrow F_k(\sigma)$ .*

The proof of Proposition 3.1 can be deduced as a direct consequence of the following:

**Proposition 3.2.** *The map  $F_k$  realizes a homeomorphism (with respect to the Kantorovich-Rubinstein metric) between  $(S^1)_k$  and a topological sphere  $\mathcal{S}_k$  in  $\mathbb{C}^k$  which bounds a neighborhood  $\mathcal{U}_k$  of  $0 \in \mathbb{C}^k$ .*

We first use Proposition 3.2 to prove Proposition 3.1.

*Proof of Proposition 3.1.* It is straightforward to check that for  $\delta_k, \tau_k > 0$  small enough and for any  $\delta \leq \delta_k$  then any  $f$  satisfying (31) is close (with respect to the Kantorovich-Rubinstein metric) to a  $k$ -barycenter of  $S^1$ , and hence it is mapped in some neighborhood  $\mathcal{N}_k$  of  $\mathcal{S}_k$ .

By Theorem E.3 in [12], since  $\mathcal{S}_k$  is a topological sphere, it is a retract of some neighborhood of it in  $\mathbb{C}^k$ . Choosing  $\delta_k$  and  $\tau_k$  possibly smaller, we find that  $\mathcal{N}_k$  will be contained in this neighborhood. The map  $\Xi_k$  is finally obtained as the composition of  $F_k$  with the above retraction.  $\square$

*Proof of Proposition 3.2.* For  $t_k \in \mathbb{R}_k^{(+)} \cup \mathcal{S}_k$ , let  $\Psi_{k,t_k} : \mathbb{C}^k \mapsto \mathbb{C}^k$  be defined by

$$\Psi_{k,t_k}(z_k) = \begin{pmatrix} t_1 z_1 + t_2 z_2 + \dots + t_k z_k \\ t_1 z_1^2 + t_2 z_2^2 + \dots + t_k z_k^2 \\ \vdots \\ t_1 z_1^k + t_2 z_2^k + \dots + t_k z_k^k \end{pmatrix}.$$

Hence,

$$F_k \left( (S^1)_k \right) = \{ \Psi_{k,t_k}(\mathcal{D}_k) \}_{t_k \in \mathcal{S}_k}.$$

Let  $\deg(\Psi, B_R^{(k)}, \underline{y}_k)$  denote the topological degree (see for example [73]) of a map  $\Psi : \mathbb{C}^k \mapsto \mathbb{C}^k$  relative to  $B_R^{(k)}$  with respect to  $\underline{y}_k \in \mathbb{C}^k$ . We have the following:

**Lemma 3.3** (see Lemma 3.1 in [4]). *For fixed  $t_k \in \mathbb{R}_k^{(+)}$ ,*

$$\deg(\Psi_{k,t_k}, B_R^{(k)}, \underline{0}_k) = k! \text{ holds.} \tag{32}$$

Although Lemma 3.1 in [4] concerns the case  $R = 1$  the proof provided there works indeed for general  $R$ .

Let us consider the new variables  $w_i = t_i z_i \in B_1^{(1)}$ ,  $\forall i \in \{1, \dots, k\}$ , so that for  $\underline{t}_k \in \overset{\circ}{S}_k$  and  $\underline{z}_k \in \mathcal{D}_k$  we have in particular  $t_i = |w_i| \in (0, 1)$  and  $\underline{w}_k \in B_1^{(k)}$ . Hence  $\Psi_{k, \underline{t}_k}$  takes the form

$$\Phi_k(\underline{w}_k) := \Psi_{k, \underline{t}_k}(\underline{z}_k) = \begin{pmatrix} w_1 & + & w_2 & + \dots + & w_k \\ \vdots & & \vdots & & \vdots \\ |w_1|^{(1-j)} w_1^j + |w_2|^{(1-j)} w_2^j + \dots + |w_k|^{(1-j)} w_k^j \\ \vdots & & \vdots & & \vdots \\ |w_1|^{(1-k)} w_1^k + |w_2|^{(1-k)} w_2^k + \dots + |w_k|^{(1-k)} w_k^k \end{pmatrix},$$

and we conclude that

$$F_k \left( (S^1)_k \right) = \{ \Psi_{k, \underline{t}_k}(\mathcal{D}_k) \}_{\underline{t}_k \in S_k} \equiv \Phi_k(\partial \mathcal{R}_k),$$

where

$$\mathcal{R}_k = \{ \underline{w}_k \in \mathbb{C}^k \mid |w_1| + |w_2| + \dots + |w_k| < 1 \}.$$

We compute next a topological degree related to (32).

**Lemma 3.4.** *We have*

$$\deg(\Phi_k, \mathcal{R}_k, \underline{0}_k) = k!, \tag{33}$$

and in particular  $\Phi_k(\underline{w}_k) = 0 \iff \underline{w}_k = \underline{0}_k$ .

*Proof.* By using Lemma 3.2 in [4] we see that it is enough to prove the assertion with  $\mathcal{R}_k$  replaced by  $B_1^{(k)}$ , that is

$$\deg(\Phi_k, B_1^{(k)}, \underline{0}_k) = k!.$$

In view of (32), for  $s \in [0, 1]$  we set

$$\mathcal{H}_k(\underline{0}_k, 0) := \underline{0}_k, \\ \mathcal{H}_k(\underline{w}_k, s)$$

$$:= \frac{1}{sk + (1-s)} \begin{pmatrix} w_1 & + & w_2 & + \dots + & w_k \\ \vdots & & \vdots & & \vdots \\ w_1 \left( \frac{w_1}{\sqrt{s+(1-s)|w_1|}} \right)^{j-1} + w_2 \left( \frac{w_2}{\sqrt{s+(1-s)|w_2|}} \right)^{j-1} + \dots + w_k \left( \frac{w_k}{\sqrt{s+(1-s)|w_k|}} \right)^{j-1} \\ \vdots & & \vdots & & \vdots \\ w_1 \left( \frac{w_1}{\sqrt{s+(1-s)|w_1|}} \right)^{k-1} + w_2 \left( \frac{w_2}{\sqrt{s+(1-s)|w_2|}} \right)^{k-1} + \dots + w_k \left( \frac{w_k}{\sqrt{s+(1-s)|w_k|^2}} \right)^{k-1} \end{pmatrix},$$

the latter being defined for  $\underline{w}_k \neq \underline{0}_k$ . Clearly  $\mathcal{H}_k$  is continuous and

$$\mathcal{H}_k(\underline{w}_k, 1) = \Psi_k(\underline{w}_k) := \Psi_{k, \underline{t}_k}(\underline{w}_k) \Big|_{\underline{t}_k = \frac{1}{k}}, \quad \mathcal{H}_k(\underline{w}_k, 0) = \Phi_k(\underline{w}_k).$$

Therefore, to obtain (33), we are left to prove that  $\mathcal{H}_k(\underline{w}_k, s) \neq \underline{0}_k$  for any  $s \in [0, 1]$  and for any  $\underline{w}_k \in \partial B_1^{(k)}$ . More generally, we will show by induction that

if for some  $\underline{w}_k \in \mathbb{C}^k$  and  $s \in [0, 1]$  it holds  $H_k(\underline{w}_k, s) = \underline{0}_k$ , then  $\underline{w}_k = \underline{0}_k$ .  $(P)_k$

We notice that for  $s = 1$  this is Lemma 3.2 in [4].  $(P)_1$  is trivially true, being  $\mathcal{H}_1(\underline{w}_1, s) = w_1$ . Now, suppose that for some  $k \geq 2$   $(P)_m$  holds for any  $m \leq k - 1$ : then if by contradiction there exist  $s \in [0, 1]$  and  $\underline{w}_k^{(1)} \in \mathbb{C}^k$  such that  $\mathcal{H}_k(\underline{w}_k^{(1)}, s) = \underline{0}_k$ , we would have that  $w_i^{(1)} \neq 0$  for all  $i \in \{1, \dots, k\}$ . Indeed otherwise, if for some  $i$ ,  $w_i^{(1)} = 0$ , holds then  $\mathcal{H}_{k-1}(\underline{w}_{k-1,i}^{(1)}) = \underline{0}_k$ , where  $\underline{w}_{k-1,i}^{(1)} = (w_1^{(1)}, \dots, w_{i-1}^{(1)}, w_{i+1}^{(1)}, \dots, w_k^{(1)})$  is the  $(k - 1)$ -tuple not including  $w_i^{(1)}$ . Hence  $(P)_{k-1}$  implies  $\underline{w}_{k-1,i}^{(1)} = \underline{0}_{k-1}$  and we would conclude that  $\underline{w}_k^{(1)} = \underline{0}_k$  which is of course a contradiction.

At this point we are allowed to define:

$$\tilde{w}_i^{(1)} := \frac{w_i^{(1)}}{\sqrt{s + (1 - s) |w_i^{(1)}|}} \quad \text{and} \quad t_i := \frac{\sqrt{s + (1 - s) |w_i^{(1)}|}}{sk + (1 - s)}, \quad i = 1, \dots, k, \tag{34}$$

and conclude from the previous considerations that  $\tilde{\underline{w}}_k^{(1)} := (\tilde{w}_1^{(1)}, \dots, \tilde{w}_k^{(1)}) \in \mathbb{C}^k \setminus \{\underline{0}_k\}$ . Of course  $t_k \in \mathbb{R}_k^{(+)}$  and since  $\mathcal{H}_k(\underline{w}_k, s) = \underline{0}_k$  if and only if  $\Psi_k(\tilde{\underline{w}}_k^{(1)}) = \underline{0}_k$  we deduce from Lemma 3.2 in [4] that  $\tilde{\underline{w}}_k^{(1)} = \underline{0}_k$  which is the desired contradiction.  $\square$

Let  $\Phi_k^* : \mathbb{R}^{2k} \mapsto \mathbb{R}^{2k}$  be the map  $\Phi_k$  when expressed in real coordinates and set

$$\begin{aligned} \Upsilon_k &= \{\underline{w}_k \in \mathbb{C}^k \mid w_i = w_j, \text{ for some } i \neq j, \{i, j\} \subseteq \{1, \dots, k\}\}, \\ \Upsilon_k^{(0)} &= \{\underline{w}_k \in \mathbb{C}^k \mid w_j = 0, \text{ for some } j \in \{1, \dots, k\}\}. \end{aligned}$$

We have the following result:

