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Asymptotic Density of Eigenvalue Clusters for the Perturbed Landau Hamiltonian

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Abstract: We consider the Landau Hamiltonian (i.e. the 2D Schrödinger operator with constant magnetic field) perturbed by an electric potential V which decays sufficiently fast at infinity. The spectrum of the perturbed Hamiltonian consists of clusters of eigenvalues which accumulate to the Landau levels. Applying a suitable version of the anti-Wick quantization, we investigate the asymptotic distribution of the eigenvalues within a given cluster as the number of the cluster tends to infinity. We obtain an explicit description of the asymptotic density of the eigenvalues in terms of the Radon transform of the perturbation potential V.

1. Introduction and Main Results

1.1. Introduction. Let

$$H_0 := \left(-i\frac{\partial}{\partial x} + \frac{B}{2}y\right)^2 + \left(-i\frac{\partial}{\partial y} - \frac{B}{2}x\right)^2$$

be the self-adjoint operator defined initially on $C_0^\infty(\mathbb{R}^2)$, and then closed in $L^2(\mathbb{R}^2)$. The operator H_0 is the Hamiltonian of a non-relativistic spinless 2D quantum particle subject to a constant magnetic field of strength $B > 0$. It is often called the Landau Hamiltonian in honor of the author of the pioneering paper [23]. The spectrum of H_0 consists of the eigenvalues (called Landau levels) $\lambda_q = B(2q+1)$, $q \in \mathbb{Z}_+ = 0, 1, 2, \ldots$. The multiplicity of each of these eigenvalues is infinite, and so

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = \bigcup_{q=0}^{\infty} \{\lambda_q\}, \quad \lambda_q = B(2q+1).$$

Next, let $V \in C(\mathbb{R}^2; \mathbb{R})$ satisfy the estimate

$$|V(\mathbf{x})| \le C\langle \mathbf{x}\rangle^{-\rho}, \quad \mathbf{x} \in \mathbb{R}^2, \quad \rho > 1, \tag{1.1}$$

where $\langle \mathbf{x} \rangle := (1 + |\mathbf{x}|^2)^{1/2}$. We also denote by V the operator of multiplication by V in $L^2(\mathbb{R}^2)$. Consider the perturbed Landau Hamiltonian $H = H_0 + V$. The spectrum of H consists of eigenvalue clusters around the Landau levels. More precisely, we have $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0)$ and so all eigenvalues of H in $\mathbb{R}\backslash\sigma_{\mathrm{ess}}(H)$ have finite multiplicities and can accumulate only to the Landau levels λ_q. Our first preliminary result says that the eigenvalue clusters shrink towards the Landau levels as $O(q^{-1/2})$ for $q \to \infty$:

Proposition 1.1. *Assume* (1.1)*; then there exists* $C_1 > 0$ *such that for all* $q \in \mathbb{Z}_+$ *one has*

$$\sigma(H) \cap [\lambda_q - B, \lambda_q + B] \subset (\lambda_q - C_1\lambda_q^{-1/2}, \lambda_q + C_1\lambda_q^{-1/2}). \tag{1.2}$$

The proof is given in Sect. 4.

Remark 1.2. (i) Obviously, the above estimate $O(\lambda_q^{-1/2})$ for the width of the qth cluster can also be written as $O(q^{-1/2})$; however, as we will see, λ_q provides a more natural scale than q.

(ii) Simple considerations (see Remark 3.2 in [22]) show that the estimate $O(\lambda_q^{-1/2})$ cannot be improved: the eigenvalue clusters have width $\geq c\lambda_q^{-1/2}$ with $c > 0$ (unless $V \equiv 0$). This will also follow from the main result of this paper.

(iii) Proposition 1.1 was proven in [22] for $V \in C_0^\infty(\mathbb{R}^2)$. The proof we give here not only covers the case of more general potentials V, but also is based on different ideas from those of [22].

1.2. Main result. Our purpose is to describe the asymptotic density of eigenvalues in the qth cluster as $q \to \infty$. Let $\mathbb{1}_{\mathcal{O}}$ denote the characteristic function of the set $\mathcal{O} \subset \mathbb{R}$. For $q \in \mathbb{Z}_+$ and $\mathcal{O} \in \mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$, set

$$\mu_q(\mathcal{O}) := \operatorname{rank} \mathbb{1}_{\lambda_q^{-1/2}\mathcal{O}+\lambda_q}(H).$$

The measure μ_q is not finite, but if $\mathcal{O}$ is bounded, and its closure does not contain the origin, we have $\mu_q(\mathcal{O}) < \infty$ for q sufficiently large. In particular, for any fixed bounded interval $[\alpha, \beta] \subset \mathbb{R}\backslash\{0\}$ we have

$$\mu_q([\alpha, \beta]) = \sum_{\lambda_q+\alpha\lambda_q^{-1/2} \leq \lambda \leq \lambda_q+\beta\lambda_q^{-1/2}} \dim \operatorname{Ker}(H - \lambda I) < \infty \tag{1.3}$$

for all sufficiently large q. Below we study the asymptotics of the counting measure μ_q as $q \to \infty$. In order to describe the limiting measure, we need to fix some notation. We denote by $\mathbb{T} \subset \mathbb{R}^2$ the circle of radius one, centered at the origin. The circle $\mathbb{T}$ is endowed with the usual Lebesgue measure normalized so that $\int_{\mathbb{T}} d\omega = 2\pi$. For $\omega = (\omega_1, \omega_2) \in \mathbb{T}$, we denote $\omega^\perp = (-\omega_2, \omega_1)$. We set

$$\widetilde{V}(\omega, b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(b\omega + t\omega^\perp)dt, \quad \omega \in \mathbb{T}, \quad b \in \mathbb{R}.$$

Thus, $\widetilde{V}$ is (up to a factor) the Radon transform of V. In order to make our notation more concise, we find convenient to introduce the Banach space X_ρ of all potentials $V \in C(\mathbb{R}^2, \mathbb{R})$ that satisfy (1.1) equipped with the norm

$$\|V\|_{X_\rho} = \sup_{\mathbf{x}\in\mathbb{R}^2} \langle \mathbf{x} \rangle^\rho |V(\mathbf{x})|. \tag{1.4}$$

Using this notation, by an elementary calculation one finds

$$|\widetilde{V}(\omega, b)| \le C_\rho \|V\|_{X_\rho} \langle b\rangle^{1-\rho}, \quad b \in \mathbb{R}. \tag{1.5}$$

Define the measure μ by

$$\mu(\mathcal{O}) = \frac{1}{2\pi} \left|\widetilde{V}^{-1}(B^{-1}\mathcal{O})\right|, \quad \mathcal{O} \in \mathcal{B}(\mathbb{R}),$$

where $|\cdot|$ stands for the Lebesgue measure (on $\mathbb{T} \times \mathbb{R}$). Evidently, for any interval $[\alpha, \beta] \subset \mathbb{R}\backslash\{0\}$ we have $\mu([\alpha, \beta]) < \infty$. Moreover, estimate (1.5) implies that μ has a bounded support in $\mathbb{R}$, and

$$\int_{\mathbb{R}} |t|^\ell d\mu(t) < \infty, \quad \forall \ell > 1/(\rho - 1). \tag{1.6}$$

Our main result is:

Theorem 1.3. *Let $V \in C(\mathbb{R}^2)$ be a continuous function that satisfies (1.1). Then, for any function $\varrho \in C_0^\infty(\mathbb{R}\backslash\{0\})$, we have*

$$\lim_{q\to\infty} \lambda_q^{-1/2} \int_{\mathbb{R}} \varrho(\lambda) d\mu_q(\lambda) = \int_{\mathbb{R}} \varrho(\lambda) d\mu(\lambda). \tag{1.7}$$

Remark 1.4. (i) The asymptotics (1.7) can be more explicitly written as

$$\lim_{q\to\infty} \lambda_q^{-1/2} \operatorname{Tr} \varrho(\sqrt{\lambda}_q(H - \lambda_q)) = \frac{1}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \varrho(B\widetilde{V}(\omega, b))\, db\, d\omega. \tag{1.8}$$

(ii) By standard approximation arguments, the asymptotics (1.7) can be extended to a wider class of continuous functions ϱ. Further, it follows from Theorem 1.3 that if $[\alpha, \beta] \subset \mathbb{R}\backslash\{0\}$, and $\mu(\{\alpha\}) = \mu(\{\beta\}) = 0$, then

$$\lim_{q\to\infty} \lambda_q^{-1/2} \mu_q([\alpha, \beta]) = \mu([\alpha, \beta]).$$

However, the assumption $\mu(\{\alpha\}) = \mu(\{\beta\}) = 0$ does not automatically hold, i.e. in general the measure μ may have atoms. Indeed, a description of the class of all Radon transforms $\widetilde{V}(\omega, b)$ of functions $V \in C_0^\infty(\mathbb{R}^2)$ is well known, see e.g. [19, Thm. 2.10]. According to this description, if $a \in C_0^\infty(\mathbb{R})$ is an even real-valued function, then $\widetilde{V}(\omega, b) := a(b)$, $b \in \mathbb{R}$, $\omega \in \mathbb{T}$, is a Radon transform in this class. Of course, if the derivative $a'(b)$ vanishes on some open interval, then the corresponding measure μ has an atom.

(iii) If $V \in C_0^\infty(\mathbb{R}^2)$, one can prove (see [22]) that the trace in the l.h.s. of (1.8) has a complete asymptotic expansion in inverse powers of $\lambda_q^{1/2}$, but the formulae for the higher order coefficients of this expansion are not known.

1.3. Method of proof. Let P_q be the orthogonal projection in $L^2(\mathbb{R}^2)$ onto the subspace $\operatorname{Ker}(H_0 - \lambda_q I)$. For $\ell \ge 1$, let S_ℓ be the Schatten-von Neumann class, with the norm $\|\cdot\|_\ell$; the usual operator norm is denoted by $\|\cdot\|$.

We first fix a natural number ℓ and examine the asymptotics of the trace in the l.h.s. of (1.8) for functions $\varrho \in C_0^\infty(\mathbb{R})$ such that $\varrho(\lambda) = \lambda^\ell$ for small λ. We have the following (fairly standard) technical result:

Lemma 1.5. *For any integer $\ell > 1/(\rho - 1)$, the operators $(H - \lambda_q)^\ell \mathbb{1}_{(\lambda_q - B, \lambda_q + B)}(H)$ and $(P_q V P_q)^\ell$ belong to the trace class and*

$$\mathrm{Tr}\{(H - \lambda_q)^\ell \mathbb{1}_{(\lambda_q - B, \lambda_q + B)}(H)\} = \mathrm{Tr}(P_q V P_q)^\ell + o(\lambda_q^{-(\ell-1)/2}), \quad q \to \infty. \tag{1.9}$$

The proof of this lemma is given in Subsect. 4.3. This lemma essentially reduces the question to the study of the asymptotics of traces of $(P_q V P_q)^\ell$. Our main technical result is the following statement:

Theorem 1.6. *Let V satisfy (1.1) and let $B_0 > 0$.*

(i) *For some $C = C(B_0)$, one has*

$$\sup_{q \geq 0} \sup_{B \geq B_0} \lambda_q^{1/2} B^{-1} \| P_q V P_q \| \leq C \| V \|_{X_\rho}. \tag{1.10}$$

(ii) *For any real $\ell > 1/(\rho - 1)$, we have $P_q V P_q \in S_\ell$, and for some $C = C(B_0, \ell)$, the estimate*

$$\sup_{q \geq 0} \sup_{B \geq B_0} \lambda_q^{(\ell-1)/(2\ell)} B^{-1} \| P_q V P_q \|_\ell \leq C \| V \|_{X_\rho} \tag{1.11}$$

holds true.

(iii) *For any integer $\ell > 1/(\rho - 1)$, we have*

$$\lim_{q \to \infty} \lambda_q^{(\ell-1)/2} \mathrm{Tr}(P_q V P_q)^\ell = \frac{B^\ell}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \widetilde{V}(\omega, b)^\ell \, db \, d\omega. \tag{1.12}$$

Although in our main Theorem 1.3 the strength B of the magnetic field is assumed to be fixed, we make the dependence on B explicit in the estimates (1.10), (1.11), as these estimates are of an independent interest (see e.g. [10]), and can be used in the study of other asymptotic regimes.

The proof of Theorem 1.6 consists of two steps. In Sect. 2 we establish the unitary equivalence of the Berezin-Toeplitz operator $P_q V P_q$ to a certain generalized anti-Wick pseudodifferential operator (ΨDO) whose symbol V_B is defined explicitly below in (2.27). Further, in Sect. 3 we study this ΨDO, prove appropriate estimates, and analyze its asymptotic behavior as $q \to \infty$.

A combination of (1.9) and (1.12) essentially yields (1.7) for a function $\varrho \in C_0^\infty(\mathbb{R})$ such that $\varrho(\lambda) = \lambda^\ell$ for small λ. After this, the main result follows by an application of the Weierstrass' approximation theorem; this argument is given in Subsect. 4.4.

Remark 1.7. In [22] the limit (1.12) was computed for $\ell = 1, 2$, but the result was written in a form not suggestive of the general formula.

