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# Renormalization Horseshoe and Rigidity for Circle Diffeomorphisms with Breaks

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**Abstract:** In this paper we build the renormalization horseshoe for the circle homeomorphisms, which are $C^{2+\alpha}$-smooth everywhere except for one point, and at that point have a jump in first derivative. We also show that two such homeomorphisms are $C^1$-smoothly conjugate for a certain class of rotation numbers, which include non-Diophantine numbers with arbitrarily high rate of growth.

## 1. Introduction

The main aim of this paper is to present an almost complete renormalization theory for circle homeomorphisms with break points. Nonlinear circle homeomorphisms with breaks were introduced about 20 years ago. Those are homeomorphisms which are smooth everywhere except at a single point where the first derivative has a jump discontinuity. The motivation was mainly based on a rich renormalization behavior such maps exhibit, having many properties normally associated with "criticality": singularity of the invariant measure, nontrivial scalings, prevalence of periodic trajectories in one-parameter families etc (see [1] for more details). Another motivation is related to a recent interest in so-called generalized, or nonlinear, interval exchange transformations [2]. It is well-known that circle rotation can be considered as interval exchange of two intervals. The "generalized" version of it will be a circle homeomorphism. While matching conditions for the images of the end points are natural, the matching conditions on the derivatives are not. Thus one ends up with a circle homeomorphism with two break points. However, since two break points belong to the same trajectory, the map can be easily conjugated to a break map with exactly one break point. This shows that break homeomorphisms can be considered as first non-trivial examples of generalized interval exchanges. While rigidity analysis in the break case can provide certain intuition for the case of "genuine" interval exchanges, we must also emphasize a significant difference. Namely, two piecewise-smooth irrational circle homeomorphisms are topologically conjugate provided that they are combinatorially equivalent. This is in general not true for interval exchange transformations [3]. In other words, Denjoy's theory holds only in the case of circle homeomorphisms.

Renormalization approach, which plays a central role in the modern theory of dynamical systems, can be considered as the main tool in establishing rigidity results. The rigidity theory aims at proving the smoothness of conjugacy between two dynamical systems, which a priori are only topologically equivalent. In the context of circle dynamics it means that two maps with the same irrational rotation number and the same local structure of their singular points must be smoothly conjugate to each other. In most cases one also has to impose certain Diophantine conditions on the rotation number. The main step in proving rigidity is convergence of renormalizations. Herman's theory [4] deals with the classical case of smooth diffeomorphisms, where renormalizations approach a family of linear maps with slope 1. On the other end, a highly nontrivial renormalization behavior was discovered for critical circle maps. There is strong numerical evidence that renormalizations behave universally for maps with the same order of critical points and the same irrational rotation number, although at present rigorous results are available only in the case when the order of the critical points is an odd integer ([5,6]). By a rather standard argument, the convergence of renormalizations implies the smoothness of conjugacy for rotation numbers of bounded type, and can be extended further to a broader class of rotation numbers. In the case of $C^\infty$-smooth circle diffeomorphisms this class consists of Diophantine numbers [7]. It is interesting that in the case of critical circle maps the $C^1$-rigidity holds for all irrational rotation numbers [8].

Circle maps with breaks form another interesting setting. In this case the local structure of the break point is determined by fixing the ratio of left and right derivatives at it. This parameter, which is obviously invariant under smooth changes of coordinates, play the same role as the order of critical point for critical circle maps. Namely, one can expect that renormalizations of two maps with the same irrational rotation number and the same ratio of the corresponding left and right derivatives, are getting exponentially close to each other. It was known for a long time [9] that in the case of maps with breaks renormalizations converge to a two-parameter family of linear-fractional maps. This essentially reduces analysis to a study of renormalizations for this canonical family. Such an analysis is the main result of the present paper. We prove that the corresponding two-dimensional transformation has strong hyperbolic properties, which allows us to construct the full renormalization horseshoe. Previously, hyperbolicity has been proved only in the case of rotation numbers with periodic continued fraction expansion [1]. Such maps correspond to periodic orbits for renormalizations. Here we prove that maps with a given irrational rotation number form a stable manifold in terms of renormalizations. Our approach is based on two ingredients. We first construct cones containing the stable manifolds, and then use the exact symmetry to prove uniform convergence of renormalizations. Finally, we prove that under certain technical conditions on the rotation numbers renormalizations for general diffeomorphisms with breaks converge to the renormalization horseshoe, which implies rigidity for the corresponding rotation numbers.

The structure of this paper is following. In Sect. 2 we present the basic renormalization construction for orientation-preserving circle homeomorphisms and, more generally, for so-called commuting pairs. In Sect. 3 we define circle diffeomorphisms with breaks and describe the properties of their renormalizations. Section 4 gives the complete description of the uniformly hyperbolic horseshoe in the two-dimensional invariant manifold consisting of linear-fractional commuting pairs, to which the sequences of renormalizations converge. In Sect. 5 we fit together the hyperbolic dynamics inside the linear-fractional manifold and the convergence of renormalizations to that manifold in order to prove the exponential convergence of renormalizations; in Sect. 6 we derive the $C^1$-rigidity result for circle diffeomorphisms with breaks from the convergence of renormalizations by means of the Conditional Theorem proved in [8].

The results presented in this paper were first announced in [10], see also [11,12].

By the end of this Introduction, the authors feel obliged to issue a warning that the mathematics of this paper includes lots of lengthy rational expressions. We used a computer algebra system to maintain them effectively, and advise an interested reader to do the same.

## 2. Two Notions of Renormalizations

*2.1. Renormalizations of circle homeomorphisms.* Let $T$ be an orientation preserving homeomorphism of the unit circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Its rotation number $\rho$ is defined up to an integer constant as $\rho = \rho(T) = \lim_{m\to\infty}(L_T^m x)/m$, where $L_T$ is a lift of $T$ from $\mathbb{T}^1$ to the real line $\mathbb{R}$ (the limit does not depend on $x \in \mathbb{R}$). The notion of renormalization we use is related to the expansion of $\rho$ in the form of *continued fraction*. The latter is defined as

$$\rho = [k_1, k_2, \ldots, k_n, \ldots] = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\cfrac{\cdots}{k_n + \cfrac{1}{\cdots}}}}}, \tag{1}$$

where the sequence of positive integers $k_n$, $n \geq 1$, called *partial quotients*, can be either finite or infinite, in which two cases the right-hand side of (1) corresponds to either a rational number which can be calculated directly, or an irrational number given by a limit for the sequence of *rational convergents* (or, just *convergents*) $p_n/q_n = [k_1, k_2, \ldots, k_n]$ (here $p_n$ and $q_n$ are co-prime positive integers). The continued fraction expansion for rotation number

$$\rho(T) = [k_1, k_2, \ldots, k_n, \ldots] \tag{2}$$

is defined by (1) almost uniquely, where 'almost' concerns the rational numbers: they can be expanded in two different ways, namely as $[k_1, k_2, \ldots, k_n]$, $k_n \geq 2$, or as $[k_1, k_2, \ldots, k_n - 1, 1]$. For convenience, we also define $p_0 = 0$, $q_0 = 1$ and $p_{-1} = 1$, $q_{-1} = 0$.

Given a circle homeomorphism $T$ with irrational $\rho(T)$, one may consider the *marked trajectory* (i.e. the trajectory of the marked point) $\xi_i = T^i \xi_0 \in \mathbb{T}^1$, $i \geq 0$, and pick out of it the sequence of the *dynamical convergents* $\xi_{q_n}$, $n \geq 0$, indexed by the denominators of the consecutive rational convergents to $\rho(T)$. The well-understood arithmetical properties of rational convergents and the combinatorial equivalence of all the circle homeomorphisms with a fixed irrational rotation number imply that the dynamical convergents approach the marked point, alternating their order in the following way:

$$\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \cdots < \xi_{q_{2m+1}} < \cdots < \xi_0 < \cdots < \xi_{q_{2m}} < \cdots < \xi_{q_2} < \xi_{q_0}$$

(here we conventionally use $\xi_{q_{-1}} = \xi_0 - 1$). Let us define the $n^{\text{th}}$ *renormalization segment* $\Delta_0^{(n)}$ as the circle arc $[\xi_0, \xi_{q_n}]$ if $n$ is even and $[\xi_{q_n}, \xi_0]$ if $n$ is odd.

The iterates $T^{q_n}$ and $T^{q_{n-1}}$ restricted to $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ respectively are nothing else but two continuous components of the first-return map for $T$ on the segment $\overline{\Delta}_0^{(n-1)} = \Delta_0^{(n-1)} \cup \Delta_0^{(n)}$. The consecutive images of $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ until the return to $\overline{\Delta}_0^{(n-1)}$ cover the whole circle without overlapping beyond their endpoints, thus forming the $n^{\text{th}}$ *dynamical partition* of $\mathbb{T}^1$:

$$\mathbb{P}_n = \{\Delta_i^{(n-1)}, 0 \le i < q_n\} \cup \{\Delta_i^{(n)}, 0 \le i < q_{n-1}\}, \tag{3}$$

where $\Delta_i^{(n)}$ stands for $T^i \Delta_0^{(n)}$. In particular, we use the fact that

$$\sum_{i=0}^{q_n-1} |\Delta_i^{(n-1)}| < 1.$$

For $n \ge 0$, the $n^{\text{th}}$ *renormalization* of an orientation-preserving homeomorphism $T$ of the unit circle $\mathbb{T}^1$ with rotation number (2) with respect to a marked point $\xi_0 \in \mathbb{T}^1$ is a pair of functions $(f_n, g_n)$ obtained from the mappings $T^{q_n}$ and $T^{q_{n-1}}$, restricted to $\Delta_0^{(n-1)}$ and $\Delta_0^{(n)}$ respectively, by *rescaling* the coordinates:

$$f_n = \mathfrak{r}_n \circ T^{q_n} \circ \mathfrak{r}_n^{-1} \qquad g_n = \mathfrak{r}_n \circ T^{q_{n-1}} \circ \mathfrak{r}_n^{-1},$$

where $\mathfrak{r}_n$ is an affine change of coordinates that sends $\xi_{q_{n-1}}$ to $-1$ and $\xi_0$ to 0.

*2.2. Renormalization of commuting pairs.* The sequence of renormalizations defined in previous paragraph may be obtained by means of another approach.

We say that two real functions $F$ and $G$ form a *commuting pair* $(F, G)$, if the following conditions hold: $F(0) \ge 0$, $G(0) \le 0$, $F$ and $G$ are defined, continuous and strictly increasing on the segments $[G(0), 0]$ and $[0, F(0)]$ respectively, and commute at zero, i.e. $F(G(0)) = G(F(0))$.

The *Farey iteration* of a commuting pair $(F, G)$ such that $F(G(0)) < 0$ is the commuting pair $(F, F \circ G)$.

A commuting pair $(F, G)$ is said to be *normalized* if $G(0) = -1$, so that $F$ is defined on $[-1, 0]$. For a pair of functions $(F, G)$ such that $G(0) \ne 0$ we define its *normalization* as $\overline{(F, G)} = (\nu^{-1} \circ F \circ \nu, \nu^{-1} \circ G \circ \nu)$, where $\nu(z) = -G(0)z$. A normalized commuting pair $(F, G)$ is called *degenerate* if $F(0) = 0$ (then $G$ is defined at zero only, and $G(0) = F(-1) = -1$).

*The main notations for sets.* The set of all normalized commuting pairs $(F, G)$ (in the sequel we call them just 'pairs') is denoted by $\mathfrak{P}$. Let $\dot{\mathfrak{P}} = \mathfrak{P} \backslash \mathfrak{Z}$, where $\mathfrak{Z}$ is the set of all degenerate pairs. Define the 'layers' $\Pi_k \subset \mathfrak{P}$, $0 \le k < \infty$, by the condition $F^k(-1) \le 0 < F^{k+1}(-1)$; their 'interiors' $\dot{\Pi}_k$, $k \ge 0$, by the condition $F^k(-1) < 0 < F^{k+1}(-1)$; and the separating 'hypersurfaces' $\gamma_{1/k}$, $1 \le k < \infty$, by the condition $F^k(-1) = 0$. Also define the set $\Pi_\infty$ of all pairs $(F, G) \in \mathfrak{P}$ such that $F^i(-1) < 0$ for all $i \ge 0$, the set $\gamma_0 = \gamma_{1/\infty}$ of all pairs $(F, G) \in \Pi_\infty$ such that $\min\{F(z) - z\} = 0$, and the set $\dot{\Pi}_\infty$ such that $\min\{F(z) - z\} < 0$. In terms of graphs of functions, the sets $\Pi_\infty$, $\gamma_0$ and $\dot{\Pi}_\infty$ correspond to the cases when the graph of $F$ has a common point with, touches, or crosses the diagonal (i.e. the graph of the identity map) respectively. Obviously, the decomposition $\mathfrak{P} = \bigcup_{0 \le k \le \infty} \Pi_k$ is disjoint, as are the decompositions $\Pi_k = \dot{\Pi}_k \cup \gamma_{1/k}$, $1 \le k \le \infty$ (and for $k = 0$ we have $\Pi_0 = \dot{\Pi}_0$). Let $\mathfrak{S} = \mathfrak{P} \backslash (\Pi_0 \cup \gamma_1)$ (the set of pairs $(F, G)$ such that $F(-1) < 0$), $\overline{\mathfrak{S}} = \mathfrak{S} \cup \gamma_1$ (pairs such that $F(-1) \le 0$) and $\dot{\mathfrak{S}} = \mathfrak{S} \backslash \mathfrak{Z}$ (pairs such that $F(-1) < 0$, $F(0) > 0$).

The *switch* operator $\mathcal{S} : \dot{\mathfrak{P}} \to \dot{\mathfrak{P}}$ defined as $\mathcal{S}(F, G) = \overline{(G, F)}$ effectively swaps $F$ and $G$. It is easy to see that this operator is an involution ($\mathcal{S}^2 = \mathbb{I}\mathrm{d}$) and maps the sets $\dot{\mathfrak{S}}$ and $\Pi_0$ onto each other.

The *Farey step* operator $\mathcal{F} : \mathfrak{S} \to \mathfrak{P}$ is defined as $\mathcal{F}(F, G) = \overline{(F, F \circ G)}$. Obviously, $\mathcal{F}(\dot{\Pi}_k) \subset \dot{\Pi}_{k-1}, k \ge 1$; $\mathcal{F}(\gamma_{1/k}) \subset \gamma_{1/(k-1)}, k \ge 2$; $\mathcal{F}(\Pi_\infty) \subset \Pi_\infty$; and $\mathcal{F} = \mathbb{I}\mathrm{d}$ on $\mathfrak{Z}$. Starting with a pair in $\mathfrak{S}$, the operator $\mathcal{F}$ can be iterated a number of times, but on some step the trajectory may fall out of $\mathfrak{S}$. The Farey step is not defined for the 'runaway' pair $(F, G) \in \mathfrak{P} \backslash \mathfrak{S} = \Pi_0 \cup \gamma_1$, but in the generic case this will mean that $(F, G) \in \Pi_0$, and the switch operator $\mathcal{S}$ will carry the trajectory back into $\mathfrak{S}$. This consideration suggests the following concept of renormalization in $\overline{\mathfrak{S}}$ that is principal for our work.

The *renormalizing operator on pairs* (shortened as 'renorm-operator') $\mathcal{R} : \overline{\mathfrak{S}} \backslash \Pi_\infty \to \mathfrak{S}$ is defined as $\mathcal{R}(F, G) = \overline{(F^k \circ G, F)}$ for $(F, G) \in \Pi_k$, $k \ge 1$. Indeed, the domain of $\mathcal{R}$ is decomposed as $\overline{\mathfrak{S}} \backslash \Pi_\infty = \bigcup_{k \ge 1} \Pi_k$. A pair from the domain of $\mathcal{R}$ is called *renormalizable*, while the positive integer $k = k(F, G) \ge 1$ such that $(F, G) \in \Pi_k$ is referred to as the renormalization *height* of this pair. Naturally, the pairs from $\Pi_\infty$ are called *non-renormalizable*, and their height is $\infty$. Notice, that renormalizable pairs are those for which $F$ has no fixed points in $[-1, 0]$.

It is easy to see that on $\dot{\Pi}_k$, $1 \le k < \infty$, the following representation is valid: $\mathcal{R} = \mathcal{S} \circ \mathcal{F}^k$. On the other hand, $\mathcal{R}(\gamma_{1/k}) \subset \mathfrak{Z}$, $1 \le k < \infty$.

A pair $(F, G)$ is called *infinitely renormalizable*, if $\mathcal{R}^n(F, G) \in \mathfrak{S} \backslash \Pi_\infty$ for all $n \ge 0$. The set of all such pairs we denote by $\mathfrak{R}$. Now, it is easy to check that every renormalization $(f_n, g_n)$ (defined in the previous subsection) of a circle homeomorphism $T$ with irrational rotation number belongs to $\mathfrak{R}$, and that the sequence of those renormalizations forms a trajectory of the renorm-operator:

$$\mathcal{R}(f_n, g_n) = (f_{n+1}, g_{n+1}). \tag{4}$$

It is possible to define *rotation number* $\rho(F, G) \in [0, 1]$ for a pair $(F, G) \in \overline{\mathfrak{S}}$, substituting its consecutive renormalization heights for partial quotients due in a continued fraction expansion: $\rho(F, G) = [k_1, k_2, ...]$, where $k_n = k(\mathcal{R}^{n-1}(F, G))$. This expansion lasts while $\mathcal{R}^{n-1}(F, G) \notin \Pi_\infty$, so it can appear to be empty, finite or infinite assigning zero, rational or irrational value to $\rho(F, G)$ respectively. Notice that rotation numbers $\rho = 0$ and $\rho = 1$ are different in this context: the first one corresponds to the non-renormalizable set $\Pi_\infty$, while the second one determines a subset in $\Pi_1$.

For irrational $\rho \in (0, 1)$ let us denote by $\gamma_\rho$ the set of all pairs with rotation number $\rho$. (Notice, that all the pairs from $\gamma_{1/k}$ have rotation number $1/k$, $1 \le k \le \infty$, but not all pairs with $\rho = 1/k$ belong to $\gamma_{1/k}$.) Obviously, $\mathfrak{R} = \bigcup_{\rho \in (0,1) \backslash \mathbb{Q}} \gamma_\rho$.

Since Gauss transformation $\mathcal{G}$ acts as the left unit shift on the continued fraction: $\mathcal{G}[k_1, k_2, \dots, k_n, \dots] = [k_2, k_3, \dots, k_n, \dots]$, we may state that

$$\rho(\mathcal{R}(F, G)) = \mathcal{G}\rho(F, G), \quad (F, G) \in \overline{\mathfrak{S}} \backslash \Pi_\infty.$$

For a renormalization $(f_n, g_n)$ we have $\rho(f_n, g_n) = \rho_n = \mathcal{G}^n(\rho(T))$.

