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Characteristic Classes and Hitchin Systems. General Construction

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Abstract: We consider topologically non-trivial Higgs G-bundles over Riemann surfaces Σ_g with marked points and the corresponding Hitchin systems. We show that if G is not simply-connected, then there exists a finite number of different sectors of the Higgs bundles endowed with the Hitchin Hamiltonians. They correspond to different characteristic classes of the underlying bundles defined as elements of $H^2(\Sigma_g, \mathcal{Z}(G))$, ($\mathcal{Z}(G)$ is a center of G). We define the conformal version CG of G - an analog of GL(N) for SL(N), and relate the characteristic classes with degrees of CG-bundles. We describe explicitly bundles in the genus one ($g = 1$) case. If Σ_1 has one marked point and the bundles are trivial then the Hitchin systems coincide with Calogero-Moser (CM) systems. For the nontrivial bundles we call the corresponding systems the modified Calogero-Moser (MCM) systems. Their phase space has the same dimension as the phase space of the CM systems with spin variables, but less number of particles and greater number of spin variables. Starting with these bundles we construct Lax operators, quadratic Hamiltonians, and define the phase spaces and the Poisson structure using dynamical r-matrices. The latter are completion of the classification list of Etingof-Varchenko corresponding to the trivial bundles. To describe the systems we use a special basis in the Lie algebras that generalizes the basis of 't Hooft matrices for sl(N). We find that the MCM systems contain the standard CM subsystems related to some (unbroken) subalgebras. The configuration space of the CM particles is the moduli space of the stable holomorphic bundles with non-trivial characteristic classes.

Contents

1. Introduction

The paper conventionally speaking contains two types of results. First, we construct topologically nontrivial holomorphic G-bundles over Riemann surfaces Σ_g, where G is a complex non-simply-connected Lie group. The topological types of the bundles are characterized by elements of $H^2(\Sigma_g, \mathcal{Z}(G))$, where $\mathcal{Z}(G))$ is a center of G. We call them the *characteristic classes* of the bundles, since for G =SO$_n$ they coincide with the Shtiefel-Whitney classes. We define the conformal version CG of G - an analog of GL(N) for SL(N), and relate the characteristic classes with degrees of CG-bundles. For genus one surfaces with a single marked point we describe a big cell in the moduli space of stable holomorphic bundles with arbitrary characteristic classes.

Then, on the basis of these results, we construct a family of classical integrable systems - the Hitchin systems. The phase spaces of these systems are the Higgs G-bundles. For non-simply-connected groups the Higgs bundles have different sectors corresponding to the characteristic classes of the underlying bundles. The similar phenomena was observed in the WZW theory in [21]. For bundles over elliptic curves with one marked point the corresponding systems are analogues of the elliptic Calogero-Moser systems. The standard Calogero-Moser systems are related to the trivial bundles. We define the Lax operators, quadratic Hamiltonians and the classical dynamical elliptic r-matrices. The latter completes the classification list of classical elliptic dynamical r- matrices [17], where the underlying bundles are topologically trivial.

1. Non-trivial bundles over Riemann surfaces Σ_g*.* Let $\mathcal{P}$ be a principal G-bundle over Σ_g, π is a representation of G in V, and $E = \mathcal{P} \times_G V$. According with [52] the stable holomorphic G-bundles can be defined using representations of the fundamental group $\pi_1(\Sigma_g)$. This group has $2g$ generators $\{a_\alpha, b_\alpha\}$ with the relation

$$\prod_{\alpha=1}^{g} [b_\alpha, a_\alpha] = 1, \quad [b_\alpha, a_\alpha] = b_\alpha a_\alpha b_\alpha^{-1} a_\alpha^{-1}. \tag{1.1}$$

Let ρ be a representation of π_1 in V such that $\rho(\pi_1) \subset \pi(G)$. Due to (1.1) we have

$$\prod_{\alpha=1}^{g} [\rho(b_\alpha), \rho(a_\alpha)] = Id. \tag{1.2}$$

The G-bundles described in this way are topologically trivial. To consider a less trivial situation assume that G has a non-trivial center $\mathcal{Z}(G)$. It means that G is a classical simply-connected group, or some of its subgroups, or a simply-connected group of type E_6 or E_7. In what follows we use the following notations:

$$\bar{G} - \text{simply} - \text{connected group}, \;\; G^{ad} - \text{adjoint group}, \;\; \bar{G} \supseteq G \supseteq G^{ad},$$

and $\mathcal{Z}(G)$ is a center of G. Let $\zeta \in \mathcal{Z}(G)$. Replace (1.2) by

$$\prod_{\alpha=1}^{g} [\rho(b_\alpha), \rho(a_\alpha)] = \zeta. \tag{1.3}$$

Table 1. Λ^0-invariant subgroups and subalgebras

G	ord (Λ^0)	$\tilde{\mathfrak{g}}_0$	$\mathfrak{g}_0$
$\mathrm{SL}(N,\mathbb{C})$ $(N=pl)$	N/p	$\mathbf{sl_p}$	$\mathbf{sl_p}\oplus_{j=1}^{l-1}\mathbf{gl_p}$
$\mathrm{SO}(2n+1)$	2	$\mathbf{so}(2n-1)$	$\mathbf{so}(2n)$
$\mathrm{Sp}(2l)$	2	$\mathbf{so}(2l)$	$\mathbf{gl}_{2l}$
$\mathrm{Sp}(2l+1)$	2	$\mathbf{so}(2l+1)$	$\mathbf{gl}_{2l+1}$
$\mathrm{SO}(4l+2)$	4	$\mathbf{so}(2l-1)$	$\mathbf{so}(2l)\oplus\mathbf{so}(2l)\oplus\underline{1}$
$\mathrm{SO}(4l+2)$	2	$\mathbf{so}(4l-1)$	$\mathbf{so}(4l)\oplus\underline{1}$
$\mathrm{SO}(4l)$	2	$\mathbf{so}(2l)$	$\mathbf{so}(2l)\oplus\mathbf{so}(2l)$
$\mathrm{SO}(4l)$	2	$\mathbf{so}(4l-3)$	$\mathbf{so}(4l-2)\oplus\underline{1}$
E_6	3	$\mathbf{g}_2$	$\mathbf{so}(8)\oplus 2\cdot\underline{1}$
E_7	2	$\mathbf{f}_4$	$\mathbf{e}_6\oplus\underline{1}$

Since $\mathcal{Z}(\mathrm{SO}(4l))=\mu_2\oplus\mu_2$ we take two different Λ^0_a, $(a=1,2)$
Sp(n) is a group preserving the antisymmetric bilinear form in $\mathbb{C}^{2n}$

Then the pairs $(\hat{\rho}(a_\alpha),\hat{\rho}(b_\beta))$, satisfying (1.3), cannot describe transition matrices of G-bundle, but can serve as transition matrices of the $G^{ad}=G/\mathcal{Z}(G)$-bundle. The bundle E in this case is topologically non-trivial and ζ represents *the characteristic class* of E. It is an obstruction to lift the G^{ad} bundle to the G bundle. The topologically non-trivial G-bundles are characterized by elements of $H^2(\Sigma,\mathcal{Z}(G))$.

If $g>1$ we cannot find general solutions of (1.2), but in the case $g=1$ we found almost all solutions. In this case we deal with the elliptic curve $\Sigma_1\sim\Sigma_\tau=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and the bundle E can be described by two operators $\mathcal{Q}(z)$ and $\Lambda(z)$, that satisfy the equation $\mathcal{Q}(z+\tau)\Lambda(z)\mathcal{Q}(z)^{-1}\Lambda^{-1}(z+1)=\zeta$. It follows from [52] that it is possible to choose the constant transition operators. Then we come to the equation on G,

$$\mathcal{Q}\Lambda\mathcal{Q}^{-1}\Lambda^{-1}=\zeta. \tag{1.4}$$

As in [52] we define the moduli space of stable holomorphic G-bundles as

$$\mathcal{M}(G)=(\text{solutions of (1.4)})/(\text{conjugation}). \tag{1.5}$$

Assume that $\mathcal{Q}$ is a semisimple element ($\mathcal{Q}\in\mathcal{H}_{\bar{G}}$, where $\mathcal{H}_{\bar{G}}$ is a Cartan subgroup). It means that we consider an open subset $\mathcal{M}(G)\supset\mathcal{M}(G)^0=\{(\mathcal{Q},\Lambda)\}$. The elements $\mathcal{Q}$ and Λ can be represented as

$$\mathcal{Q}=\exp\left(2\pi i\frac{\rho^\vee}{h}\right)U,\quad \Lambda=\Lambda^0 V,$$

$\rho^\vee$ is a half-sum of positive coroots, h is the Coxeter number, Λ^0 is an element of the Weyl group defined by ζ. It is a symmetry of the extended Dynkin diagram of $\mathfrak{g}=\mathrm{Lie}(\bar{G})$. V and U are arbitrary elements of the Cartan subgroup $\tilde{\mathcal{H}}_0\subset\mathcal{H}_{\bar{G}}$ commuting with Λ^0 and $\tilde{\mathfrak{H}}_0=\mathrm{Lie}\,(\tilde{\mathcal{H}}_0)$ is a Cartan subalgebra corresponding to a simple Lie subgroup $\tilde{G}_0\subset\bar{G}$.

Since $(\Lambda^0)^l=1$ for some l, the adjoint action of Λ^0 on $\mathfrak{g}$ is an automorphism of order l. All such automorphisms are described in [30]. $\mathrm{Ad}_{(\Lambda^0)}$ induces a $\mu_l=\mathbb{Z}/l\mathbb{Z}$ gradation in $\mathfrak{g}$ $\mathfrak{g}=\oplus_{k=0}^{l-1}\mathfrak{g}_k$, where $\mathfrak{g}_0$ is a reductive subalgebra. The Lie algebra $\tilde{\mathfrak{g}}_0=\mathrm{Lie}(\tilde{G}_0)$ in its turn is a subalgebra of $\mathfrak{g}_0$. The concrete forms of invariant subalgebras are presented in Table 1. They are calculated in [48–50].

For trivial holomorphic bundles over an elliptic curve $\mathcal{M}(G)^0$ is a quotient of the Cartan subalgebra $\mathfrak{H}$ of G under the action of some discrete group. For $G = GL_N$ the moduli space was described by M.Atiyah [1]. For trivial G-bundles, where G is a complex simple group, it was done in [6,44]. Nontrivial G-bundles and their moduli spaces were considered in [22,23,60].

It is important for applications to consider the holomorphic bundles with quasi-parabolic structures at marked points at Σ_τ. It means that the automorphisms of the bundles (the gauge transformations) preserve flags Fl_a located at n marked points [61]. The structure of a big cell $\mathcal{M}^0_{g,n}$ $(g = 1)$ in the moduli space of these bundles can be extracted from the moduli space $\mathcal{M}(G)$ of solutions of (1.4). In the simplest case $n = 1$,

$$\tilde{\mathcal{M}}^0_{1,1} = \big(\tilde{\mathfrak{H}}_0/ \times (Fl/\tilde{\mathcal{H}}_0)\big)/\tilde{W}_{BS}, \tag{1.6}$$

where $\tilde{W}_{BS}$ are the Bernstein-Schwarzman generalizations [6] of the affine Weyl groups $W^{aff}(\tilde{G}_0)$ corresponding to different sublattices of the coweight lattice. $\tilde{W}_{BS}$ acts on $(Fl/\tilde{\mathcal{H}}_0)$ only by the Weyl subgroup $\tilde{W}$. Note that for the trivial bundles Λ^0 can be chosen as Id. In this case

$$\mathcal{M}^0_{1,1} = (\mathfrak{H} \times Fl/\mathcal{H}_{\bar{G}})/W_{BS}, \tag{1.7}$$

where $\mathfrak{H} = \mathrm{Lie}(\mathcal{H}_{\bar{G}})$. Thus, the big cell $\tilde{\mathcal{M}}^0_{1,1}$ for the nontrivial G bundles is the same as the big cell $\mathcal{M}^0_{1,1}$ for trivial $\tilde{G}_0$-bundles. A detailed description is given in Sect. 3.2.

As by product, we obtained some additional results related to this subject. We describe a relation between the characteristic classes and degrees of some bundles. In the A_{N-1} case this relation is simple. The center of $G = \mathrm{SL}(N, \mathbb{C})$ is the cyclic group $\mu_N = \mathbb{Z}/N\mathbb{Z}$. The cohomology group $H^2(\Sigma, \mathcal{Z}(\mathrm{SL}(N, \mathbb{C})))$ is isomorphic to μ_N. Represent elements of μ_N as $\exp \frac{2\pi i}{N} j,\ j = 1\ldots, N-1$. Let ζ be a generator of μ_N. Consider a principal $\mathrm{PGL}(N, \mathbb{C})$ bundle with the characteristic classes ζ. It cannot be lifted to a $\mathrm{SL}(N, \mathbb{C})$-bundle, but can be lifted to a $\mathrm{GL}(N, \mathbb{C})$ bundle. The degree of its determinant bundle $deg E$ is -1 and $\zeta = \exp(-\frac{2\pi i}{N}) = \exp(\frac{2\pi i\, deg E}{rank\ E})$. We generalize this construction to other simple groups. To this end for a simple group G we define its conformal version CG (Definition 3.2). In particular, for the symplectic and orthogonal groups their conformal versions are groups preserving (anti)symmetric forms up to dilatations. It allows us to relate the characteristic classes of G-bundles to degrees of the determinant bundles of CG (Theorem 3.1).

We introduce a special basis in $\mathfrak{g} = Lie\, G$. In the A_{N-1} case it is the basis of the finite-dimensional sin-algebra [18], generated by the t'Hooft matrices $\mathcal{Q}, \Lambda$ $(\mathcal{Q}\Lambda\mathcal{Q}^{-1}\Lambda^{-1} = \exp(\frac{2\pi i}{N}))$. We call it the generalized sin (GS) basis and use it in the context of integrable systems.

2. Integrable systems. Generically, the Hitchin systems come up as a result of the Hamiltonian reduction of the cotangent bundles to the stable holomorphic bundles [27]. The Higgs bundles are a result of the reduction. If the Riemann surface has not marked points the Higgs bundles are cotangent bundles to the moduli space of stable holomorphic bundles. For bundles with quasi-parabolic structures the Higgs bundles are principal homogeneous spaces over cotangent bundles to the moduli spaces. After the reduction we obtain an integrable system in the Lax form, with the Lax operator given by the Higgs field where the spectral parameter plays the role of a local coordinate on the Riemann surface.

Using the above construction we find that the Hitchin systems have different sectors corresponding to characteristic classes. It means down to earth that the Lax operators have different quasi-periodicities, corresponding to (1.3).

Table 2. Integrable systems corresponding to different characteristic classes of SL(N) bundles

	1	2	3
ζ	1	$\exp\left(-\frac{2\pi ip}{N}\right),\ \ N=pl$	$\exp\left(\frac{-2\pi i}{N}\right)$
System	SL_N-CM system	SL_p-CM-system + interacting SL_l EA-tops	SL_N-EA-top

In the elliptic case we describe the Lax operators explicitly.[1] Using the above construction we describe a new class of the finite-dimensional classical completely integrable systems related to simple Lie groups with nontrivial centers. They are generalizations of the elliptic Calogero-Moser systems, in general with spin degrees of freedom. Calogero-Moser systems (CM) were originally defined in quantum case by Francesco Calogero [11] and in classical case by Jurgen Moser [47], as an integrable model of one-dimensional nuclei. Now they play an essential role both in mathematics and in theoretical physics.[2]

Their generalizations as integrable systems related to simple Lie groups has a long history. It was started more than thirty years ago [54], but the classical integrability was proved there only for the classical groups. It was done later in [8,28]. They are the so-called spinless CM systems. The case of the A_{n-1} type (SL(n)) systems is very special. The integrability of these systems for rational and trigonometric potentials has a natural explanation in terms of Hamiltonian reduction [32,55]. Later this approach was generalized for a wide class of classical integrable systems - the so-called Hitchin systems [27]. It was realized in [14,26,36,46,53] that the A_{n-1} type CM systems with elliptic potential are particular examples of the Hitchin systems. Note, that long before these works the Lax matrix with a spectral parameter for the elliptic CM system was constructed by Krichever [34].

From the point of view of the Hitchin construction it is more natural to consider CM systems with spin, introduced in the A_{n-1} case in [24,64].[3] Their description for all simple Lie algebras can be found in [45].

As we said, the standard classification of the CM systems is based on topologically trivial bundles. As a result we obtain a classification of the Modified Calogero Moser (MCM) systems related to topologically non-trivial bundles. Some particular examples related to $SL(N,\mathbb{C})$ are known. If the characteristic class of the bundle $\zeta=\exp\left(-\frac{2\pi i}{N}\right)$, then instead of the interacting CM particles we get the Euler-Arnold (EA) top [2] related to $SL(N,\mathbb{C})$ [33,35,57]. This top describes the classical degrees of freedom on a vertex in the vertex spin chain. The corresponding classical r matrix is non-dynamical [5]. But if $N=pl$ there exists an intermediate situation [43] described in column 2 (Table 2):

In this paper we construct Lax operators, quadratic Hamiltonians and corresponding classical dynamical r-matrices for any simple complex Lie group G with a non-trivial center and arbitrary characteristic classes $\zeta\in H^2(\Sigma_\tau,\mathcal{Z}(G))$. The obtained elliptic r-matrices complete the list [17,45], because the dynamical parameters belong to the Cartan subalgebra $\tilde{\mathfrak{H}}_0\subset\mathfrak{H}_G$. This type of r-matrices in the trigonometric case were constructed in [16,59], using an algebraic approach.

In fact, $\tilde{\mathfrak{H}}_0$ is the same Cartan subalgebra that participates in the definition of the moduli space (1.6). Let us explain this phenomena. The phase space of the Hitchin systems

[1] For simplicity we consider only one mark point. For many marked points we come to generalized Gaudin systems.

[2] The mathematical aspects of the systems are discussed in [15].

[3] The spinless CM systems considered in [8,28] were described as some sort of Hitchin systems in [29].

is the moduli space $\mathcal{M}^H_{\Sigma_n}$ of the Higgs bundles over a curve Σ_n with the quasi-parabolic structure at n marked points. It is a fibration over the moduli space $\mathcal{M}^H_{\Sigma}$ of the Higgs bundles over the compact curve Σ. The base $\mathcal{M}^H_{\Sigma}$ can be interpreted as the phase space of interacting particles. It is the cotangent bundle to the moduli space $\mathcal{M}_{\Sigma}$ holomorphic bundles over Σ. The fibers $\mathcal{M}^H_{\Sigma_n} \to \mathcal{M}^H_{\Sigma}$ are coadjoint G-orbits located at the marked points. The coordinates on the orbits are called the spin variables.[4] If the number of the marked points $n = 1$ and the G-bundle over the elliptic curve has a trivial characteristic class, then the spin variables can be identified with angular velocities of the EA top related to G. The inertia tensor of the top depends on the coordinates of CM particles related to the same group G. The configuration spaces of particles are the quotient of the Cartan algebra as in (1.7). It is the space of dynamical parameters of r.

For the non-trivial bundles the configuration space of particles is a quotient of the Cartan subalgebra $\tilde{\mathfrak{H}}_0 \subset \mathfrak{H}$ and the dynamical r-matrix depends on variables belonging to $\tilde{\mathfrak{H}}_0$. The integrable system looks like interacting EA tops with parameters depending on coordinates of the CM system related to $\tilde{G}_0$. For this reason we call $\tilde{G}_0$ the unbroken subgroup (see Table 1).

Solutions of (1.4) allow us to define the Lax operators and the classical dynamical r-matrix for non-trivial bundles. We prove that the r-matrices satisfy the classical dynamical Yang-Baxter equation [20]. We describe the Poisson brackets for matrix elements of the Lax operators in terms of the classical dynamical r-matrix as was done in [3,7,10,19,20,45,62], where the systems corresponding to the trivial bundles were considered.

It is worthwhile to emphasize that for the standard CM systems we deal in fact with a few different systems. More exactly, we have as many configuration spaces as a number of non-isomorphic moduli spaces. It amounts to existence of different sublattices in the coweight lattice containing the coroot lattice. A naive explanation of this fact is as follows. The potential of the system has the form $\sum \wp(\langle \mathbf{u}, \alpha\rangle)$, where $\mathbf{u}$ is a coordinate vector, the sum is taken over positive roots α, and $\wp$ is the Weierstrass function.[5] Adding to $\mathbf{u}$ any combination $\gamma_1 + \gamma_2\tau$, where $\gamma_j \in Q^\vee$-coroot lattice, does not change the potential, because $\wp(\langle \mathbf{u}, \alpha\rangle)$ is doubly-periodic on the lattice $\tau\mathbb{Z} \oplus \mathbb{Z}$ and $\langle \gamma, \alpha\rangle$ is an integer. Thus, the configuration space is the quotient $\mathfrak{H}/(\tau Q^\vee \oplus Q^\vee)$. It is the largest configuration space. But we can harmlessly shift as well by the coweight lattice $\tau P^\vee \oplus P^\vee$. Then we come to a different configuration space (the smallest one). For A_{N-1} root systems we describe in this way the $\mathrm{SL}(N, \mathbb{C})$ and $\mathrm{PSL}(N, \mathbb{C})$ CM systems. Their configuration spaces are different, while the Hamiltonians are the same. Evidently, this fact becomes important for the quantum systems. The same is valid for the systems with non-trivial characteristic class. But now one should consider the lattices related to the unbroken subgroups.

