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The Yang-Mills Heat Semigroup on Three-Manifolds with Boundary

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Abstract: Long time existence and uniqueness of solutions to the Yang-Mills heat equation is proven over a compact 3-manifold with smooth boundary. The initial data is taken to be a Lie algebra valued connection form in the Sobolev space H_1. Three kinds of boundary conditions are explored, Dirichlet type, Neumann type and Marini boundary conditions. The last is a nonlinear boundary condition, specified by setting the normal component of the curvature to zero on the boundary. The Yang-Mills heat equation is a weakly parabolic nonlinear equation. We use gauge symmetry breaking to convert it to a parabolic equation and then gauge transform the solution of the parabolic equation back to a solution of the original equation. Apriori estimates are developed by first establishing a gauge invariant version of the Gaffney-Friedrichs inequality. A gauge invariant regularization procedure for solutions is also established. Uniqueness holds upon imposition of boundary conditions on only two of the three components of the connection form because of weak parabolicity. This work is motivated by possible applications to quantum field theory.

Contents

1. Introduction

1.1. Nonlinear distribution spaces. This paper is intended as a first step in constructing nonlinear distribution spaces for Yang-Mills fields over three dimensional space.

Heat equations have been used to characterize various function spaces by identifying these function spaces with the initial data space for a parabolic equation. This method of characterizing function spaces goes back at least to the 1961 paper of Lions [27][Sect. 5], the 1960s papers, [54–56], of Taibleson and to the 1980s papers, [34–36], of Matsuzawa. The papers of Matsuzawa characterize an ultradistribution u on a compact subset of $\mathbb{R}^n$ by properties of the solution to the heat equation with initial data u. See the classic book [5] for early work and the paper [1] for some recent history.

By way of a simple example, consider a non-negative unbounded self-adjoint operator A acting on a Hilbert space H. Assume for simplicity that $A \geq I$. Let $\alpha > 0$. The easily verified identity,

$$\|A^{-\alpha}u_0\|^2 = C_\alpha \int_0^\infty s^{2\alpha-1}\|e^{-sA}u_0\|^2 ds, \;\; C_\alpha = \text{constant} \tag{1.1}$$

shows that the norm $u_0 \to \|A^{-\alpha}u_0\|$ on H can be characterized in terms of solutions to the initial value problem

$$u'(s) = -Au(s), \quad \text{for } s > 0, \quad u(0) = u_0, \tag{1.2}$$

since the solution is just $u(s) = e^{-sA}u_0$. In fact it is clear that the initial value problem (1.2) sets up a one-to-one correspondence between the space of those solutions of the equation $u'(s) = -Au(s)$ for which the right side of (1.1) is finite, and the large initial data space consisting of the completion of H in the norm $\|A^{-\alpha}u_0\|$. If H is an L^2 space over some Riemannian manifold and $-A$ is a second order elliptic operator then these completed spaces are just negative Sobolev spaces and the correspondence $u_0 \leftrightarrow u(\cdot)$, set up by (1.2), identifies these Sobolev spaces with certain spaces of solutions of the heat equation for $-A$. In general H may be some other kind of Banach space or Frechet space, and the completion spaces need not be Sobolev spaces, [1].

Some quantum field theories seem to require use of large completions of spaces which are not linear spaces. Most important is the example in which the space to be completed is a space $\mathscr{A}$ of connections on $\mathbb{R}^3$ modulo a gauge group $\mathscr{G}$. Whatever smoothness one imposes on $\mathscr{A}$ and $\mathscr{G}$, the space $\mathscr{A}/\mathscr{G}$ is not a linear space in generic cases of interest. See e.g. [51,52,42] for discussions of the geometry of this space in case $\mathbb{R}^3$ is replaced by a compact manifold.

The reason for the need to complete such a quotient space is that the quantum theory requires a space large enough to support certain measures of physical interest. Typically, the measures arising in quantum field theory need some negative Sobolev space to live on. In the preceding gauge field example some kind of *nonlinear* negative Sobolev space seems to be required. We are going to explore the (nonlinear) Yang-Mills heat equation as a replacement for the linear equation (1.2). The measure theoretic difficulties increase with spatial dimension as does the difficulty in proving existence of solutions to the Yang-Mills heat equation. For example no completion is necessary for addressing the measure theory in one spatial dimension, i.e., two space-time dimensions, even though study of the associated stochastic process presents severe problems of its own. See, e.g., A. Sengupta, [49,50]. We are going to address the Yang-Mills heat equation in three space dimensions only, with intended application to the canonical formalism over $\mathbb{R}^3$ or the Euclidean formalism over four dimensional space-time. The corresponding existence and uniqueness theorems are simpler in two space dimensions and follow easily from our techniques.