1.4. Semiclassical interpretation. Consider the classical Hamiltonian function

$$\mathcal{H}(\boldsymbol{\xi}, \mathrm{x}) = (\xi + \frac{1}{2} B y)^2 + (\eta - \frac{1}{2} B x)^2, \quad \boldsymbol{\xi} := (\xi, \eta) \in \mathbb{R}^2, \quad \mathrm{x} := (x, y) \in \mathbb{R}^2, \tag{1.13}$$

in the phase space $T^*\mathbb{R}^2 = \mathbb{R}^4$ with the standard symplectic form. The projections onto the configuration space of the orbits of the Hamiltonian flow of $\mathcal{H}$ are circles of radius $\sqrt{E}/B$, where $E > 0$ is the value of the energy corresponding to the orbit. The classical

particles move along these circles with period $T_B = \pi/B$. The set of these orbits can be parameterized by the energy $E > 0$ and the center $\mathbf{c} \in \mathbb{R}^2$ of a circle. Let us denote the path in the configuration space corresponding to such an orbit by $\gamma(\mathbf{c}, E, t),\ t \in [0, T_B)$, and set

$$\langle V\rangle(\mathbf{c}, E) = \frac{1}{T_B}\int_0^{T_B} V(\gamma(\mathbf{c}, E, t))dt, \quad T_B = \pi/B. \tag{1.14}$$

For an energy $E > 0$, consider the set M_E of all orbits with this energy. The set M_E is a smooth manifold with coordinates $\mathbf{c} \in \mathbb{R}^2$. It can be considered as the quotient of the constant energy surface

$$\Sigma_E = \{(\boldsymbol{\xi}, \mathrm{x}) \in \mathbb{R}^4 \mid \mathcal{H}(\boldsymbol{\xi}, \mathrm{x}) = E\}$$

with respect to the flow of $\mathcal{H}$. Restricting the standard Lebesgue measure of $\mathbb{R}^4$ to Σ_E and then taking the quotient, we obtain the measure $B\,dc_1\,dc_2$ on M_E. An elementary calculation shows that the r.h.s. of (1.8) can be rewritten as

$$\frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \varrho(B\widetilde{V}(\omega, b))db\,d\omega = \frac{1}{2\pi}\lim_{E\to\infty}\frac{1}{\sqrt{E}}\int_{\mathbb{R}^2}\varrho(\sqrt{E}\,\langle V\rangle(\mathbf{c}, E))\,B\,dc_1\,dc_2. \tag{1.15}$$

The basis of this calculation is the fact that as $E \to \infty$, the radius $\sqrt{E}/B$ of the circles representing the classical orbits tends to infinity. Thus, the classical orbits approximate straight lines on any compact domain of the configuration space.

Given (1.15), we can rewrite the main result as

$$\lim_{q\to\infty}\frac{1}{\sqrt{\lambda_q}}\operatorname{Tr}\varrho(\sqrt{\lambda_q}(H-\lambda_q)) = \frac{1}{2\pi}\lim_{E\to\infty}\frac{1}{\sqrt{E}}\int_{\mathbb{R}^2}\varrho(\sqrt{E}\,\langle V\rangle(\mathbf{c}, E))\,B\,dc_1\,dc_2. \tag{1.16}$$

This agrees with the semiclassical intuition. Formula (1.16) corresponds to the well known "averaging principle" for systems close to integrable ones. This principle states that a good approximation is obtained if one replaces the original perturbation by the one which results by averaging the original perturbation along the orbits of the free dynamics. This method is very old; quoting V. Arnold [2, Sect. 52]: "In studying the perturbations of planets on one another, Gauss proposed to distribute the mass of each planet around its orbit proportionally to time and to replace the attraction of each planet by the attraction of the ring so obtained".

1.5. Related results.

1.5.1. Asymptotics for eigenvalue clusters for manifolds with closed geodesics. In spectral theory, results of this type originate from the classical work by A. Weinstein [37] (see also [9]). Weinstein considered the operator $-\Delta_{\mathcal{M}} + V$, where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on a compact Riemannian manifold $\mathcal{M}$ with periodic bicharacteristic flow (e.g. a sphere), and $V \in C(\mathcal{M}; \mathbb{R})$. In this case, all eigenvalues of $\Delta_{\mathcal{M}}$ have finite multiplicities which however grow with the eigenvalue number. Adding the perturbation V creates clusters of eigenvalues. Weinstein proved that the asymptotic density of eigenvalues in these clusters can be described by the density function obtained by averaging V

along the closed geodesics on $\mathcal{M}$. Let us illustrate these results with the case $\mathcal{M} = \mathbb{S}^2$. It is well known that the eigenvalues of $-\Delta_{\mathbb{S}^2}$ are $\lambda_q = q(q+1)$, $q \in \mathbb{Z}_+$, and their multiplicities are $d_q = 2q+1$. For $V \in C(\mathbb{S}^2; \mathbb{R})$ set

$$\widetilde{V}(\omega) := \frac{1}{2\pi} \int_0^{2\pi} V(\mathcal{C}_\omega(s))ds, \quad \omega \in \mathbb{S}^2,$$

where $\mathcal{C}_\omega(s) \in \mathbb{S}^2$ is the oriented great circle orthogonal to ω, and s is the arc length on this circle. Then for each $\varrho \in C_0^\infty(\mathbb{S}^2; \mathbb{R})$ we have

$$\lim_{q\to\infty} \frac{\operatorname{Tr} \varrho(-\Delta_{\mathbb{S}^2} + V - \lambda_q)}{d_q} = \int_{\mathbb{S}^2} \varrho(\widetilde{V}(\omega))dS(\omega), \tag{1.17}$$

where dS is the normalized Lebesgue measure on $\mathbb{S}^2$. Since $\mathbb{S}^2$ can be identified with its set of oriented geodesics $\mathcal{G}$, the r.h.s. of (1.17) can be interpreted as an integral with respect to the $SO(3)$–invariant normalized measure on $\mathcal{G}$. This result admits extensions to the case $\mathcal{M} = \mathbb{S}^n$ with $n > 2$, and, more generally, to the case where $\mathcal{M}$ is a compact symmetric manifold of rank 1 (see [9,37]).

In more recent works [33,35,36], the relation between the quantum Hamiltonian of the hydrogen atom and the Laplace-Beltrami operator on the unit sphere is exploited, and the asymptotic distribution within the eigenvalue clusters of the perturbed Hamiltonian hydrogen atom is investigated. The asymptotic density of eigenvalues in these clusters was described in terms of the perturbation averaged along the trajectories of the unperturbed dynamics (i.e. the solutions to the Kepler problem).

Among the main technical tools used in [33,35,36] which originate from [15,34], are the Bargmann-type representations of the particular quantum Hamiltonians considered, implemented via the so-called Segal-Bargmann transforms. In our analysis generalized coherent states and associated anti-Wick ΨDOs closely related to the Bargmann representation and the Segal-Bargmann transform appear again in a natural way (see Sect. 2), although their role is different from the one of their counterparts in [33,35,36].

Although this paper is inspired by [33,35–37], much of our construction (see Sect. 4) is based on the analysis of [22]. In [22] it was proven that for $V \in C_0^\infty(\mathbb{R}^2)$ the trace in the l.h.s. of (1.12) has complete asymptotic expansions in inverse powers of $\lambda_q^{1/2}$. However, the coefficients of this expansion have not been computed explicitly; see Remark 1.7 above.

1.5.2. Strong magnetic field asymptotics. It is useful to compare our main result with the asymptotics as $B \to \infty$ of the eigenvalues of H. It has been found in [26] (see also [21]) that

$$\lim_{B\to\infty} B^{-1} \operatorname{Tr} \varrho(H - \lambda_q) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \varrho(V(\mathrm{x}))d\mathrm{x} = \int_{\mathbb{R}} \varrho(t)dm(t), \tag{1.18}$$

where $\varrho \in C_0^\infty(\mathbb{R}\backslash\{0\})$, $q \in \mathbb{Z}_+$, $V \in L^p(\mathbb{R}^2)$, $p > 1$, and $m(\mathcal{O}) := \frac{1}{2\pi}\left|V^{-1}(\mathcal{O})\right|$, $\mathcal{O} \in \mathcal{B}(\mathbb{R})$. Similarly to Theorem 1.3, the proof of (1.18) is based on an analogue of Theorem 1.6 (i)–(ii) (see Lemma 2.12 below), and the asymptotic relations

$$\lim_{B\to\infty} B^{-1} \operatorname{Tr}(P_q V P_q)^\ell = \lim_{B\to\infty} B^{-1} \operatorname{Tr} P_q V^\ell P_q = \frac{1}{2\pi} \int_{\mathbb{R}^2} V(\mathrm{x})^\ell d\mathrm{x}, \quad q \in \mathbb{Z}_+, \tag{1.19}$$

with $V \in C_0^\infty(\mathbb{R}^2)$ and $\ell \in \mathbb{N}$, close in spirit to (1.12). Since $Bm([\alpha,\beta]) = \frac{1}{2\pi}\left|V_B^{-1}([\alpha,\beta])\right|$, $[\alpha,\beta] \subset \mathbb{R}\backslash\{0\}$, where V_B (see (2.27)) is the symbol of the generalized anti-Wick ΨDO to which $P_q V P_q$ is unitarily equivalent, we find that (1.18) is again a result of semiclassical nature. However, (1.19) implies that in the strong magnetic field regime the main asymptotic terms of $\mathrm{Tr}(P_q V P_q)^\ell$ and $\mathrm{Tr}\, P_q V^\ell P_q$ coincide, and hence in the first approximation the commutators $[V, P_q]$ are negligible, while (1.12) shows that obviously this is not the case in the high energy regime considered in the present article. Therefore, Theorem 1.3 retains "more quantum flavor" than (1.18), and hence its proof is technically much more involved.

1.5.3. The spectral density of the scattering matrix for high energies. In the recent work [8] inspired by this paper, D. Bulger and A. Pushnitski considered the scattering matrix $S(\lambda)$, $\lambda > 0$, for the operator pair $(-\Delta + V, -\Delta)$, where Δ is the standard Laplacian acting in $L^2(\mathbb{R}^d)$, $d \geq 2$, and $V \in C(\mathbb{R}^d;\mathbb{R})$ is an electric potential which satisfies an estimate analogous to (1.1). Although the methods applied in [8] are different from ours, it turned out that the asymptotics as $\lambda \to \infty$ of the eigenvalue clusters for $S(\lambda)$ are written in terms of the X-ray transform of V in a manner similar to (1.7).

2. Unitary Equivalence of Berezin-Toeplitz Operators and Generalized Anti-Wick ΨDOs

2.1. Outline of the section. From a methodological point of view, this section plays a central role in the proof of Theorem 1.3. Its principal goal is to establish the unitary equivalence between the Berezin-Toeplitz operators $P_q V P_q$, $q \in \mathbb{Z}_+$, and some generalized anti-Wick ΨDOs $\mathrm{Op}_q^{aw}(V_B)$ whose symbol V_B is defined explicitly in (2.27). This equivalence is proved in Theorem 2.11 below. The ΨDOs Op_q^{aw} introduced in Subsect. 2.3, are quite similar to the classical anti-Wick operators Op_0^{aw} (see [4, Chap. V, Sect. 2], [29, Sect. 24]); the only difference is that the quantization Op_0^{aw} is related to coherent states built on the first eigenfunction φ_0 of the harmonic oscillator (2.4), while Op_q^{aw}, $q \in \mathbb{N}$, is related to coherent states built on its qth eigenfunction φ_q.

In our further analysis of the operator $\mathrm{Op}_q^{aw}(V_B)$ performed in Sect. 3, we also heavily use the properties of the Weyl symbol of this operator. Thus, in Subsect. 2.1 – 2.2 we introduce the Weyl quantization Op^w, and in Subsect. 2.4 we briefly discuss its relation to Op_q^{aw}. In particular, we show that $\mathrm{Op}_q^{aw}(s) = \mathrm{Op}^w(s * \Psi_q)$, where s is a symbol from an appropriate class, and $2\pi\Psi_q$ is the Wigner function associated with φ_q, defined explicitly in (2.13). Therefore, the Berezin-Toeplitz operator $P_q V P_q$, $q \in \mathbb{Z}_+$, with domain $P_q L^2(\mathbb{R}^2)$, is unitarily equivalent to $\mathrm{Op}^w(V_B * \Psi_q)$ (see Corollary 2.13).

2.2. Weyl ΨDOs . Let $d \geq 1$. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz class, and by $\mathcal{S}'(\mathbb{R}^d)$ its dual class.

