

# A Renormalized Rough Path over Fractional Brownian Motion

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**Abstract:** We construct in this article a rough path over fractional Brownian motion with arbitrary Hurst index by (i) using the Fourier normal ordering algorithm introduced in (Unterberger, Commun Math Phy 298(1):1–36, 2010) to reduce the problem to that of regularizing tree iterated integrals and (ii) applying the Bogolioubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization algorithm to Feynman diagrams representing tree iterated integrals.

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## 0. Introduction

Consider a  $d$ -dimensional continuous path  $t \mapsto \Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d)), t \in \mathbb{R}$ . Assume  $\Gamma$  is *not* differentiable, but only  $\alpha$ -H lder for some  $\alpha \in (0, 1)$ . Rough path

theory answers positively the following related two questions, in particular: (i) can one integrate a (sufficiently regular) one-form along  $\Gamma$ ? (ii) can one solve differential equations driven by  $\Gamma$ ? The solution of these relies on the definition of a so-called *rough path over  $\Gamma$* , denoted by  $\mathbf{\Gamma} = (\Gamma^{ts}(i_1, \dots, i_n))_{1 \leq n \leq \lfloor 1/\alpha \rfloor, 1 \leq i_1, \dots, i_n \leq d}$ , which is a *substitute for iterated integrals*  $\int_s^t d\Gamma_{t_1}(i_1) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n)$  for  $n = 1, \dots, \lfloor 1/\alpha \rfloor$ , defined both by *algebraic* and *regularity* properties (see Definition 1.1). Rough path solutions are remarkably well-behaved with respect to controllability and numerical schemes, and the construction is robust enough to extend to a variety of settings.

Given this, it is important to know how to construct a rough path. A first answer to this problem has been given by T. Lyons and N. Victoir [23]. However, their construction is non-canonical (actually, it uses the axiom of choice) and does not provide a closed formula, which bars the way to applications to non-pathwise results for stochastic processes for instance. Among these, fractional Brownian motion (fBm for short) is probably the one which has drawn most attention, probably because it is the simplest non-trivial example. This Gaussian, self-similar process, depending on a regularity index  $\alpha \in (0, 1)$  called Hurst index, has  $\alpha^-$ -Hölder (i.e.  $(\alpha - \varepsilon)$ -Hölder for every  $\varepsilon > 0$ ) paths. Consider a  $d$ -dimensional fBm,  $B_t = (B_t(1), \dots, B_t(d))$ , with  $d \geq 2$  (the one-dimensional case is much simpler and has been solved earlier [15]). Classical results imply that the natural iterated integrals of the piecewise linear [10] or analytic [30] approximation of fBm converge to a rough path over  $B$  if and only if  $\alpha > 1/4$ . The search for other Gaussian approximations with converging iterated integrals has failed up to now, and recent investigations have turned (i) either to *non-Gaussian approximations*, using the tools of *constructive quantum field theory* [24] (including renormalization); or (ii) to “*algebraic*” *rough paths*, i.e. substitute for iterated integrals in the above sense, satisfying the required algebraic and regularity properties, but not given by any explicit approximation.<sup>1</sup> It is the second approach that we pursue in this article, but always keeping an eye on the first one, as we shall see.

This approach relies on a combinatorial algorithm called *Fourier normal ordering*. Initially, it was conceived as a splitting into sectors of the domain of integration in Fourier coordinates which produces naturally Hölder bounds [32]. For iterated integrals of lowest orders at least, it appeared clearly that recombining regularized iterated integrals defined within each sector gave a quantity satisfying the algebraic properties required for a rough path. With the time, it became clear that Fourier normal ordering made it possible to separate the rough path construction problem into two questions of a totally different nature:

- the first one consists in *regularizing tree iterated integrals* or more precisely *tree skeleton integrals* – restricted to the above Fourier sectors – which are natural combinatorial extensions of iterated integrals indexed by decorated trees;
- the second one consists in showing that one may reconstruct in a canonical way a rough path out of these data.

It turns out that rough path construction is a *very undetermined problem*, since in some sense *any* regularization scheme (including the brutal-force regularization by zero, except for first-order integrals) gives in the end a *formal rough path*, i.e. a set of quantities satisfying the algebraic requirements. It seems also rather clear without pretending to make this a formal statement – that regularized tree skeleton integrals with the correct Hölder regularity should yield by recombination a rough path with the correct Hölder regularity.

<sup>1</sup> Such approximations – using pieces of sub-riemannian geodesics – have been shown to exist in general, but are not very explicit [13].

Taking for granted the combinatorial part of Fourier normal ordering – which we briefly recall in Sect. 1 for completeness – one is naturally led to decide *which* regularization scheme is most natural. We believe that the only possible answer to this question is to provide a natural approximation scheme leading to the corresponding rough path, which leads us back to the first approach – still under way – using quantum field theory methods [24]. Its perturbative formulation is based on the Bogolioubov-Parasiuk-Hepp-Zimmermann (BPHZ for short) renormalization scheme for Feynman diagrams [18]. To say things shortly, this is a recursive method to discard nested divergences, depending on the choice of a regularization scheme for diagrams without subdivergences. Usually, the renormalization is implemented by a change of the parameters of the measure. Here, however, the theory is a priori free, i.e. Gaussian, and such an implementation is impossible without changing the definition of the underlying process, see again [24] for a way out of this. Hence any Gaussian renormalization is in some sense arbitrary. Nevertheless it seems natural to mimic the renormalization schemes of quantum field theory in the following way. The *variance of iterated integrals* may be represented as *Feynman diagrams*; iterated integrals themselves are represented by *Feynman “half-diagrams”* and evaluated by integrating some *deterministic kernel* against a multi-dimensional Brownian motion. Renormalizing directly Feynman diagrams, as mentioned above, leads us to the non-Gaussian constructive field theory approach. Instead, we choose here to renormalize the *kernel*, still by the same BPHZ algorithm, which is a non-conventional approach. This yields directly a renormalized *random variable* in the same chaos as the original, unrenormalized quantity, which is proved to enjoy the required Hölder regularity.

Our main result may be stated as follows.

**Theorem 0.1.** *Let  $\alpha \in (0, 1)$  be such that  $1/\alpha \notin \mathbb{N}$ . Let  $\mathbf{B}^{ts}(i_1, \dots, i_n) := J_B^{ts}(i_1, \dots, i_n), n = 1, \dots, \lfloor 1/\alpha \rfloor$  be the random variable in the  $n^{\text{th}}$  chaos of fBm, defined in Proposition 1.11 and Definition 3.2. Then:*

1.  $\|\mathbf{B}^{ts}(i_1, \dots, i_n)\|_{2, n\alpha} := \sup_{s, t \in [0, T]} \frac{|\mathbf{B}^{ts}(i_1, \dots, i_n)|}{|t - s|^{n\alpha}}$  is an  $L^2$  random variable.
2.  $\mathbf{B} := (\mathbf{B}^{ts}(i_1, \dots, i_n))_{1 \leq n \leq \lfloor 1/\alpha \rfloor, 1 \leq i_1, \dots, i_n \leq d}$  satisfies the Chen and shuffle properties (see Definition 1.1).

Hence  $\mathbf{B}$  is an  $\alpha^-$ -Hölder rough path over  $B$ .

*Remarks.* 1. If  $1/\alpha \in \mathbb{N}$ , and  $\kappa < \alpha$  is chosen as close to  $\alpha$  as desired, then Theorem 0.1 applies to  $B$  seen as a  $\kappa$ -Hölder path, and yields a  $\kappa^-$ -Hölder rough path over  $B$ .

2. Property (1) in Theorem 0.1 is a consequence of the estimates

$$\mathbb{E}|\mathbf{B}^{ts}(i_1, \dots, i_n)|^2 \leq C|t - s|^{2n\alpha}, \quad n \leq \lfloor 1/\alpha \rfloor \tag{0.1}$$

proved in Sect. 5, as follows from the Garsia-Rodemich-Rumsey lemma [14] and from the equivalence of  $L^p$ -norms for variables in a fixed Gaussian chaos, see [31], Sect. 1, for details.

Here is a plan of the article. We start in Sect. 1 by recalling the fundamentals of the Fourier normal ordering algorithm, referring to [12, 33] for a complete treatment. The correspondence with Feynman diagrams and half-diagrams is explained in Sect. 2. The systematics of renormalization, including its multi-scale version which has been acknowledged as the quickest way to get estimates, is recalled in Sect. 3. We use a classical multi-scale expansion to derive a general bound for Feynman diagrams in Sect. 4.

We conclude in Sect. 5 by proving the Hölder estimates for the rough path and adding some remarks on related previous attempts and on possible extensions to general Hölder paths.

### 1. The Fourier Normal Ordering Algorithm

Let  $\Gamma = (\Gamma_t(1), \dots, \Gamma_t(d)) : \mathbb{R} \rightarrow \mathbb{R}^d$  be some continuous path, compactly supported in  $[0, T]$ . Assume that  $\Gamma$  is not differentiable, but only  $\alpha$ -Hölder for some  $0 < \alpha < 1$ , i.e. bounded in the  $\mathcal{C}^\alpha$ -norm,

$$\|\gamma\|_{\mathcal{C}^\alpha} := \sup_{t \in [0, T]} \|\Gamma_t\| + \sup_{s, t \in [0, T]} \frac{\|\Gamma_t - \Gamma_s\|}{|t - s|^\alpha}. \tag{1.1}$$

Then iterated integrals of  $\Gamma$  are not canonically defined. As explained in the Introduction, rough path theory may be seen as a black box taking as input some lift of  $\Gamma$  called *rough path over  $\Gamma$* , producing e.g. solutions of differential equations driven by  $\Gamma$ .

*1.1. Rough paths and iterated integrals.* The usual definition of a rough path is the following. We let in the sequel  $\lfloor 1/\alpha \rfloor$  be the entire part of  $1/\alpha$ .

**Definition 1.1.** *A rough path over  $\Gamma$  is a functional  $J_\Gamma^{TS}(i_1, \dots, i_n)$ ,  $n \leq \lfloor 1/\alpha \rfloor$ ,  $i_1, \dots, i_n \in \{1, \dots, d\}$ , such that  $J_\Gamma^{TS}(i) = \Gamma_t(i) - \Gamma_s(i)$  are the increments of  $\Gamma$ , and the following 3 properties are satisfied:*

- (i) (Hölder continuity)  $J_\Gamma^{TS}(i_1, \dots, i_n)$  is  $n\alpha$ -Hölder continuous as a function of two variables, namely,  $\sup_{s, t \in \mathbb{R}} \frac{|J_\Gamma^{TS}(i_1, \dots, i_n)|}{|t - s|^\alpha} < \infty$ .
- (ii) (Chen property)

$$J_\Gamma^{TS}(i_1, \dots, i_n) = J_\Gamma^{TU}(i_1, \dots, i_n) + J_\Gamma^{US}(i_1, \dots, i_n) + \sum_{n_1+n_2=n} J_\Gamma^{TU}(i_1, \dots, i_{n_1}) J_\Gamma^{US}(i_{n_1+1}, \dots, i_n); \tag{1.2}$$

- (iii) (Shuffle property)

$$J_\Gamma^{TS}(i_1, \dots, i_{n_1}) J_\Gamma^{TS}(j_1, \dots, j_{n_2}) = \sum_{\vec{k} \in Sh(\vec{i}, \vec{j})} J_\Gamma^{TS}(k_1, \dots, k_{n_1+n_2}), \tag{1.3}$$

where  $Sh(\vec{i}, \vec{j})$  – the set of shuffles of the words  $\vec{i}$  and  $\vec{j}$  – is the subset of permutations of the union of the lists  $\vec{i}, \vec{j}$  leaving unchanged the order of the sublists  $\vec{i}$  and  $\vec{j}$ . For instance,  $J_\Gamma^{TS}(i_1, i_2) J_\Gamma^{TS}(j_1) = J_\Gamma^{TS}(i_1, i_2, j_1) + J_\Gamma^{TS}(i_1, j_1, i_2) + J_\Gamma^{TS}(j_1, i_1, i_2)$ .

A formal rough path over  $\Gamma$  is a functional satisfying all the above properties except Hölder continuity (i).

In particular, if  $\Gamma$  is smooth, then its natural iterated integrals

$$I_\Gamma^{TS}(i_1, \dots, i_n) := \int_s^t d\Gamma_{t_1}(i_1) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n) \tag{1.4}$$

satisfy properties (ii) and (iii).

These two algebraic axioms may be rewritten in a Hopf algebraic language. Let us say a few words about it. The reader who is allergic to algebra may just read Definition 1.2 and Proposition 1.8, skip the rest of the section and jump to the end of Subsect. 1.4. However, this language has proved to be very useful both from a theoretic and a practical point of view [12,33].

**Definition 1.2** (Hopf algebra of decorated rooted trees).

- (i) A decorated rooted tree is a tree with a distinguished vertex called root (drawn growing up from the root to the top), provided with a decoration for each vertex. In this article, decorations are always assumed to range in the set  $\{1, \dots, d\}$ . The set of trees is denoted by  $\mathcal{T}$ . The commutative product  $\mathbb{T}_1 \cdot \mathbb{T}_2$  of two trees yields the forest with the two connected components  $\mathbb{T}_1$  and  $\mathbb{T}_2$ . The algebra over  $\mathbb{R}$  generated by trees is denoted by  $\mathbf{H}$ , and the linear subspace of forests with  $n$  vertices by  $\mathbf{H}(n)$ .
- (ii) If  $w$  is a descendant of  $v$  (i.e.  $w$  is above  $v$ ) then one writes  $w \rightarrow v$ . One says that  $v$  is connected to  $w$  ( $w = v$ ,  $w \rightarrow v$  or  $v \rightarrow w$ ). A subset of vertices  $\vec{v} \subset V(\mathbb{T})$  is an **admissible cut** if  $(v, w \in \vec{v}, v \neq w) \Rightarrow (v \text{ is not connected to } w)$ . If  $\vec{v}$  is admissible, which we write  $\vec{v} \models V(\mathbb{T})$ , then  $\text{Roo}_{\vec{v}}\mathbb{T}$  is the subforest with vertices  $\{w \in V(\mathbb{T}); \exists v \in \vec{v}, v \rightarrow w\}$ , while  $\text{Lea}_{\vec{v}}\mathbb{T}$  is the subforest with the complementary set of vertices. Note that  $\text{Roo}_{\vec{v}}\mathbb{T}$  is a tree if  $\mathbb{T}$  is a tree.
- (iii) Define

$$\Delta(\mathbb{T}) = \sum_{\vec{v} \models V(\mathbb{T})} \text{Roo}_{\vec{v}}\mathbb{T} \otimes \text{Lea}_{\vec{v}}\mathbb{T}. \tag{1.5}$$

Then  $\mathbf{H}$  equipped with  $\Delta : \mathbf{H} \rightarrow \mathbf{H} \otimes \mathbf{H}$  is a coproduct. For instance,

$$\Delta(\mathbf{V}_a^b \mathbf{V}_c^d) = \mathbf{V}_a^b \otimes \mathbf{V}_c^d + \mathbf{1} \otimes \mathbf{V}_a^b \mathbf{V}_c^d + \mathbf{!}_a^b \otimes \cdot_c + \mathbf{!}_a^b \otimes \cdot_b + \cdot_a \otimes \cdot_b \cdot_c \tag{1.6}$$

- (iv)  $\mathbf{H}$  has an antipode  $\bar{S}$ , defined inductively by

$$\bar{S}(\mathbf{1}) = 1, \quad \bar{S}(\mathbb{T}) = -\mathbb{T} - \sum_{\vec{v} \models V(\mathbb{T}), \vec{v} \neq \emptyset} \text{Roo}_{\vec{v}}\mathbb{T} \cdot \bar{S}(\text{Lea}_{\vec{v}}\mathbb{T}). \tag{1.7}$$

We shall also need the following Hopf algebra in order to encode the shuffle property.

**Definition 1.3** (Shuffle algebra).

- (i) Let  $\mathbf{Sh}$  be the shuffle algebra with decorations in  $\{1, \dots, d\}$ , i.e. the set of words  $(i_1 \dots i_n)$ ,  $i_1, \dots, i_n \in \{1, \dots, d\}$ , with product

$$(i_1 \dots i_{n_1}) \uplus (j_1 \dots j_{n_2}) = \sum_{\vec{k} \in \text{Sh}(\vec{i}, \vec{j})} (k_1 \dots k_{n_1+n_2}). \tag{1.8}$$

An element of  $\mathbf{Sh}$  is naturally represented as a trunk tree decorated by  $\ell = (\ell(1), \dots, \ell(n))$  from the root to the top. For instance  $(i_1 i_2 i_3) = \mathbf{!}_{i_1}^{i_2} \mathbf{!}_{i_1}^{i_3}$  is decorated by  $\ell(j) = i_j$ ,  $j = 1, 2, 3$ .

- (ii)  $\mathbf{Sh}$  equipped with the restriction of the coproduct  $\Delta$  of  $\mathbf{H}$  to trunk trees, and with the antipode  $S((i_1 \dots i_n)) = -(i_n \dots i_1)$ , is a Hopf algebra.  $\Delta((i_1 \dots i_n)) = \sum_{k=0}^n (i_1 \dots i_k) \otimes (i_{k+1} \dots i_n)$ , holds.

One of the links between these two algebras is given by the following proposition.

**Proposition 1.4** (Projection morphism). *Let  $\theta : \mathbf{H} \rightarrow \mathbf{Sh}$  be the projection Hopf morphism given by associating to a tree  $\mathbb{T}$  the sum of the trunk trees  $\mathfrak{t}$  with same decorations such that*

$$(v \twoheadrightarrow w \text{ in } \mathbb{T}) \Rightarrow (v \twoheadrightarrow w \text{ in } \mathfrak{t}). \tag{1.9}$$

For instance  $\theta(\mathfrak{b} \vee_a^c) = \mathfrak{!}_a^c + \mathfrak{!}_a^b$ .