In contrast with the simple example of (1.2), the flow equation associated to such a nonlinear distribution space will itself be nonlinear. In the case of a Yang-Mills field the natural equation is the gradient flow equation of the magnetic energy (which is the square of the L^2 norm of the curvature). Elsewhere, the nonlinear sigma model will be investigated from this same point of view and the nonlinear equation will again be a gradient flow equation of a non-quadratic energy. Thus in each of the examples of interest the flow equation is a geometric flow given as the gradient flow of some natural energy functional on some nonlinear manifold. It is the intention of this program to realize the required nonlinear distribution spaces as complete "Riemannian" infinite dimensional manifolds whose elements are geometric flows and which support genuine functions, such as gauge invariantly regularized Wilson loop variables.

In order to understand the spaces of flows for which there is no identifiable initial data it is first necessary to understand those flows for which there is an identifiable initial value. Unlike the linear case a proper understanding of the space of initial data for some class of flows requires treating both the space of flows and the initial data space as infinite dimensional Riemannian manifolds: one needs to know not only which initial data propagates to a flow but also which variations of the initial data propagate to a solution of the variational equation along the flow. In the linear case there is no distinction between the flow equation and its variational equation. In the nonlinear case, when the initial data is singular, the variational equation will have singular coefficients at time zero, and a variation of the initial data may not propagate past the singularity. This issue will be treated in a separate work. In the present paper we are going to prove existence and uniqueness of solutions to the Yang-Mills heat equation, (1.3), with initial data in Sobolev class 1.

We will also establish some apriori estimates aimed at extending the class of initial data to connection forms of Sobolev class 1/2. Sobolev class 1/2 is the natural class for initial data from the point of view of relativity theory because it is the unique class whose hyperbolic flow by Maxwell's equations is Lorentz invariant. Furthermore, for initial data of Sobolev class 1/2, three spatial dimensions is the critical dimension for the Yang-Mills heat equation. We will pursue the extension of our results to initial data of Sobolev class 1/2 in a future work.

1.2. Manifolds with boundary and local observables. We are going to consider the Yang-Mills heat equation in a product bundle over a compact Riemannian 3-manifold M with smooth boundary. The case of interest for quantum field theory is that in which M is the closure of a bounded open set O in $\mathbb{R}^3$ with smooth boundary. Roughly, our main theorem asserts that if K is a compact, connected Lie group with Lie algebra $\mathfrak{k}$ and if A_0 is a $\mathfrak{k}$ valued connection form over M, lying in the first order Sobolev space $W_1(M)$, then there exists a unique solution to the Yang-Mills heat equation

$$\partial A(t)/\partial t = -d^*_{A(t)} B(t), \quad t > 0 \quad \text{with} \quad A(0) = A_0, \tag{1.3}$$

satisfying Dirichlet type or Neumann type boundary conditions. Here $B(t)$ is the curvature 2-form, $B(t) = dA(t) + A(t) \wedge A(t)$, of the connection form $A(t)$ and $d^*_{A(t)}$ is the gauge covariant coderivative. Equation (1.3) is the gradient flow equation for the magnetic energy $\|B\|^2_{L^2(M)}$.

In addition to Neumann and Dirichlet boundary conditions, we are also going to examine a purely nonlinear boundary condition, which is specified by requiring that the normal component of the curvature be zero. Such a boundary condition was first studied by A. Marini [31–33] in the context of elliptic boundary value problems for Yang-Mills connections over four dimensional manifolds. We will henceforth refer to this boundary condition as Marini boundary conditions.