Proposition 2.1 [20, Lem. 18.6.1]. *Let $s \in \mathcal{S}'(\mathbb{R}^{2d})$. Assume that $\hat{s} \in L^1(\mathbb{R}^{2d})$, where $\hat{s}$ is the Fourier transform of s, introduced explicitly in (3.1) below. Then the operator $\mathrm{Op}^w(s)$ defined initially as a mapping from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by*

$$\left(\mathrm{Op}^w(s)u\right)(x) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} s\left(\frac{x+x'}{2},\xi\right) e^{i(x-x')\cdot\xi} u(x')dx'd\xi, \quad x \in \mathbb{R}^d, \quad (2.1)$$

extends uniquely to an operator bounded in $L^2(\mathbb{R}^d)$. *Moreover,*

$$\|\mathrm{Op}^w(s)\| \leq (2\pi)^{-d}\|\hat{s}\|_{L^1(\mathbb{R}^{2d})}. \tag{2.2}$$

Some of the arguments of our proofs require estimates which are more sophisticated than (2.2). Let $\Gamma(\mathbb{R}^{2d})$, $d \geq 1$, denote the set of functions $s : \mathbb{R}^{2d} \to \mathbb{C}$ such that

$$\|s\|_{\Gamma(\mathbb{R}^{2d})} := \sup_{\{\alpha,\beta\in\mathbb{Z}_+^d \mid |\alpha|,|\beta|\leq[\frac{d}{2}]+1\}} \sup_{(x,\xi)\in\mathbb{R}^{2d}} |\partial_x^\alpha \partial_\xi^\beta s(x,\xi)| < \infty.$$

Proposition 2.2 [6, Cor. 2.5 (i)]. *There exists a constant* c_0 *such that for any* $s \in \Gamma(\mathbb{R}^{2d})$, $d \geq 1$, *we have*

$$\|\mathrm{Op}^w(s)\| \leq c_0\|s\|_{\Gamma(\mathbb{R}^{2d})}.$$

Further, if $s \in L^2(\mathbb{R}^{2d})$, then, obviously, the operator $\mathrm{Op}^w(s)$ belongs to the Hilbert-Schmidt class, and

$$\|\mathrm{Op}^w(s)\|_2^2 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} |s(x,\xi)|^2\,dx\,d\xi. \tag{2.3}$$

Next, we describe the well known metaplectic unitary equivalence of Weyl ΨDOs whose symbols are mapped into each other by a linear symplectic change of the variables.

Proposition 2.3 [11, Chap. 7, Thm. A.2]. *Let* $\kappa : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, $d \geq 1$, *be a linear symplectic transformation,* $s_1 \in \Gamma(\mathbb{R}^{2d})$, *and* $s_2 := s_1 \circ \kappa$. *Then there exists a unitary operator* $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *such that*

$$\mathrm{Op}^w(s_2) = U^*\mathrm{Op}^w(s_1)U.$$

Remark 2.4. (i) The operator U is called *the metaplectic operator* corresponding to the linear symplectic transformation κ. There exists a one-to-one correspondence between metaplectic operators and linear symplectic transformations, apart from a constant factor of modulus 1 (see e.g. [20, Thm. 18.5.9]). Moreover, every linear symplectic transformation κ is a composition of a finite number of elementary linear symplectic maps (see e.g. [20, Lem. 18.5.8]), and for each elementary linear symplectic map there exists an explicit simple metaplectic operator (see e.g. the proof of [20, Thm. 18.5.9]).

(ii) Proposition 2.3 extends to a large class of not necessarily bounded operators. In particular, it holds for Weyl ΨDOs with quadratic symbols.

2.3. Generalized anti-Wick ΨDOs. In this subsection we introduce generalized anti-Wick ΨDOs . These operators are a special case of the ΨDOs with contravariant symbols whose theory has been developed in [3]. Introduce the harmonic oscillator

$$h := -\frac{d^2}{dx^2} + x^2, \tag{2.4}$$

self-adjoint in $L^2(\mathbb{R})$. It is well known that the spectrum of h is purely discrete and simple, and consists of the eigenvalues $2q+1$, $q \in \mathbb{Z}_+$, while its associated real-valued eigenfunctions φ_q, normalized in $L^2(\mathbb{R})$, could be written as

$$\varphi_q(x) := \frac{\mathrm{H}_q(x)e^{-x^2/2}}{(\sqrt{\pi}2^q q!)^{1/2}}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \tag{2.5}$$

where

$$\mathrm{H}_q(x) := (-1)^q e^{x^2/2}\left(\frac{d}{dx} - x\right)^q e^{-x^2/2}, \quad x \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \tag{2.6}$$

are the Hermite polynomials. Introduce the generalized coherent states (see e.g. [28])

$$\varphi_{q;x,\xi}(y) := e^{i\xi y}\varphi_q(y - x), \quad y \in \mathbb{R}, \quad (x, \xi) \in \mathbb{R}^2, \tag{2.7}$$

so that $\varphi_q = \varphi_{q;0,0}$. Note that if $f \in L^2(\mathbb{R})$, then

$$\|f\|^2_{L^2(\mathbb{R})} = (2\pi)^{-1}\int_{\mathbb{R}^2} |\langle f, \varphi_{q;x,\xi}\rangle|^2 dx d\xi, \tag{2.8}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R})$. Introduce the orthogonal projection

$$p_{q;x,\xi} := |\varphi_{q;x,\xi}\rangle\langle\varphi_{q;x,\xi}| : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad q \in \mathbb{Z}_+, \quad (x, \xi) \in \mathbb{R}^2. \tag{2.9}$$

Let $s \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$. Define

$$\mathrm{Op}^{aw}_q(s) := (2\pi)^{-1}\int_{\mathbb{R}^2} s(x, \xi) p_{q;x,\xi} dx d\xi$$

as the operator generated in $L^2(\mathbb{R})$ by the bounded sesquilinear form

$$F_{q,s}[f, g] := (2\pi)^{-1}\int_{\mathbb{R}^2} s(x, \xi)\langle f, \varphi_{q;x,\xi}\rangle\overline{\langle g, \varphi_{q;x,\xi}\rangle} dx d\xi, \quad f, g \in L^2(\mathbb{R}). \tag{2.10}$$

We will call $\mathrm{Op}^{aw}_q(s)$ the operator with *anti-Wick symbol of order* q equal to s. We introduce these operators only in the case of dimension $d = 1$ since this is sufficient for our purposes; of course, their definition extends easily to any dimension $d \geq 1$.

Note that the quantization Op^{aw}_q with $q = 0$ coincides with the standard anti-Wick one (see e.g. [29, Sect. 24]). In the following lemma we summarize some elementary basic properties of generalized anti-Wick ΨDOs which follow immediately from the corresponding properties of general contravariant ΨDOs .

Lemma 2.5 [29, Sect. 24], [4, Sect. 5.3].

(i) *Let* $s \in L^\infty(\mathbb{R}^2)$. *Then we have*

$$\|\mathrm{Op}^{aw}_q(s)\| \leq \|s\|_{L^\infty(\mathbb{R}^2)}. \tag{2.11}$$

(ii) *Let* $s \in L^\ell(\mathbb{R}^2)$ *with* $\ell \in [1, \infty)$. *Then we have*

$$\|\mathrm{Op}^{aw}_q(s)\|^\ell_\ell \leq (2\pi)^{-1}\|s\|^\ell_{L^\ell(\mathbb{R}^2)}. \tag{2.12}$$

2.4. Relation between generalized anti-Wick and Weyl ΨDOs. For $q \in \mathbb{Z}_+$ set

$$\Psi_q(x, \xi) = \frac{(-1)^q}{\pi}\mathrm{L}_q(2(x^2 + \xi^2))e^{-(x^2+\xi^2)}, \quad (x, \xi) \in \mathbb{R}^2, \tag{2.13}$$

where

$$\mathrm{L}_q(t) := \frac{1}{q!}e^t\frac{d^q(t^q e^{-t})}{dt^q} = \sum_{k=0}^q \binom{q}{k}\frac{(-t)^k}{k!}, \quad t \in \mathbb{R}, \tag{2.14}$$

are the Laguerre polynomials.

Lemma 2.6. *For a fixed* $(x,\xi)\in\mathbb{R}^2$ *we have* $p_{q;x,\xi}=\mathrm{Op}^w(\varsigma_{q;x,\xi})$, *where* $p_{q;x,\xi}$ *is the orthogonal projection defined in* (2.9), *and*

$$\varsigma_{q;x,\xi}(x',\xi'):=2\pi\Psi_q(x'-x,\xi'-\xi),\quad (x',\xi')\in\mathbb{R}^2. \tag{2.15}$$

Proof. Using the well-known relation between the Schwartz kernel of a linear operator and its Weyl symbol (see e.g [20, Eq. (18.5.4)"]), we find that the Weyl symbol $\varsigma_{q;x,\xi}$ of the projection $p_{q;x,\xi}$ satisfies

$$\varsigma_{q;x,\xi}(x',\xi')=\int_{\mathbb{R}}e^{-iv\xi'}\varphi_{q;x,\xi}(x'+v/2)\overline{\varphi_{q;x,\xi}(x'-v/2)}dv. \tag{2.16}$$

By (2.5) and (2.7),

$$\begin{aligned}&\int_{\mathbb{R}}e^{-iv\xi'}\varphi_{q;x,\xi}(x'+v/2)\overline{\varphi_{q;x,\xi}(x'-v/2)}dv\\&=\frac{1}{\sqrt{\pi}2^q q!}\int_{\mathbb{R}}e^{iv(\xi-\xi')}\mathrm{H}_q\left(x'\frac{1}{2}v-x\right)\mathrm{H}_q\left(x'-\frac{1}{2}v-x\right)\\&\quad\times e^{-(x'+\frac{1}{2}v-x)^2/2}e^{-(x'-\frac{1}{2}v-x)^2/2}dv.\end{aligned} \tag{2.17}$$

Changing the variable of integration $v=2(t+i(\xi-\xi'))$, and bearing in mind the parity of the Hermite polynomial H_q, we get

$$\begin{aligned}&\int_{\mathbb{R}}e^{iv(\xi-\xi')}\mathrm{H}_q(x'+\frac{1}{2}v-x)\mathrm{H}_q(x'-\frac{1}{2}v-x)e^{-(x'+\frac{1}{2}v-x)^2/2}e^{-(x'-\frac{1}{2}v-x)^2/2}dv\\&=2(-1)^q e^{-(x'-x)^2-(\xi'-\xi)^2}\int_{\mathbb{R}}e^{-t^2}\mathrm{H}_q(t-(x-x'-i(\xi-\xi')))\\&\quad\times\mathrm{H}_q(t+x-x'+i(\xi-\xi'))dt.\end{aligned} \tag{2.18}$$

Employing the relation between the Laguerre polynomials and the integrals of Hermite polynomials (see e.g. [14, Eq. 7.377]), we obtain

$$\begin{aligned}&\int_{\mathbb{R}}e^{-t^2}\mathrm{H}_q(t-(x-x'-i(\xi-\xi')))\mathrm{H}_q(t+x-x'+i(\xi-\xi'))dt\\&=\sqrt{\pi}2^q q!\mathrm{L}_q(2((x'-x)^2+(\xi-\xi')^2)).\end{aligned} \tag{2.19}$$

Putting together (2.16)–(2.19), we obtain (2.15). □

Remark 2.7. Let $\psi\in L^2(\mathbb{R})$ and $\|\psi\|_{L^2(\mathbb{R})}=1$. Then the Weyl symbol of the rank-one orthogonal projection $|\psi\rangle\langle\psi|$, is called the Wigner function associated with ψ (see e.g. [24, Def. 2.2]). Thus, Lemma 2.6 tells us, in particular, that $2\pi\Psi_q$ is the Wigner function associated with φ_q.

Lemma 2.6 immediately entails the following

Corollary 2.8. *Let* $s\in L^1(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)$. *Then we have*

$$\mathrm{Op}_q^{aw}(s)=\mathrm{Op}^w(\Psi_q*s). \tag{2.20}$$

2.5. Metaplectic mapping of the operators H_0, P_q and V. For $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $\boldsymbol{\xi} = (\xi, \eta) \in \mathbb{R}^2$, set

$$\varkappa_B(\mathbf{x}, \boldsymbol{\xi}) := \left(\frac{1}{\sqrt{B}}(x - \eta), \frac{1}{\sqrt{B}}(\xi - y), \frac{\sqrt{B}}{2}(\xi + y), -\frac{\sqrt{B}}{2}(\eta + x) \right). \quad (2.21)$$

Evidently, the transformation $\varkappa_B$ is linear and symplectic. Define the unitary operator $\mathcal{U}_B : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$(\mathcal{U}_B u)(x, y) := \frac{\sqrt{B}}{2\pi} \int_{\mathbb{R}^2} e^{i\phi_B(x,y;x',y')} u(x', y') dx' dy', \quad (2.22)$$

where

$$\phi_B(x, y; x', y') := B\frac{xy}{2} + B^{1/2}(xy' - yx') - x'y'.$$

Writing $\varkappa_B$ as a product of elementary linear symplectic transformations (see e.g. [20, Lem. 18.5.8]), we can easily check that $\mathcal{U}_B$ is a metaplectic operator corresponding to $\varkappa_B$. Note that

$$\mathcal{H} \circ \varkappa_B(\mathrm{x}, \boldsymbol{\xi}) = B(\xi^2 + x^2), \quad \mathrm{x} = (x, y) \in \mathbb{R}^2, \quad \boldsymbol{\xi} = (\xi, \eta) \in \mathbb{R}^2,$$

where $\mathcal{H}$ is the Weyl symbol of the operator H_0 defined in (1.13). On the other hand, $B(\xi^2 + x^2)$ is the Weyl symbol of the operator $B(h \otimes I_y)$ self-adjoint in $L^2(\mathbb{R}^2_{x,y})$, where h is the harmonic oscillator (2.4), acting in $L^2(\mathbb{R}_x)$, and I_y is the identity operator in $L^2(\mathbb{R}_y)$. Denote by $p_q = |\varphi_q\rangle\langle\varphi_q| = p_{q;0,0}$ the orthogonal projection onto Ker $(h - 2q - 1)$, $q \in \mathbb{Z}_+$. Applying Proposition 2.3 with $\kappa = \varkappa_B$, and bearing in mind Remark 2.4 (ii), we obtain the following

Corollary 2.9. *(i) We have*

$$\mathcal{U}_B^* H_0 \mathcal{U}_B = B\left(h \otimes I_y\right), \quad (2.23)$$

$$\mathcal{U}_B^* P_q \mathcal{U}_B = p_q \otimes I_y, \quad q \in \mathbb{Z}_+. \quad (2.24)$$

(ii) If $V \in \Gamma(\mathbb{R}^2)$, then

$$\mathcal{U}_B^* V \mathcal{U}_B = \mathrm{Op}^w(\mathbf{V}_B), \quad (2.25)$$

where

$$\mathbf{V}_B(x, y; \xi, \eta) := V(B^{-1/2}(x - \eta), B^{-1/2}(\xi - y)), \quad (x, y; \xi, \eta) \in \mathbb{R}^4. \quad (2.26)$$

Remark 2.10. Various versions of the symplectic transformation $\varkappa_B$ in (2.21) and the corresponding metaplectic operator $\mathcal{U}_B$ in (2.22) have been used in the spectral theory of the perturbations of the Landau Hamiltonian (see e.g. [18]). Of course, the close relation between the Landau Hamiltonian H_0 and the harmonic oscillator h is well-known since the seminal work [23] where the basic spectral properties of H_0 were first described.