Indexing the  $J_\Gamma^{ts}(i_1, \dots, i_n)$  by trunk trees  $\mathbb{T} \in \mathbf{Sh}$  with decoration  $\ell(j) = i_j, j = 1, \dots, n$ , properties (ii) and (iii) in Definition 1.1 are equivalent to

(ii)bis

$$J_\Gamma^{ts}(\mathbb{T}) = \sum_{\bar{v} \models V(\mathbb{T})} J_\Gamma^{tu}(\text{Roo}_{\bar{v}}(\mathbb{T})) J_\Gamma^{us}(\text{Lea}_{\bar{v}}(\mathbb{T})), \quad \mathbb{T} \in \mathbf{Sh}; \tag{1.10}$$

in other words,  $J_\Gamma^{ts} = J_\Gamma^{tu} * J_\Gamma^{us}$  for the shuffle convolution defined in Subsect. 5.2;

(iii)bis

$$J_\Gamma^{ts}(\mathbb{T}) J_\Gamma^{ts}(\mathbb{T}') = J_\Gamma^{ts}(\mathbb{T} \pitchfork \mathbb{T}'), \quad \mathbb{T}, \mathbb{T}' \in \mathbf{Sh}. \tag{1.11}$$

In other words,  $J_\Gamma^{ts}$  is a character of  $\mathbf{Sh}$ .

Such a functional indexed by *trunk trees* extends easily to a general *tree-indexed functional* or *tree-indexed rough path* by setting  $\bar{J}_\Gamma^{ts}(\mathbb{T}) := J_\Gamma^{ts} \circ \theta(\mathbb{T})$ . Since  $\theta$  is a Hopf algebra morphism, one gets immediately the generalized properties

(ii)ter  $\bar{J}_\Gamma^{ts} = \bar{J}_\Gamma^{tu} * \bar{J}_\Gamma^{us}$  for the convolution of  $\mathbf{H}$ , i.e.

$$\bar{J}_\Gamma^{ts}(\mathbb{T}) = \sum_{\bar{v} \models V(\mathbb{T})} \bar{J}_\Gamma^{tu}(\text{Roo}_{\bar{v}}\mathbb{T}) \bar{J}_\Gamma^{us}(\text{Lea}_{\bar{v}}\mathbb{T}); \tag{1.12}$$

(iii)ter  $\bar{J}_\Gamma^{ts}(\mathbb{T}) \bar{J}_\Gamma^{ts}(\mathbb{T}') = \bar{J}_\Gamma^{ts}(\mathbb{T} \pitchfork \mathbb{T}')$ , in other words,  $\bar{J}_\Gamma^{ts}$  is a character of  $\mathbf{H}$ .

Properties (ii), (iii) and their generalizations are satisfied for the usual integration operators  $I_\Gamma^{ts}$  and their tree extension  $\bar{I}_\Gamma^{ts}$ , provided  $\Gamma$  is a smooth path so that iterated integrals make sense [16].

Let us give an explicit formula for tree iterated integrals. Let  $\mathbb{T}$  be e.g. a tree, and index its vertices as  $1, \dots, n$ , so that  $(i \twoheadrightarrow j) \Rightarrow (i > j)$ . Denoting by  $i^-$  the ancestor of the vertex  $i$  in  $\mathbb{T}$ , one has

$$\bar{I}_\Gamma^{ts}(\mathbb{T}) = \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_{2^-}} d\Gamma_{x_2}(\ell(2)) \dots \int_s^{x_{n^-}} d\Gamma_{x_n}(\ell(n)). \tag{1.13}$$

*Remark 1.5.* Note that (1.13) obviously does not depend on the choice of the vertex indexation. We call this **invariance under indexation of the vertices**, or **naturality property**. We may rephrase it saying that  $\bar{I}_\Gamma^{ts}(\mathbb{T})$  depends only on the **topology** of  $\mathbb{T}$ . The same property applies to every natural construction and is required in Definition 1.10.

Suppose now one wishes to construct a rough path over  $\Gamma$ , and concentrate on the algebraic properties (ii), (iii) of Definition 1.1. Assume one has constructed characters of  $\mathbf{Sh}$ ,  $J_\Gamma^{t s_0}$ ,  $t \in [0, T]$  with  $s_0$  fixed – in other words, a one-time functional satisfying the usual shuffle property (iii) – such that  $J_\Gamma^{t s_0}(i) = \Gamma_t(i) - \Gamma_{s_0}(i)$ , then one immediately checks that  $J_\Gamma^{t s} := J_\Gamma^{t s_0} * (J_\Gamma^{s s_0} \circ \bar{S})$  satisfies properties (ii)bis and (iii)bis. Namely (by the definition of the antipode)  $J_\Gamma^{t s} := J_\Gamma^{t s_0} * (J_\Gamma^{s s_0} \circ \bar{S})$  is equivalent to the Chen property  $J_\Gamma^{t s} * J_\Gamma^{s s_0} = J_\Gamma^{t s_0}$ . So the only difficult part consists in defining some regularized character of  $\mathbf{Sh}$  satisfying the regularity properties (i).

1.2. *Fourier transform and skeleton integrals.* Instead of regularizing iterated integrals,  $J_\Gamma^{t s_0} \rightsquigarrow J_\Gamma^{t s_0}$  with  $s_0$  fixed, we choose to regularize *skeleton integrals*,  $\text{SkI}_\Gamma^t$ , which are analogues of iterated integrals but depending naturally on a single argument, defined by using Fourier transform.

**Definition 1.6** (Skeleton integral). *Let*

$$\begin{aligned} \text{SkI}_\Gamma^t(a_1 \dots a_n) &:= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \prod_{j=1}^n \mathcal{F}\Gamma'_{\xi_j}(a_j) d\xi_j \cdot \int^t dx_1 \int^{x_1} dx_2 \dots \int^{x_{n-1}} dx_n e^{i(x_1 \xi_1 + \dots + x_n \xi_n)}, \end{aligned} \tag{1.14}$$

where, by definition,  $\int^x e^{iy\xi} dy = \frac{e^{ix\xi}}{i\xi}$ . It may be checked that  $\text{SkI}_\Gamma^t$  is a character of  $\mathbf{Sh}$  – or, in other words, satisfies the shuffle property – just as for usual iterated integrals.

The projection  $\theta$  yields immediately a generalization of this notion to tree skeleton integrals, compare with Eq. (1.13),

$$\overline{\text{SkI}}_\Gamma^t(\mathbb{T}) = \text{SkI}_\Gamma^t \circ \theta(\mathbb{T}) = \int^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_2^-} d\Gamma_{x_2}(\ell(2)) \dots \int^{x_n^-} d\Gamma_{x_n}(\ell(n)). \tag{1.15}$$

An explicit computation yields ([33], Lem. 4.5):

$$\overline{\text{SkI}}_\Gamma^t(\mathbb{T}) = (2\pi)^{-n/2} i^{-n} \int_{\mathbb{R}^n} \prod_{j=1}^n \mathcal{F}\Gamma'_{\xi_j}(\ell(j)) d\xi_j \cdot \frac{e^{it(\xi_1 + \dots + \xi_n)}}{\prod_{i=1}^n [i\xi_i + \sum_{j \rightarrow i} \xi_j]}. \tag{1.16}$$

1.3. *Fourier normal ordering for smooth paths.* We begin by the following

**Definition 1.7** (Fourier projections and measure-splitting).

- (i) Let  $\mu$  be some signed measure with compact support, typically,  $\mu = \mu_{(\Gamma, \ell)}$  ( $dx_1, \dots, dx_n$ ) =  $\otimes_{j=1}^n d\Gamma_{x_j}(\ell(j))$ . Then

$$\mu = \sum_{\sigma \in \Sigma_n} \mu^\sigma \circ \sigma^{-1}, \tag{1.17}$$

where

$$\mathcal{P}^\sigma : \mu \mapsto \mathcal{F}^{-1}(\mathbf{1}_{|\xi_{\sigma(1)}| \leq \dots \leq |\xi_{\sigma(n)}|} \mathcal{F}\mu(\xi_1, \dots, \xi_n)) \tag{1.18}$$

is a Fourier projection, and  $\mu^\sigma$  is defined by

$$\mu^\sigma := \mathcal{P}^{\text{Id}}(\mu \circ \sigma) = (\mathcal{P}^\sigma \mu) \circ \sigma. \tag{1.19}$$

The set of all measures whose Fourier transform is supported in  $\{(\xi_1, \dots, \xi_n); |\xi_1| \leq \dots \leq |\xi_n|\}$  will be denoted by  $\mathcal{P}^+ \text{Meas}(\mathbb{R}^n)$ . Thus  $\mu^\sigma \in \mathcal{P}^+ \text{Meas}(\mathbb{R}^n)$ .

(ii) More generally, if  $T$  is a tree,

$$\mathcal{P}^{\mathbb{T}} \text{Meas}(\mathbb{R}^n) = \left\{ \mu; \bar{\xi} \in \text{supp}(\mathcal{F}\mu) \Rightarrow ((i \rightarrow j) \Rightarrow (|\xi_i| > |\xi_j|)) \right\}. \tag{1.20}$$

This definition applies in particular to the tensor measures  $\mu = \mu_{(\Gamma, \ell)} = \otimes_{i=1, \dots, n} d\Gamma_{x_i}(\ell(i)) = \otimes_{i=1, \dots, n} \Gamma'_{x_i}(\ell(i)) dx_i$  if  $\ell = (\ell(1), \dots, \ell(n))$  is the decoration of a trunk tree. Note that even though  $\mu$  is a *tensor measure* in this case, the projected measures  $\mu^\sigma$  are *not*. This forces us to extend the previous definitions of  $I_\Gamma^{ts}, J_\Gamma^{ts}, \bar{I}_\Gamma^{ts}, \bar{J}_\Gamma^{ts}, \text{Sk}I_\Gamma^{ts}, \overline{\text{Sk}}I_\Gamma^{ts}$  to measure-indexed characters. This is straightforward. However, one must then trade decorated trees (or forests) for so-called *heap-ordered trees* (or *forests*), i.e. trees without decoration but with indexed vertices  $1, \dots, n$  such that

$$(i \rightarrow j) \Rightarrow (i > j). \tag{1.21}$$

For instance,

$$\bar{I}_\mu^{ts}(\mathbb{T}) = \int_s^t \int_s^{x_2^-} \dots \int_s^{x_n^-} d\mu(x_1, \dots, x_n). \tag{1.22}$$

*Remark.* Recall from Remark 1.5 that iterated integrals depend only on the topology of the tree, which means that

$$\bar{I}_\mu^{ts}(\mathbb{T}) = \bar{I}_{\mu \circ \sigma}^{ts}(\sigma^{-1} \cdot \mathbb{T}) \tag{1.23}$$

if  $\sigma \in \Sigma_n$  is a reindexation of the vertices preserving the topology of  $\mathbb{T}$ , i.e. such that

$$(i \rightarrow j \text{ in } \mathbb{T}) \Rightarrow (i \rightarrow j \text{ in } \sigma^{-1}(\mathbb{T})). \tag{1.24}$$

To say things shortly, *skeleton integrals* are convenient when using Fourier coordinates, since they avoid awkward boundary terms such as those generated by usual integrals,  $\int_0^x e^{iy\xi} dy = \frac{e^{ix\xi}}{i\xi} - \frac{1}{i\xi}$ , which create terms with different homogeneity degree in  $\xi$  by iterated integrations. *Measure splitting* gives the *relative scales* of the Fourier coordinates; orders of magnitude of the corresponding integrals may be obtained separately in each sector  $|\xi_{\sigma(1)}| \leq \dots \leq |\xi_{\sigma(n)}|$ . It turns out that these are easiest to get after a permutation of the integrations (applying Fubini's theorem) such that *innermost (or rightmost) integrals bear highest Fourier frequencies*. This is the essence of *Fourier normal ordering*.

**Proposition 1.8** (Permutation graph). *Let  $\mathfrak{T}_n \in \mathbf{Sh}$  be a trunk tree with  $n$  vertices, and  $\sigma \in \Sigma_n$  a permutation of  $\{1, \dots, n\}$ . Then there exists a unique element  $\mathbb{T}^\sigma \in \mathbf{H}$  called permutation graph such that*

$$I_\Gamma^{ts}(\mathfrak{T}_n) = I_\Gamma^{ts}(\mathbb{T}^\sigma). \tag{1.25}$$



Let us give an example. Let  $\mathfrak{T}_n = \mathfrak{!}_{a_1}^{a_3}$  and  $\sigma : (1, 2, 3) \rightarrow (2, 3, 1)$ . Then

$$\begin{aligned} I_\Gamma^{ts}(\mathfrak{T}_n) &= \int_s^t d\Gamma_{a_1}(x_3) \int_s^{x_3} d\Gamma_{a_2}(x_1) \int_s^{x_1} d\Gamma_{a_3}(x_2) \\ &= \int_s^t d\Gamma_{a_2}(x_1) \int_s^{x_1} d\Gamma_{a_3}(x_2) \int_{x_1}^t d\Gamma_{a_1}(x_3) \\ &= \int_s^t d\Gamma_{a_2}(x_1) \int_s^{x_1} d\Gamma_{a_3}(x_2) \int_s^t d\Gamma_{a_1}(x_3) \\ &\quad - \int_s^t d\Gamma_{a_2}(x_1) \int_s^{x_1} d\Gamma_{a_3}(x_2) \int_s^{x_1} d\Gamma_{a_1}(x_3) \\ &= I_\Gamma^{ts}(\mathfrak{!}_{a_2}^{a_3} \cdot a_1) - I_\Gamma^{ts}(\mathfrak{!}_{a_2}^{a_3} \mathfrak{!}_{a_2}^{a_1}), \end{aligned}$$

so  $\mathbb{T}^\sigma = \mathfrak{!}_{a_2}^{a_3} \cdot a_1 - \mathfrak{!}_{a_2}^{a_3} \mathfrak{!}_{a_2}^{a_1}$ . Note that all permutation graphs  $\mathbb{T}^\sigma$  with  $\sigma$  fixed are obtained from the same sum of heap-ordered forests (also denoted by  $\mathbb{T}^\sigma$ , by abuse of notation) by including the decorations of  $\mathfrak{T}_n$  permuted by  $\sigma$ .

As an elementary Corollary of Definition 1.7 and Proposition 1.8, one obtains:

**Corollary 1.9** (Fourier normal ordering for smooth paths). *Let  $\Gamma$  be a smooth path and  $\mathfrak{T}_n \in \mathbf{Sh}$  a trunk tree with  $n$  vertices and decoration  $\ell$ , then*

$$I_\Gamma^{ts}(\mathfrak{T}_n) = \sum_{\sigma \in \Sigma_n} I_{\mu^\sigma}^{ts}(\mathbb{T}^\sigma). \tag{1.26}$$

1.4. *Fourier normal ordering and regularization.* Formal rough paths over  $\Gamma$  will be reconstructed out of tree data  $\phi_\mathbb{T}^t$  defined arbitrarily for each tree  $\mathbb{T}$ , and then extended by multiplication to forests, as we shall now see.

**Definition 1.10.** (i) *For every heap-ordered  $\mathbb{T}$  with  $n$  vertices, and  $t \in \mathbb{R}$ , let  $\phi_\mathbb{T}^t : \mathcal{P}^\mathbb{T} \text{Meas}(\mathbb{R}^n) \rightarrow \mathbb{R}$ ,  $\mu \mapsto \phi_\mathbb{T}^t(\mu)$ , also written  $\phi_\mu^t(\mathbb{T})$  be a family of linear forms such that:*

- (a)  $\phi_{d\Gamma(i)}^t(\mathbb{T}_1) - \phi_{d\Gamma(i)}^s(\cdot_i) = I_\Gamma^{ts}(\mathbb{T}_1) = \Gamma_t(i) - \Gamma_s(i)$  if  $\mathbb{T}_1$  is the trivial heap-ordered tree with one vertex;
- (b) if  $\mathbb{T}_i, i = 1, 2$  are heap-ordered trees with  $n_i$  vertices, and  $\mu_i \in \mathcal{P}^{\mathbb{T}_i} \text{Meas}(\mathbb{R}^{n_i})$ ,  $i = 1, 2$ , the following multiplicative property holds,

$$\phi_{\mu_1}^t(\mathbb{T}_1)\phi_{\mu_2}^t(\mathbb{T}_2) = \phi_{\mu_1 \otimes \mu_2}^t(\mathbb{T}_1 \wedge \mathbb{T}_2), \tag{1.27}$$

where  $\mathbb{T}_1 \wedge \mathbb{T}_2$  is the non-decorated product  $\mathbb{T}_1 \cdot \mathbb{T}_2$  with labels of  $\mathbb{T}_2$  shifted by  $n_1$ .<sup>2</sup>

- (c) (Naturality property) the following invariance condition under reindexation of the vertices holds, see the preceding two remarks,

$$\phi_\mu^t(\mathbb{T}) = \phi_{\mu \circ \sigma}^t(\sigma^{-1} \cdot \mathbb{F}), \tag{1.28}$$

if  $\sigma$  – which acts by permuting the vertices of  $\mathbb{T}$  – is such that

$$(i \rightarrow j \text{ in } \mathbb{T}) \Rightarrow (i \rightarrow j \text{ in } \sigma^{-1}(\mathbb{T})). \tag{1.29}$$

<sup>2</sup> The product  $\mathbb{T}_1 \wedge \mathbb{T}_2$  defines actually the product of the Hopf algebra of heap-ordered trees [12].