**Lemma 3.5.**

$$\det(D\Phi_k^*(\text{Re}(\underline{w}_k), \text{Im}(\underline{w}_k))) \neq 0, \quad \forall \underline{w}_k \notin \Upsilon_k \cup \Upsilon_k^{(0)} \text{ holds.} \tag{35}$$

*Proof.* Setting  $w_j = r_j e^{i\theta_j}, \forall j = 1, \dots, k$ , it is straightforward to check that  $\det(D\Phi_k^*)$  takes the form

$$\det(D\Phi_k^*(\text{Re}(\underline{w}_k), \text{Im}(\underline{w}_k))) = r_1^k r_2^k \cdots r_k^k \det(A_{2k}(\underline{\theta}_k)),$$

where

$$A_{2k}(\underline{\theta}_k) = \begin{pmatrix} \cos(\theta_1) & \cos(\theta_2) & \cdots & \cos(\theta_k) & -\sin(\theta_1) & -\sin(\theta_2) & \cdots & -\sin(\theta_k) \\ \sin(\theta_1) & \sin(\theta_2) & \cdots & \sin(\theta_k) & \cos(\theta_1) & \cos(\theta_2) & \cdots & \cos(\theta_k) \\ \cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cos(2\theta_k) & -2\sin(2\theta_1) & -2\sin(2\theta_2) & \cdots & -2\sin(2\theta_k) \\ \sin(2\theta_1) & \sin(2\theta_2) & \cdots & \sin(2\theta_k) & 2\cos(2\theta_1) & 2\cos(2\theta_2) & \cdots & 2\cos(2\theta_k) \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \cos(k\theta_1) & \cos(k\theta_2) & \cdots & \cos(k\theta_k) & -k\sin(k\theta_1) & -k\sin(k\theta_2) & \cdots & -k\sin(k\theta_k) \\ \sin(k\theta_1) & \sin(k\theta_2) & \cdots & \sin(k\theta_k) & k\cos(k\theta_1) & k\cos(k\theta_2) & \cdots & k\cos(k\theta_k) \end{pmatrix}.$$



It follows immediately that  $\underline{w}_k \notin \Upsilon_k^{(0)}$  is a necessary condition for (35) to be satisfied.

Next observe that  $\det(A_{2k}(\underline{\theta}_k)) \neq 0$  whenever  $\theta_i \neq \theta_j, \forall i \neq j$ . In fact, letting  $\{\underline{v}_m\}_{m=1, \dots, k}$  be the complex row vectors

$$\underline{v}_m = \left( e^{mi\theta_1}, \dots, e^{mi\theta_k} \right),$$

we see that  $A_{2k}(\underline{\theta}_k)$  takes the form

$$A_{2k}(\underline{\theta}_k) = \begin{pmatrix} \underline{v}_1 & i \underline{v}_1 \\ \underline{v}_2 & 2i \underline{v}_2 \\ \vdots & \vdots \\ \underline{v}_k & k i \underline{v}_k \end{pmatrix}.$$

Hence, if  $\det(A_{2k}(\underline{\theta}_k)) = 0$ , then there exists  $\underline{\lambda}_k \in \mathbb{C}^k$  such that,

$$\sum_{m=1}^k \lambda_m \begin{pmatrix} \underline{v}_m \\ m i \underline{v}_m \end{pmatrix} = \underline{0}_{2k},$$

which readily implies that the complex vectors  $\{\underline{v}_m\}_{m=1, \dots, k}$  must be linearly dependent. Since the matrix whose rows are  $\{\underline{v}_m\}_{m=1, \dots, k}$  is the restriction to  $\mathcal{D}_k$  of the Vandermonde-type matrix defined by  $\Psi_{k, \underline{l}_k}(\underline{z}_k) |_{\underline{l}_k = \underline{1}_k}$ , where  $\underline{1}_k$  is the vector whose entries are all 1, then a well known argument shows that necessarily  $\theta_i = \theta_j$  for some  $i \neq j$ .

Since the region  $\mathbb{C}^k \setminus (\Upsilon_k \cup \Upsilon_k^{(0)})$  is connected and  $\det(D\Phi_k^*(\text{Re}(\underline{w}_k), \text{Im}(\underline{w}_k)))$  is continuous, it follows from Lemma 3.5 that  $\det(D\Phi_k^*)$  has in fact constant sign (unless it is zero). In this situation, and by using (33), one can conclude (see [73] Vol. I, p. 639, Prob. 14.3d and also Theorem 14A and Corollary 14.8) that for fixed  $\underline{b}_k \in \mathbb{C}^k$ , the Vandermonde-type system

$$\Phi_k(\underline{w}_k) = \underline{b}_k, \quad (V)_k(\underline{b}_k)$$

admits at least one and at most  $k!$  distinct solutions.

Hence  $\Phi_k(\mathcal{R}_k)$  is open and we define  $\mathcal{U}_k := \Phi_k(\mathcal{R}_k)$  and  $\mathcal{S}_k := \partial\mathcal{U}_k$ . The underlying idea toward the conclusion of the proof is that  $\sigma_k \in (S^1)_k \setminus (S^1)_{k-1}$  if and only if  $\underline{t}_k \in \mathring{S}_k$  and  $\theta_i \neq \theta_j \forall i \neq j$ , that in terms of  $\underline{w}_k$  variables is equivalent to  $\underline{w}_k \in \partial\mathcal{R}_k \setminus (\Upsilon_k \cup \Upsilon_k^{(0)})$ .

For  $i \in \{1, \dots, k\}$  let  $\Pi_i : \mathbb{C}^k \mapsto \mathbb{C}^k$  be the standard permutation map

$$\Pi_i(\underline{w}_k) = \Pi_i((w_1, \dots, w_i, w_{i+1}, \dots, w_k)) = (w_1, \dots, w_{i+1}, w_i, \dots, w_k),$$

which is defined with the periodic condition (for fixed  $k \in \mathbb{N}$ )  $k + 1 = 1$ . Clearly, for fixed  $\underline{b}_k \in \mathbb{C}^k$ , each one of the  $k!$  permutations of a given solution  $\underline{w}_k$  of  $(V)_k(\underline{b}_k)$  will yield a distinct solution whenever  $\underline{w}_k \notin \Upsilon_k$ . If  $\underline{b}_k \notin \Phi_k(\Upsilon_k \cup \Upsilon_k^{(0)})$ , then to each of the corresponding  $k!$  distinct solutions  $\underline{w}_k^{(m)} \in \partial\mathcal{R}_k, \forall m = 1, \dots, k!$ , there correspond the same  $k$ -barycenter  $\sigma_k = F_k^{-1}(\underline{b}_k) = \sum_{j=1}^k t_j \delta_{\theta_j}$ , where  $t_j = |w_j^{(1)}|, \theta_j = \arg(w_j^{(1)})$ .

Clearly  $F_k$  is continuous, so Lemma 3.5 and the Inverse Function Theorem together

imply that  $F_k^{-1}$  is locally of class  $C^1$  in  $\mathcal{R}_k \setminus \Phi_k \left( \Upsilon_k \cup \Upsilon_k^{(0)} \right)$ . We conclude in particular that  $F_k^{-1}$  is well defined and continuous in  $\mathcal{S}_k \setminus \Phi_k \left( \Upsilon_k \cup \Upsilon_k^{(0)} \right)$ . Next, we analyze the case  $\underline{a}_k \in \Phi_k \left( \partial\mathcal{R}_k \cap \left( \Upsilon_k \cup \Upsilon_k^{(0)} \right) \right)$ .

*Claim.* If  $\underline{a}_k \in \Phi_k \left( \partial\mathcal{R}_k \cap \left( \Upsilon_k \cup \Upsilon_k^{(0)} \right) \right)$ , then to every solution of  $\Phi_k(\underline{w}_k) = \underline{a}_k$  there corresponds a unique  $k$ -barycenter  $\sigma_k \in (S^1)_{k-1} \subset (S^1)_k$ . In particular  $F_k^{-1}$  is well defined and continuous on  $\Phi_k \left( \partial\mathcal{R}_k \cap \left( \Upsilon_k \cup \Upsilon_k^{(0)} \right) \right)$ .

*Proof.* We argue by induction and observe that for  $k = 2$ ,

$$\Phi_2 \left( \partial\mathcal{R}_2 \cap \left( \Upsilon_2 \cup \Upsilon_2^{(0)} \right) \right) = \left\{ \begin{pmatrix} e^{i\theta} \\ e^{2i\theta} \end{pmatrix} \right\}_{\theta \in [0, 2\pi)} \quad \text{holds.}$$

It is readily seen that in fact to each  $\underline{a}_2 \in \Phi_2 \left( \partial\mathcal{R}_2 \cap \left( \Upsilon_2 \cup \Upsilon_2^{(0)} \right) \right)$  there corresponds a unique 2-barycenter  $\sigma_2 \in (S^1)_1 \subset (S^1)_2$ . In particular  $F_2^{-1}$  is continuous on  $\left\{ \begin{pmatrix} e^{i\theta} \\ e^{2i\theta} \end{pmatrix} \right\}_{\theta \in [0, 2\pi)}$  with respect to the Kantorovich-Rubinstein metric since in this case we have

$$F^{-1} \left( \begin{pmatrix} e^{i\theta} \\ e^{2i\theta} \end{pmatrix} \right) = \delta_\theta.$$

Therefore, let us assume that the property in the statement of the claim holds for any  $m \in \{1, \dots, k - 1\}$  and let us prove that it holds for  $m = k$  as well. Let  $\underline{w}_k \in \partial\mathcal{R}_k \cap \left( \Upsilon_k \setminus \Upsilon_k^{(0)} \right)$  be such that  $w_1 = w_2$ . Then set

$$\tilde{w}_1 = 2w_1, \quad \tilde{w}_{\ell-1} = w_\ell, \quad \forall \ell = 3, \dots, k.$$

Hence a  $k - 1$ -dimensional vector  $\tilde{w}_{k-1}$  is well defined satisfying  $\tilde{w}_{k-1} \in \partial\mathcal{R}_{k-1}$ . However it is not too difficult to verify that any other  $\underline{w}_k$  satisfying either  $\underline{w}_k \in \partial\mathcal{R}_k \cap \left( \Upsilon_k \setminus \Upsilon_k^{(0)} \right)$  and one of the  $\binom{k}{2}$  constraints  $w_i = w_j$  for some  $i \neq j$  or  $\underline{w}_k \in \partial\mathcal{R}_k \cap \Upsilon_k \cap \Upsilon_k^{(0)}$  can be transformed in this way after an appropriate relabelling of the indices. In particular a similar argument works for  $\underline{w}_k \in \partial\mathcal{R}_k \cap \left( \Upsilon_k^{(0)} \setminus \Upsilon_k \right)$ . Hence

$$\Phi_k \left( \partial\mathcal{R}_k \cap \left( \Upsilon_k \cup \Upsilon_k^{(0)} \right) \right) = \left\{ \Phi_k(\tilde{w}_{k-1}) \right\}_{\tilde{w}_{k-1} \in \partial\mathcal{R}_{k-1}}.$$