The above constructions allow one to consider a sequence of renormalizations $(f_n, g_n)$ built for a circle homeomorphism $T$ as a trajectory of the (infinite-dimensional) dynamical system $(\overline{\mathfrak{S}}; \mathcal{R})$. In the next section we will investigate the hyperbolic properties of this operator being restricted to the subspace of pairs, which correspond to circle diffeomorphisms with breaks.

Notice that $(\overline{\mathfrak{S}}; \mathcal{R})$ is not quite a dynamical system in the classical sense, because $\mathcal{R}$ is not defined over the whole set $\overline{\mathfrak{S}}$. Nevertheless, classical concepts are being easily generalized to this type of "leaking" systems. Just keep in mind that trajectories of our system can be finite: they terminate when falling into $\Pi_\infty$.

## 3. Diffeomorphisms with Breaks: the Settings

A circle homeomorphism $T$ is called a *diffeomorphism* of smoothness $C^{2+\alpha}$, $\alpha \in (0, 1)$, *with a break* at a point $\xi_0$, if the following conditions hold:

1) $T \in C^{2+\alpha}(\xi_0, \xi_0 + 1)$;
2) $\inf_{\xi \neq \xi_0} T'(\xi) > 0$;
3) there exist non-equal one-sided derivatives $T'(\xi_0+)$ and $T'(\xi_0-)$.

The term *size of break* refers to the number $c = \sqrt{\frac{T'(\xi_0-)}{T'(\xi_0+)}}$. For a diffeomorphism with a break, the size of break is positive and not equal to 1 by the definition above.

Let us define the set of pairs that corresponds to the set of diffeomorphisms of smoothness $C^{2+\alpha}$ with breaks of size $c$. To do this, consider firstly the set $\dot{\mathfrak{P}}_c^{2+\alpha} \subset \dot{\mathfrak{P}}$ of all non-degenerate pairs $(F, G)$, both entries of which are $C^{2+\alpha}$-smooth, their first derivatives are positive over the segments of definition, and satisfy the condition $c^2 = \frac{F'(0)G'(F(0))}{G'(0)F'(-1)}$. Secondly, consider the set $\mathfrak{Z}_c^{2+\alpha} \subset \mathfrak{Z}$ of all degenerate pairs $(F, G)$ such that $F$ is smooth on $[-1, 0]$, has positive derivative and satisfies $c^2 = \frac{F'(0)}{F'(-1)}$. It is natural to put $\mathfrak{P}_c^{2+\alpha} = \dot{\mathfrak{P}}_c^{2+\alpha} \cup \mathfrak{Z}_c^{2+\alpha}$ and introduce the whole line of notations by the template $*_c^{2+\alpha} = * \cap \mathfrak{P}_c^{2+\alpha}$, where $*$ stands for a subset of $\mathfrak{P}$ defined in the previous section. It is easy to check that $\mathcal{F}(\mathfrak{S}_c^{2+\alpha}) \subset \mathfrak{P}_c^{2+\alpha}$, $\mathcal{S}(\dot{\mathfrak{P}}_c^{2+\alpha}) = \dot{\mathfrak{P}}_{1/c}^{2+\alpha}$, $\mathcal{R}(\overline{\mathfrak{S}}_c^{2+\alpha} \backslash \Pi_{\infty,c}^{2+\alpha}) \subset \mathfrak{S}_{1/c}^{2+\alpha}$ and $\mathcal{R}(\gamma_{1/k,c}^{2+\alpha}) \subset \mathfrak{Z}_{1/c}^{2+\alpha}$. The switch from size of break $c$ to $1/c$ each time $\mathcal{R}$ acts on a pair with break implies the necessity either to consider the spaces $\mathfrak{P}_c^{2+\alpha}$ and $\mathfrak{P}_{1/c}^{2+\alpha}$ together or study the action of $\mathcal{R}^2$ on (twice renormalizable pairs from) $\mathfrak{S}_c^{2+\alpha}$. Notice that the set $\mathfrak{R}_c^{2+\alpha}$ is invariant w.r.t. $\mathcal{R}^2$.

*The canonical lift.* For a given pair $(F, G) \in \overline{\mathfrak{S}}_c^{2+\alpha} \backslash \mathfrak{Z}_c^{2+\alpha}$, consider a real function determined by the equalities

$$H_{F,G}(w) = \begin{cases} m + H_{F,G}^{(1)}(w - m), & w \in [m - 1, m + \phi(F^{-1}(0))), \\ m + 1 + H_{F,G}^{(2)}(w - m), & w \in [m + \phi(F^{-1}(0)), m), \end{cases} \qquad m \in \mathbb{Z},$$

where

$$H_{F,G}^{(1)} = \phi \circ F \circ \phi^{-1}, \qquad H_{F,G}^{(2)} = \phi \circ G \circ F \circ \phi^{-1},$$

and $\phi(z) = \frac{(F(0)+1)z}{2F(0)+(F(0)-1)z}$ is a linear-fractional map that sends $-1, 0$ and $F(0)$ to $-1, 0$ and 1 respectively. It is easy to see that $H_{F,G}^{(2)}(0) = H_{F,G}^{(1)}(-1)$ and $H_{F,G}^{(2)}(\phi(F^{-1}(0))) = H_{F,G}^{(1)}(\phi(F^{-1}(0))) - 1$, therefore $H_{F,G}$ is continuous on $\mathbb{R}$ and satisfies the equivalence $H_{F,G}(w+1) = H_{F,G}(w) + 1$, thus it is a lift for a certain circle homeomorphism. Let us call $H_{F,G}$ the *canonical lift* for a pair $(F, G)$. We will denote the circle homeomorphism determined by the canonical lift by the same letter $H_{F,G}$ and refer to it as the homeomorphism *generated* by the pair $(F, G)$. Due to combinatorial properties of trajectories (which are the same for a homeomorphism as for the corresponding rigid rotation), we have $\rho(H_{F,G}) = \lim_{i\to+\infty} H^i_{F,G}(0)/i = \rho(\phi \circ F \circ \phi^{-1}, \phi \circ G \circ \phi^{-1}) = \rho(F, G) \in [0, 1]$.

*Remark 1.* Notice that a homeomorphism $H_{F,G}$ generated by $(F, G) \in \dot{\mathfrak{S}}^{2+\alpha}_c$ has, generally speaking, two breaks rather than one, but they belong to the same trajectory, and the product of their sizes is equal to $c$.

*Remark 2.* We cannot give a conceptual explanation why is it so convenient to consider this particular lift as the canonical one for the case of circle diffeomorphisms with breaks. Our techniques work with this lift, while they do not work with other constructions, which we have tested (including the one used in [1]). The problem of producing a convenient circle homeomorphism from a given commuting pair seems to be important. We would like to mention in this connection a natural construction by Yampolsky [13] related to the so-called cylinder renormalization for holomorphic commuting pairs.

Next we introduce the family of pairs of linear-fractional functions given in the form

$$F_{a,v,c}(z) = \frac{a + cz}{1 - vz}, \qquad G_{a,v,c}(z) = \frac{-c + z}{c - \frac{c-1-v}{a} z}. \tag{5}$$

It is also convenient to assume $G_{0,c-1,c}(0) \equiv -1$, so that for each size of break $c$ we have a single degenerate pair in the family (5), namely with $a = 0$, $v = c - 1$.

This family plays an important role in the theory of circle diffeomorphisms with breaks because it is invariant w.r.t. $\mathcal{R}$ (up to non-renormalizable pairs), and the renormalizations $(f_n, g_n)$ of such a diffeomorphism with an irrational rotation number converge exponentially fast to the family (5) as $n \to +\infty$ (we will give the precise statement in a moment).

Let $T$ be an arbitrary circle diffeomorphism of smoothness $C^{2+\alpha}$ with a break of the size $c$ and irrational rotation number. Denote $c^{(n)} = c$ for $n$ even, $c^{(n)} = 1/c$ for $n$ odd and

$$a^{(n)} = \frac{|\xi_{q_n} - \xi_0|}{|\xi_{q_{n-1}} - \xi_0|}, \quad b^{(n)} = \frac{|\xi_{q_n+q_{n-1}} - \xi_0|}{|\xi_{q_{n-1}} - \xi_0|}, \quad v^{(n)} = \frac{c^{(n)} - a^{(n)} - b^{(n)}}{b^{(n)}}. \tag{6}$$

The following statements are proved in [9]:

**(A)** $|\log(T^{q_n})'(\xi)| \le \mathrm{Var}_{\mathbb{T}^1} \log T'$ (the total variation of $\log T'$ over $\mathbb{T}^1$);
**(B)** $|a^{(n)} + b^{(n)} m_n - c^{(n)}| \le C\lambda^n$, where $m_n \in [C^{-1}, C]$;
**(C)** $\|f_n - F_{a^{(n)},v^{(n)},c^{(n)}}\|_{C^2([-1,0])} \le C\lambda^n$, $\|g_n - G_{a^{(n)},v^{(n)},c^{(n)}}\|_{C^1([0,a^{(n)}])} \le C\lambda^n$,

where the constants $C > 0$ and $\lambda \in (0, 1)$ do not depend on $n$.

## 4. Renormalization in the Linear-Fractional Family

*4.1. Basic two-dimensional coordinate system.* Since renormalizations of diffeomorphisms with breaks converge to the family (5), we need to study the action of renorm-operator on pairs from that family. For every particular $c$ it is convenient to identify a point $(a, v)$ on the $\mathbb{R}^2$ plane with the corresponding pair of functions $(F_{a,v,c}, G_{a,v,c})$, as soon as the latter is defined. Denote by $\mathfrak{P}_c$ the sets of all points $(a, v) \in \mathbb{R}^2$ such that $(F_{a,v,c}, G_{a,v,c})$ belongs to $\mathfrak{P}$, and similarly define the other sets with index $c$ such as $\mathfrak{S}_c$, $\mathfrak{R}_c$ and $\gamma_{1/k,c}$, $1 \le k \ge \infty$, etc. For points in $\overline{\mathfrak{S}}_c$, the rotation number $\rho(a, v, c) = \rho(F_{a,v,c}, G_{a,v,c})$ is defined.

One can easily calculate that $\mathfrak{Z}_c = \{(0, c-1)\}$, $\dot{\mathfrak{S}}_c = \{(a, v) : 0 < a < c, v + a - c + 1 > 0\}$, $\gamma_{1,c} = \{(a, v) : a = c, v > -1\}$, $\Pi_{0,c} = \{(a, v) : a > c, v > -c/a\}$ and accordingly construct $\mathfrak{S}_c = \dot{\mathfrak{S}}_c \cup \mathfrak{Z}_c$, $\overline{\mathfrak{S}}_c = \mathfrak{S}_c \cup \gamma_{1,c}$ and $\mathfrak{P}_c = \overline{\mathfrak{S}}_c \cup \Pi_{0,c}$. (Notice that $\overline{\mathfrak{S}}_c$ is the closure of $\mathfrak{S}_c$ in $\mathfrak{P}_c$, but not in $\mathbb{R}^2$.)

In Fig. 1, we show the key lines in the $(a, v)$-plane used to define the sets mentioned above and those that we will mention in the following subsections.

It is easy to check that the Farey step operator maps $\mathfrak{S}_c$ to $\mathfrak{P}_c$, the switch operator maps $\dot{\mathfrak{P}}_c$ onto $\dot{\mathfrak{P}}_{1/c}$, and the renorm-operator maps $\overline{\mathfrak{S}}_c \setminus \Pi_{\infty,c}$ to $\mathfrak{S}_{1/c}$ (we shall presently check that $\mathcal{R}_c$ maps the whole set $\bigcup_{1 \le k < \infty} \gamma_{1/k,c}$ into the single degenerate point $(0, 1/c - 1) \in \mathfrak{S}_{1/c}$). Let us denote these operators, restricted to the above-listed sets (in $\mathbb{R}^2$), by $\mathcal{F}_c$, $\mathcal{S}_c$, and $\mathcal{R}_c$ respectively. In this section, we shall study the appropriate 2D dynamics and describe the horseshoe hyperbolic structure they produce.

It is easy to check that

$$\mathcal{F}_c(a, v) = \left(\frac{v+1}{c-a} a, \frac{c-a}{v+1} v\right), \qquad \mathcal{S}_c(a, v) = \left(\frac{1}{a}, \frac{v+1-c}{c}\right).$$

A direct calculation shows that their Jacobians

$$\det \frac{\partial \mathcal{F}_c(a, v)}{\partial (a, v)} = \frac{c + av}{(v+1)(c-a)}, \qquad \det \frac{\partial \mathcal{S}_c(a, v)}{\partial (a, v)} = -\frac{1}{ca^2} \tag{7}$$

are positive and negative respectively. This important property allows us to calculate images of given domains under the action of the Farey step or the switch operator by calculating just the images of boundaries, and state that the restricted mappings are one-to-one as soon as it is true for the boundaries.

The renorm-operator acts ([9,1]) as

$$\mathcal{R}_c(a, v) = \left(-\frac{1}{a} F^k_{a,v,c}(-1), \frac{1}{c}\left(1 - v F^k_{a,v,c}(-1)\right) - 1\right),$$
$$(a, v) \in \Pi_{k,c}, \quad 1 \le k < \infty. \tag{8}$$

We have $F^k_{a,v,c}(-1) = 0$ for $(a, v) \in \gamma_{1/k,c}$, hence $\mathcal{R}_c\left(\bigcup_{1 \le k < \infty} \gamma_{1/k,c}\right) = \left\{\left(0, \frac{1}{c} - 1\right)\right\}$, and this justifies our setup of the degenerate point in $\mathfrak{S}_c$. One may think of $\mathcal{R}$ acting on the family (5) as the two operators $\mathcal{R}_c$ and $\mathcal{R}_{1/c}$ acting in turns, sending a point from $\mathfrak{S}_c$ to $\mathfrak{S}_{1/c}$ and back respectively. The composition $\mathcal{R}_{1/c} \circ \mathcal{R}_c = \mathcal{R}^2$ maps twice-renormalizable points from $\overline{\mathfrak{S}}_c$ to $\mathfrak{S}_c$, and the set $\mathfrak{R}_c$ is invariant w.r.t. $\mathcal{R}^2$.

Denote the canonical lift for the linear-fractional family as $H_c(w; a, v) = H_{F_{a,v,c}, G_{a,v,c}}(w)$. It is obvious that $H_c : \mathbb{R} \times (\overline{\mathfrak{S}}_c \setminus \{(0, c-1)\}) \mapsto \mathbb{R}$ is a continuous function, and therefore the rotation number $\rho(a, v, c) = \rho(H_c(\cdot; a, v))$ is continuous on $\overline{\mathfrak{S}}_c \setminus \{(0, c-1)\}$. (In fact, it is continuous on the whole set $\overline{\mathfrak{S}}_c$, as it follows from our analysis below.)

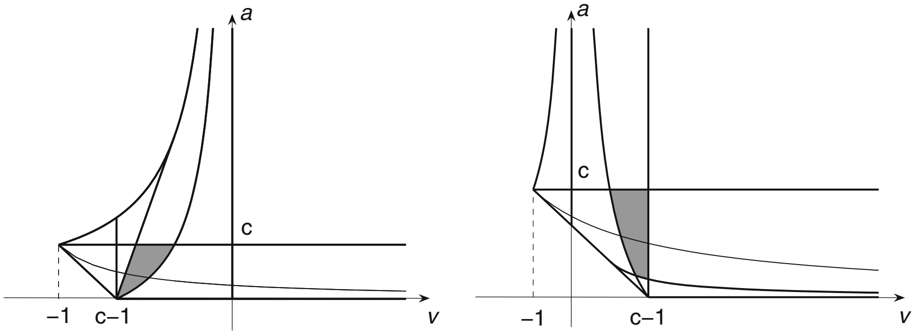


**Fig. 1.** The boundaries of the domains mentioned throughout Sect. 4 in the cases $c < 1$ (left) and $c > 1$ (right), namely $a = 0, a = c, v = 0, v = c - 1, a = c - 1 - v, av = -c, av = c(c - 1 - v), 4av = (c - 1)^2$ (for $c > 1$ only) and $4c(v - c + 1) = (1 - c)^2 a$ (for $c < 1$ only). Also shown is one of the curves $\gamma$ (the thinner one), for which the inequalities (9) hold. The gray domains are the sets $\check{\mathfrak{D}}_c$ (see Subsect. 4.5)

For the smooth components of $H_c$ we get the following expressions:

$$H_c^{(1)}(w; a, v) = \frac{A_1 + B_1 w}{C_1 + D_1 w}, \quad w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right],$$
$$H_c^{(2)}(w; a, v) = \frac{A_2 + B_2 w}{C_2 + D_2 w}, \quad w \in \left[-\frac{a+1}{c+1+(c-a)}, 0\right],$$

where $A_1 = (a+1)^2$, $B_1 = (a+1)(c+1+(c-a))$, $C_1 = (a+1)^2$, $D_1 = 1 - 4va - a^2 + 2ca - 2c$; $A_2 = (a+1)^2(c-a)$, $B_2 = (a+1)(c - a + a^2 - 3ca - 2cva)$, $C_2 = (a+1)(-a^2 + ac - a - 2av - c)$, $D_2 = a^3 + 2va^2 - 3ca^2 + 2cva^2 + 4c^2 a - a - 2va - 2cva - c$.

*4.2. The geometry of the main sets in* $(a, v)$*-coordinates.* It follows from the results of [1], Sect. 3.2, that the set $\Pi_{\infty,c}$ of points in $\overline{\mathfrak{S}}_c$ with rotation number 0 is the single degenerate point $(0, c-1)$ for $c < 1$ and

$$\Pi_{\infty,c} = \left\{(a, v) : \max\{0, c - v - 1\} < a \le \frac{(c-1)^2}{4v}, v > \frac{c-1}{2}\right\} \cup \{(0, c-1)\}$$

for $c > 1$. Accordingly, $\gamma_{0,c} = \{(0, c-1)\}$ for $c < 1$ and $\gamma_{0,c} = \{(a, v) : a = (c-1)^2/(4v), v > (c-1)/2\}$ for $c > 1$ (Fig. 1).

The following proposition lists the rest of the information we get about the geometry of the sets of points $(a, v) \in \overline{\mathfrak{S}}_c$ with constant rotation numbers and the sets $\gamma_{1/k,c}$. In what follows we define some sets in $\overline{\mathfrak{S}}_c$, which can be given as graphs of functions in the form $a = \gamma(v)$, $v > -1$. With a slight abuse of notations, we denote such sets the same as those functions (namely as $\gamma$ with different indices).

**Proposition 1.** *The set* $\gamma_{\rho,c}$ *of points with irrational rotation number* $\rho \in (0, 1)$ *is a continuous graph of the form* $a = \gamma_{\rho,c}(v)$*,* $v > -1$*.*

*The set of points with rational rotation number* $p/q \in (0, 1)$ *is a strip*

$$\Gamma_{p/q,c} = \{(a, v) : \gamma^{(1)}_{p/q,c}(v) \le a \le \gamma^{(2)}_{p/q,c}(v), v > -1\},$$

*where* $\gamma^{(1)}_{p/q,c} < \gamma^{(2)}_{p/q,c}$ *are continuous functions of* $v > -1$. *Moreover, we have* $\gamma_{1/k,c} = \gamma^{(2)}_{1/k,c}$ *for* $c < 1$ *and* $\gamma_{1/k,c} = \gamma^{(1)}_{1/k,c}$ *for* $c > 1$, $2 \le k < \infty$.