There is a fundamental conceptual reason for considering the Yang-Mills heat equation over a bounded open set O in $\mathbb{R}^3$ rather than over all of $\mathbb{R}^3$ or over a closed 3-manifold such as T^3: Suppose that γ is a piecewise smooth closed curve in $\mathbb{R}^3$. Denote by $W_\gamma(A)$ the composition of a character of K with the parallel transport around γ by a connection form A defined in a neighborhood of γ. That is, $W_\gamma(A) \equiv trace\ (//^A_\gamma)$, where the trace is computed in some finite dimensional unitary representation of K and $//^A_\gamma$ denotes parallel transport. Then the holonomy function $A \mapsto W_\gamma(A)$ (the Wilson loop variable) is gauge invariant and descends to a function on a quotient manifold $\mathscr{A}/\mathscr{G}$ such as discussed above. In the sought for space of connection forms, on whose moduli space the desired ground state measure lives, a typical connection form A is not even an almost everywhere defined form, let alone continuous, and the function $W_\gamma(A)$ is therefore not well defined. This is known from the electromagnetic case, $K = U(1)$, for which the measure theory is explicitly solvable. Nevertheless similar holonomy functions on $\mathscr{A}/\mathscr{G}$ have been used extensively both for formulation of a mathematical theory [52, Chap. 8, 48], and for computational comparisons with experiment [26,28,29]. If $A(\cdot)$ solves the Yang-Mills heat equation (1.3), with initial data A_0, which, in the spirit of Sect. 1.1 we take to be some kind of generalized connection form on $\mathbb{R}^3$, then, for any $t > 0$, $A(t)$ will be (essentially) a C^∞ 1-form and the map $A_0 \mapsto W_\gamma(A(t))$ will be well defined and gauge invariant. Thus the Yang-Mills heat equation offers a gauge invariant regularization procedure for some class of irregular connection forms.

The regularizing effect of the flow has already been pointed out in both the lattice and continuum quantized theories, [28–30], where it also appears as a first order approximation in a method aimed at implementing a Monte Carlo computational protocol for lattice gauge theory.

However, since the (weakly) parabolic equation (1.3) propagates information with infinite speed, the map $A_0 \mapsto W_\gamma(A(t))$ depends on A_0 over all of $\mathbb{R}^3$. This is unsatisfactory from the point of view of local quantum field theory, which requires use of "local observables", [18,53], that is, functions of A_0 which depend only on the behavior of A_0 in some specified (say bounded) open set $O \subset \mathbb{R}^3$. Now solving Eq. (1.3) over O, rather than over $\mathbb{R}^3$, with initial data $A_0|O$, produces a function $W_\gamma(A(t))$ depending only on $A_0|O$, when $\gamma \subset O$. In this way one can hope to construct useful "local observables". Of course it is essential that this regularization procedure commutes with gauge transformations in the sense that, for a (think C^∞) connection form A_0 on $\mathbb{R}^3$, and any function $g \in C^\infty(\mathbb{R}^3; K)$, one has

$$(A_0^g|O)(t) = (A_0|O)(t)^g,$$

where $(A_0|O)(t)$ refers to the solution in O of (1.3) at time $t > 0$ with initial condition $A_0|O$. The superscript g refers to the usual gauge transformation of the connection form. (See e.g. after Eq. (2.17).) Such commutativity will hold for Marini boundary conditions but not for Dirichlet or Neumann boundary conditions. For this reason we expect that Marini boundary conditions will be the most important ones for our purposes.

We anticipate that the conventional lattice regularization of Yang-Mills quantum field theory, [19,25,28,29,59,48], will mesh well with the present continuum regularization.

1.3. Technical description and history. The Yang-Mills heat equation has a long history [3,6,9,10,21–24,44,46]. While most of these works were aimed at immediate application in mathematics, some, e.g. [46], were aimed primarily at application to physics.

Standard methods for proving existence and uniqueness for nonlinear parabolic equations do not seem applicable to Eq. (1.3) because the equation is only weakly parabolic and the functional $A \mapsto \|dA + A \wedge A\|^2_{L^2(M)}$, whose flow we are following, is not (even weakly) convex. We are going to adapt a method that has its origin in papers of Zwanziger, [60], Donaldson, [9], and Sadun, [46]. This consists in adding a term $-d_A d^* A$ to the right side of (1.3), which makes the equation parabolic. A time dependent gauge transformation can then be constructed which changes the solution of the modified equation into a solution of the original equation, (1.3).

Techniques of proof of existence for solutions of parabolic equations over closed manifolds extend in a well understood way to manifolds with boundary when the solutions sought are real valued. But in our setting the components of the connection form A are mixed up by the nonlinear differential equation, and, in the case of Marini boundary conditions, are also mixed up by these boundary conditions. Moreover, the apriori energy estimates that we will need must be formulated in terms of gauge covariant derivatives because neither the connection form nor its curvature is smoothed by the flow. To this end, it is necessary to express Sobolev inequalities in terms of the gauge covariant exterior derivative d_A and its adjoint. For real valued functions this is accomplished by the Gaffney-Friedrichs inequality [13,12,39–41,57]. In our case we will need to prove a gauge invariant version of the Gaffney-Friedrichs inequality. Not surprisingly, the curvature of the connection form A enters these inequalities in a substantial way and contributes to some of the technical problems to be resolved. It is the need to adhere to gauge invariant estimates that is responsible for much of the novelty in this work.