Finally, we should mention the following fact. As we said the obtained integrable systems correspond to different sectors of the Higgs bundles. It turns out that in spite of the apparent distinction, corresponding to Lax operators and the Hamiltonians, the integrable systems are symplectomorphic. These symplectomorphisms are not smooth but singular, and in this way change the topological type of the bundle. In particular, the MCM systems are symplectomorphic to the standard spin CM systems. The symplectomorphisms are provided by the so-called Symplectic Hecke Transformation [35].

[4] For elliptic curves the phase space of the spin variables is a result of a Hamiltonian reduction of the coadjoint orbits with respect to the action of the Cartan subgroup.

[5] In what follows we use the second Eisenstein function $E_2(z)$. It differs from $\wp$ on a constant.

In terms of the Lax operators the symplectomorphisms are defined by acting on them by special singular gauge transformations. A particular example of such transformation establishing an equivalence of the SL(N, $\mathbb{C}$) CM system and the SL(N, $\mathbb{C}$) EA top was given in [35].

In the theory of integrable models of statistical mechanics this Hecke transformation defines a twist providing a passage from the so-called IRF type models to the Vertex type models. The isomonodromic deformations problem corresponding to the Hitchin systems [36] on elliptic curves relates the Painlevé VI equation and the nonautonomous Zhukovsky-Volterra gyrostat [37,38]. The field (1+1) generalizations of the Hitchin-Nekrasov (Gaudin) models are discussed in [35,39]. In terms of a gauge field theory the Hecke transformation can be explained as a monopole solution of the Bogomolny equation [31]. Details can be found in [40].

In our next publications we plan to obtain the quantum dynamical elliptic R-matrices [41] and Knizhnik-Zamolodchikov-Bernard equations [42] corresponding to the non-trivial characteristic classes of GL(N, $\mathbb{C}$)-bundles.

2. Holomorphic Bundle

Global description of holomorphic bundles. Let G be a complex simple Lie group and K its maximal compact subgroup. According to Narasimhan and Seshadri [52] (see also [58]) stable holomorphic G bundles over a Riemann surface Σ_g of genus g arise from flat K-bundles over Σ_g. Then the stable holomorphic bundles can be described in the following way.

Let $\pi_1(\Sigma_g)$ be a fundamental group of Σ_g. It has $2g$ generators $\{a_\alpha, b_\alpha\}$, corresponding to the fundamental cycles of Σ_g with the relation

$$\prod_{\alpha=1}^{g}[b_\alpha, a_\alpha] = 1, \tag{2.1}$$

where $[b_\alpha, a_\alpha] = b_\alpha a_\alpha b_\alpha^{-1} a_\alpha^{-1}$ is the group commutator.

Consider a finite-dimensional representation π of G in a space V. Let $\mathcal{P}$ be a principal G-bundle over Σ_g. We define a holomorphic vector G-bundle $E = \mathcal{P} \times_G V$ (or in more detail E_G or $E_G(V)$) over Σ_g. The bundle E_G has the space of sections $\Gamma(E_G) = \{s\}$, where s takes values in V. Let ρ be a homomorphism from $\pi_1(\Sigma_g)$ to G. The bundle E_G is defined by transition matrices of its sections. Let $z \in \Sigma_g$ be a fixed point. Then

$$s(a_\alpha z) = \pi(\rho(a_\alpha))s(z), \quad s(b_\beta z) = \pi(\rho(b_\beta))s(z). \tag{2.2}$$

Thus, the sections are defined by their quasi-periodicities. Due to (2.1) we have

$$\prod_{\alpha=1}^{g}[\rho(b_\alpha), \rho(a_\alpha)] = Id. \tag{2.3}$$

The G-bundles described in this way are topologically trivial. To consider a less trivial situation assume that G has a non-trivial center $\mathcal{Z}(G)$. Let $\zeta \in \mathcal{Z}(G)$. Replace (2.3) by

$$\prod_{\alpha=1}^{g}[\rho(b_\alpha), \rho(a_\alpha)] = \zeta. \tag{2.4}$$

Then the pairs $(\rho(a_\alpha), \rho(b_\beta))$, satisfying (2.4), cannot describe transition matrices of the G-bundle, but can serve as transition matrices of the $G^{ad} = G/\mathcal{Z}(G)$-bundle. The bundle E_G in this case is topologically non-trivial and ζ represents *the characteristic class* of E_G. It is an obstruction to lift the G^{ad} bundle to the G bundle. We will give a formal definition in Sect. 4.

The transition matrices can be deformed without breaking (2.3) or (2.4). Among these deformations are the gauge transformations

$$\rho(a_\alpha) \to f^{-1}\rho(a_\alpha)f, \quad \rho(b_\beta) \to f^{-1}\rho(b_\beta)f. \tag{2.5}$$

The moduli space of stable holomorphic bundles $\mathcal{M}_g$ is the space of transition matrices defined up to the gauge transformations. Its dimension is independent on the characteristic class and is equal to

$$\dim(\mathcal{M}_g) = (g-1)\dim(G). \tag{2.6}$$

It means that the nonempty moduli spaces arise for the holomorphic bundles over surfaces of genus $g > 1$.

To include into the construction the surfaces with $g = 0, 1$ consider a Riemann surface with n marked points and attribute E with what is called the quasi-parabolic structure at the marked points. Let B be a Borel subgroup of G. We assume that the gauge transformation f preserves the flag variety $Fl = G/B$. It means that $f \in B$ at the marked points. It follows from (A.25),

$$\begin{aligned}\dim(\mathcal{M}_{g,n}) &= (g-1)\dim(G) + n\dim(Fl)\\ &= (g-1)\dim(G) + n\sum_{j=1}^{rank\,G}(d_j - 1).\end{aligned} \tag{2.7}$$

In the important for applications case, $g = 1, n = 1$ $\dim(\mathcal{M}_{1,1}) = \dim(Fl)$.

Local description of holomorphic bundles and modification. There exists another description of holomorphic bundles over Σ_g. Let w_0 be a fixed point on Σ_g and D_{w_0} ($D^\times_{w_0}$) be a disc (punctured disc) with a center w_0 with a local coordinate z. Consider a G-bundle E_G over Σ_g. It can be trivialized over D and over $\Sigma_g\backslash w_0$. These two trivializations are related by a G transformation $\pi(g)$ holomorphic in $D^\times_{w_0}$, where D_{w_0} and $\Sigma_g\backslash w_0$ overlap. If we consider another trivialization over D then g is multiplied from the right by $h \in G$. Likewise, a trivialization over $\Sigma_g\backslash w_0$ is determined up to the multiplication on the left $g \to hg$, where $h \in G$ is holomorphic on $\Sigma_g\backslash w_0$. Thus, the set of isomorphism classes of G-bundles is described as a double-coset

$$G(\Sigma_g\backslash w_0)\backslash G(D^\times_{w_0})/G(D_{w_0}), \tag{2.8}$$

where $G(U)$ denotes the group of G-valued holomorphic functions on U.

To define a G-bundle over Σ_g the transition matrix g should have a trivial monodromy around w_0 $g(ze^{2\pi i}) = g(z)$ on the punctured disc $D^\times_{w_0}$. But if the monodromy is nontrivial

$$g(ze^{2\pi i}) = \zeta g(z), \quad \zeta \in \mathcal{Z}(G),$$

then $g(z)$ is not a transition matrix. But it can be considered as a transition matrix for the G^{ad}-bundle, since $G^{ad} = G/\mathcal{Z}(G)$. This relation is similar to (2.4).

Our aim is to construct from a bundle E with a fixed characteristic class a new bundle $\tilde{E}$ which can have a different characteristic class. This procedure is called *a modification* of the bundle E. The modification is trivial if the characteristic class is not changed. The modification is defined by a singular gauge transformation at some point, say w_0. Since it is a local transformation we replace Σ_g by a sphere $\Sigma_0 = \mathbb{C}P^1$, where w_0 corresponds to the point $z = 0$ on $\mathbb{C}P^1$. Since z is a local coordinate, we can replace $G(\Sigma_g \setminus w_0)$ in (2.8) by the group $G(\mathbb{C}((z)))$. It is the group of Laurent series with G valued coefficients. Similarly, $G(D_{w_0})$ is replaced by the power series $G(\mathbb{C}[[z]])$. Then instead of (2.8) we have

$$G(\mathbb{C}[[z^{-1}]])\backslash G(\mathbb{C}((z)))/G(\mathbb{C}[[z]]). \tag{2.9}$$

Replace $g(z)$ by $g(z)h(z)$, where $h(z)$ can be singular at $z = 0$. It is a singular gauge transformation mentioned above. Due to (2.9), $h(z)$ is defined up to the multiplication from the right by $f(z) \in G(\mathbb{C}[[z]])$. On the other hand, as the original $g(z)$ is defined up to the multiplication from the right by an element from $G(\mathbb{C}[[z]])$, $h(z)$ is an element from the double coset

$$G(\mathbb{C}[[z]])\backslash G(\mathbb{C}((z)))/G(\mathbb{C}[[z]]).$$

In particular, $h(z)$ is defined up to a conjugation. It means that as a representative of this double coset one can take a co-character (A.35) $h(z) \in t(G)$,

$$g(z) \to g(z)z^{\gamma}, \quad (z^{\gamma} = \mathbf{e}(\ln(z\gamma))), \quad (\mathbf{e}(x) = \exp(2\pi i x)), \tag{2.10}$$

where γ belongs to the coweight lattice ($\gamma = (m_1, m_2, \ldots, m_l) \in P^{\vee}$) (A.12). The monodromy of z^{γ} is $\exp -(2\pi \imath\gamma)$. Since $\langle \alpha, \gamma \rangle \in \mathbb{Z}$ for any $\mathbf{x} \in \mathfrak{g}$ we have $\mathrm{Ad}_{\exp -(2\pi\imath\gamma)}\mathbf{x} = \mathbf{x}$. Then $\exp -(2\pi\imath\gamma)$ is an element of $\mathcal{Z}(\bar{G})$ (A.39). If the transition matrix $g(z)$ defining E has a trivial monodromy, the new transition matrix (2.10) acquires a nontrivial monodromy. In this way we come to a new bundle $\tilde{E}$ with a nontrivial characteristic class. The bundle $\tilde{E}$ is called *the modified bundle*. It is defined by the new transition matrix (2.10). If $\gamma \in Q^{\vee}$, then $\zeta = 1$ and the modified bundle $\tilde{E}$ has the same type as E.

This transformation of the bundle E corresponds to the following transformations of its sections $\tilde{E}$:

$$\Gamma(E) \xrightarrow{\Xi(\gamma)} \Gamma(\tilde{E}), \quad (\Xi(\gamma) \sim \pi(z^{m_1}, z^{m_2} \ldots z^{m_l})). \tag{2.11}$$

We say that this modification has a type $\gamma = (m_1, m_2, \ldots, m_l)$. Another name of the modification is *the Hecke transformation*. It acts on the characteristic classes of bundles as follows:

$$\Xi(\gamma) : \ln\zeta(E) \to \ln\zeta(\tilde{E}) = \ln\zeta(E) + 2\pi i\gamma, \quad \gamma \in P^{\vee}. \tag{2.12}$$

Consider the action of modification on sections (2.11) in more details. Let V be a space of a finite-dimensional representation π of G with a highest weight ν and ν_j ($j = 1, \ldots, N$) is a set of its weights

$$\nu_j = \nu - \sum_{\alpha_m \in \Pi} c_j^m \alpha_m, \quad c_j^k \in \mathbb{Z}, \; c_j^k \geq 0. \tag{2.13}$$

It means that for $\mathbf{x} \in \mathfrak{H}$ $\pi(\mathbf{x})|\nu_j\rangle = \langle \mathbf{x}, \nu_j\rangle|\nu_j\rangle$. The weights belong to the weight diagram defined by the highest weight $\nu \in P$ of π. The space V has the weight basis $(|\nu_1^{s_1}\rangle, \ldots, |\nu_N^{s_N}\rangle)$ in V, where $s_1 = 1, \ldots, m_1, \ldots s_N = 1, \ldots, m_N$ and $m_1, \ldots, m_N$ are multiplicities of weights. Thus, $M = \dim V = \sum m_j$.

Let us choose a trivialization of E over D by fixing this basis. Thereby, the bundle E over D is represented by a sum of M line bundles $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \ldots \oplus \mathcal{L}_M$. The Cartan subgroup $\mathcal{H}$ acts in this basis in a diagonal way: for $\mathbf{s} = (|\nu_1^{s_1}\rangle, \ldots, |\nu_N^{s_N}\rangle)$,

$$\pi(h) : |\nu_j^{s_j}\rangle \to \mathbf{e}\langle \mathbf{x}, \nu_j\rangle|\nu_j^{s_j}\rangle, \quad h = \mathbf{e}(\mathbf{x}), \quad \mathbf{x} \in \mathfrak{H}, \quad (\mathbf{e}(x) = \exp(2\pi i \mathbf{x})).$$

Assume for simplicity that in (2.10) $g(z) = 1$. Then the modification transformation (2.11) of the sections assumes the form

$$\Xi(\gamma) : |\nu_j^{s_j}\rangle \to z^{\langle \gamma, \nu_j\rangle}|\nu_j^{s_j}\rangle, \quad j = 1, \ldots, M.$$

It means that away from the point $z = 0$, where the transformations are singular, the sections of $\tilde{E}$ are the same as of E. But near $z = 0$ they are singular with the leading terms $|\nu_j^{s_j}\rangle \sim z^{-\langle \gamma, \nu_j\rangle}$.

It is sufficient to consider the case when $\gamma = \varpi_i^\vee$ is a fundamental coweight and π is a fundamental representation $\nu = \varpi_k$. Then from (2.13) we have

$$z^{\langle \gamma, \nu_j\rangle} = z^{\langle \varpi_i^\vee, \varpi_k - \sum_{\alpha_m \in \Pi} c_j^m \alpha_m\rangle}.$$

The weight ϖ_k can be expanded in the basis of simple roots $\varpi_k = \sum_k A_{km}\alpha_m$, where A_{jk} is the inverse Cartan matrix $(A_{jk}a_{ki} = \delta_{ji})$. Its matrix elements are rational numbers with the denominator $N = ord(\mathcal{Z})$. Then from (A.12),

$$z^{\langle \gamma, \nu_j\rangle} \sim z^{\frac{l}{N}+m}, \quad l, m \in \mathbb{Z}.$$

Note, that the branching does not happen for G^{ad}-bundles, because the corresponding weights ν_j belong to the root lattice Q and thereby $\langle \gamma, \nu_j\rangle \in \mathbb{Z}$.

It is possible to go around the branching by multiplying the sections on a scalar matrix of the form $\mathrm{diag}(z^{-A_{ik}}, \ldots, z^{-A_{ik}})$. This matrix no longer belongs to the representation of $\bar{G}$, because it has the determinant $z^{-MA_{ik}}$ $(M = \dim V)$. It can be checked that MA_{ik} is an integer number.

If $G = \mathrm{SL}(N, \mathbb{C})$ the scalar matrix belongs to $\mathrm{GL}(N, \mathbb{C})$. Thereby, after this transformation we come to a $\mathrm{GL}(N, \mathbb{C})$-bundle. But this bundle is topologically non-trivial, because it has a non-trivial degree. In this way the characteristic classes for the $\mathrm{SL}(N, \mathbb{C})$-bundles are related to another topological characteristic, namely to degrees of the $\mathrm{GL}(N, \mathbb{C})$-bundles. We describe below the similar construction for other simple groups.

3. Holomorphic Bundles over Elliptic Curves

Hereinafter we consider the bundles over an elliptic curve, described as the quotient $\Sigma_\tau \sim \mathbb{C}/(\tau\mathbb{C} \oplus \mathbb{C})$, $(Im\,\tau > 0)$. There are two generators of the fundamental group corresponding to the shifts $z \to z + 1$ and $z \to z + \tau$. Let G be a complex simple Lie group. Sections of a G-bundle $E_G(V)$ over Σ_τ satisfy the quasi-periodicity conditions (2.2)

$$s(z + 1) = \pi(\mathcal{Q})\, s(z), \quad s(z + \tau) = \pi(\Lambda)\, s(z), \tag{3.1}$$

where $\mathcal{Q}$, Λ take values in G. A bundle $\tilde{E}$ is equivalent to E if its sections $\tilde{s}$ are related to s as $\tilde{s}(z) = f(z)s(z)$, where $f(z)$ is an invertible operator in V. It follows from (3.1) that the transition operators, have the form

$$\tilde{\mathcal{Q}} = f(z+1)\mathcal{Q}f^{-1}(z), \quad \tilde{\Lambda} = f(z+\tau)\Lambda f^{-1}(z). \tag{3.2}$$

As we have mentioned, the moduli space $\mathcal{M}_{1,n}$ is the quotient space of pairs $(\mathcal{Q}, \Lambda)$ with respect to this action. In what follows we consider the simplest case $n = 1$, though our construction is applicable for arbitrary n.

The transition operators define a trivial bundle if $[\mathcal{Q}, \Lambda] = Id$. Let ζ be an element of $\mathcal{Z}(\bar{G})$. To come to a nontrivial bundle we should find solutions $\Lambda, \mathcal{Q} \in \bar{G}$ of the equation

$$\Lambda\mathcal{Q}\Lambda^{-1}\mathcal{Q}^{-1} = \zeta. \tag{3.3}$$

It follows from (A.38) that the r.h.s. can be represented as $\zeta = \mathbf{e}(-\varpi^\vee)$, where $\varpi^\vee \in P^\vee$ (A.12). Then (3.3) takes the form

$$\Lambda\mathcal{Q}\Lambda^{-1}\mathcal{Q}^{-1} = \mathbf{e}(-\varpi^\vee), \quad (\mathbf{e}\,(x) = \exp\,(2\pi i x)). \tag{3.4}$$

It follows from [52] that the transition operators can be chosen as constants. Therefore, to describe the moduli space of stable holomorphic bundles we should find a pair $\mathcal{Q}, \Lambda \in G$ satisfying (3.4) and defined up to the conjugation

$$\Lambda \to f\Lambda f^{-1}, \quad \mathcal{Q} \to f\mathcal{Q}f^{-1}. \tag{3.5}$$

Let $G = \bar{G}$ be a simply-connected group. Let us fix a Cartan subgroup $\mathcal{H}_{\bar{G}} \subset \bar{G}$. Assume that $\mathcal{Q}$ is semisimple, and therefore is conjugated to an element from $\mathcal{H}_{\bar{G}}$. We will see that by neglecting non-semisimple transition operators we still define a big cell in the moduli space. Our goal is to find solutions of (3.3), where $\mathcal{Q}$ is a generic element of a fixed Cartan subgroup $\mathcal{H} \subset \bar{G}$.

Algebraic equation.

Proposition 3.1. *Solutions of (3.4) up to the conjugations have the following description:*

- *The element Λ has the form $\Lambda = \Lambda^0 V$, where Λ^0 is defined uniquely by the coweight $\varpi_j^\vee$ ($\Lambda^0 = \Lambda_j^0$). It is an element from the Weyl group W preserving the extended coroot system $\Pi^{\vee ext} = \Pi^\vee \cup \alpha_0^\vee$, and in this way is a symmetry of the extended Dynkin diagram. $V \in \mathcal{H}_{\bar{G}}$ commutes with Λ^0.*

•• *The element $\mathcal{Q}$ has the form $\mathcal{Q} = \mathcal{Q}^0 U$, where*

$$\mathcal{Q}^0 = \exp\,2\pi i\kappa, \quad \kappa = \frac{\rho^\vee}{h} \in \mathfrak{H}, \tag{3.6}$$

where h is the Coxeter number, $\rho^\vee = \frac{1}{2}\sum_{\alpha^\vee \in (R^\vee)^+} \alpha^\vee$ and U commutes with Λ^0.[6]

Proof. In (3.6) κ can be chosen from a fixed Weyl chamber (A.9). From (A.34) and (A.36) we find that if $\kappa_0 \in Q^\vee$ then $\mathbf{e}\,(\kappa_0) = Id$. Therefore, by shifting $\kappa \to \kappa + \gamma$, $\gamma \in Q^\vee$, κ can be put in C_{alc} (A.16). Rewrite (3.3) as

[6] The first statement can be found in [9] (Prop. 5 in VI.3.2). We give another proof because it elucidates the proof of the second statement.

$$\Lambda \mathcal{Q} \Lambda^{-1} = \zeta \mathcal{Q}, \quad \zeta = \mathbf{e}(-\xi). \tag{3.7}$$

Here Λ is defined up to multiplication from $\mathcal{H}_{\bar{G}}$ and we write it in the form $\Lambda^0 V$, $V \in \mathcal{H}_{\bar{G}}$.

Lemma 3.1. *There exists a conjugation f (3.5) such that $\mathcal{Q} \to \mathcal{Q}$ and $\Lambda^0 V \to \Lambda^0 V_\lambda$, and $\Lambda^0 V_\lambda = V_\lambda \Lambda^0$.*

Proof. Let us take $f \in \mathcal{H}_{\bar{G}}$. Then f preserves $\mathcal{Q}$. It acts on the second transition operator as

$$\Lambda^0 V \to f \Lambda^0 f^{-1} V = \Lambda^0 (\Lambda^0)^{-1} f \Lambda^0 f^{-1} V.$$

Define V_λ as $V_\lambda = (\Lambda^0)^{-1} f \Lambda^0 f^{-1} V$. Our goal is to prove that there exists such f that V_λ commutes with Λ^0. In other words, $f \Lambda^0 f^{-1} V = \Lambda^0)^{-1} f \Lambda^0 f^{-1} V \Lambda^0$. Let $V = \mathbf{e}(\mathbf{x})$, $f = \mathbf{e}(\mathbf{y})$, $\mathbf{x}$, $\mathbf{y} \in \mathfrak{H}$, $\lambda = Ad_{\Lambda^0}$. Then the commutativity condition takes the form $(\lambda - 1)\mathbf{x} = (\lambda^{-1} - 1)\mathbf{y} + (\lambda - 1)\mathbf{y}$.