*The set of points with rotation number 1 is the ray* $\gamma_{1,c}$ *for* $c > 1$ *and the strip* $\Gamma_{1,c} = \{(a, v) : \gamma^{(1)}_{1,c}(v) \le a \le c, v > -1\}$ *for* $c < 1$, *where* $\gamma^{(1)}_{1,c}(v) < c$ *is a continuous function of* $v > -1$.

*Let* $\gamma$ *stand for one of the following sets:* $\gamma_{\rho,c}$, *where* $\rho \in (0, 1)\backslash\mathbb{Q}$; $\gamma^{(1)}_{p/q,c}$, *where* $p/q \in (0, 1)$; $\gamma^{(2)}_{p/q,c}$, *where* $p/q \in (0, 1)$; $\gamma_{1/k,c}$, *where* $2 \le k < \infty$; *or* $\gamma^{(1)}_{1,c}$, *where* $c < 1$. *For any two different points* $(a_1, v_1), (a_2, v_2) \in \gamma$, *the following Lipschits-type inequalities hold:*

$$-1 < \frac{a_2 - a_1}{v_2 - v_1} < 0. \tag{9}$$

*The described graphs* $a = \gamma(v)$, $v > -1$, *are all ordered in accordance with their first lower index, i.e.* $\gamma_{\rho,c} < \gamma^{(1)}_{p/q,c}$ *for* $\rho < p/q$ *etc.*

We will prove this proposition in the next subsection.

*4.3. Proof of Proposition 1.* To study the geometry of the sets $\gamma_{\rho,c}$, we use the following idea. It is a well-known fact that the rotation number is a monotone function of circle homeomorphism in the following sense: if $H_\Omega$, $\Omega \in [A, B]$, is a continuous one-parameter family of lifts of circle homeomorphisms such that $H_\Omega(w)$ is non-decreasing w.r.t. $\Omega$ for any fixed $w \in \mathbb{R}$, then $\rho(H_\Omega)$ is a continuous and non-decreasing function of $\Omega$. Moreover, if $H_\Omega(w)$ is strictly increasing w.r.t. $\Omega$ for any fixed $w \in \mathbb{T}^1$, then $\rho(H_\Omega)$ is strictly increasing at its irrational values. Thus, if the canonical lift $H_c(w; a, v)$ is strictly monotone along a certain curve in $\dot{\mathfrak{S}}_c$, then this curve cannot have more than one common point with any of the sets $\gamma_{\rho,c}$, $\rho \in (0, 1)\backslash\mathbb{Q}$.

To establish monotonicity of the canonical lift, we will differentiate its smooth components in certain directions.

**Lemma 1.** *For* $(a, v) \in \dot{\mathfrak{S}}_c$, *we have*

$$\frac{\partial H^{(1)}_c(w; a, v)}{\partial v} > 0 \quad \textit{for all } w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right),$$

$$\frac{\partial H^{(2)}_c(w; a, v)}{\partial v} > 0 \quad \textit{for all } w \in \left(-\frac{a+1}{c+1+(c-a)}, 0\right],$$

*while at* $w = -\frac{a+1}{c+1+(c-a)}$ *both derivatives vanish.*

*Proof.* The derivative $\frac{\partial H^{(1)}_c(w;a,v)}{\partial v} = \frac{-4wa(a+1)P_1}{Q_1^2}$, where $P_1 = (-1+a-2c)w-(a+1)$ and $Q_1 = (a^2 + 4va - 2ca + 2c - 1)w - (a + 1)^2$. Obviously, $-4wa(a + 1)/Q_1^2 > 0$ for all $w \in [-1, -(a + 1)/(c + 1 + (c - a))]$, $(a, v) \in \dot{\mathfrak{S}}_c$. The value of $P_1$ at $w = -1$ is $2a(a + 1)(c - a) > 0$, and its value at $w = -(a + 1)/(c + 1 + (c - a))$ is 0. Since $P_1$ is linear w.r.t. $w$, the statement follows.

Next, $\frac{\partial H^{(2)}_c(w;a,v)}{\partial a} = \frac{2a(a+1)(c-a)P_2}{Q_2^2}$, where $P_2 = (a + 2ca - 1)(-1 + a - 2c)w^2 - 2(a-1)(a+1)(c+1)w+(a+1)^2$ and $Q_2 = (a^3-3ca^2+2a^2v+2a^2vc+4c^2a-a-2va-$

$2vac - c)w - a^3 - 2a^2 + ca^2 - 2a^2v - a - 2va - c$. Since $2a(a+1)(c-a)/Q_2^2$ is positive, it is enough to check the sign of $P_2$. The value of $P_2$ at $w = -(a+1)/(c+1+(c-a))$ is 0, and its value at $w = 0$ is $(a+1)^2$. Now notice that $P_2/w^2$ is a quadratic polynomial w.r.t. $1/w$, the parabola looks upwards, and the vertex $1/w = (c+1)(a-1)/(a+1)$ lies to the right from the interval $(-\infty, -(c+1+(c-a))/(a+1)]$ of our interest. The proof is complete.

**Corollary 1.** *$H_c(w; a, v)$ is non-decreasing, and its second iterate $H_c(H_c(w; a, v); a, v)$ is strictly increasing, for every $w \in \mathbb{R}$ along every line $a =$ const in $\dot{\mathfrak{S}}_c$ as $v$ increases.*

In the sequel, $\nabla$ denotes the gradient vector w.r.t. the variables $(a, v)$.

**Lemma 2.** *For $(a, v) \in \dot{\mathfrak{S}}_c$, we have*

$$\overrightarrow{(1,-1)} \cdot \nabla_{(a,v)} H_c^{(1)}(w; a, v) > 0 \quad \text{for all } w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right],$$
$$\overrightarrow{(1,-1)} \cdot \nabla_{(a,v)} H_c^{(2)}(w; a, v) > 0 \quad \text{for all } w \in \left[-\frac{a+1}{c+1+(c-a)}, 0\right].$$

*Proof.* The derivative $\overrightarrow{(1,-1)} \cdot \nabla H_c^{(1)}(w; a, v) = \frac{-4wP_1}{Q_1^2}$, where $P_1 = (2ca^2 + a - a^2v - a^3 + 2c^2 - v - 2vc)w + a^3 + 2a^2 + 2ca + a^2v - v + a + 2c$ and $Q_1 = (-1 + 4va + a^2 - 2ca + 2c)w - 1 - 2a - a^2$. We have $-4w/Q_1^2 > 0$ for all $w \in [-1, -(a+1)/(c+1+(c-a))]$. The value of $P_1$ at the point $w = -1$ is $2(a^2+c)(v+a-c+1) > 0$, and its value at $w = -(a+1)/(c+1+(c-a))$ is $2(c+1)(a+1)(c+va)/(c+1+(c-a)) > 0$. Since $P_1$ is linear w.r.t. $w$, the statement follows.

Next, $\overrightarrow{(1,-1)} \cdot \nabla H_c^{(2)}(w; a, v) = \frac{2(v+a-c+1)P_2}{Q_2^2}$, where $P_2 = (-2ca + a^4 + c + 4a^2vc + a^2 - 4c^2a^3 + 2c^2a^2 + 2c^2 + 2a^4c + 4a^2vc^2 - 4ca^3 + 3ca^2 - 2a^3)w^2 - 2(c+1)(a-1)(a+1)(a^2+c)w + (a+1)^2(a^2+c)$ and $Q_2 = (a^3 - 3ca^2 + 2a^2v + 2a^2vc - a + 4c^2a - 2va - 2vca - c)w - a^3 - 2a^2v - 2a^2 + ca^2 - a - 2va - c$. Since $2(v+a-c+1)/Q_2^2 > 0$, it is enough to check that $P_2 > 0$. The value of $P_2$ at $w = -(a+1)/(c+1+(c-a))$ is $4ca(a+1)^2(c+1)(c+va)/(c+1+(c-a))^2 > 0$, and its value at $w = 0$ is $(a+1)^2(a^2+c) > 0$. Now notice that $P_2/w^2$ is again a quadratic polynomial w.r.t. $1/w$, the parabola looks upwards, and the vertex $1/w = (c+1)(a-1)/(a+1)$ lies to the right from the interval $(-\infty, -(c+1+(c-a))/(a+1)]$ of our interest. The proof is complete.

**Corollary 2.** *$H_c(w; a, v)$ is strictly increasing for every $w \in \mathbb{R}$ along every line $a + v =$ const in $\mathfrak{S}_c$ as $a$ increases.*

Since $\overrightarrow{(1,0)} = \overrightarrow{(0,1)} + \overrightarrow{(1,-1)}$, the next statement follows easily from Lemmas 1 and 2.

**Corollary 3.** *$H_c(w; a, v)$ is strictly increasing w.r.t. $a$ in $\dot{\mathfrak{S}}_c$ for any fixed $v > -1$, $w \in \mathbb{R}$.*

We need one more lemma, this time concerning the 'devil staircase' structure of dependence of rotation number $\rho(a, v, c)$ on $a$.

**Lemma 3.** *Assume the values of $c \neq 1$ and $v > -1$ to be fixed.*

*The rotation number $\rho(a, v, c)$ is a continuous non-decreasing function in $a$ such that the preimages of irrational numbers $\rho \in (0, 1)$ are single values $a = \gamma_{\rho,c}(v)$, and the preimages of rational numbers $p/q \in (0, 1)$ are closed segments $a \in [\gamma^{(1)}_{p/q,c}(v), \gamma^{(2)}_{p/q,c}(v)]$ of non-zero length. The preimage of rotation number 1 is the single value $a = c$ for $c > 1$ and a segment $a \in [\gamma^{(1)}_{1,c}(v), c]$ of non-zero length for $c < 1$.*

*For every $2 \le k < \infty$, there exists a single value $a = \gamma_{1/k,c}(v)$ such that $(a, v) \in \gamma_{1/k,c}$; moreover, we have $\gamma_{1/k,c}(v) = \gamma^{(2)}_{1/k,c}(v)$ for $c < 1$ and $\gamma_{1/k,c}(v) = \gamma^{(1)}_{1/k,c}(v)$ for $c > 1$.*

*The values $\gamma_{\rho,c}(v)$ and $\gamma^{(i)}_{p/q,c}(v)$, $i \in \{1, 2\}$, are ordered in accordance with their first lower index.*

*Proof.* Let us fix some $c \neq 1$ and $v > -1$.

From the description of $\gamma_{1,c}$ we know that the upper boundary for $a$ in $\overline{\mathfrak{S}}_c$ is $c$, and that $\rho(c, v, c) = 1$. We start with showing that $\rho(a, v, c) \to 0$ as $a$ tends to its lower boundary in $\overline{\mathfrak{S}}_c$, which is $\max\{0, c - 1 - v\}$ (notice that for $c > 1$ and $v > (c - 1)/2$ a stronger property holds: $\rho(a, v, c) = 0$ as soon as $a \le (c - 1)^2/(4v)$ due to the description of $\gamma_{0,c}$). The difference

$$H^{(1)}_c(-1; a, v) - (-1) = \frac{2a(a + 1 + v - c)}{(c + av) + a(v + a - c + 1)} > 0$$

tends to zero as $a \to \max\{0, c - 1 - v\}$. On the other hand, the derivative

$$\frac{\partial H^{(1)}_c(w; a, v)}{\partial w} = \frac{4(a + 1)^2(c + av)}{((-1 + 4av + a^2 - 2ac + 2c)w - 1 - 2a - a^2)^2} > 0$$

is bounded in $\dot{\mathfrak{S}}_c$ for every fixed $v$. Indeed, the nominator is obviously bounded, while the denominator is separated from zero by a positive constant (for $w = -1$ the denominator is equal to $4((c+av)+a(v+a-c+1))^2 \ge 4(c+av)^2 \ge \min\{4c^2(v+1)^2, 4c^2\} > 0$ and for $w = 0$ it is $(a + 1)^4 \ge 1 > 0$). It follows that the renormalization height of $(a, v)$, which is less by one than a number of iterates of $H^{(1)}_c(\cdot; a, v)$ required to move the point $-1$ beyond 0, tends to $+\infty$ as $a \to \max\{0, c - 1 - v\}$, and thus the rotation number tends to zero indeed.

The continuity of $\rho(\cdot, v, c)$, the equality $\rho(c, v, c) = 1$ and the fact we proved in the previous paragraph imply that all numbers from $(0, 1]$ have non-empty preimages in $(\max\{0, c - 1 - v\}, c]$. Corollary 3 implies that $\rho(a, v, c)$ is non-decreasing for $a \in (\max\{0, c - 1 - v\}, c]$, and that it is strictly increasing at its irrational values. Thus the preimages of irrational numbers are single values, while the preimages of rational numbers from $(0, 1]$ are either single values or closed segments.

We have $\rho(a, v, c) = p/q \in (0, 1)$ if and only if the $q^{\text{th}}$ iterate $(H_c(\cdot; a, v))^q - p$ has a fixed point. Let it have a fixed point. Since $(H_c(\cdot; a, v))^q - p$ has a break of non-unit size, its graph does not coincide with the identity line. Assume that this graph has a point located above (below) the identity line. Corollary 3 implies that $(H_c(\cdot; a, v))^q - p$ is strictly increasing w.r.t. $a$, hence for small enough negative (positive) increment $\delta a$ the function $(H_c(\cdot; a + \delta a, v))^q - p$ has a fixed point too. Thus the preimages of rational numbers from $(0, 1)$ are closed segments $[\gamma^{(1)}_{p/q,c}(v), \gamma^{(2)}_{p/q,c}(v)]$ of non-zero length. For the case of rotation number 1 we already know a value of $a$ that has it, namely $a = c$, since $(c, v) \in \gamma_{1,c}$. For $w \in [-1, 0]$ we have

$$H_c(w; c, v) - 1 = H_c^{(2)}(w; c, v) = \frac{1 + w}{1 + (1 - c)w},$$

and the second derivative

$$\frac{\partial^2 H_c(w; c, v)}{\partial w^2} = \frac{2(c - 1)c}{(1 + (1 - c)w)^3}$$

has the sign of $c - 1$. It follows that $w = -1$ and $w = 0$ are fixed points for the map $H_c(\cdot; c, v) - 1$, while for $w \in (-1, 0)$ its graph lies below the identity line in the case of $c > 1$ and above that line in the case of $c < 1$. Therefore, due to Corollary 3, in the first case a small decrease in $a$ will leave no fixed points on $\mathbb{R}$, while in the second case at least two fixed points will be located on $(-1, 0)$.

Now let us look at the sets $\gamma_{1/k,c}$, $2 \le k < +\infty$, defined by the property $F^k_{a,v,c}(-1) = 0$. The fixed points of the lift $H = (H_c(\cdot; a, v))^k - 1$ in this case are $w_i = (H_c(\cdot; a, v))^i(0)$, $i \in \mathbb{Z}$, the function $H$ is linear-fractional on every segment $[w_i, w_{i+1}]$, the one-sided derivatives of $H$ at $w_i$, $i \in \mathbb{Z}$, do not depend on $i$ (they are equal to the product of the corresponding one-sided derivatives of $H_c(\cdot; a, v)$ at $w_j$, $0 \le j < k$), and at each point $w_i$, $i \in \mathbb{Z}$, the function $H$ has a break of size $c$. It follows that between each pair of its neighboring fixed points the function $H$ is convex (i.e. $H'' > 0$) in the case of $c > 1$ and concave in the case of $c < 1$. Corollary 3 hereby implies the second statement of the lemma.

The last one follows from the same corollary.

*Proof (Proof of Proposition 1).* Let us notice that all the sets $\gamma$ listed in the last statement of the proposition share the following common property: if a point $(a, v)$ belongs to $\gamma$, and $(a', v')$ is another point such that the second iterate of the canonical lift $(H_c(w; a', v'))^2$ is greater (smaller) than $(H_c(w; a, v))^2$ for every $w \in \mathbb{T}^1$, then $(a', v') \notin \gamma$.

Corollaries 1 and 2 taken together imply that, given an arbitrary point $(a, v) \in \gamma$, for any point $(a', v') \neq (a, v)$ inside the angle $\{(a', v') : a' \le a, a' + v' \le a + v\}$ we have $(H_c(w; a', v'))^2 < (H_c(w; a, v))^2$ and therefore $(a', v') \notin \gamma$. Similarly, for any point $(a'', v'') \neq (a, v)$ inside the angle $\{(a'', v'') : a'' \ge a, a'' + v'' \ge a + v\}$ we have $(H_c(w; a'', v''))^2 > (H_c(w; a, v))^2$ and therefore $(a'', v'') \notin \gamma$ as well. The inequalities (9) follow.

The rest of the statements of this proposition are implied by Lemma 3 and the continuity of $\rho$ in $v$.

*4.4. Absorbing areas.* Let us introduce the set $\mathfrak{D}_c = \{(a, v) : 1/2 \le v/(c - 1) < 1, c(c - v - 1)/v \le a \le c\}$ and its closure $\overline{\mathfrak{D}}_c = \mathfrak{D}_c \cup \{(a, c - 1) : a \in [0, c]\} \subset \mathfrak{S}_c$ (see Fig. 1 for easy visualization of those sets). The following proposition shows that these sets are absorbing areas for the dynamics determined by the renorm-operator $\mathcal{R}$.

**Proposition 2.** *For any $c \neq 1$ we have $\mathcal{R}_c(\overline{\mathfrak{D}}_c \backslash \Pi_{\infty,c}) \subset \overline{\mathfrak{D}}_{1/c}$. Any trajectory of $\mathcal{R}^2 = \mathcal{R}_{1/c} \circ \mathcal{R}_c$ in $\mathfrak{R}_c$ eventually falls into $\mathfrak{D}_c$ and stays there forever afterwards.*

*Proof.* In [1], Sect. 3.4, it was proved that any trajectory of $\mathcal{R}^2$ in $\mathfrak{R}_c$ eventually falls into the set $\Phi_c = \{(a, v) \in \overline{\mathfrak{S}}_c : 0 \le v/(c-1) \le 1\}$ (to be precise, they did not include $(0, c-1)$ into $\Phi_c$, while we do) and stays there forever afterwards. Since $\overline{\mathfrak{D}}_c \subset \Phi_c$, showing that $\mathcal{R}_c(\Phi_c \backslash \Pi_{\infty,c}) \subset \overline{\mathfrak{D}}_{1/c}$ and $\mathcal{R}_c(\mathfrak{R}_c \cap \overline{\mathfrak{D}}_c) \subset \mathfrak{D}_{1/c}$ will prove the proposition.