(ii) Let, for  $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ ,  $\chi_\Gamma^t : \mathbf{Sh} \rightarrow \mathbb{R}$  be the linear form on  $\mathbf{Sh}$  defined by

$$\chi_\Gamma^t(\mathfrak{T}_n) := \sum_{\sigma \in \Sigma_n} \phi_{\mu_{(\Gamma, \ell)}^\sigma}^t(\mathbb{T}^\sigma), \quad \mathfrak{T}_n = (\ell(1) \dots \ell(n)) \tag{1.30}$$

as in Proposition 1.8.

The main result is the following.

**Proposition 1.11** (Rough path construction by Fourier normal ordering). *For every path  $\Gamma$  such that  $\chi_\Gamma^t$  is well-defined,  $\chi_\Gamma^t$  is a character of  $\mathbf{Sh}$ .*

Consequently, the following formula for  $\mathfrak{T}_n \in \mathbf{Sh}$ ,  $n \geq 1$ , with  $n$  vertices and decoration  $\ell$ ,

$$J_\Gamma^{tS}(\ell(1), \dots, \ell(n)) := \chi_\Gamma^t * (\chi_\Gamma^s \circ S)(\mathfrak{T}_n) \tag{1.31}$$

defines a formal rough path over  $\Gamma$ .

Furthermore, the following equivalent definition holds:

$$J_\Gamma^{tS}(\mathfrak{T}_n) := \sum_{\sigma \in \Sigma_n} (\phi^t * (\phi^s \circ \bar{S}))_{\mu_{(\Gamma, \ell)}^\sigma}(\mathbb{T}^\sigma), \tag{1.32}$$

where the convolution in the right equation is defined by reference to the (heap-ordered) tree coproduct, namely, one sets

$$(\phi^t * (\phi^s \circ \bar{S}))_\nu(\mathbb{T}) = \sum_{\bar{v} \models V(\mathbb{T})} \phi_{\otimes_{v \in V(\text{Root}_{\bar{v}}\mathbb{T})} \nu_{\bar{v}}}^t(\text{Root}_{\bar{v}}\mathbb{T}) \phi_{\otimes_{v \in V(\text{Lea}_{\bar{v}}\mathbb{T})} \nu_{\bar{v}}}^s(\bar{S}(\text{Lea}_{\bar{v}}\mathbb{T})) \tag{1.33}$$

for a tensor measure  $\nu = \nu_1 \otimes \dots \otimes \nu_n$ , and by multilinear extension

$$\begin{aligned} (\phi^t * (\phi^s \circ \bar{S}))_\nu(\mathbb{T}) &= (2\pi)^{-n/2} \int \mathcal{F}\nu(\xi_1, \dots, \xi_n) d\xi_1 \dots d\xi_n \cdot \\ &\cdot \sum_{\bar{v} \models V(\mathbb{T})} \phi_{\otimes_{v \in V(\text{Root}_{\bar{v}}(\mathbb{T}))} e^{ix\nu} dx_{\bar{v}}}^t(\text{Root}_{\bar{v}}\mathbb{T}) \phi_{\otimes_{v \in V(\text{Lea}_{\bar{v}}(\mathbb{T}))} e^{ix\nu} dx_{\bar{v}}}^s(\bar{S}(\text{Lea}_{\bar{v}}\mathbb{T})), \mathbb{T} \in \mathcal{F}_{ho}(n). \end{aligned} \tag{1.34}$$

for an arbitrary measure  $\nu \in \text{Meas}(\mathbb{R}^n)$ ,  $\nu = (2\pi)^{-n/2} \int d\vec{\xi} \mathcal{F}\nu(\vec{\xi}) \otimes_{j=1}^n e^{ix^j \xi_j} dx_j$ .

Quite naturally, we shall call formula (1.31), resp. (1.32) the *shuffle convolution*, resp. *tree convolution definition* of  $J$ .

Assuming  $\Gamma$  is smooth, then defining  $\phi^t$  as the skeleton integral  $\text{Sk}I^t$  yields trivially by recombination  $\chi_\Gamma^t = \text{Sk}I^t$  too, and then  $J_\Gamma^{tS} = I_\Gamma^{tS}$  is the canonical rough path over  $\Gamma$ . Proposition 1.11 shows that the same recombination algorithm yields a rough path over  $\Gamma$  whenever  $\phi^t$  satisfies conditions (a), (b) and (c) of Definition 1.10. It is actually clear from Definition 1.10 that *any* rough path over  $\Gamma$  may be obtained in this way [12].

The enormous advantage now with respect to the original problem is that one may construct as many linear forms  $\phi^t$  as one wishes by assigning some arbitrary value to  $\phi_\mu^t(\mathbb{T})$ ,  $\mathbb{T}$  ranging over all (heap-ordered) trees with  $\geq 2$  vertices, and extending to forests by multiplication following condition (b).

It is now natural to try and define  $\phi^t$  as some regularized skeleton integral in such a way that  $J_\Gamma^{tS}$  satisfies the Hölder continuity property (i) in Definition 1.1. We shall do so in the next sections by *renormalizing* skeleton integrals.

For the sequel, we shall start from the *tree convolution definition* (1.32) of  $J$ , which will be used in the following guise. Assume  $\nu = \mu_{(\Gamma, \ell)}^\sigma$  and  $\mathbb{T} = \mathbb{T}_1 \wedge \dots \wedge \mathbb{T}_p$  is the (heap-ordered) product of  $p$  trees. Set  $\hat{\nu}_{\mathbb{T}'}(\vec{\xi}) = \otimes_{v \in V(\mathbb{T}')} \mathcal{F}(\Gamma'(\ell \circ \sigma(v))) (\xi_v) e^{ix_v \xi_v} dx_v$  for  $\mathbb{T}'$  subtree of  $\mathbb{T}$  and  $\vec{\xi} = (\xi_v)_{v \in V(\mathbb{T}')}.$  Then, by the multiplicative property (b) for  $\phi^s$  and  $\phi^t$ , see Definition 1.10,

$$\begin{aligned} & (\phi^t * (\phi^s \circ \bar{S}))_{\nu}(\mathbb{T}) \\ &= (2\pi)^{-n/2} \int d\xi_1 \dots d\xi_n \mathbf{1}_{|\xi_1| \leq \dots \leq |\xi_n|} \prod_{q=1}^p (\phi^t * (\phi^s \circ \bar{S}))_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v \in V(\mathbb{T}_q)})}(\mathbb{T}_q). \end{aligned} \quad (1.35)$$

Now the inductive definition of the antipode implies

$$\begin{aligned} & (\phi^t * (\phi^s \circ \bar{S}))_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v \in V(\mathbb{T}_q)})}(\mathbb{T}_q) \\ &= \phi_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v \in V(\mathbb{T}_q)})}^t(\mathbb{T}_q) + \phi_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v \in V(\mathbb{T}_q)})}^s(\bar{S}(\mathbb{T}_q)) \\ &+ \sum_{\vec{v} \in V(\mathbb{T}_q)} \phi_{\hat{\nu}_{\text{Roov}_{\vec{v}}\mathbb{T}_q}((\xi_v)_{v \in V(\text{Roov}_{\vec{v}}\mathbb{T}_q)})}^t(\text{Roov}_{\vec{v}}\mathbb{T}_q) \phi_{\hat{\nu}_{\text{Leav}_{\vec{v}}\mathbb{T}_q}((\xi_v)_{v \in V(\text{Leav}_{\vec{v}}\mathbb{T}_q)})}^s(\bar{S}(\text{Leav}_{\vec{v}}\mathbb{T}_q)) \\ &= (\phi^t - \phi^s)_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v \in V(\mathbb{T}_q)})}(\mathbb{T}_q) \\ &+ \sum_{\vec{v} \models V(\mathbb{T}_q), \vec{v} \neq \emptyset} (\phi^t - \phi^s)_{\hat{\nu}_{\text{Roov}_{\vec{v}}\mathbb{T}_q}((\xi_v)_{v \in V(\text{Roov}_{\vec{v}}\mathbb{T}_q)})} \\ &\times (\text{Roov}_{\vec{v}}\mathbb{T}_q) \phi_{\hat{\nu}_{\text{Leav}_{\vec{v}}\mathbb{T}_q}((\xi_v)_{v \in V(\text{Leav}_{\vec{v}}\mathbb{T}_q)})}^s(\bar{S}(\text{Leav}_{\vec{v}}\mathbb{T}_q)). \end{aligned} \quad (1.36)$$

Finally, applying iteratively the inductive definition of the antipode leads to an expression of  $\bar{S}(\text{Leav}_{\vec{v}}\mathbb{T}_q)$  in terms of a sum of forests obtained by multiple cuts an in [8]. Applying once again the multiplicative property to  $\phi^s$  yields  $(\phi^t * (\phi^s \circ \bar{S}))_{\hat{\nu}_{\mathbb{T}_q}((\xi_v)_{v \in V(\mathbb{T}_q)})}(\mathbb{T}_q)$  as a sum of terms of the form

$$\begin{aligned} & \Phi^{tS}(\mathbb{T}_q; \vec{\xi}; \vec{v}, (\mathbb{T}'_j)) \\ &:= (\phi^t - \phi^s)_{\hat{\nu}_{\text{Roov}_{\vec{v}}\mathbb{T}_q}((\xi_v)_{v \in V(\text{Roov}_{\vec{v}}\mathbb{T}_q)})}(\text{Roov}_{\vec{v}}\mathbb{T}_q) \prod_{j=1}^J \phi_{\hat{\nu}_{\mathbb{T}'_j}((\xi_v)_{v \in V(\mathbb{T}'_j)})}^s(\mathbb{T}'_j), \end{aligned}$$

with  $V(\mathbb{T}_q) = V(\text{Roov}_{\vec{v}}\mathbb{T}_q) \cup \uplus_{j=1}^J V(\mathbb{T}'_j).$

## 2. Feynman Diagram Reformulation

Let  $\mathbb{T}$  be a forest. We shall show in this section how to compute tree skeleton integrals  $\text{SkI}_B(\mathbb{T})$  of fractional Brownian motion by means of Feynman diagrams of a particular type. Computations are based on the *harmonizable representation* of fBm,

$$B_t(i) = (2\pi c_\alpha)^{-\frac{1}{2}} \int \frac{e^{it\xi} - 1}{i\xi} |\xi|^{\frac{1}{2}-\alpha} dW_\xi(i), \quad 1 \leq i \leq d, \quad (2.1)$$

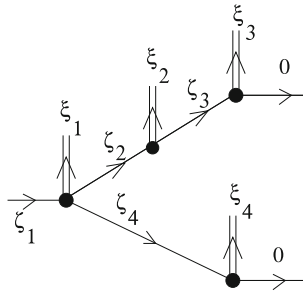


Fig. 1. Feynman half-diagram  $G^{\frac{1}{2}}(\mathbb{T})$  associated to  $\mathbb{T}$

where  $(W_\xi(1), \dots, W_\xi(d))$  are  $d$  independent, identically distributed complex Brownian motions such that  $W_{-\xi}(i) = \overline{W_\xi(i)}$ . With the usual normalization choice  $(2\pi c_\alpha)^{-\frac{1}{2}} = \frac{1}{2} \sqrt{-\frac{\alpha}{\cos \pi \alpha \Gamma(-2\alpha)}}$ ,  $(B_t)_{t \in \mathbb{R}}$  is the unique centered Gaussian process with covariance

$$\mathbb{E} B_s B_t = \frac{1}{2} (|s|^{2\alpha} + |t|^{2\alpha} - |t - s|^{2\alpha}). \tag{2.2}$$

Quite generally, the associated physical theory contains particles of 2 types, corresponding to two Gaussian fields,  $\sigma$ , resp.  $\phi$ , whose propagators are represented by simple, resp. double lines. Vertices are of type  $(\phi \sigma^n)_{n \geq 2}$ , namely, at each vertex meet  $n \geq 2$  simple lines and exactly 1 double line. More specifically, we shall only need to consider *tree Feynman diagrams* in an unusual sense, namely, Feynman diagrams such that the subset of *simple* lines contains no loops.

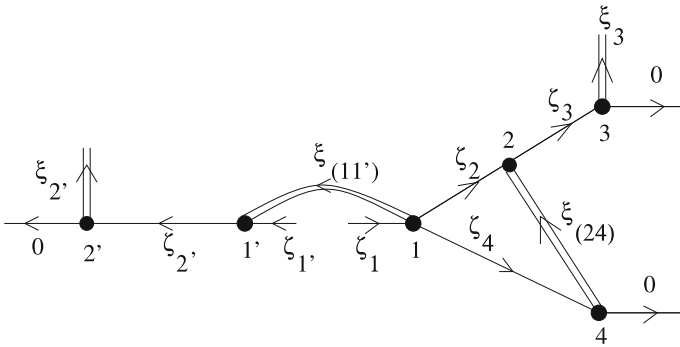
We shall also speak for convenience of Feynman *half-diagrams*, which are Feynman diagrams in the above sense, except that it also possibly admits – besides *true* external  $\phi$ -legs – *uncontracted*  $\phi$ -legs, which are assumed to be cut in the middle (this implies special evaluation rules as we shall see). On the other hand, *contracted*  $\phi$ -legs are always internal lines. Gluing a Feynman half-diagram  $G^{\frac{1}{2}}$  with its image in a mirror along the middle of its external double lines yields a *symmetric Feynman diagram*  $G = (G^{\frac{1}{2}})^2$ .

A tree (or more generally a forest)  $\mathbb{T}$  determines a unique tree Feynman half-diagram  $G^{\frac{1}{2}}(\mathbb{T})$  (called: *uncontracted tree Feynman half-diagram associated to*  $\mathbb{T}$ ), admitting only uncontracted  $\phi$ -legs, whose underlying tree structure of simple lines is that of  $\mathbb{T}$ , see Fig. 1 and Fig. 3. One always assigns *zero momentum* to the simple external lines attached to the *leaves* of  $\mathbb{T}$ . All other tree Feynman half-diagrams are obtained from some  $G^{\frac{1}{2}}(\mathbb{T})$  by pairwise contracting some of the uncontracted  $\phi$ -legs, and denoted

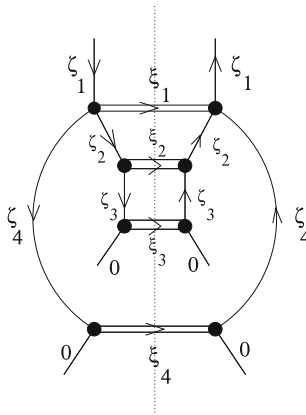
accordingly, see Fig. 2 and Fig. 4. In these diagrams,  $\mathbb{T} = \begin{matrix} & & 3 \\ & & \downarrow \\ & & \vee \\ & & 1 \end{matrix}$  and  $\mathbb{T}' = \begin{matrix} & & 2 \\ & & \downarrow \\ & & \vee \\ & & 1 \end{matrix}$ . If  $G^{\frac{1}{2}}$  is a tree Feynman half-diagram (in the same sense as for Feynman diagrams) then  $G$  is called a *symmetric tree Feynman diagram*. By definition, a symmetric tree Feynman diagram has no external double line.

Let us now define Feynman rules. If  $G$  is a diagram or half-diagram, the set of vertices, resp. internal lines shall be denoted by  $V(G)$ , resp.  $L(G)$ . The set of external lines is denoted by  $L_{ext}(G)$ .

**Definition 2.1** (Feynman rules). *Let  $G$  be a Feynman diagram or half-diagram consisting of simple lines, double lines and vertices  $v$  connecting one double line with 2, 3, ...*



**Fig. 2.** Feynman half-diagram  $G^{\frac{1}{2}}(\mathbb{T}', \mathbb{T}; (11'), (24))$  associated to  $\mathbb{T}', \mathbb{T}$ . By momentum conservation,  $\xi_{2'} + \xi_3 = \zeta_{1'} + \zeta_1$

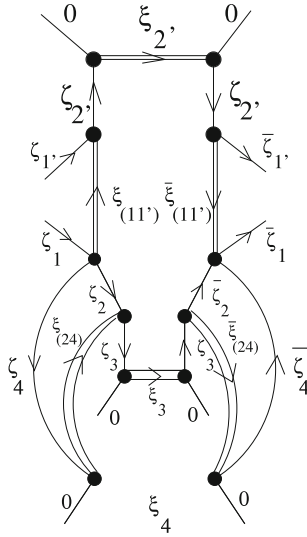


**Fig. 3.** Full Feynman diagram  $G(\mathbb{T}) = (G^{\frac{1}{2}}(\mathbb{T}))^2$  associated to Fig. 1

double lines, and a certain number of external lines. Each line is oriented and decorated by a real-valued momentum, conventionally denoted by  $\zeta_i$ , resp.  $\xi_i$  or  $\xi_{(ij)}$  for some index  $i$  or pair contraction  $(ij)$  for simple, resp. double lines; reversing the orientation of the line is equivalent to changing the sign of the momentum. The momentum preservation relation holds, namely, the sum of all momenta at any vertex is zero. We denote by  $I_\sigma(G)$ , resp.  $I_\phi(G)$ , the number of internal simple, resp. double lines, so that simple, resp. double lines may be thought of as propagators of some field denoted by  $\sigma$ , resp.  $\phi$ . We also let  $I(G) := I_\sigma(G) + I_\phi(G)$  be the total number of internal lines.