2.6. Unitary equivalence of $P_q V P_q$ and $\mathrm{Op}_q^{aw}(V_B)$. Set

$$V_B(x, y) = V(-B^{-1/2}y, -B^{-1/2}x), \quad (x, y) \in \mathbb{R}^2. \quad (2.27)$$

Theorem 2.11. *For any $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and $q \in \mathbb{Z}_+$, we have*

$$\mathcal{U}_B^* P_q V P_q \mathcal{U}_B = p_q \otimes \mathrm{Op}_q^{aw}(V_B). \quad (2.28)$$

For the proof of Theorem 2.11 we need some well known estimates for Berezin-Toeplitz operators:

Lemma 2.12 [25, Lem. 5.1], [13, Lem. 3.1]. *Let* $V \in L^\ell(\mathbb{R}^2)$, $\ell \in [1,\infty)$. *Then* $P_q V P_q \in S_\ell(L^2(\mathbb{R}^2))$, *and* $\|P_q V P_q\|_\ell^\ell \le \frac{B}{2\pi}\|V\|_{L^\ell(\mathbb{R}^2)}^\ell$, $q \in \mathbb{Z}_+$. *Moreover, if* $V \in L^1(\mathbb{R}^2)$, *then*

$$\operatorname{Tr} P_q V P_q = \frac{B}{2\pi}\int_{\mathbb{R}^2} V(\mathrm{x})d\mathrm{x}, \quad q \in \mathbb{Z}_+. \tag{2.29}$$

Proof of Theorem 2.11. Assume at first $V \in C_0^\infty(\mathbb{R}^2)$. Then, by (2.24), (2.25), and (2.26),

$$\mathcal{U}_B^* P_q V P_q \mathcal{U}_B = (p_q \otimes I_y)\mathrm{Op}^w(\mathbf{V}_B)(p_q \otimes I_y). \tag{2.30}$$

Let $u \in \mathcal{S}(\mathbb{R}^2)$. Set

$$u_q(y) := \int_{\mathbb{R}^2} u(x,y)\varphi_q(x)dx.$$

Then we have

$$\begin{aligned}
&\langle \mathcal{U}_B^* P_q V P_q \mathcal{U}_B u, u\rangle_{L^2(\mathbb{R}^2)} = \langle \mathrm{Op}^w(\mathbf{V}_B)(\varphi_q \otimes u_q), (\varphi_q \otimes u_q)\rangle_{L^2(\mathbb{R}^2)}\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^6} V_B((y_1+y_2)/2-\xi, \eta-(x_1+x_2)/2)\, e^{i[(x_1-x_2)\xi+(y_1-y_2)\eta]}\\
&\quad \times \varphi_q(x_1)u_q(y_1)\ \varphi_q(x_2)\overline{u_q(y_2)}\ dx_1dx_2\, dy_1dy_2\, d\xi d\eta\\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^5} V_B((y_1+y_2)/2-y', \eta-\eta')\, e^{i(y_1-y_2)\eta}\\
&\quad \times \left(\int_{\mathbb{R}} \varphi_q(\eta'+v/2)\varphi_q(\eta'-v/2)e^{ivy'}dv\right) u_q(y_1)\overline{u_q(y_2)}d\eta' d\eta dy' dy_1 dy_2\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^5} \Psi_q(y',\eta')V_B((y_1+y_2)/2-y', \eta-\eta')\\
&\quad \times e^{i(y_1-y_2)\eta}u_q(y_1)\overline{u_q(y_2)}dy_1dy_2dy'd\eta d\eta'\\
&= \langle \mathrm{Op}^w(V_B * \Psi_q)u_q, u_q\rangle_{L^2(\mathbb{R})} = \langle \mathrm{Op}_q^{aw}(V_B)u_q, u_q\rangle_{L^2(\mathbb{R})}\\
&= \langle (p_q \otimes \mathrm{Op}_q^{aw}(V_B))u, u\rangle_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{2.31}$$

To obtain the first identity, we have utilized Corollary 2.9. To establish the second identity, we have used (2.1), (2.26), and (2.27). To get the third identity, we have changed the variables $x_1 = \eta' + v/2$, $x_2 = \eta' - v/2$, $\xi = y'$. To obtain the fourth identity, we have used (2.15)–(2.16) with $\xi' = y'$, $x' = \eta'$, and $x = 0$, $\xi = 0$, taking into account that $\Psi_q(\eta', -y') = \Psi_q(y', \eta')$. To deduce the fifth identity, we have applied (2.1), bearing in mind the symmetry of the convolution $\Psi_q * V_B = V_B * \Psi_q$, and for the sixth identity, we have applied (2.20) with $s = V_B$. Finally, the last identity is obvious. Now, (2.31) entails (2.28) in the case $V \in C_0^\infty(\mathbb{R}^2)$.

Further, let $V \in L^1(\mathbb{R}^2)$, and pick a sequence $\{V_m\}$ of functions $V_m \in C_0^\infty(\mathbb{R}^2)$ such that $V_m \to V$ in $L^1(\mathbb{R}^2)$ as $m \to \infty$. Then by Lemma 2.12 and the unitarity of $\mathcal{U}_B$, we have

$$\lim_{m\to\infty} \|\mathcal{U}_B^* P_q V_m P_q \mathcal{U}_B - \mathcal{U}_B^* P_q V P_q \mathcal{U}_B\|_1 = 0.$$

Similarly, it follows from (2.12) with $\ell = 1$ and (2.20) that

$$\lim_{m\to\infty} \|p_q \otimes \mathrm{Op}_q^{aw}(V_{m,B}) - p_q \otimes \mathrm{Op}_q^{aw}(V_B)\|_1 = 0.$$

Hence, (2.28) is valid for $V \in L^1(\mathbb{R}^2)$.

Finally, let now $V = V_1 + V_2$ with $V_1 \in L^1(\mathbb{R}^2)$ and $V_2 \in L^\infty(\mathbb{R}^2)$. Denote by χ_R the characteristic function of a disk of radius $R > 0$ centered at the origin. Then $V_1 + \chi_R V_2 \in L^1(\mathbb{R}^2)$. Evidently,

$$\operatorname*{w-lim}_{R\to\infty} \mathcal{U}_B^* P_q (V_1 + \chi_R V_2) P_q \mathcal{U}_B = \mathcal{U}_B^* P_q V P_q \mathcal{U}_B,$$

while (2.10) entails

$$\operatorname*{w-lim}_{R\to\infty} p_q \otimes \mathrm{Op}_q^{aw}((V_1 + \chi_R V_2)_B) = p_q \otimes \mathrm{Op}_q^{aw}(V_B),$$

which yields (2.28) in the general case. □

Combining Theorem 2.11 and Corollary 2.8, we obtain the following

Corollary 2.13. *Let $V \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and $q \in \mathbb{Z}_+$. Then we have*

$$\mathcal{U}_B^* P_q V P_q \mathcal{U}_B = p_q \otimes \mathrm{Op}^w(V_B * \Psi_q). \tag{2.32}$$

Remark 2.14. To the authors' best knowledge, the unitary equivalence between the Toeplitz operators $P_q V P_q$, $q \in \mathbb{Z}_+$, and ΨDO with generalized anti-Wick symbols in the context of the spectral theory of perturbations of the Landau Hamiltonian, was first shown in [25]. Related heuristic arguments can be found in [7,28]. In the case $q = 0$ this equivalence is closely related to the Segal-Bargmann transform in appropriate holomorphic spaces which, in one form or another, plays an important role in the semiclassical analysis performed in [33–36]. Let us comment in more detail on this relation. The Hilbert space $P_0 L^2(\mathbb{R}^2)$ coincides with the classical Bargmann space

$$\left\{ f \in L^2(\mathbb{R}^2) \mid f(\mathrm{x}) = e^{-B|\mathrm{x}|^2/4} g(\mathrm{x}), \quad \frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y} = 0, \quad \mathrm{x} = (x, y) \in \mathbb{R}^2 \right\}.$$

Then the Segal-Bargmann transform $T_0 : L^2(\mathbb{R}) \to P_0 L^2(\mathbb{R}^2)$ is a unitary operator with integral kernel

$$\mathcal{T}_0 := \frac{1}{\sqrt{2}} \left(\frac{B}{\pi}\right)^{3/4} e^{-B((x+iy+2t)^2 - 2t^2 + |\mathrm{x}|^2)/4}, \quad \mathrm{x} = (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

(see [24, Lem. 3.1]). Fix $q \in \mathbb{Z}_+$. Denote by $M_q : L^2(\mathbb{R}) \to (p_q \otimes I_y) L^2(\mathbb{R}^2)$ the unitary operator which maps $u \in L^2(\mathbb{R})$ into $B^{1/4}\varphi_q(x) u(B^{1/2} y)$, $(x, y) \in \mathbb{R}^2$, and by $\mathcal{R} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ the unitary operator generated by the rotation by angle $\pi/2$, i.e. $(\mathcal{R}u)(x, y) = u(y, -x)$, $(x, y) \in \mathbb{R}^2$; note that $[\mathcal{R}, P_q] = 0$. Then we have

$$T_0 = \mathcal{R}\mathcal{U}_B M_0.$$

From this point of view the operators $T_q := \mathcal{R}\mathcal{U}_B M_q$, $q \in \mathbb{N}$, could be called generalized Segal-Bargmann transforms.

3. Analysis of $Op^w(V_B * \Psi_q)$ and Proof of Theorem 1.6

3.1. Reduction of $Op^w(V_B * \Psi_q)$ to $Op^w(V_B * \delta_{\sqrt{2q+1}})$. In the sequel we will use the following notations. For $k > 0$, let δ_k be the δ-function in $\mathbb{R}^2$ supported on the circle of radius k centered at the origin. More precisely, the distribution $\delta_k \in \mathcal{S}'(\mathbb{R}^2)$ is defined by

$$\delta_k(\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \varphi(k\cos\theta, k\sin\theta)d\theta, \quad \varphi \in \mathcal{S}(\mathbb{R}^2).$$

Next, we denote by $\hat{f}$ the Fourier transform of the distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, unitary in $L^2(\mathbb{R}^d)$, i.e.

$$\hat{f}(\xi) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \mathbb{R}^d, \tag{3.1}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$.