A direct calculation shows that $\mathcal{F}_c(\Phi_c \cap \mathfrak{S}_c) = \{(a, v) \in \mathfrak{P}_c : 0 \le v/(c-1) \le 1, a \le c(c-v-1)/v\}$, and the intersection of this set with $\mathfrak{S}_c$ is still included into $\Phi_c$. Therefore, for $1 \le k < \infty$ we have $\mathcal{F}_c^k(\dot{\Pi}_{k,c}) \subset \mathcal{F}_c(\Phi_c \cap \mathfrak{S}_c) \cap \Pi_{0,c} = \{(a, v) : 0 \le v/(c-1) \le 1, c < a \le c(c-v-1)/v\} = \{(a, v) : 0 \le v/(c-1) \le 1/2, c < a \le c(c-v-1)/v\}$. Another direct calculation shows that $\mathcal{S}_c(\{(a, v) : 0 \le v/(c-1) \le 1/2, c \le a \le c(c-v-1)/v\}) = \overline{\mathfrak{D}}_{1/c} \backslash \{(0, 1/c-1)\}$, hence $\mathcal{R}_c(a, v) = \mathcal{S} \circ \mathcal{F}^k(a, v) \in \overline{\mathfrak{D}}_{1/c}$ as soon as $(a, v) \in \dot{\Pi}_{k,c}$, $1 \le k < \infty$. For $(a, v) \in \bigcup_{1 \le k < \infty} \gamma_{1/k,c}$ we have already shown that $\mathcal{R}_c(a, v) = (0, 1/c-1) \in \overline{\mathfrak{D}}_{1/c}$. thus we have proved that $\mathcal{R}_c(\Phi_c \backslash \Pi_{\infty,c}) \subset \overline{\mathfrak{D}}_{1/c}$.

To prove that $\mathcal{R}_c(\mathfrak{R}_c \cap \overline{\mathfrak{D}}_c) \subset \mathfrak{D}_{1/c}$, it is enough to look at the expression (8) for $(a', v') = \mathcal{R}_c(a, v)$ and notice that $v' = 1/c - 1$ would imply $vF^k_{a,v,c}(-1) = 0$, but neither $v = 0$ nor $(a, v) \in \gamma_{1/k,c}$ can happen in $\mathfrak{R}_c \cap \overline{\mathfrak{D}}_c$.

We have shown that, after a finite number of steps the trajectory of the renorm-operator enters one of the two triangular-shaped domains $\mathfrak{D}_c$ and $\mathfrak{D}_{1/c}$, and stays in their union forever. The renorm-operators $\mathcal{R}_c$ and $\mathcal{R}_{1/c}$ map $\mathfrak{D}_c$ and $\mathfrak{D}_{1/c}$ inside each other. In the next subsection we will uncover a surprising symmetry in their actions that produces the hyperbolic horseshoe.

*4.5. Symmetric properties of the renorm-operator.* In this subsection we introduce an explicit time-reverse symmetry provided by an involution $\mathcal{T}_c$ (see below). This symmetry plays a very important role in our further analysis. It represents a hidden symmetry in the renormalization dynamics, which looks rather mysterious in the coordinates $(a, v)$. It has much more transparent meaning in the nonlinearity coordinates $(x, y)$ introduced later in Subsect. 4.6.

On the domain $\Psi_c = \{(a, v) : a > 0, 0 < v/(c-1) < 1\}$, consider the map

$$\mathcal{T}_c(a, v) = \left(\frac{av}{c-1-v}, c-1-v\right), \quad \det \frac{\partial \mathcal{T}_c(a, v)}{\partial(a, v)} = \frac{v}{v-c+1} < 0.$$

Obviously, $\mathcal{T}_c$ is an involution of $\Psi_c$ in the usual sense: $\mathcal{T}_c^2 = \mathbb{I}\mathrm{d}$.

Notice that $\mathcal{T}_c$ maps the domain $\Psi_c \cap \mathfrak{P}_c$ onto itself, too. The Farey step is one-to-one on $\Psi_c \cap \mathfrak{S}_c$, and a direct calculation shows that the equality

$$\mathcal{F}_c^{-1} = \mathcal{T}_c \circ \mathcal{F}_c \circ \mathcal{T}_c \tag{10}$$

holds on the set $\mathcal{T}_c(\Psi_c \cap \mathfrak{S}_c) = \mathcal{F}_c(\Psi_c \cap \mathfrak{S}_c) = \{(a, v) \in \mathfrak{P}_c : a < c(c-1-v)/v\}$.

In accordance with the subdivision of $\Psi_c \cap \mathfrak{P}_c$ into the sequence of domains $\Psi_c \cap \dot{\Pi}_{k,c}$, $0 \le k \le \infty$, and separating arcs $\Psi_c \cap \gamma_{1/k,c}$, $1 \le k \le \infty$, the involution $\mathcal{T}_c$ induces the subdivision of $\Psi_c \cap \mathfrak{P}_c$ into the sequence of domains $\dot{\Omega}_{l,c} = \mathcal{T}_c(\Psi_c \cap \dot{\Pi}_{l,c})$, $0 \le l \le \infty$, and separating curves $\beta_{1/l,c} = \mathcal{T}_c(\Psi_c \cap \gamma_{1/l,c})$, $1 \le l \le \infty$. (Notice, that the sets $\Psi_c \cap \dot{\Pi}_{\infty,c}$; $\Psi_c \cap \gamma_{0,c}$; $\dot{\Omega}_{\infty,c}$; and $\beta_{0,c}$ are empty in the case $c < 1$.)

**Lemma 4.** *$\mathcal{F}_c(\dot{\Pi}_{k,c} \cap \dot{\Omega}_{l,c}) = \dot{\Pi}_{k-1,c} \cap \dot{\Omega}_{l+1,c}$ for $1 \le k < \infty$, $0 \le l < \infty$.*

*Proof.* It is easy to see that $\mathcal{F}_c(\dot{\Pi}_{k,c} \cap \Psi_c) = \dot{\Pi}_{k-1,c} \cap \mathcal{F}_c(\Psi_c \cap \mathfrak{S}_c) = \dot{\Pi}_{k-1,c} \cap \Psi_c \backslash (\dot{\Omega}_{0,c} \cup \beta_{0,c})$, and (10) implies that $\mathcal{F}_c(\dot{\Omega}_{k-1,c} \cap \mathfrak{S}_c) = \dot{\Omega}_{k,c}$, $1 \le k < \infty$. The statement of the lemma follows.

*Remark 3.* Proposition 4 that will be proved later in this section implies in particular that the curve $\beta_{1,c} = \{(a, v) : a = c(c - v - 1)/v, 0 < v/(c-1) < 1\}$ (which includes one of the boundaries of $\mathfrak{D}_c$) intersects each of the curves $\gamma_{1/k,c}$, $1 \le k < \infty$, at a single point. This fact implies that all the sets $\dot{\Pi}_{k,c} \cap \dot{\Omega}_{l,c}$, $1 \le k < \infty$, $0 \le l < \infty$ are quadrilateral cells of a planar grid created by two transversal sequences of simple curves $\left\{\gamma_{1/k,c}\right\}_k$ and $\left\{\beta_{1/l,c}\right\}_l$.

To formulate the next proposition, we introduce some further notations.

A direct calculation shows that $\mathcal{S}_c \circ \mathcal{T}_c = \mathcal{T}_{1/c} \circ \mathcal{S}_c$. It is easy to check that $\mathcal{T}_c$ maps $\mathfrak{D}_c$ onto $\mathcal{S}_{1/c}\mathfrak{D}_{1/c}$, and the composition $\mathcal{I}_c = \mathcal{S}_c \circ \mathcal{T}_c : \Psi_c \to \Psi_{1/c}$ given by the expressions

$$\mathcal{I}_c(a, v) = \left(\frac{c-1-v}{av}, -\frac{v}{c}\right), \quad \det \frac{\partial \mathcal{I}_c(a, v)}{\partial(a, v)} = \frac{c-1-v}{a^2 cv} > 0,$$

maps $\mathfrak{D}_c$ onto $\mathfrak{D}_{1/c}$ and is an involution in the sense that $\mathcal{I}_{1/c} \circ \mathcal{I}_c = \mathbb{I}\mathrm{d}$.

The set $\mathfrak{D}_c$ is subdivided into the sequence of curvilinear quadrilaterals $\dot{\Pi}^+_{k,c} = \mathfrak{D}_c \cap \dot{\Pi}_{k,c}$, $1 \le k \le \infty$, and the separating arcs $\gamma^+_{1/k,c} = \mathfrak{D}_c \cap \gamma_{1/k,c}$, $2 \le k \le \infty$. On the other hand, it is subdivided into the transversal sequence of curvilinear triangulars $\dot{\Pi}^-_{k,c} = \mathcal{I}_{1/c}(\dot{\Pi}^+_{k,1/c}) = \mathcal{S}_{1/c}(\Pi_{0,c} \cap \dot{\Omega}_{k,c})$, $1 \le k \le \infty$, and the separating arcs $\gamma^-_{1/k,c} = \mathcal{I}_{1/c}(\gamma^+_{1/k,1/c})$, $2 \le k \le \infty$.

Notice that the sets $\dot{\Pi}^+_{\infty,c}$ and $\gamma^+_{0,c}$ are empty for $c < 1$, while $\dot{\Pi}^-_{\infty,c}$ and $\gamma^-_{0,c}$ are empty for $c > 1$. For $c < 1$ the set $\gamma^-_{0,c} = \{(a, v) : a \in (0, c], v = \frac{(1-c)^2}{4c} a + c - 1\}$ is a straight segment and $\dot{\Pi}^-_{\infty,c}$ is a right triangle with sides $\gamma^-_{0,c}$, $v = c - 1$ and $a = c$.

Generally speaking, a point $(a, v) \in \mathfrak{D}_{1/c}$ has infinitely many preimages with respect to $\mathcal{R}_c$ in $\mathfrak{S}_c$, but only one of them lies in $\mathfrak{D}_c$. Since $\mathfrak{D}_c$ is an absorbing area, we are interested exactly in that preimage. In this sense, the inverse map for $\mathcal{R}_c$ may be defined uniquely. The following proposition states that the restriction of the renorm-operator $\mathcal{R}$ to the (non-connected) area $\bigcup_{1\le k<\infty} \dot{\Pi}^+_{k,c}$ is invertible and conjugate with the inverse one by an involution. Notice that the involution that conjugates $\mathcal{R}_c$ with $\mathcal{R}_c^{-1}$ is written explicitly in surprisingly simple form and does not depend on renormalization height at a particular point, while the operators themselves do depend on that height, and the explicit expressions for them become more and more complicated as the height grows.

**Proposition 3.** *For every $1 \le k < \infty$, the renorm-operator $\mathcal{R}_c$ is a one-to-one map from $\dot{\Pi}^+_{k,c}$ onto $\dot{\Pi}^-_{k,1/c}$; moreover, $\mathcal{R}_c^{-1} = \mathcal{I}_{1/c} \circ \mathcal{R}_c \circ \mathcal{I}_{1/c}$, where by $\mathcal{R}_c^{-1}$ we mean the uniquely defined inverse operator from $\bigcup_{k\ge1} \dot{\Pi}^-_{k,1/c} \subset \mathfrak{D}_{1/c}$ onto $\bigcup_{k\ge1} \dot{\Pi}^+_{k,c} \subset \mathfrak{D}_c$.*

*Proof.* Applying $k$ times Lemma 4 for a set $\dot{\Pi}^+_{k,c} = \dot{\Pi}_{k,c} \cap \dot{\Omega}_{0,c}$, we get $\mathcal{F}_c^k(\dot{\Pi}^+_{k,c}) = \Pi_{0,c} \cap \dot{\Omega}_{k,c} = \mathcal{S}_{1/c}(\Pi^-_{k,1/c})$, hence $\mathcal{R}_c(\dot{\Pi}^+_{k,c}) = \dot{\Pi}^-_{k,1/c}$ is indeed a one-to-one map.

The conjugacy follows from (10): $\mathcal{R}_c^{-1} = (\mathcal{S}_c \circ \mathcal{F}_c^k)^{-1} = \mathcal{F}_c^{-k} \circ \mathcal{S}_{1/c} = (\mathcal{T}_c \circ \mathcal{F}_c^k \circ \mathcal{T}_c) \circ \mathcal{S}_{1/c} = (\mathcal{T}_c \circ \mathcal{S}_{1/c}) \circ (\mathcal{S}_c \circ \mathcal{F}_c^k) \circ (\mathcal{T}_c \circ \mathcal{S}_{1/c}) = \mathcal{I}_{1/c} \circ \mathcal{R}_c \circ \mathcal{I}_{1/c}$ independently on $k \ge 1$.

This symmetry allows us to define for any point $(a, v) \in \mathfrak{D}_c$ its 'two-sided' rotation number. The *generalized rotation number* for $(a, v) \in \mathfrak{D}_c$ is the pair of numbers $(\rho^-(a, v, c), \rho^+(a, v, c))$ (which we call the forward and backward rotation numbers respectively), where $\rho^+(a, v, c) = \rho(a, v, c)$ and $\rho^-(a, v, c) = \rho(\mathcal{I}_c(a, v), 1/c)$. It is convenient to write down a generalized rotation number as a bi-infinite (in irrational case) sequence of positive integers

$$(\rho^-(a, v, c), \rho^+(a, v, c)) = [\dots, k_{-2}, k_{-1}, k_0, k_1, k_2, \dots],$$

where $\rho(a, v, c) = [k_1, k_2, \dots]$ is the forward and $\rho^-(a, v, c) = [k_0, k_{-1}, k_{-2}, \dots]$ is the backward rotation number.

By the end of this subsection let us strengthen the second statement of Proposition 2. Denote by $\check{\mathfrak{D}}_c$ the set $\mathfrak{D}_c \backslash \dot{\Pi}^-_{\infty,c}$ in the case of $c < 1$ and the set $\mathfrak{D}_c \backslash \dot{\Pi}^+_{\infty,c}$ in the case of $c > 1$. It is easy to check that $\mathcal{I}_c(\check{\mathfrak{D}}_c) = \check{\mathfrak{D}}_{1/c}$. The next statement easily follows from Propositions 3 and 2.

**Corollary 4.** *For any $c \neq 1$ we have $\mathcal{R}_c(\mathfrak{R}_c \cap \check{\mathfrak{D}}_c) \subset \check{\mathfrak{D}}_{1/c}$. Any trajectory of $\mathcal{R}^2 = \mathcal{R}_{1/c} \circ \mathcal{R}_c$ in $\mathfrak{R}_c$ eventually falls into $\check{\mathfrak{D}}_c$ and stays there forever afterwards.*

*4.6. The alternative $(x, y)$-coordinate system in $\mathfrak{D}_c$.* Here we introduce the alternative coordinates

$$(x, y) = \pi_c(a, v) = \left(av, \frac{v + 1 - c}{ca}\right), \qquad \det \frac{\partial(x, y)}{\partial(a, v)} = \frac{v - (c - 1)/2}{ac}. \tag{11}$$

The geometric sense of the variables $x$ and $y$ is the following. If we normalize the first entry $F_{a,v,c}$ of a pair $(F_{a,v,c}, G_{a,v,c}) \in \dot{\mathfrak{P}}_c$ by means of a linear change of variable that sends 0 and $F(0)$ into 0 and 1 respectively, we obtain a linear-fractional function

$$M_{x,c} : t \mapsto \frac{1 + ct}{1 - xt}.$$

The similar manipulation with the second entry $G_{a,v,c}$ produces a function $M_{y,1/c} : t \mapsto \frac{1+t/c}{1-yt}$. Thus, $x$ and $y$ may be considered as independent parameters characterizing the nonlinearity for the two entries of a commuting pair from $\dot{\mathfrak{P}}_c$. It is therefore not surprising that the Farey step $\mathcal{F}$ preserves $x$ and the switch $\mathcal{S}$ interchanges the values of $x$ and $y$. What is surprising, however, is that the involution $\mathcal{T}_c$ on $\Psi_c$ preserves both $x$ and $y$! (All three properties can be checked by direct calculation.) Since $\pi_c$ is one-to-one on both sets $\{(a, v) \in \Psi_c, 0 < v/(c - 1) < 1/2\}$ and $\{(a, v) \in \Psi_c, 1/2 < v/(c - 1) < 1\}$, which is easy to check in view of (11), and $\mathcal{T}_c$ maps them one onto another, then $\pi_c$ on $\Psi_c$ is two-fold, bent along the ray $v = (c - 1)/2$, $a > 0$.

It is easy to check that $\pi_c$ maps $\mathfrak{D}_c$ one-to-one onto the right triangle with sides $x = 0$, $y = 0$ and $x - c^3 y = c(c - 1)$ (notice that $\pi_c$ gets 'corrupted' at the degenerate point $a = 0$, $v = c - 1$, stretching it onto the whole $x = 0$, $y \in [0, (1 - c)/c^2]$ side of that triangle). We will preserve the notations for the sets and maps inside $\mathfrak{D}_c$ in the coordinates $(x, y)$ with omission of '$\pi_c$', indicating instead the current set of variables as the set of arguments when needed (for example, $\mathcal{R}_c(x, y)$ will mean $\pi_c(\mathcal{R}_c(\pi_c^{-1}(x, y)))$ etc).

One can calculate that $\pi_c^{-1}$ on $\mathfrak{D}_c$ is given by

$$a = \frac{c-1}{2cy}\left(-1+\sqrt{1+\frac{4cxy}{(c-1)^2}}\right), \qquad v = \frac{c-1}{2}\left(1+\sqrt{1+\frac{4cxy}{(c-1)^2}}\right).$$

**Lemma 5.** *For any* $(x, y) \in \mathfrak{D}_c$ *we have* $\mathcal{I}_c(x, y) = (y, x)$. *For any* $(x, y) \in \mathfrak{D}_c \backslash \Pi_{\infty,c}$ *we have* $\mathcal{R}_c(x, y) = (x', y')$ *with* $y' = x$.

*Proof.* The statement concerning $\mathcal{I}_c$ can be checked by direct calculation, and the one about $\mathcal{R}_c$ follows from the above-mentioned properties of $\mathcal{F}$ and $\mathcal{S}$: $\mathcal{F}$ preserves $x$, and $\mathcal{S}$ interchanges $x$ and $y$.

Notice that $\gamma_{0,c}$ for $c > 1$ in $(x, y)$-coordinates is a straight line given by $x = (c-1)^2/4$. Hence the domain $\check{\mathfrak{D}}_c$ is a triangle with the sides $x - c^3 y = c(c-1)$, $x = (c-1)^2/4$ and $y = 0$ in the case $c > 1$ and a triangle with the sides $x - c^3 y = c(c-1)$, $x = 0$ and $y = (1/c - 1)^2/4$ (which is $\gamma_{0,c}^-$) in the case $c < 1$. The two triangles are, of course, symmetric w.r.t. the identity line, and $\mathcal{I}_c$ is the symmetry.