(i) Feynman half-diagrams

Let  $G^{\frac{1}{2}}$  be a half-diagram. Associate  $\zeta_i^{-1}$  to each internal simple line with momentum  $\zeta_i$ ,  $|\xi_i|^{\frac{1}{2}-\alpha}$  to each uncontracted  $\phi$ -leg with momentum  $\xi_i$ , and  $|\xi_{(ij)}|^{1-2\alpha}$  to each contracted double line with momentum  $\xi_{(ij)}$ . The result is a function of the momenta of the external lines,  $\zeta_{ext}$  and  $\xi_{ext}$ , denoted by  $A_{G^{\frac{1}{2}}}(\zeta_{ext}, \xi_{ext})$ . We shall denote by  $\zeta_{ext}$ , resp.  $\xi_{ext}$  the sum of the momenta of the external simple, resp. double lines.



**Fig. 4.** Full Feynman diagram  $G(\mathbb{T}'\cdot\mathbb{T}; (11'), (24)) = (G^{\frac{1}{2}}(\mathbb{T}'\cdot\mathbb{T}; (11'), (24)))^2$  associated to Fig. 2

In the particular case when  $G^{\frac{1}{2}} = G^{\frac{1}{2}}(\mathbb{T}; (i_1 i_2), \dots, (i_{2p-1} i_{2p}))$ ,  $p \geq 0$ , comes from a tree, one has  $\xi_{ext} = \{\xi_v \mid \xi_v \text{ uncontracted}\}$ , and

$$A_{G^{\frac{1}{2}}}(\zeta_{ext}, \xi_{ext}) = \delta(\zeta_{ext} + \xi_{ext}) \prod_{v \in V(\mathbb{T}) \mid \xi_v \text{ uncontracted}} |\xi_v|^{\frac{1}{2}-\alpha} \cdot \int \prod_{v \in V(\mathbb{T}) \setminus \{\text{roots}\}} \frac{1}{\zeta_v} \cdot \prod_{q=1}^p |\xi_{(i_{2q-1} i_{2q})}|^{1-2\alpha} d\xi_{(i_{2q-1} i_{2q})}, \tag{2.3}$$

all external  $\zeta$ -momenta attached to the leaves of  $\mathbb{T}$  vanishing, as explained before.

(ii) Feynman diagrams

Associate  $\zeta_i^{-1}$  to each internal simple line with momentum  $\zeta_i$ , and  $|\xi_i|^{1-2\alpha}$ , resp.  $|\xi_{(ij)}|^{1-2\alpha}$  to each internal double line with momentum  $\xi_i$ , resp.  $\xi_{(ij)}$ .

The resulting amplitude of the amputated diagram (i.e., shorn of its external legs), function of the momenta of the external lines,  $\zeta_{ext}$  and  $\xi_{ext}$ , is denoted by  $A_G(\zeta_{ext}, \xi_{ext})$ .

In the particular case when  $G = (G^{\frac{1}{2}})^2$  is a symmetric tree Feynman diagram, denoting by  $\bar{\zeta}_i, \bar{\xi}_{(ij)}$  the momenta of the mirror lines, and by  $\zeta_{ext}$ , resp.  $\bar{\zeta}_{ext}$  the sum of the external  $\zeta$ -, resp.  $\bar{\zeta}$ -momenta, one has

$$A_G(\zeta_{ext}; \bar{\zeta}_{ext}) = \int \prod_{v \in V(\mathbb{T}) \mid \xi_v \text{ uncontracted}} d\xi_v \left( A_{G^{\frac{1}{2}}}(\zeta_{ext}, \xi_{ext}) \right) \left( A_{G^{\frac{1}{2}}}(\bar{\zeta}_{ext}, \xi_{ext}) \right), \tag{2.4}$$

where  $\xi_{ext} = \{\xi_v; \xi_v \text{ uncontracted}\}$  as in (i). Equation (2.4) contains a hidden  $\delta$ -function  $\delta(\zeta_{ext} + \bar{\zeta}_{ext})$  due to overall momentum conservation.

*Remark.* The relation between the  $\zeta$ - and  $\xi$ -coordinates for a half- or full diagram coming from a tree is simply  $\zeta_v = \xi_v + \sum_{w \rightarrow v} \xi_w$  or conversely,  $\xi_v = \zeta_v - \sum_{w \rightarrow v} \zeta_w$ , where  $\{w : w \rightarrow v\}$ , resp.  $\{w : w \rightarrow v\}$  are the descendants, resp. children of  $v$ , and one has set  $\xi_i = -\xi_j = \xi_{(ij)}$  for contracted double lines.

Let  $G$  be a connected Feynman diagram. It contains  $I(G) - |V(G)| + 1$  independent momenta: namely, there is one momentum constraint at each vertex, which gives altogether  $|V(G)| - 1$  independent constraints, because the global translation invariance has already been taken into account by demanding that the sum of the external momenta be zero. Remove one internal line at each vertex, so that all remaining momenta are independent. The set of all lines which have been removed, together with the vertices at the end of the lines, constitute a subdiagram of  $G$  with no loops, hence a sub-forest. For such a choice of lines,  $L'(G)$ , say, we let  $(z_\ell)_{\ell \in L(G) \setminus L'(G)}$ ,  $z = \zeta$  or  $\xi$ , be the set of remaining, independent momenta. Each  $z_{\ell'}$ ,  $\ell' \in L'(G)$ , may be written uniquely as some linear combination  $z_{\ell'} = z_{\ell'}((z_\ell)_{\ell \in L_{ext}(G) \cup (L(G) \setminus L'(G))})$ , which yields an explicit formula for  $A_G$ ,

$$A_G(\mathbf{z}_{ext}) = \delta(z_{ext}) \int \prod_{\ell \in L(G) \setminus L'(G)} dz_\ell \cdot \prod_{\ell \in L_\phi(G)} |\xi_\ell|^{1-2\alpha} \prod_{\ell \in L_\sigma(G)} \zeta_\ell^{-1}, \tag{2.5}$$

where  $z_{ext}$  is the sum of the external momenta.

The relation with iterated integrals of fractional Brownian motion is the following.

**Lemma 2.2. 1.** *Let  $\mathbb{T}$  be a tree with  $n$  vertices and root indexed by 1. Then, see (1.16) and (2.1),*

$$\begin{aligned} \overline{\text{SkI}}_B^t(\mathbb{T}) &= (i\sqrt{2\pi c_\alpha})^{-n} \int \prod_{v \in V(\mathbb{T})} dW_{\xi_v}(\ell(v)) \\ &\int \frac{e^{it\zeta_1}}{\zeta_1} d\zeta_1 A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_{ext} = (\zeta_1, 0), \xi_{ext} = (\xi_v)_{v \in V(\mathbb{T})}), \end{aligned} \tag{2.6}$$

with

$$\begin{aligned} A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_{ext} = (\zeta_1, 0), \xi_{ext} = (\xi_v)_{v \in V(\mathbb{T})}) \\ = \delta(\zeta_1 + \xi_{ext}) \prod_{v \in V(\mathbb{T})} |\xi_v|^{\frac{1}{2}-\alpha} \cdot \prod_{v \in V(\mathbb{T}) \setminus \{1\}} \frac{1}{\zeta_v}. \end{aligned} \tag{2.7}$$

Hence, provided the decorations  $(\ell(v))_{v \in V(\mathbb{T})}$  are all distinct,

$$\text{Var}(\overline{\text{SkI}}_B^t(\mathbb{T}) - \overline{\text{SkI}}_B^s(\mathbb{T})) = (2\pi c_\alpha)^{-n} \int \frac{d\zeta_1}{\zeta_1^2} |e^{it\zeta_1} - e^{is\zeta_1}|^2 A_{G(\mathbb{T})}(\zeta_{ext} = (\zeta_1, 0)), \tag{2.8}$$

with

$$A_{G(\mathbb{T})}(\zeta_{ext} = (\zeta_1, 0)) = \int \prod_{v \in V(\mathbb{T})} d\xi_v \left| A_{G^{\frac{1}{2}}((\zeta_1, 0), \xi)} \right|^2 \tag{2.9}$$

$$= \int \prod_{v \in V(\mathbb{T})} d\xi_v \delta(\zeta_1 + \sum_{v \in V(\mathbb{T})} \xi_v) \prod_{v \in V(\mathbb{T})} |\xi_v|^{1-2\alpha} \cdot \prod_{v \in V(\mathbb{T}) \setminus \{1\}} \frac{1}{\zeta_v^2}. \tag{2.10}$$

2. Let more generally  $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_q$  and  $\mathbb{T}' = \mathbb{T}'_1 \dots \mathbb{T}'_{q'}$ ,  $q, q' \geq 1$ , with roots  $r_1, \dots, r_q, r'_1, \dots, r'_{q'}$ . Consider some multiple contraction  $(i_1 i_2), \dots, (i_{2p-1} i_{2p})$  of  $\prod_{m=1}^q (\overline{\text{SkI}}_B^t(\mathbb{T}_m) - \overline{\text{SkI}}_B^s(\mathbb{T}_m)) \prod_{m'=1}^{q'} \overline{\text{SkI}}_B^s(\mathbb{T}'_{m'})$  connecting the vertices of  $\mathbb{T}$  and  $\mathbb{T}'$ , which we write for short  $\delta \overline{\text{SkI}}_B^t \overline{\text{SkI}}_B^s(\mathbb{T}, \mathbb{T}'; (i_1 i_2), \dots, (i_{2p-1} i_{2p}))$ , and let  $G^{\frac{1}{2}} := G^{\frac{1}{2}}(\mathbb{T}, \mathbb{T}'; (i_1 i_2), \dots, (i_{2p-1} i_{2p}))$  be the corresponding (connected) Feynman half-diagram. Then

$$\begin{aligned} \delta \overline{\text{SkI}}_B^t \overline{\text{SkI}}_B^s(\mathbb{T}, \mathbb{T}'; (i_1 i_2), \dots, (i_{2p-1} i_{2p})) &= (i\sqrt{2\pi c_\alpha})^{-(|V(\mathbb{T})|+|V(\mathbb{T}')|)} \\ &\int \prod_{v \in V(\mathbb{T}, \mathbb{T}') \mid \xi_v \text{ uncontracted}} dW_{\xi_v}(\ell(v)) \prod_{m=1}^q \frac{e^{it\zeta_{r_m}} - e^{is\zeta_{r_m}}}{\zeta_{r_m}} d\zeta_{r_m} \\ &\prod_{m'=1}^{q'} \frac{e^{is\zeta_{r'_{m'}}}}{\zeta_{r'_{m'}}} d\zeta_{r'_{m'}} A_{G^{\frac{1}{2}}}(\zeta_{ext} = ((\zeta_{r_m}), (\zeta_{r'_{m'}}), 0), \xi_{ext}), \end{aligned} \tag{2.11}$$

where  $\xi_{ext} := \{(\xi_v)_{v \in V(\mathbb{T})} \mid \xi_v \text{ uncontracted}\}$ .

Assume furthermore all non-contracted indices  $\ell(i), i \neq i_1, \dots, i_{2p}$  are distinct. Then

$$\begin{aligned} \text{Var} \delta \overline{\text{SkI}}_B^t \overline{\text{SkI}}_B^s(\mathbb{T}, \mathbb{T}'; (i_1 i_2), \dots, (i_{2p-1} i_{2p})) &= (2\pi c_\alpha)^{-(|V(\mathbb{T})|+|V(\mathbb{T}')|)} \\ &\int \prod_{m=1}^q d\zeta_{r_m} d\bar{\zeta}_{r_m} \left( \frac{e^{it\zeta_{r_m}} - e^{is\zeta_{r_m}}}{\zeta_{r_m}} \right) \left( \frac{e^{it\bar{\zeta}_{r_m}} - e^{is\bar{\zeta}_{r_m}}}{\bar{\zeta}_{r_m}} \right) \\ &\int \prod_{m'=1}^{q'} \frac{d\zeta_{r'_{m'}} d\bar{\zeta}_{r'_{m'}}}{\zeta_{r'_{m'}} \bar{\zeta}_{r'_{m'}}} A_{(G^{\frac{1}{2}})^2}(\zeta_{ext} = ((\zeta_{r_m}), (\zeta_{r'_{m'}}), 0); \bar{\zeta}_{ext} = ((\bar{\zeta}_{r_m}), (\bar{\zeta}_{r'_{m'}}), 0)). \end{aligned} \tag{2.12}$$

Example. Let  $\mathbb{T}, \mathbb{T}'$  be as in Fig. 2, 4. Then:

$$\begin{aligned} \delta \overline{\text{SkI}}_B^t \overline{\text{SkI}}_B^s(\mathbb{T}, \mathbb{T}'; (11'), (24)) &= (i\sqrt{2\pi c_\alpha})^{-6} \int dW_{\xi_{2'}}(\ell(2')) dW_{\xi_3}(\ell(3)) \left[ \frac{e^{it\zeta_1} - e^{is\zeta_1}}{\zeta_1} d\zeta_1 \right] \\ &\times \left[ \frac{e^{is\zeta_{1'}}}{\zeta_{1'}} d\zeta_{1'} \right] A_{G^{\frac{1}{2}}}(\zeta_1, \zeta_{1'}, \xi_{2'}, \xi_3) \end{aligned} \tag{2.13}$$

with

$$A_{G^{\frac{1}{2}}}(\zeta_1, \zeta_{1'}, \xi_{2'}, \xi_3) = \delta(\zeta_1 + \zeta_{1'} + \xi_{2'} + \xi_3) |\xi_{2'} \xi_3|^{\frac{1}{2}-\alpha} \frac{|\xi_{(11')} \xi_{(24)}|^{1-2\alpha} d\xi_{(11')} d\xi_{(24)}}{(\xi_{(24)} + \xi_3) \xi_3 \xi_{(24)} \xi_{2'}}. \tag{2.14}$$

As for its variance, assuming  $\ell(2') \neq \ell(3)$ ,

$$\begin{aligned} \text{Var} \delta \overline{\text{SkI}}_B^t \overline{\text{SkI}}_B^s(\mathbb{T}, \mathbb{T}'; (11'), (24)) &= (2\pi c_\alpha)^{-6} \int \frac{d\zeta_1 d\zeta_{1'} d\bar{\zeta}_1 d\bar{\zeta}_{1'}}{\zeta_{1'} \bar{\zeta}_{1'}} \left[ \frac{e^{it\zeta_1} - e^{is\zeta_1}}{\zeta_1} \right] \left[ \frac{e^{it\bar{\zeta}_1} - e^{is\bar{\zeta}_1}}{\bar{\zeta}_1} \right] A_{(G^{\frac{1}{2}})^2}(\zeta_1, \zeta_{1'}; \bar{\zeta}_1, \bar{\zeta}_{1'}) \end{aligned} \tag{2.15}$$



with

$$A_{(G^{\frac{1}{2}})^2}(\zeta_1, \zeta_{1'}; \bar{\zeta}_1, \bar{\zeta}_{1'}) = \int d\xi_{2'} d\xi_3 d\xi_{(24)} d\xi_{(11')} d\bar{\xi}_{(24)} d\bar{\xi}_{(11')} \delta(\zeta_1 + \zeta_{1'} + \xi_{2'} + \xi_3) \delta(\bar{\zeta}_1 + \bar{\zeta}_{1'} + \xi_2 + \xi_3) |\xi_{2'} \xi_3|^{1-2\alpha} \frac{|\xi_{(24)} \xi_{(11')} \bar{\xi}_{(24)} \bar{\xi}_{(11')}|^{1-2\alpha}}{(\xi_3 \xi_{2'})^2 (\xi_{(24)} + \xi_3) (\bar{\xi}_{(24)} + \xi_3) \xi_{(24)} \bar{\xi}_{(24)}}. \tag{2.16}$$

### 3. Definition of Renormalization Scheme

We present here the general features of the BPHZ renormalization scheme, together with its multi-scale formulation which will allow us to prove Hölder regularity. It relies

- (i) on the choice of a set of graphs called *diverging graphs*. In general (see Subsect. 3.1 below) it is simply the subset of Feynman graphs  $G$  such that  $\omega(G) > 0$ , where  $\omega$  is the *overall degree of divergence* (or simply degree of homogeneity) of the graph.
- (ii) on a choice of *regularization scheme*. Here we choose the Taylor evaluation at zero external momenta, denoted by  $\tau$ . To be definite, if  $A_g(z_{ext,1}, \dots, z_{ext,N_{ext}})$  is the amplitude of the graph  $g$  with  $N_{ext}$  external momenta, then  $\tau_g A_g(z_{ext,1}, \dots, z_{ext,N_{ext}}) = A_g(0, \dots, 0)$ .

Consider now a subdiagram  $g^{\frac{1}{2}}$  of a Feynman half-diagram  $G^{\frac{1}{2}}$ , with external legs  $\mathbf{z}_{ext} := \mathbf{z}'_{ext} \uplus \{\xi_v : \xi_v \text{ uncontracted}\}$ . The uncontracted  $\phi$ -legs ( $\xi_v$ ) are not considered as *true*, free external legs since they are attached on the mirror and must eventually be integrated, see e.g. Eq. (2.9). Hence one sets

$$\tau_{g^{\frac{1}{2}}} A_{g^{\frac{1}{2}}}(\mathbf{z}_{ext}) := A_{g^{\frac{1}{2}}}(\mathbf{z}'_{ext} = \mathbf{0}, \{\xi_v : \xi_v \text{ uncontracted}\}). \tag{3.1}$$

For this reason, it is more natural to write  $\tau_g A_{g^{\frac{1}{2}}}$  instead of  $\tau_{g^{\frac{1}{2}}} A_{g^{\frac{1}{2}}}$ , where in the symmetric graph  $g := (g^{\frac{1}{2}})^2$ , the uncontracted  $\phi$ -legs have now become *internal legs*.