Lemma 3.1. *Let $V \in C_0^\infty(\mathbb{R}^2)$ and $B_0 > 0$. Then for some constant $C = C(B_0)$ one has*

$$\begin{aligned}
&\sup_{q>0}\sup_{B\geq B_0} \lambda_q^{3/4} B^{-1}\|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\| \\
&\quad \leq C\int_{\mathbb{R}^2} (|\zeta|^{5/2} + |\zeta|^6)|\widehat{V}(\zeta)|d\zeta, \qquad (3.2)\\
&\sup_{q>0}\sup_{B\geq B_0} \lambda_q^{3/4} B^{-1}\|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\|_2 \\
&\quad \leq C\left(\int_{\mathbb{R}^2} (|\zeta|^5 + |\zeta|^{12})|\widehat{V}(\zeta)|^2 d\zeta\right)^{1/2}. \qquad (3.3)
\end{aligned}$$

The intuition behind this lemma is the convergence of Ψ_q to $\delta_{\sqrt{2q+1}}$ in an appropriate sense as $q \to \infty$. We also note the similarity between the definition of $V_B * \delta_k$ and the "classical" formula (1.14). For brevity, we introduce the short-hand notations

$$s_q := V_B * \Psi_q, \quad q \in \mathbb{Z}_+, \quad t_k := V_B * \delta_k, \quad k \in (0, \infty). \tag{3.4}$$

Proof. First we represent the symbols s_q, t_k in a form convenient for our purposes. For s_q we have

$$s_q(z) = \int_{\mathbb{R}^2} e^{iz\zeta}\widehat{\Psi}_q(\zeta)\widehat{V}_B(\zeta)d\zeta, \quad z \in \mathbb{R}^2. \tag{3.5}$$

In the Appendix we will prove the formula

$$\widehat{\Psi_q}(\zeta) = (-1)^q \Psi_q(2^{-1}\zeta)/2, \quad q \in \mathbb{Z}_+, \quad \zeta \in \mathbb{R}^2. \tag{3.6}$$

By (3.5), (3.6), and the definition (2.13) of Ψ_q,

$$s_q(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{iz\zeta}\,\mathrm{L}_q(|\zeta|^2/2)e^{-|\zeta|^2/4}\widehat{V}_B(\zeta)d\zeta.$$

For t_k we can write

$$t_k(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{iz\zeta} J_0(k|\zeta|)\widehat{V}_B(\zeta)d\zeta, \quad z \in \mathbb{R}^2, \tag{3.7}$$

since the integral representation for the Bessel function J_0 can be written as

$$J_0(k|\zeta|) = 2\pi\hat{\delta}_k(\zeta), \quad \zeta \in \mathbb{R}^2. \tag{3.8}$$

Thus (3.2) (resp. (3.3)) reduces to estimating the operator norm (resp. the Hilbert-Schmidt norm) of the operator with the Weyl symbol

$$s_q(z) - t_{\sqrt{2q+1}}(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{iz\zeta}\left(\mathrm{L}_q(|\zeta|^2/2)e^{-|\zeta|^2/4} - J_0(\sqrt{2q+1}|\zeta|)\right)\widehat{V}_B(\zeta)d\zeta, \quad q \in \mathbb{Z}_+. \tag{3.9}$$

In what follows the estimate

$$\left|\mathrm{L}_q(x)e^{-x/2} - J_0(\sqrt{(4q+2)x})\right| \le C(q^{-3/4}x^{5/4} + q^{-1}x^3), \quad q \in \mathbb{N}, \quad x > 0, \tag{3.10}$$

plays a key role. This estimate is probably well known to experts, but since we could not find it explicitly in the literature, we include its proof in the Appendix.

Let us prove the estimate (3.2). Using the estimates (2.2) and (3.10), we obtain:

$$\begin{aligned}
&\|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\| \le (2\pi)^{-1}\|\hat{s_q} - \hat{t}_{\sqrt{2q+1}}\|_{L^1(\mathbb{R}^2)} \\
&= (2\pi)^{-1}\int_{\mathbb{R}^2} |\mathrm{L}_q(|\zeta|^2/2)e^{-|\zeta|^2/4} - J_0(\sqrt{2q+1}|\zeta|)||\widehat{V}_B(\zeta)|d\zeta \\
&\le C\int_{\mathbb{R}^2} (q^{-3/4}|\zeta|^{5/2} + q^{-1}|\zeta|^6)|\widehat{V}_B(\zeta)|d\zeta.
\end{aligned} \tag{3.11}$$

Recalling the definition (2.27) of V_B, we obtain $\widehat{V}_B(\zeta) = B\widehat{V}_1(B^{1/2}\zeta)$, and so the l.h.s. of (3.11) can be estimated by

$$\begin{aligned}
&CBq^{-3/4}\int_{\mathbb{R}^2} |\zeta|^{5/2}|\widehat{V}_1(B^{1/2}\zeta)|d\zeta + CBq^{-1}\int_{\mathbb{R}^2} |\zeta|^6|\widehat{V}_1(B^{1/2}\zeta)|d\zeta \\
&= CB^{-5/4}q^{-3/4}\int_{\mathbb{R}^2} |\zeta|^{5/2}|\widehat{V}_1(\zeta)|d\zeta + CB^{-3}q^{-1}\int_{\mathbb{R}^2} |\zeta|^6|\widehat{V}_1(\zeta)|d\zeta.
\end{aligned}$$

This yields (3.2).

Next, let us prove the estimate (3.3). By (2.3) and the unitarity of the Fourier transform,

$$\begin{aligned}
\|\mathrm{Op}^w(s_q - t_k)\|_2^2 &= (2\pi)^{-1}\int_{\mathbb{R}^2} |\hat{s}_q(\zeta) - \hat{t}_{\sqrt{2q+1}}(\zeta)|^2 d\zeta \\
&= (2\pi)^{-1}\int_{\mathbb{R}^2} |\mathrm{L}_q(|\zeta|^2/2)e^{-|\zeta|^2/4} - J_0(\sqrt{2q+1}|\zeta|)|^2|\widehat{V}_B(\zeta)|^2 d\zeta.
\end{aligned}$$

Now using the estimate (3.10) again, we obtain (3.3) in a similar way to the previous step of the proof. □

3.2. Norm estimate of $Op^w(V_B * \delta_k)$.

Lemma 3.2. *Let $V(\mathbf{x}) = \langle \mathbf{x} \rangle^{-\rho}$, $\rho > 1$, and $B_0 > 0$. Then*

$$\sup_{k>0} \sup_{B>B_0} kB^{-1/2} \|\mathrm{Op}^w(V_B * \delta_k)\| < \infty.$$

Proof. By Proposition 2.2, it suffices to prove that for any differential operator L with constant coefficients we have

$$\sup_{k>0} \sup_{B>B_0} kB^{-1/2} \sup_{z\in\mathbb{R}^2} |(LV_B * \delta_k)(z)| < \infty. \tag{3.12}$$

Note that, by the standard symbol properties of $\langle \mathrm{x} \rangle^{-\rho}$, we have

$$|LV_B(\mathrm{x})| \le CV_B(\mathrm{x}), \quad \mathrm{x} \in \mathbb{R}^2,$$

where C depends only on B_0 and L. Thus, it remains to prove that

$$\sup_{k>0} \sup_{B>B_0} kB^{-1/2} \sup_{z\in\mathbb{R}^2} |(V_B * \delta_k)(z)| < \infty. \tag{3.13}$$

We have

$$V_B(z) = (B^{-1}|z|^2 + 1)^{-\rho/2}.$$

Take $z = (r, 0)$, $r \ge 0$. Then

$$\begin{aligned}
(V_B * \delta_k)(z) &= \frac{1}{2\pi} \int_0^{2\pi} (B^{-1}(k\cos\theta - r)^2 + B^{-1}(k\sin\theta)^2 + 1)^{-\rho/2} d\theta \\
&\le \frac{1}{2\pi} \int_0^{2\pi} (B^{-1}k^2(\sin\theta)^2 + 1)^{-\rho/2} d\theta = \frac{2}{\pi} \int_0^{\pi/2} (B^{-1}k^2(\sin\theta)^2 + 1)^{-\rho/2} d\theta \\
&\le \frac{2}{\pi} \int_0^{\pi/2} (B^{-1}k^2(2\theta/\pi)^2 + 1)^{-\rho/2} d\theta \\
&\le \frac{2}{\pi} \int_0^{\infty} (B^{-1}k^2(2\theta/\pi)^2 + 1)^{-\rho/2} d\theta \\
&= \frac{2B^{1/2}}{\pi k} \int_0^{\infty} ((2\theta/\pi)^2 + 1)^{-\rho/2} d\theta = CB^{1/2}/k.
\end{aligned}$$

This yields

$$\sup_{z\in\mathbb{R}^2} |(V_B * \delta_k)(z)| \le CB^{1/2}k^{-1},$$

and (3.13) follows. □

3.3. Asymptotics of traces.

Theorem 3.3. *Let $V \in C_0^\infty(\mathbb{R}^2)$. Then for each $\ell \in \mathbb{N}$, $\ell \ge 2$, we have*

$$\lim_{q\to\infty} \lambda_q^{(\ell-1)/2} \mathrm{Tr}\big(\mathrm{Op}^w(t_{\sqrt{2q+1}})\big)^\ell = \frac{B^\ell}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \widetilde{V}(\omega, b)^\ell \, db \, d\omega. \tag{3.14}$$

The proof is based on the following technical lemma.

Lemma 3.4. *Let $\ell \in \mathbb{N}$, $\ell \ge 2$, $f \in \mathcal{S}(\mathbb{R}^{2(\ell-1)})$, and let the function $\varphi : \mathbb{T}^\ell \times \mathbb{R}^{2(\ell-1)} \to \mathbb{R}$ be given by*

$$\varphi(\boldsymbol{\omega}, \mathbf{z}) = \sum_{j=1}^{\ell-1} z_j \cdot (\omega_{j+1} - \omega_j), \tag{3.15}$$

where $\mathbf{z} = (z_1, \dots, z_{\ell-1}) \in \mathbb{R}^{2(\ell-1)}$, $\boldsymbol{\omega} = (\omega_1, \dots, \omega_\ell) \in \mathbb{T}^\ell \subset \mathbb{R}^{2\ell}$, *and* $\cdot$ *denotes the scalar product in* $\mathbb{R}^2$. *Then*

$$\begin{aligned}\lim_{k\to\infty} k^{\ell-1} &\int_{\mathbb{R}^{2(\ell-1)}} \int_{\mathbb{T}^\ell} f(\mathbf{z}) e^{ik\varphi(\boldsymbol{\omega},\mathbf{z})}\, d\boldsymbol{\omega}\, d\mathbf{z} \\ &= (2\pi)^{\ell-1} \int_{\mathbb{T}} \int_{\mathbb{R}^{\ell-1}} f(\alpha_1\omega, \alpha_2\omega, \dots, \alpha_{\ell-1}\omega)\, d\alpha_1 d\alpha_2 \cdots d\alpha_{\ell-1}\, d\omega. \end{aligned} \tag{3.16}$$

Proof. The proof consists in an application of the stationary phase method.

We use the following parametrisation of the variables $\boldsymbol{\omega}$, $\mathbf{z}$:

$$\begin{aligned} &\omega_\ell = (\cos\theta, \sin\theta), \quad \theta \in [-\pi, \pi); \\ &\omega_j = (\cos(\theta + \theta_j), \sin(\theta + \theta_j)), \quad \theta_j \in [-\pi, \pi), \quad j = 1, \dots, \ell - 1; \\ &z_j = \alpha_j \omega_\ell + \beta_j \omega_\ell^\perp, \quad \alpha_j, \beta_j \in \mathbb{R}, \quad j = 1, \dots, \ell - 1. \end{aligned}$$

We write $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{\ell-1}) \in \mathbb{R}^{\ell-1}$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{\ell-1}) \in \mathbb{R}^{\ell-1}$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{\ell-1}) \in [-\pi, \pi)^{\ell-1}$. Using this notation, we can rewrite the integral in the l.h.s. of (3.16) as

$$\begin{aligned}&\int_{\mathbb{R}^{2(\ell-1)}} \int_{\mathbb{T}^\ell} f(\mathbf{z}) e^{ik\varphi(\boldsymbol{\omega},\mathbf{z})}\, d\boldsymbol{\omega}\, d\mathbf{z} \\ &= \int_{-\pi}^{\pi} \int_{(-\pi,\pi)^{\ell-1}} \int_{\mathbb{R}^{\ell-1}} \int_{\mathbb{R}^{\ell-1}} F(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}, \theta) e^{ik\Phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})}\, d\boldsymbol{\beta}\, d\boldsymbol{\alpha}\, d\boldsymbol{\theta}\, d\theta, \end{aligned} \tag{3.17}$$

where

$$F(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}, \theta) = f(\alpha_1\omega_\ell + \beta_1\omega_\ell^\perp, \dots, \alpha_{\ell-1}\omega_\ell + \beta_{\ell-1}\omega_\ell^\perp),$$

and

$$\begin{aligned} \Phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}) &= \alpha_1(1 - \cos\theta_1) - \beta_1 \sin\theta_1, \quad \text{if } \ell = 2, \\ \Phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}) &= \alpha_{\ell-1} - (\alpha_1 \cos\theta_1 + \beta_1 \sin\theta_1) \\ &\quad + \sum_{j=2}^{\ell-1} ((\alpha_{j-1} - \alpha_j)\cos\theta_j + (\beta_{j-1} - \beta_j)\sin\theta_j), \quad \text{if } \ell \geq 3. \end{aligned}$$

Let us consider the stationary points of the phase function Φ. By a direct calculation, $\nabla\Phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}) = 0$ if and only if $\boldsymbol{\beta} = 0$ and $\boldsymbol{\theta} = 0$. By a standard localisation argument, it follows that the asymptotics of the integral (3.17) will not change if we multiply F by a function $\chi = \chi(\boldsymbol{\beta}, \boldsymbol{\theta})$, $\chi \in C^\infty(\mathbb{R}^{\ell-1} \times [-\pi, \pi)^{\ell-1})$, such that $\chi(\boldsymbol{\beta}, \boldsymbol{\theta}) = 1$ in an open neighbourhood of the origin $\boldsymbol{\beta} = 0$, $\boldsymbol{\theta} = 0$, and $\chi(\boldsymbol{\beta}, \boldsymbol{\theta}) = 0$ if $|\boldsymbol{\beta}| \geq 1/2$ or $|\boldsymbol{\theta}| \geq \pi/2$.