J. Råde, [44], has proven existence and uniqueness of solutions for the Yang-Mills heat equation on a closed 3-manifold and investigated the longtime behavior of the solutions. The method used by Råde to solve the problem of lack of parabolicity is quite different from the method of Donaldson and Sadun. The curvature, F_A, of the 1-form A is taken as an unknown, L, independent of A, and a joint system of equations for A and L is solved. The joint system is parabolic. Råde proved that the solution $L(t)$ agrees with $F_{A(t)}$ for all time if they agree at time zero. This method seems to go back to Ginibre and Velo, [16,17], in the context of the hyperbolic Yang-Mills equations and to De Turck, [8], in the context of the parabolic Ricci flow problem. Råde's method might offer some advantages in our circumstance. But the presence of boundary conditions seems to add considerable difficulty.

The transition from short time existence to long time existence is carried out in different ways in the various works [9,10,44,46] and in the present paper. In addition, semi-probabilistic methods have also been used: See, e.g., Arnaudon et al. [2], and Pulemotov, [43], for a very different approach to long time existence.

Compactness of the manifold M is not really needed. We have included it as a hypothesis to simplify some statements and arguments. However we want to emphasize that all estimates derived here will also hold for a complete open manifold without boundary as long as the Bochner-Weitzenboch tensor (which is zero on $\mathbb{R}^3$) is bounded and the appropriate Sobolev inequalities and heat kernel bounds hold, which they do on $\mathbb{R}^3$. The same estimates will also hold on a manifold with boundary if the second fundamental form is bounded below.

2. Statement of Results

Notation 2.1. M will denote a compact Riemannian 3-manifold with smooth boundary. K will denote a compact connected Lie group. Without loss of generality we may and will identify K with a subgroup of the orthogonal group, respectively unitary group, of some finite dimensional real, respectively complex, inner product space $\mathcal{V}$. The Lie algebra of K, denoted $\mathfrak{k}$, may then be identified with a real subspace of $End\,\mathcal{V}$. We will be concerned only with a product bundle $M \times \mathcal{V} \to M$ over M. We assume given an $Ad\ K$ invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ with norm denoted by $|\xi|_{\mathfrak{k}}$ for $\xi \in \mathfrak{k}$. We will not distinguish between $|\xi|_{\mathfrak{k}}$ and $|\xi|_{End\,\mathcal{V}}$, which are equivalent norms.

If ω and ϕ are $\mathfrak{k}$ valued p-forms define $(\omega, \phi) = \int_M \langle \omega(x), \phi(x) \rangle_{\Lambda^p \otimes \mathfrak{k}} dx$ and $\|\omega\|_2^2 = (\omega, \omega)$. Define also $\|\omega\|_\infty = \sup_{x \in M} |\omega(x)|_{\Lambda^p \otimes \mathfrak{k}}$ and

$$\|\omega\|_{W_1(M)}^2 = \int_M |\nabla \omega|^2_{\Lambda^p \otimes \mathfrak{k}} d\,\mathrm{Vol}\ + \|\omega\|_2^2, \tag{2.1}$$

where ∇ is the Riemannian gradient on forms. Define $W_1 = W_1(M) = \{\omega : \|\omega\|_{W_1(M)} < \infty\}$. The notation H_1 will be used later for forms in W_1 which satisfy specified boundary conditions. Since we are concerned only with a product bundle, a connection form can be identified with a $\mathfrak{k}$ valued 1-form. For a connection form A, given in local coordinates by $A = \sum_{j=1}^3 A_j(x) dx^j$, its curvature (magnetic field) is given by

$$B = dA + (1/2)[A \wedge A], \tag{2.2}$$

where $[A \wedge A] = \sum_{i,j}[A_i, A_j]dx^i \wedge dx^j$ and $[A_i(x), A_j(x)]$ is the commutator in $\mathfrak{k}$. B is a $\mathfrak{k}$ valued 2-form. For $\omega \in W_1$ we define $d_A\omega = d\omega + (ad\ A) \wedge \omega$ and $d_A^*\omega = d^*\omega + (ad\ A\wedge)^*\omega$. No boundary conditions are implied on these operators in this section. The domains of these operators will be discussed further in Sect. 3.