Let l be an order of Λ^0, $((\Lambda^0)^l = 1)$. Then a solution of this equation is given by a sum

$$\mathbf{y} = \frac{1}{l} \sum_{i=1}^{l} i \lambda^i (\mathbf{x}).$$

Thus Λ^0 and V defines $V_\lambda = \mathbf{e}(\mathbf{p})$ commuting with Λ^0, where $\mathbf{p}$ is the average along the λ-orbit

$$\mathbf{p} = \frac{1}{l} \sum_{i=0}^{l-1} \lambda^i (\mathbf{x}). \tag{3.8}$$

□

On the next step we find Λ^0. Rewrite (3.4) in the form

$$\lambda(\kappa) = \kappa - \xi, \;\; \xi = \varpi_j^\vee, \quad \lambda = \mathrm{Ad}_\Lambda, \tag{3.9}$$

where $\kappa \in C_{alc}$. Define a subgroup $\Gamma_{C_{alc}}$ of the affine Weyl group W'_a (A.18) $\Gamma_{C_{alc}} \subset W'_a$ that preserves C_{alc} (A.17). It acts by permutations on its vertices (A.17). Equivalently, $\Gamma_{C_{alc}}$ acts by permutations of nodes of the extended Dynkin graph. The face of C_{alc} belonging to the hyperplane $\langle \alpha_i, x \rangle = 0$ contains all vertices except $\varpi_i^\vee / n_i$. Similarly, the face belonging to the hyperplane $(\alpha_0, x) = 1$ contains all vertices except 0. By this duality the permutations of vertices by $g = (\lambda, \xi) \in \Gamma_{C_{alc}}$ correspond to permutations of the faces, and in this way to permutations of the coroots $\Pi^{\vee ext}$.

Instead of (3.9) consider $\lambda(C_{alc}) + \xi = C_{alc}$. The left-hand side of this equation is a transformation $g = (\lambda, \xi) \in \Gamma_{C_{alc}}$. Let us take $\xi = \varpi_j^\vee$, where $\varpi_j^\vee$ is a fundamental coweight that is a vertex of C_{alc} ($n_j = 1$ in (A.17)). Remember, that only these $\varpi_j^\vee$ define nontrivial elements of the quotient $P^\vee / Q^\vee$. Then we have

$$\lambda_j(C_{alc}) = C_{alc} - \varpi_j^\vee \equiv C'_{alc}. \tag{3.10}$$

The node 0 of C'_{alc} is an image of the node $\varpi_j^\vee$ of C_{alc} after the shift. Let us define λ_j. The Weyl group W action on the Weyl alcoves that contains 0 is simple transitive. Therefore, there exists a unique $\lambda_j \in W$ such that $\lambda_j(C_{alc}) = C'_{alc}$. Then $(\lambda_j, \varpi_j^\vee) \in \Gamma_{C_{alc}}$

defines a transformation of C_{alc}, which is a permutation of its vertices (A.17) such that $\varpi_j^\vee \to 0$. Taking into account the action of $\Gamma_{C_{alc}}$ on the extended Dynkin graph we find

$$\lambda_j^*(\alpha_k) = \begin{cases} \alpha_m & k \neq j \\ -\alpha_0 & k = j \end{cases} \quad \alpha_k, \alpha_m \in \Pi. \tag{3.11}$$

Thus, taking $\xi = \varpi_j^\vee$ we find Λ_j.

Fixed points of the Γ_{alc}-action are solutions of (3.9). It will give us κ and in this way $\mathcal{Q}$. Let us prove that a particular solution of (3.9) is

$$\kappa = \frac{\rho^\vee}{h}, \tag{3.12}$$

where h is the Coxeter number (A.8). Equation (3.9) is equivalent to

$$\langle \kappa, \lambda_j^*(\alpha_k) \rangle = \langle \kappa, \alpha_k \rangle - \delta_{jk}, \quad \alpha_k \in \Pi, k = 1, \ldots, l. \tag{3.13}$$

Since $\rho^\vee = \sum_{m=1}^l \varpi_m^\vee$ (see (A.13)) for $k \neq j$ (3.13) becomes a trivial identity. For $k = j$ using (A.7) we obtain $-\frac{1}{h}\sum_{m=1}^l n_m - \frac{1}{h} = -1$. It follows from (A.8) that it is again identity.

An arbitrary solution of (3.9) takes the form

$$\kappa = \frac{\rho^\vee}{h} + \mathbf{q}, \quad \mathbf{q} \in Ker(\lambda_j - 1).$$

In other words, the Weyl transformation λ_j should preserve $\mathbf{q}$.

Thus, taking in (3.3) $\zeta = \exp -(2\pi i \varpi_j^\vee)$ we find solutions $(\Lambda_j = \Lambda_j^0 V_\lambda, \mathcal{Q})$, where Λ_j^0 is a symmetry of the extended Dynkin graph corresponding to $\varpi_j^\vee$ and

$$\mathcal{Q} = \exp 2\pi i (\frac{\rho^\vee}{h} + \mathbf{q}). \tag{3.14}$$

The pair $(\mathbf{p}, \mathbf{q})$ (3.8), belonging to the Cartan subalgebra $\mathfrak{H}$, plays the role of the moduli parameters of solutions to (3.4). □

Remark 3.1. For $Spin(4n)$ there are two generators ζ_1 and ζ_2 of $\mathcal{Z}(Spin(4n)) \sim \mu_2 \oplus \mu_2$ corresponding to the fundamental weights ϖ_a, ϖ_b of the left and the right spinor representations. Arguing as above we will find two solutions Λ_a and Λ_b of (3.4), while $\mathcal{Q}$ is the same in both cases.

Consider a group G, $(\bar{G} \supset G \supseteq G_{ad})$ and let $\Lambda, \mathcal{Q} \in G$. Let us choose $\xi = \varpi$ such that it generates the group $t(G)$ of co-characters $t(G) = P_l^\vee$ (A.33), (A.34) $t(G) = \varpi + Q^\vee$, $l\varpi \in Q^\vee$. Then $\zeta = \mathbf{e}(-\varpi)$ is a generator of center $\mathcal{Z}(G) \sim P^\vee / t(G) = \mu_l$ (see (A.38)). Arguing as above we come to

Proposition 3.2. • *The element Λ is defined by the coweight $\varpi^\vee \in W$. It is a symmetry of the extended Dynkin diagram. Λ is defined up to invariant elements from $\mathcal{H}_G$.*
•• *Let*

$$(\lambda(G) - 1)\mathbf{q} = 0, \quad \mathbf{q} \in \mathfrak{H}, \ \lambda(G) = Ad_{\Lambda(G)}. \tag{3.15}$$

A general solution of (3.4) is

$$\kappa = \frac{\rho^\vee}{h} + \mathbf{q}, \tag{3.16}$$

Therefore, the group of cocharacters $t(G)$ defines a Weyl symmetry $\Lambda^0(G)$ of the extended Dynkin diagram $\Pi^{\vee ext}$ such that $(\Lambda^0)^l(G) = Id$.

$\Lambda(G)$ and $\mathcal{Q}$ play the role of transition operators of G-bundles over Σ_τ. A generator $\zeta = \mathbf{e}(\varpi)$ defines a characteristic class of the bundles. It is an obstruction to lift the G-bundle to the $\bar{G}$-bundle.

Remark 3.2. If $\xi \in Q^\vee$ then $\zeta = Id$. It means that we can take $\xi = 0$ as a representative of $P^\vee/Q^\vee$. Then $\lambda = 1$ (see (3.10) and $Ker\,(\lambda - 1) = \mathfrak{H}$. In this case the bundle has a trivial characteristic class, but has holomorphic moduli defined by the vector $\mathbf{q} \in \mathfrak{H}$. The corresponding Higgs bundle over $\Sigma_\tau/(z = 0)$ defines the elliptic spin Calogero-Moser system.

The moduli space. We have described a G-bundle $E_G(V)$ by the transition operators $(\Lambda = \Lambda^0\mathbf{e}(\mathbf{p}), \mathcal{Q} = \mathbf{e}(\frac{\rho^\vee}{h} + \mathbf{q})$, where Λ^0 corresponds to the coweight $\varpi^\vee \in P^\vee$. The topological type of E is defined by an element of the quotient $P^\vee/t(G)$. Let us transform $(\Lambda, \mathcal{Q})$ taking in (3.2) $f = \mathbf{e}(-\mathbf{q}z)$. Since f commutes with Λ^0 we come to new transition operators $\mathcal{Q} = \mathbf{e}(\kappa + \mathbf{q}) \to \mathcal{Q} = \mathbf{e}(\kappa)$, $\Lambda \to \Lambda^0\mathbf{e}(\mathbf{p} - \mathbf{q}\tau)$. Denote $\mathbf{p} - \mathbf{q}\tau = \tilde{\mathbf{u}}$.[7] Then sections of $E_G(V)$ assume the quasi-periodicities

$$s(z+1) = \pi(\mathbf{e}(\kappa))\, s(z), \quad s(z+\tau) = \pi(\mathbf{e}(\tilde{\mathbf{u}})\Lambda^0)\, s(z). \tag{3.17}$$

Thus, we come to the transition operators

$$\mathcal{Q} = \mathbf{e}(\kappa), \qquad \Lambda = \mathbf{e}(\tilde{\mathbf{u}})\Lambda^0. \tag{3.18}$$

Here $\tilde{\mathbf{u}}$ plays the role of a parameter in the moduli space. In this subsection we describe it in details.

Trivial bundles. Consider first the simplest case $\Lambda = Id$ and $\mathbf{u} \in \mathfrak{H}$ (see Remark 3.2)). It means that E has a trivial characteristic class. The transition transformations $\pi(\mathbf{e}(\kappa))$, $\pi(\mathbf{e}(\mathbf{u}))$ lie in the Cartan subgroup $\mathcal{H}_G$ of G.

Consider first a bundle $E_{\bar{G}}$ for a simply-connected group $\bar{G}$. Since $t(\bar{G}) \sim Q^\vee$ (A.36) and due to (A.34), $\mathbf{e}(\mathbf{u} + \gamma) = \mathbf{e}(\mathbf{u})$ for $\gamma \in Q^\vee$. Taking into account that $\mathbf{u}$ lies in a Weyl chamber we conclude that in fact $\mathbf{u} \in C_{alc}$ as it was already established. Now apply the transformation $\mathbf{e}(\gamma z)$,

$$s(z) \to \pi(\mathbf{e}(\gamma z))s(z), \quad \gamma \in Q^\vee. \tag{3.19}$$

The sections are transformed as

$$s(z+1) = \pi(\mathbf{e}(\kappa))\, s(z), \quad s(z+\tau) = \pi(\mathbf{e}(\mathbf{u} + \gamma\tau))\, s(z). \tag{3.20}$$

Thus, transition operators, defined by parameters $\mathbf{u}$ and $\mathbf{u} + \tau\gamma_1 + \gamma_2$ $(\gamma_{1,2} \in Q^\vee)$, describe equivalent bundles. The semidirect product of the Weyl group W and the lattice $\tau Q^\vee \oplus Q^\vee$ is called *the Bernstein-Schwarzman group* [6],

$$W_{BS} = W \ltimes (\tau Q^\vee \oplus Q^\vee).$$

[7] We will write $\tilde{\mathbf{u}}$ for nontrivial bundles reserving $\mathbf{u}$ for trivial bundles.

Thereby, $\mathbf{u}$ can be taken from the fundamental domain $C^{(sc)}$ of W_{BS}. Thus,

$$\boxed{C^{(sc)} = \mathfrak{H}/W_{BS} \text{ is the moduli space of trivial } \bar{G} - \text{bundles.}} \tag{3.21}$$

Consider the G^{ad} bundle and let $\mathbf{e}(\mathbf{u}) \in G^{ad}$. In this case $\mathbf{e}(\gamma) = 1$ if $\gamma \in P^\vee$ (A.36), (A.34). Define the group

$$W_{BS}^{ad} = W \ltimes (\tau P^\vee \oplus P^\vee).$$

As above, we come to the similar conclusion:

$$\boxed{C^{(ad)} = \mathfrak{H}/W_{BS}^{ad} \text{ is the moduli space of trivial } G_{ad} - \text{bundles.}} \tag{3.22}$$

Consider a coweight $\varpi^\vee \in P^\vee$ such that $l\varpi^\vee \in Q^\vee$, the coweight lattice $P_l^\vee = \mathbb{Z}\varpi^\vee \oplus Q^\vee$ (A.37). Thus, $P^\vee/P_l^\vee \sim \mu_l$. Consider a group G_l (A.29). The coweight sublattice $P_l^\vee$ is the group of its cocharacters $t(G_l)$ (A.33). Representations of G_l are defined by the dual to $t(G_l)$ groups of characters $\Gamma(G_l)$ (A.30). The dual to $P_l^\vee$ lattice $P_p \subset P$ has the form

$$P_p = \mathbb{Z}\varpi + Q, \quad p\varpi \in Q.$$

By means of $P_l^\vee$ define the affine group of the Bernstein-Schwarzman type

$$W_{BS}^{(l)} = W \ltimes (\tau P_l^\vee \oplus P_l^\vee).$$

Making use of the gauge transform $\mathbf{e}(\gamma z) \in G_l$, $(\gamma \in P_l^\vee)$ we find that

$$\boxed{C^{(l)} = \mathfrak{H}/W_{BS}^{(l)} \text{ is the moduli space of trivial } G_l - \text{bundles.}} \tag{3.23}$$

Consider the dual picture and the lattice $P_p^\vee$. It is formed by $Q^\vee$ and a coweight $\varpi^\vee$,

$$P_p^\vee = \mathbb{Z}\varpi^\vee + Q^\vee, \quad p\varpi^\vee \in Q^\vee.$$

The lattice $P_p^\vee$ plays the role of the group of cocharacters for the dual group ${}^L G_l = G_p = \bar{G}/\mu_p$, (A.29), while P_l defines characters of G_p. Again by means of the group,

$$W_{BS}^{(p)} = W \ltimes (P_p^\vee \oplus P_p^\vee \tau),$$

we find that

$$\boxed{C^{(p)} = \mathfrak{H}/W_{BS}^{(p)} \text{ is the moduli space of trivial } G_p - \text{bundles.}} \tag{3.24}$$

Thus, for the $\bar{G}, G_l, G_p, G_{ad}$ trivial bundles we have the following interrelations between their moduli space

$$\begin{array}{ccccc} & & C^{(sc)} & & \\ & \swarrow & \mid & \searrow & \\ C^{(l)} & & \mid & & C^{(p)} \\ & \searrow & \downarrow & \swarrow & \\ & & C^{(ad)} & & \end{array} \tag{3.25}$$

Here arrows mean coverings. Note that $C^{(sc)}$, $C^{(ad)}$ and as well as $C^{(l)}$, $C^{(p)}$ are dual to each other in the sense that in defining them the lattices are dual.

Let Fl be a flag variety located at the marked point. In this way we have defined a space $\tilde{\mathcal{M}}_{1,1} = (C^a, Fl)$ $(a = (sc), (l), (ad))$ related to the moduli space of trivial bundles over Σ_τ with one marked point. But we still have freedom to act on Fl by constant conjugations from the Cartan subgroup $\mathcal{H}^a$. Thus, eventually we come to $\mathcal{M}^0_{1,1} = (C^a, Fl/\mathcal{H}^a)$. It has dimension $\mathcal{M}_{1,1}$ (2.7). It is a big cell in $\mathcal{M}_{1,1}$. In our construction we have excluded non-semisimple elements $\mathcal{Q}$.

Nontrivial bundles. Consider a general case $\Lambda^0 \neq Id$. It was explained above that Λ^0 corresponds to some characteristic class related to $\varpi^\vee \in P^\vee$, and $\varpi^\vee \notin Q^\vee$. In this case $\tilde{\mathbf{u}} \in Ker(\lambda - 1)$, and in fact $\tilde{\mathbf{u}} \in C_{alc} \cap \tilde{\mathfrak{H}}_0$, where $\tilde{\mathfrak{H}}_0$ is the invariant subalgebra $\lambda(\tilde{\mathfrak{H}}_0) = \tilde{\mathfrak{H}}_0$. There is a basis in $\tilde{\mathfrak{H}}_0$ defined by a system of simple coroots $\tilde{\Pi}^\vee$ (see Sect. 5.4). Moreover, the corresponding root system defines a simple Lie algebra $\tilde{\mathfrak{g}}_0$.

Let $\tilde{W}$ be the Weyl group W of the root system $\tilde{R} = \tilde{R}(\tilde{\Pi})$,

$$\tilde{W} = \{w \in \tilde{W} \mid w(\tilde{R}) = \tilde{R}\}, \tag{3.26}$$

and

$$\tilde{Q}^\vee = \{\gamma = \sum_{j=1}^{p} m_j \tilde{\alpha}_j^\vee,\ m_j \in \mathbb{Z}\} \tag{3.27}$$

is the coroot lattice generated by $\tilde{\Pi}^\vee$ (5.26). Consider first $E_{\bar{G}}$ bundles. As above, $\mathbf{e}(\tilde{\mathbf{u}} + \gamma) = \mathbf{e}(\tilde{\mathbf{u}})$, $\gamma \in \tilde{Q}^\vee$. The automorphism (3.19) for $\gamma \in \tilde{Q}^\vee$ commutes with Λ. Thus, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{u}} + \tau\gamma_1 + \gamma_2$ $\gamma_{1,2} \in \tilde{Q}^\vee$ define equivalent $\bar{G}$-bundles. Consider the semidirect products

$$\tilde{W}_{BS} = \tilde{W} \ltimes (\tau\tilde{Q}^\vee \oplus \tilde{Q}^\vee). \tag{3.28}$$

The fundamental domain in $\tilde{\mathfrak{H}}$ under the $\tilde{W}_{BS}$ action is the moduli space of $\bar{G}$-bundles with characteristic classes defined by $\varpi^\vee$,

$$\boxed{\tilde{C}^{sc} = \tilde{\mathfrak{H}}/\tilde{W}_{BS} \text{ is the moduli space of nontrivial } \bar{G} - \text{bundles}}. \tag{3.29}$$

Consider $E_{G^{ad}}$-bundles. Let $\tilde{\varpi}_j^\vee$ be fundamental coweights ($\langle\tilde{\varpi}_j^\vee, \tilde{\alpha}_k\rangle = \delta_{jk}$) and

$$\tilde{P}^\vee = \{\gamma = \sum_{j=1}^{p} m_j \tilde{\varpi}_j^\vee,\ m_j \in \mathbb{Z}\} \tag{3.30}$$

is the coweight lattice in $\tilde{\mathfrak{H}}_0$. Define the semidirect product

$$\tilde{W}^{ad}_{BS} = \tilde{W} \ltimes (\tau\tilde{P}^\vee \oplus \tilde{P}^\vee). \tag{3.31}$$

A fundamental domain under its action

$$\boxed{\tilde{C}^{ad} = \tilde{\mathfrak{H}}_0/\tilde{W}^{ad}_{BS} \text{ is the moduli space of nontrivial } G^{ad} - \text{bundles}} \tag{3.32}$$

is a moduli space of a G^{ad}-bundle with characteristic class defined by $\varpi^\vee$. If $ord(\mathcal{Z}(\bar{\tilde{G}}))$ is not a primitive number then we again come to the hierarchy of the moduli spaces similar to (3.25).

As above the space $\mathcal{M}^0_{1,1} = (C^a, Fl/\tilde{\mathcal{H}}_0)$ is a big cell in the moduli space of nontrivial bundles.

4. Characteristic Classes and Conformal Groups

Characteristic classes. Let $\mathcal{E}_G$ be is a principal G-bundle over Σ. Consider a finite-dimensional representation of complex group G in a space V and let $E_G(V)$ be the vector bundle $E_G(V) = \mathcal{E}_G \times_G V$ induced by V.