At this point let  $\underline{a}_k \in \Phi_k \left( \partial\mathcal{R}_k \cap \left( \Upsilon_k \cup \Upsilon_k^{(0)} \right) \right)$  and define  $\tilde{a}_{k-1} \in \mathbb{C}^{k-1}$  to be the vector whose entries are the first  $k - 1$  entries of  $\underline{a}_k$  and  $\tilde{\Phi}_{k-1}(\tilde{w}_{k-1})$  the map whose rows are the first  $k - 1$  rows of  $\Phi_k$ . By our discussion above and the induction assumption, to any such  $\tilde{a}_{k-1}$  there corresponds a unique  $(k - 1)$ -barycenter  $\sigma_{k-1} = \sigma_{k-1}(\tilde{a}_{k-1}) \in (S^1)_{k-1}$  which can be obtained via any fixed solution of the system

$$\tilde{\Phi}_{k-1}(\tilde{w}_{k-1}) = \tilde{a}_{k-1}.$$

In particular the inverse map determined in this way is continuous. Finally, since  $\tilde{w}_{k-1} \in \partial\mathcal{R}_{k-1}$ , it follows from Lemma 3.4 that  $\tilde{a}_{k-1} \neq \underline{0}_{k-1}$ . Hence, both the  $k^{\text{th}}$  component of  $\underline{a}_k$  and the  $(k - 1)$ -barycenter  $\sigma_k(\underline{a}_k) \equiv \sigma_{k-1}(\tilde{a}_{k-1})$  in the pre-image of  $\underline{a}_k$  are fixed by  $\underline{a}_{k-1}$ , and therefore in particular the inverse map determined in this way is continuous.  $\square$

We can now conclude the proof of Proposition 3.2. Putting  $\underline{a}_k \in \partial\mathcal{R}_k$ , we set  $\Gamma_{s\underline{a}_k} = \{s\underline{a}_k\}_{s \in [0,1]}$  to be the ray joining together the origin  $\underline{0}_k$  and  $\underline{a}_k$ . We conclude that if  $\Gamma_{s\underline{a}_k} \cap \Upsilon_k = \emptyset$  (which is satisfied if and only if  $\underline{a}_k \notin \Upsilon_k$ ), then each  $\underline{b}_k \in \Phi_k(\Gamma_{s\underline{a}_k})$  admits exactly  $k!$  distinct pre images and in particular that on any such  $\Gamma_{s\underline{a}_k}$ ,  $\Phi_k$  is injective. It is straightforward to check by an induction argument as in the claim that in fact  $\Phi_k$  is injective on  $\Gamma_{s\underline{a}_k}$  for  $\underline{a}_k \in \Upsilon_k$  as well. Since  $\Gamma_{s\underline{a}_k}$  is a continuous curve, we see that  $\Phi_k(\mathcal{R}_k)$  is foliated by  $\{\Phi_k(\Gamma_{s\underline{a}_k})\}_{\underline{a}_k \in \partial\mathcal{R}_k}$ . Hence  $\mathcal{S}_k \equiv \{\Phi_k(\Gamma_{\underline{a}_k})\}_{\underline{a}_k \in \partial\mathcal{R}_k} \equiv \Phi_k(\partial\mathcal{R}_k) = F_k((S^1)_k)$  is homeomorphic to a  $2k - 1$  dimensional sphere embedded in  $\mathbb{C}^k$ .  $\square$

### 4. A General Improved Inequality

Throughout this section we work on the unit ball  $B$  and we assume that

$$m = 1, \quad p_1 = 0 \in B, \quad \alpha = \alpha_1 > 0.$$

The main result of this section is the following proposition, which will be useful to obtain lower bounds on  $I_{\rho, \alpha}$  (see Corollary 4.5).

**Proposition 4.1.** *Let  $\varepsilon > 0$  and  $k \in \{1, \dots, k_\alpha\}$ . Suppose that  $\delta_k$  and  $\tau_k$  are so small that Proposition 3.1 applies, and let  $\delta \leq \delta_k$ . Then there exists a constant  $C_{\varepsilon, \alpha, \delta_k, \tau_k}$ , depending only on  $\varepsilon, \alpha, \delta_k, \tau_k$ , such that*

$$\log \int_B f_u dx \leq \frac{1 + \varepsilon}{4\pi \min\{1 + k, 1 + \alpha\}} \int_B |\nabla u|^2 dx + C_{\varepsilon, \alpha, \delta_k, \tau_k},$$

for all functions  $u \in H_0^1(B)$  such that

$$J_{k, \delta}(\tilde{f}_u) \leq 1 - \tau_k.$$

The proof of the proposition is divided into several steps. We begin by choosing a large constant  $C_1$ , depending on  $\varepsilon$  and  $\alpha$ , such that

$$\frac{1}{\log C_1} = \frac{1}{32(k_\alpha + 1)^2} \frac{\varepsilon}{1 + C_0^2}, \tag{36}$$

where  $C_0$  is the constant in Lemma 2.1. First, we derive an alternative in case we are under the assumptions of Proposition 4.1. Consider the cylindrical metric as described in the Introduction, after Eq. (12). Proposition 4.2 asserts that if the conformal volume is not concentrated near  $k$  points of the cylinder obtained from the blown-up metric, then either part of it accumulates near  $k + 1$  well separated regions, or part of it *vanishes*. By this we mean that its integral over bounded sets in some region of the cylinder becomes arbitrarily small. The division of the volume into  $N$  parts in  $(jj)$  is technical and will be needed in the next subsections.

**Proposition 4.2.** *Let  $C_1$  be as in (36), let  $k \in \{1, \dots, k_\alpha\}$  and let  $f \in L^1(B)$  be such that  $\int_B f \, dx = 1$  and  $J_{k,\delta}(f) \leq 1 - \tau_k$  ( $\tau_k$  and  $\delta$  are as in Proposition 4.1). Then for any  $\sigma_0 > 0$  there exists  $\sigma \in (0, \sigma_0]$ , depending on  $\sigma_0, \alpha$  and  $\varepsilon$ , but not on  $u$ , such that the following alternative holds: either*

(j) *there exist  $k + 1$  points  $\{p_1, \dots, p_{k+1}\} \subset B \setminus \{0\}$  such that*

$$\int_{B_{(10C_1k)^{-8}|p_i|}(p_i)} f \, dx \geq \sigma, \quad \forall i = 1, \dots, k + 1, \quad \text{and}$$

$$B_{(10C_1k)^{-4}|p_i|}(p_i) \cap B_{(10C_1k)^{-4}|p_j|}(p_j) = \emptyset, \quad \forall i \neq j, \tag{37}$$

or

(jj) *there exist  $0 < r < R \leq 1$  such that*

$$\int_{A(r,R)} f \, dx \geq \frac{\tau_k}{(10k)^2}, \tag{38}$$

and for any  $N \in \mathbb{N}$ ,  $N \geq 4(k + 1)$ , there exist  $r \leq s_1 < s_2 < \dots < s_{N+1} \leq R$  such that

$$\int_{A(s_i, s_{i+1})} f \, dx = \frac{1}{N} \int_{A(r,R)} f \, dx \quad \forall i = 1, \dots, N, \tag{39}$$

and

$$\int_{A(\frac{r}{C_1}, C_1 s)} f \, dx < \sigma_0, \quad \forall s \in \left( C_1 r, \frac{R}{C_1} \right). \tag{40}$$

*Proof.* We define for convenience

$$\mathcal{A}_k = \{f \in L^1(B) \mid f > 0 \text{ a.e., } \int_B f \, dx = 1, J_{k,\delta}(f) \leq 1 - \tau_k\},$$

and let

$$\mathcal{A}_{k,0} = \left\{ f \in L^1(B) \mid (jj) \text{ holds for some } 0 < r < R \leq 1 \right\}.$$

For each  $y \neq 0$  we denote  $m_\delta(y; f)$  the integral

$$m_\delta(y; f) = \int_{B_{\delta(10C_1k)^{-6}|y|}(y)} f \, dx.$$

Consider the set

$$\Lambda_k := \left\{ \{x_1, \dots, x_{k+1}\} \subset B \setminus \{0\} \mid x_i \in B \setminus \bigcup_{\ell \neq i} B_{\frac{\delta|x_\ell|}{2}}(x_\ell), \quad i = 1, \dots, k. \right\},$$

and the number

$$\sigma_k(\delta, \sigma_0) := \inf_{f \in \mathcal{A}_k \setminus \mathcal{A}_{k,0}} \sup \left\{ \min_{i=1, \dots, k+1} m_\delta(x_i; f) \mid \{x_1, \dots, x_{k+1}\} \in \Lambda_k \right\}.$$

□

A main step in our proof is the following

*Claim.*  $\sigma_k(\delta, \sigma_0) > 0$ .

*Proof of the Claim.* Arguing by contradiction, for every  $n \in \mathbb{N}$  there exists  $f_n \in \mathcal{A}_k \setminus \mathcal{A}_{k,0}$  such that

$$\min_{i=1, \dots, k+1} m_\delta(x_i; f_n) \leq \frac{1}{n}, \quad \forall \{x_1, \dots, x_{k+1}\} \in \Lambda_k. \tag{41}$$

□

For later use we fix here a positive number  $0 < \varepsilon_0 \ll \frac{\tau_k}{(10k)^2 10^k}$ . In the rest of this proof we will freely pass to subsequences which will not be relabelled and make use of the following:

**Lemma 4.3.** *Suppose that*

$$\int_{A(r_{1,n}, r_{2,n})} f_n \, dx \geq \frac{\tau_k}{(10k)^2} > 0, \quad \forall n > \nu_0, \tag{42}$$

for some  $r_{1,n} < r_{2,n}$  and  $\nu_0 \in \mathbb{N}$ . If there exists  $0 < \delta_0 \leq \frac{1}{2}$  such that

$$\int_{B_{\delta_0|x_n|}(x_n)} f_n \, dx \rightarrow 0, \quad n \rightarrow +\infty, \quad \forall x_n \in A(r_{1,n}, r_{2,n}), \tag{43}$$

then there exists  $\nu_1 \in \mathbb{N}$  such that  $f_n \in \mathcal{A}_{k,0}$  for all  $n > \nu_1$ .