In view of Lemma 5 it is natural to study the metrical properties of $\mathcal{R}_c$ on $\mathfrak{D}_c$ in the metric

$$\mathbf{d}_c[(x, y), (\tilde{x}, \tilde{y})] = |x - \tilde{x}| + |y - \tilde{y}|.$$

The following statement establishes crucial Lipschits properties for the curves listed in Proposition 1 inside the domain $\check{\mathfrak{D}}_c$ expressed in $(x, y)$-coordinates.

**Proposition 4.** *Consider any two points* $(x_1, y_1), (x_2, y_2) \in \check{\mathfrak{D}}_c$ *belonging to one of the following sets:* $\gamma_{\rho,c}$, *where* $\rho \in (0, 1)\backslash\mathbb{Q}$; $\gamma^{(1)}_{p/q,c}$, *where* $p/q \in (0, 1)$; $\gamma^{(2)}_{p/q,c}$, *where* $p/q \in (0, 1)$; $\gamma_{1/k,c}$, *where* $2 \le k < \infty$; *or* $\gamma^{(1)}_{1,c}$, *where* $c < 1$. *Then* $|x_2 - x_1| < B_c|y_2 - y_1|$, *where* $B_c = c^3$ *for* $c > 1$ *and* $B_c \in (0, c^3)$ *for* $c < 1$ *does not depend on the choice of the points.*

*4.7. Proof of Proposition 4.* We will exploit the same idea as in the proof of Proposition 1. Let us do some settings before we start. The set $\check{\mathfrak{D}}_c$ can be parameterized in different ways. With parameters $(a, v)$ the commuting pairs from $\check{\mathfrak{D}}_c$ are written in terms of rational functions. An alternative parameterization for $\check{\mathfrak{D}}_c$ is $(x, y)$. Let us introduce two more variables:

$$s = x - c^3 y, \qquad \bar{s} = x + c^3 y.$$

Any two of the coordinate systems $(a, v)$, $(x, y)$, $(a, s)$, and $(a, \bar{s})$ on $\check{\mathfrak{D}}_c$ are conjugate by a $C^\infty$ bijective change of variables. We will use the same letter $H_c$ to denote the corresponding functions in different coordinate systems on $\check{\mathfrak{D}}_c$, specifying the set of variables we are looking at as the set of arguments of $H_c(w; \cdot, \cdot)$. Notice, that the 3D manifold $\mathbb{R} \times \check{\mathfrak{D}}_c$ is split into domains by the surfaces $w = m$ and $w = \phi(F^{-1}_{a,v,c}(0)) + m$, $m \in \mathbb{Z}$, and $H_c$ is $C^{2+\alpha}$-smooth on the closures of these domains.

**Lemma 6.** *For a point* $(a,s)\in\check{\mathfrak{D}}_c\backslash\gamma_{1,c}$ *we have*

$$\frac{\partial H_c^{(1)}(w;a,s)}{\partial a}>0\quad \textit{for all}\quad w\in\left[-1,-\frac{a+1}{c+1+(c-a)}\right],$$
$$\frac{\partial H_c^{(2)}(w;a,s)}{\partial a}>0\quad \textit{for all}\quad w\in\left[-\frac{a+1}{c+1+(c-a)},0\right].$$

*Proof.* We have $v=\frac{c^3-c^2-sa}{c^2-a^2}$, hence $H_c^{(1)}(w;a,s)=\frac{A_1+B_1w}{C_1+D_1w}$, $H_c^{(2)}(w;a,s)=\frac{A_2+B_2w}{C_2+D_2w}$, where $A_1=(c^2-a^2)(a+1)^2$, $B_1=(c^2-a^2)(a+1)(c+1+(c-a))$, $C_1=(c^2-a^2)(a+1)^2$, $D_1=-a^2-2c^3a+c^2+4c^2a+a^4-a^2c^2-2a^3c+2a^2c-2c^3+4a^2s$; $A_2=(a+1)^2(a+c)(c-a)^2$, $B_2=(a+1)(3a^3c+a^2c^2-2c^4a-c^3a+a^3-a^4-c^2a+c^3-a^2c+2a^2cs)$, $C_2=(a+1)(a^4-a^3c+a^3-a^2c^2+a^2c-c^3a+c^2a-c^3+2a^2s)$, $D_2=c^2a-3c^3a^2-3a^3c^2+3a^4c-a^5+2c^4a+2c^4a^2+a^3+a^2c-c^3-2a^2c^2+2a^2(c+1)(1-a)s$.

The derivative $\frac{\partial H_c^{(1)}(w;a,s)}{\partial a}=\frac{-4wP_1}{Q_1^2}$, where $P_1=2a(-a^4+c^2+c^3a+2c^3+a^3c)sw+c(-5c^3a^2+a^2c+3a^4c^2-a^4c+c^4-4c^3a^3-2a^5+c^3-2c^4a+a^2c^2+8a^3c^2-c^4a^2)w+2a(a+1)(a^3+c^2)s+c(a+1)(2a^4-3a^3c^2+3a^3c-5a^2c^2+a^2c-c^3a+c^4a+c^3+c^4)$ and $Q_1=4a^2sw+(-a^2-2c^3a+c^2+4c^2a+a^4-a^2c^2-2a^3c+2a^2c-2c^3)w+(a+1)^2(c-a)(a+c)$. Let us fix arbitrary $a\in(0,c)$. It is easy to check that for points in $\check{\mathfrak{D}}_c\backslash\gamma_{1,c}$ we have $s\in[a(c-1),c(c-1)]$ for $c>1$ and $s\in\left[c(c-1),(a+(1-c)(c^2-a^2)/(4c))(c-1)\right]$ for $c<1$. Since $-4w/Q_1^2$ is strictly positive, and $P_1$ is linear w.r.t. both $s$ and $w$, it is enough to check that $P_1>0$ at the following corner points: $w=-1$, $s=c(c-1)$; $w=-(a+1)/(c+1+(c-a))$, $s=c(c-1)$; $w=-1$, $s=a(c-1)$ for $c>1$; $w=-1$, $s=(a+(1-c)(c^2-a^2)/(4c))(c-1)$ for $c<1$; and $w=-(a+1)/(c+1+(c-a))$, $s=a(c-1)$ (accidentally, this point works for $c<1$ as well). Substitution shows that indeed at those points $P_1$ gets the positive values: $4a(c-a)^2c^2(a+1)^2$; $2(1+c)(a+1)^2(c-a)^2(a+c)c^2/(c+1+(c-a))$; $2a(c-a)^2(a+c)(2a^2(c-1)+a(2c-1)+c^2a+2c^2)$; $a(c-a)^2(ac+(c-a)+c)(a+c)(2a^2c+2(c-a)(c+a)+c^2a+4ac+(c-a)+c^3)/(2c)$; and $2(c+1)(a+1)(c-a)^2(a+c)^2(ac+c-a)/(c+1+(c-a))$ respectively.

Next, $\frac{\partial H_c^{(2)}(w;a,s)}{\partial a}=2aP_2/Q_2^2$, where $P_2=(a+1)^2(c-a)(c(-2a^3-a^2+3a^2c^2-4a^2c-3ac+c^3a+4c^2a+2c^3)-(2a^3+a^2+a^2c+ac+c^2a+2c^2)s)-2(c+1)(a-1)(a+1)(c-a)(c(-2a^3-a^2+3a^2c^2-4a^2c-3ac+c^3a+4c^2a+2c^3)-(2a^3+a^2+a^2c+ac+c^2a+2c^2)s)w+(c(10a^3c^2-6a^4c^2+a^3+2a^2c+2a^4c-3a^5c^2+12c^4a^4-6a^5c^3+2a^2c^2-3a^5-3c^2a-2a^3c^5-4c^6a+4c^5-4c^3a+2a^6-2c^3a^3+4ca^6-14a^4c^3+2c^4-10c^3a^2+4c^4a^2+17c^4a^3-2c^5a+c^4a-4a^2c^5)+(-3a^5c-2c^3+a^3+c^2a+4c^5a-4c^4a^2+a^3c+5c^3a+c^3a^3-2a^2c^2+2a^6-6a^5c^2-3a^5-6c^4a^3+3a^3c^2-4c^4+2c^4a+4ca^6+6a^4c)s+4a^3c(c+1)s^2)w^2$ and $Q_2=-(a+1)(a^4+a^3-a^3c-a^2c^2+a^2c-c^3a+c^2a-c^3)+(-3a^4c-2c^4a+2a^2c^2-c^2a+3a^3c^2+c^3-a^2c-a^3-2c^4a^2+3c^3a^2+a^5)w-2a^2(a+1)s+2a^2(c+1)(a-1)sw$. Since $2a/Q_2^2$ is positive, it is enough to check that $P_2>0$. Let us fix arbitrary $a\in(0,c)$. The value of $P_2$ at $w=-(a+1)/(c+1+(c-a))$ is $4ac(c+1)(a+1)^2L/(c+1+(c-a))^2$, where $L=(s-c^2+a-ac+a^2)(a^2c+a^2s-c^3a+c^2a-c^3)$; the value of $P_2$ at $w=0$ is $(a+1)^2R/(2a)$, where $R=2a(c-a)(-(2a^3+a^2+a^2c+ac+c^2a+2c^2)s-c(2a^3-3a^2c^2+a^2+4a^2c+3ac-c^3a-4c^2a-2c^3))$ is exactly the value of $P_1$ at $w=-1$, which was shown in the previous paragraph to be positive. Notice that $L$ is quadratic w.r.t. $s$. The derivative $\frac{dL}{ds}$ is equal to $-(a+1)(c-a)(a^2+c^2)$ at $s=c(c-1)$, and $-(a^2+ca+c-a)(c-a)(a+c)$ at $s=a(c-1)$; since these both values are negative, then the vertex of $L$ lies beyond the segment of our interest. At the endpoints $s = c(c-1)$ and $s = a(c-1)$ we have $L = (a+1)^2(c-a)^2c^2 > 0$ and $L = (c-a)^2(a+c)^2(ac+c-a) > 0$ respectively. Thus, both $L$ and $R$ are positive at any point $(a, s)$ in $\check{\mathfrak{D}}_c \backslash \gamma_{1,c}$. Now we fix $s$, too, and notice that $P_2/w^2$ is a quadratic polynomial w.r.t. $1/w$, whose vertex $1/w = (1+c)(a-1)/(a+1)$ lies beyond the interval $(-\infty, -(c+1+(c-a))/(a+1)]$ of our interest. The proof is complete.

**Lemma 7.** *Let $A = A(w, a, c)$ be the linear function in $w$ that is equal to $a$ at $w = -1$ and equal to $1$ at $w = -(a+1)/(c+1+(c-a))$.*

*For a point $(a, s) \in \check{\mathfrak{D}}_c \backslash (\Gamma_{1,c} \backslash \gamma_{1,c}^{(1)})$, $c < 1$, the quantities*

$$\frac{1}{(-w)A} \cdot \frac{\partial H_c^{(1)}(w; a, s)}{\partial a} \quad \textit{for } w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right],$$
$$\frac{1}{a} \cdot \frac{\partial H_c^{(2)}(w; a, s)}{\partial a} \quad \textit{for } w \in \left[-\frac{a+1}{c+1+(c-a)}, 0\right]$$

*are bounded and separated from zero by positive constants depending on $c$ only.*

*Proof.* It follows from Proposition 1 that inside the set $\check{\mathfrak{D}}_c \backslash (\Gamma_{1,c} \backslash \gamma_{1,c}^{(1)})$, $c < 1$, the quantity $c - a$ is separated from zero by a positive constant depending on $c$ only. Having that in mind, we will look through the proof of Lemma 6 once again.

First, let us check that $Q_1$ and $Q_2$ from that proof are bounded and separated from zero. They are linear w.r.t. both $s$ and $w$, hence it is enough to check the corners. At the points $w = -1$, $s = c(c-1)$; $w = -(a+1)/(c+1+(c-a))$, $s = c(c-1)$; $w = -1$, $s = a(c-1)$; and $w = -(a+1)/(c+1+(c-a))$, $s = a(c-1)$ the values of $Q_1$ are $2(a+1)(a^2+c^2)(c-a)$; $4(a+1)^2c^2(c-a)/(c+1+(c-a))$; $2(c+a)(a^2+ac+c-a)(c-a)$; and $4(a+1)(a+c)(ac+c-a)(c-a)/(c+1+(c-a))$ respectively. At the points $w = 0$, $s = c(c-1)$; $w = -(a+1)/(c+1+(c-a))$, $s = c(c-1)$; $w = 0$, $s = a(c-1)$; and $w = -(a+1)/(c+1+(c-a))$, $s = a(c-1)$ the values of $Q_2$ are $(a+1)^2(a^2+c^2)(c-a)$; $2c^3(a+1)^3(c-a)/(c+1+(c-a))$; $(a+1)(a^2+ac+c-a)(a+c)(c-a)$; and $2c(a+1)^2(a+c)(ac+c-a)(c-a)/(c+1+(c-a))$ respectively. It is easy to see that all the listed values are indeed bounded and separated from zero.

Next, let us show that $P_1/A$ is bounded and separated from zero. We do not need to write down the expression for $A$ explicitly. Since both $P_1$ and $A$ are linear w.r.t. $w$, it is enough to check the endpoints $w = -1$ and $w = -(a+1)/(c+1+(c-a))$. From the proof of Lemma 6 it is evident that $P_1/a$ at $w = -1$ and $P_1$ at $w = -(a+1)/(c+1+(c-a))$ are bounded and separated from zero.

Finally, it is easy to check along the lines of the proof of Lemma 6 that $P_2$ is bounded and separated from zero. (The only tricky step appears when we revisit the last statement in that proof, namely considering $P_2/w^2$ as a parabola w.r.t. $1/w$, since we have divided and multiplied $P_2$ by $w^2$, which may be close to zero. It is true however that $P_2$ at $w = 0$ is separated from zero, and the same is true for the parabola $P_2/w^2$, $1/w \in (-\infty, -(c+1+(c-a))/(a+1)]$. Since the derivative $\frac{\partial}{\partial w} P_2$ is bounded, we can find a constant $\delta = \delta(c) > 0$ such that $P_2$ is separated from zero on the whole interval $w \in [-\delta, 0]$, and for $w \in [-(a+1)/(c+1+(c-a)), -\delta)$ it will be greater than $\delta^2 \cdot P_2/w^2$.)

**Lemma 8.** *For a point* $(a, \bar{s}) \in \check{\mathfrak{D}}_c \setminus \gamma_{1,c}$ *we have*

$$\frac{\partial H_c^{(1)}(w; a, \bar{s})}{\partial a} > 0 \quad \textit{for all } w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right],$$
$$\frac{\partial H_c^{(2)}(w; a, \bar{s})}{\partial a} > 0 \quad \textit{for all } w \in \left[-\frac{a+1}{c+1+(c-a)}, 0\right].$$

*Proof.* We have $v = (c^3 - c^2 + \bar{s}a)/(c^2 + a^2)$, hence $H_c^{(1)}(w; a, \bar{s}) = \frac{A_1+B_1 w}{C_1+D_1 w}$, $H_c^{(2)}(w; a, \bar{s}) = \frac{A_2+B_2 w}{C_2+D_2 w}$, where $A_1 = (c^2+a^2)(a+1)^2$, $B_1 = (c^2+a^2)(a+1)(c+1+(c-a))$, $C_1 = (c^2+a^2)(a+1)^2$, $D_1 = a^2-2ac^3+c^2+4c^2a-a^4-a^2c^2+2ca^3-2ca^2-2c^3-4a^2\bar{s}$; $A_2 = (a+1)^2(c^2+a^2)(c-a)$, $B_2 = (a+1)(-3ca^3+a^2c^2-2c^4a-c^3a-a^3+a^4-c^2a+c^3+a^2c-2ca^2\bar{s})$, $C_2 = (a+1)(-a^4+a^3c-a^3-a^2c^2-a^2c-c^3a+c^2a-c^3-2a^2\bar{s})$, $D_2 = c^2a - 3c^3a^2 + 5a^3c^2 - 3a^4c + a^5 + 2c^4a + 2c^4a^2 - a^3 - a^2c - c^3 - 2a^2c^2 + 2a^2(c+1)(a-1)\bar{s}$.

The derivative $\frac{\partial H_c^{(1)}(w;a,\bar{s})}{\partial a} = -4wP_1/Q_1^2$, where $P_1 = 2a(-a^4 - c^2 - c^3a - 2c^3 + a^3c)\bar{s}w + c(c-a)(2a^4 + 3a^3c^2 - 3a^3c - a^2c^3 + 5a^2c^2 + ca - 2c^3a + c^2a + c^2 + c^3)w + 2a(a+1)(a^3-c^2)\bar{s} + c(a+1)(2a^4+3a^3c^2-3a^3c+5a^2c^2-a^2c-c^3a+c^4a+c^3+c^4)$ and $Q_1 = 4a^2\bar{s}w+(-a^2+2c^3a-c^2-4c^2a+a^4+a^2c^2-2a^3c+2a^2c+2c^3)w-(a+1)^2(a^2+c^2)$. Let us fix arbitrary $a \in (0, c)$. It is easy to check that for points in $\check{\mathfrak{D}}_c \setminus \gamma_{1,c}$ we have $\bar{s} \in [a(c-1), -c(c-1)]$ for $c < 1$ and $\bar{s} \in [-(c+1)(c-1)/2, a(c-1)]$ for $c > 1$. Since $-4w/Q_1^2$ is strictly positive, and $P_1$ is linear w.r.t. both $\bar{s}$ and $w$, it is enough to check that $P_1 > 0$ at the following corner points: $w = -1$, $\bar{s} = a(c-1)$; $w = -(a+1)/(c+1+(c-a))$, $\bar{s} = a(c-1)$; $w = -1$, $\bar{s} = -c(c-1)$ for $c < 1$; $w = -1$, $\bar{s} = -(c+1)(c-1)/2$ for $c > 1$; and $w = -(a+1)/(c+1+(c-a))$, $\bar{s} = -c(c-1)$ (accidentally, this point works for $c > 1$ as well). Substitution shows that at those points $P_1$ gets the positive values: $2a(a^2+c^2)(2a^3c + 2a^2c + (2a^2 + c^2a + a + 2c^2)(c-a))$; $2(c+1)(a+1)(a^2+c^2)^2(ac+c-a)/(c+1+(c-a))$; $4ac((ac^2+a^2c^2+a^4+a^3)(1-c) + a^2(c-a)^2 + c(c^2-a^2) + 2a^2c^2(a+1))$; $a(2a^4c + a^4c^2 + c^3 + a^3 + 2a^4 + 3a^2c^3 + a^3c^3 + 3c^3a + a^3c + c^4a + (2+a)c(c-a)^4 + c(c + a^4 + 4ca + a^2c)(c-a))$; and $2(c+1)(a+1)(a^2+c^2)c(2a^2 + (a+1)c(c-a))$ respectively.