The first cohomology $H^1(\Sigma_g, G(\mathcal{O}_\Sigma))$ of Σ with coefficients in analytic sheaves defines the moduli space $\mathcal{M}(G, \Sigma)$ of holomorphic G-bundles. Let $\bar{G}$ be a simply-connected group and G^{ad} be an adjoint group. Using (A.28) and (A.29) we write three exact sequences:

$$
\begin{aligned}
&1 \to \mathcal{Z}(\bar{G})) \to \bar{G}(\mathcal{O}_\Sigma) \to G^{ad}(\mathcal{O}_\Sigma) \to 1,\\
&1 \to \mathcal{Z}_l \to \bar{G}(\mathcal{O}_\Sigma) \to G_l(\mathcal{O}_\Sigma) \to 1,\\
&1 \to \mathcal{Z}(G_l) \to G_l(\mathcal{O}_\Sigma) \to G^{ad}(\mathcal{O}_\Sigma) \to 1,
\end{aligned}
$$

where $G_l = \bar{G}/\mathcal{Z}_l$. Then we come to the long exact sequences

$$\to H^1(\Sigma_g, \bar{G}(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G^{ad}(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, \mathcal{Z}(\bar{G})) \sim \mathcal{Z}(\bar{G})) \to 0, \tag{4.1}$$

$$\to H^1(\Sigma_g, \bar{G}(\mathcal{O}_\Sigma)) \to H^1(\Sigma_g, G_l(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, \mathcal{Z}_l) \sim \mu_l \to 0, \tag{4.2}$$

$$\to H^1(\Sigma_g, G_l(\mathcal{O}_\Sigma) \to H^1(\Sigma_g, G^{ad}(\mathcal{O}_\Sigma)) \to H^2(\Sigma_g, \mathcal{Z}(G_l)) \sim \mu_p \to 0. \tag{4.3}$$

The elements from H^2 are obstructions to lift bundles, namely

$$
\begin{aligned}
\zeta(E_{G^{ad}}) &\in H^2(\Sigma_g, \mathcal{Z}(\bar{G})) - \text{ obstructions to lift } E_{G^{ad}} - \text{bundle to } E_{\bar{G}} - \text{bundle},\\
\zeta(E_{G_l}) &\in H^2(\Sigma_g, \mathcal{Z}_l) - \text{ obstructions to lift } E_{G_l} - \text{bundle to } E_{\bar{G}} - \text{bundle},\\
\zeta^\vee(E_{G^{ad}}) &\in H^2(\Sigma_g, \mathcal{Z}(G_l)) - \text{ obstructions to lift } E_{G^{ad}} - \text{bundle to } E_{G^l} - \text{bundle}.
\end{aligned}
$$

Definition 4.1. *Images of $H^1(\Sigma_g, G(\mathcal{O}_\Sigma))$ in $H^2(\Sigma_g, \mathcal{Z})$ are called the characteristic classes $\zeta(E_G)$ of G-bundles.*

Since $\mathcal{Z}_l \to \mathcal{Z}(\bar{G}) \to \mathcal{Z}(G_l)$. we have the following relations between these characteristic classes $\zeta^\vee(E_{G^{ad}}) = \zeta(E_{G^{ad}})\ mod\ \mathcal{Z}_l$, and the characteristic class $\zeta(E_{G_l})$ coincides with $\zeta(E_{G^{ad}})$ as an obstruction to lift a E_{G_l}-bundle, treated as a $E_{G^{ad}}$-bundle to a $E_{\bar{G}}$-bundle.

Consider a particular case $\bar{G} = \mathrm{SL}(N, \mathbb{C})$, $G^{ad} = \mathrm{PSL}(N, \mathbb{C})$. Then the elements $\zeta \in \mathcal{Z}(\mathrm{SL}(N, \mathbb{C})) \sim \mu_N$ are obstructions to lift $\mathrm{PSL}(N, \mathbb{C})$-bundles to $\mathrm{SL}(N, \mathbb{C})$-bundles. They represent the characteristic classes of $\mathrm{PSL}(N, \mathbb{C})$-bundles. On the other hand, the exact sequence

$$1 \to \mathcal{O}^* \to \mathrm{GL}(N, \mathbb{C}) \to \mathrm{PGL}(N, \mathbb{C}) \to Id \tag{4.4}$$

gives rise to the exact sequence of cohomology

$$H^1(\Sigma_g, \mathrm{GL}(N, \mathbb{C})) \to H^1(\Sigma_g, \mathrm{PGL}(N, \mathbb{C})) \to H^2(\Sigma_g, \mathcal{O}^*). \tag{4.5}$$

The Brauer group $H^2(\Sigma, \mathcal{O}^*)$ vanishes and, therefore, there are no obstructions to lift $\mathrm{PGL}(N, \mathbb{C}) \sim \mathrm{PSL}(N, \mathbb{C})$-bundles to $\mathrm{GL}(N, \mathbb{C})$-bundles. A topological characteristic of a $\mathrm{GL}(N, \mathbb{C})$-bundle is the degree of its determinant bundle. In the following subsections we will construct an analog of $\mathrm{GL}(N, \mathbb{C})$ for other simple groups. We call them the *conformal groups*. The main goal is to relate the characteristic classes to the degrees of some line bundles connected to the conformal groups.

Conformal groups. Here we introduce an analog of the group $\mathrm{GL}(N,\mathbb{C})$ for other simple groups apart from $\mathrm{SL}(N,\mathbb{C})$. Let

$$\phi\,:\,\mathcal{Z}(\bar{G})\hookrightarrow\left(\mathbb{C}^*\right)^r \tag{4.6}$$

be an embedding of the center $\mathcal{Z}(\bar{G})$ into an algebraic torus $(\mathbb{C}^*)^r$ of minimal dimension ($r=1$ for a cyclic center and $r=2$ for $\mu_2\times\mu_2$). Note that any two embeddings for $\mathcal{Z}(\bar{G})$, ($\bar{G}\neq\mathrm{SL}(N,\mathbb{C})$) are conjugate from the left: $\phi_1=A\phi_2$ for some automorphism A of the torus $(\mathbb{C}^*)^r$. For these groups we deal with μ_2,μ_3,μ_4 or $\mu_2\times\mu_2$. In these cases nontrivial roots of unity coincide or they are inverse to each other. In the latter case $A\,:\,x\to x^{-1}$.

Consider the "anti-diagonal" embedding $\mathcal{Z}(\bar{G})\to\bar{G}\times(\mathbb{C}^*)^r$, $\zeta\mapsto(\zeta,\phi(\zeta)^{-1})$, $\zeta\in\mathcal{Z}(\bar{G})$. The image of this map is a normal subgroup since $\mathcal{Z}$ is the center of $\bar{G}$.

Definition 4.2. *The quotient*

$$C\bar{G}=\left(\bar{G}\times\left(\mathbb{C}^*\right)^r\right)/\mathcal{Z}(\bar{G})$$

is called the conformal version of $\bar{G}$.

Similarly the conformal version can be defined for any G with a non-trivial center. If the center of G is trivial as for G^{ad} then $CG=G\times\mathbb{C}^*$.

The group $C\bar{G}$ does not depend on embedding in $\mathbb{C}^r$ due to above remark about conjugacy of ϕ's. We have a natural inclusion $\bar{G}\subset C\bar{G}$.

Consider the quotient torus $Z^\vee=(\mathbb{C}^*)^r/\mathcal{Z}(\bar{G})\sim(\mathbb{C}^*)^r$. The last isomorphism is defined by $\lambda\to\lambda^N$ for cyclic center and $(\lambda_1,\lambda_2)\to(\lambda_1^2,\lambda_2^2)$ for D_{even}. The sequence

$$1\to\bar{G}\to C\bar{G}\to Z^\vee\to 1 \tag{4.7}$$

is the analogue of

$$1\to\mathrm{SL}(N,\mathbb{C})\to\mathrm{GL}(N,\mathbb{C})\to\mathbb{C}^*\to 1.$$

On the other hand, we have embedding of $(\mathbb{C}^*)^r\to C\bar{G}$ with the quotient $C\bar{G}/(\mathbb{C}^*)^r=G^{ad}$. Then the sequence

$$1\to\left(\mathbb{C}^*\right)^r\to C\bar{G}\to G^{ad}\to 1 \tag{4.8}$$

is similar to the sequence

$$1\to\mathbb{C}^*\to\mathrm{GL}(N,\mathbb{C})\to\mathrm{PGL}(N,\mathbb{C})\to 1.$$

Let π be an irreducible representation of $\bar{G}$ and χ be a character of the torus $(\mathbb{C}^*)^r$. It follows from (4.7) that an irreducible representation $\tilde{\pi}$ of $C\bar{G}$ is defined as

$$\tilde{\pi}=\pi\boxtimes\chi((\mathbb{C}^*)^r),\ \text{such that}\ \pi|_{\mathcal{Z}(\bar{G})}=\chi\phi,\quad(\phi\ (4.6)). \tag{4.9}$$

Assume for simplicity that π is a fundamental representation. It means that the highest weight ν of π is a fundamental weight. Let $\varpi^\vee$ be a fundamental coweight generating $\mathcal{Z}(\bar{G})$ for $r=1$. In other words, $\zeta=\mathbf{e}(\varpi^\vee)$ is a generator of $\mathcal{Z}(\bar{G})$ ($\zeta^N=1$, $N=\mathrm{ord}(\mathcal{Z}(\bar{G}))$. Then $\pi|_{\mathcal{Z}(\bar{G})}$ acts as a scalar $\mathbf{e}\langle\varpi^\vee,\nu\rangle$. The highest weight can be expanded in the basis of simple roots $\nu=\sum_{\alpha\in\Pi}c_\alpha^\nu\alpha$. Then the coefficients c_α^ν are rows of the inverse Cartan matrix. They have the form k/N, where k is an integer. Therefore the scalar

$$\mathbf{e}\langle\varpi^\vee,\nu\rangle=\mathbf{e}\left(\sum_{\alpha\in\Pi}c_\alpha^\nu\delta_{\langle\varpi^\vee,\alpha\rangle}\right) \tag{4.10}$$

is a root of unity. On the other hand, let $\chi_m(\mathbb{C}^*)=w^m$ ($w\in\mathbb{C}^*$) be a character of $\mathbb{C}^*$, and $\phi(\zeta)=\mathbf{e}\,(l/N)$. In terms of weights the definition of $\tilde{\pi}$ (4.9) takes the form $\mathbf{e}\langle\varpi^\vee,\nu\rangle=\mathbf{e}\left(\frac{ml}{N}\right)$. It follows from this construction that characters of $C\bar{G}$ are defined by the weight lattice P and the integer lattice $\mathbb{Z}$ with an additional restriction

$$\chi_{(\gamma,m)}(\mathbf{x},w)=\exp\,2\pi i\langle\gamma,\mathbf{x}\rangle w^m,\ \ \langle\gamma,\varpi^\vee\rangle=\frac{ml}{N}+j,\ \ \gamma\in P,\ \ m,j\in\mathbb{Z},\ \ \mathbf{x}\in\mathfrak{H}.$$

The case D_{even} ($r=2$) can be considered similarly.

Remark 4.1. Simple groups can be defined as subgroups of $GL(V)$ preserving some multi-linear forms in V. For examples, in the defining representations these forms are bilinear symmetric forms for SO, bilinear antisymmetric forms for Sp, a trilinear form for E_6 and a form of fourth order for E_7. In a generic situation G is defined as a subgroup of $GL(V)$ preserving a three tensor in $V^*\otimes V^*\otimes V$ [25]. The conformal versions of these groups can be alternatively defined as transformations preserving the forms up to dilatations. We prefer to use here the algebraic construction, but this approach justifies the name "conformal version".

The conformal versions can be also defined in terms of exact representations of $\bar{G}$. Let V be such a representation and assume that $\mathcal{Z}(\bar{G})$ is a cyclic group. Then $C\bar{G}$ is a subgroup of $GL(V)$ generated by G and dilatations $\mathbb{C}^*$. The character det V is equal to $\lambda^{\dim(V)}$, where λ is equal to (4.10) for fundamental representations.

For D_{even} we use two representations, f.e. the left and right spinors $Spin^{L,R}$. The conformal group $CSpin_{4k}$ is a subgroup of $GL(Spin^L\oplus Spin^R)$ generated by $Spin_{4k}$ and $\mathbb{C}^*\times\mathbb{C}^*$, where the first factor $\mathbb{C}^*$ acts by dilatations on $Spin^L$ and the second factor acts on $Spin^R$. The character det $Spin^L$ (det $Spin^R$) is equal to $\lambda_1^{\dim(Spin^L_{4k})}$ ($\lambda_2^{\dim(Spin^R_{4k})}$), ($\dim(Spin^{L,R}_{4k})=2^{2k-1}$).

Characteristic classes and degrees of vector bundles. From the exact sequence (4.8) and vanishing of the second cohomology of a curve $H^2(\Sigma,\mathcal{O}^*)=0$ with coefficients in the analytic sheaf we get that any $G^{ad}(\mathcal{O})$-bundle (even a topological non-trivial one with $\zeta(G^{ad}(\mathcal{O})\neq 0)$ can be lifted to a $C\bar{G}(\mathcal{O})$-bundle.

Let V be an exact representation either irreducible or with the sum $Spin^L\oplus Spin^R$ for D_{2k}. Then from (4.6) one has an embedding of $\mathcal{Z}(\bar{G})$ to the automorphisms of V

$$\phi_V\ :\ \mathcal{Z}(\bar{G})\hookrightarrow\left(\mathbb{C}^*\right)^r=\mathrm{Aut}_{\bar{G}}(V). \tag{4.11}$$

In a particular case, when V is a fundamental defining representation the center acts by multiplication on (4.10).

Let $\mathcal{P}_{C\bar{G}}$ be a principal $C\bar{G}(\mathcal{O})$-bundle. Denote by $E(V)=\mathcal{P}_{C\bar{G}}\otimes_{C\bar{G}}V$ (or $E(Spin^{L,R})$) a vector bundle induced by a representation V ($Spin^{L,R}$ for D_{even}).

Theorem 4.1.[8] *Let $E_{ad} = E(Ad)$ be the adjoint bundle with the characteristic class $\zeta(E_{ad})$. The image of $\zeta(E_{ad})$ under ϕ_V (4.11) is*

$$\phi_V(\zeta(E_{ad})) = \begin{cases} \exp\big(-2\pi i \deg(E_{\bar{G}}(V))/\dim V\big) & \mathcal{Z}(\bar{G}) - \text{is cyclic}, \\ \exp\big(-2\pi i \deg(E_{Spin^{L,R}_{4k}})/2^{2k-1}\big). \end{cases}$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & Z^{\vee}(\mathcal{O}_\Sigma) & \xrightarrow{\sim} & Z^{\vee}(\mathcal{O}_\Sigma) & \longrightarrow & 1 & & \\
 & & \uparrow{\scriptstyle [N]} & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \left(\mathcal{O}^*_\Sigma\right)^r & \longrightarrow & C\bar{G}(\mathcal{O}_\Sigma) & \longrightarrow & G^{ad}(\mathcal{O}_\Sigma) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathcal{Z}(\bar{G}) & \longrightarrow & \bar{G}(\mathcal{O}_\Sigma) & \longrightarrow & G^{ad}(\mathcal{O}_\Sigma) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

and the corresponding diagram of Ĉech cochains. Let ψ be a 1-cocycle with values in $G^{ad}(\mathcal{O}_\Sigma)$. Consider its preimage as a cocycle with values in $C\bar{G}(\mathcal{O}_\Sigma)$. Due to the definition of $C\bar{G}$ this cocycle is a pair of cochains (Ψ, ν) with values in $\bar{G}(\mathcal{O}_\Sigma)$ and $\left(\mathcal{O}^*_\Sigma\right)^r$ such that $\phi_V(d\Psi)d\nu = 1 \in (\mathcal{O}^*)^r$, where d is the Ĉech coboundary operator. The cohomology class of $d\Psi$ by definition is the characteristic class **c**, so ϕ_V is opposite to the class of $d\nu$: $\phi_V(\zeta(E_{ad})) = (d\nu)^{-1}$. Since ν acts in V as a scalar $\nu^{\dim V}$, it is a one-cocycle as a determinant of this action. It represents the determinant of the bundle $E(V)$. In this way ν is a preimage of the cocycle $\nu^{\dim V}$, taking $N = \dim(V)$ power $\mathcal{O}^* \stackrel{[N]}{\to} \mathcal{O}^*,\ \nu \to \nu^N,\ N = \dim(V)$. □

Consider the long exact sequence

$$1 \to \mu_N \to \mathcal{O}^*_\Sigma \stackrel{[N]}{\to} \mathcal{O}^*_\Sigma \to 1, \quad (\mu_N = \mathbb{Z}/N\mathbb{Z}).$$

It induces the map $H^1(\Sigma, \mathcal{O}^*_\Sigma) \to H^2(\Sigma, \mu_N)$. The cocycle $d\nu$ lies in the cohomology class which is an image of the class of det $E(V) = \nu^N$ under the coboundary map $H^1(\Sigma, \mathcal{O}^*) \to H^2(\Sigma, \mu_N)$. Denote it by Inv_N =Image(det $E(V)$). Thus, by the definition, the class of $d\nu$ equals $\mathrm{Inv}_N(\det E(V)) = \mathrm{Inv}_N(\zeta_1(E(V)))$.

The statement of the theorem follows from the following proposition

Proposition 4.1. *Let γ be a 1-cocycle with values in $\mathcal{O}^*$. Then* $\mathrm{Inv}_N(\gamma) = \exp\left(\frac{1}{N} 2\pi i \deg(\gamma)\right)$.

[8] For $G = \mathrm{GL}(N, \mathbb{C})$ this theorem was proved in [52].

Table 3. Degrees of bundles for conformal groups

$\bar{G}$	$\nu,$	V	$\deg\,(E(V))$
$\mathrm{SL}(n,\mathbb{C})$	$\varpi_1^\vee$	$\underline{n}$	$-1+kn$
$\mathrm{Spin}_{2n+1}(\mathbb{C})$	$\varpi_n^\vee$	$\underline{2^n}$	$2^{n-1}(1+2k)$
$\mathrm{Mp}_n(\mathbb{C})$	$\varpi_1^\vee$	$\underline{2n}$	$n(1+2k)$
$\mathrm{Spin}_{4n}^{L,R}(\mathbb{C})$	$\varpi_{n,n-1}^\vee$	$\underline{2^{2n-1}}$	$2^{2n-2}(1+2k)$
$\mathrm{Spin}_{4n+2}(\mathbb{C})$	$\varpi_n^\vee$	$\underline{2^n}$	$2^{n-2}(1+4k)$
$E_6(\mathbb{C})$	$\varpi_1^\vee$	$\underline{27}$	$9(1+3k)$
$E_7(\mathbb{C})$	$\varpi_1^\vee$	$\underline{56}$	$28(1+2k)$

$\mathrm{Mp}_n(\mathbb{C})$ is the universal covering of $\mathrm{Sp}_n(\mathbb{C})$
$(k \in \mathbb{Z})$

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_N & \longrightarrow & \mathcal{O}_\Sigma^* & \xrightarrow{[N]} & \mathcal{O}_\Sigma^* & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} & & \\
 & & 0 & \longrightarrow & \mathcal{O}_\Sigma & \xrightarrow{\times N} & \mathcal{O}_\Sigma & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 2\pi i\mathbb{Z} & \xrightarrow{\times N} & 2\pi i\mathbb{Z} & &
\end{array}$$

Let γ be a 1-cocycle of $\mathcal{O}_\Sigma^*$. By definition its image in $H^2(X,\mu_N)$ is equal to the coboundary of 1-cochain $\gamma^{1/N}$ of $\mathcal{O}_\Sigma^*$, $(\gamma^{1/N})^N = \gamma$. Let $\log(\gamma)$ be a preimage of the cycle γ under an exponential map; $\log(\gamma)$ is a 1-cochain of $\mathcal{O}_\Sigma$ and its coboundary equals the degree of γ times $2\pi i$. As the multiplication by N is invertible on $\mathcal{O}_\Sigma$, the cochain $\frac{1}{N}\log(\gamma)$ is well-defined; due to commutativity of the diagram we can choose $\exp\left(\frac{1}{N}\log(\gamma)\right)$ as $\gamma^{1/N}$. Hence, the image of γ in $H^2(X,\mu_N)$ equals the coboundary of $\exp\left(\frac{1}{N}\log(\gamma)\right)$ equals exponential of coboundary of $\frac{1}{N}\log(\gamma)$ equals the exponential of degree of γ times $\frac{2\pi i}{N}$.

The case $r = 2$, can be analyzed in the same way. The theorem is proved. □

Let as above $\varpi^\vee$ be a fundamental coweight generating a center $\mathcal{Z}(\bar{G})$ and ν is weight of the representation of $\bar{G}$ in V. Then it follows from Theorem 4.1 and (4.10) that

$$deg\,(E(V)) = \dim\,(V)(\langle\varpi^\vee,\nu\rangle + k), \quad k \in \mathbb{Z}. \tag{4.12}$$

Then for the fundamental representations of $\bar{G}$ we have the following realization of this formula.

It follows from our considerations that replacing the transition matrix

$$\Lambda \to \tilde{\Lambda} = \mathbf{e}\,(\langle\varpi^\vee,\nu\rangle(z+\frac{\tau}{2}))\Lambda$$

defines the bundle of conformal group CG of degree (4.12) (Table 3).

5. GS-Basis in Simple Lie Algebras

We pass from the Chevalley basis (A.20) to a new basis that is more convenient to define bundles corresponding to nontrivial characteristic classes. We call it *the generalized sin basis* (GS-basis), because for the A_n case and degree one bundles it coincides with the sin-algebra basis (see, for example, [18]).

Let us take an element $\zeta \in \mathcal{Z}(\bar{G})$ of order l and the corresponding $\Lambda^0 \in W$ from (3.3). Then Λ^0 generates a cyclic group $\mu_l = (\Lambda^0, (\Lambda^0)^2, \ldots, (\Lambda^0)^l = 1)$ isomorphic to a subgroup of $\mathcal{Z}(\bar{G})$. Note that l is a divisor of ord$(\mathcal{Z}(\bar{G}))$. Consider the action of Λ^0 on $\mathfrak{g}$. Since $(\Lambda^0)^l = Id$ we have a l-periodic gradation

$$\mathfrak{g} = \oplus_{a=0}^{l-1}\mathfrak{g}_a, \quad \lambda(\mathfrak{g}_a) = \omega^a \mathfrak{g}_a, \quad \omega = \exp\frac{2\pi i}{l}, \quad \lambda = Ad_{\Lambda^0}, \tag{5.1}$$

$$[\mathfrak{g}_a, \mathfrak{g}_b] = \mathfrak{g}_{a+b}\ (mod\, l), \tag{5.2}$$

where $\mathfrak{g}_0$ is a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ and the subspaces $\mathfrak{g}_a$ are its representations.