*Proof of Lemma 4.3.* We first prove that necessarily

$$\frac{r_{1,n}}{r_{2,n}} \rightarrow +\infty, \quad n \rightarrow +\infty. \tag{44}$$

We argue by contradiction and observe that then, up to the extraction of a subsequence, we could find  $C > 0$  such that

$$A(r_{1,n}, r_{2,n}) \subseteq A(r_{1,n}, Cr_{1,n}).$$

Observe that there exists  $m = m(C) \in \mathbb{N}$  depending only on  $C$  such that

$$A(r, Cr) \subseteq \bigcup_{i=1}^m B_{\delta_0|y_i|}(y_i), \quad \{y_i\}_{i=1, \dots, m} \subset A(r, Cr).$$

Therefore, by using (42), (43) we obtain

$$\frac{\tau_k}{(10k)^2} \leq \int_{A(r_{1,n}, r_{2,n})} f_n \, dx \leq \sum_{i=1}^m \int_{B_{\delta_0|x_{i,n}|}(x_{i,n})} f_n \, dx \leq m o(1), \quad n \rightarrow +\infty,$$

which is the desired contradiction.

Next observe that  $\int_{A(s,t)} f_n \, dx$  is a continuous function of  $s$  and  $t$ . Hence, for any  $N \in \mathbb{N}$  there exist  $\{s_{1,n}, s_{2,n}, \dots, s_{N+1,n}\}$  such that

$$r_{1,n} \leq s_{1,n} \leq \dots \leq s_{N+1,n} \leq r_{2,n},$$

and

$$\int_{A(s_{i,n}, s_{i+1,n})} f_n dx = \frac{1}{N} \int_{A(r_{1,n}, r_{2,n})} f_n dx \geq \frac{\tau_k}{(10k)^2}, \quad \forall i = 1, \dots, N.$$

If for some  $i \in \{1, \dots, N + 1\}$ , along a subsequence we had

$$\int_{A(\frac{s_{i,n}}{C_1}, C_1 s_{i,n})} f_n dx \geq \sigma_0, \quad \forall n \in \mathbb{N}$$

then, by applying (44) on  $A(\frac{s_{i,n}}{C_1}, C_1 s_{i,n})$ , we would obtain

$$(C_1)^2 = \frac{C_1 s_{i,n}}{\frac{s_{i,n}}{C_1}} \rightarrow +\infty, \quad n \rightarrow \infty,$$

which is the desired contradiction. Therefore there exists  $\nu_1 \in \mathbb{N}$  such that  $f_n \in \mathcal{A}_{k,0}$  for any  $n > \nu_1$ .  $\square$

*Proof of the Claim Continued.* There is no loss of generality in assuming

$$m_\delta(x_{k+1,n}, f_n) = \min_{i=1, \dots, k+1} m_\delta(x_i, f_n). \tag{45}$$

Clearly (41), (45) and the definition of  $\sigma_k(\delta, \sigma_0)$  imply that for any  $n \in \mathbb{N}$ ,

$$m_\delta(x_{k+1}; f_n) \leq \frac{1}{n}, \quad \forall x_{k+1} \neq 0 \text{ s.t. } x_{k+1} \in B \setminus \bigcup_{i=1, \dots, k} B_{\frac{\delta |x_{i,n}|}{2}}(x_{i,n}), \tag{46}$$

with  $\{x_{1,n}, \dots, x_{k,n}, x_{k+1}\} \in \Lambda_k$ . Set

$$R_{n,(-)} = \min_{i=1, \dots, k} |x_{i,n}| \left(1 - \frac{\delta}{2}\right), \quad r_{n,(+)} = \max_{i=1, \dots, k} |x_{i,n}| \left(1 + \frac{\delta}{2}\right), \quad \forall n \in \mathbb{N},$$

and pick  $0 < r_{n,(-)} < R_{n,(-)}$ ,  $r_{n,(+)} < R_{n,(+)}$  and  $\nu \in \mathbb{N}$  such that

$$\int_{B_{r_{n,(-)}}} f_n dx < \frac{\varepsilon_0}{2}, \quad \int_{B \setminus B_{R_{n,(+)}}} f_n dx < \frac{\varepsilon_0}{2}, \quad \forall n > \nu.$$

If either  $\int_{A(r_{n,(-)}, R_{n,(-)})} f_n dx \geq \frac{\tau_k}{(10k)^2}$  or  $\int_{A(r_{n,(+)}, R_{n,(+)})} f_n dx \geq \frac{\tau_k}{(10k)^2}$  for all  $n > \nu_0$  for some  $\nu_0 \in \mathbb{N}$ , since (46) ensures that (43) holds on both  $A(r_{n,(-)}, R_{n,(-)})$  and  $A(r_{n,(+)}, R_{n,(+)})$ , then Lemma 4.3 implies that  $f_n \in \mathcal{A}_{k,0}$  for all  $n > \nu_1$ , which is a contradiction.  $\square$

Therefore, passing to a further subsequence if necessary, we can assume that

$$\int_{A(r_{n,(-)}, R_{n,(-)})} f_n dx < \frac{\tau_k}{(10k)^2}, \quad \text{and} \quad \int_{A(r_{n,(+)}, R_{n,(+)})} f_n dx < \frac{\tau_k}{(10k)^2},$$

for any  $n \in \mathbb{N}$ , and in particular

$$\int_{A(R_{n,(-)}, r_{n,(+)})} f_n dx \geq 1 - 2 \frac{\tau_k}{(10k)^2} - \varepsilon_0 > 1 - \frac{\tau_k}{10k}, \quad \forall n \in \mathbb{N}.$$

Hence we conclude that

$$\begin{aligned} \int_{A(R_{n,(-)}, r_{n,(+)}) \setminus \bigcup_{i=1, \dots, k} B_{\frac{\delta|x_{i,n}|}{2}}(x_{i,n})} f_n dx &\geq \int_{A(R_{n,(-)}, r_{n,(+)}) \setminus \bigcup_{i=1, \dots, k} B_{\delta|x_{i,n}|}(x_{i,n})} f_n dx \\ &\geq 1 - \frac{\tau_k}{10k} - (1 - \tau_k) > \frac{\tau_k}{2}, \end{aligned} \tag{47}$$

for all  $n \in \mathbb{N}$ . On the other hand, we have the following:

**Lemma 4.4.** *There exists  $\tilde{C} \geq 1$  such that*

$$\frac{r_{n,(+)}}{R_{n,(-)}} \leq \tilde{C}, \quad \forall n \in \mathbb{N}.$$

*Proof of Lemma 4.4.* If the claim were false then, up to a relabelling of the indices, we could find at least one index  $i = i_n \in \{1, \dots, k - 1\}$  such that

$$|x_{\ell,n}| \leq |x_{i_n,n}| < |x_{i_n+1,n}| \leq |x_{m,n}|, \quad \forall \ell \leq i_n, \forall m \geq i_n + 1,$$

with the property that, passing to a subsequence if necessary,

$$\lim_{n \rightarrow +\infty} \frac{|x_{i_n+1,n}|}{|x_{i_n,n}|} = +\infty. \tag{48}$$

Set

$$R_{n,0} = |x_{i_n+1,n}| \left(1 - \frac{\delta}{2}\right), \quad r_{n,0} = |x_{i_n,n}| \left(1 + \frac{\delta}{2}\right), \quad \forall n \in \mathbb{N}.$$

If  $\int_{A(r_{n,0}, R_{n,0})} f_n dx \geq \frac{\tau_k}{(10k)^2}$  for all  $n > \nu_0$  for some  $\nu_0 \in \mathbb{N}$ , since (46) and (48) together ensure that (43) holds on  $A(r_{n,0}, R_{n,0})$ , then once more Lemma 4.3 implies that  $f_n \in \mathcal{A}_{k,0}$  for all  $n > \nu_1$ , which is the desired contradiction.  $\square$

*End of the Proof of the Claim.* We are going to use Lemma 4.4 together with (47) to obtain a contradiction. In fact, observe that there exists  $\ell = \ell(\delta, \tilde{C}) \in \mathbb{N}$  depending only on  $\delta$  and  $\tilde{C}$  such that

$$A(R_{n,(-)}, r_{n,(+)}) \setminus \bigcup_{i=1, \dots, k} B_{\frac{\delta|x_{i,n}|}{2}}(x_{i,n}) \subseteq \bigcup_{i=1}^{\ell} B_{\delta(10C_1k)^{-6}|y_{i,n}|}(y_{i,n}),$$

where

$$\{y_{i,n}\}_{i=1, \dots, \ell} \subset A(R_{n,(-)}, r_{n,(+)}) \setminus \bigcup_{i=1, \dots, k} B_{\frac{\delta|x_{i,n}|}{2}}(x_{i,n}).$$

Hence (46) and (47) imply

$$\frac{\tau_k}{2} \leq \int_{A(R_{n,(-)}, r_{n,(+)}) \setminus \bigcup_{i=1, \dots, k} B_{\frac{\delta|x_{i,n}|}{2}}(x_{i,n})} f_n dx \leq \sum_{i=1}^{\ell} m_{\delta}(y_{i,n}; f_n) \leq \ell o(1), \quad n \rightarrow +\infty,$$

which is the desired contradiction.  $\square$

*End of the Proof of Proposition 4.2.* We claim that  $\sigma = \min\{\frac{\sigma_k(\delta, \sigma_0)}{2}, \sigma_0\}$  with  $\delta = \tilde{\delta}_k := 8(10C_1k)^{-4}$  satisfies the required properties. In fact, let us first assume that

$$\frac{\sigma_k(\tilde{\delta}_k, \sigma_0)}{2} \leq \sigma_0, \quad \text{that is } \sigma = \frac{\sigma_k(\tilde{\delta}_k, \sigma_0)}{2}.$$

In this case, if  $f$  does not satisfy  $(jj)$ , then by definition of  $\sigma_k(\tilde{\delta}_k, \sigma_0)$  there exist  $\{x_1, \dots, x_{k+1}\} \in \Lambda_k$ , such that

$$\int_{B_{(10C_1k)^{-8}|x_i|}(x_i)} f \, dx \geq \int_{B_{\tilde{\delta}_k(10C_1k)^{-6}|x_i|}(x_i)} f \, dx \geq \frac{\sigma_k(\tilde{\delta}_k, \sigma_0)}{2} = \sigma, \quad i = 1, \dots, k+1. \tag{49}$$

Next, let us prove that

$$B_{\frac{\delta|x_\ell|}{8}}(x_\ell) \cap B_{\frac{\delta|x_m|}{8}}(x_m) = \emptyset, \quad \forall \{\ell, m\} \subset \{1, \dots, k\}, \quad \ell \neq m. \tag{50}$$

If  $|x_\ell| \leq 2|x_m|$  and  $x \in B_{\frac{\delta|x_m|}{8}}(x_m)$ , then

$$d_g(x, x_\ell) > \frac{\delta|x_m|}{2} - \frac{\delta|x_m|}{8} = \frac{3\delta|x_m|}{8} > \frac{\delta|x_m|}{4} \geq \frac{\delta|x_\ell|}{8},$$

while if  $|x_\ell| > 2|x_m|$  and  $x \in B_{\frac{\delta|x_m|}{8}}(x_m)$ , then

$$d_g(x, x_\ell) > |x_\ell| - |x_m| - \frac{\delta|x_m|}{8} > \frac{\delta|x_m|}{8},$$

that is, (50) holds. Hence, if  $\delta = \delta_k$  we see that  $(j)$  is satisfied with  $\{p_1, \dots, p_{k+1}\} = \{x_1, \dots, x_{k+1}\}$  and the desired property holds with  $\sigma \leq \sigma_0$ .