Next, $\frac{\partial H_c^{(2)}(w;a,\bar{s})}{\partial a} = 2aP_2/Q_2^2$, where $P_2 = (a+1)^2(c(2a^4 - 4ca^3 + a^3 + 3a^3c^2 - 2ca^2 - 2a^2c^3 + 8a^2c^2 + 3c^2a + ac^4 - 4ac^3 + 2c^4) + (2a^4 + a^3 - ca^3 - c^2a + ac^3 + 2c^3)\bar{s}) - 2(1+c)(a-1)(a+1)(c(2a^4 - 4ca^3 + a^3 + 3a^3c^2 - 2ca^2 - 2a^2c^3 + 8a^2c^2 + 3c^2a + ac^4 - 4ac^3 + 2c^4) + (2a^4 + a^3 - ca^3 - c^2a + ac^3 + 2c^3)\bar{s})w + (c(-4a^2c^4 - 19a^3c^4 - 11ac^4 + 3c^2a + a^3 - 2a^2c^2 + 2ac^3 + 2ca^3 - 2ca^2 - 3a^5 - 14a^3c^2 + 10ca^4 + 14a^2c^3 + 6a^4c^2 - 6ca^5 + 2a^6 + 2c^4 + 6a^3c^5 - 9c^2a^5 - 2c^5a + 12a^2c^5 + 14a^4c^3 + 6a^5c^3 - 12a^4c^4 + 4ca^6 + 4c^5 - 4c^6a) + (7a^3c^3 + 2a^2c^2 - 2c^4a + 4a^2c^4 + 6a^3c^4 + a^3 + 2a^6 + a^3c^2 - 6c^2a^5 - 5c^3a + 2c^3 - c^2a - 4c^5a + 4c^4 - 3a^5 + 4ca^6 + a^3c + 6a^4c - 3a^5c)\bar{s} + 4a^3c(c+1)\bar{s}^2)w^2$ and $Q_2 = -(a+1)(a^4-ca^3+a^3+a^2c^2+ca^2+ac^3-c^2a+c^3)+(2ac^4+2a^2c^4-3ca^4-ca^2-3a^2c^3+c^2a-2a^2c^2-a^3+5a^3c^2+a^5-c^3)w-2a^2(a+1)\bar{s}+2a^2(1+c)(a-1)\bar{s}w$. Since $2a/Q_2^2$ is positive, it is enough to check that $P_2 > 0$. Let us fix arbitrary $a \in (0, c)$. The value of $P_2$ at $w = -(a+1)/(c+1+(c-a))$ is $4ac(c+1)(a+1)^2L/(c+1+(c-a))^2$, where $L = (\bar{s}+c^2+a-ac+a^2)(a^2\bar{s}+c^3-c^2a+ac^3+ca^2)$; the value of $P_2$ at $w = 0$ is $(a+1)^2R/(2a)$, where $R = 2a(c(-4ac^3+2c^4-2a^2c^3+3a^3c^2-2a^2c^1+ca^2+3ac^2+8a^2c^2-4a^3c^1+2a^4+c^4a)+(a^3-c^2a+ac^3-ca^3+2a^4+2c^3)\bar{s})$ is exactly the value of $P_1$ at $w = -1$, which was shown in the previous paragraph to be positive. Notice that $L$ is quadratic w.r.t. $s$. The derivative $\frac{dL}{d\bar{s}}$ is equal to $(ac+c-a+a^2)(a^2+c^2)$ at $\bar{s} = a(c-1)$, and $3ca^2 + a^3 + a(c+a)(c-a)^2 + c^2(c-a)$ at $\bar{s} = -c(c-1)$; since both values are positive, then the vertex of $L$ lies beyond the segment of our interest. At the endpoints $\bar{s} = a(c-1)$; $\bar{s} = -c(c-1)$ for $c < 1$; and $\bar{s} = -(c+1)(c-1)/2$ for $c > 1$ the values of $L$ are positive: $(a^2+c^2)^2(ac+c-a)$; $(c+a(1-c)+a^2)(a^2(1-c)+a^2+c(c-a)+c^2a)$ and $((a+1)^2+(a+c)^2)(2c^2(c-a)+ac^2(c-a)+a^2+ac^3+2ca^2)/4$ respectively. Thus, both $L$ and $R$ are positive at any point $(a,s)$ in $\check{\mathfrak{D}}_c\backslash\gamma_{1,c}$. Now we fix $s$, too, and notice that $P_2/w^2$ is again a quadratic polynomial w.r.t. $1/w$, whose vertex $1/w = (1+c)(a-1)/(a+1)$ lies beyond the interval $(-\infty, -(c+1+(c-a))/(a+1)]$ of our interest. The proof is finished.

**Lemma 9.** *For a point* $(a,\bar{s}) \in \check{\mathfrak{D}}_c\backslash(\Gamma_{1,c}\backslash\gamma_{1,c}^{(1)})$, $c < 1$, *the quantities*

$$\frac{1}{(-w)A}\cdot\frac{\partial H_c^{(1)}(w;a,\bar{s})}{\partial a} \quad \text{for } w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right],$$
$$\frac{1}{a}\cdot\frac{\partial H_c^{(2)}(w;a,\bar{s})}{\partial a} \quad \text{for } w \in \left[-\frac{a+1}{c+1+(c-a)}, 0\right],$$

*are bounded and separated from zero by positive constants depending on c only (here A is the same as in Lemma 7).*

*Proof.* We look through the proof of Lemma 8 again, having in mind that $c-a$ is separated from zero.

Let us check that $Q_1$ and $Q_2$ from the proof of Lemma 8 are bounded and separated from zero. They are linear w.r.t. both $s$ and $w$, hence it is enough to check the corners. At the points $w=-1$, $s=-c(c-1)$; $w=-(a+1)/(c+1+(c-a))$, $s=-c(c-1)$; $w=-1$, $s=a(c-1)$; and $w=-(a+1)/(c+1+(c-a))$, $s=a(c-1)$ the values of $Q_1$ are $8ca^2+2(a+1)(a+c)(c-a)^2$; $(8a^2+c(a+1)(c-a))/(c+1+(c-a))$; $2(ac+(c-a)+a^2)(c^2+a^2)$; and $4(a+1)(ac+(c-a))(c^2+a^2)/(c+1+(c-a))$ respectively. At the points $w=0$, $s=-c(c-1)$; $w=-(a+1)/(c+1+(c-a))$, $s=-c(c-1)$; $w=0$, $s=a(c-1)$; and $w=-(a+1)/(c+1+(c-a))$, $s=a(c-1)$ the values of $Q_2$ are $(a+1)(4ca^2+(a+1)(a+c)(c-a)^2)$; $2c^2(a+1)^2(2a^2+c(a+1)(c-a))/(c+1+(c-a))$; $(a+1)(ac+(c-a)+a^2)(c^2+a^2)$; and $2c(a+1)^2(ac+(c-a))(c^2+a^2)/(c+1+(c-a))$ respectively. It is easy to see that all the listed values are indeed bounded and separated from zero.

Showing that $P_1/A$ and $P_2$ are bounded and separated from zero is similar to the proof of the corresponding fact in Lemma 7.

**Lemma 10.** *Let* $(x,y) \in \check{\mathfrak{D}}_c\backslash\gamma_{1,c}$, $c > 1$, *or* $(x,y) \in \check{\mathfrak{D}}_c\backslash(\Gamma_{1,c}\backslash\gamma_{1,c}^{(1)})$, $c < 1$. *There exist constants* $B_c \in (0, c^3)$ *for* $c < 1$ *and* $B_c = c^3$ *for* $c > 1$ *such that the derivatives*

$$\overrightarrow{(B_c, 1)}\cdot\nabla_{(x,y)}H_c^{(1)}(w;x,y) \quad \text{for } w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right],$$
$$\overrightarrow{(B_c, 1)}\cdot\nabla_{(x,y)}H_c^{(2)}(w;x,y) \quad \text{for } w \in \left[-\frac{a+1}{c+1+(c-a)}, 0\right],$$
$$\overrightarrow{(B_c, -1)}\cdot\nabla_{(x,y)}H_c^{(1)}(w;x,y) \quad \text{for } w \in \left[-1, -\frac{a+1}{c+1+(c-a)}\right],$$
$$\overrightarrow{(B_c, -1)}\cdot\nabla_{(x,y)}H_c^{(2)}(w;x,y) \quad \text{for } w \in \left[-\frac{a+1}{c+1+(c-a)}, 0\right]$$

*are positive in the case of* $c > 1$ *and negative in the case of* $c < 1$.

*Proof.* It is easy to see that

$$\overrightarrow{(c^3,1)} \cdot \nabla_{(x,y)} H_c^{(i)}(w;x,y) = c^3 \frac{\partial H_c^{(i)}(w;x,s)}{\partial x} = c^3 \frac{\partial H_c^{(i)}(w;a,s)}{\partial a} : \frac{\partial x(a,s)}{\partial a}, \quad i \in \{1,2\};$$

$$\overrightarrow{(c^3,-1)} \cdot \nabla_{(x,y)} H_c^{(i)}(w;x,y) = c^3 \frac{\partial H_c^{(i)}(w;x,\bar{s})}{\partial x} = c^3 \frac{\partial H_c^{(i)}(w;a,\bar{s})}{\partial a} : \frac{\partial x(a,\bar{s})}{\partial a}, \quad i \in \{1,2\}.$$

We have $x = a(c^2(c-1) - as)/(c^2 - a^2)$ and $x = a(c^2(c-1) + a\bar{s})/(c^2 + a^2)$, therefore $\frac{\partial x(a,s)}{\partial a} = c^2((c-1)(c^2+a^2) - 2as)/(c^2-a^2)^2$ and $\frac{\partial x(a,\bar{s})}{\partial a} = c^2(2a\bar{s} + (c-1)(c-a)(c+a))/(c^2+a^2)^2$.

In the case of $c > 1$, we have $s \in [a(c-1), c(c-1)]$ and $\bar{s} \in [-c(c-1)(c-a)/(c+a), a(c-1)]$, hence $\frac{\partial x(a,s)}{\partial a} \geq c^2(c-1)/(c+a)^2 > 0$ and $\frac{\partial x(a,\bar{s})}{\partial a} \geq c^2(c-1)(c-a)/((c^2+a^2)(c+a)) > 0$. The result follows from Lemmas 6 and 8.

In the case of $c < 1$, the quantity $c - a$ is separated from zero. We similarly obtain the inequalities $\frac{\partial x(a,s)}{\partial a} \leq c^2(c-1)/(c+a)^2 < 0$ and $\frac{\partial x(a,\bar{s})}{\partial a} \leq c^2(c-1)(c-a)/((c^2+a^2)(c+a)) < 0$, but now they imply that both $\frac{\partial x(a,s)}{\partial a}$ and $\frac{\partial x(a,\bar{s})}{\partial a}$ are separated from zero (by negative constants). It is obvious that these derivatives are (negatively) bounded, too. It follows therefore from Lemmas 7 and 9 that there exists such $\varepsilon_c > 0$ that the quantities

$$c^3(-w)A\left((1-\varepsilon_c)\frac{1}{(-w)A}\frac{\partial H_c^{(1)}(w;a,s)}{\partial a} : \frac{\partial x(a,s)}{\partial a} - \varepsilon_c \frac{1}{(-w)A}\frac{\partial H_c^{(1)}(w;a,\bar{s})}{\partial a} : \frac{\partial x(a,\bar{s})}{\partial a}\right)$$

for $w \in [-1, -(a+1)/(c+1+(c-a))]$ and

$$c^3 a\left((1-\varepsilon_c)\frac{1}{a}\frac{\partial H_c^{(2)}(w;a,s)}{\partial a} : \frac{\partial x(a,s)}{\partial a} - \varepsilon_c \frac{1}{a}\frac{\partial H_c^{(2)}(w;a,\bar{s})}{\partial a} : \frac{\partial x(a,\bar{s})}{\partial a}\right)$$

for $w \in [-(a+1)/(c+1+(c-a)), 0]$ are both negative. But they are exactly equal to $\overrightarrow{(B_c,1)} \cdot \nabla_{(x,y)} H_c^{(i)}(w;x,y)$, $i \in \{1,2\}$, where $B_c = (1-2\varepsilon_c)c^3$. For $\overrightarrow{(B_c,-1)} \cdot \nabla_{(x,y)} H_c^{(i)}(w;x,y)$, $i \in \{1,2\}$, the proof is similar.

**Corollary 5.** *In the case of* $c > 1$*,* $H_c(w;x,y)$ *is strictly increasing for every* $w \in \mathbb{R}$ *along every line* $x + c^3 y = \text{const}$ *and every line* $x - c^3 y = \text{const}$ *in* $\check{\mathfrak{D}}_c \setminus \gamma_{1,c}$ *as* $x$ *increases.*

*In the case of* $c < 1$*,* $H_c(w;x,y)$ *is strictly decreasing for every* $w \in \mathbb{R}$ *along every line* $x + B_c y = \text{const}$ *and every line* $x - B_c y = \text{const}$ *in* $\check{\mathfrak{D}}_c \setminus (\Gamma_{1,c} \setminus \gamma_{1,c}^{(1)})$ *as* $x$ *increases, where* $B_c \in (0, c^3)$.

*Proof (Proof of Proposition 4).* This proof is similar to the proof of the inequalities in Proposition 1. Just notice that the condition $(x, y) \in \gamma$, where $\gamma$ is one of the listed sets, implies that $(x, y) \in \check{\mathfrak{D}}_c \backslash \gamma_{1,c}$ in the case of $c > 1$ and $(x, y) \in \check{\mathfrak{D}}_c \backslash (\Gamma_{1,c} \backslash \gamma_{1,c}^{(1)})$ in the case of $c < 1$. It follows from Corollary 5 that, given an arbitrary point $(x, y) \in \gamma$, for any point $(x', y') \neq (x, y)$ such that either $x' - B_c y' \leq x - B_c y$ and $x' + B_c y' \leq x + B_c y$, or $x' - B_c y' \geq x - B_c y$ and $x' + B_c y' \geq x + B_c y$, we have either $H_c(w; x', y') < H_c(w; x, y)$ for all $w \in \mathbb{R}$, or $H_c(w; x', y') > H_c(w; x, y)$ for all $w \in \mathbb{R}$, and therefore $(x', y') \notin \gamma$. The inequalities follow.

*4.8. Hyperbolicity.* The transformation $\mathcal{R}^2 = \mathcal{R}_{1/c} \circ \mathcal{R}_c$ of the domain $\check{\mathfrak{D}}_c$ has strong hyperbolic properties. Namely, it contracts the domain along the family of curves $\{\gamma_{\rho,c}\}$ and stretches it along the transversal family $\mathcal{I}_{1/c}(\{\gamma_{\rho,1/c}\})$.

**Theorem 1.** *For every* $0 < c \neq 1$ *there exists a constant* $\mu_c \in (0, 1)$ *such that for any two points* $(x, y), (\tilde{x}, \tilde{y}) \in \check{\mathfrak{D}}_c$ *with the same irrational rotation number the following inequality holds:*

$$\mathbf{d}_c[\mathcal{R}_c^2(x, y), \mathcal{R}_c^2(x, y)] \leq \mu_c\, \mathbf{d}_c[(x, y), (\tilde{x}, \tilde{y})].$$

*Proof.* Denote $(x_0, y_0) = (x, y)$, $(x_1, y_1) = \mathcal{R}_c(x_0, y_0)$, $(x_2, y_2) = \mathcal{R}_{1/c}(x_1, y_1)$, and $(\tilde{x}_0, \tilde{y}_0) = (\tilde{x}, \tilde{y})$, $(\tilde{x}_1, \tilde{y}_1) = \mathcal{R}_c(\tilde{x}_0, \tilde{y}_0)$, $(\tilde{x}_2, \tilde{y}_2) = \mathcal{R}_{1/c}(\tilde{x}_1, \tilde{y}_1)$. Due to Lemma 5 and Proposition 4, we have

$$|x_2 - \tilde{x}_2| < B_c|y_2 - \tilde{y}_2| = B_c|x_1 - \tilde{x}_1| < B_c B_{1/c}|y_1 - \tilde{y}_1| = B_c B_{1/c}|x_0 - \tilde{x}_0|,$$
$$|y_2 - \tilde{y}_2| = |x_1 - \tilde{x}_1| < B_{1/c}|y_1 - \tilde{y}_1| = B_{1/c}|x_0 - \tilde{x}_0| < B_{1/c} B_c|y_0 - \tilde{y}_0|,$$

therefore $\mathbf{d}_c[(x_2, y_2), (\tilde{x}_2, \tilde{y}_2)] \leq \mu_c \mathbf{d}_c[(x_0, y_0), (\tilde{x}_0, \tilde{y}_0)]$, where $\mu_c = B_c B_{1/c} < 1$.

Theorem 1 means that for every $c \neq 1$ the operator $\mathcal{R}^2$ is uniformly hyperbolic on $\check{\mathfrak{D}}_c$ in the metric $\mathbf{d}_c$. The curves $\{\gamma_{\rho,c}\}_{\rho \notin \mathbb{Q}}$ are its stable manifolds and $\mathcal{I}_{1/c}\{\gamma_{\rho,1/c} \cap \dot{\mathfrak{D}}_{1/c}\}_{\rho \notin \mathbb{Q}}$ are the unstable ones. The intersection points of stable and unstable curves, i.e., elements of the set $\mathfrak{A}_c = (\mathfrak{R}_c \cap \check{\mathfrak{D}}_c) \cap \mathcal{I}_{1/c}(\mathfrak{R}_{1/c} \cap \check{\mathfrak{D}}_{1/c})$, are exactly those points, whose generalized rotation numbers expansions are infinite both to the left and to the right. Thus, the set $\mathfrak{A}_c$ is mapped onto $\mathfrak{A}_{1/c}$ by $\mathcal{R}_c$ in one-to-one fashion, and the corresponding left shift in generalized rotation numbers represents classical symbolic dynamics on the horseshoe-type hyperbolic set. The closure $\bar{\mathfrak{A}}_c$ is a Cantor set of zero Lebesgue measure. The latter statement follows easily from the fact proved in [9]: for every vertical line $v = const$, the set of values of $a$ corresponding to irrational rotation numbers has zero one-dimensional Lebesgue measure.

## 5. Beyond the Linear-Fractional Family

In this section we consider renormalizations for general circle maps with breaks. As we have pointed out, such renormalizations approach the linear-fractional family (5) (see the statements (**B**) and (**C**) in Sect. 3).

*5.1. Half-bounded rotation numbers.* Let us denote by $M_{\mathrm{o}}$ and $M_{\mathrm{e}}$ the classes of all rotation numbers $\rho = [k_1, k_2, \ldots]$, for which the subsequences of partial quotients $k_n$ with odd and with even indices $n$ respectively are bounded:

$$M_{\mathrm{o}} = \{\rho : (\exists C > 0)\ (\forall m \in \mathbb{Z}_+)\ k_{2m-1} \leq C\},$$
$$M_{\mathrm{e}} = \{\rho : (\exists C > 0)\ (\forall m \in \mathbb{Z}_+)\ k_{2m} \leq C\}.$$

In this section we consider a circle diffeomorphism $T$ with a break of size $c$ and rotation number $\rho$ such that either $c > 1$ and $\rho \in M_{\mathrm{e}}$, or $c < 1$ and $\rho \in M_{\mathrm{o}}$. The subsequent renormalizations $(f_n, g_n) \in \mathfrak{S}^{2+\alpha}_{c^{(n)}}$. The renormalization height $k_n$ can be arbitrary large for the values of $n$ such that $c^{(n)} > 1$, while it is bounded for $n$ such that $c^{(n)} < 1$.