Since $\Lambda^0 \in W$ it preserves the root system R. Define the quotient set $\mathcal{T}_l = R/\mu_l$. Then R is represented as a union of μ_l-orbits $R = \cup_{\mathcal{T}_l}\mathcal{O}$. We denote by $\mathcal{O}(\bar{\beta})$ an orbit starting from the root β,

$$\mathcal{O}(\bar{\beta}) = \{\beta, \lambda(\beta), \ldots, \lambda^{l-1}(\beta)\}, \quad \bar{\beta} \in \mathcal{T}_l.$$

The number of elements in an orbit $\mathcal{O}$ (the length of $\mathcal{O}$) is $l/p_\alpha = l_\alpha$, where p_α is a divisor of l. Let ν_α be a number of orbits $\mathcal{O}_{\bar{\alpha}}$ of the length l_α. Then $\sharp R = \sum \nu_\alpha l_\alpha$. Note, that if $\mathcal{O}(\bar{\beta})$ has length l_β $(l_\beta \neq 1)$, then the elements $\lambda^k\beta$ and $\lambda^{k+l_\beta}\beta$ coincide.

Basis in $\mathfrak{L}$ (A.19). Transform first the root basis $\mathcal{E} = \{E_\beta,\ \beta \in R\}$ in $\mathfrak{L}$. Define an orbit in $\mathcal{E}$,

$$\mathcal{E}_{\bar{\beta}} = \{E_\beta, E_{\lambda(\beta)}, \ldots, E_{\lambda^{l-1}(\beta)}\},$$

corresponding to $\mathcal{O}(\bar{\beta})$. Again $\mathcal{E} = \cup_{\bar{\beta}\in\mathcal{T}_l}\mathcal{E}_{\bar{\beta}}$.

For $\mathcal{O}(\bar{\beta})$ define the set of integers

$$J_{p_\alpha} = \{a = mp_\alpha \,|\, m \in \mathbb{Z},\ \ a \text{ is defined } mod\, l\,\}, \quad (p_\alpha = l/l_\alpha). \tag{5.3}$$

"The Fourier transform" of the root basis on the orbit $\mathcal{O}(\bar{\beta})$ is defined as

$$\mathfrak{t}^a_{\bar{\beta}} = \frac{1}{\sqrt{l}}\sum_{m=0}^{l-1}\omega^{ma}E_{\lambda^m(\beta)}, \quad \omega = \exp\frac{2\pi i}{l}, \quad a \in J_\beta. \tag{5.4}$$

This transformation is invertible $E_{\lambda^k(\beta)} = \frac{1}{\sqrt{l}}\sum_{a\in J_l}\omega^{-ka}\mathfrak{t}^a_{\bar{\beta}}$, and therefore there is the one-to-one map $\mathcal{E}_\beta \leftrightarrow \{\mathfrak{t}^a_{\bar{\beta}},\ a \in J_\beta\}$. In this way we have defined the new basis

$$\{\mathfrak{t}^a_{\bar{\beta}},\ (a \in J_l,\ \bar{\beta} \in \mathcal{T}_l)\}. \tag{5.5}$$

Since $\lambda(E_\alpha) = E_{\lambda(\alpha)}$, we have for $\Lambda\mathbf{e}(\tilde{\mathbf{u}})$ $(\tilde{\mathbf{u}} \in \tilde{\mathfrak{H}}_0)$,

$$Ad_\Lambda(\mathfrak{t}^a_{\bar{\beta}}) = \mathbf{e}(\langle\tilde{\mathbf{u}}, \beta\rangle - \frac{a}{l})\mathfrak{t}^a_{\bar{\beta}}, \quad \mathbf{e}(x) = \exp(2\pi i x). \tag{5.6}$$

It means that $\mathfrak{t}^a_{\bar\beta}$ $(\bar\beta \in \mathcal{T}_l)$ is a part of basis in $\mathfrak{g}_{l-a}$ (5.1). Moreover,

$$Ad_{\mathcal{Q}}(\mathfrak{t}^a_{\bar\beta}) = \mathbf{e}(\langle\kappa,\beta\rangle)\mathfrak{t}^a_{\bar\beta}. \tag{5.7}$$

This relation follows from (3.7) and (3.14). We also take into account that $\mathcal{Q}$ and Λ commute in the adjoint representation and $\mathbf{e}\,(x)E_\alpha\mathbf{e}\,(-x) = \mathbf{e}\,\langle x,\alpha\rangle E_\alpha$ for $x \in \tilde{\mathfrak{H}}_0$.

Picking another element Λ' generating a subgroup $\mathcal{Z}_{l'_1}$ $(l' \neq l)$ we come to another set of orbits and to another basis. We have as many types of bases as non-isomorphic subgroups in $\mathcal{Z}(\bar G)$.

The Killing form. Consider two orbits $\mathcal{O}(\bar\alpha)$ and $\mathcal{O}(\bar\beta)$, passing through E_α and E_β. Assume that there exists such integer r that $\alpha = -\lambda^r(\beta)$. It implies that elements of two orbits are related as $\lambda^n(\alpha) = -\lambda^m(\beta)$ if $m - n = r$. In other words, $-\beta \in \mathcal{O}(\bar\alpha)$. In particular, it means that orbits have the same length. It follows from (5.4) and (A.24) that

$$(\mathfrak{t}^{c_1}_{\bar\alpha}, \mathfrak{t}^{c_2}_{\bar\beta}) = \delta_{\alpha,-\lambda^r(\beta)}\delta^{(c_1+c_2,0\ (mod\, l))}\omega^{-rc_1}\frac{2p_\alpha}{(\alpha,\alpha)}, \tag{5.8}$$

where $p_\alpha = l/l_\alpha$, and l_α is the length of $\mathcal{O}(\bar\alpha)$. In particular, $(\mathfrak{t}^a_{\bar\alpha}, \mathfrak{t}^{-a}_{-\bar\alpha}) = \frac{2p_\alpha}{(\alpha,\alpha)}$.

In what follows we need a dual basis $\mathfrak{T}^b_{\bar\alpha}$,

$$(\mathfrak{T}^{b_1}_{\bar\alpha_1}, \mathfrak{t}^{b_2}_{\bar\alpha_2}) = \delta^{(b_1+b_2,0\ (mod\, l))}\delta_{\bar\alpha_1,-\bar\alpha_2}, \quad \mathfrak{T}^b_{\bar\alpha} = \mathfrak{t}^{-b}_{-\bar\alpha}\frac{(\alpha,\alpha)}{2p_\alpha}. \tag{5.9}$$

The Killing form in this basis is inverse to (5.8),

$$(\mathfrak{T}^{a_1}_{\bar\alpha_1}, \mathfrak{T}^{a_2}_{\bar\alpha_2}) = \delta_{\alpha_1,-\lambda^r(\alpha_2)}\delta^{(a_1+a_2,0\ (mod\, l))}\omega^{ra_1}\frac{(\alpha_1,\alpha_1)}{2p_{\alpha_1}}.$$

In particular,

$$(\mathfrak{T}^a_\alpha, \mathfrak{T}^{-a}_{-\alpha}) = \frac{(\alpha,\alpha)}{2p_\alpha}. \tag{5.10}$$

A basis in the Cartan subalgebra. Almost the same construction exists in $\mathfrak{H}$. Again let Λ^0 generate the group μ_l. Since Λ^0 preserves the extended Dynkin diagram, its action preserves the extended coroot system $\Pi^{\vee ext} = \Pi^\vee \cup \alpha_0^\vee$ in $\mathfrak{H}$. Consider the quotient $\mathcal{K}_l = \Pi^{\vee ext}/\mu_l$. Define an orbit $\mathcal{H}(\bar\alpha)$ of length $l_\alpha = l/p_\alpha$ in $\Pi^{\vee ext}$ passing through $H_\alpha \in \Pi^{\vee ext}$,

$$\mathcal{H}(\bar\alpha) = \{H_\alpha, H_{\lambda(\alpha)}, \ldots, H_{\lambda^{l-1}(\alpha)}\}, \quad \bar\alpha \in \mathcal{K}_l = \Pi^{\vee ext}/\mu_l.$$

The set $\Pi^{\vee ext}$ is a union of $\mathcal{H}(\bar\alpha)$,

$$(\Pi^\vee)^{ext} = \cup_{\bar\alpha\in\mathcal{K}_l}\mathcal{H}(\bar\alpha).$$

Define "the Fourier transform"

$$\mathfrak{h}^c_{\bar\alpha} = \frac{1}{\sqrt{l}}\sum_{m=0}^{l-1}\omega^{mc}H_{\lambda^m(\alpha)}, \quad \omega = \exp\frac{2\pi i}{l}, \quad c \in J_\alpha\ (5.3). \tag{5.11}$$

The basis $\mathfrak{h}^c_{\bar\alpha}$, $(c \in J_\alpha,\ \bar\alpha \in \mathcal{K}_l)$ is over-complete in $\mathfrak{H}$. Namely, let $\mathcal{H}(\bar\alpha_0)$ be an orbit passing through the minimal coroot $\{H_{\alpha_0}, H_{\lambda(\alpha_0)}, \dots, H_{\lambda^{l-1}(\alpha_0)}\}$. Then the element $\mathfrak{h}^0_{\bar\alpha_0}$ is a linear combination of elements $\mathfrak{h}^0_{-\bar\alpha}$, $(\alpha \in \Pi)$ and we should exclude it from the basis. We replace the basis $\Pi^\vee$ in $\mathfrak{H}$ by

$$\mathfrak{h}^c_{\bar\alpha},\ (c \in J_\alpha),\quad \begin{cases} \bar\alpha \in \tilde{\mathcal{K}}_l = \mathcal{K}_l \backslash \mathcal{H}(\bar\alpha_0), & c = 0 \\ \bar\alpha \in \mathcal{K}_l, & c \neq 0. \end{cases} \tag{5.12}$$

As before there is a one-to-one map $\Pi^\vee \leftrightarrow \{\mathfrak{h}^c_{\bar\alpha}\}$.

The elements $(\mathfrak{h}^a_{\bar\alpha}, \mathfrak{t}^a_{\bar\alpha})$ form the GS basis in $\mathfrak{g}_{l-a}$ (5.1).

The Killing form. The Killing form in the basis (5.12) can be found from (A.23),

$$(\mathfrak{h}^a_{\bar\alpha}, \mathfrak{h}^b_{\bar\beta}) = \delta^{(a+b,0\,(mod\,l))}\mathcal{A}^a_{\alpha,\beta},\quad \mathcal{A}^a_{\alpha,\beta} = \frac{2}{(\beta,\beta)}\sum_{s=0}^{l-1}\omega^{-sa}a_{\beta,\lambda^s(\alpha)}, \tag{5.13}$$

where $a_{\alpha,\beta}$ is the Cartan matrix (A.4).

The dual basis is generated by elements $\mathfrak{H}^a_{\bar\alpha}$,

$$(\mathfrak{H}^a_{\bar\alpha}, \mathfrak{h}^b_{\bar\beta}) = \delta^{(a+b,0\,(mod\,l))}\delta_{\alpha,\beta},\quad \mathfrak{H}^a_{\bar\alpha} = \sum_{\beta\in\Pi}(\mathcal{A}^a_{\alpha,\beta})^{-1}\mathfrak{h}^{-a}_{\bar\beta},\quad \mathfrak{h}^a_{\bar\beta} = \sum_{\alpha\in\Pi}(\mathcal{A}^{-a}_{\alpha,\beta})\mathfrak{H}^{-a}_{\bar\alpha}. \tag{5.14}$$

The Killing form in the dual basis takes the form

$$(\mathfrak{H}^{a_1}_{\bar\alpha_1}, \mathfrak{H}^{a_2}_{\bar\alpha_2}) = \delta^{(a_1+a_2,0\,(mod\,l))}(\mathcal{A}^{a_1}_{\bar\alpha_1,\bar\alpha_2})^{-1}. \tag{5.15}$$

In summary, we have defined the GS-basis in $\mathfrak{g}$,

$$\{\mathfrak{t}^a_{\bar\beta}, \mathfrak{h}^c_{\bar\alpha},\quad (a, \bar\beta, c, \bar\alpha) \text{ are defined in (5.5), (5.12)}\}, \tag{5.16}$$

and the dual basis

$$\{\mathfrak{T}^a_{\bar\beta}, \mathfrak{H}^c_{\bar\alpha},\quad (a, \bar\beta, c, \bar\alpha) \text{ are defined in (5.9), (5.14)}\}, \tag{5.17}$$

along with the Killing forms.

Commutation relations. The commutation relations in the GS basis can be found from the commutation relations in the Chevalley basis (A.21). Taking into account the invariance of the structure constants with respect to the Weyl group action $C_{\lambda\alpha,\lambda\beta} = C_{\alpha,\beta}$ it is not difficult to derive the commutation relations in the GS basis using its definition in the Chevalley basis (5.4), (5.11). In the case of root-root commutators we come to the following relations:

$$[\mathfrak{t}^a_\alpha, \mathfrak{t}^b_\beta] = \begin{cases} \frac{1}{\sqrt{l}}\sum_{s=0}^{l-1}\omega^{bs}C_{\alpha,\lambda^s\beta}\,\mathfrak{t}^{a+b}_{\alpha+\lambda^s\beta}, & \alpha \neq -\lambda^s\beta \\ \frac{p_\alpha}{\sqrt{l}}\,\omega^{s\,b}\,\mathfrak{h}^{a+b}_\alpha & \alpha = -\lambda^s\beta. \end{cases} \tag{5.18}$$

The Cartan-root commutators are:

$$\left[\mathfrak{h}^k_\alpha, \mathfrak{t}^m_\beta\right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} \frac{2(\alpha, \lambda^s \beta)}{(\alpha, \alpha)} \mathfrak{t}^{k+m}_\beta,$$
$$\left[\mathfrak{H}^k_\alpha, \mathfrak{t}^m_\beta\right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} \frac{(\alpha, \alpha)}{2} (\hat{\alpha}, \lambda^s \beta)\, \mathfrak{t}^{k+m}_\beta. \tag{5.19}$$

Here we denote by $\hat{\alpha}$ the dual to the simple roots elements in the Cartan subalgebra:

$$(\hat{\alpha}_i, \beta_j) = \delta_{ij}. \tag{5.20}$$

In Sect. 7 for explicit computations with Lax operators and r-matrices, it will be much more convenient to use the following normalized basis for Cartan subalgebra:

$$\bar{\mathfrak{h}}^k_\alpha = \frac{(\alpha, \alpha)}{2} \mathfrak{h}^k_\alpha, \qquad \bar{\mathfrak{H}}^k_\alpha = \frac{2}{(\alpha, \alpha)} \mathfrak{H}^k_\alpha. \tag{5.21}$$

This reparametrization leads to the following commutation relations:

$$\left[\bar{\mathfrak{h}}^k_\alpha, \mathfrak{t}^m_\beta\right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} (\alpha, \lambda^s \beta)\, \mathfrak{t}^{k+m}_\beta,$$
$$\left[\bar{\mathfrak{H}}^k_\alpha, \mathfrak{t}^m_\beta\right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} (\hat{\alpha}, \lambda^s \beta)\, \mathfrak{t}^{k+m}_\beta. \tag{5.22}$$

The following simple formula expresses the decomposition of Cartan element in the basis of simple roots:

$$\bar{\mathfrak{h}}^k_\beta = \sum_{\alpha \in \Pi} (\hat{\alpha}, \beta)\, \bar{\mathfrak{h}}^k_\alpha, \quad \beta \in R, \tag{5.23}$$

the connection of dual bases is clear from the following expression:

$$\sum_{\beta \in \Pi} (\hat{\alpha}, \beta)\, \bar{\mathfrak{h}}^k_\beta = \sum_{\beta \in \Pi} (\alpha, \beta)\, \bar{\mathfrak{H}}^k_\beta. \tag{5.24}$$

The Cartan elements have the following symmetry property:

$$\bar{\mathfrak{h}}^k_{-\alpha} = -\bar{\mathfrak{h}}^k_\alpha, \quad \bar{\mathfrak{H}}^k_{-\alpha} = -\bar{\mathfrak{H}}^k_\alpha, \tag{5.25}$$

Invariant subalgebra. Consider the invariant subalgebra $\mathfrak{g}_0$. It is generated by the basis $(\mathfrak{t}^0_{\bar{\beta}}, \mathfrak{h}^0_{\bar{\alpha}})$ (5.16). In particular, $\{\mathfrak{h}^0_{\bar{\alpha}}\}$ (5.11), (5.12) form a basis in the Cartan subalgebra $\tilde{\mathfrak{H}}_0 \subset \mathfrak{H}$ ($\dim \tilde{\mathfrak{H}}_0 = p < n$).

We pass from $\{\mathfrak{h}^0_{\bar{\alpha}}\}$ to a special basis in $\tilde{\mathfrak{H}}_0$,

$$\tilde{\Pi}^\vee = \{\tilde{\alpha_k}^\vee \,|\, k = 1, \ldots, p\}. \tag{5.26}$$

It is constructed in the following way. Consider a subsystem of simple coroots

$$\Pi^\vee_1 = \Pi^{ext\vee} \backslash \mathcal{O}(\bar{\alpha}^\vee_0) \tag{5.27}$$

(see (5.12)). In other words, $\Pi_1^\vee$ is a subset of simple coroots that does not contain simple coroots from the orbit passing through α_0. For A_{N-1}, B_n, E_6 and E_7 the coroot basis $\tilde{\Pi}^\vee$ (5.26) is a result of averaging along the λ orbits in $\Pi_1^\vee$,

$$\tilde{\alpha}^\vee = \sum_{m=1}^{l-1} H_{\lambda^m(\alpha)}, \quad H_\alpha \in \Pi_1^\vee. \tag{5.28}$$

In the C_n and D_n cases this construction is valid for almost all coroots except the last on the Dynkin diagram (see Remark 10.1 in [48–50]). Consider the dual vectors $\tilde{\Pi} = \{\tilde{\alpha_k} \mid k = 1, \dots, p,\ \langle \tilde{\alpha_k}, \tilde{\alpha_k}^\vee \rangle = 2\}$ in $\tilde{\mathfrak{H}}_0^*$.

Proposition 5.1. *The set of vectors in* $\tilde{\mathfrak{H}}_0^*$,

$$\tilde{\Pi} = \{\tilde{\alpha_k} \mid k = 1, \dots, p\}, \tag{5.29}$$

is a system of simple roots of a simple Lie subalgebra $\tilde{\mathfrak{g}}_0 \subset \mathfrak{g}_0$ *defined by the root system* $\tilde{R} = \tilde{R}(\tilde{\Pi})$ *and the Cartan matrix* $\langle \tilde{\alpha_k}, \tilde{\alpha_j}^\vee \rangle$.

The check this statement case by case is done in [48–50].

Let $R_1 = R_1(\Pi_1)$ be a subset of roots generated by simple roots $\Pi_1 = \Pi^{ext} \backslash \mathcal{O}(\alpha_0)$. It is invariant under the λ action. The root system $\tilde{R}$ of $\tilde{\mathfrak{g}}_0$ corresponds to the λ invariant set of R_1. Consider the complementary set of roots $R \backslash R_1$ and the set of orbits

$$\mathcal{T}_l' = (R \backslash R_1)/\mu_l. \tag{5.30}$$

It is a subset of all orbits $\mathcal{T}_l = R/\mu_l$. Therefore, $\mathcal{T}_l = \tilde{R} \cup \mathcal{T}_l'$. The λ-invariant subalgebra $\mathfrak{g}_0$ contains the subspace

$$V = \{\sum_{\bar{\beta} \in \mathcal{T}_l'} a_{\bar{\beta}} \mathfrak{t}^0_{\bar{\beta}},\ \ a_{\bar{\beta}} \in \mathbb{C}\}. \tag{5.31}$$

Then $\mathfrak{g}_0$ is a sum of $\tilde{\mathfrak{g}}_0$ and V,

$$\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \oplus V. \tag{5.32}$$

The components of this decomposition are orthogonal with respect to the Killing form (5.13), and V is a representation of $\tilde{\mathfrak{g}}_0$. We find below the explicit forms of $\mathfrak{g}_0$ for all simple algebras from our list.

Let $\mathfrak{H}'$ be a subalgebra of $\mathfrak{H}$ with the basis $\mathfrak{h}^c_{\bar{\alpha}}$ $c \neq 0$ (5.11) and $\tilde{\mathfrak{H}}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}_0$. Then

$$\mathfrak{H} = \tilde{\mathfrak{H}}_0 \oplus \mathfrak{H}'. \tag{5.33}$$

We summarize the information about invariant subalgebras in Table 4.