**3.1. Diverging graphs.** Consider a connected Feynman diagram  $G$ . In order to decide whether to renormalize it or not, we compute its degree of divergence  $\omega(G)$ . It is simply obtained as the sum of the overall degree of homogeneity of the integrand,  $(1 - 2\alpha)I_\phi(G) - I_\sigma(G)$ , and of the number,  $I(G) - |V(G)| + 1$ , of independent momenta, with respect to which the integrand is integrated; hence it is simply the overall homogeneity degree of the Feynman integral. Taking into account the relation  $|V(G)| = 2I_\phi(G) + N_\phi(G)$  (obtained by counting one half double line per vertex, except for external double lines which are only connected to one vertex), yields

$$\omega(G) = 1 - \alpha|V(G)| - (1 - \alpha)N_\phi(G). \tag{3.2}$$

**Definition 3.1** (Diverging graphs). *We call a Feynman graph  $G$  diverging if and only if it has no external  $\phi$ -legs.*

Clearly enough, with this definition, small graphs (i.e. with  $\alpha|V(G)| < 1$ ) are diverging if and only if  $\omega(G) > 0$  (which is the usual definition). It is natural to extend this notion to Feynman *half-diagrams* by letting  $\omega(G^{\frac{1}{2}}) := 1 - \alpha|V(G^{\frac{1}{2}})| - (1 - \alpha)N_\phi(G^{\frac{1}{2}})$ , where  $N_\phi(G^{\frac{1}{2}})$  is the number of *true* external  $\phi$ -legs. Then Feynman half-diagrams

$G^{\frac{1}{2}}(\mathbb{T})$  associated to skeleton integrals of order  $n = |V(\mathbb{T})| < \lfloor 1/\alpha \rfloor$  are diverging if and only if  $\omega(G^{\frac{1}{2}}) > 0$ .

Consider a connected half-diagram  $g^{\frac{1}{2}} \subset G^{\frac{1}{2}}$  and its symmetric double  $g := (g^{\frac{1}{2}})^2$ . If  $g^{\frac{1}{2}}$  has uncontracted  $\phi$ -legs, then  $g^{\frac{1}{2}}$  is connected to its image in the mirror by some “bridge”, hence  $g$  is also connected;  $g$  is then called a *bilateral diagram*. Otherwise,  $g$  is made up of two *unilateral (full) diagrams*. As we shall see in Sect. 4, renormalizing  $g^{\frac{1}{2}}$  if  $g^{\frac{1}{2}}$  is divergent amounts to replacing  $\omega(g^{\frac{1}{2}})$  with  $\omega^*(g^{\frac{1}{2}}) = \omega(g^{\frac{1}{2}}) - 1$ . On the other hand, if  $g^{\frac{1}{2}}$  is convergent, no renormalization is performed, hence simply  $\omega^*(g^{\frac{1}{2}}) = \omega(g^{\frac{1}{2}})$ . Now power-counting must really be understood in terms of *half-diagrams*, which implies the following rules:

- if  $g^{\frac{1}{2}}$  is a unilateral diagram, then  $\omega^*(g^{\frac{1}{2}}) = \omega(g^{\frac{1}{2}})$  if  $g^{\frac{1}{2}}$  has external  $\phi$ -legs,  $\omega(g^{\frac{1}{2}}) - 1$  otherwise. Hence (considering that any non-empty diagram contains at least one line and two vertices)  $\omega^*(g^{\frac{1}{2}}) \leq -\alpha$  in any case;
- if  $g^{\frac{1}{2}}$  is connected to its image by some bridge, then  $\omega^*(g) = \omega(g)$  if  $g$  has external  $\phi$ -legs,  $\omega(g) - 2$  otherwise. But since  $g$  is symmetric,  $N_\phi(g)$  is *even* and  $|V(g)| \geq 4$ . Hence, in both cases,  $\omega(g) \leq -1 - 2\alpha$ .

These elementary power-counting arguments are essential for Sect. 4.

3.2. *The multiscale BPHZ algorithm.* We denote hereafter by  $\mathcal{F}^{div}(G^{\frac{1}{2}})$  the set of forests of diverging subgraphs of  $G^{\frac{1}{2}}$ . Equivalently,

$$\mathcal{F}^{div}(G) := \{g = (g^{\frac{1}{2}})^2 \mid g^{\frac{1}{2}} \in \mathcal{F}^{div}(G^{\frac{1}{2}})\}, \tag{3.3}$$

if  $G = (G^{\frac{1}{2}})^2$  is a symmetric graph, is the set of diverging *symmetric* subgraphs.

We refer to [28] or [35] for the whole paragraph.

**Definition 3.2** (Bogolioubov’s non-recursive definition of renormalization).

(i) *Let*

$$\mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}((\zeta_1, 0), \xi) := \sum_{\mathbb{F} \in \mathcal{F}^{div}(G(\mathbb{T}))} \prod_{g \in \mathbb{F}} (-\tau_g) A_{G^{\frac{1}{2}}(\mathbb{T})}((\zeta_1, 0), \xi). \tag{3.4}$$

(ii) *Define correspondingly, for  $\nu := D(f)\mu = \mathcal{F}^{-1}(f \cdot (\mathcal{F}\mu))$ , with  $\mu = \otimes_{v \in V(\mathbb{T})} dB_{x_v}(\ell(v))$  and  $f = f(\xi_1, \dots, \xi_n)$  such that  $\text{supp}(f) \subset \mathbb{R}_+^{\mathbb{T}}$ ,*

$$\begin{aligned} \phi'_\nu(\mathbb{T}) &:= (2\pi c_\alpha)^{-n/2} \int \prod_{v \in V(\mathbb{T})} dW_{\xi_v}(\ell(v)) \frac{e^{i\tau \zeta_1}}{[i\zeta_1]} d\zeta_1, \\ f(\xi) \mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_{ext} = (\zeta_1, 0), \xi_{ext} = (\xi_v)_{v \in V(\mathbb{T})}), \end{aligned} \tag{3.5}$$

so that, assuming all decorations  $(\ell(v))_{v \in V(\mathbb{T})}$  are distinct,

$$\begin{aligned} \text{Var}(\phi_v^t(\mathbb{T}) - \phi_v^s(\mathbb{T})) &= (2\pi c_\alpha)^{-n} \int \frac{d\xi_1}{\xi_1^2} |e^{it\xi_1} - e^{is\xi_1}|^2 \text{Var}\hat{\phi}_v^{\xi_1}(\mathbb{T}), \\ \text{Var}\hat{\phi}_v^{\xi_1}(\mathbb{T}) &= D(f)\mathcal{R}A_{G(\mathbb{T})}(\zeta_{ext} = (\zeta_1, 0)), \end{aligned} \tag{3.6}$$

where

$$D(f)\mathcal{R}A_{G(\mathbb{T})}(\zeta_1, 0) := \int \prod_{v \in V(\mathbb{T})} d\xi_v f^2(\xi) \left| \mathcal{R}A_{G^{\frac{1}{2}}}((\zeta_1, 0), \xi) \right|^2. \tag{3.7}$$

Now come two essential remarks, based on the fact that divergent subgraphs have *no* external  $\phi$ -leg by definition.

1. Since renormalization leaves  $\xi$ -momenta unchanged, one may consider the integration measure  $f(\xi) \prod_{v \in V(\mathbb{T})} dW_{\xi_v}(\ell(v))$  in Eq. (3.5) as a simple decoration of the vertices. In this sense  $\phi_v^t(\mathbb{T})$  may be considered as a *renormalized skeleton integral*, denoted by  $\mathcal{R}\overline{\text{Sk}}_v^t(\mathbb{T})$ .
2. Consider some multiple contraction  $\phi_v^t(\mathbb{T}; (i_1 i_2), \dots, (i_{2p-1}, i_{2p}))$  of  $\phi_v^t(\mathbb{T})$ . Then

$$\begin{aligned} \phi_v^t(\mathbb{T}; (i_1 i_2), \dots, (i_{2p-1} i_{2p})) &:= (2\pi c_\alpha)^{-n/2} \int \prod_{v \in V(\mathbb{T})} dW_{\xi_v}(\ell(v)) \frac{e^{it\xi_1}}{[i\xi_1]} d\xi_1 \\ f(\xi)\mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T}; (i_1 i_2), \dots, (i_{2p-1} i_{2p}))}(\zeta_{ext} = (\zeta_1, 0), \xi_{ext} = (\xi_v)_{v \in V(\mathbb{T})}). \end{aligned} \tag{3.8}$$

In other words, contractions and renormalization *commute*. This remark extends in a straightforward way to contractions between different trees as in Lemma 2.2 (2). This allows us to extend the BPHZ construction to contracted graphs. Namely, consider the Feynman diagram  $G = (G^{\frac{1}{2}})^2$  obtained by gluing two identical Feynman half-diagrams *with the same external structure*, i.e. such that  $\bar{z} = z$  whenever  $z$  is a *true* external leg. Then *all* (internal or external) momenta  $\zeta$  or  $\xi$  are equal to their image  $\bar{\zeta}$  or  $\bar{\xi}$  in the mirror. Now one defines

$$\mathcal{R}A_G(z_{ext}) = \int \prod_{\xi \mid \bar{\xi} \text{ uncontracted}} d\xi |\mathcal{R}A_{G^{\frac{1}{2}}}(z_{ext}, \xi)|^2, \tag{3.9}$$

where  $\mathcal{R}A_{G^{\frac{1}{2}}}(\cdot) = \sum_{\mathbb{F} \in \mathcal{F}^{div}(G)} \prod_{g \in \mathbb{F}} (-\tau_g) A_{G^{\frac{1}{2}}}(\cdot)$  is defined by the BPHZ formula as in Eq. (3.4).

3. Let  $G = (G^{\frac{1}{2}})^2$  be as in 2. As already mentioned,  $\mathcal{R}A_G(\cdot)$  differs from the usual BPHZ renormalized graph amplitude since (due to the square in the right-hand side of Eq. (3.9)) divergent bilateral subgraphs are in some sense renormalized *twice* from the point of view of power-counting.

Choose some constant  $M > 1$ . An *attribution of momenta*  $\mu$  for a Feynman diagram  $G$  is a choice of  $M$ -adic scale for each momentum of  $G$ , i.e. a function  $\mu : L(G) \cup L_{ext}(G) \rightarrow \mathbb{Z}$  and an associated restriction of the momentum  $|z_\ell| = |\zeta_\ell|$  or

$|\xi_\ell|$ ,  $\ell \in L(G) \cup L_{ext}(G)$  to the  $M$ -adic interval  $[M^{\mu(\ell)}, M^{\mu(\ell)+1})$ . Thus one may define, e.g. for a tree Feynman half-diagram  $G^{\frac{1}{2}}$ , compare with Eq. (2.3),

$$\begin{aligned}
 A_{G^{\frac{1}{2}}}^\mu(\zeta_{ext}, \xi_{ext}) &:= \delta(\zeta_{ext} + \xi_{ext}) \\
 &\prod_{v \in V(\mathbb{T}) \mid \xi_v \text{ uncontracted}} |\xi_v|^{\frac{1}{2}-\alpha} \cdot \int \prod_{q=1}^p |\xi_{(i_{2q-1}i_{2q})}|^{1-2\alpha} d\xi_{(i_{2q-1}i_{2q})} \\
 &\prod_{v \in V(\mathbb{T}) \setminus \{\text{roots}\}} \frac{1}{\zeta_v} \prod_{v \in V(\mathbb{T})} (\mathbf{1}_{|\zeta_v| \in [M^{\mu(\zeta_v)}, M^{\mu(\zeta_v)+1})}) (\mathbf{1}_{|\xi_v| \in [M^{\mu(\xi_v)}, M^{\mu(\xi_v)+1})}),
 \end{aligned} \tag{3.10}$$

where by definition  $M^{\mu(\xi_{i_{2q-1}})} = M^{\mu(\xi_{i_{2q}})} = M^{\mu(\xi_{(i_{2q-1}i_{2q})})}$  for contracted lines, and similarly for an arbitrary Feynman diagram  $G$ , compare with Eq. (2.5),

$$\begin{aligned}
 A_G^\mu(\mathbf{z}_{ext}) &:= \delta(\mathbf{z}_{ext}) \int \prod_{\ell \in L(G) \setminus L'(G)} dz_\ell \prod_{\ell \in L(G) \cup L_{ext}(G)} \mathbf{1}_{z_\ell \in [M^{\mu(\ell)}, M^{\mu(\ell)+1})} \\
 &\prod_{\ell \in L_\phi(G)} |\xi_\ell|^{1-2\alpha} \prod_{\ell \in L_\sigma(G)} \zeta_\ell^{-1}.
 \end{aligned} \tag{3.11}$$

Feynman diagrams with a fixed scale attribution are called *multiscale diagrams*. In the corresponding graphical representation (see below), vertices are split according to the scales of the lines attached to them.

**Definition 3.3** (Gallavotti-Nicolò tree). *Let  $G^j \subset G$ ,  $j \in \mathbb{Z}$  be the subdiagram with set of lines  $L(G^j) \cup L_{ext}(G^j) := \{\ell \in L(G) \cup L_{ext}(G); \mu(\ell) \geq j\}$ , and  $(G_k^j)_{k=1,2,\dots}$  the connected components of  $G^j$ .*

*The set of connected subgraphs  $(G_k^j)_{j,k}$  – called **local subgraphs** – makes up a tree of subgraphs of  $G$ , called a **Gallavotti-Nicolò tree**.*

Two instances of Gallavotti-Nicolò trees are represented on Figs. 5, 6. By shifting slightly the  $M$ -adic intervals, it is possible to manage to have both lines of highest momentum of any given vertex in the same interval.

**Definition 3.4.** *Let  $\mathbb{F} \in \mathcal{F}^{div}(G)$  be a forest of diverging subgraphs of  $G$ .*

- (i) *Let  $g \in G$  be a subgraph of  $G$ . Then  $g$  is **compatible** with  $\mathbb{F}$  if and only if  $\mathbb{F} \cup \{g\}$  is a forest.*
- (ii) *Assume  $g \in G$  is compatible with  $\mathbb{F}$ . We let  $g_{\mathbb{F}}^-$  be the ancestor of  $g$  in the forest of graphs  $\mathbb{F} \cup \{G\}$ , and  $g_{\mathbb{F}}^\uparrow$  be the union of its children, namely,*

$$g_{\mathbb{F}}^\uparrow = \bigcup_{h \subsetneq g, h \in \mathbb{F}} h. \tag{3.12}$$

- (iii) *Let  $\mu$  be a momentum scale attribution. The **dangerous forest**  $D^\mu(\mathbb{F}) \subset \mathbb{F}$  associated to the forest  $\mathbb{F}$  and the momentum scale attribution  $\mu$  is the sub-forest defined by*

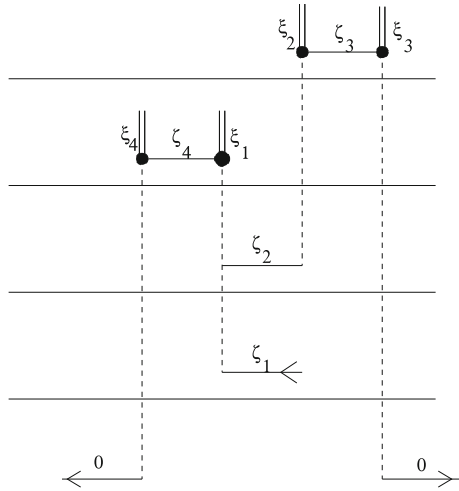


Fig. 5. A Gallavotti-Nicolò tree (Case 1)

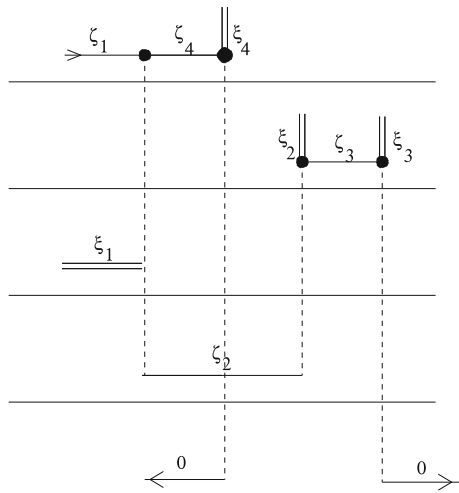


Fig. 6. A Gallavotti-Nicolò tree (Case 2)

$$\begin{aligned}
 &(g \in D^\mu(\mathbb{F})) \\
 &\iff \left( \min\{i_\ell(\mu) : \ell \in L(g \setminus g_{\mathbb{F}}^\uparrow)\} > \max\{i_\ell(\mu) : \ell \in L_{ext}(g) \cap L(g_{\mathbb{F}}^-)\} \right).
 \end{aligned}
 \tag{3.13}$$

(iv) Call the sub-forest  $ND^\mu(\mathbb{F}) := \mathbb{F} \setminus D^\mu(\mathbb{F}) \subset \mathbb{F}$  the **non-dangerous or harmless forest** associated to  $\mathbb{F}$  and  $\mu$ .

One can prove that  $ND^\mu \circ ND^\mu = ND^\mu$ . Hence

$$\mathcal{F}^{div}(G) = \cup_{\mathbb{F} \in \mathcal{F}^{div}(G)} | T_\mu(\mathbb{F}) = \mathbb{F} \{ \mathbb{F}' \supset \mathbb{F} : ND_\mu(\mathbb{F}') = \mathbb{F} \}.
 \tag{3.14}$$

One obtains the following classification of forests:

**Proposition 3.5.** *Let*

- (i)  $\text{Safe}^\mu(G) \subset \mathcal{F}^{\text{div}}(G)$  be the set of forests of diverging graphs which are invariant under the projection operator  $ND^\mu$  and thus harmless, namely,  $\text{Safe}^\mu(G) := \{\mathbb{F} \in \mathcal{F}^{\text{div}}(G) : ND^\mu(\mathbb{F}) = \mathbb{F}\}$ ;
- (ii)  $\text{Ext}^\mu(\mathbb{F}) \subset \mathcal{F}^{\text{div}}(G)$ , with  $\mathbb{F} \in \text{Safe}^\mu(G)$ , be the “maximal dangerous extension” of the harmless forest  $\mathbb{F}$  within the  $ND^\mu$ -equivalence class of  $\mathbb{F}$ , namely,  $\mathbb{F} \uplus \text{Ext}^\mu(\mathbb{F})$  is the maximal forest such that  $ND^\mu(\mathbb{F} \uplus \text{Ext}^\mu(\mathbb{F})) = \mathbb{F}$ .