Let us write

$$\begin{aligned}&\int_{-\pi}^{\pi} \int_{(-\pi,\pi)^{\ell-1}} \int_{\mathbb{R}^{\ell-1}} \int_{\mathbb{R}^{\ell-1}} F(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\theta}, \theta) \chi(\boldsymbol{\beta}, \boldsymbol{\theta}) e^{ik\Phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})}\, d\boldsymbol{\beta}\, d\boldsymbol{\alpha}\, d\boldsymbol{\theta}\, d\theta \\ &= \int_{-\pi}^{\pi} \int_{\mathbb{R}^{\ell-1}} I(k; \boldsymbol{\alpha}, \theta)\, d\boldsymbol{\alpha}\, d\theta, \end{aligned} \tag{3.18}$$

where

$$I(k;\boldsymbol{\alpha},\theta)=\int_{(-\pi,\pi)^{\ell-1}}\int_{\mathbb{R}^{\ell-1}}F(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta},\theta)\chi(\boldsymbol{\beta},\boldsymbol{\theta})e^{ik\Phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})}\,d\boldsymbol{\beta}\,d\boldsymbol{\theta}.$$

Let us fix $\boldsymbol{\alpha}$, θ and compute the asymptotics of the integral $I(k;\boldsymbol{\alpha},\theta)$ as $k\to\infty$. A direct calculation shows that the stationary phase equations

$$\frac{\partial\Phi}{\partial\beta_j}=0,\quad \frac{\partial\Phi}{\partial\theta_j}=0,\quad j=1,\dots,\ell-1$$

are simultaneously satisfied on the support of χ if and only if $\boldsymbol{\beta}=0$, $\boldsymbol{\theta}=0$. In order to apply the stationary phase method, we need to compute the determinant and the signature (i.e. the difference between the number of positive and negative eigenvalues) of the Hessian of $\Phi(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})$ with respect to the variables $\boldsymbol{\beta}$, $\boldsymbol{\theta}$. Let us denote this Hessian by $H(\boldsymbol{\alpha})$. The $2(\ell-1)\times 2(\ell-1)$ matrix $H(\boldsymbol{\alpha})$ can be represented in a block form

$$H(\boldsymbol{\alpha})=\begin{pmatrix}H_{11}(\boldsymbol{\alpha}) & H_{12}(\boldsymbol{\alpha})\\ H_{21}(\boldsymbol{\alpha}) & H_{22}(\boldsymbol{\alpha})\end{pmatrix},$$

where

$$H_{11}(\boldsymbol{\alpha}):=\left\{\frac{\partial^2\Phi}{\partial\beta_p\partial\beta_q}(\boldsymbol{\alpha},0,0)\right\}_{p,q=1}^{\ell-1},\quad H_{12}(\boldsymbol{\alpha}):=\left\{\frac{\partial^2\Phi}{\partial\beta_p\partial\theta_q}(\boldsymbol{\alpha},0,0)\right\}_{p,q=1}^{\ell-1},$$
$$H_{21}(\boldsymbol{\alpha}):=\left\{\frac{\partial^2\Phi}{\partial\theta_p\partial\beta_q}(\boldsymbol{\alpha},0,0)\right\}_{p,q=1}^{\ell-1},\quad H_{22}(\boldsymbol{\alpha}):=\left\{\frac{\partial^2\Phi}{\partial\theta_p\partial\theta_q}(\boldsymbol{\alpha},0,0)\right\}_{p,q=1}^{\ell-1}.$$

An explicit computation shows that

$$H_{11}=0,\quad H_{12}=\left\{-\delta_{q,p}+\delta_{q,p+1}\right\}_{p,q=1}^{\ell-1},$$
$$H_{21}=H_{12}^T,\quad H_{22}(\boldsymbol{\alpha})=\operatorname{diag}\{\alpha_1,\alpha_2-\alpha_1,\dots,\alpha_{\ell-1}-\alpha_{\ell-2}\}.$$

Hence,

$$\det_{2(\ell-1)}H(\boldsymbol{\alpha})=\det_{\ell-1}(-H_{12}H_{21})=(-1)^{\ell-1};$$

in particular, our stationary point is non-degenerate.

In order to calculate the signature of $H(\boldsymbol{\alpha})$, note that since $\det H(\boldsymbol{\alpha})\neq 0$ for all $\boldsymbol{\alpha}$ and $H(\boldsymbol{\alpha})$ depends smoothly (in fact, polynomially) on $\boldsymbol{\alpha}$, the signature is independent of $\boldsymbol{\alpha}$. Some elementary analysis shows that $\operatorname{sign}H(0)=0$.

Now we can apply a suitable version of the stationary phase method (see e.g. [16, Chap. 1] or [12, Chap. III, Sect. 2]) to calculate the asymptotics of $I(k;\boldsymbol{\alpha},\theta)$. This yields

$$\lim_{k\to\infty}k^{\ell-1}I(k;\boldsymbol{\alpha},\theta)=(2\pi)^{\ell-1}F(\boldsymbol{\alpha},0,0,\theta).$$

Using, for example, the Lebesgue dominated convergence theorem, one concludes that the above asymptotics can be integrated over $\boldsymbol{\alpha}$ and θ, see (3.18). This yields the required result (3.16). □

Proof of Theorem 3.3. We use the notation t_k, see (3.4). Denote by $t_{k,\ell}$ the Weyl symbol of the operator $(\mathrm{Op}^w(t_k))^\ell$, i.e.

$$\mathrm{Op}^w(t_{k,\ell}) = (\mathrm{Op}^w(t_k))^\ell, \quad \ell = 2, 3, \dots.$$

By the standard Weyl pseudodifferential calculus (see e.g. [32, Chap. 7, Eq. (14.21)]), for $\zeta_\ell \in \mathbb{R}^2$ we have

$$\hat{t}_{k,\ell}(\zeta_\ell) = (2\pi)^{-\ell+1} \int_{\mathbb{R}^{2(\ell-1)}} \hat{t}_k(\zeta_\ell - \zeta_{\ell-1})\hat{t}_k(\zeta_{\ell-1} - \zeta_{\ell-2}) \cdots \hat{t}_k(\zeta_1) e^{\frac{i}{2}\sum_{j=2}^{\ell} \sigma(\zeta_j, \zeta_{j-1})} d\boldsymbol{\zeta},$$

where $\sigma(\cdot, \cdot)$ is the symplectic form in $\mathbb{R}^2 \times \mathbb{R}^2$, and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_{\ell-1})$. It follows that

$$\begin{aligned}
\mathrm{Tr}\big(\mathrm{Op}^w(t_k)\big)^\ell &= \mathrm{Tr}\,\mathrm{Op}^w(t_{k,\ell}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} t_{k,\ell}(x)dx = \hat{t}_{k,\ell}(0) \\
&= (2\pi)^{-\ell+1} \int_{\mathbb{R}^{2(\ell-1)}} \hat{t}_k(-\zeta_{\ell-1})\hat{t}_k(\zeta_{\ell-1} - \zeta_{\ell-2}) \cdots \hat{t}_k(\zeta_1) \\
&\quad \times \exp\Big(\frac{i}{2}\sum_{j=2}^{\ell-1} \sigma(\zeta_j, \zeta_{j-1})\Big) d\boldsymbol{\zeta},
\end{aligned} \tag{3.19}$$

where we use the convention that $\sum_{j=2}^{\ell-1} = 0$ if $\ell = 2$.

Recalling (3.8), (3.7), we get

$$\hat{t}_k(\zeta) = \frac{1}{2\pi}\widehat{V}_B(\zeta)\int_{\mathbb{T}} e^{-ik\omega\zeta} d\omega, \quad \zeta \in \mathbb{R}^2,$$

and so, substituting into (3.19), we get

$$\mathrm{Tr}\big(\mathrm{Op}^w(t_k)\big)^\ell = \int_{\mathbb{R}^{2(\ell-1)}} \int_{\mathbb{T}^\ell} f(\boldsymbol{\zeta}) e^{ik\varphi(\boldsymbol{\omega}, \boldsymbol{\zeta})}\, d\boldsymbol{\omega}\, d\boldsymbol{\zeta},$$

where

$$f(\boldsymbol{\zeta}) = (2\pi)^{-2\ell+1}\widehat{V}_B(-\zeta_{\ell-1})\widehat{V}_B(\zeta_{\ell-1} - \zeta_{\ell-2}) \cdots \widehat{V}_B(\zeta_1) \exp\left(\frac{i}{2}\sum_{j=2}^{\ell-1} \sigma(\zeta_j, \zeta_{j-1})\right),$$

and φ is given by (3.15). Applying Lemma 3.4, we obtain

$$\begin{aligned}
&\lim_{k\to\infty} k^{\ell-1}\,\mathrm{Tr}\big(\mathrm{Op}^w(t_k)\big)^\ell \\
&\quad = (2\pi)^{-\ell} \int_{\mathbb{T}} \int_{\mathbb{R}^{\ell-1}} \widehat{V}_B(-\alpha_{\ell-1}\omega)\widehat{V}_B((\alpha_{\ell-1} - \alpha_{\ell-2})\omega) \cdots \widehat{V}_B((\alpha_2 - \alpha_1)\omega) \\
&\qquad \times \widehat{V}_B(\alpha_1\omega)\, d\boldsymbol{\alpha}\, d\omega.
\end{aligned} \tag{3.20}$$

It remains to transform the last identity into (3.14). We have

$$\widehat{V}_B(\alpha\omega) = B\int_{\mathbb{R}} e^{-i\alpha b B^{1/2}}\, \widetilde{V}_1(\omega, b)\, db,$$

where, in accordance with (2.27), we use the notation $V_1(x, y) = V(-y, -x),\ (x, y) \in \mathbb{R}^2$. Therefore,

$$(2\pi)^{-\ell} \int_{\mathbb{T}} \int_{\mathbb{R}^{\ell-1}} \widehat{V}_B(-\alpha_{\ell-1}\omega)\widehat{V}_B((\alpha_{\ell-1}-\alpha_{\ell-2})\omega) \cdots \widehat{V}_B((\alpha_2-\alpha_1)\omega)\widehat{V}_B(\alpha_1\omega)\, d\boldsymbol{\alpha}\, d\omega$$
$$= \frac{B^{(1+\ell)/2}}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \widetilde{V}_1(\omega, b)^{\ell}\, db\, d\omega = \frac{B^{(1+\ell)/2}}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} \widetilde{V}(\omega, b)^{\ell}\, db\, d\omega. \tag{3.21}$$

Now (3.14) follows from (3.20) and (3.21). □

3.4. Proof of Theorem 1.6.

(i) First we observe that

$$\begin{aligned}\|P_q V P_q\| &\le \|P_q \langle\cdot\rangle^{-\rho/2}\| \|\langle\cdot\rangle^{\rho} V\| \|\langle\cdot\rangle^{-\rho/2} P_q\| \\ &\le \|V\|_{X_\rho} \|P_q \langle\cdot\rangle^{-\rho/2}\| \|\langle\cdot\rangle^{-\rho/2} P_q\| = \|V\|_{X_\rho} \|P_q \langle\cdot\rangle^{-\rho} P_q\|,\end{aligned}$$

and so it suffices to consider the case $V(\mathrm{x}) = \langle \mathrm{x}\rangle^{-\rho}$.

Next, by Corollary 2.13, we have

$$\|P_q V P_q\| = \|\mathrm{Op}^w(V_B * \Psi_q)\|.$$

In order to estimate the norm of $\mathrm{Op}^w(V_B * \Psi_q)$, we use Lemmas 3.1 and 3.2. We note that for $V(\mathrm{x}) = \langle \mathrm{x}\rangle^{-\rho},\ \rho > 1$, we have $\widehat{V} \in L^1$ and

$$|\widehat{V}(\zeta)| \le C_N |\zeta|^{-N}, \quad |\zeta| \ge 1,$$

for all $N \ge 1$ (see e.g. [31, Chap. XII, Lem. 3.1]). Thus, the integral in the r.h.s. of (3.2) is convergent, and so the proof of (3.2) applies to $V(\mathrm{x}) = \langle \mathrm{x}\rangle^{-\rho},\ \rho > 1$. Now, combining Lemmas 3.1 and 3.2, we get

$$\begin{aligned}&\sup_{q\ge 0} \sup_{B\ge B_0} \lambda_q^{1/2} B^{-1} \|\mathrm{Op}^w(V_B * \Psi_q)\| \\ &\quad \le \sup_{q\ge 0} \sup_{B\ge B_0} \lambda_q^{3/4} B^{-1} \|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\| \\ &\qquad + \sup_{q\ge 0} \sup_{B\ge B_0} \lambda_q^{1/2} B^{-1} \|\mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\| < \infty,\end{aligned}$$

which proves the required estimate.

(ii) As in the proof of part (i), we may assume $V(\mathrm{x}) = \langle \mathrm{x}\rangle^{-\rho}$. First let us consider the case $\rho > 2,\ \ell = 1$. By Lemma 2.12 with $\ell = 1$ we have

$$B^{-1} \|P_q V P_q\|_1 = \frac{1}{2\pi} \int_{\mathbb{R}^2} V(\mathrm{x}) d\mathrm{x} \le \frac{1}{2\pi} \|V\|_{X_\rho} \int_{\mathbb{R}^2} \langle \mathrm{x}\rangle^{-\rho} d\mathrm{x},$$

which proves (1.11) in this case.