In the invariant simple algebra $\tilde{\mathfrak{g}}_0$ instead of the basis $(\mathfrak{h}^0_{\bar{\alpha}}, \mathfrak{t}^0_{\bar{\beta}})$ we can use the Chevalley basis and incorporate it in the GS-basis,

$$\{\mathfrak{h}^0_{\bar{\alpha}}, \mathfrak{t}^0_{\bar{\beta}}\} \to \{\tilde{\mathfrak{g}}_0 = (H_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{\Pi},\ E_{\tilde{\beta}}, \tilde{\beta} \in \tilde{R}),\ V = (\mathfrak{t}^0_{\bar{\beta}}, \bar{\beta} \in \mathcal{T}')\}. \tag{5.34}$$

Remark 5.1. For any $\xi \in Q^\vee$ a solution of (3.10) is $\Lambda = Id$. In this case $\tilde{\mathfrak{g}}_0 = \mathfrak{g}$ and the GS-basis is the Chevalley basis.

The GS basis from a canonical basis in $\mathfrak{H}$. Let $(e_1, e_2, \dots, e_n)$ be a canonical basis in $\mathfrak{H}$, $((e_j, e_k) = \delta_{jk})$.[9] Since Λ preserves $\mathfrak{H}$ we can consider the action of μ_l on

[9] For A_n and E_6 root systems it is convenient to choose canonical bases in $\mathfrak{H} \oplus \mathbb{C}$.

Table 4. Invariant subalgebras $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_{\tilde{\Pi}}$ and $\mathfrak{g}_0$ of simple Lie algebras

Π	$\mathcal{Z}(\bar{G})$	$\varpi_j^\vee$	Π_1	$l=\mathrm{ord}\,(\Lambda)$	$\tilde{\mathfrak{g}}_0$	$\mathfrak{g}_0$
1	2	3	4	5	6	7
$A_{N-1}, (N = pl)$	μ_N	$\varpi_{N-1}^\vee$	$\cup_1^l A_{p-1}$	N/p	$\mathbf{sl_p}$	$\mathbf{sl_p} \oplus_{j=1}^{l-1} \mathbf{gl_p}$
B_n	μ_2	$\varpi_n^\vee$	$\mathbf{so_{2n-1}}$	2	$\mathbf{so(2n-1)}$	$\mathbf{so(2n)}$
$C_{2l},\ (l > 1)$	μ_2	$\varpi_{2l}^\vee$	A_{2l-1}	2	$\mathbf{so(2l)}$	$\mathbf{gl_{2l}}$
C_{2l+1}	μ_2	$\varpi_{2l+1}^\vee$	A_{2l}	2	$\mathbf{so(2l+1)}$	$\mathbf{gl_{2l+1}}$
$D_{2l+1},\ (l > 1)$	μ_4	$\varpi_{2l+1}^\vee$	A_{2l-2}	4	$\mathbf{so(2l-1)}$	$\mathbf{so(2l)} \oplus \mathbf{so(2l)} \oplus \underline{1}$
$D_{2l+1},\ (l > 1)$	μ_4	$\varpi_1^\vee$	D_{2l}	2	$\mathbf{so(4l-1)}$	$\mathbf{so(4l)} \oplus \underline{1}$
$D_{2l},\ (l > 2)$	$\mu_2 \oplus \mu_2$	$\varpi_{2l}^\vee$	A_{2l-1}	2	$\mathbf{so(2l)}$	$\mathbf{so(2l)} \oplus \mathbf{so(2l)}$
$D_{2l},\ (l > 2)$	$\mu_2 \oplus \mu_2$	$\varpi_1^\vee$	D_{2l-1}	2	$\mathbf{so(4l-3)}$	$\mathbf{so(4l-2)} \oplus \underline{1}$
E_6	μ_3	$\varpi_1^\vee$	D_4	3	$\mathbf{g_2}$	$\mathbf{so(8)} \oplus 2 \cdot \underline{1}$
E_7	μ_2	$\varpi_7^\vee$	$\mathbf{e_6}$	2	$\mathbf{f_4}$	$\mathbf{e_6} \oplus \underline{1}$

The coweights generating central elements are displaced in column 3

the canonical basis. Define an orbit of length $l_s = l/p_s$ passing through e_s $\mathcal{O}(s) = \{e_s, \lambda(e_s), \ldots, \lambda^{(l-1)}e_{s)}\}$.

The Fourier transform along $\mathcal{O}(s)$ takes the form

$$\mathfrak{h}_s^c = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{mc} \lambda^m(e_s), \quad c \in J_{p_s}, \quad \omega = \exp\left(\frac{2\pi i}{l}\right), \tag{5.35}$$

where $J_{p_s} = \{c = mp_s \, mod(l) \mid m \in \mathbb{Z}\}$. Consider the quotient $\mathcal{C}_l = (e_1, e_2, \ldots, e_n)/\mu_l$. Then we can pass from the canonical basis to the GS basis,

$$(e_1, e_2, \ldots, e_n) \longleftrightarrow \{\mathfrak{h}_s^c,\ s \in \mathcal{C}_l\}.$$

The Killing form is read of from (5.35),

$$(\mathfrak{h}_{s_1}^{c_1}, \mathfrak{h}_{s_2}^{c_2}) = \delta_{(s_1,s_2)} \delta^{(c_1,-c_2)}. \tag{5.36}$$

Then the dual generators are

$$\mathfrak{H}_s^c = \mathfrak{h}_s^{-c}. \tag{5.37}$$

The commutation relations in $\mathfrak{g}$ in this form of GS basis take the form

$$[\mathfrak{h}_s^{k_1}, \mathfrak{t}_{\bar{\beta}}^{k_2}] = \frac{1}{\sqrt{l}} \sum_{r=0}^{l-1} \omega^{-rk_1} \langle \lambda^r(\beta), e_s \rangle \mathfrak{t}_{\bar{\beta}}^{k_1+k_2}, \tag{5.38}$$

$$[\mathfrak{t}_{\bar{\alpha}}^{k_1}, \mathfrak{t}_{\bar{\beta}}^{k_2}] = \frac{1}{p_\alpha \sqrt{l}} \omega^{rk_2} \sum_s (\alpha^\vee, e_s) \mathfrak{h}_s^{k_1+k_2}, \quad \text{if } \alpha = -\lambda^r(\beta) \text{ for some } r.$$

We obtain the last relation from (5.4) and from the expansion $\mathfrak{h}_\alpha^k = \sum_s (\alpha^\vee, e_s) \mathfrak{h}_s^k$. Alternatively, the same relations can be written as given in (5.18)–(5.19):

$$[\mathfrak{t}^a_\alpha, \mathfrak{t}^b_\beta] = \begin{cases} \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{bs} C_{\alpha,\lambda^s\beta} \mathfrak{t}^{a+b}_{\alpha+\lambda^s\beta}, & \alpha \neq -\lambda^s\beta \\ \frac{p_\alpha}{\sqrt{l}} \omega^{s\,b} \mathfrak{h}^{a+b}_\alpha & \alpha = -\lambda^s\beta, \end{cases}$$

$$\left[\mathfrak{h}^k_\alpha, \mathfrak{t}^m_\beta\right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} \frac{2(\alpha, \lambda^s\beta)}{(\alpha,\alpha)} \mathfrak{t}^{k+m}_\beta,$$

$$\left[\mathfrak{H}^k_\alpha, \mathfrak{t}^m_\beta\right] = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks} \frac{(\alpha,\alpha)}{2} (\hat{\alpha}, \lambda^s\beta)\, \mathfrak{t}^{k+m}_\beta.$$

6. General Description of Systems with Non-trivial Characteristic Classes

The Lax operators and symplectic Hecke correspondence. Consider a meromorphic section Φ of the adjoint bundle $End\, E_G \otimes K$, where K is a canonical class. Φ is called a Higgs field. The set of pairs (Φ, E_G) defines a cotangent bundle to the space of holomorphic bundles equipped with a canonical symplectic structure. Evidently, the gauge transformations (2.5) can be lifted to the cotangent bundle as canonical transformations with respect to the symplectic form. The hamiltonian reduction of the cotangent bundle under this action leads to the Higgs bundles $Hig(\mathcal{M}_g)$. The Higgs bundles are principle homogeneous spaces over the cotangent bundles to the moduli space of holomorphic bundles over Σ_g. The Higgs bundles are phase spaces of the Hitchin integrable systems [27], and the Higgs field becomes the Lax operator L. This construction is valid for curves with marked points. In this case we deal with the Higgs bundle with quasi-parabolic structures at the marked points. It implies that the Lax operators have first order poles at the marked poles with residues belonging to generic coadjoint orbits $\mathcal{O}$. The coadjoint orbits are affine spaces over the flag varieties mentioned in Sect. 2. The dimension of the Higgs bundle $Hig(\mathcal{M}_{g,n})$ is twice that of dim $\mathcal{M}_{g,n}$ (2.7),

$$\dim(Hig(\mathcal{M}_g)) = 2(g-1)\dim(G) + n\dim(\mathcal{O}).$$

Below we consider the case $g = 1, n = 1$. Then the phase space has dimension of a coadjoint orbit (A.26) $2\sum_{j=1}^{rankG}(d_j - 1)$. In this case L satisfies the conditions

$$L(z+1) = \mathcal{Q}L(z)\mathcal{Q}^{-1}, \quad L(z+\tau) = \Lambda L(z)\Lambda^{-1}, \tag{6.1}$$

where $\mathcal{Q}$ and Λ are solutions of (3.3), and

$$\bar{\partial}L(z) = \mathbf{S}\delta(z,\bar{z}). \tag{6.2}$$

In other words $Res|_{z=0}\, L(z) = \mathbf{S}$. These conditions fix L.

To make dependence on the characteristic class $\zeta(E_G)$ explicit we will write $L(z)^{\varpi_j^\vee}$, if the Lax matrix satisfies the quasi-periodicity conditions with $\Lambda = \Lambda_{\varpi_j^\vee}$, $\mathcal{Q}_{\varpi_j^\vee}$, where $\Lambda_{\varpi_j^\vee}$ $\mathcal{Q}_{\varpi_j^\vee}$ are solutions of (3.3) with $\zeta = \mathbf{e}(-\varpi_j^\vee)$, $\varpi_j^\vee \in P^\vee$.

The modification $\Xi(\gamma)$ of E_G changes the characteristic class (2.12). It acts on $L^{\varpi_j^\vee}$ as follows

$$L^{\varpi_j^\vee}\,\Xi(\gamma) = \Xi(\gamma)L^{\varpi_j^\vee+\gamma}. \tag{6.3}$$

It is the singular symplectic transformation mentioned in the Introduction.

The action (6.3) allows one to write down the condition on $\Xi(\gamma)$. Since $L^{\varpi_j^\vee}$ has a simple pole at $z=0$ the modified Lax matrix $L^{\varpi_j^\vee+\gamma}$ should have also a simple pole at $z=0$. Decompose $L^{\varpi_j^\vee}$ and $L^{\varpi_j^\vee+\gamma}$ in the Chevalley basis (A.19), (A.20),

$$L^{\varpi_j^\vee} = L_{\mathfrak{H}}(z) + \sum_{\alpha\in R} L_\alpha(z)E_\alpha, \quad L^{\varpi_j^\vee+\gamma} = \tilde{L}_{\mathfrak{H}}(z) + \sum_{\alpha\in R} \tilde{L}_\alpha(z)E_\alpha.$$

Expand α in the basis of simple roots (A.2) $\alpha = \sum_{j=1}^l f_j^\alpha \alpha_j$ and γ in the basis of fundamental coweights $\gamma = \sum_{j=1}^l m_j\varpi_j^\vee$. Assume that $\langle\gamma,\alpha_j\rangle \geq 0$ for simple α_j. In other words γ is a dominant coweight. Then $\langle\gamma,\alpha\rangle = \sum_{j=1}^l m_j n_j^\alpha$ is an integer number, positive for $\alpha\in R^+$ and negative for $\alpha\in R^-$. From (2.11) and (6.3) we find

$$L_{\mathfrak{H}}^{\varpi_j^\vee+\gamma}(z) = L_{\mathfrak{H}}^{\varpi_j^\vee}(z), \quad L_\alpha^{\varpi_j^\vee+\gamma}(z) = z^{\langle\gamma,\alpha\rangle}L_\alpha^{\varpi_j^\vee}(z). \tag{6.4}$$

In a neighborhood of $z=0$, $L_\alpha(z)$ should have the form

$$L_\alpha^{\varpi_j^\vee}(z) = a_{\langle\gamma,\alpha\rangle}z^{-\langle\gamma,\alpha\rangle} + a_{\langle\gamma,\alpha\rangle+1}z^{-\langle\gamma,\alpha\rangle+1} + \cdots, \quad (\alpha\in R^-), \tag{6.5}$$

otherwise the transformed Lax operator becomes singular. It means that the type of the modification γ is not arbitrary, but depends on the local behavior of the Lax operator. It allows one to find the dimension of the space of the Hecke transformation. We do not need it here.

Now consider a global behavior of $L(z)$ (6.1). Then we find that $\Xi(\gamma)$ should intertwine the quasi-periodicity conditions

$$\Xi(\gamma,z+1)\mathcal{Q}_{\varpi_j^\vee} = \mathcal{Q}_{\varpi_j^\vee+\gamma}\Xi(\gamma,z), \quad \Xi(\gamma,z+\tau)\Lambda_{\varpi_j^\vee} = \Lambda_{\varpi_j^\vee+\gamma}\Xi(\gamma,z).$$

For $G=\mathrm{SL}(N,\mathbb{C})$, $\gamma=\varpi_1^\vee$ and the special residue of L solutions of these equations were found in [35].

The Lax matrix. Explicit form. Assume that L has a residue at $z=0$ taking values in a coadjoint orbit $\mathcal{O}\subset\mathfrak{g}^*$,

$$\begin{aligned} Res\, L|_{z=0} = \mathbf{S} &= \sum_{\alpha\in\Pi}\frac{1}{2}(\alpha,\alpha)S_\alpha^{\mathfrak{H}}\sum_{\beta\in\Pi}a_{\alpha,\beta}^{-1}H_\beta + \sum_{\beta\in R}S_\beta^{\mathfrak{L}}\frac{(\beta,\beta)}{2}E_{-\beta} \\ &= \sum_{j=1}^n S_je_j + \sum_{\beta\in R}S_\beta^{\mathfrak{L}}\frac{(\beta,\beta)}{2}E_{-\beta}. \end{aligned} \tag{6.6}$$

We identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by means the Killing form (A.23), (A.24). Then the coordinates are linear functionals on $\mathfrak{g}$,

$$S_\alpha^{\mathfrak{H}} = (\mathbf{S},H_\alpha), \text{ or } S_j = (\mathbf{S},e_j)), \; S_\beta^{\mathfrak{L}} = (\mathbf{S},E_\beta). \tag{6.7}$$

The Poisson brackets for $S_\alpha^{\mathfrak{H}}$, (S_j), $S_\beta^{\mathfrak{L}}$ have the same structure constants as $\mathfrak{g}$ (A.21). To define a generic orbit $\mathcal{O}$ we fix the Casimir functions $C_j(\mathbf{S})$.

Rewrite (6.6) in the dual GS-basis. We use the gradation (5.1) to define the Lax matrix

$$L(z)=\sum_{a=0}^{l-1}L_a(z), \tag{6.8}$$

where the zero component is decomposed according to (5.32) $L_0(z)=\tilde{L}_0(z)+L'_0(z)$. Then

$$\mathbf{S}=Res\,L|_{z=0}=Res\,\tilde{L}_0|_{z=0}+Res\,L'_0|_{z=0}+\sum_{a=1}^{l-1}Res\,L_a|_{z=0}=\tilde{\mathbf{S}}_0+\mathbf{S}'_0+\sum_{a=1}^{l-1}\mathbf{S}_a,$$

where

$$\tilde{\mathbf{S}}_0=\sum_{\tilde{\alpha},\tilde{\beta}\in\tilde{\Pi}}\tilde{S}^{\mathfrak{H}}_{\tilde{\alpha}}\frac{(\alpha,\alpha)(\beta,\beta)}{4(\tilde{\alpha},\tilde{\beta})}H_{\tilde{\beta}}+\sum_{\tilde{\beta}\in\tilde{R}}\tilde{S}^{\mathfrak{L}}_{\tilde{\beta}}\frac{(\tilde{\beta},\tilde{\beta})}{2}E_{-\tilde{\beta}},$$
$$\mathbf{S}'_0=\sum_{\bar{\beta}\in\mathcal{T}'_l}S'_{\bar{\beta}}\mathfrak{T}^0_{\bar{\beta}},\quad \mathbf{S}_a=\sum_{\bar{\alpha}\in\mathcal{K}_l}S^{\mathfrak{H},a}_{\bar{\alpha}}\mathfrak{H}^{l-a}_{\bar{\alpha}}+\sum_{\bar{\beta}\in\mathcal{T}_l}S^{\mathfrak{L},a}_{\bar{\beta}}\mathfrak{T}^{l-a}_{-\bar{\beta}}, \tag{6.9}$$

(see (5.30), (5.31)). Again, as in (6.7), the coordinates are defined as

$$S^{\mathfrak{H},a}_{\bar{\alpha}}=(\mathbf{S},\mathfrak{h}^a_{\bar{\alpha}}),\quad S'_{\bar{\beta}}=(\mathbf{S},\mathfrak{t}^0_{\bar{\beta}}),\quad S^{\mathfrak{L},a}_{\bar{\beta}}=(\mathbf{S},\mathfrak{t}^a_{\bar{\beta}})\quad \tilde{S}^{\mathfrak{H}}_{\tilde{\alpha}}=(\mathbf{S},H_{\tilde{\alpha}}),\quad S^{\mathfrak{L}}_{\tilde{\beta}}=(\mathbf{S},E_{\tilde{\beta}}). \tag{6.10}$$

We also will use another basis in $\mathfrak{H}$ (5.21). Then

$$\bar{S}^{\mathfrak{H},a}_{\bar{\alpha}}=(\mathbf{S},\bar{\mathfrak{h}}^a_{\bar{\alpha}}) \tag{6.11}$$

have the structure constants of the Poisson brackets as in (5.18), (5.19), (5.38). We can pass from one data to another by the Fourier transform introduced above. We rewrite $\tilde{\mathbf{S}}_0$ in terms of a canonical basis $(e_1,\ldots,e_p)$ in the invariant Cartan algebra $\tilde{\mathfrak{h}}_0$,

$$\tilde{\mathbf{S}}_0=\sum_{j=1}^{p}\tilde{S}^{\mathfrak{H}}_j e_j+\sum_{\tilde{\beta}\in\tilde{R}}\tilde{S}^{\mathfrak{L}}_{\tilde{\beta}}\frac{(\tilde{\beta},\tilde{\beta})}{2}E_{-\tilde{\beta}}. \tag{6.12}$$

It follows from (5.6), (5.7) and from the definition of the dual basis (5.9), (5.14) that

$$Ad_{\Lambda}(\mathfrak{T}^c_{\bar{\beta}})=\mathbf{e}\,(\frac{c}{l}-\langle\tilde{\mathbf{u}},\beta\rangle)\mathfrak{T}^c_{\bar{\beta}},\quad Ad_{\Lambda}(\mathfrak{H}^c_{\bar{\beta}})=\mathbf{e}\,(\frac{c}{l})\mathfrak{H}^c_{\bar{\beta}},\quad (\mathbf{e}\,(x)=\exp\,(2\pi i x)).$$

In addition, we have

$$Ad_{\mathcal{Q}}(\mathfrak{H}^c_{\bar{\beta}})=\mathfrak{H}^c_{\bar{\beta}},\quad Ad_{\mathcal{Q}}(H_{\tilde{\alpha}})=H_{\tilde{\alpha}}, \tag{6.13}$$
$$Ad_{\mathcal{Q}}(\mathfrak{T}^c_{\bar{\beta}})=\mathbf{e}\,(-\langle\kappa,\beta\rangle)\mathfrak{T}^c_{\bar{\beta}},\quad Ad_{\mathcal{Q}}(E_{\tilde{\alpha}})=\mathbf{e}\,\langle\kappa,\tilde{\alpha}\rangle E_{\tilde{\alpha}}. \tag{6.14}$$

Using (A.14) we obtain $\langle\kappa,\alpha\rangle=f_\alpha/h$. Then the last relation assumes the form

$$Ad_{\mathcal{Q}}(\mathfrak{T}^c_{\bar{\beta}})=\mathbf{e}\,(-f_\beta/h)\mathfrak{T}^c_{\bar{\beta}},\quad Ad_{\mathcal{Q}}(E_{\tilde{\alpha}})=\mathbf{e}\,(f_\alpha/h)E_{\tilde{\alpha}}. \tag{6.15}$$

There are also the evident relations:

$$Ad_\Lambda(E_{\tilde\alpha}) = \mathbf{e}\,(\langle\tilde{\mathbf{u}},\tilde\alpha\rangle)E_{\tilde\alpha},\quad Ad_\Lambda(H_{\tilde\alpha}) = H_{\tilde\alpha},\quad \tilde{\mathbf{u}}\in\tilde{\mathfrak{H}}.$$

The quasi-periodicity conditions and the existence of pole at $z = 0$ dictate the form of the components for $a \neq 0$. We define the matrix element of Lax the operator using $\phi(u, z)$ (B.3). Let

$$\varphi^a_\beta(\mathbf{x}, z) = \mathbf{e}\,(\langle\kappa,\beta\rangle z)\phi(\langle\mathbf{x}+\kappa\tau,\beta\rangle + \frac{a}{l}, z) = \mathbf{e}\,(zf_\beta/h)\phi(\langle\mathbf{x},\beta\rangle + \tau f_\beta/h + \frac{a}{l}, z). \tag{6.16}$$

The last equality follows from the identity $\langle\kappa,\alpha\rangle = \frac{1}{h}\langle\rho^\vee,\alpha\rangle = f_\alpha/h$ (see (A.14)). It follows from (B.6) that $\varphi^a_\beta(\mathbf{x}, z+1) = \mathbf{e}\,(\langle\kappa,\beta\rangle)\varphi^a_\beta(\mathbf{x}, z)$, $\varphi^a_\beta(\mathbf{x}, z+\tau) = \mathbf{e}\,(-\langle x,\beta\rangle - \frac{a}{l})\varphi^a_\beta(\mathbf{x}, z)$.