Then:

(i)

$$(ND^\mu(\mathbb{F}') = \mathbb{F}) \iff (\mathbb{F} \subset \mathbb{F}' \subset \mathbb{F} \uplus \text{Ext}^\mu(\mathbb{F})); \tag{3.15}$$

(ii)  $\text{Ext}^\mu(\mathbb{F})$  is the set of subgraphs  $g \in G$ , compatible with  $\mathbb{F}$ , such that  $g \in D^\mu(\mathbb{F} \cup \{g\})$ .

In particular,  $\text{Ext}^\mu(\emptyset)$  is the forest of local subgraphs of  $G$ , or in other words the Gallavotti-Nicolò tree, see Definition 3.3.

**Corollary 3.6.**

$$\mathcal{R}A_{G^{\frac{1}{2}}} = \sum_{\mathbb{F} \in \mathcal{F}^{\text{div}}(G)} \mathcal{R}A_{G^{\frac{1}{2}}, \mathbb{F}}, \tag{3.16}$$

where

$$\mathcal{R}A_{G^{\frac{1}{2}}, \mathbb{F}} := \sum_{\mu \mid \mathbb{F} \in \text{Safe}^\mu(G)} \prod_{g \in \mathbb{F}} (-\tau_g) \prod_{h \in \text{Ext}^\mu(\mathbb{F})} (1 - \tau_h) A_{G^{\frac{1}{2}}}^\mu. \tag{3.17}$$

The BPHZ renormalization scheme is perfect in perturbative field theory, but experts of constructive field theory scorn it because it leads to unwanted combinatorial factors of order  $O(n!)$  for large Feynman diagrams with  $O(n)$  vertices, called renormalons (see [35], Eq. (1.1.12)), which ruin any hope of resumming the series of perturbations. These may be avoided by considering only *useful renormalizations* associated to *local subgraphs* in the sense of Definition 3.3, at the price of introducing scale-dependent renormalized coupling constants, see [35], §1.4. This gives another possible renormalization formula, which is however scale-dependent,

$$\mathcal{R}^{\text{useful}} A_{G^{\frac{1}{2}}} = \mathcal{R}A_{G^{\frac{1}{2}}, \emptyset} = \sum_{\mu} \prod_{j,k} (1 - \tau_{G_k^j}) A_{G^{\frac{1}{2}}}^\mu. \tag{3.18}$$

In our context, the whole discussion seems a priori pointless since (i) required Feynman diagrams have at most  $2\lfloor 1/\alpha \rfloor < \infty$  vertices; (ii) there are no coupling constants at all. Purely esthetic reasons plead for the scale-independent renormalization  $\mathcal{R}A_{G^{\frac{1}{2}}}$ . However, it may be that using  $\mathcal{R}^{\text{useful}} A_{G^{\frac{1}{2}}}$  instead of  $\mathcal{R}A_{G^{\frac{1}{2}}}$  gives better bounds for higher-order iterated integrals, which may after all also be rewritten as Feynman diagrams. Good bounds are notoriously difficult to obtain for general rough paths, which is a major problem when solving stochastic differential equations, see [13] for a general discussion, or [34] in the particular case of linear stochastic differential equations, in connection with the Magnus series.

### 4. Main Bound for Feynman Diagrams

This section is devoted to the proof by classical multi-scale arguments [28,35] of the following theorem.

**Definition 4.1** (Highest bridge) (see end of Subsect. 3.1). *Let  $g = (g^{\frac{1}{2}})^2$  be a multi-scale symmetric Feynman diagram, such that  $g^{\frac{1}{2}}$  is connected. If  $g^{\frac{1}{2}}$  has at least one uncontracted  $\xi$ -leg, then  $g$  is connected by “bridges”. Then the **highest bridge** is the uncontracted  $\xi$ -leg of highest scale.*

Let  $G = G(\mathbb{T}; (i_1 i_2) \dots (i_{2p-1} i_{2p}))$  be a symmetric tree Feynman diagram with  $2n$  vertices: then  $G = (G^{\frac{1}{2}})^2$  is made up of two disconnected unilateral Feynman diagrams if and only if  $G$  has been totally contracted, i.e.  $2p = n$ , in which case the momentum conservation condition implies that  $\zeta_{ext} = 0$ . Then there is no bridge and hence no highest bridge. In particular, if  $\mathbb{T}$  is connected, so that  $\zeta_{ext} = \{\zeta_1\}$ , the diagram evaluation  $A_G$  vanishes by symmetry (namely,  $\zeta_1 = 0$ , and the denominator  $\frac{1}{\zeta_2 \dots \zeta_{2n}}$  changes sign when all momenta are changed to their opposites). On the other hand, assuming  $n < \lfloor 1/\alpha \rfloor$ ,  $\omega(G^{\frac{1}{2}}) = 1 - n\alpha > 0$ , whereas  $\omega(G) = 1 - 2n\alpha < 0$  if  $\frac{1}{2\alpha} < n < \frac{1}{\alpha}$  for a connected, symmetric tree Feynman diagram.

Estimates for Feynman diagrams with  $2n$  vertices must be expressed in terms of a *reference scale*. It turns out that any (internal or external) momentum may be chosen as a reference scale when  $2n < \lfloor 1/\alpha \rfloor$ , because the renormalized amplitude is then both ultra-violet and infra-red convergent. On the other hand, diagrams with  $1/\alpha < 2n < 2/\alpha$  vertices (thus not unilateral) increase indefinitely when external momenta go to zero, and computations show that momenta above the highest bridge are too “loosely” attached to those below to control the infra-red behaviour of the whole diagram. In that case, the most appropriate reference scale is that of the highest bridge. This is the content of the following theorem.

**Theorem 4.1.** *Let  $G := G(\mathbb{T}; (i_1 i_2) \dots (i_{2p-1} i_{2p}))$  be a symmetric tree Feynman diagram with  $2n < 2/\alpha$  vertices. Write  $\zeta_{ext} = (\zeta_{r_1}, \dots, \zeta_{r_q}, \bar{\zeta}_{r_1}, \dots, \bar{\zeta}_{r_q})$  as in Lemma 2.2. Assume  $\zeta_{r_m} = \bar{\zeta}_{r_m}$ ,  $m = 1, \dots, q$ , so that each  $\zeta$ -momentum and each contracted  $\xi$ -momentum is equal to the corresponding  $\bar{\zeta}$ - or  $\bar{\xi}$ -momentum on the other side of the mirror.*

*Label  $\zeta_{ext}$  so that  $|\zeta_{r_1}| < \dots < |\zeta_{r_q}|$ .*

1. (Bilateral diagrams). *Assume  $G$  is bilateral, so  $G$  is connected. Let  $\xi_{ref}$  be the highest bridge. Fix  $j_{ref} := j(\xi_{ref})$  and sum over all scale attributions  $\mu$  such that  $\mu(\xi_{ref}) = j_{ref}$ . Replace one of the  $\xi$ -propagators,  $|\xi_1|^{1-2\alpha}$ , say, by  $|\xi_1|^{(1-2\alpha)-2n'\alpha}$  in the integrand, with  $n' \geq 0$ ,  $n + n' < 1/\alpha$ . Denote by  $\mathcal{R}A_{G\xi_1 n'}^{j_{ref}}(\zeta_{ext}) := \sum_{\mu} \mathcal{R}A_{G\xi_1 n'}^{\mu}(\zeta_{ext})$  the result. Then:*

$$\text{Var} \mathcal{R}A_{G\xi_1 n'}^{j_{ref}}(\zeta_{ext}) \lesssim M^{(1-2(n+n')\alpha)j_{ref}} \left( \frac{\min(|\zeta_{r_1}|, M^{j_{ref}})}{\max(|\zeta_{r_q}|, M^{j_{ref}})} \right)^{\alpha^-} \tag{4.1}$$

*whenever  $\alpha^- < \alpha$ .*

2. (Diagrams with  $2n < \lfloor 1/\alpha \rfloor$  vertices). Let  $G$  be indifferently a unilateral diagram, or a bilateral with  $2n < \lfloor 1/\alpha \rfloor$  vertices. Let  $\xi_{ref}$  be one of the  $\xi$ -lines of  $G$ . Fix  $j_{ref} := j(\xi_{ref})$  and sum over all scale attributions  $\mu$  such that  $\mu(\xi_{ref}) = j_{ref}$ . Replace the propagator  $|\xi_{ref}|^{1-2\alpha}$ , say, by  $|\xi_{ref}|^{(1-2\alpha)-2n'\alpha}$  in the integrand, with  $n' \geq 0, n + n' < 1/\alpha$ . Denote by  $\mathcal{RA}_{G\xi_{ref}n'}^{j_{ref}}(\zeta_{ext})$  the result. Then Eq. (4.1) holds.

- Remarks 1. The factor  $\left(\frac{\min(|\zeta_{r_1}|, M^{j_{ref}})}{\max(|\zeta_{r_q}|, M^{j_{ref}})}\right)^{\alpha^-}$  in Eq. (4.1) is obtained and shall be used as a product of *spring factors*,  $\prod_{m=1}^q \left|\frac{u_m}{u_{m+1}}\right|^{\alpha^-}$ , where  $|u_1| < \dots < |u_{q+1}|$  is the ordered list of momenta  $(M^{j_{ref}}, |\zeta_{r_1}|, \dots, |\zeta_{r_q}|)$ .
2. The supplementary factors  $|\xi_1|^{-2n'\alpha}$  or  $|\xi_{ref}|^{-2n'\alpha}$  may be seen as a ‘‘grafting’’ of another tree  $\mathbb{T}'$  on  $\mathbb{T}$ . It will be used for  $G = G_1$  and unrooted diagrams  $G'_i, i = 1, \dots, I'$  (see introduction to Sect. 5). The term ‘‘grafting’’ is only approximate since  $\mathbb{T}$  and  $\mathbb{T}'$  remain disjoint.

Using the Cauchy-Schwarz inequality in Eq. (2.4), this result yields immediately

**Corollary 4.2.** Consider a bilateral diagram  $G$ . Then

$$\begin{aligned} & \text{Var} \mathcal{RA}_{G\xi_1 n'}^{j_{ref}}(\zeta_{ext}, \bar{\zeta}_{ext}) \\ & \lesssim M^{(1-2(n+n')\alpha)j_{ref}} \left(\frac{\min(|\zeta_{r_1}|, M^{j_{ref}})}{\max(|\zeta_{r_q}|, M^{j_{ref}})}\right)^{\alpha^-/2} \left(\frac{\min(|\bar{\zeta}_{r_1}|, M^{j_{ref}})}{\max(|\bar{\zeta}_{r_q}|, M^{j_{ref}})}\right)^{\alpha^-/2}. \end{aligned} \quad (4.2)$$

*Proof of Theorem 4.1.* Let  $\mu$  be any attribution of momenta. We shall consider only *useful* renormalizations in the proof. Namely, as shown in [35], §1.3, the operations  $\prod_{g \in \mathbb{F}} (-\tau_g), \mathbb{F} \in \text{Saf}e^\mu(G)$ , see Eq. (3.17), are equivalent to displacing all external  $\zeta$ -legs to the same point, and do not change the power-counting rules in the proof.

Choose inductively, starting from the highest momentum scale, a subset of lines  $L'(G) \subset L(G)$  so that  $(z_\ell)_{\ell \in L(G_k^j) \setminus L'(G_k^j)}$ , where  $L'(G_k^j) := L'(G) \cap L(G_k^j)$ , make up a maximal set of independent momenta of the graph  $G_k^j$  shorn of its external legs. Note that  $G_k^j$  is necessarily symmetric. The degree of divergence  $\omega(G_k^j) = 1 - 2|V(G_k^j)|\alpha - (1 - \alpha)N_\phi(G_k^j)$  has been defined in Subsect. 3.1.

Assume for a moment that all  $(z_\ell)_{\ell \in L(G_k^j)}$  are of the same order,  $M^{j'}$ , say  $(j' \geq j)$ .

Then the previous power-counting arguments show that  $A_{G_k^j}$  is of order  $M^{j'\omega(G_k^j)}$ . If  $\omega(G_k^j) \geq 0$ , then clearly the sum  $\sum_{j'=j}^\infty M^{j'\omega(G_k^j)}$  diverges, so the sum over all momenta attributions diverges. On the other hand, if  $\omega(G_k^j) < 0$ , then the sum over all momenta attributions may still diverge because of so-called *sub-divergences* due to the higher subgraphs  $G_k^{j'}, j' > j$ ; the graph is a priori only *overall convergent*.

Let us now see how renormalization will make all symmetric subgraphs convergent. Consider any of the local subgraphs  $G_k^j$ . Assume  $\omega(G_k^j) > 0$ , so that  $G_k^j$  must be



renormalized. We introduce some notations for the sake of clarity. Let quite generally  $V_{ext}(g)$  be the set of external vertices of a graph  $g$ , and  $L_v$ , resp.  $L_{v,ext}$  ( $v \in V(G)$ ) be the set of internal, resp. external lines of  $g$  attached to  $v$ . Now, to each  $v \in V_{ext}(G_k^j)$ , one associates the unique line  $\ell'_v \in L'(G_k^j) \cap L_v(G_k^j)$ , and let  $z_v^* := \sum_{\ell \in L_{v,ext}(G_k^j) \setminus \{\ell'_v\}} z_\ell$  and  $z_{v,ext} := \sum_{\ell \in L_{v,ext}(G_k^j)} z_\ell$ , so that  $z_{\ell'_v} + z_v^* + z_{v,ext} = 0$ . Choose some arbitrary ordering of the external legs of  $G$ ,  $L_{ext}(G_k^j) = \{\ell_1, \dots, \ell_{|L_{ext}(G_k^j)|}\}$ . Renormalization changes only the values of the external momenta, so it acts really on the product  $A_{ext}(G_k^j) := \prod_{v \in V_{ext}(G_k^j)} |z_v^* + z_{v,ext}|^{\beta_{\ell'_v}}$ , with  $\beta_{\ell'_v} = 1 - 2\alpha$  or  $-1$ ,

$$\begin{aligned} A_{ext}(G_k^j) &\rightsquigarrow \mathcal{R}A_{ext}(G_k^j) := \prod_{v \in V_{ext}(G_k^j)} |z_v^* + z_{v,ext}|^{\beta_{\ell'_v}} - \prod_{v \in V_{ext}(G_k^j)} |z_v^*|^{\beta_{\ell'_v}} \\ &= \sum_{i=1}^{|L_{ext}(G_k^j)|} \int_0^1 z_{\ell_i} \partial_{z_{\ell_i}} \prod_{v \in V_{ext}(G_k^j)} |z_v^* + s z_{v,ext}|^{\beta_{\ell'_v}} ds. \end{aligned} \tag{4.3}$$

Now only one or two factors in the product over  $V_{ext}(G_k^j)$  depend on  $z_{\ell_i}$ ; the derivative  $\partial_{z_{\ell_i}}$  acting on each of these, generically denoted by  $|z_v^* + s z_{v,ext}|^{\beta_{\ell'_v}}$ , generates an extra multiplicative factor called *spring factor*,  $\frac{z_{\ell_i}}{z_v^* + s z_{v,ext}}$ , up to a constant. This spring factor is at most  $O(M^{\min \mu(G_k^j) - \max \mu(\partial G_k^j)})$ , where  $\min \mu(G_k^j)$  is the minimal scale index of all *internal* lines of  $G_k^j$ , and  $\max \mu(\partial G_k^j)$  the maximal scale index of all *external* lines of  $G_k^j$ .

We shall now rewrite  $\mathcal{R}A(G)$  by using the local graph decomposition of  $G$ . First, each factor  $M^{\beta_\ell}$ ,  $\ell \in L(G)$  may be rewritten as  $\prod_{(j,k); \ell \in L(G_k^j)} M^{\beta_\ell}$ . Similarly, the integration over the independent momenta yields  $\prod_{(j,k)} M^{|L(G_k^j)| - |V(G_k^j)| + 1}$ . Multiplying these two expressions, one gets  $\prod_{(j,k)} M^{\omega(G_k^j)}$ . Now, the spring factors due to renormalization contribute – due to the fact that  $A_G$  and  $\mathcal{R}A_G$  are *squared amplitudes* – a factor  $M^{-2}$  per scale until  $G_k^j$  absorbs one external line, so all together  $\prod_{(j,k); \omega(G_k^j) > 0} M^{-2}$ , where unilateral diagrams are only counted *once*. Finally, “grafting”  $|\xi_1|^{-2n'\alpha}$  into the graph is equivalent to subtracting  $2n'\alpha$  to all  $\omega(G^j)$  with  $j \leq j(\xi_1)$ , with  $\xi_1 = \xi_{ref}$  in case (2). All together, one has proved that

$$\mathcal{R}A^\mu(G \xrightarrow{\xi_1} n') \leq K^{2n} \prod_{(j,k)} M^{\omega^*(G_k^j)} \cdot \prod_{j \leq j(\xi_1)} M^{-2n'\alpha} \tag{4.4}$$

for some constant  $K$ , where  $\omega^*(G_k^j) = \omega(G_k^j)$  if  $\omega(G_k^j) < 0$ , and  $\omega(G_k^j) - 1$ , resp.  $\omega(G_k^j) - 2$  otherwise for unilateral, resp. bilateral subdiagrams, *except* for the total graph  $G$  which is not renormalized (having only external legs of zero momentum), so that  $\omega^*(G) = \omega(G) = 1 - 2n\alpha$ . As noted at the end of Subsect. 3.1,  $\omega^*(G_k^j) \leq -\alpha$ , resp.  $\leq -1 - \alpha$  for unilateral, resp. bilateral subdiagrams others than  $G$ . Summing up the divergence degrees of a given scale  $j$ ,  $\omega^*(G^j) = \sum_k \omega^*(G_k^j)$ , yields a quantity  $\leq -1 - \alpha$

if  $j \leq j_{ref}$  in Case 1 because the subdiagram containing the highest bridge is bilateral. Finally,  $\mathcal{R}A^\mu(G \xrightarrow{\xi_1} n') \leq K^{2n} \prod_j M^{\omega_{gr}^*(G^j)}$  if one lets  $\omega_{gr}^*(G^j) := \omega^*(G^j) - 2n'\alpha$  ( $j \leq j(\xi_1)$ ),  $\omega^*(G^j)$  ( $j > j(\xi_1)$ ) be the equivalent degree of divergence of  $G^j$  after renormalization and grafting.