Let us consider the case of a general ℓ. For a fixed $s > 1$ and any $\ell \in [1, \infty]$, let

$$M_q^{(\ell)} = B^{-1} \lambda_q^{\frac{1}{2}-\frac{1}{2\ell}} P_q \langle\cdot\rangle^{-s(1+\frac{1}{\ell})} P_q;$$

for $\ell = \infty$, one should replace $1/\ell$ by 0. By the previous step of the proof and part (i) of the theorem,

$$\sup_{q\geq 0}\sup_{B\geq B_0}\|M_q^{(1)}\|_1 \leq C_1 < \infty, \quad \sup_{q\geq 0}\sup_{B\geq B_0}\|M_q^{(\infty)}\| \leq C_\infty < \infty,$$

where the constants $C_1,\ C_\infty$ depend only on B_0 and s. Applying the Calderon-Lions interpolation theorem (see e.g. [27, Thm. IX.20]), we get

$$\sup_{q\geq 0}\sup_{B\geq B_0}\|M_q^{(\ell)}\|_\ell \leq C_1^{1/\ell}C_\infty^{(\ell-1)/\ell} < \infty$$

for all $\ell \geq 1$. It is easy to see that the last statement is equivalent to (1.11).

(iii) First let us note that the case $\ell = 1$ is straightforward. Indeed, if the integer $\ell = 1$ is admissible, i.e. if $\ell = 1 > 1/(\rho - 1)$, then $\rho > 2$, $V \in L^1(\mathbb{R}^2)$, and (2.29) yields the identity

$$\operatorname{Tr} P_q V P_q = \frac{B}{2\pi}\int_{\mathbb{R}^2} V(\mathrm{x})d\mathrm{x} = \frac{B}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \widetilde{V}(\omega, b)\, db\, d\omega.$$

Thus, we may now assume $\ell \geq 2$. We will first prove the required identity (1.12) for $V \in C_0^\infty(\mathbb{R}^2)$ and then use a limiting argument to extend it to all $V \in X_\rho$. Denote

$$\gamma_\ell(V) = \frac{B^\ell}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} \widetilde{V}(\omega, b)^\ell\, db\, d\omega. \tag{3.22}$$

By Corollary 2.13, we have

$$\operatorname{Tr}(P_q V P_q)^\ell = \operatorname{Tr}\big(\operatorname{Op}^w(V_B * \Psi_q)\big)^\ell,$$

and Theorem 3.3 says

$$\lim_{q\to\infty}\lambda_q^{(\ell-1)/2}\operatorname{Tr}\big(\operatorname{Op}^w(V_B * \delta_{\sqrt{2q+1}})\big)^\ell = \gamma_\ell(V).$$

Thus, it suffices to prove that

$$\lim_{q\to\infty}\lambda_q^{(\ell-1)/2}\big|\operatorname{Tr}\big(\operatorname{Op}^w(V_B * \Psi_q)\big)^\ell - \operatorname{Tr}\big(\operatorname{Op}^w(V_B * \delta_{\sqrt{2q+1}})\big)^\ell\big| = 0. \tag{3.23}$$

In order to prove (3.23), let us first display an elementary estimate

$$\left|\operatorname{Tr}\left(A_1^\ell\right) - \operatorname{Tr}\left(A_2^\ell\right)\right| \leq \ell\max\{\|A_1\|_\ell^{\ell-1}, \|A_2\|_\ell^{\ell-1}\}\|A_1 - A_2\|_\ell; \tag{3.24}$$

here $\ell \in \mathbb{N}$ and $A_n \in S_\ell,\ n = 1, 2$. The estimate follows from the formula

$$A_1^\ell - A_2^\ell = \sum_{j=0}^{\ell-1} A_1^{\ell-j-1}(A_1 - A_2)A_2^j$$

and the Hölder type inequality for the S_ℓ classes.

Next, using Corollary 2.13 and part (ii) of the theorem, we get

$$\limsup_{q\to\infty} \lambda_q^{(\ell-1)/(2\ell)} \|\mathrm{Op}^w(V_B * \Psi_q)\|_\ell = \limsup_{q\to\infty} \lambda_q^{(\ell-1)/(2\ell)} \|P_q V P_q\|_\ell < \infty. \quad (3.25)$$

Further, by estimate (3.3), using the assumption $\ell \geq 2$, we get

$$\begin{aligned}\limsup_{q\to\infty} &\lambda_q^{(\ell-1)/(2\ell)} \|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\|_\ell \\ &\leq \limsup_{q\to\infty} \lambda_q^{1/2} \|\mathrm{Op}^w(V_B * \Psi_q) - \mathrm{Op}^w(V_B * \delta_{\sqrt{2q+1}})\|_2 = 0. \end{aligned} \quad (3.26)$$

Combining (3.24), (3.25) and (3.26), we obtain (3.23) for $V \in C_0^\infty$; thus, (1.12) is proven for this class of potentials.

It remains to extend (1.12) to all potentials $V \in C(\mathbb{R}^2)$ that satisfy (1.1). For $\ell > 1/(\rho - 1)$, denote

$$\begin{aligned}\Delta_\ell(V) &= \limsup_{q\to\infty} \lambda_q^{(\ell-1)/2} \operatorname{Tr}(P_q V P_q)^\ell, \\ \delta_\ell(V) &= \liminf_{q\to\infty} \lambda_q^{(\ell-1)/2} \operatorname{Tr}(P_q V P_q)^\ell.\end{aligned}$$

Above we have proven that

$$\Delta_\ell(V) = \delta_\ell(V) = \gamma_\ell(V) \quad (3.27)$$

for all potentials $V \in C_0^\infty(\mathbb{R}^2)$; now we need to extend this identity to all $V \in X_\rho$. From (1.5) we obtain, similarly to (3.24),

$$\begin{aligned}|\gamma_\ell(V_1) - \gamma_\ell(V_2)| &\leq \frac{B^\ell}{2\pi} \int_{\mathbb{T}} \int_{\mathbb{R}} |\widetilde{V}_1(\omega, b)^\ell - \widetilde{V}_2(\omega, b)^\ell| db d\omega \\ &\leq \frac{B^\ell}{2\pi} C \max\{\|V_1\|_{X_\rho}^{\ell-1}, \|V_2\|_{X_\rho}^{\ell-1}\} \|V_1 - V_2\|_{X_\rho} \int_{\mathbb{T}} \int_{\mathbb{R}} \langle b\rangle^{(1-\rho)\ell} db\, d\omega.\end{aligned}$$

It follows that γ_ℓ is a continuous functional on X_ρ. Similarly, using (3.24) and part (ii) of the theorem, we get

$$\begin{aligned}\limsup_{q\to\infty} &\lambda_q^{(\ell-1)/2} |\operatorname{Tr}(P_q V_1 P_q)^\ell - \operatorname{Tr}(P_q V_2 P_q)^\ell| \\ &\leq C \max\{\|V_1\|_{X_\rho}^{\ell-1}, \|V_2\|_{X_\rho}^{\ell-1}\} \|V_1 - V_2\|_{X_\rho},\end{aligned}$$

and so the functionals Δ_ℓ, δ_ℓ are continuous on X_ρ. It follows that (3.27) extends by continuity from C_0^∞ to the closure X_ρ^0 of C_0^∞ in X_ρ. In order to prove (3.27) for all $V \in X_\rho$, one can argue as follows. For a given $\ell > 1/(\rho - 1)$, choose ρ_1 such that $1 < \rho_1 < \rho$ and $\ell > 1/(\rho_1 - 1)$. Then $X_\rho \subset X_{\rho_1}^0$ and by the same argument as above, (3.27) holds true for all $V \in X_{\rho_1}^0$. □

4. Proof of Proposition 1.1 and Theorem 1.3

As already indicated, this section heavily uses the construction of [22].

4.1. Proof of Proposition 1.1. Set $R_0(z) := (H_0 - zI)^{-1}$. By the Birman-Schwinger principle, if $\lambda \in \mathbb{R}\setminus \cup_{q=0}^{\infty}\{\lambda_q\}$ is an eigenvalue of the operator H, then -1 is an eigenvalue of the operator $|V|^{1/2}R_0(\lambda)V^{1/2}$. Hence, it suffices to show that for some $C > 0$ and all sufficiently large q, we have

$$\||V|^{1/2}R_0(\lambda)|V|^{1/2}\| < 1, \quad \text{for all } \lambda \in [\lambda_q - B, \lambda_q + B], \quad |\lambda - \lambda_q| > \frac{C}{\sqrt{q}}. \tag{4.1}$$

Choose $m \in \mathbb{N}$ sufficiently large so that $\|V\|/\lambda_m < 1/2$, and write $R_0(\lambda)$ as

$$R_0(\lambda) = \sum_{k=q-m}^{q+m} \frac{P_k}{\lambda_k - \lambda} + \widetilde{R}_0(\lambda).$$

Then, for $\lambda \in [\lambda_q - B, \lambda_q + B]$,

$$\||V|^{1/2}R_0(\lambda)|V|^{1/2}\| \le \sum_{k=q-m}^{q+m} \frac{\||V|^{1/2}P_k|V|^{1/2}\|}{|\lambda_k - \lambda|} + \||V|^{1/2}\widetilde{R}_0(\lambda)|V|^{1/2}\|.$$

By the choice of m, one has

$$\||V|^{1/2}\widetilde{R}_0(\lambda)|V|^{1/2}\| \le \||V|^{1/2}\|(1/\lambda_m)\||V|^{1/2}\| = \|V\|/\lambda_m < 1/2.$$

On the other hand, by Theorem 1.6(i),

$$\sum_{k=q-m}^{q+m} \frac{\||V|^{1/2}P_k|V|^{1/2}\|}{|\lambda_k - \lambda|} \le (2m+1)O(q^{-1/2}) \max_{q-m\le k\le q+m} |\lambda_k - \lambda|^{-1} = O(q^{-1/2})|\lambda_q - \lambda|^{-1}.$$

Thus, we get (4.1) for sufficiently large $C > 0$. □

4.2. Resolvent estimates. Let Γ_q be a positively oriented circle of center λ_q and radius B.

Lemma 4.1. *Let V satisfy (1.1). Then for any $\ell > 1$, $\ell > 1/(\rho - 1)$, one has*

$$\sup_{z\in\Gamma_q} \||V|^{1/2}R_0(z)|V|^{1/2}\|_\ell = O(q^{-(\ell-1)/2\ell}\log q), \quad q \to \infty, \tag{4.2}$$

$$\sup_{z\in\Gamma_q} \||V|^{1/2}R_0(z)\|_{2\ell} = O(q^{-(\ell-1)/4\ell}\log q), \quad q \to \infty. \tag{4.3}$$

Proof. Let us prove (4.2). Using the estimate (1.11), we get for $z \in \Gamma_q$:

$$\begin{aligned}
\||V|^{1/2} R_0(z)|V|^{1/2}\|_\ell &\le \sum_{k=0}^{\infty} \frac{\||V|^{1/2} P_k |V|^{1/2}\|_\ell}{|\lambda_k - z|} \le \sum_{k=0}^{\infty} \frac{C(1+k)^{-(\ell-1)/2\ell}}{|\lambda_k - z|} \\
&\le C \int_0^{q-1} \frac{(1+x)^{-(\ell-1)/2\ell}}{|B(2x+1) - z|} dx \\
&\quad + C \int_{q+1}^{\infty} \frac{(1+x)^{-(\ell-1)/2\ell}}{|B(2x+1) - z|} dx + O(q^{-(\ell-1)/2\ell}) \\
&= O(q^{-(\ell-1)/2\ell} \log q),
\end{aligned}$$

as $q \to \infty$. This proves (4.2). Using the fact that

$$\||V|^{1/2} P_q\|_{2\ell}^2 = \||V|^{1/2} P_q |V|^{1/2}\|_\ell,$$

one proves the estimate (4.3) in the same way. □

4.3. Proof of Lemma 1.5. The fact that $(P_q V P_q)^\ell \in S_1$ follows directly from Theorem 1.6. Let, as above, Γ_q be a positively oriented circle with the centre λ_q and radius B. Let q be sufficiently large so that (see Proposition 1.1) the contour Γ_q does not intersect the spectrum of H. We will use the formula

$$(H - \lambda_q)^\ell \mathbb{1}_{(\lambda_q - B, \lambda_q + B)}(H) = -\frac{1}{2\pi i} \int_{\Gamma_q} (z - \lambda_q)^\ell R(z) dz, \tag{4.4}$$

where $R(z) = (H - zI)^{-1}$. Let us expand the resolvent $R(z)$ in the r.h.s. of (4.4) in the standard perturbation series:

$$R(z) = R_0(z) + \sum_{j=1}^{\infty} (-1)^j R_0(z)(V R_0(z))^j. \tag{4.5}$$

Let us discuss the convergence of these series for $z \in \Gamma_q$, q large. Denote $W = |V|^{1/2}$, $W_0 = \operatorname{sign}(V)$. For $j \ge \ell$, we have

$$\begin{aligned}
\|R_0(z)(V R_0(z))^j\|_1 &= \|(R_0(z)W)(W_0 W R_0(z) W)^{j-1} W_0 (W R_0(z))\|_1 \\
&\le \|R_0(z)W\|_{2\ell} \|W R_0(z) W\|_\ell^{j-1} \|W R_0(z)\|_{2\ell}, \quad j \ge \ell.
\end{aligned} \tag{4.6}$$

Applying Lemma 4.1, we get that the series in the r.h.s. of (4.5) converges in the trace norm for $z \in \Gamma_q$ and q sufficiently large (note that although the tail of the series converges in the trace class, the series itself is not necessarily trace class).