On the other hand, if

$$\frac{\sigma_k(\tilde{\delta}_k, \sigma_0)}{2} > \sigma_0, \quad \text{that is } \sigma = \sigma_0,$$

and  $(jj)$  is not satisfied then by definition of  $\sigma_k(\tilde{\delta}_k, \sigma_0)$  we can find  $\{x_1, \dots, x_{k+1}\}$  as above such that both (49) and (50) with  $\delta = \tilde{\delta}_k$  hold, so that  $(j)$  is satisfied with  $\{p_1, \dots, p_{k+1}\} = \{x_1, \dots, x_{k+1}\}$  and  $\sigma = \sigma_0$ . This fact concludes the proof.  $\square$

In the next subsections we prove Proposition 4.1 in both alternatives of Proposition 4.2 choosing  $\sigma_0$  as

$$\sigma_0 = \frac{\tau_k}{100k^2} \frac{\varepsilon}{4(k+1) \log C_1}. \tag{51}$$



4.1. *Proof of Proposition 4.1 in case (jj).* We will argue that if a certain fixed amount of conformal volume is *diluted* in a large portion of the cylinder, then we can divide it into  $N$  parts, with  $N$  large enough, and choose the one with smallest Dirichlet energy for using Lemma 2.1.

Letting  $r, R$  be as in (jj), we choose a large number  $N$ , depending on  $k_\alpha$  and  $\varepsilon$ , such that

$$N = \left\lceil \frac{8(k_\alpha + 1)}{\varepsilon} \right\rceil,$$

where the square bracket stands for the integer part.

We next choose  $r \leq s_1 < \dots < s_{N+1} \leq R$  such that (39) and (40) hold. We notice immediately that by the choice of  $N$  one has

$$\int_{A(s_i, s_{i+1})} \tilde{f}_u dx \geq \frac{\varepsilon}{16(k_\alpha + 1)} \frac{\tau_k}{(10k)^2}; \quad i = 1, \dots, N. \tag{52}$$

We also claim that for every index  $i$  the intersection  $A\left(\frac{s_i}{C_1}, C_1 s_i\right) \cap A\left(\frac{s_{i+1}}{C_1}, C_1 s_{i+1}\right)$  is empty. In fact, if this were not the case we would have by (40) and (51)

$$\begin{aligned} \frac{\varepsilon}{16(k_\alpha + 1)} \frac{\tau_k}{(10k)^2} &\leq \int_{A(s_i, s_{i+1})} \tilde{f}_u dx \leq \int_{A\left(\frac{s_i}{C_1}, C_1 s_i\right)} \tilde{f}_u dx + \int_{A\left(\frac{s_{i+1}}{C_1}, C_1 s_{i+1}\right)} \tilde{f}_u dx \\ &\leq \frac{\varepsilon}{16(k_\alpha + 1)} \frac{2\tau_k}{(10k)^2 \log C_1}, \end{aligned}$$

which is a contradiction by the choice of  $C_1$ . We can now choose an index  $i$  for which

$$\int_{A(s_i/C_1, C_1 s_{i+1})} |\nabla u|^2 dx \leq \frac{2}{N} \int_B |\nabla u|^2 dx \leq \frac{\varepsilon}{4(k_\alpha + 1)} \int_B |\nabla u|^2 dx. \tag{53}$$

We can also choose  $\tilde{s}_i \in \left[\frac{2s_i}{C_1}, \frac{s_i}{2}\right]$  and  $\tilde{s}_{i+1} \in \left[2s_{i+1}, \frac{1}{2}C_1 s_{i+1}\right]$  such that

$$\begin{aligned} \int_{A(\tilde{s}_i/2, 2\tilde{s}_i)} |\nabla u|^2 dx &\leq \frac{4}{\log C_1} \int_{A(s_i/C_1, C_1 s_{i+1})} |\nabla u|^2 dx; \\ \int_{A(\tilde{s}_{i+1}/2, 2\tilde{s}_{i+1})} |\nabla u|^2 dx &\leq \frac{4}{\log C_1} \int_{A(s_i/C_1, C_1 s_{i+1})} |\nabla u|^2 dx, \end{aligned}$$

so by (53) we have

$$\int_{A(\tilde{s}_i/2, 2\tilde{s}_i)} |\nabla u|^2 dx \leq \frac{4}{\log C_1} \frac{\varepsilon}{4(k_\alpha + 1)} \int_B |\nabla u|^2 dx; \tag{54}$$

$$\int_{A(\tilde{s}_{i+1}/2, 2\tilde{s}_{i+1})} |\nabla u|^2 dx \leq \frac{4}{\log C_1} \frac{\varepsilon}{4(k_\alpha + 1)} \int_B |\nabla u|^2 dx. \tag{55}$$

We define then a new function  $\tilde{u}$  as the harmonic lifting of  $u$  inside  $B_{\tilde{s}_i}(0)$ : recalling the definition in (19), we set

$$\tilde{u}(x) = \mathcal{H}_{B_{\tilde{s}_i}}(u).$$

By Lemma 2.1, (53) and (54) we have that

$$\int_{B_{2\tilde{s}_{i+1}}} |\nabla \tilde{u}|^2 dx \leq \frac{\varepsilon}{2(k_\alpha + 1)} \left( 1 + \frac{4C_0}{\log C_1} \right) \int_B |\nabla u|^2 dx. \tag{56}$$

We then apply Proposition 2.7 with  $s = \tilde{s}_{i+1}/2$ ,  $\eta = \frac{4}{\log C_1} \frac{\varepsilon}{4(k_\alpha + 1)}$  and  $\tau = \frac{\varepsilon}{16(k_\alpha + 1)} \frac{\tau_k}{(10k)^2}$  (see (52), (55) and the choice of  $\tilde{s}_{i+1}$ ) and use (56) to find

$$\begin{aligned} & (1 + \alpha) \log \int_B f_u dx \\ & \leq \frac{1}{4\pi} \left( \alpha \int_{B_{\tilde{s}_{i+1}/2}} |\nabla u|^2 dx + \int_{B \setminus B_{\tilde{s}_{i+1}/2}} |\nabla u|^2 dx + C_0 \eta \int_B |\nabla u|^2 dx \right) + \tilde{C} \\ & \leq \frac{1}{4\pi} \left( 1 + \frac{\alpha\varepsilon}{2(k_\alpha + 1)} \left( 1 + \frac{4C_0}{\log C_1} \right) + C_0 \frac{4\varepsilon}{4(k_\alpha + 1) \log C_1} \right) \int_B |\nabla u|^2 dx + \tilde{C} \\ & \leq \left( 1 + \frac{\alpha\varepsilon}{2(k_\alpha + 1)} + C_0 \frac{4\varepsilon(\alpha + 1)}{4(k_\alpha + 1) \log C_1} \right) \int_B |\nabla u|^2 dx + \tilde{C}, \end{aligned}$$

where  $\tilde{C}$  depends on  $\alpha$  and  $\varepsilon$ . By the choice of  $C_1$  the last formula implies

$$(1 + \alpha) \log \int_B f_u dx \leq \frac{1}{4\pi} (1 + \varepsilon) \int_B |\nabla u|^2 dx + \tilde{C},$$

and in turn

$$\min\{1 + k, 1 + \alpha\} \log \int_B f_u dx \leq \frac{1}{4\pi} (1 + \varepsilon) \int_B |\nabla u|^2 dx + \tilde{C},$$

which concludes the proof.

4.2. *Proof of Proposition 4.1 in case (j).* If  $p_1, \dots, p_{k+1}$  are as in (37), then there exist  $\theta_i$ ,  $i = 1, \dots, k + 1$  such that

$$(10C_1k)^{-6} |p_i| \leq \theta_i \leq (10C_1k)^{-5} |p_i|; \quad \int_{B_{4\theta_i}(p_i) \setminus B_{\theta_i}(p_i)} |\nabla u|^2 dx \leq \eta \int_B |\nabla u|^2 dx, \tag{57}$$

where

$$\eta = \frac{1}{\log(10kC_1)}.$$

We can also assume that either  $B_{4\theta_i}(p_i) \subseteq B$  or that  $d(p_i, \partial B) \leq \frac{1}{4}\theta_i$  (we require these conditions in view of Remark 2.8).