*5.2. Limiting bounds on renormalizations.*

**Proposition 5.** *There exists a constant $\varepsilon = \varepsilon(T) > 0$ such that for large enough n the following bounds hold: 1) $a^{(n)} \in (\varepsilon, c^{(n)} - \varepsilon)$, 2) $\frac{v^{(n)}}{c^{(n)}-1} \in (\varepsilon, 1-\varepsilon)$.*

*Proof.* 1) In the case $c^{(n)} < 1$ the value of $a^{(n)}$ cannot be too small since $f_n'$ and $g_n'$ are bounded due to (**A**) (see Sect. 3), and the renormalization height $k_{n+1}$ is bounded due to our settings.

Suppose that $c^{(n)} > 1$ and the value of $a^{(n)}$ is very small (in other words, $f_n(0)$ is very close to zero). Then $b^{(n)}$ is separated from zero due to (**B**), and therefore $v^{(n)}$ is bounded. The first derivative $f_n'(0)$ is close to $F'_{a^{(n)},v^{(n)},c^{(n)}}(0) = c^{(n)} + a^{(n)}v^{(n)}$, which is in turn close to $c^{(n)} > 1$. The second derivative $f_n''(0)$ is close to $F''_{a^{(n)},v^{(n)},c^{(n)}}(0) = 2v^{(n)}(c^{(n)} + a^{(n)}v^{(n)})$, which is bounded. It follows that $f_n(z^*) = z^*$ at some point $z^*$ in vicinity of zero, so $\rho(f_n, g_n) = 0$, which contradicts irrationality of $\rho(T)$.

We have proved in both cases that $a^{(n)}$ is separated from zero.

Suppose that $c^{(n)} - a^{(n)}$ is negative or close to zero. We have $c^{(n)} - a^{(n)} > -C\lambda^n$ due to (**B**), hence $c^{(n)} - a^{(n)}$ is close to zero. Again due to (**B**), $b^{(n)}$ is small. Notice now that $a^{(n+1)} \leq b^{(n)}/a^{(n)}$ by definition (6), therefore $a^{(n+1)}$ appears to be small, which contradicts what we have shown in the first part of this proof.

Thus the statement is proved. Its obvious corollary is that

$$a^{(n+1)}a^{(n)} \in (\mu, 1-\mu) \tag{12}$$

with some $\mu > 0$.

2) One may calculate that $v^{(n+1)} = a^{(n)}(c^{(n+1)} - a^{(n+1)})/f_n(-a^{(n+1)}a^{(n)}) - 1 = c^{(n+1)}(1 + a^{(n+1)}a^{(n)}v^{(n)}) - 1 + \mathcal{O}(\lambda^n)$ due to (**C**) and the statement 1).

Hence, the following property takes place:

$$\frac{v^{(n+2)}}{c^{(n+2)}-1} = (a^{(n+2)}a^{(n+1)})(a^{(n+1)}a^{(n)})\frac{v^{(n)}}{c^{(n)}-1} + \Big(1 - (a^{(n+2)}a^{(n+1)})\Big) + \mathcal{O}(\lambda^n). \tag{13}$$

In view of (12), it is easy to see that if the value of $\frac{v^{(n)}}{c^{(n)}-1}$ is negative, then in two steps it will be raised by an increment greater than a positive constant (which can be taken arbitrary close to $1-\mu$). And once $\frac{v^{(n)}}{c^{(n)}-1}$ is non-negative, in two steps it will become (and stay forever) greater than that positive constant.

Having rewritten the property (13) as

$$1-\frac{v^{(n+2)}}{c^{(n+2)}-1}=(a^{(n+2)}a^{(n+1)})\left((a^{(n+1)}a^{(n)})\left(1-\frac{v^{(n)}}{c^{(n)}-1}\right)\right.$$
$$\left.+(1-(a^{(n+1)}a^{(n)}))\right)+\mathcal{O}(\lambda^n),$$

we similarly prove that $1-\frac{v^{(n)}}{c^{(n)}-1}$ is separated from zero by a positive constant.

**Proposition 6.** *The point* $(a^{(n)},v^{(n)})$ *belongs to the* $\mathcal{O}(\lambda^n)$*-neighborhood of the set* $\check{\mathfrak{D}}_{c^{(n)}}$, *where* $\lambda\in(0,1)$ *is from* (**C**).

*Proof.* We assume that $n$ is large enough, so that the estimates from Proposition 5 hold. First let us derive that $(a^{(n+1)},v^{(n+1)})$ belongs to the $\mathcal{O}(\lambda^n)$-neighborhood of $\mathfrak{D}_{c^{(n+1)}}$. We have to show that

$$a^{(n+1)}\geq\frac{c^{(n+1)}(c^{(n+1)}-v^{(n+1)}-1)}{v^{(n+1)}}+\mathcal{O}(\lambda^n). \tag{14}$$

Let $c^{(n)}>1$. Since $b^{(n+1)}=f_n(-a^{(n)}a^{(n+1)})/a^{(n)}$, the statement (**C**) implies that $v^{(n+1)}=c^{(n+1)}-1+c^{(n+1)}a^{(n+1)}a^{(n)}v^{(n)}+\mathcal{O}(\lambda^n)$. It follows that $\frac{c^{(n+1)}-v^{(n+1)}-1}{a^{(n+1)}}-\frac{v^{(n+1)}}{c^{(n+1)}}=c^{(n)}-1-a^{(n)}v^{(n)}(c^{(n+1)}+a^{(n+1)})+\mathcal{O}(\lambda^n)$. Since $a^{(n+1)}=-\frac{f_n^{k_{n+1}}(-1)}{a^{(n)}}\leq-\frac{f_n(-1)}{a^{(n)}}=\frac{c^{(n)}-a^{(n)}}{a^{(n)}(1+v^{(n)})}+\mathcal{O}(\lambda^n)$, one has $a^{(n)}v^{(n)}(c^{(n+1)}+a^{(n+1)})\leq\frac{v^{(n)}}{1+v^{(n)}}(c^{(n)}-a^{(n)})+v^{(n)}\frac{a^{(n)}}{c^{(n)}}+\mathcal{O}(\lambda^n)$. The last expression (the terms before $\mathcal{O}(\lambda^n)$) is strictly increasing in $v^{(n)}$, and its value at $v^{(n)}=c^{(n)}-1$ is $c^{(n)}-1$ therefore $\frac{c^{(n+1)}-v^{(n+1)}-1}{a^{(n+1)}}-\frac{v^{(n+1)}}{c^{(n+1)}}\geq\mathcal{O}(\lambda^n)$, and (14) follows (because $v^{(n+1)}<0$). For $c^{(n)}<1$ the proof is the same up to the change of the inequality signs.

Thus we proved that $(a^{(n)},v^{(n)})$ is $\mathcal{O}(\lambda^n)$-close to $\mathfrak{D}_{c^{(n)}}$. In view of the bounds given by Proposition 5 this implies that for large enough $n$ one has $\frac{v^{(n)}}{c^{(n)}-1}\in(\frac{1}{2}+\varepsilon,1-\varepsilon)$ for some $\varepsilon>0$. Let $c^{(n)}>1$. Since $\rho_n$ is irrational and due to (**C**) one has $z<f_n(z)\leq F_{a^{(n)},v^{(n)},c^{(n)}}(z)+\mathcal{O}(\lambda^n)$ for all $z\in[-1,0]$, in particular for $z_0=-\frac{c^{(n)}-1}{2v^{(n)}}$. But $F_{a^{(n)},v^{(n)},c^{(n)}}(z_0)=z_0+(\frac{c^{(n)}+1}{2})^{-1}(a^{(n)}-\frac{(c^{(n)}-1)^2}{4v^{(n)}})$, hence $a^{(n)}-\frac{(c^{(n)}-1)^2}{4v^{(n)}}\geq\mathcal{O}(\lambda^n)$, which implies that $(a^{(n)},v^{(n)})$ is $\mathcal{O}(\lambda^n)$-close to $\check{\mathfrak{D}}_{c^{(n)}}$.

Finally, for $c^{(n+1)}<1$ one has $\frac{v^{(n+1)}-c^{(n+1)}+1}{c^{(n+1)}a^{(n+1)}}=a^{(n)}v^{(n)}+\mathcal{O}(\lambda^n)\geq\frac{(c^{(n)}-1)^2}{4}+\mathcal{O}(\lambda^n)=\frac{(c^{(n+1)}-1)^2}{4(c^{(n+1)})^2}+\mathcal{O}(\lambda^n)$, which implies that $(a^{(n+1)},v^{(n+1)})$ is $\mathcal{O}(\lambda^n)$-close to $\check{\mathfrak{D}}_{c^{(n+1)}}$.

*5.3. Silly and clever projections.* Let us denote $\Phi_c^\varepsilon=\{(a,v):\varepsilon<a<c-\varepsilon,\varepsilon<v/(c-1)<1-\varepsilon,v+a-c+1>\varepsilon\}$, $\varepsilon>0$. It follows from Propositions 5 and 6 that for arbitrary $T$ (satisfying our settings for this section), after certain relaxation time $n_0=n_0(T)$, the projection $(a^{(n)},v^{(n)})$ of a renormalization $(f_n,g_n)$ to the linear-fractional subspace $\mathfrak{S}_{c^{(n)}}$ belongs to $\Phi^\varepsilon_{c^{(n)}}$. However, the rotation number $\rho(a^{(n)},v^{(n)},c^{(n)})$, generally speaking, does not equal $\rho_n=\rho(f_n,g_n)=\mathcal{G}^n(\rho(T))$. This is the reason why we call $(a^{(n)},v^{(n)})\in\mathfrak{S}_{c^{(n)}}$ the *silly* projection of $(f_n,g_n)\in\mathfrak{S}^{2+\alpha}_{c^{(n)}}$. The idea is to point out the *clever* projection $(a_*^{(n)}, v_*^{(n)}) \in \gamma_{\rho_n, c^{(n)}} \cap \check{\mathfrak{D}}_{c^{(n)}}$ by shifting the point $(a^{(n)}, v^{(n)})$ a little bit.

Namely, let us define the projector operator $\mathcal{P} f_n = (a_*^{(n)}, v_*^{(n)})$ by the following rule: $a_*^{(n)} = \gamma_{\rho_n, c^{(n)}}(v^{(n)})$, $v_*^{(n)} = v^{(n)}$ in the case when $(\gamma_{\rho_n, c^{(n)}}(v^{(n)}), v^{(n)}) \in \check{\mathfrak{D}}_{c^{(n)}}$, otherwise let $(a_*^{(n)}, v_*^{(n)})$ be the closest to $(\gamma_{\rho_n, c^{(n)}}(v^{(n)}), v^{(n)})$ intersection point of the curve $a = \gamma_{\rho_n, c^{(n)}}(v)$ with the boundary of $\check{\mathfrak{D}}_{c^{(n)}}$.

*Remark 4.* Notice that every curve $a = \gamma_{\rho, c^{(n)}}(v)$ with irrational $\rho$ intersects the set $\check{\mathfrak{D}}_{c^{(n)}}$ over some interval of non-zero length in $v$, so there are exactly two intersection points of such a curve with the boundary of $\check{\mathfrak{D}}_{c^{(n)}}$, and the intersection is uniformly transversal. Since we are going to prove presently that $\gamma_{\rho_n, c^{(n)}}(v^{(n)})$ is $\mathcal{O}(\lambda^n)$-close to $a^{(n)}$, and by Proposition 6 this implies that the point $(\gamma_{\rho_n, c^{(n)}}(v^{(n)}), v^{(n)})$ is $\mathcal{O}(\lambda^n)$-close to $\check{\mathfrak{D}}_{c^{(n)}}$, there will be no doubt of which intersection point is the closest one, and the closest point is always uniquely defined.

*Remark 5.* Formally, $\mathcal{P}$ should be defined as an operator acting from $\mathfrak{S}_c^{2+\alpha}$ to $\mathfrak{S}_c$, but the image of a pair $(F, G) \in \mathfrak{S}_c^{2+\alpha}$ does not depend on $G$. Also note that we will actually apply $\mathcal{P}$ only to the renormalizations of given $T$ with large enough $n$.

In this subsection we prove that the silly and clever projections are $\mathcal{O}(\lambda^n)$-close.

**Lemma 11.** *There exists a constant $h_{c,\varepsilon} > 0$ such that for any $(a, v) \in \Phi_c^\varepsilon$ we have $\frac{\partial H_c^{(i)}(w; a, v)}{\partial a} \geq h_{c,\varepsilon}$, $i \in \{1, 2\}$.*

*Proof.* The explicit expressions for $H_c^{(i)}(w; a, v)$, $i \in \{1, 2\}$, were given in Subsect. 4.1.

The derivative $\frac{\partial H_c^{(1)}(w;a,v)}{\partial a} = \frac{-4wP_1}{Q_1^2}$, where $P_1 = ((-a^2 - 1 - 2c)v - 2ca + 2c^2)w + (-1 + a^2)v + 2ca + 2c$ and $Q_1 = (-1 + 4va + a^2 - 2ca + 2c)w - 1 - 2a - a^2$. Obviously, $Q_1$ is bounded and so $-4w/Q_1^2 > 0$ is separated from zero for $w \in [-1, -(a+1)/(c + 1 + (c - a))]$. At the point $w = -1$ we have $P_1 = 2a(c + av) + 2c(v + a - c + 1)$, and at $w = -(a+1)/(c + 1 + (c - a)$ we have $P_1 = 2(c+1)(a+1)(c+av)/(c+1+(c-a))$. Both are separated from zero by positive constants depending on $c$ and $\varepsilon$ only (notice that $c + av$ lies between $c$ and $c^2$). Since $P_1$ is linear w.r.t. $w$, the statement of the lemma for $i = 1$ is proved.

Next, $\frac{\partial H_c^{(2)}(w;a,v)}{\partial a} = \frac{2P_2}{Q_2^2}$, where $P_2 = 4a^2c(c+1)v^2w^2 + (2c^2a^2 + 2c^2 + 2a^4c - 4c^3a^2 + a^2 + a^4 + c - 2a^3 + 7a^2c - 2ca)vw^2 + c(4a^3c + c - 6c^2a^2 - 2c^2 - 3ca^2 + 1 - 3a^2 + 2a^3 + 6ca)w^2 - 2(c+1)(a-1)(a+1)(a^2+c)vw - 2c(c+1)(a-1)(a+1)(2a - c + 1)w + (a+1)^2(a^2+c)v + c(a+1)^2(2a - c + 1)$ and $Q_2 = -(a+1)(a^2 - ca + a + c) + (3a^2c - a + a^3 + 4c^2a - c)w - 2a(a+1)v + 2a(1+c)(a-1)vw$. Since $Q_2$ is bounded, then $2/Q_2^2 > 0$ is separated from zero. At the point $w = -(a+1)/(c+1+(c-a))$ we have $P_2 = 4ca(1+c)(1+a)^2(v + a - c + 1)(c + av)/(c + 1 + (c - a))^2$, and at $w = 0$ we have $P_2 = (a+1)^2(a(c+av) + c(v + a - c + 1))$. Both values are positive and separated from zero. Now we fix $v$ and notice that $P_2/w^2$ is a quadratic polynomial w.r.t. $1/w$, the parabola looks upwards, and the vertex $1/w = (1+c)(a-1)/(a+1)$ lies beyond the interval $(-\infty, -(c + 1 + (c - a))/(a + 1)]$ of our interest, therefore $1/w = -(c+1+(c-a))/(a+1)$ is the point of its global minimum. To finish the proof, we notice that $\frac{\partial}{\partial w} P_2$ is bounded, hence there exists a constant $\delta > 0$ such that $P_2$ is separated from zero over $w \in [-\delta, 0]$, while for $w \in [-(a+1)/(c+1+(c-a)), -\delta]$ we have $P_2 \geq \delta^2 \cdot P_2/w^2$ with the same result. The proof is finished.

**Corollary 6.** *For any $a_1 < a_2$ and $v$ such that $(a_1, v), (a_2, v) \in \Phi_c^\varepsilon$ the inequality*

$$H_c(w; a_2, v) - H_c(w; a_1, v) \geq h_{c,\varepsilon}(a_2 - a_1) \tag{15}$$

*holds true for all $w \in \mathbb{T}^1$.*

**Lemma 12.** *There exists $C_1 = C_1(T) > 0$ such that*

$$|H_{f_n,g_n}(w) - H_{c^{(n)}}(w; a^{(n)}, v^{(n)})| \leq C_1\lambda^n.$$

*Proof.* Recall that $H_{c^{(n)}}(w; a^{(n)}, v^{(n)})$ is equal to $\phi \circ F_{a^{(n)},v^{(n)},c^{(n)}} \circ \phi^{-1}$ over $[-1, \phi(F^{-1}_{a^{(n)},v^{(n)},c^{(n)}}(0))]$ and equals $1+\phi\circ G_{a^{(n)},v^{(n)},c^{(n)}} \circ F_{a^{(n)},v^{(n)},c^{(n)}} \circ\phi^{-1}$ over $[\phi(F^{-1}_{a^{(n)},v^{(n)},c^{(n)}}(0)), 0]$, with $\phi(z) = \frac{(a^{(n)}+1)z}{(2a^{(n)}+(a^{(n)}-1))z}$.

On the other hand, $H_{f_n,g_n}$ equals $\phi \circ f_n \circ \phi^{-1}$ on $[-1, \phi(f_n^{-1}(0))]$ and $1 + \phi \circ g_n \circ f_n \circ \phi^{-1}$ on $[\phi(f_n^{-1}(0)), 0]$ (here $\phi$ is the same since $f_n(0) = a^{(n)}$).

The statement of the lemma follows from (**C**) and Proposition 5 due to the fact that the derivatives of $\phi$, $F_{a^{(n)},v^{(n)},c^{(n)}}$ and $G_{a^{(n)},v^{(n)},c^{(n)}}$ are bounded and separated from zero by positive constants. Notice that the interval between $\phi(F^{-1}_{a^{(n)},v^{(n)},c^{(n)}}(0))$ and $\phi(f_n^{-1}(0))$ is less than $C\lambda^n$, while $H_{c^{(n)}}(w; a^{(n)}, v^{(n)})$ and $H_{f_n,g_n}(w)$ are both continuous and increasing, therefore their closeness over that interval follows from their closeness at its endpoints.