Then from (6.1) we find

$$L_a(z) = \sum_{\tilde\alpha\in\mathcal{K}_l} S^{\mathfrak{H},a}_{\tilde\alpha}\phi(\frac{a}{l}, z)\mathfrak{H}^{l-a}_{\tilde\alpha} + \sum_{\bar\beta\in\mathcal{T}_l} S^{\mathfrak{L},a}_{\bar\beta}\varphi^a_\beta(-\tilde{\mathbf{u}}, z)\mathfrak{T}^{l-a}_{-\bar\beta}, \tag{6.17}$$

and $L'_0(z) = \sum_{\bar\alpha\in\mathcal{T}'_l} S'_{\bar\alpha}\varphi^0_\alpha(-\tilde{\mathbf{u}}, z)\mathfrak{T}^0_{-\bar\alpha}$.

In the canonical basis in $\mathfrak{H}$ (5.35), $L_a(z)$ takes the form

$$L_a(z) = \sum_{s\in\mathcal{C}_l} S^{\mathfrak{H},a}_s\phi(\frac{a}{l}, z)\mathfrak{h}^{-a}_s + \sum_{\bar\beta\in\mathcal{T}_l} S^{\mathfrak{L},a}_{\bar\beta}\varphi^a_\beta(-\tilde{\mathbf{u}}, z)\mathfrak{T}^{l-a}_{-\bar\beta}.$$

It follows from (6.16), (B.4) and (B.5), and that $L_a(z)$ has the required quasi-periodicities and the residues.

We replace the basis on the dual basis using (5.9) and (5.37) and finally obtain

$$L_a(z) = \sum_{s\in\mathcal{C}_l} S^{\mathfrak{H},a}_s\phi(\frac{a}{l}, z)\mathfrak{h}^{-a}_s + \sum_{\bar\beta\in\mathcal{T}_l} S^{\mathfrak{L},a}_{\bar\beta}\varphi^a_\beta(-\tilde{\mathbf{u}}, z)\mathfrak{t}^a_{\bar\beta}\frac{(\beta,\beta)}{2p_\beta}, \tag{6.18}$$

$$L'_0(z) = \sum_{\bar\alpha\in\mathcal{T}'_l} S'_{\bar\alpha}\varphi^0_\alpha(-\tilde{\mathbf{u}}, z)\mathfrak{t}^0_{\bar\alpha}\frac{(\alpha,\alpha)}{2p_\alpha}. \tag{6.19}$$

Consider the invariant subalgebra $\tilde{\mathfrak{g}}_0$. For $\tilde{\mathfrak{g}}_0$ we write down the Lax matrix in the Chevalley basis. Let $p \leq n$ be a rank of $\tilde{\mathfrak{g}}_0$, $(e_1, \ldots, e_p)$ is a canonical basis in $\tilde{\mathfrak{H}}_0$, and $E_{\tilde\alpha}$ are generators of the root subspaces. The matrix elements of $\tilde{L}_0$ are constructed by means of φ^0_β (6.16) and the Eisenstein functions (B.9):

$$\tilde{L}_0(z) = \sum_{j=1}^{p}(v_j + \tilde{S}^{\mathfrak{H}}_j E_1(z))e_j + \sum_{\tilde\beta\in\tilde{R}} \tilde{S}^{\mathfrak{L}}_{\tilde\beta}\varphi^0_{\tilde\alpha}(-\tilde{\mathbf{u}}, z)E_{\tilde\beta}. \tag{6.20}$$

Here $\tilde{\mathbf{v}} = (v_1, \ldots, v_p)$ are momenta vectors dual to $\tilde{\mathbf{u}} = (u_1, \ldots, u_p)$. The Lax operator (6.20) differs from the standard Lax operator related to $\tilde{\mathfrak{g}}_0$:

$\tilde{L}_0(z) = \sum_{j=1}^{p}(v_j + \tilde{S}^{\mathfrak{H}}_j E_1(z))e_j + \sum_{\tilde\alpha\in\tilde{R}} \tilde{S}^{\mathfrak{L}}_{\tilde\alpha}\phi(\langle-\tilde{\mathbf{u}},\tilde\alpha\rangle, z)E_{\tilde\alpha}$. It is gauge equivalent to the previous one after $\tilde{\mathbf{u}} \to \tilde{\mathbf{u}} + \kappa$. For this reason we call $\tilde{\mathfrak{g}}_0$ *the unbroken subalgebra*. The operator $\tilde{L}_0(z)$ has the needed residue (see (B.4) and (B.14)). However, the Cartan term containing $E_1(z)$ breaks the quasi-periodicities (see (B.13)), because there are no double-periodic functions with one pole on Σ_τ. To go around this problem we use the Poisson reduction procedure.

The Lax matrix. Poisson reduction. The Lax element we have defined depends on the spin variables representing an orbit $\mathcal{O}$, on the vector $\tilde{\mathbf{u}}$ in the moduli space described in Sect. 5, and the dual covector $\tilde{\mathbf{v}}$. It is a Poisson manifold $\mathbf{P}$ with the canonical brackets for $\tilde{\mathbf{v}}$, $\tilde{\mathbf{u}}$ and the Poisson-Lie brackets for $\mathbf{S}$.

$$\mathbf{P} = T^*C \times \mathcal{O} = \{\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S}\}, \quad \tilde{\mathbf{u}} \in C, \quad \mathbf{S} \in \mathcal{O}. \tag{6.21}$$

It has dimension $\dim(\mathcal{O}) + 2\dim(\tilde{\mathfrak{H}}_0)$.

Consider the Poisson algebra $\mathcal{A} = C^\infty(\mathbf{P})$ of smooth function on $\mathbf{P}$. Let $\epsilon \in \tilde{\mathfrak{H}}_0$ and γ be a small contour around $z = 0$. Consider the following function $\mu_\epsilon = \oint_\gamma (\epsilon, L(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S})) = (\epsilon, \mathbf{S}_0^{\mathfrak{H}})$, $\mathbf{S}_0^{\mathfrak{H}} = \sum_{j=1}^p S_j^{\mathfrak{H}} e_j$. It generates the vector field on $\mathbf{P}$ $V_\epsilon : L(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S}) \to \{\mu_\epsilon, L(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S})\} = [\epsilon, L(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S})]$.

Let $\mathcal{A}^{inv}$ be an invariant subalgebra of $\mathcal{A}$ under the V_ϵ action. Then $I = \{\mu_\epsilon F(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S}) \mid F(\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S}) \in \mathcal{A}\}$ is the Poisson ideal in $\mathcal{A}^{inv}$. The reduced Poisson algebra is the factor-algebra

$$\mathcal{A}^{red} = \mathcal{A}^{inv}/I = \mathcal{A}//\tilde{\mathcal{H}}_0, \quad (\tilde{\mathcal{H}}_0 = \exp \tilde{\mathfrak{H}}_0).$$

The reduced Poisson manifold $\mathbf{P}^{red}$ is defined by the moment constraint $\tilde{S}_s^{\mathfrak{H}} = 0$ and $\dim \tilde{\mathfrak{H}}$ gauge fixing constraints on the spin variables that we do not specify,

$$\mathbf{P}^{red} = \mathbf{P}//\tilde{\mathcal{H}}_0 = \mathbf{P}(\tilde{S}_s^{\mathfrak{H}} = 0)/\tilde{\mathcal{H}}_0, \quad \dim(\mathbf{P}^{red}) = \dim(\mathbf{P}) - 2\dim(\tilde{\mathcal{H}}_0) = \dim(\mathcal{O}). \tag{6.22}$$

Due to the moment constraints we come from (6.20) to the Lax operator that has the correct periodicity. It depends on variables $\{\tilde{\mathbf{v}}, \tilde{\mathbf{u}}, \mathbf{S}\} \in \mathbf{P}^{red}$,

$$\tilde{L}_0^{red}(z) = \sum_{j=1}^p v_j e_j + \sum_{\tilde{\beta} \in \tilde{R}} S_{\tilde{\beta}}^{\mathfrak{L}} \varphi_{\tilde{\alpha}}^0(-\tilde{\mathbf{u}}, z) E_{\tilde{\beta}}. \tag{6.23}$$

Here $S_{\tilde{\beta}}^{\mathfrak{L}}$ are not free due to the gauge fixing.

Thus, after the reduction we come to the Poisson manifold that has dimension of the coadjoint orbit $\mathcal{O}$, but the Poisson structure on $\mathbf{P}^{red}$ is not the Lie-Poisson structure. The Poisson brackets on $\mathbf{P}^{red}$ are the Dirac brackets [12,13].

Hamiltonians. To find an integrable hierarchy we construct on the phase space $\mathbf{P}^{red}$ a family of independent commuting integrals. For this purpose consider the ring S^W of invariant polynomials on $\mathfrak{H}$ with the basis $P_1, P_2, \ldots, P_n$ (A.5). It follows from the RLL relations (see below (7.8)) that $P_j(L(z))$ generate commuting integrals. They are double periodic meromorphic functions on Σ_τ and thereby can be expanded in the basis of elliptic functions,

$$\frac{1}{m_j} P_j(L(z)) = I_{j,0} + I_{j,2} E_2(z) + \cdots + I_{j,j} E_j(z).$$

The coefficients $I_{j,k}$ $(0 \leq k \leq m_j,\ k \neq 1)$ become commuting independent integrals. The highest order coefficients $I_{j,j}$ are the Casimir functions fixing the orbits. The coefficient $I_{j,1}$ vanishes, because there are no double periodic functions with one simple pole.

The number of remaining coefficients is equal to $\frac{1}{2}\dim(\mathcal{O})$. Thus, on $\mathbf{P}^{red}$ the system becomes completely integrable.

Consider the Hamiltonian $H = I_{1,0}$ coming from the expansion $\frac{1}{2}P_1(L(z)) = H + I_{2,2}E_2(z)$. We represent it in the form

$$H = \tilde{H}_0 + H' + \sum_{a=1}^{M} H_a, \quad M = \left[\frac{l}{2}\right]. \tag{6.24}$$

Due to the orthogonality of L_a and L_b ($a \neq -b \bmod l$) with respect to the Killing form the Hamiltonians $\tilde{H}$, H' and H_k are determined by pairing of the corresponding Lax operators,

$$\tilde{H}_0 = \frac{1}{2}(\tilde{L}_0(z), \tilde{L}_0(z))|_{const}, \quad H' = \frac{1}{2}(L_0'(z), L_0'(z))|_{const},$$
$$H_a = \frac{1}{2}(L_a(z), L_{l-a}(z))|_{const}.$$

To calculate the Hamiltonians we use (5.10), (5.36) (B.15). Then we come to the following expressions[10]

$$\tilde{H}_0 = \frac{1}{2}\sum_{s=1}^{\tilde{n}} v_s^2 - \sum_{\tilde{\beta}\in\tilde{R}} \frac{1}{(\tilde{\beta}, \tilde{\beta})} \tilde{S}_{\tilde{\beta}}^{\mathfrak{L}} \tilde{S}_{-\tilde{\beta}}^{\mathfrak{L}} E_2(\langle \tilde{\mathbf{u}} - \kappa\tau, \tilde{\beta}\rangle). \tag{6.25}$$

As it was noted above $\tilde{H}_0$ is the elliptic CM Hamiltonian related to $\tilde{\mathfrak{g}}_0$.

Using (5.8) we find

$$H' = -\sum_{\bar{\alpha},\bar{\beta}\in\mathcal{T}_l',} \delta_{\beta=-\lambda^r(\alpha)} \frac{(\beta,\beta)}{p_\beta} S_{\bar{\alpha}}' S_{\bar{\beta}}' E_2(\langle \tilde{\mathbf{u}} - \kappa\tau, \beta\rangle). \tag{6.26}$$

Similarly, from (5.15), (5.10) and (5.13),

$$H_a = -\frac{1}{2}\sum_{s\in\mathcal{C}_l} E_2\left(\frac{a}{l}\right) S_s^{\mathfrak{H},a} S_s^{\mathfrak{H},l-a}$$
$$- \sum_{\bar{\alpha},\bar{\beta}\in\mathfrak{T}_l} \delta_{\alpha,-\lambda^r(\beta)} \omega^{-ar} \frac{(\alpha,\alpha)}{p_\alpha} S_{\bar{\beta}}^{\mathfrak{L},a} S_{\bar{\alpha}}^{\mathfrak{L},l-a} E_2(\langle \tilde{\mathbf{u}} - \kappa\tau, \beta\rangle). \tag{6.27}$$

The Hamiltonians H', H_a are the Hamiltonians of the EA tops with the inertia tensors depending on $\tilde{\mathbf{u}}$.

On the reduced space $\mathbf{P}^{red}$ the equations of motion corresponding to integrals I_{jk} acquire the Lax form $\partial_{t_{jk}} L = [L, M_{jk}]$. The operator M_{jk} is reconstructed from L and the r-matrix defined below as in [4].

7. Classical RLL-Relation and Classical Dynamical Yang-Baxter Equation

Using the commutation relations in the GS basis (5.18)–(5.19) we find the Poisson-Lie brackets on $\mathbf{P}$ (6.21):

---

[10] In what follows we shall use the standard CM Hamiltonians, where the coordinate vector is shifted $\tilde{\mathbf{u}} \to \tilde{\mathbf{u}} + \kappa\tau$.

$$
\begin{aligned}
&\left\{S^{\mathfrak{L},a}_{\alpha}, S^{\mathfrak{L},b}_{\beta}\right\} = \begin{cases} \frac{1}{\sqrt{l}} \sum\limits_{s=0}^{l-1} \omega^{bs}\, C_{\alpha,\lambda^s\beta}\, S^{\mathfrak{L},a+b}_{\alpha+\lambda^s\beta}, & \alpha \neq -\lambda^s\beta \\ \frac{p_\alpha}{\sqrt{l}}\, \omega^{s\,b}\, S^{\mathfrak{h},a+b}_{\alpha}, & \alpha = -\lambda^s\beta \end{cases} \\
&\left\{\bar{S}^{\mathfrak{H}\,k}_{\alpha}, S^{\mathfrak{L},m}_{\beta}\right\} = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} \omega^{-ks}\, (\hat{\alpha}, \lambda^s\beta)\, S^{\mathfrak{L},k+m}_{\beta} \\
&\left\{\bar{v}^{\mathfrak{H}}_{\alpha}, u_\beta\right\} = \frac{1}{\sqrt{l}} \sum_{s=0}^{l-1} (\hat{\alpha}, \lambda^s\beta) \\
&\left\{v_\alpha, S^{\mathfrak{L},a}_{\alpha}\right\} = \left\{v_\alpha, S^{\mathfrak{H},a}_{\alpha}\right\} = \left\{v_\alpha, S^{\mathfrak{h},a}_{\alpha}\right\} = 0 \\
&\left\{u_\alpha, S^{\mathfrak{L},a}_{\alpha}\right\} = \left\{u_\alpha, S^{\mathfrak{H},a}_{\alpha}\right\} = \left\{u_\alpha, S^{\mathfrak{h},a}_{\alpha}\right\} = 0
\end{aligned}
\tag{7.1}
$$

We demonstrate here that these relations can be reformulated in the form of the RLL relations. To this end define the classical dynamical r-matrix using GS basis:

$$
r(z,w) = r_{\mathcal{L}}(z,w) + r_{\mathfrak{H}}(z,w), \tag{7.2}
$$

where

$$
\begin{aligned}
r_{\mathcal{L}}(z,w) &= \frac{1}{2} \sum_{a=0}^{l-1} \sum_{\alpha \in R} |\alpha|^2\, \varphi^a_\alpha(\tilde{\mathbf{u}}, z-w)\, \mathfrak{t}^a_\alpha \otimes \mathfrak{t}^{-a}_{-\alpha}, \\
r_{\mathfrak{H}}(z,w) &= \sum_{a=0}^{l-1} \sum_{\alpha \in \Pi} \phi(\frac{a}{l}, z-w)\, \mathfrak{H}^a_\alpha \otimes \mathfrak{h}^{-a}_\alpha.
\end{aligned}
\tag{7.3}
$$

The defined above L-operator has a form:

$$
L(z) = L_R(z) + L_{\mathfrak{H}}(z) + L^0_{\mathfrak{H}}(z), \tag{7.4}
$$

with

$$
\begin{aligned}
&L_R(z) = \frac{1}{2} \sum_{a=0}^{l-1} \sum_{\beta \in R} |\beta|^2\, \varphi^a_\beta(\tilde{\mathbf{u}}, z)\, S^{\mathfrak{L},-a}_{-\beta}\, \mathfrak{t}^a_\beta, \\
&L_{\mathfrak{H}}(z) = \sum_{a=1}^{l-1} \sum_{\alpha \in \Pi} \phi(\frac{a}{l}, z)\, S^{\mathfrak{H},-a}_\alpha\, \mathfrak{h}^a_\alpha, \quad L^0_{\mathfrak{H}}(z) = \sum_{\alpha \in \Pi} \left(v^{\mathfrak{H}}_\alpha + E_1(z)\, S^{\mathfrak{H},0}_\alpha\right) \mathfrak{h}^0_\alpha.
\end{aligned}
\tag{7.5}
$$

We prove two statements:

Proposition 7.1. *The r-matrix (7.2)–(7.3) and the Lax operator (7.4)–(7.5) described above define the Poisson brackets (7.1) via RLL-equation:*

$$\{L(z)\otimes 1,\ 1\otimes L(w)\} = [L(z)\otimes 1 + 1\otimes L(w), r(z,w)]$$
$$-\frac{\sqrt{l}}{2}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\,\partial_{\tilde{\mathbf{u}}}\,\varphi_\alpha^k(\tilde{\mathbf{u}}, z-w)\,\bar{S}_\alpha^{\mathfrak{h}0}\,t_\alpha^k\otimes t_{-\alpha}^{-k}. \qquad (7.6)$$

Here L has the $\tilde{\mathfrak{g}}_0$ part $\tilde{L}_0$ (6.20).

The Jacoby identity for the brackets (7.6) is provided by the following statement:

Proposition 7.2. *The r-matrix (7.3) satisfies the classical dynamical Yang-Baxter equation:*

$$[r_{12}(z,w), r_{13}(z,x)] + [r_{12}(z,w), r_{23}(w,x)] + [r_{13}(z,x), r_{23}(w,x)]$$
$$-\sqrt{l}\sum_{k=0}^{l-1}\sum_{\alpha\in R}\frac{|\alpha|^2}{2}\mathfrak{t}_\alpha^k\otimes\mathfrak{t}_{-\alpha}^{-k}\otimes\bar{\mathfrak{h}}_\alpha^0\,\partial_{\tilde{\mathbf{u}}}\varphi_\alpha^k(\tilde{\mathbf{u}}, z-w) - \frac{|\alpha|^2}{2}\mathfrak{t}_\alpha^k\otimes\bar{\mathfrak{h}}_\alpha^0\otimes\mathfrak{t}_{-\alpha}^{-k}\,\partial_{\tilde{\mathbf{u}}}\varphi_\alpha^k(\tilde{\mathbf{u}}, z-x)$$
$$+\frac{|\alpha|^2}{2}\bar{\mathfrak{h}}_\alpha^0\otimes\mathfrak{t}_\alpha^k\otimes\mathfrak{t}_{-\alpha}^{-k}\,\partial_{\tilde{\mathbf{u}}}\varphi_\alpha^k(\tilde{\mathbf{u}}, w-x) = 0. \qquad (7.7)$$

The last term in (7.6) prevents the system to be integrable on $\mathbf{P}$. As explained, after reduction with respect to $\tilde{\mathcal{H}}_0$ (6.22) we come to $\mathbf{P}^{red}$. On $\mathbf{P}^{red}$ this term vanishes. Then (7.7) becomes the standard classical Yang-Baxter equation, providing integrability

$$\left\{L^{red}(z)\otimes 1,\ 1\otimes L^{red}(w)\right\} = \left[L^{red}(z)\otimes 1 + 1\otimes L^{red}(w), \tilde{r}(z,w)\right]. \qquad (7.8)$$

Here the r- matrix is replaced on $\tilde{r}$, because the Poisson structure on $\mathbf{P}^{red}$ differs from the Poisson structure on $\mathbf{P}$. We don't need its explicit form. Note that $L^{red}(z)$ has the $\tilde{\mathfrak{g}}_0$ part $\tilde{L}_0^{red}$ (6.20).

The classical dynamical r-matrices corresponding to trivial bundles were found in [17]. In this case the dynamical parameter $\mathbf{u}$ belongs to the Cartan subalgebra $\mathfrak{H}\subset\mathfrak{g}$. The problem of classifications of r-matrices if $\mathbf{u}\in\tilde{\mathfrak{H}}\subset\mathfrak{H}$ was formulated in [17]. For trigonometric r-matrices without the spectral parameter it was done in [59]. Here we give a classification of such types of r-matrices based on a topological classification of stable holomorphic bundles.

We omit the proofs of these statements because they are long and straightforward. They can be found in [48–50].