We next select an index  $\bar{i}$  such that

$$\bar{D} := \frac{1}{4\pi} \int_{B_{\theta_{\bar{i}}}(p_{\bar{i}})} |\nabla u|^2 dx = \min_{i \in \{1, \dots, k+1\}} \frac{1}{4\pi} \int_{B_{\theta_i}(p_i)} |\nabla u|^2 dx,$$

and then another index  $\tilde{i}$  for which

$$\tilde{D} := \frac{1}{4\pi} \int_{B_{\theta_{\tilde{i}}}(p_{\tilde{i}})} |\nabla u|^2 dx = \min_{i \neq \tilde{i}} \frac{1}{4\pi} \int_{B_{\theta_i}(p_i)} |\nabla u|^2 dx.$$

Below, we set for convenience

$$D_1 = \frac{1}{4\pi} \int_{\cup_{i=1}^{k+1} B_{\theta_i}(p_i)} |\nabla u|^2 dx; \quad D_2 = \frac{1}{4\pi} \int_{B \setminus \cup_{i=1}^{k+1} B_{\theta_i}(p_i)} |\nabla u|^2 dx,$$

and

$$D = \frac{1}{4\pi} \int_B |\nabla u|^2 dx = D_1 + D_2.$$

Notice that, by our choices of  $\bar{i}$  and  $\tilde{i}$ ,  $\bar{D}$  and  $\tilde{D}$  satisfy

$$\bar{D} \leq \frac{D_1}{k+1}; \quad \tilde{D} \leq \frac{D_1 - \bar{D}}{k}. \tag{58}$$

We then consider a modified function  $\check{u}$  defined as

$$\check{u} = \mathcal{H}_{\cup_{i \neq \tilde{i}} B_{\theta_i}(p_i)}(u).$$

Notice that, by construction and by Lemma 2.1, one has

$$\frac{1}{4\pi} \int_{B \setminus B_{\theta_{\tilde{i}}}(p_{\tilde{i}})} |\nabla \check{u}|^2 dx \leq \tilde{D} + D_2 + C_0 k \eta D, \tag{59}$$

and also, by (37),

$$\log \int_B f_u dx \leq \log \int_{B_{\theta_{\tilde{i}}}(p_{\tilde{i}})} f_{\check{u}} dx + C_\sigma; \quad \log \int_B f_u dx \leq \log \int_{B \setminus B_{4\theta_{\tilde{i}}}(p_{\tilde{i}})} f_{\check{u}} dx + C_\sigma. \tag{60}$$

Using then (27) and (30) for  $\check{u}$ , with  $B_s(p_{\tilde{i}})$ ,  $s = \theta_i$ ,  $\tau = \sigma$ , and taking Remark 2.8 into account (in which we allow  $\tilde{C}$  to depend on  $\varepsilon$ ,  $\alpha$  but not on the  $p_i$ 's), from (60) we get

$$\log \int_B f_u dx \leq \frac{1}{4\pi} \left( \int_B |\nabla \check{u}|^2 dx + 2C_0 \eta \int_B |\nabla u|^2 dx \right) + \tilde{C}, \tag{61}$$

where  $\tilde{C}$  depends on  $\eta$ ,  $\tau$  and  $\alpha$ . From (59) we obtain

$$(\alpha + 1) \log \int_B f_u dx \leq \alpha \bar{D} + \tilde{D} + D_2 + C_0(k+2)\eta D + \tilde{C}.$$

Then by the second inequality in (58) one finds

$$(\alpha + 1) \log \int_B f_u dx \leq \alpha \bar{D} + \frac{D_1 - \bar{D}}{k} + D_2 + C_0(k+2)\eta D + \tilde{C},$$

which implies

$$(\alpha + 1) \log \int_B f_u dx \leq \frac{1}{k} D_1 + \left( \alpha - \frac{1}{k} \right) \bar{D} + D_2 + C_0 k \eta D + \tilde{C}. \tag{62}$$

If  $\alpha \leq 1$  then necessarily  $k = 1$ , so the coefficient of  $\bar{D}$  in the latter formula is negative and can be discarded, yielding

$$(\alpha + 1) \log \int_B f_u dx \leq D_1 + D_2 + C_0(k+2)\eta D + \tilde{C} \leq D + C_0 k \eta D + \tilde{C} \leq (1 + \varepsilon) D + \tilde{C},$$

which gives the conclusion, by our choices of  $\eta$  and  $C_1$ .

On the other hand, if  $\alpha > 1$  we have that  $\alpha - \frac{1}{k} > 0$  and hence, since  $\bar{D} < \frac{1}{k+1} D_1$  (see the first inequality in (58)), (62) gives

$$\begin{aligned} (\alpha + 1) \log \int_B f_u dx &\leq \left( \frac{1}{k} + \frac{\alpha k - 1}{k} \frac{1}{k+1} \right) D_1 + D_2 + C_0 k \eta D + \tilde{C} \\ &\leq \frac{k + 1 + \alpha k - 1}{k(k+1)} D_1 + D_2 + C_0 k \eta D + \tilde{C} \\ &\leq \frac{\alpha + 1}{k+1} D_1 + D_2 + \varepsilon D + \tilde{C}. \end{aligned}$$

If  $\alpha \leq k$  this implies

$$(\alpha + 1) \log \int_B f_u dx \leq D_1 + D_2 + \varepsilon D \leq (1 + \varepsilon) D + \tilde{C},$$

as desired.

If instead  $k < \alpha$  we obtain

$$(k + 1) \log \int_B f_u dx \leq D_1 + \frac{k + 1}{\alpha + 1} D_2 + \varepsilon D \leq (1 + \varepsilon) D + \tilde{C},$$

which still gives the conclusion.

From the latter proposition we immediately deduce the following lower bound on  $I_\rho$ , which can be obtained choosing  $\varepsilon > 0$  small enough.

**Corollary 4.5.** *Let  $\delta_k$  and  $\tau_k$  be so small that Proposition 3.1 applies, and let  $\delta \leq \delta_k$ . Let  $k \in \{1, \dots, k_\alpha\}$ : then there exists a constant  $C_{k,\alpha}$ , depending only on  $k$  and  $\alpha$ , such that,*

$$\forall \rho < 4\pi \min\{1 + k, 1 + \alpha\} \text{ it holds } I_{\rho,\alpha}(u) \geq -C_{k,\alpha},$$

for all functions  $u$  such that  $J_{k,\delta}(\tilde{f}_u) \leq 1 - \tau_k$ .

As a consequence of the last corollary and of Proposition 3.2 we obtain an explicit condition which guarantees lower bounds on  $I_{\rho,\alpha}$ .

**Corollary 4.6.** *Let  $\delta_k$  and  $\tau_k$  be so small that Proposition 3.1 applies, and let  $\delta \leq \delta_k$ . Let  $k \in \{1, \dots, k_\alpha\}$ , and let  $F_k$  denote the map in (15). Then there exists a constant  $C_{k,\alpha}$ , depending only on  $k$  and  $\alpha$ , such that,*

$$\forall \rho < 4\pi \min\{1 + k, 1 + \alpha\} \text{ it holds } I_{\rho,\alpha}(u) \geq -C_{k,\alpha},$$

whenever  $F_k(\tilde{f}_u) = 0$ .

### 5. Proof of the Existence and Non-existence Results

In this section we provide applications of the improved inequality in Proposition 4.1 to the existence of solutions to (1). We give full details in two simple cases, namely in the unit ball with Dirichlet boundary data and one singularity, as well as on the sphere with two singularities, see Remark 5.6 for more general situations. The variational argument combines different known strategies, therefore we will be quite sketchy in some parts.

We then prove one may have non existence of solutions in case the assumptions on  $\rho$  are dropped, showing that the hypotheses of Theorems 1.1 and 1.2 are sharp.

*5.1. Proof of Theorem 1.1 (for  $m = 1$  in simply connected domains).* First of all, through a Riemann map we can reduce ourselves to the case of the unit ball  $B$  with the singularity at the origin. We let  $k$  be the unique integer for which  $\rho \in (4k\pi, 4(k + 1)\pi)$ , and we let  $F_k$  denote the map in (15), which realizes a homeomorphism between  $(S^1)_k$  and  $S_k$ , see Proposition 3.2.

Choose a non negative cut-off function  $\chi$  such that

$$\begin{cases} \chi \in C_c^\infty(B); \\ \chi(x) \equiv 1 & \text{in } B_{\frac{3}{4}}, \end{cases}$$

and for  $\sigma = \sum_{i=1}^k t_i \delta_{\theta_i} \in (S^1)_k$ ,  $\lambda > 0$ , we define the test function

$$\varphi_{\lambda,\sigma}(x) = \chi(x) \log \sum_{i=1}^k t_i \left( \frac{\lambda}{1 + \lambda^2 |y - \frac{1}{2}x_i|^2} \right)^2, \quad x_i = (\cos \theta_i, \sin \theta_i). \tag{63}$$

Reasoning as in [34] (see also [52] for a simpler proof of this estimate) one can obtain the following result with minor modifications of the proof.

**Lemma 5.1.** *Let  $\varphi_{\lambda,\sigma}$  be defined as in (63). Then as  $\lambda \rightarrow +\infty$  one has*

$$d_{K-R}(\tilde{f}_{\varphi_{\lambda,\sigma}}, \tilde{\sigma}) \rightarrow 0, \quad \tilde{\sigma} = \sum_{i=1}^k t_i \delta_{\frac{1}{2}x_i}, \tag{64}$$

and

$$I_{\rho,\underline{\alpha}}(\varphi_{\lambda,\sigma}) \rightarrow -\infty$$

uniformly for  $\sigma \in (S^1)_k$ .

We next define the variational scheme which will allow us to find existence of solutions. Recalling that  $\mathcal{U}_k$  denotes the interior of  $S_k$  in  $\mathbb{C}^k$ , consider the family of continuous maps

$$\mathcal{K}_{\lambda,\rho} = \left\{ \mathfrak{h} : \mathcal{U}_k \rightarrow H_0^1(B) : \mathfrak{h}(y) = \varphi_{\lambda,F_k^{-1}(y)} \text{ for every } y \in S_k = \partial\mathcal{U}_k \right\}.$$

We define also the min-max value

$$\overline{\mathcal{K}}_{\lambda,\rho} = \inf_{\mathfrak{h} \in \mathcal{K}_{\lambda,\rho}} \sup_{z \in \mathcal{U}_k} I_{\rho,\underline{\alpha}}(\mathfrak{h}(z)).$$

We have then the following result, which implies the conclusion of Theorem 1.1.

**Proposition 5.2.** *Under the assumptions of Theorem 1.1, if  $\lambda$  is sufficiently large then*

$$\bar{\mathcal{K}}_{\lambda,\rho} > \sup_{y \in \mathcal{S}_k} I_{\rho,\underline{\alpha}}(\varphi_{\lambda, F_k^{-1}(y)}).$$

Moreover  $\bar{\mathcal{K}}_{\lambda,\rho}$  is a critical value of  $I_{\rho,\underline{\alpha}}$ .