**Lemma 13.** *There exists $C_2 = C_2(T) > 0$ such that $|\gamma_{\rho_n,c^{(n)}}(v^{(n)}) - a^{(n)}| \leq C_2\lambda^n$.*

*Proof.* Let us denote $\Delta a = a^{(n)} - \gamma_{\rho_n,c^{(n)}}(v^{(n)}) > 0$ and assume that $\Delta a > 0$ (for the opposite case the proof is similar). Since $(a^{(n)}, v^{(n)}) \in \Phi^\varepsilon_{c^{(n)}}$ for large enough $n$, at least a half of the segment $[\gamma_{\rho_n,c^{(n)}}(v^{(n)}), a^{(n)}] \times \{v^{(n)}\}$ lies inside $\Phi^{\varepsilon/2}_{c^{(n)}}$, therefore Lemma 12 and Corollary 6 imply

$$\begin{aligned} H_{f_n,g_n}(w) &\geq H_{c^{(n)}}(w; a^{(n)}, v^{(n)}) - C_1\lambda^n \\ &\geq H_{c^{(n)}}(w; \gamma_{\rho_n,c^{(n)}}(v^{(n)}), v^{(n)}) + h_{c^{(n)},\frac{\varepsilon}{2}} \frac{\Delta a}{2} - C_1\lambda^n \end{aligned}$$

for all $w \in [-1, 0]$. It follows that $\Delta a \leq \frac{2}{h} C_1\lambda^n$, where $h = h(T) = \min\{h_{c,\frac{\varepsilon}{2}}, h_{\frac{1}{c},\frac{\varepsilon}{2}}\} > 0$, since otherwise there would be $H_{f_n,g_n}(w) > H_{c^{(n)}}(w; \gamma_{\rho_n,c^{(n)}}(v^{(n)}), v^{(n)})$ for all $w$, and therefore $\rho(f_n, g_n) > \rho(\gamma_{\rho_n,c^{(n)}}(v^{(n)}), v^{(n)})$, while in fact both values are equal. The statement of Lemma holds with $C_2 = 2C_1/h$.

**Proposition 7.** $a_*^{(n)} - a^{(n)} = \mathcal{O}(\lambda^n)$, $v_*^{(n)} - v^{(n)} = \mathcal{O}(\lambda^n)$.

*Proof.* Follows immediately from the definition of $\mathcal{P}$, Lemma 13 and Proposition 6.

*5.4. Renormalization of projection and projection of renormalization.*

**Proposition 8.** *The points $\mathcal{P} f_{n+1}$ and $\mathcal{R}_{c^{(n)}} \mathcal{P} f_n$ in $\check{\mathfrak{D}}_{c^{(n+1)}}$ are $\mathcal{O}(\lambda^n)$-close.*

*Proof.* We restrict our attention to $n$ large enough, so that both $(a^{(n)}, v^{(n)})$ and $(a_*^{(n)}, v_*^{(n)})$ lie within $\Phi^\varepsilon_{c^{(n)}}$ for some $\varepsilon = \varepsilon(T) > 0$ due to Propositions 5 and 7. Hence the values $a^{(n)}$, $c^{(n)} - a^{(n)}$, $a_*^{(n)}$, $c^{(n)} - a_*^{(n)}$ and $v^{(n)}/(c^{(n)} - 1)$ are positive, bounded and separated from zero by constants independent on $n$. Let us denote $(\bar{a}^{(n+1)}, \bar{v}^{(n+1)}) = \mathcal{R}_{c^{(n)}}(a_*^{(n)}, v_*^{(n)}) \in \check{\mathfrak{D}}_{c^{(n+1)}}$. We intend to show that the points $(\bar{a}^{(n+1)}, \bar{v}^{(n+1)})$ and $(a_*^{(n+1)}, v_*^{(n+1)}) = \mathcal{P} f_{n+1}$ are $\mathcal{O}(\lambda^n)$-close.

Denote $(x^{(n)}, y^{(n)}) = \pi_c(a^{(n)}, v^{(n)})$, $(\bar{x}^{(n+1)}, \bar{y}^{(n+1)}) = \pi_c(\bar{a}^{(n+1)}, \bar{v}^{(n+1)})$, $(x_*^{(n)}, y_*^{(n)}) = \pi_c(a_*^{(n)}, v_*^{(n)})$, $(x_*^{(n+1)}, y_*^{(n+1)}) = \pi_c(a_*^{(n+1)}, v_*^{(n+1)})$.

In the proof of Proposition 6 we saw that $v^{(n+1)} = c^{(n+1)}(1 + a^{(n)} a^{(n+1)} v^{(n)}) - 1 + \mathcal{O}(\lambda^n)$, hence $y^{(n+1)} = x^{(n)} + \mathcal{O}(\lambda^n)$. By Lemma 5, $\bar{y}^{(n+1)} = x_*^{(n)}$, therefore Proposition 7 (used once for $n$ and once for $n+1$) implies $y_*^{(n+1)} - \bar{y}^{(n+1)} = \mathcal{O}(\lambda^n)$.

Now, the most important 'geometric' step. On the one hand, the slope of the curve $\gamma_{\rho_{n+1}, c^{(n+1)}}$ satisfies the bound $\frac{\Delta v}{\Delta a} < -1$ due to Proposition 1). On the other, the line $y = \bar{y}^{(n+1)}$, i.e. $\frac{v - c^{(n+1)} + 1}{c^{(n+1)} a} = \bar{y}^{(n+1)}$, has the constant slope $\frac{dv}{da} = \bar{y}^{(n+1)} c^{(n+1)}$ that lies between 0 and $c^{(n)} - 1$ since $(\bar{x}^{(n+1)}, \bar{y}^{(n+1)}) \in \mathfrak{D}_{c^{(n+1)}}$, and thus is separated from $-1$ by a constant independent on $n$. The slope of the line $y = y_*^{(n+1)}$ is close to that of $y = \bar{y}^{(n+1)}$ and thus is separated from $-1$, too. So the lines $y = \bar{y}^{(n+1)}$ and $y = y_*^{(n+1)}$ are transversal to $\gamma_{\rho_{n+1}, c^{(n+1)}}$ uniformly in $n$. It follows that the estimate $y_*^{(n+1)} - \bar{y}^{(n+1)} = \mathcal{O}(\lambda^n)$ implies the same order of closeness for the intersection points of those lines with $\gamma_{\rho_{n+1}, c^{(n+1)}}$, which are exactly $(\bar{a}^{(n+1)}, \bar{v}^{(n+1)})$ and $(a_*^{(n+1)}, v_*^{(n+1)})$.

Since the metrics in coordinates $(a, v)$ and $(x, y)$ are equivalent on the set $\mathfrak{D}_{c^{(n)}} \cap \Phi^\varepsilon_{c^{(n)}}$, we have the following

**Corollary 7.** *There exists $C = C(T) > 0$ such that*

$$\mathbf{d}_{c^{(n+1)}}[\mathcal{P} f_{n+1}, \mathcal{R}_{c^{(n)}} \mathcal{P} f_n] \le C \lambda^n.$$

*5.5. Exponential convergence of renormalizations.* Let $\tilde{T}$ be a circle diffeomorphism with break of the same size $c$ and the same (half-bounded) rotation number $\rho$ as $T$. Denote $(\tilde{f}_n, \tilde{g}_n)$ the $n^{\text{th}}$ renormalization of $\tilde{T}$.

**Lemma 14.** *There exists $A = A(\varepsilon, c) > 0$ such that*

$$\|F_{a,v,c} - F_{\tilde{a},\tilde{v},c}\|_{C^2} \le A(|a - \tilde{a}| + |v - \tilde{v}|)$$

*for any $(a, v), (\tilde{a}, \tilde{v}) \in \Phi^\varepsilon_c$.*

*Proof.* The claim of the lemma follows from the representation

$$F_{a,v,c}(z) - F_{\tilde{a},\tilde{v},c}(z) = (a - \tilde{a}) \frac{1}{1 - \tilde{v} z} + (v - \tilde{v}) \frac{z(a + cz)}{(1 - \tilde{v} z)(1 - vz)},$$

since $1 - vz$ and $1 - \tilde{v} z$ are both separated from zero, and all the variables are bounded.

**Theorem 2.** *There exist constants* $C = C(T, \tilde{T}) > 0$ *and* $\lambda_1 \in (0, 1)$ *such that*

$$\|f_n - \tilde{f}_n\|_{C^2} \leq C\lambda_1^n.$$

*Proof.* Due to the statement (**C**) and Lemma 14 we have

$$\|f_n - \tilde{f}_n\|_{C^2} \leq C\left(\lambda^n + \mathbf{d}_{c^{(n)}}[\mathcal{P}f_n, \mathcal{P}\tilde{f}_n]\right).$$

Theorem 1 and Corollary 7 imply

$$\mathbf{d}_{c^{(n)}}[\mathcal{P}f_n, \mathcal{P}\tilde{f}_n] \leq C\lambda^n + \mu_{c^{(n)}}\mathbf{d}_{c^{(n)}}[\mathcal{P}f_{n-2}, \mathcal{P}\tilde{f}_{n-2}].$$

Telescoping the last estimate leads to $\mathbf{d}_{c^{(n)}}[\mathcal{P}f_n, \mathcal{P}\tilde{f}_n] = \mathcal{O}\left(\sum_{i=0}^{n/2} \mu_c^i \lambda^{n-2i}\right)$, which is $\mathcal{O}(\lambda_1^n)$ for any $\lambda_1 > \max\{\lambda, \sqrt{\mu_c}\}$.

## 6. The Rigidity Theorem

Finally we are going to show the smoothness of the conjugacy of two circle diffeomorphisms with breaks with the same size of break and the same irrational rotation number of a half-bounded type. This result was announced first in [10].

**Theorem 3.** *Let* $T$ *and* $\tilde{T}$ *be two circle diffeomorphisms with breaks of the same size* $c$ *and the same rotation number* $\rho \in M_{\mathrm{e}}$ *in the case of* $c > 1$*, or* $\rho \in M_{\mathrm{o}}$ *in the case of* $0 < c < 1$*. There exists an orientation-preserving* $C^1$*-smooth circle diffeomorphism* $\phi$ *such that*

$$\phi \circ T \circ \phi^{-1} = \tilde{T}. \tag{16}$$

To prove this theorem, we use the theory developed in [8], namely the following statement proved there:

**Conditional Theorem.** *Suppose that for two circle diffeomorphisms* $T$ *and* $\tilde{T}$ *with singularities (in particular of the break type) the following conditions hold:*

1) $\rho(T) = \rho(\tilde{T})$ *is irrational;*
2) *there exists a vector* $\mathbf{K} = (K_1, K_2, K_3, K_4)$ *such that the renormalizations* $f_n$ *and* $\tilde{f}_n$ *are* $\mathbf{K}$*-regular uniformly in* $n$*;*
3) *the systems of dynamical partitions* $\mathbb{P}_n$ *and* $\tilde{\mathbb{P}}_n$ *are exponentially refining;*
4) *there exist constants* $C > 0$ *and* $\lambda_1 \in (0, 1)$ *such that* $\|f_n - \tilde{f}_n\|_{C^2} \leq C\lambda_1^n$.

*Then* $T$ *and* $\tilde{T}$ *are* $C^1$*-smoothly conjugate in the sense of (16).*

The *regularity conditions* of 2) are the following. Given a vector with positive components $\mathbf{K} = (K_1, K_2, K_3, K_4)$ and a strictly increasing real function $f \in C^{2+\alpha}([-1, 0])$ such that $f(z) > z$ for each $z \in [-1, 0]$, we say that $f$ is $\mathbf{K}$-*regular*, if:

*i)* $\|f\|_{2+\alpha} \leq K_1$;
*ii)* the set $M_{f,K_2} = \{z \in [-1, 0], f(z) - z < K_2\}$ is either an open interval or empty (in particular, this implies $f(-1) \geq K_2 - 1$ and $f(0) \geq K_2$);
*iii)* $\frac{d^2 f}{dz^2}(z) > K_3$ for each $z \in M_{f,K_2}$;
*iv)* $\frac{df}{dz}(z) > K_4$ for each $-1 \leq z < -K_2^2$.

The *refining partitions* of 3) are defined as follows. A system of dynamical partitions of the circle (3) is called *exponentially refining* if there exist constants $C > 0$ and $0 < \beta < 1$ such that $|I| \le C\beta^{n-m}|J|$ for any $I \in \mathbb{P}_n$ and $J \in \mathbb{P}_m$ such that $I \subset J$.

Let us check the conditions of Conditional Theorem.

1) This one is a part of our settings.
2) The regularity conditions obviously hold for $F_{a,v,c}$ such that $(a, v) \in \Phi_c^{\varepsilon}$. Therefore the statement (**C**) and Propositions 5 and 6 imply that both $f_n$ and $\tilde{f}_n$ are bounded in $C^2$-norm and satisfy *ii*)-*iv*). The only thing remaining to be proved is the Hölder condition on $f_n''$ and $\tilde{f}_n''$, which is the subject of Lemma 15 just below.
3) Follows from (**A**).
4) This is our Theorem 2.

So, all that remains to prove Theorem 3 is the following statement.

**Lemma 15.** *There exists* $C_{\mathrm{H}} > 0$ *such that the estimate* $|f_n''(z) - f_n''(w)| \le C_{\mathrm{H}}|z - w|^{\alpha}$ *holds for all* $z, w \in [0, 1]$.

*Proof.* In terms of $T$ the estimate of the lemma is equivalent to the following:

$$|(T^{q_n})''(\eta_0) - (T^{q_n})''(\zeta_0)| \le C_{\mathrm{H}} \frac{|\eta_0 - \zeta_0|^{\alpha}}{|\Delta_0^{(n-1)}|^{1+\alpha}}$$

for all $\eta_0, \zeta_0 \in \Delta_0^{(n-1)}$. Let $\eta_i = T^i\eta_0$, $\zeta_i = T^i\zeta_0$, $i \ge 0$. From the formal expansion $(T^j)''(\theta_0) = (T^j)'(\theta_0)\sum_{i=0}^{j-1}\frac{T''(\theta_i)}{T'(\theta_i)}(T^i)'(\theta_0)$, $\theta_i = T^i\theta_0$, one gets the equality

$$\begin{aligned}(T^{q_n})''(\eta_0) - (T^{q_n})''(\zeta_0) &= \big((T^{q_n})'(\eta_0) - (T^{q_n})'(\zeta_0)\big)\sum_{i=0}^{q_n-1}\frac{T''(\eta_i)}{T'(\eta_i)}(T^i)'(\eta_0) \\ &+ (T^{q_n})'(\zeta_0)\sum_{i=0}^{q_n-1}\left(\frac{T''(\eta_i)}{T'(\eta_i)} - \frac{T''(\zeta_i)}{T'(\zeta_i)}\right)(T^i)'(\eta_0) \\ &+ (T^{q_n})'(\zeta_0)\sum_{i=0}^{q_n-1}\frac{T''(\zeta_i)}{T'(\zeta_i)}\Big((T^i)'(\eta_0) - (T^i)'(\zeta_0)\Big). \quad (17)\end{aligned}$$

Let us estimate the three summands in (17) separately. Since $f_n''$ is bounded, one has $|(T^{q_n})'(\eta_0) - (T^{q_n})'(\zeta_0)| = |\eta_0 - \zeta_0|\mathcal{O}(|\Delta_0^{(n-1)}|^{-1})$. It is easy to see that $|\log(T^i)'(\theta_0) - \log(T^i)'(\theta_0')| \le \mathrm{Var}_{\mathbb{T}^1}\log T'$ for any $\theta_0, \theta_0' \in \Delta_0^{(n-1)}$, hence $(T^i)'(\theta_0) \sim \frac{|\Delta_i^{(n-1)}|}{|\Delta_0^{(n-1)}|}$, $0 \le i < q_n$. Also, $T''/T'$ is bounded on $\mathbb{T}^1$. It follows that $\left|\sum_{i=0}^{q_n-1}\frac{T''(\eta_i)}{T'(\eta_i)}(T^i)'(\eta_0)\right| = \mathcal{O}(|\Delta_0^{(n-1)}|^{-1})$, and so for the first summand in (17) we have $\left|\big((T^{q_n})'(\eta_0) - (T^{q_n})'(\zeta_0)\big)\sum_{i=0}^{q_n-1}\frac{T''(\eta_i)}{T'(\eta_i)}(T^i)'(\eta_0)\right| = \mathcal{O}\Big(\frac{|\eta_0-\zeta_0|}{|\Delta_0^{(n-1)}|^2}\Big)$, which is also $\mathcal{O}\Big(\frac{|\eta_0-\zeta_0|^{\alpha}}{|\Delta_0^{(n-1)}|^{1+\alpha}}\Big)$ since $|\eta_0 - \zeta_0| \le |\Delta_0^{(n-1)}|$.

For the terms in the second summand we have $\left|\frac{T''(\eta_i)}{T'(\eta_i)} - \frac{T''(\zeta_i)}{T'(\zeta_i)}\right| = \mathcal{O}(|\eta_i - \zeta_i|^{\alpha})$ because $T''/T'$ is $C^{2+\alpha}$-smooth beyond the break; and $\frac{|\eta_i-\zeta_i|}{|\eta_0-\zeta_0|} \sim \frac{|\Delta_i^{(n-1)}|}{|\Delta_0^{(n-1)}|} \sim (T^i)'(\eta_0)$

as before, hence $\left|(T^{q_n})'(\zeta_0)\sum_{i=0}^{q_n-1}\left(\frac{T''(\eta_i)}{T'(\eta_i)}-\frac{T''(\zeta_i)}{T'(\zeta_i)}\right)(T^i)'(\eta_0)\right| = \mathcal{O}\left(\frac{|\eta_0-\zeta_0|^\alpha}{|\Delta_0^{(n-1)}|^{1+\alpha}}\max_i |\Delta_i^{(n-1)}|^\alpha\right)$, which is obviously $\mathcal{O}\left(\frac{|\eta_0-\zeta_0|^\alpha}{|\Delta_0^{(n-1)}|^{1+\alpha}}\right)$.

Concerning the last summand in (17), one has $(T^i)'(\eta_0) - (T^i)'(\zeta_0) = (T^i)''(\theta_0)(\eta_0-\zeta_0)$ for some $\theta_0 \in \Delta_0^{(n-1)}$, but we have already proved that $|(T^i)''(\theta_0)| = |(T^i)'(\theta_0)| \cdot \left|\sum_{k=0}^{i-1}\frac{T''(\theta_k)}{T'(\theta_k)}(T^k)'(\theta_0)\right| = \mathcal{O}\left(\frac{|\Delta_i^{(n-1)}|}{|\Delta_0^{(n-1)}|^2}\right)$, $0 \le i < q_n$. Now it follows easily that the last summand in (17) satisfies $\left|(T^{q_n})'(\zeta_0)\sum_{i=0}^{q_n-1}\frac{T''(\zeta_i)}{T'(\zeta_i)}((T^i)'(\eta_0)-(T^i)'(\zeta_0))\right| = \mathcal{O}\left(\frac{|\eta_0-\zeta_0|}{|\Delta_0^{(n-1)}|^2}\right)$, which is also $\mathcal{O}\left(\frac{|\eta_0-\zeta_0|^\alpha}{|\Delta_0^{(n-1)}|^{1+\alpha}}\right)$.

Thus Theorem 3 is proved.

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