Fix the scales of  $\mu$ , say,  $j_1 < j_2 < \dots, j_l = j_{max}$ , with  $j_{l_1} = j$ , and let  $j'_1 < \dots < j'_{q+1}$  be the scales of  $|\zeta_{r_1}|, \dots, |\zeta_{r_q}|, M^{j_{ref}}$  put into increasing order. Then the renormalized amplitude is bounded up to a constant by

$$\begin{aligned} & \sum_{j_1 > -\infty} M^{j_1 \omega_{gr}^*(G)} \left( \sum_{j_2 \geq j_1} M^{(j_2 - j_1) \omega_{gr}^*(G^{j_2})} \left( \dots \left( \sum_{j_l \geq j_{l-1}} M^{(j_l - j_{l-1}) \omega_{gr}^*(G^{j_l})} \dots \right) \right) \right) \\ & \leq \prod_{m=1}^q M^{-(j'_{m+1} - j'_m) \alpha^-} \cdot \sum_{j_1 > -\infty} M^{j_1 \omega_{gr}^*(G)} \left( \sum_{j_2 \geq j_1} M^{(j_2 - j_1) (\omega_{gr}^*(G^{j_2}) + \alpha^-)} \right. \\ & \quad \left. \left( \dots \left( \sum_{j_l \geq j_{l-1}} M^{(j_l - j_{l-1}) (\omega_{gr}^*(G^{j_l}) + \alpha^-)} \dots \right) \right) \right), \end{aligned} \tag{4.5}$$

the scale  $j_{l_1}$  being fixed, and the scales  $j_1, \dots, j_{l_1-1}$  constrained to be below  $j_{ref}$ . Since all  $\omega_{gr}^*(G^{j_i})$  except possibly  $\omega_{gr}^*(G) = \omega(G) - 2n'\alpha$  are  $\leq -\alpha < -\alpha^-$ , one may sum down to scale  $j_{l_1}$ , which (discarding the  $\alpha^-$ -spring prefactors) leads to the following bound:

$$\begin{aligned} & \sum_{j_1 > -\infty} M^{j_1 \omega_{gr}^*(G)} \left( \sum_{j_2 \geq j_1} M^{(j_2 - j_1) (\omega_{gr}^*(G^{j_2}) + \alpha^-)} \right. \\ & \quad \left. \left( \dots \left( \sum_{j_{l_1} \geq j_{l_1-1}} M^{(j_{l_1} - j_{l_1-1}) (\omega_{gr}^*(G^{j_{l_1}}) + \alpha^-)} \dots \right) \right) \right). \end{aligned} \tag{4.6}$$

However,  $j_{ref}$  is fixed, hence this expression must be computed as

$$\begin{aligned} & M^{j_{l_1} (\omega_{gr}^*(G^{j_{l_1}}) + \alpha^-)} \cdot \sum_{j_1 \leq j_{l_1}} M^{j_1 (\omega_{gr}^*(G) - \omega_{gr}^*(G^{j_2}) - \alpha^-)} \sum_{j_2 = j_1}^{j_{l_1}} M^{j_2 (\omega_{gr}^*(G^{j_2}) - \omega_{gr}^*(G^{j_3}))} \dots \\ & \quad \sum_{j_{l_1-1} = j_{l_1-2}}^{j_{l_1}} M^{j_{l_1-1} (\omega_{gr}^*(G^{j_{l_1-1}}) - \omega_{gr}^*(G^{j_{l_1}}))}, \end{aligned} \tag{4.7}$$

or (integrating from the lowest to the highest scale instead)

$$\begin{aligned} & M^{j_{l_1} (\omega_{gr}^*(G^{j_{l_1}}) + \alpha^-)} \cdot \sum_{j_{l_1-1} < j_{l_1}} M^{j_{l_1-1} (\omega_{gr}^*(G^{j_{l_1-1}}) - \omega_{gr}^*(G^{j_{l_1}}))} \\ & \quad \dots \sum_{j_2 < j_3} M^{j_2 (\omega_{gr}^*(G^{j_2}) - \omega_{gr}^*(G^{j_3}))} \sum_{j_1 < j_2} M^{j_1 (\omega_{gr}^*(G) - \omega_{gr}^*(G^{j_2}) - \alpha^-)}. \end{aligned} \tag{4.8}$$

This is convergent if and only if  $\omega_{gr}^*(G) - \omega_{gr}^*(G^{j_2}) - \alpha^-$ ,  $(\omega_{gr}^*(G) - \omega_{gr}^*(G^{j_2}) - \alpha^-) + (\omega_{gr}^*(G^{j_2}) - \omega_{gr}^*(G^{j_3})) = \omega_{gr}^*(G) - \omega_{gr}^*(G^{j_3}) - \alpha^-$ ,  $\dots$ ,  $\omega_{gr}^*(G) - \omega_{gr}^*(G^{j_{ref}})$  are  $> 0$ . This holds true since  $\omega_{gr}^*(G) - \omega_{gr}^*(G^{j_i}) - \alpha^- \geq (1 - 2(n + n')\alpha) + (1 + \alpha) - \alpha^- > 0$  in Case 1, and  $\geq (1 - 2n\alpha) + \alpha - \alpha^- > 0$  in Case 2. Hence one gets in the end a bound of order  $O(M^{(1-2n\alpha)j_{ref}})$ .  $\square$

*Examples* In the two examples below, we use as reference scale that of the external  $\zeta$ -leg, called  $\zeta_1$  here by reference to the root of the corresponding tree and let  $n' = 0$  to simplify. Taking for reference scale some internal  $\xi$ -line as in Theorem 4.1 would of course be possible, with minor differences.

1. Consider the first Gallavotti-Nicolò tree of Fig. 5. One may choose as integration variables  $L(G) \setminus L'(G) = \{\zeta_2, \zeta_3, \zeta_4\}$ , so that  $\xi_2 = \zeta_2 - \zeta_3$ ,  $\xi_3 = \zeta_3$ ,  $\xi_4 = \zeta_4$ ,  $\xi_1 = \zeta_1 - \zeta_2 - \zeta_4$ . Hence

$$A(G) = \int d\zeta_2 d\zeta_3 d\zeta_4 \left( |\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} |\zeta_3|^{-\frac{1}{2}-\alpha} \cdot |\zeta_1 - \zeta_2 - \zeta_4|^{\frac{1}{2}-\alpha} |\zeta_4|^{-\frac{1}{2}-\alpha} \cdot \zeta_2^{-1} \right)^2. \tag{4.9}$$

The subdiagrams with lines  $(\xi_2, \zeta_3, \xi_3)$ ,  $(\xi_4, \zeta_4, \xi_1)$  are renormalized by subtracting their value at  $\zeta_2 = 0$ , and then the larger subdiagram  $(\xi_4, \zeta_4, \xi_1, \zeta_2, \xi_2, \zeta_3, \xi_3)$  is further renormalized by subtracting its value at  $\zeta_1 = 0$ . Hence  $|\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha}$  is replaced with  $|\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} - |\zeta_3|^{\frac{1}{2}-\alpha} = O(\zeta_2 \cdot |\zeta_3|^{-\frac{1}{2}-\alpha})$ , and  $|\zeta_1 - \zeta_2 - \zeta_4|^{\frac{1}{2}-\alpha}$  by

$$\begin{aligned} & \left( |\zeta_1 - \zeta_2 - \zeta_4|^{\frac{1}{2}-\alpha} - |\zeta_1 - \zeta_4|^{\frac{1}{2}-\alpha} \right) - \left( |\zeta_2 + \zeta_4|^{\frac{1}{2}-\alpha} - |\zeta_4|^{\frac{1}{2}-\alpha} \right) \\ & = O(\zeta_1 \zeta_2 \cdot |\zeta_4|^{-3/2-\alpha}). \end{aligned} \tag{4.10}$$

Integrating the square of the renormalized amplitude yields (going *down* the scales *above*  $\zeta_1$ )

$$\begin{aligned} & \zeta_1^2 \left( \int_{|\zeta_1|}^\infty \zeta_2^2 d\zeta_2 \left( \int_{|\zeta_2|}^\infty |\zeta_4|^{-4-4\alpha} d\zeta_4 \left( \int_{|\zeta_4|}^\infty |\zeta_3|^{-2-4\alpha} d\zeta_3 \right) \right) \right) \\ & \lesssim \zeta_1^2 \int_{|\zeta_1|}^\infty \zeta_2^2 d\zeta_2 \int_{|\zeta_2|}^\infty |\zeta_4|^{-5-8\alpha} d\zeta_4 \\ & \lesssim \zeta_1^2 \int_{|\zeta_1|}^\infty |\zeta_2|^{-2-8\alpha} d\zeta_2 \\ & = O(|\zeta_1|^{1-8\alpha}). \end{aligned} \tag{4.11}$$

Note that the exponents are sufficiently *negative* so that these ultra-violet integrals converge.

The computation of the integrals yields the same bound as

$$\begin{aligned} & M^{j(\zeta_1)\omega^*(G)} \sum_{j(\zeta_2) > j(\zeta_1)} M^{(j(\zeta_2)-j(\zeta_1))\omega^*(G^{j(\zeta_2)})} \cdot \\ & \cdot \sum_{j(\zeta_4) > j(\zeta_2)} M^{(j(\zeta_4)-j(\zeta_2))\omega^*(G^{j(\zeta_4)})} \sum_{j(\zeta_3) > j(\zeta_4)} M^{(j(\zeta_3)-j(\zeta_4))\omega^*(G^{j(\zeta_3)})}, \end{aligned} \tag{4.12}$$

see Eq. (4.5), since  $\omega^*(G^{j(\zeta_3)}) = (1 - 4\alpha) - 2 = -1 - 4\alpha$ ,  $\omega^*(G^{j(\zeta_4)}) = -4 - 8\alpha$  (due to the fact that the subdiagram with lines  $(\xi_4, \zeta_4, \xi_1)$  is renormalized twice),  $\omega^*(G^{j(\zeta_2)}) = (1 - 8\alpha) - 2 = -1 - 8\alpha$  and  $\omega^*(G) = \omega(G) = 1 - 8\alpha$ .

2. Consider now the second Gallavotti-Nicolò tree, see Fig. 6. One may choose as integration variables  $L(G) \setminus L'(G) = \{\xi_1, \zeta_2, \zeta_3\}$ , so that  $\zeta_4 = \xi_4 = \zeta_1 - \zeta_2 - \xi_1$ ,  $\xi_2 = \zeta_2 - \zeta_3$ ,  $\xi_3 = \zeta_3$ . Hence

$$A(G) = \int d\xi_1 d\zeta_2 d\zeta_3 \left( |\zeta_1 - \zeta_2 - \xi_1|^{-\frac{1}{2}-\alpha} \cdot |\zeta_3|^{-\frac{1}{2}-\alpha} |\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} \cdot |\xi_1|^{\frac{1}{2}-\alpha} \cdot \zeta_2^{-1} \right)^2. \tag{4.13}$$

The subdiagram with lines  $(\zeta_1, \zeta_4, \xi_4)$  has one external  $\phi$ -leg,  $\xi_1$ , hence needs not be renormalized. On the other hand, the subdiagrams with lines  $(\zeta_1, \zeta_4, \xi_4, \xi_1)$  and  $(\xi_2, \zeta_3, \xi_3)$  must be renormalized by subtracting their values at  $\zeta_2 = 0$ . Hence  $|\zeta_1 - \zeta_2 - \xi_1|^{\frac{1}{2}-\alpha}$  is replaced with  $|\zeta_1 - \zeta_2 - \xi_1|^{-\frac{1}{2}-\alpha} - |\zeta_1 - \xi_1|^{-\frac{1}{2}-\alpha} = O(\zeta_2 \cdot |\zeta_1|^{-\frac{3}{2}-\alpha})$ , and  $|\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha}$  by  $|\zeta_2 - \zeta_3|^{\frac{1}{2}-\alpha} - |\zeta_3|^{\frac{1}{2}-\alpha} = O(\zeta_2 \cdot |\zeta_3|^{\frac{1}{2}-\alpha})$ .

Integrating the square of the renormalized amplitude yields (going up the scales below  $\zeta_1$ )

$$|\zeta_1|^{-3-2\alpha} \left( \int_0^{\zeta_1} |\zeta_3|^{-2-4\alpha} d\zeta_3 \left( \int_0^{\zeta_3} |\xi_1|^{1-2\alpha} d\xi_1 \left( \int_0^{|\xi_1|} \zeta_2^2 d\zeta_2 \right) \right) \right) = O(|\zeta_1|^{1-8\alpha}). \tag{4.14}$$

Note that the exponents are sufficiently positive so that these infra-red integrals converge.

In order to make the connection with Eq. (4.8), we replace  $(|\zeta_1|^{-3/2-\alpha} \zeta_2 \cdot |\xi_1|^{\frac{1}{2}-\alpha})^2 = (|\zeta_1|^{-1/2-\alpha} |\xi_1|^{1/2-\alpha} \cdot \frac{\zeta_2}{\xi_1})^2$  with  $(|\zeta_1|^{-1/2-\alpha} |\xi_1|^{1/2-\alpha} \cdot \frac{\zeta_2}{\xi_1})^2 = |\zeta_1|^{-1-2\alpha} |\xi_1|^{-1-2\alpha} \zeta_2^2$ . The reduced spring factor  $\frac{\zeta_2}{\xi_1}$  takes into account the difference between the *minimum scale* of the diagram with lines  $(\zeta_1, \zeta_4, \xi_4, \xi_1)$  and its external leg  $\zeta_2$ , corresponding to the lifetime of this diagram; it is the factor which is counted in the multi-scale estimates. The actual spring factor  $\frac{\zeta_2}{\xi_1}$ , which is better, is due to the difference of scales between the scale where the vertex connecting  $\zeta_1, \zeta_4, \xi_1$  and  $\zeta_2$  appears and the scale of the external leg  $\zeta_2$ . With this slight modification, one gets

$$|\zeta_1|^{-1-2\alpha} \int_0^{\zeta_1} |\zeta_3|^{-2-4\alpha} d\zeta_3 \int_0^{\zeta_3} |\xi_1|^{-1-2\alpha} d\xi_1 \int_0^{|\xi_1|} \zeta_2^2 d\zeta_2. \tag{4.15}$$

This is equivalent to the bound given in Eq. (4.8),

$$M^{j(\zeta_1)\omega^*(G^{j(\zeta_4)})} \sum_{j(\zeta_3) < j(\zeta_1)} M^{j(\zeta_3)(\omega^*(G^{j(\zeta_3)}) - \omega^*(G^{j(\zeta_1)}))} \cdot \sum_{j(\xi_1) < j(\zeta_3)} M^{j(\xi_1)(\omega^*(G^{j(\xi_1)}) - \omega^*(G^{j(\zeta_3)}))} \sum_{j(\zeta_2) < j(\xi_1)} M^{j(\zeta_2)(\omega^*(G) - \omega^*(G^{j(\xi_1)}))}, \tag{4.16}$$

since  $\omega^*(G^{j(\zeta_4)}) = \omega^*(G^{j(\zeta_1)}) = -1 - 2\alpha$ ,  $\omega^*(G^{j(\zeta_3)}) = -6\alpha - 2$ ,  $\omega^*(G^{j(\xi_1)}) = -8\alpha - 2$  and  $\omega^*(G) = \omega(G) = 1 - 8\alpha$ .

### 5. Proof of Hölder Regularity for Renormalized Skeleton Integrals

We want to prove that, for any indices  $(\ell(1), \dots, \ell(n))$  and  $n \leq \lfloor 1/\alpha \rfloor$ ,

$$\text{Var} J_B^{ts}(\ell(1), \dots, \ell(n)) \lesssim |t - s|^{2n\alpha}, \tag{5.1}$$

where  $J_B^{ts}$  is defined in Proposition 1.11 and Definition 3.2.