*Proof.* If  $C := C_{k,\alpha}$  is as in Corollary 4.6, we let  $L = 4C$ , and choose  $\lambda$  to be so large that

$$\sup_{y \in \mathcal{S}_k} I_{\rho,\underline{\alpha}}(\varphi_{\lambda, F_k^{-1}(y)}) < -L,$$

which is possible in view of Lemma 5.1.

We are going to show that  $\bar{\mathcal{K}}_{\lambda,\rho} > -\frac{L}{2}$ . Indeed, assume by contradiction that there exists a continuous  $\mathfrak{h}_0$  such that

$$\mathfrak{h}_0 \in \bar{\mathcal{K}}_{\lambda,\rho} \quad \text{and} \quad \sup_{z \in \mathcal{U}_k} I_{\rho,\underline{\alpha}}(\mathfrak{h}_0(z)) \leq -\frac{1}{2}L. \tag{65}$$

Then, by our choice of  $L$ , Corollary 4.5 and Proposition 3.1 would apply, yielding a continuous map  $F_{\lambda,\rho} : \mathcal{U}_k \rightarrow \mathcal{S}_k$  defined as the composition

$$F_{\lambda,\rho} = \Xi_k \circ \mathfrak{h}_0.$$

Notice that, since  $\mathfrak{h}_0 \in \bar{\mathcal{K}}_{\lambda,\rho}$ ,  $\mathfrak{h}_0(\cdot)$  coincides with  $\varphi_{\lambda, F_k^{-1}(\cdot)}$  on  $\mathcal{S}_k = \partial\mathcal{U}_k$ , so by (64) we deduce that

$$F_{\lambda,\rho}|_{\mathcal{S}_k} \text{ is homotopic to } Id|_{\mathcal{S}_k} : \tag{66}$$

the homotopy is obtained by letting the parameter  $\lambda$  tend to  $+\infty$ . Since  $\mathcal{S}_k$  is homeomorphic to  $S^{2k-1}$ , it is non contractible, and we obtain a contradiction to (66). This proves  $\bar{\mathcal{K}}_{\lambda,\rho} > \sup_{y \in \mathcal{S}_k} I_{\rho,\underline{\alpha}}(\varphi_{\lambda, F_k^{-1}(y)})$ .

To check that  $\bar{\mathcal{K}}_{\lambda,\rho}$  is a critical level is rather standard, as one can use a monotonicity method from [49,63]. Consider a sequence  $\rho_n \rightarrow \rho$  and the corresponding functionals  $I_{\rho_n,\underline{\alpha}}$ . All the above estimates, including also those from the previous sections, can be worked out for  $I_{\rho_n,\underline{\alpha}}$  as well with minor changes, if  $n$  is large enough.

We then define the min-max value  $\tilde{\mathcal{K}}_{\lambda,\rho} := \frac{\bar{\mathcal{K}}_{\lambda,\rho}}{\rho}$ , which corresponds to the functional  $\frac{I_{\rho,\underline{\alpha}}}{\rho}$ . It is immediate to see that

$$\rho \mapsto \tilde{\mathcal{K}}_{\lambda,\rho} \quad \text{is monotone,}$$

and that, reasoning as in [32], there exists a subsequence of  $(\rho_n)_n$  such that  $I_{\rho_n,\underline{\alpha}}$  has a critical point  $u_n$  at level  $\bar{\mathcal{K}}_{\lambda,\rho_n}$ . Then, applying Corollary 2.10 and passing to a further subsequence, we obtain that  $u_n$  converges to a critical point  $u$  of  $I_{\rho,\underline{\alpha}}$  at level  $\bar{\mathcal{K}}_{\lambda,\rho}$ .  $\square$

*5.2. Proof of Theorem 1.2 (for  $m = 2$ ).* The argument is very similar in spirit to the previous case. We list the main changes which are necessary to deal with this situation, especially for what concerns the improved Moser-Trudinger inequality.

First of all, using again a Möbius map on  $S^2$ , we can assume that the two singularities  $p_1$  and  $p_2$  are antipodal, and coincide respectively with the south and the north pole of  $S^2$ , viewed as the standard sphere embedded in  $\mathbb{R}^3$ .

Given a small  $\delta > 0$  we can define the following quantity, analogous to the one in (13)

$$\tilde{J}_{k,\delta}(\tilde{f}_u) = \sup_{x_1, \dots, x_k \neq \{p_1, p_2\}} \int_{\cup_{i=1}^k B_{\delta \min\{d(x_i, p_1), d(x_i, p_2)\}}(x_i)} \tilde{f}_u dV_g, \tag{67}$$

as well as the measure on the unit circle (viewed as the  $(x, y)$  plane in  $\mathbb{R}^3$  intersected with  $S^2$ )

$$\tilde{\mu}_f(A) = \int_{\tilde{\pi}^{-1}(A)} \tilde{f}_u dV_g; \quad A \subseteq S^1,$$

where  $\tilde{\pi} : S^2 \setminus \{p_1, p_2\} \rightarrow S^1$  stands for the projection onto the equator along the meridians.

Reasoning as in Sect. 3, if  $\tilde{J}_{k,\delta}(\tilde{f}_u) > 1 - \tau_k$ ,  $\delta \leq \delta_k$ , we can project continuously  $u$  onto the  $k$ -barycenters of  $S^1$ ,  $(S^1)_k$ . For the case  $\tilde{J}_{k,\delta}(\tilde{f}_u) \leq 1 - \tau_k$  we have a counterpart of Proposition 4.1.

**Proposition 5.3.** *Let  $\varepsilon > 0$ , and let  $k \in \{1, \dots, k_\alpha\}$ . Let  $\delta_k$  and  $\tau_k$  be so small that Proposition 3.1 applies, and let  $\delta \leq \delta_k$ . Then there exists a constant  $C_{\varepsilon, \alpha_1, \alpha_2}$ , depending only on  $\varepsilon, \alpha_1$  and  $\alpha_2$ , such that*

$$\log \int_{S^2} f_u dV_g \leq \frac{1 + \varepsilon}{4\pi \min\{1 + k, 1 + \alpha_1, 1 + \alpha_2\}} \int_{S^2} |\nabla u|^2 dV_g + C_{\varepsilon, \alpha_1, \alpha_2} + 2 \int_{S^2} u dV_g$$

for all functions  $u$  verifying

$$\tilde{J}_{k,\delta}(\tilde{f}_u) \leq 1 - \tau_k.$$

To check this statement, one can reason as in the proof of Proposition 4.1, with two main differences. The first is that the average of  $u$  on  $S^2$  should be added to the right-hand side of the inequality, since there are no boundary data in this case (compare (22) and (23)): it will not be a loss of generality to assume that  $\int_{S^2} u dV_g = 0$ . The second is that in case (j) (resp. in case (jj)) the points  $x_i$  (resp. the region of vanishing for the measure  $\tilde{f}_u$ ) can lie near either  $p_1$  or  $p_2$ .

Suppose that (jj) holds, and let  $p_i$  be a point near which vanishing occurs. Then the previous arguments yield the inequality

$$\min\{1 + k, 1 + \alpha_i\} \log \int_{S^2} f_u dV_g \leq \frac{1 + \varepsilon}{4\pi} \int_{S^2} |\nabla u|^2 dV_g + C_{\varepsilon, \alpha_1, \alpha_2},$$

which implies

$$\min\{1 + k, 1 + \alpha_1, 1 + \alpha_2\} \log \int_{S^2} f_u dV_g \leq \frac{1 + \varepsilon}{4\pi} \int_{S^2} |\nabla u|^2 dV_g + C_{\varepsilon, \alpha_1, \alpha_2},$$

namely the desired conclusion.

If (j) holds instead, there will be  $k_1$  points among the  $x_i$ 's approaching  $p_1$ , and  $k_2$  points which either lie in a fixed compact set of  $S^2 \setminus \{p_1, p_2\}$  or approaching  $p_2$ , with

$k_1 + k_2 = 1 + k$ . Applying Proposition 4.1 (twice, with  $B$  replaced by two spherical caps whose union is  $S^2$  and whose boundaries are well separated from the points  $x_i$ ) and Lemma 2.3 one finds that

$$(\min\{k_1, 1 + \alpha_1\} + \min\{k_2, 1 + \alpha_2\}) \log \int_{S^2} f_u dV_g \leq \frac{1 + \varepsilon}{4\pi} \int_{S^2} |\nabla u|^2 dV_g + C_{\varepsilon, \alpha_1, \alpha_2}.$$

Then it is enough to use the elementary inequality

$$\min\{1 + k, 1 + \alpha_1, 1 + \alpha_2\} \leq \min\{k_1, 1 + \alpha_1\} + \min\{k_2, 1 + \alpha_2\},$$

to obtain again the conclusion.

For  $\theta_1, \dots, \theta_k \in S^1$  and for  $\hat{\sigma} = \sum_{i=1}^k t_i \delta_{\theta_i}$ , the following test function replaces  $\varphi_{\lambda, \sigma}$  in (63):

$$\hat{\varphi}_{\lambda, \hat{\sigma}}(x) = \log \sum_{i=1}^k t_i \left( \frac{\lambda}{1 + \lambda^2 d(y, x_i)^2} \right)^2, \quad x_i = (\cos \theta_i, \sin \theta_i, 0) \in S^2. \quad (68)$$

One can then prove the counterpart of Lemma 5.1.

**Lemma 5.4.** *Let  $\hat{\varphi}_{\lambda, \hat{\sigma}}$  be defined as in (68). Then as  $\lambda \rightarrow +\infty$  one has*

$$d_{K-R}(\tilde{f}_{\hat{\varphi}_{\lambda, \hat{\sigma}}}, \hat{\sigma}) \rightarrow 0, \quad I_{\rho, \underline{\alpha}}(\hat{\varphi}_{\lambda, \hat{\sigma}}) \rightarrow -\infty \quad (69)$$

uniformly for  $\hat{\sigma} \in (S^1)_k$ .

In the above lemma, with an abuse of notation, we are identifying  $S^1$  as the equator of  $S^2$ . Considering now the class of continuous maps,

$$\hat{\mathcal{K}}_{\lambda, \rho} = \left\{ \mathfrak{h} : \mathcal{U}_k \rightarrow H_0^1(B) : \mathfrak{h}(y) = \hat{\varphi}_{\lambda, F_k^{-1}(y)} \text{ for every } y \in \mathcal{S}_k = \partial \mathcal{U}_k \right\},$$

and the min-max value

$$\hat{\mathcal{K}}_{\lambda, \rho} = \inf_{\mathfrak{h} \in \hat{\mathcal{K}}_{\lambda, \rho}} \sup_{z \in \mathcal{U}_k} I_{\rho, \underline{\alpha}}(\mathfrak{h}(z)),$$

one has the counterpart of Proposition 5.2.