Next, it is easy to see that the integrals

$$\int_{\Gamma_q} (z - \lambda_q)^\ell R_0(z)(V R_0(z))^j dz$$

with $j < \ell$ vanish (since the integrand is analytic inside Γ_q). Thus, recalling (4.6), we obtain that the operator $(H - \lambda_q)^\ell \mathbb{1}_{(\lambda_q - B, \lambda_q + B)}(H)$ belongs to the trace class and

$$\begin{aligned} &\mathrm{Tr}\{(H - \lambda_q)^\ell \mathbb{1}_{(\lambda_q - B, \lambda_q + B)}(H)\} \\ &\quad = -\frac{1}{2\pi i} \sum_{j=\ell}^{\infty} (-1)^j \int_{\Gamma_q} (z - \lambda_q)^\ell \,\mathrm{Tr}[R_0(z)(V R_0(z))^j] dz. \end{aligned} \tag{4.7}$$

Integrating by parts in each term of this series and computing the term with $j = \ell$ by the residue theorem, we obtain

$$\begin{aligned} &\mathrm{Tr}\{(H - \lambda_q)^\ell \mathbb{1}_{(\lambda_q - B, \lambda_q + B)}(H)\} \\ &\quad = \mathrm{Tr}(P_q V P_q)^\ell + \frac{\ell}{2\pi i} \sum_{j=\ell+1}^{\infty} \frac{(-1)^j}{j} \int_{\Gamma_q} (z - \lambda_q)^{\ell-1} \,\mathrm{Tr}(V R_0(z))^j dz. \end{aligned} \tag{4.8}$$

It remains to estimate the series in the r.h.s. of (4.8). This can be easily done by using Lemma 4.1. Similarly to (4.6), we have

$$|\mathrm{Tr}(V R_0(z))^j| \le \|W R_0(z) W\|_j^j \le \|W R_0(z) W\|_\ell^j, \quad j \ge \ell,$$

and so

$$\left| \int_{\Gamma_q} (z - \lambda_q)^{\ell-1} \,\mathrm{Tr}(V R_0(z))^j dz \right| \le C_1 (C_2 q^{-(\ell-1)/2\ell} \log q)^j,$$

for all sufficiently large q. Thus, the series in the r.h.s. of (4.8) can be estimated by

$$C_1 \sum_{j=\ell+1}^{\infty} C_2^j q^{-\frac{\ell-1}{2\ell} j} (\log q)^j.$$

For all sufficiently large q, this series converges and can be estimated as $o(q^{-(\ell-1)/2})$. □

4.4. Proof of Theorem 1.3. Let $R \ge C_1$ where C_1 is the constant from Proposition 1.1. Then

$$\mathbb{1}_{[-R,R]}(\lambda_q^{1/2}(H - \lambda_q)) = \mathbb{1}_{(\lambda_q - B, \lambda_q + B)}(H), \quad q \in \mathbb{Z}_+. \tag{4.9}$$

Next, choose $R \ge C_1$ so large that $\mathrm{supp}\varrho \subset [-R, R]$. Let ℓ_0 be an even natural number satisfying $\ell_0 > 1/(\rho - 1)$. Since $\varrho(\lambda)$ by assumption vanishes near $\lambda = 0$, the function $\varrho(\lambda)/\lambda^{\ell_0}$ is smooth. Applying the Weierstrass approximation theorem to this function on the interval $[-R, R]$, we obtain that for any $\varepsilon > 0$ there exist polynomials P_+, P_- such that

$$P_\pm(0) = P'_\pm(0) = \cdots = P_\pm^{(\ell_0 - 1)}(0) = 0, \tag{4.10}$$

$$P_-(\lambda) \le \varrho(\lambda) \le P_+(\lambda), \quad \forall \lambda \in [-R, R], \tag{4.11}$$

$$P_+(\lambda) - P_-(\lambda) \le \varepsilon \lambda^{\ell_0}, \quad \forall \lambda \in [-R, R]. \tag{4.12}$$

Thus, we can write

$$\mathbb{1}_{[-R,R]}(\lambda)P_-(\lambda) \le \varrho(\lambda) \le \mathbb{1}_{[-R,R]}(\lambda)P_+(\lambda),$$

for any $\lambda \in [-R, R]$, and therefore

$$\begin{aligned}\operatorname{Tr}\{\mathbb{1}_{[-R,R]}(\lambda_q^{1/2}(H-\lambda_q))\, P_-(\lambda_q^{1/2}(H-\lambda_q))\} &\le \operatorname{Tr}\, \varrho(\lambda_q^{1/2}(H-\lambda_q))\\ &\le \operatorname{Tr}\{\mathbb{1}_{[-R,R]}(\lambda_q^{1/2}(H-\lambda_q))\, P_+(\lambda_q^{1/2}(H-\lambda_q))\}.\end{aligned}$$

By (4.9) it follows that for all sufficiently large q,

$$\begin{aligned}\operatorname{Tr}\{\mathbb{1}_{(\lambda_q-B,\lambda_q+B)}(H)\, P_-(\lambda_q^{1/2}(H-\lambda_q))\} &\le \operatorname{Tr}\varrho(\lambda_q^{1/2}(H-\lambda_q))\\ &\le \operatorname{Tr}\{\mathbb{1}_{(\lambda_q-B,\lambda_q+B)}(H)\, P_+(\lambda_q^{1/2}(H-\lambda_q))\}.\end{aligned} \tag{4.13}$$

By Lemma 1.5 and Theorem 1.6(iii), we have

$$\begin{aligned}&\lim_{q\to\infty} \lambda_q^{-1/2} \operatorname{Tr}\{\mathbb{1}_{(\lambda_q-B,\lambda_q+B)}(H)\, P_\pm(\lambda_q^{1/2}(H-\lambda_q))\}\\ &\quad = \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{R}} P_\pm(\widetilde{V}(\omega,b))\, db\, d\omega = \int_{\mathbb{R}} P_\pm(t)d\mu(t).\end{aligned}$$

Combining this with (4.13), we get

$$\begin{aligned}\limsup_{q\to\infty} \lambda_q^{-1/2} \operatorname{Tr}\varrho(\lambda_q^{1/2}(H-\lambda_q)) &\le \int_{\mathbb{R}} P_+(\lambda)d\mu(\lambda),\\ \liminf_{q\to\infty} \lambda_q^{-1/2} \operatorname{Tr}\varrho(\lambda_q^{1/2}(H-\lambda_q)) &\ge \int_{\mathbb{R}} P_-(\lambda)d\mu(\lambda).\end{aligned}$$

Finally, by (4.12),

$$\int_{\mathbb{R}} (P_+(\lambda) - P_-(\lambda))d\mu(\lambda) \le \varepsilon \int_{\mathbb{R}} \lambda^{\ell_0} d\mu(\lambda).$$

By (1.6), the integral in the r.h.s. is finite. Since $\varepsilon > 0$ can be taken arbitrary small, we obtain the required statement. □

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Appendix A

A.1. Proof of formula (3.6). By definition,

$$\widehat{\Psi_q}(\zeta)=\frac{2(-1)^q}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-iz\zeta}\mathrm{L}_q(2|z|^2)e^{-|z|^2}dz=\frac{(-1)^q}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-iu\zeta/\sqrt{2}}\mathrm{L}_q(|u|^2)e^{-|u|^2/2}du$$

(see (2.13)). Further, by [1, Eq. 22.12.6] we have

$$\mathrm{L}_q(|u|^2)=\mathrm{L}_q(u_1^2+u_2^2)=\sum_{m=0}^{q}\mathrm{L}_m^{(-1/2)}(u_1^2)\mathrm{L}_{q-m}^{(-1/2)}(u_2^2),\quad u\in\mathbb{R}^2,$$

where $\mathrm{L}_m^{(-1/2)}$, $m\in\mathbb{Z}_+$, are the generalized Laguerre polynomials of order $-1/2$. By [1, Eq. 22.5.38] we have

$$\mathrm{L}_m^{(-1/2)}(t^2)=\frac{(-1)^m}{m!2^{2m}}\mathrm{H}_{2m}(t),\quad t\in\mathbb{R},\quad m\in\mathbb{Z}_+,$$

where H_m are the Hermite polynomials. Therefore,

$$\mathrm{L}_q(|u|^2)=\frac{(-1)^q}{2^{2q}}\sum_{m=0}^{q}\frac{\mathrm{H}_{2m}(u_1)\mathrm{H}_{2q-2m}(u_2)}{m!(q-m)!},$$

and

$$\begin{aligned}\widehat{\Psi_q}(\zeta)&=\frac{1}{2^{2q+2}\pi^2}\sum_{m=0}^{q}\frac{1}{m!(q-m)!}\\&\quad\times\int_{\mathbb{R}}e^{-iu_1\zeta_1/\sqrt{2}}\mathrm{H}_{2m}(u_1)e^{-u_1^2/2}du_1\int_{\mathbb{R}}e^{-iu_2\zeta_2/\sqrt{2}}\mathrm{H}_{2q-2m}(u_2)e^{-u_2^2/2}du_2.\end{aligned}$$

It is well known that the functions $\mathrm{H}_m(t)e^{-t^2/2}$, $t\in\mathbb{R}$, $m\in\mathbb{Z}_+$, are eigenfunctions of the unitary Fourier transform with eigenvalues equal to $(-i)^m$ (see e.g. [5]). Hence,

$$\begin{aligned}\widehat{\Psi_q}(\zeta)&=\frac{(-1)^q}{2^{2q+1}\pi}\sum_{m=0}^{q}\frac{1}{m!(q-m)!}\mathrm{H}_{2m}(2^{-1/2}\zeta_1)\mathrm{H}_{2q-2m}(2^{-1/2}\zeta_2)e^{-|\zeta|^2/4}\\&=(2\pi)^{-1}\mathrm{L}_q(2^{-1}|\zeta|^2)e^{-|\zeta|^2/4}=(-1)^q\Psi_q(2^{-1}\zeta)/2.\end{aligned}$$

A.2. Proof of estimate (3.10). Denote $u_q(x)=e^{-x/2}\,\mathrm{L}_q(x)$, $v_q(x)=J_0(\sqrt{(4q+2)x})$. Using the differential equations for the Laguerre polynomials and for the Bessel functions, one easily checks that u_q and v_q satisfy

$$\begin{aligned}xu_q''(x)+u_q'(x)+(q+\frac{1}{2})u_q(x)&=\frac{x}{4}u_q(x),\\xv_q''(x)+v_q'(x)+(q+\frac{1}{2})v_q(x)&=0.\end{aligned}$$

Using these differential equations and the initial conditions for $u_q(x)$, $v_q(x)$ at $x = 0$, it is easy to verify that u_q satisfies the integral equation

$$u_q = v_q + K_q u_q, \quad (K_q f)(x) = \int_0^x F_q(x, y) f(y) dy, \tag{A.1}$$

$$F_q(x, y) = -\frac{\pi}{4} y \big(J_0(\sqrt{(4q+2)x}) Y_0(\sqrt{(4q+2)y}) - Y_0(\sqrt{(4q+2)x}) J_0(\sqrt{(4q+2)y}) \big).$$

This argument is borrowed from [30]. Iterating (A.1), we obtain

$$u_q - v_q = K_q v_q + K_q^2 u_q. \tag{A.2}$$

Now it remains to estimate the two terms in the r.h.s. of (A.2) in an appropriate way. Using the estimates

$$|J_0(x)| \le C/\sqrt{x}, \qquad |Y_0(x)| \le C/\sqrt{x}, \qquad x > 0,$$

we obtain

$$|F_q(x, y)| \le C q^{-1/2} x^{-1/4} y^{3/4}, \qquad q \in \mathbb{N}, \quad x > 0.$$

This yields

$$\left| \int_0^x F_q(x, y) v_q(y) dy \right| \le C q^{-3/4} x^{-1/4} \int_0^x y^{1/2} dy = C q^{-3/4} x^{5/4}. \tag{A.3}$$

Next, using the estimate $|u_q(x)| \le 1$ (see [1, Eq. 22.14.12]), we obtain

$$|(K_q u_q)(x)| \le C q^{-1/2} x^{-1/4} \int_0^x y^{3/4} dy = C q^{-1/2} x^{3/2},$$

and so

$$|(K_q^2 u_q)(x)| \le C q^{-1} x^{-1/4} \int_0^x y^{\frac{3}{2}+\frac{3}{4}} dy = C q^{-1} x^3. \tag{A.4}$$

Combining (A.2) with (A.3) and (A.4), we obtain the required estimate (3.10).

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