Definition 2.2. *Let* $0 < T \le \infty$. *By a* **strong solution** *to the Yang-Mills heat equation over* $[0, T)$ *we mean a continuous function*

$$A(\cdot) : [0, T) \to W_1 \subset \mathfrak{k}\text{-valued 1-forms} \tag{2.3}$$

such that

$$a)\quad B(t) \in W_1 \text{ for each } t \in (0, T), \text{ where } B(t) = \text{curvature of } A(t), \tag{2.4}$$

$$b)\quad \text{the strong } L^2(M) \text{ derivative } A'(t) \equiv dA(t)/dt \text{ exists on } (0, T), \tag{2.5}$$

$$c)\quad A'(t) = -d_{A(t)}^* B(t) \text{ for each } t \in (0, T). \tag{2.6}$$

A strong solution will be called **locally bounded** *if*

$$d)\quad \|B(t)\|_\infty \text{ is bounded on each bounded interval } [a, b) \subset (0, T) \text{ and} \tag{2.7}$$

$$e)\quad t^{3/4}\|B(t)\|_\infty \text{ is bounded on some interval } (0, b) \text{ with } 0 < b < T. \tag{2.8}$$

Remark 2.3. The condition e) allows the degree of singular behavior near $t = 0$ that is to be expected in three dimensions. We will prove long time existence and uniqueness of locally bounded strong solutions under various boundary conditions. The local boundedness is a vital ingredient in our uniqueness proof. We don't know if uniqueness holds in the absence of some such regularity condition. See Remark 8.18 for further discussion of this point.

Usually $A'(t)$ will signify $\partial A(t)/\partial t$. But in b) we are regarding $A(\cdot)$ as a function into $L^2(M; \Lambda^1 \otimes \mathfrak{k})$.

2.1. Dirichlet, Neumann and Marini boundary conditions.

Notation 2.4 (Tangential and normal components). At a point $x \in \partial M$ denote by $\mathbf{n}$ the outward drawn unit normal and by ν the dual unit conormal. Any p-form γ over $T_x(M)$ can be written uniquely as $\gamma = \alpha \wedge \nu + \beta$, where $\beta(\mathbf{n}, X_1, \ldots, X_{p-1}) = \alpha(\mathbf{n}, X_1, \ldots, X_{p-2}) = 0$ for all $X_j \in T_x(M)$. As is customary, we will write $\gamma_{norm} = \alpha \wedge \nu$ and $\gamma_{tan} = \beta$. The restriction maps $\alpha \to i^*\alpha$ and $\beta \to i^*\beta$ are clearly isomorphisms on these classes of forms when $i : T_x(\partial M) \to T_x(M)$ is the inclusion map. Moreover $\gamma_{tan} = 0$ if and only if $\gamma \wedge \nu = 0$. A coordinate based description of these two components of γ will be given in Sect. 4.

Theorem 2.5 (Neumann boundary conditions). *Suppose that* $A_0 \in W_1$ *and* $(A_0)_{norm} = 0$. *Then there is a locally bounded strong solution* $A(\cdot)$ *over* $[0, \infty)$ *such that* $A(0) = A_0$ *and that satisfies the boundary conditions*

$$i)\quad A(t)_{norm} = 0 \quad \text{for all } t \ge 0 \quad \text{and} \tag{2.9}$$

$$ii)\quad B(t)_{norm} = 0 \quad \text{for all } t > 0. \tag{2.10}$$

Uniqueness: If A_1 *and* A_2 *are two locally bounded strong solutions which agree at time zero and satisfy* (2.10) *then* $A_1 = A_2$ *on* $[0, \infty)$.

Remark 2.6. Notice that for uniqueness the condition (2.9) is not required, even for $t = 0$. For an explanation of the terminology "Neumann boundary conditions" for the pair of conditions (2.9) and (2.10) see Remark 2.11.

Theorem 2.7 (Dirichlet boundary conditions). *Suppose that $A_0 \in W_1$ and $(A_0)_{tan} = 0$. Then there is a locally bounded strong solution over* $[0, \infty)$ *such that* $A(0) = A_0$ *and that satisfies the boundary conditions*

$$i) \quad A(t)_{tan} = 0 \quad \text{for all } t \geq 0 \quad \text{and} \tag{2.11}$$

$$ii) \quad B(t)_{tan} = 0 \quad \text{for all } t > 0. \tag{2.12}$$

Uniqueness: If A_1 and A_2 are two locally bounded strong solutions which agree at time zero and satisfy (2.11) *then $A_1 = A_2$ on* $[0, \infty)$.