**Proposition 5.5.** *Under the assumptions of Theorem 1.1, if  $\lambda$  is sufficiently large then*

$$\hat{\mathcal{K}}_{\lambda, \rho} > \sup_{y \in \mathcal{S}_k} I_{\rho, \underline{\alpha}}(\hat{\varphi}_{\lambda, F_k^{-1}(y)}).$$

Moreover  $\hat{\mathcal{K}}_{\lambda, \rho}$  is a critical value of  $I_{\rho, \underline{\alpha}}$ .

*Remark 5.6.* As anticipated in Remark 1.3, the above min-max method can also be applied to the case of more singularities, multiply connected domains or to surfaces with positive genus, combining the present approach to the one in [5].

Regarding Theorem 1.1 for  $m \geq 2$  or domain  $\Omega$  of  $\mathbb{R}^2$  which is not simply connected we argue as follows. Choosing an index  $\bar{i}$  for which  $\alpha_{\bar{i}} = \min_{i=1, \dots, m} \alpha_i$ , one can find a simple curve  $\gamma$  in  $\Omega \setminus \cup_i p_i$  non contractible in  $\Omega \setminus p_{\bar{i}}$  such that there is a continuous map  $\Gamma : \Omega \setminus p_{\bar{i}} \rightarrow \gamma$  satisfying  $\Gamma|_{\gamma} = Id|_{\gamma}$ .



On one hand, it is possible to associate to each  $\tilde{f}_u, u \in H_0^1(\Omega)$ , a unit measure on  $S^1$  via the push-forward of  $\Gamma$ , and hence introduce a counterpart of the map  $F_k$ . On the other hand, one can use for the min-max scheme a test function as in (63), but with the points  $x_i$  distributed on  $\gamma$ .

The combination of these two facts allows to repeat the above procedure: in the case of the sphere with  $m > 2$  or for compact surfaces with positive genus one can argue similarly.

5.3. *A non existence result on the unit ball.* Here we show that our theorem on simply connected domains is sharp. Actually we provide a sketchy proof of the following well known fact:

**Proposition 5.7.** *If the following problem admits a solution  $u \in H_0^1(B)$ ,*

$$\begin{cases} -\Delta u = \rho \frac{|x|^{2\alpha} e^{2u}}{\int_B |x|^{2\alpha} e^{2u} dx} & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{70}$$

then necessarily  $\rho < 4\pi(1 + \alpha)$ .

*Proof.* Set  $V(x) = \rho |x|^{2\alpha} \left(\int_B |x|^{2\alpha} e^{2u} dx\right)^{-1}$ . By using the fact that  $u = 0$  on  $\partial B$ , then the Pohožaev identity for the equation in (70) reads

$$-\frac{1}{2} \int_{\partial B} (x, \nu)(u_\nu)^2 d\sigma = \frac{1}{2} \int_{\partial B} (x, \nu) V e^{2u} d\sigma - \frac{1}{2} \int_B \left[ 2V e^{2u} + \langle x, \nabla \log V \rangle V e^{2u} \right] dx,$$

where  $\nu$  is the unit outer normal to  $\partial B$  and  $u_\nu = (v, \nabla u)$ . Since  $(x, \nu) = 1$  on  $\partial B$ , the Cauchy-Schwarz inequality then yields

$$\begin{aligned} -\frac{1}{4\pi} \left( \int_{\partial B} u_\nu d\sigma \right)^2 &= -\frac{1}{2} \left( \int_{\partial B} \frac{d\sigma}{(x, \nu)} \right)^{-1} \left( \int_{\partial B} u_\nu d\sigma \right)^2 \geq -\frac{1}{2} \int_{\partial B} (x, \nu)(u_\nu)^2 d\sigma \\ &= \frac{1}{2} \int_{\partial B} (x, \nu) V e^{2u} d\sigma - \frac{1}{2} \int_B \left[ 2V e^{2u} + \langle x, \nabla \log V \rangle V e^{2u} \right] dx. \end{aligned}$$

However (70) readily implies  $(\int_{\partial B} u_\nu d\sigma)^2 = \rho^2$ , while we clearly have  $\int_B V e^{2u} dx = \rho$ . At this point an explicit calculation yields

$$\frac{1}{4\pi} \rho^2 \leq -\frac{1}{2} \int_{\partial B} (x, \nu) V e^{2u} d\sigma + \frac{1}{2} 2\rho + \frac{1}{2} 2\alpha\rho < (1 + \alpha)\rho,$$

and the conclusion follows. Observe that the sharpness of the strict inequality is due to the negative sign of the first term on the right in the first inequality which in fact vanishes along the well known radial and explicit solutions blowing up at the origin.  $\square$

5.4. *A non existence result on  $S^2$  with two antipodal singularities.* We generalize an argument in [68] to obtain a non existence result for (1) in the case of the sphere with two antipodal singularities.

**Proposition 5.8.** *Let  $(\Sigma, g) = (S^2, g_0)$ , where  $g_0$  is the standard round metric, let  $a \equiv \frac{1}{|S^2|}, h \equiv 1, m = 2$ , let  $0 < \alpha_1 < \alpha_2 < +\infty$  be the weights of two antipodal singularities  $\{p_1, p_2\} \subset S^2$  which we assume to coincide with the south and north pole respectively  $p_1 = S, p_2 = N$ .*

*Then a necessary condition for the solvability of (1) is that*

$$\text{either } 0 < \rho < 4\pi(1 + \alpha_1), \quad \text{or } 4\pi(1 + \alpha_2) < \rho < +\infty. \tag{71}$$

*Proof.* We will work in isothermal coordinates induced by the stereographic projection  $\Pi : S^2 \mapsto \mathbb{R}^2$  satisfying  $\Pi(S) = 0$ . The local expression of the unique solution of (2) with  $p = p_1 = S$  takes the form

$$G_S(\Pi^{-1}(z)) = \frac{1}{4\pi} \log \left( \frac{1 + |z|^2}{2|z|^2} \right) - \frac{1}{2\pi} \log \left( \frac{e}{2} \right).$$

In particular the local expression of the Laplace-Beltrami operator for the standard metric on  $S^2$  is

$$\Delta_g = e^{-v_0} \Delta,$$

where  $\Delta$  is the standard Laplace operator in cartesian coordinates in  $\mathbb{R}^2$  and  $v_0$  satisfies

$$v_0(z) = 2 \log \left( \frac{2}{1 + |z|^2} \right), \quad -\Delta v_0 = 2e^{v_0} \text{ in } \mathbb{R}^2.$$

Using these facts, and setting  $\rho = 2\pi\beta$ , it is straightforward to check that  $u$  solves (1) if and only if

$$\begin{aligned} v(z) = & 2u(\Pi^{-1}(z)) + G_S(\Pi^{-1}(z)) + \frac{\beta - \alpha_2}{2} v_0(z) + 2\alpha_1 \log \left( \frac{e}{2} \right) \\ & + (2 + \alpha_2 + \alpha_1 - \beta) \log 2 + \log(2\rho) - \log \int_{S^2} e^{2u} dV_{g_0}, \end{aligned}$$

satisfies

$$\begin{cases} -\Delta v = K(z)e^v & \text{in } \mathbb{R}^2; \\ \int_{\mathbb{R}^2} K(z)e^v dx = 4\pi\beta, \end{cases} \quad \text{where } K(z) = \frac{|z|^{2\alpha_1}}{(1 + |z|^2)^{2+\alpha_1+\alpha_2-\beta}}. \tag{72}$$

Therefore we see that the results in [27] can be applied to  $v$  to conclude

$$\int_{\mathbb{R}^2} \langle z, \nabla K(z) \rangle e^v dx = 4\pi\beta(\beta - 2),$$

so that, by using the integral constraint in (72), an explicit evaluation shows

$$2(2 + \alpha_1 + \alpha_2 - \beta) \int_{\mathbb{R}^2} \frac{|z|^2}{1 + |z|^2} K(z)e^v dx = 4\pi\beta(2(1 + \alpha_1) - \beta). \tag{73}$$

Next, by writing  $\frac{|z|^2}{1+|z|^2} = 1 - \frac{1}{1+|z|^2}$ , and by using (73), we obtain the independent constraint

$$2(2 + \alpha_1 + \alpha_2 - \beta) \int_{\mathbb{R}^2} \frac{1}{1 + |z|^2} K(z) e^v dx = 4\pi\beta(2(1 + \alpha_2) - \beta). \quad (74)$$

By using (73) and (74) together and by discussing the cases  $2 + \alpha_1 + \alpha_2 - \beta \stackrel{\leq}{=} 0$  it is readily seen that if  $\alpha_1 < \alpha_2$  then a necessary condition for the solvability of (72) (and then of (1)) is (71).

Of course, by setting  $\alpha_1 = 0$  we recover the non existence result obtained in [68] for the case where only one singularity is considered.  $\square$

*Remark 5.9.* (a) Concerning the case  $\alpha_1 = 0$  it has been already observed in [68] that in particular one obtains in this way another proof of the non-existence of conformal metrics with constant Gaussian curvature on  $S^2$  with one conical singularity, see [69] and the more recent paper [2]. Indeed we obtain another proof of the non-existence of conformal metrics with constant Gaussian curvature on  $S^2$  with two conical singularities of different orders  $\alpha_1 \neq \alpha_2$  which corresponds to the case  $2 + \alpha_1 + \alpha_2 - \beta = 0$ . In fact in this situation we see that (73) and (74) together imply  $\alpha_1 = \alpha_2$ , and in this case solutions are classified explicitly, see [69] and [59]. The non-existence results for  $2 + \alpha_1 + \alpha_2 - \beta = 0$  are associated with a well known problem, see [69], corresponding to the best pinching constants for these singular surfaces. The case with negative singularities has been recently solved in [2] while, at least to our knowledge, the positive case is still open.

(b) We expect that existence should hold in some cases for which  $\rho > 4\pi \min_i \{1 + \alpha_i\}$ . For example we speculate that our method, with some extra work, could be adapted to the following situation:  $m = 2$ ,  $4k\pi \leq \alpha_1, \alpha_2 < 4(k + 1)\pi$  for some  $k \in \mathbb{N}$  and  $\rho \in (4\pi \max\{1 + \alpha_1, 1 + \alpha_2\}, 4(k + 1)\pi)$ .

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