Consider some multiple contraction  $(i_1 i_2), \dots, (i_{2p-1} i_{2p})$  – assuming that  $\ell(i_1) = \ell(i_2), \dots, \ell(i_{2p-1}) = \ell(i_{2p})$  – and the associated contracted integral  $J_B^{ts}(\mathbb{T}; (i_1 i_2), \dots, (i_{2p-1} i_{2p}))$ , where  $\mathbb{T} = (\ell(1) \dots \ell(n))$ . By arguments which may be found in [31], §4.1 (see Eq. (4.4) in particular), denoting by  $\cdot$  the Wick product of Gaussian variables,

$$\text{Var} : J_B^{ts}(\mathbb{T}; (i_1 i_2), \dots, (i_{2p-1} i_{2p})) : \leq n! \cdot \text{Var} J_B^{ts}(\mathbb{T}'; (i_1 i_2), \dots, (i_{2p-1} i_{2p})), \tag{5.2}$$

where  $\mathbb{T}'$  has decorations  $(\ell'(1) \dots \ell'(n))$  such that  $\ell'(i) \neq \ell'(j)$  if  $i \neq j$  except if  $\{i, j\} = \{i_{2m-1}, i_{2m}\}$  is a pair contraction, as in Lemma 2.2. Hence, by Wick’s lemma, it suffices to prove that  $\text{Var} J_B^{ts}(\mathbb{T}'; (i_1 i_2), \dots, (i_{2p-1} i_{2p})) \lesssim |t - s|^{2n\alpha}$ .

By Eqs. (1.35) and (1.37),  $J_B^{ts}(\mathbb{T}'; (i_1 i_2), \dots, (i_{2p-1} i_{2p}))$  is a sum of terms of the form

$$\int d\xi \left[ \prod_{q=1}^p \mathcal{R}\Phi^{ts} \right] ((\mathbb{T}_q), \xi; (\mathbf{v}_q), (\mathbb{T}'_{q,j}); (i_1 i_2), \dots, (i_{2p-1} i_{2p})), \tag{5.3}$$

with (following the notations of Lemma 2.2)

$$\begin{aligned} & \left[ \prod_{q=1}^p \mathcal{R}\Phi^{ts} \right] (\cdot) \\ &= \left[ \delta \overline{\mathcal{R}\text{SkI}}^{ts} \cdot \overline{\mathcal{R}\text{SkI}}^s \right]_{\hat{\nu}(\xi)} \left( \prod_{q=1}^p \text{RoO}_{\mathbf{v}_q} \mathbb{T}_q, \prod_{q=1}^p \prod_j \mathbb{T}'_{q,j}; (i_1 i_2), \dots, (i_{2p-1} i_{2p}) \right). \end{aligned} \tag{5.4}$$

The contractions induce links between some of the trees  $(\mathbb{T}_q)_q, (\mathbb{T}'_{q,j})_{q,j}$ . The resulting connected components may be represented by Feynman graphs of two types: (i) “rooted” Feynman diagrams  $G_1, \dots, G_l$  containing some (possibly many) root part  $\text{RoO}_{\mathbf{v}_q} \mathbb{T}_q$ ; (ii) “unrooted Feynman diagrams”  $G'_1, \dots, G'_l$  containing only leaf parts of type  $\mathbb{T}'_{q,j}$ . It turns out that the inconvenient vertex-decorating characteristic function  $\mathbf{1}_{|\xi_1| \leq \dots \leq |\xi_n|}$  may be replaced with the following much simpler characteristic function  $f$ . Let  $G_1$  be the rooted diagram containing  $\xi_1$ . For every unrooted diagram  $G'_i$  of type (ii), choose some  $\xi$ -leg  $\xi'_i$  belonging to  $G'_i$  and let  $f_i := \mathbf{1}_{j(\xi'_i) \geq j(\xi_1)}$ . Then set  $f = f^{j(\xi_1)} := \prod_{i=1}^l f_i(\xi)$ . The integral  $\int d\xi \mathbf{1}_{|\xi_1| \leq \dots \leq |\xi_n|}(\cdot)$  in (1.35) is now replaced by a simple sum  $\sum_{j=-\infty}^{+\infty}(\cdot)$ , with  $j = j(\xi_1)$ , and  $\hat{\nu}(\xi)$  by a measure depending only on the scale  $j(\xi_1)$ ,

$$\begin{aligned}
 \nu^j &:= \sum_{j'_i \geq j, i=1, \dots, I'} \int d\xi_1 \dots d\xi_n \mathbf{1}_{j(\xi_i)=j} \left[ \prod_{i=1}^{I'} \mathbf{1}_{j(\xi'_i)=j'_i} \right] \otimes_{k=1}^n \mathcal{F}(\Gamma'(\ell \circ \sigma(k)))(\xi_k) \\
 &= \int_{M^j \leq |\xi_1| \leq M^{j+1}} d\xi_1 \int d\xi_2 \dots d\xi_n \left[ \prod_{i=1}^{I'} \mathbf{1}_{|\xi'_i| \geq M^j} \right] \otimes_{k=1}^n \mathcal{F}(\Gamma'(\ell \circ \sigma(k)))(\xi_k).
 \end{aligned}
 \tag{5.5}$$

Since  $f(\xi) \leq \mathbf{1}_{|\xi_1| \geq \dots \geq |\xi_n|}$ , the associated renormalized quantity

$$\sum_{j=-\infty}^{+\infty} \left[ \prod_{q=1}^p \mathcal{R}\Phi^{ts} \right] ((\mathbb{T}_q), j; (\mathbf{v}_q), (\mathbb{T}'_{q,j}); (i_1 i_2), \dots, (i_{2p-1} i_{2p})) \tag{5.6}$$

has a larger variance than the original one, Eq. (5.3), contributing to  $J_B^{ts}$ .

The purpose of this section is to prove the estimates

$$\text{Var} \left( \sum_{j=-\infty}^{+\infty} \left[ \prod_{q=1}^p \mathcal{R}\Phi^{ts} \right] ((\mathbb{T}_q), j; (\mathbf{v}_q), (\mathbb{T}'_{q,j}); (i_1 i_2), \dots, (i_{2p-1} i_{2p})) \right) \lesssim |t - s|^{2n\alpha}, \tag{5.7}$$

from which Theorem 0.1 follows. They are a simple consequence of Theorem 4.1 and of the following two lemmas.

**Lemma 5.1** (Bound for bilateral “rooted” diagrams). (see Lemma 2.2 (2) for notations). *Let  $q \geq 1$  and  $q' \geq 0$ ,  $n := q + q'$ , and  $n' \geq 0$  such that  $n + n' < \lfloor 1/\alpha \rfloor$ . Rename  $(\zeta_{r_1}, \dots, \zeta_{r_q}, \zeta_{r'_1}, \dots, \zeta_{r'_{q'}})$ , resp.  $(\bar{\zeta}_{r_1}, \dots, \bar{\zeta}_{r_q}, \bar{\zeta}_{r'_1}, \dots, \bar{\zeta}_{r'_{q'}})$ , as  $\zeta_1, \dots, \zeta_n$ , resp.  $\tilde{\zeta}_1, \dots, \tilde{\zeta}_n$ , so that  $|\zeta_1| < \dots < |\zeta_n|$  and  $|\tilde{\zeta}_1| < \dots < |\tilde{\zeta}_n|$ , and let  $\zeta_{ext} := \sum_{m=1}^n \zeta_m$ ,  $\tilde{\zeta}_{ext} := \sum_{m=1}^n \tilde{\zeta}_m$ .*  
 Let

$$\begin{aligned}
 I(j_{ref}) &:= \int_{|\xi_{ext}| \leq M^{j_{ref}}} d\xi_{ext} \int d\zeta_{ext} \int d\tilde{\zeta}_{ext} \delta(\zeta_{ext} = \xi_{ext}) \delta(\tilde{\zeta}_{ext} = -\xi_{ext}) \\
 &M^{(1-2(n+n')\alpha)j_{ref}} \left( \frac{\min(|\zeta_1|, M^{j_{ref}})}{\max(|\zeta_n|, M^{j_{ref}})} \right)^{\alpha^{-}/2} \left( \frac{\min(|\tilde{\zeta}_1|, M^{j_{ref}})}{\max(|\tilde{\zeta}_n|, M^{j_{ref}})} \right)^{\alpha^{-}/2} \\
 &\left( \prod_{m=1}^q \left| \frac{e^{it\zeta_{r_m}} - e^{is\zeta_{r_m}}}{\zeta_{r_m}} \right| \prod_{m'=1}^{q'} \left| \frac{1}{\zeta_{r'_{m'}}} \right| \right) \left( \prod_{m=1}^q \left| \frac{e^{it\bar{\zeta}_{r_m}} - e^{is\bar{\zeta}_{r_m}}}{\bar{\zeta}_{r_m}} \right| \prod_{m'=1}^{q'} \left| \frac{1}{\bar{\zeta}_{r'_{m'}}} \right| \right).
 \end{aligned}
 \tag{5.8}$$

Then

$$\sum_{j_{ref}=-\infty}^{+\infty} I(j_{ref}) \lesssim |t - s|^{2(n+n')\alpha}. \tag{5.9}$$

*Proof* (i) ( $M^{j_{ref}} < \frac{1}{|t-s|}$ ). Integrate first over the variables *larger* than  $\frac{1}{|t-s|}$  – which defines the *ultra-violet range* in this situation –, say  $|\zeta_n| > \dots > |\zeta_{k+1}|$  and  $|\tilde{\zeta}_n| > \dots > |\tilde{\zeta}_{k+1}|$ . Let for instance  $|\zeta_n| > |\tilde{\zeta}_n|$ . The integral  $\int_{|\zeta_n| > |\zeta_{n-1}|} \frac{d\zeta_n}{|\zeta_n|} \mathbf{1}_{|\zeta_{ext}| \leq M^{j_{ref}}}$  is bounded up to a constant by  $\frac{M^{j_{ref}}}{|\zeta_{n-1}|}$ , and (due to momentum conservation) the integral over  $\tilde{\zeta}_n$  is not performed. Then

$$\int_{|\tilde{\zeta}_{k+1}| > \frac{1}{|t-s|}} \frac{d\tilde{\zeta}_{k+1}}{|\tilde{\zeta}_{k+1}|} \dots \int_{|\tilde{\zeta}_{n-1}| > |\tilde{\zeta}_{n-2}|} \frac{d\tilde{\zeta}_{n-1}}{|\tilde{\zeta}_{n-1}|} \cdot |\tilde{\zeta}_{n-1}|^{-1} = O(|t-s|), \tag{5.10}$$

and similarly for the untilded integrals, with an extra  $M^{j_{ref}}$  factor.

Integrating in the *infra-red range*, namely, over the variables smaller than  $\frac{1}{|t-s|}$  (if any) yields then, using the  $\alpha^-$ -spring factors,

$$|t-s|^{\alpha^-/2} \int_{|\tilde{\zeta}_k| < \frac{1}{|t-s|}} \frac{d\tilde{\zeta}_k}{|\tilde{\zeta}_k|} \dots \int_{|\tilde{\zeta}_2| < |\tilde{\zeta}_3|} \frac{d\tilde{\zeta}_2}{|\tilde{\zeta}_2|} \int_{|\tilde{\zeta}_1| < |\tilde{\zeta}_2|} \frac{d\tilde{\zeta}_1}{|\tilde{\zeta}_1|} \cdot |\tilde{\zeta}_1|^{\alpha^-/2} = O(1), \tag{5.11}$$

and similarly for the untilded integrals.

The above arguments do not hold if *all* variables are smaller than  $\frac{1}{|t-s|}$ . Then one must use the hypothesis that at least one of the  $\zeta$ -variables, say,  $\zeta_k$ , is accompanied by the factor  $|\frac{e^{it\zeta_k} - e^{is\zeta_k}}{\zeta_k}| = O(|t-s|)$  instead of  $O(\frac{1}{|\zeta_k|})$ , and similarly for some  $\tilde{\zeta}$ -variable, say,  $\tilde{\zeta}_k$ . One computes

$$\int_{|\tilde{\zeta}_{k-1}| < |\tilde{\zeta}_k|} \frac{d\tilde{\zeta}_{k-1}}{|\tilde{\zeta}_{k-1}|} \dots \int_{|\tilde{\zeta}_2| < |\tilde{\zeta}_3|} \frac{d\tilde{\zeta}_2}{|\tilde{\zeta}_2|} \int_{|\tilde{\zeta}_1| < |\tilde{\zeta}_2|} \frac{d\tilde{\zeta}_1}{|\tilde{\zeta}_1|} \cdot |\tilde{\zeta}_1|^{\alpha^-/2} = O(|\tilde{\zeta}_k|^{\alpha^-/2}), \tag{5.12}$$

and similarly for the untilded integrals, and

$$|\tilde{\zeta}_n|^{-1-\alpha^-/2} \int_{|\tilde{\zeta}_{n-1}| < |\tilde{\zeta}_n|} \frac{d\tilde{\zeta}_{n-1}}{|\tilde{\zeta}_{n-1}|} \int_{|\tilde{\zeta}_{n-2}| < |\tilde{\zeta}_{n-1}|} \frac{d\tilde{\zeta}_{n-2}}{|\tilde{\zeta}_{n-2}|} \dots \int_{|\tilde{\zeta}_k| < |\tilde{\zeta}_{k+1}|} d\tilde{\zeta}_k |t-s| \cdot |\tilde{\zeta}_k|^{\alpha^-/2} = O(|t-s|), \tag{5.13}$$

$$\int_{|\zeta_n| < \frac{1}{|t-s|}} \frac{d\zeta_n}{|\zeta_n|^{1+\alpha^-/2}} \mathbf{1}_{|\zeta_{ext}| \leq M^{j_{ref}}} \int_{|\zeta_{n-1}| < |\zeta_n|} \frac{d\zeta_{n-1}}{|\zeta_{n-1}|} \dots \int_{|\zeta_k| < |\zeta_{k+1}|} d\zeta_k |t-s| \cdot |\zeta_k|^{\alpha^-/2} = O(M^{j_{ref}} |t-s|). \tag{5.14}$$

All together (in both cases):  $I(j_{ref}) \lesssim M^{(1-2(n+n')\alpha)j_{ref}} \cdot |t-s| \cdot (M^{j_{ref}} |t-s|) = M^{(2-2(n+n')\alpha)j_{ref}} |t-s|^2$ . Since by assumption  $M^{j_{ref}} < \frac{1}{|t-s|}$ , this sums up to  $\sum_{j_{ref} < \log_M \frac{1}{|t-s|}} I(j_{ref}) \lesssim |t-s|^{2(n+n')\alpha}$ .

- (ii) ( $M^{j_{ref}} > \frac{1}{|t-s|}$ ) The arguments of (i) may be repeated word for word, except that the ultra-violet range is now defined by  $|\zeta|, |\tilde{\zeta}| > M^{j_{ref}}$ . Then  $I(j_{ref}) \lesssim M^{-2(n+n')\alpha j_{ref}}$  and  $\sum_{j_{ref} > \log_M \frac{1}{|t-s|}} M^{-2(n+n')\alpha j_{ref}} = O(|t-s|^{2(n+n')\alpha})$ .

The mixed cases, when e.g.  $\frac{1}{|t-s|}$  is large with respect to the  $\zeta$ -variables but small with respect to the  $\tilde{\zeta}$ -variables, are treated in the same way and left to the reader.  $\square$

**Lemma 5.2** (Bound for bilateral “unrooted” diagrams) (same notations as in Lemma 5.1). *Assume  $q = 0$ , so that  $n' := q' < \lfloor 1/\alpha \rfloor$ . Then*

$$I(j_{ref}) \lesssim M^{-2n'\alpha j_{ref}}. \tag{5.15}$$

Lemma 5.2 has already been proved, as part of Lemma 5.1 (ii).

These two lemmas extend with very minor changes to unilateral diagrams.

We may now easily finish the proof of the estimates Eq. (5.7). Lemma 5.2 yields an estimate for renormalized skeleton integrals associated to “unrooted” diagrams  $G'_i, i = 1, \dots, I'$ , where some reference scale  $j'_i = j(\xi_{ref})$  has been chosen according to the rules of Theorem 4.1. Summing over all scales  $j'_i \geq j(\xi_1)$  – where  $|\xi_1|$  is the smallest  $\xi$ -variable, as in the introduction to the present section – yields  $O(M^{-2n'\alpha j(\xi_1)})$ . Then the product of factors  $M^{-2n'\alpha j(\xi_1)} = \prod_{i=1}^{I'} M^{-2n'\alpha j(\xi_1)}$  associated to all unrooted diagrams  $(G'_i)_{i=1, \dots, I'}$  is “grafted” into the rooted diagram  $G_1$  containing  $\xi_1$ .

Turn now to the rooted diagrams  $G_1, \dots, G_I$ . Choose  $j(\xi_1)$  as reference scale for  $G_1$  if  $G_1$  is unilateral; choose some reference scale according to the rules of Theorem 4.1 for  $G_2, \dots, G_I$ , and for  $G_1$  if  $G_1$  is bilateral. Then apply Lemma 5.2.  $\square$

Let us add two comments to finish with.

1. The most “tricky” part in the story is obviously the infra-red behaviour of Feynman diagrams, particularly when  $n$  is large, i.e.  $n > \frac{1}{2\alpha}$ . The infra-red convergence of these “large” diagrams is ensured by the somewhat complicated interplay between half-diagrams and full diagrams, the key point being the existence of small enough spring factors. In a previous attempt, we tried to use the BPHZ renormalization scheme associated to the Connes-Kreimer algebra  $\mathbf{H}$ , instead of considering the associated Feynman half-diagrams. The coproduct of  $\mathbf{H}$  is much simpler than that of Feynman diagrams. Unfortunately, some “large” diagrams are infra-red divergent.
2. The results of this article may probably be extended to an arbitrary  $\alpha$ -Hölder path  $\Gamma$ , by rewriting  $\Gamma$  as  $I_{\alpha^-}(D_{\alpha^-}(\Gamma))$ , where  $I_{\alpha^-}$ , resp.  $D_{\alpha^-}$  are fractional integration, resp. derivation operators, and  $\alpha^- < \alpha$ . Then what one should really do is renormalize iterated fractional integration operators, while  $\Gamma$  would only play a “decorative” rôle; see Remark 1 after Definition 3.2. The construction would make use of Besov norms as in [33].

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