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8. Appendix A. Simple Lie Groups. Facts and Notations, [9,54]

Roots and weights.

V - a vector space over $\mathbb{R}$, dim $V = n$.

V^* is its dual and $\langle\ ,\ \rangle$ is a pairing between V and V^*. $R = \{\alpha\}$ is a root system in V^*.

The dual system $R^\vee = \{\alpha^\vee\}$ is the root system in V.

If V and V^* are identified by a scalar product $(\ ,\ )$, then $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$.

The group of automorphisms of V^* generated by reflections

$$s_\alpha \ : \ x \mapsto x - \langle x, \alpha\rangle\alpha^\vee \tag{A.1}$$

the Weyl group $W(R)$.

Simple roots $\Pi = (\alpha_1, \ldots, \alpha_l)$ form a basis in R,

$$\alpha = \sum_{j=1}^{n} f_j^\alpha \alpha_j, \quad f_j^\alpha \in \mathbb{Z}, \tag{A.2}$$

and all f_j^α are positive (in this case $\alpha \in R^+$ is a positive root), or negative (α is a negative root). $R = R^+ \cup R^-$.

The level of α is the sum

$$f_\alpha = \sum_{\alpha_j \in \Pi} f_j. \tag{A.3}$$

The Cartan matrix is

$$a_{jk} = \langle \alpha_j, \alpha_k^\vee\rangle, \ \ \alpha_j \in \Pi, \ \alpha_k^\vee \in \Pi^\vee. \tag{A.4}$$

S^W is a ring of polynomials on V invariant with respect to W-action. The ring S^W is generated by n homogeneous polynomials of degrees $d_1 = 2, d_2, \ldots, d_n$,

$$S^W = \{P_1, \ldots, P_n\}. \tag{A.5}$$

The number of roots can be read off from the degrees

$$\sharp R = 2\sum_{i=1}^{n}(d_i - 1). \tag{A.6}$$

The simple roots generate the root lattice $Q = \sum_{j=1}^n n_j\alpha_j, \ \ (n_j \in \mathbb{Z}, \ \alpha_j \in \Pi)$ in V^*.

There exists a unique *maximal root* in $-\alpha_0 \in R^+$,

$$-\alpha_0 = \sum_{\alpha_j \in \Pi} n_j \alpha_j. \tag{A.7}$$

Its level is equal to $h - 1$, where

$$h = 1 + \sum_{\alpha_j \in \Pi} n_j \tag{A.8}$$

is *the Coxeter number*.

The positive Weyl chamber is

$$C^+ = \{x \in V \mid \langle x, \alpha\rangle > 0, \ \alpha \in R^+\}. \tag{A.9}$$

The Weyl group acts simply-transitively on the set of the Weyl chambers. The simple coroots $\Pi^\vee = (\alpha_1^\vee, \ldots, \alpha_l^\vee)$ form a basis in V and generate *the coroot lattice*

$$Q^\vee = \sum_{j=1}^{n} n_j \alpha_j^\vee \subset V, \quad n_j \in \mathbb{Z}. \tag{A.10}$$

The *fundamental weights*: $\varpi_j \in V^*$, $(j = 1, \ldots, n)$ $\langle \varpi_j, \alpha_k^\vee \rangle = \delta_{jk}$, $, \alpha_k^\vee \in \Pi^\vee$.

The weight lattice $P = \sum_{j=1}^{n} m_j \varpi_j \subset V^*$, $m_j \in \mathbb{Z}$ is dual to the coroot lattice (A.10).

The fundamental coweights are

$$\langle \alpha_k, \varpi_j^\vee \rangle = \delta_{kj}. \tag{A.11}$$

The coweight lattice

$$P^\vee = \sum_{j=1}^{l} m_j \varpi_j^\vee, \quad m_j \in \mathbb{Z}, \quad \langle \varpi_j^\vee, \alpha_k \rangle = \delta_{jk} \tag{A.12}$$

is dual to the root lattice Q.

The half-sum of positive roots is $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha = \frac{1}{2} \sum_{j=1}^{n} \varpi_j$.

The dual vector in V

$$\rho^\vee = \frac{1}{2} \sum_{\alpha \in R^{\vee +}} \alpha^\vee = \sum_{j=1}^{n} \varpi_j^\vee. \tag{A.13}$$

Then from (A.2) and (A.3) the level of α is equal

$$f_\alpha = \langle \rho^\vee, \alpha \rangle. \tag{A.14}$$

Affine Weyl group.

The affine Weyl group W_a is $Q^\vee \rtimes W$,

$$x \to x - \langle \alpha, x \rangle \alpha^\vee + k\beta^\vee, \quad \alpha^\vee, \ \beta^\vee \in R^\vee \ \ k \in \mathbb{Z}. \tag{A.15}$$

The Weyl alcoves are connected components of the set $V \backslash \{\langle \alpha, x \rangle \in \mathbb{Z}\}$. Their closure are fundamental domains of the W_a-action.

An alcove belonging to C^+ (A.9),

$$C_{alc} = \{ x \in V \mid \langle \alpha, x \rangle > 0, \ \alpha \in \Pi, \ (\alpha_0, x) > -1 \}, \tag{A.16}$$

has the nodes

$$C_{alc} = \{0, \varpi_1^\vee / n_1, \ldots, \varpi_n^\vee / n_j\}. \tag{A.17}$$

Here n_j are the coefficients of expansion of the maximal root (A.7).

The shift operator $x \to x + \gamma$, $\gamma \in P^\vee$ generates a semidirect product

$$W_a' = P^\vee \rtimes W. \tag{A.18}$$

The factor group is isomorphic to the center $W_a'/W_a \sim P^\vee / Q^\vee \sim \mathcal{Z}(\bar{G})$.

Chevalley basis in $\mathfrak{g}$.
Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n and $\mathfrak{H}$ is a Cartan subalgebra. Let $\mathfrak{H} = V + iV$, where V is the vector space defined above with the root system R. The algebra $\mathfrak{g}$ has the root decomposition

$$\mathfrak{g} = \mathfrak{H} + \mathfrak{L}, \quad \mathfrak{L} = \sum_{\beta \in R} \mathfrak{R}_\beta, \quad \dim_{\mathbb{C}} \mathfrak{R}_\beta = 1. \tag{A.19}$$

The Chevalley basis in $\mathfrak{g}$ is generated by

$$\{E_{\beta_j} \in \mathfrak{R}_{\beta_j}, \ \beta_j \in R, \quad H_{\alpha_k} \in \mathfrak{H}, \ \alpha_k \in \Pi\}, \tag{A.20}$$

where H_{α_k} are defined by the commutation relations

$$\begin{gathered} [E_{\alpha_k}, E_{-\alpha_k}] = H_{\alpha_k}, \quad [H_{\alpha_k}, E_{\pm\alpha_j}] = a_{kj} E_{\pm\alpha_k}, \quad \alpha_k, \alpha_j \in \Pi, \\ [H_{\alpha_j}, E_{\alpha_k}] = a_{kj} E_{\alpha_k}, \quad [E_\alpha, E_\beta] = C_{\alpha,\beta} E_{\alpha+\beta}, \end{gathered} \tag{A.21}$$

where $C_{\alpha,\beta}$ are structure constants of $\mathfrak{g}$. They possess the properties

$$\begin{aligned} C_{\alpha,\beta} &= -C_{\beta,\alpha}, \\ C_{\lambda\alpha,\beta} &= C_{\alpha,\lambda^{-1}\beta}, \quad \lambda \in W, \\ C_{\alpha+\beta,-\alpha} &= \frac{|\beta|^2}{|\alpha+\beta|^2} C_{-\alpha,-\beta}. \end{aligned} \tag{A.22}$$

If (,) is a scalar product in $\mathfrak{H}$ then H_α can be identified with coroots as $H_\alpha = \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ and

$$(H_\alpha, H_\beta) = \frac{4(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)} = \frac{2}{(\alpha, \alpha)} a_{\alpha,\beta}. \tag{A.23}$$

The Killing form in the subspace $\mathfrak{L}$ is expressed in terms of (α, α),

$$(E_\alpha, E_\beta) = \delta_{\alpha,-\beta} \frac{2}{(\alpha, \alpha)}. \tag{A.24}$$

Flags and coadjoint orbits.
Borel subgroup B of G is generated by the Cartan subgroup $\mathcal{H}(G)$ of G and by negative root subspaces $\exp\left(\sum_{\alpha \in R^-} E_\alpha\right)$. *The flag variety* is the coset space $Fl = G/B$. It has dimension (see (A.6))

$$\dim\ Fl = \sum_{j=1}^{l} (d_j - 1). \tag{A.25}$$

The coadjoint orbits,

$$\mathcal{O} = \{Ad_g^* S_0 \mid g \in G, \ S_0 \text{ is afixed element of } \mathfrak{g}^*\}. \tag{A.26}$$

is a generalization of a cotangent bundle to the flag varieties,[11] and for generic orbits

$$\dim\ \mathcal{O} = 2 \sum_{j=1}^{l} (d_j - 1). \tag{A.27}$$

Centers of simple groups.
A simply-connected group $\bar{G}$ in all cases apart from G_2, F_4 and E_8 has a non-trivial center $\mathcal{Z}(\bar{G}) \sim P^\vee / Q^\vee$ (Table 5).

[11] It is a cotangent bundle if S_0 is a Jordan element. If S_0 is semisimple, then $\mathcal{O}$ is the torsor over Fl.

Table 5. Centers of universal covering groups ($\mu_N = \mathbb{Z}/N\mathbb{Z}$)

$\bar{G}$	Lie $(\bar{G})$	$\mathcal{Z}(\bar{G})$
$\mathrm{SL}(n, \mathbb{C})$	A_{n-1}	μ_n
$\mathrm{Spin}_{2n+1}(\mathbb{C})$	B_n	μ_2
$\mathrm{Sp}_n(\mathbb{C})$	C_n	μ_2
$\mathrm{Spin}_{4n}(\mathbb{C})$	D_{2n}	$\mu_2 \oplus \mu_2$
$\mathrm{Spin}_{4n+2}(\mathbb{C})$	D_{2n+1}	μ_4
$E_6(\mathbb{C})$	E_6	μ_3
$E_7(\mathbb{C})$	E_7	μ_2

$\mathcal{Z}(\bar{G})$ is a cyclic group except $\mathfrak{g} = D_{4l}$, and $ord\,(\mathcal{Z}(\bar{G})) = \det\,(a_{kj})$, where (a_{kj}) is the Cartan matrix,

$$G^{ad} = \bar{G}/\mathcal{Z}(\bar{G}). \tag{A.28}$$

In the cases A_{n-1} (n is non-prime), and D_n the center $\mathcal{Z}(\bar{G})$ has non-trivial subgroups $\mathcal{Z}_l \sim \mu_l = \mathbb{Z}/l\mathbb{Z}$. Then there exists the factor groups

$$G_l = \bar{G}/\mathcal{Z}_l, \quad G_p = G_l/\mathcal{Z}_p, \quad G^{ad} = G_l/\mathcal{Z}(G_l), \tag{A.29}$$

where $\mathcal{Z}(G_l)$ is the center of G_l and $\mathcal{Z}(G_l) \sim \mu_p = \mathcal{Z}(\bar{G})/\mathcal{Z}_l$.

The group $\bar{G} = Spin_{4n}(\mathbb{C})$ has a non-trivial center

$$\mathcal{Z}(Spin_{4n}) = (\mu_2^L \times \mu_2^R), \quad \mu_2 = \mathbb{Z}/2\mathbb{Z},$$

where three subgroups can be described in terms of their generators as

$$\mu_2^L = \{(1,1), (-1,1)\}, \quad \mu_2^R = \{(1,1), (1,-1)\}, \quad \mu_2^{diag} = \{(1,1), (-1,-1)\}.$$

Therefore there are three intermediate subgroups between $\bar{G} = Spin_{4n}(\mathbb{C})$ and G^{ad}

$$\begin{array}{ccccc} & & Spin_{4n} & & \\ & \swarrow & \downarrow & \searrow & \\ Spin^R_{4n} = Spin_{4n}/\Gamma^L & & SO(4n) = Spin_{4n}/\Gamma^{diag} & & Spin^L_{4n} = Spin_{4n}/\Gamma^R \\ & \searrow & \downarrow & \swarrow & \\ & & G^{ad} = Spin_{4n}/(\mu_2^L \times \mu_2^R) & & \end{array}$$

Characters and cocharacters.
Let $\mathcal{H}$ be a Cartan subgroup $\mathcal{H} \subset G$. Define the group of characters[12]

$$\Gamma(G) = \{\chi : \mathcal{H} \to \mathbb{C}^*\}. \tag{A.30}$$

This group can be identified with a lattice group in $\mathfrak{H}^*$ as follows. Let $\mathbf{x} = (x_1, z_2, \ldots, x_n)$ be an element of $\mathfrak{H}$, and $\exp 2\pi i\mathbf{x} \in \mathcal{H}$. Define $\gamma \in V^*$ such that $\chi_\gamma = \exp 2\pi i \langle \gamma, \mathbf{x} \rangle \in \Gamma(G)$. Then

$$\Gamma(\bar{G}) = P, \quad \Gamma(G^{ad}) = Q, \tag{A.31}$$

[12] The holomorphic maps of $\mathcal{H}$ to $\mathbb{C}^*$ such that $\chi(xy) = \chi(x)\chi(y)$ for $x, y \in \mathcal{H}$.

and $\Gamma(G^{ad}) \subseteq \Gamma(G_l) \subseteq \Gamma(\bar{G})$. The fundamental weights ϖ_k $(k = 1, \ldots, n)$ (simple roots α_k) form a basis in $\Gamma(\bar{G})$ ($\Gamma(G^{ad})$). Let $\mathcal{Z}(\bar{G})$) be a cyclic group and p be a divisor of $ord\,(\mathcal{Z}(\bar{G}))$ such that $l = ord\,(\mathcal{Z}(\bar{G}))/p$. Then the lattice $\Gamma(G_l)$ is defined as

$$\Gamma(G_l) = Q + \varpi\mathbb{Z}, \quad p\varpi \in Q. \tag{A.32}$$

Define the dual groups of cocharacters $t(G) = \Gamma^*(G)$ as holomorphic maps

$$t(G) = \{\mathbb{C}^* \to \mathcal{H}\}. \tag{A.33}$$

In another way

$$t(G) = \{\mathbf{x} \in \mathfrak{H} \mid \chi(e^{2\pi i\mathbf{x}}) = 1\}. \tag{A.34}$$

A generic element of $t(G)$ takes the form

$$z^{\gamma} = \exp 2\pi i\gamma \ln z \in \mathcal{H}_G, \quad \gamma \in \Gamma(G), \quad z \in \mathbb{C}^*. \tag{A.35}$$

In particular, the groups $t(\bar{G})$ and $t(G_{ad})$ are identified with the coroot and the coweight lattices

$$t(\bar{G}) = Q^{\vee}, \quad t(G_{ad}) = P^{\vee}, \tag{A.36}$$

and $t(\bar{G}) \subseteq t(G_l) \subseteq t(G^{ad})$. It follows from (A.32) that

$$t(G_l) = Q^{\vee} + \varpi^{\vee}\mathbb{Z}, \quad l\varpi^{\vee} \in Q^{\vee}. \tag{A.37}$$

The center $\mathcal{Z}(G)$ of G is isomorphic to the quotient

$$\mathcal{Z}(G) \sim P^{\vee}/t(G), \tag{A.38}$$

while $\pi_1(G) \sim t(G)/Q^{\vee}$. In particular,

$$\mathcal{Z}(\bar{G}) = P^{\vee}/t(\bar{G}) \sim P^{\vee}/Q^{\vee}. \tag{A.39}$$

Similarly, the fundamental group of G^{ad} is $\pi_1(G^{ad}) \sim t(G^{ad})/Q^{\vee} \sim P^{\vee}/Q^{\vee}$.

The triple $(R,\ t(G),\ \Gamma(G))$ is called *the root data. A Langlands dual* to G group ${}^L G$ is defined by the root data $(R^{\vee},\ t({}^L G),\ \Gamma({}^L G))$, where

$$t({}^L G) \sim \Gamma(G), \quad \Gamma({}^L G) \sim t(G). \tag{A.40}$$

In particular, in the simply-laced cases ${}^L\bar{G} = G^{ad}$ (Table 6).

9. Appendix B. Elliptic Functions, [49,61]

The basic function is the theta-function,

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i(n(n+1)\tau + 2nz)}. \tag{B.1}$$

It is a holomorphic function on $\mathbb{C}$ with simple zeroes at the lattice $\tau\mathbb{Z} + \mathbb{Z}$ and the quasi-periodicities,

Table 6. Duality in simple groups

Root system	G	${}^LG$
$A_n,\ N=n+1=pl$	$G_l=\mathrm{SL}(N,\mathbb{C})/\mu_l$	$G_p=\mathrm{SL}(N,\mathbb{C})/\mu_p$
B_n	$\mathrm{Spin}(2n+1)$	$\mathrm{Sp}(n)$
C_n	$\mathrm{Mp}(n)$	$\mathrm{SO}(2n+1)$
$D_n,\ n=2l+1$	$\mathrm{Spin}(4l+2)$	$\mathrm{SO}(4l+2)/\mu_2$
	$\mathrm{SO}(4l+2)$	$\mathrm{SO}(4l+2)$
$D_n,\ n=2l$	$\mathrm{Spin}(4l)$	$\mathrm{SO}(4l)/\mu_2$
	$\mathrm{SO}(4l)$	$\mathrm{SO}(4l)$
$l=2m$	$\mathrm{Spin}^L(8m)$	$\mathrm{Spin}^L(8m)$
	$\mathrm{Spin}^R(8m)$	$\mathrm{Spin}^R(8m)$
$l=2m+1$	$\mathrm{Spin}^L(8m+2)$	$\mathrm{Spin}^R(8m+2)$
E_6	E_6	E_6/μ_3
E_7	E_7	E_7/μ_2

$$\vartheta(z+1)=-\vartheta(z),\quad \vartheta(z+\tau)=-q^{-\frac{1}{2}}e^{-2\pi iz}\vartheta(z). \tag{B.2}$$

Define the ratio of the theta-functions

$$\phi(u,z)=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{B.3}$$

It follows from (B.1) and (B.2) that it is a meromorphic function of $z\in\mathbb{C}$ with simple poles at the lattice $\tau\mathbb{Z}+\mathbb{Z}$ and

$$Res\,\phi(u,z)|_{z\in(\tau\mathbb{Z}+\mathbb{Z})}=1, \tag{B.4}$$

and the quasi-periodicities

$$\phi(u,z+1)=\phi(u,z),\quad \phi(u,z+\tau)=e^{-2\pi iu}\phi(u,z). \tag{B.5}$$

Since $\phi(u,z)=\phi(z,u)$,

$$\phi(u+1,z)=\phi(u,z),\quad \phi(u+\tau,z)=e^{-2\pi iz}\phi(u,z). \tag{B.6}$$

We also need two Fay identities for $\phi(z,w)$, the first one:

$$\begin{aligned}\phi(u_1,z_1)\phi(u_2,z_2)-\phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1)\\ -\phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2)=0,\end{aligned} \tag{B.7}$$

and its degenerate form:

$$\phi(u_1,z)\phi(u_2,z)-\phi(u_1+u_2,z)(E_1(u_1)+E_1(u_2))-\partial_z\phi(u_1+u_2,z)=0, \tag{B.8}$$

where $E_1(z)$ is the first Eisenstein function

$$E_1(z)=\partial_z\log\vartheta(z). \tag{B.9}$$

The second Eisenstein function is

$$E_2(z)=\partial_z^2\log\vartheta(z)=-\partial_zE_1(z). \tag{B.10}$$

They are related to the Weierstrass functions as follows:

$$\zeta(z|\tau) = E_1(z|\tau) + 2\eta_1(\tau)z, \tag{B.11}$$

and

$$\wp(z|\tau) = E_2(z) - 2\eta_1(\tau). \tag{B.12}$$

Here

$$\eta_1(\tau) = \frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)}, \quad \eta(\tau) = q^{\frac{1}{24}}\prod_{n>0}(1-q^n),$$

and $\eta(\tau)$ is the Dedekind function.

$E_1(z)$ is quasi-periodic

$$E_1(z+1|\tau) = E_1(z|\tau), \quad E_1(z+\tau|\tau) = E_1(z|\tau) - 2\pi i, \tag{B.13}$$

and has simple poles at the lattice $\tau\mathbb{Z} + \mathbb{Z}$,

$$Res\, \zeta(z|\tau)|_{z\in(\tau\mathbb{Z}+\mathbb{Z})} = 1. \tag{B.14}$$

$E_2(z)$ is double-periodic with second order poles at the lattice. It is related to $\phi(u, z)$ as

$$\phi(u, z)\phi(-u, z) = E_2(z) - E_2(u). \tag{B.15}$$

$E_2(z)$ and its derivatives $\partial_z^k E_2(z)$ form a basis in a space of double periodic function on $\Sigma_\tau = \mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z})$.

The most important object for construction of Lax operators and r-matrices is the function defined as follows:

$$\varphi_\alpha^k(z) = e^{2\pi i\,\langle\kappa,\alpha\rangle\, z}\phi\left(\langle u + \kappa\,\tau,\ \alpha\rangle + \frac{k}{N},\ z\right).$$

Here u and κ are vectors defined in Proposition 3.1, α is a root of the corresponding Lie algebra. Note that to save space we omit the u-dependence of the function in its definition.

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