Remark 2.8. Notice that for uniqueness the condition (2.12) is not required. In fact $A(t)_{tan} = 0$ implies $B(t)_{tan} = 0$ when $B(t) \in W_1$. (See, e.g., (3.22)). So the latter is not an independent condition.

Remark 2.9 (Weak parabolicity and regularization). Suppose that $g \in C^2(M; K)$ and is the identity element of K in a neighborhood of ∂M. Let $A_0 = g^{-1}dg$. Then $A_0 \in C^1(M : \Lambda^1 \otimes \mathfrak{k}) \subset W_1$ and is zero in a neighborhood of ∂M. Define $A(t) = A_0$ for all $t \geq 0$. $A(t)$ has curvature zero and satisfies all of the Neumann and Dirichlet boundary conditions, (2.9), (2.10), (2.11) and (2.12), including the initial conditions. It is the unique locally bounded strong solution specified in Theorems 2.5 and 2.7. Thus the Yang-Mills heat equation does not regularize all initial data, reflecting the well known fact that it is only weakly parabolic. The weak parabolicity will be particularly visible in Eq. (7.5) and the discussion following it. There is a gain of regularity for the curvature, however, and this will allow the strong sense of solution specified in Definition 2.2. Nevertheless, for $t > 0$, the curvature $B(t)$ itself will not be smooth under our initial conditions. For example if g is as above and A_0 is any initial condition in $W_1(M)$ then the gauge transform A_0^g is also in W_1 while the curvature $B^g(t)(x) = g(x)^{-1}B(t)g(x)$, of $A^g(t)$, need not be smooth even if $B(t)$ is smooth.

Remark 2.10 (Weak parabolicity and uniqueness). Theorems 2.5 and 2.7 show that for both Dirichlet and Neumann type boundary conditions, uniqueness follows from the imposition of only two boundary conditions on the three component connection form $A(t)$, in contrast to what one expects for parabolic equations. This effect can be attributed to the fact that the Yang-Mills heat equation is only weakly parabolic. It is well known that degeneracy of an elliptic operator L on a manifold with boundary can force uniqueness on solutions of the weakly parabolic equation $\partial u/\partial t = Lu$ under fewer boundary conditions on u than usual. See [37, Sect. 7.2] for a recent work discussing this issue for scalar functions. See also Remark 8.18 for further discussion in our case.

Remark 2.11. (Neumann and Marini boundary conditions.) In Theorem 2.7 the boundary condition $A(t)_{tan} = 0$, $t \geq 0$, appears in both the existence and uniqueness portion of the theorem, whereas in Theorem 2.5 the initial boundary condition $(A_0)_{norm} = 0$ is needed for the existence proof while $A(t)_{norm} = 0$, $t > 0$ is not needed for uniqueness. If $A(t)_{norm} = 0$ for $t > 0$ then $[A(t) \wedge A(t)]_{norm} = 0$ and consequently $B(t)_{norm} = (dA(t))_{norm}$. Thus in the presence of (2.9) the nonlinear boundary condition $B(t)_{norm} = 0$ in (2.10) is equivalent to the pure Neumann boundary condition $(dA(t))_{norm} = 0$.

A. Marini, [31–33], has explored the nonlinear boundary condition $F_{norm} = 0$ in the context of the weakly elliptic boundary value problem $d_A^* F = 0$, where $F = F_A$

is the curvature of a connection A over a 4-manifold with boundary. In the context of Theorem 2.5, the corresponding Marini boundary condition, $B(t)_{norm} = 0$, is fully gauge invariant and does not depend on the choice of a fiducial gauge, unlike the pair of conditions $A_{norm} = 0$, $(dA)_{norm} = 0$, to which the pair of Eqs. (2.9) and (2.10) is equivalent.

The Marini boundary condition will ultimately be the case of interest for the intended application to quantum field theory, as explained in Sect. 1.2. Theorem 2.5 easily yields the following existence and uniqueness theorem with the pure nonlinear boundary condition $B(t)_{norm} = 0$ by itself. The restrictive regularity of the initial data will be removed in a later work.

Theorem 2.12 (Marini boundary conditions) *Suppose that* $A_0 \in C^2(M; \Lambda^1 \otimes \mathfrak{k})$. *Then there is a unique locally bounded strong solution over* $[0, \infty)$ *such that* $A(0) = A_0$ *and*

$$B(t)_{norm} = 0 \quad \textit{for all} \quad t > 0. \tag{2.13}$$

Theorems 2.5, 2.7 and 2.12 will be proven in Sect. 9.

2.2. Existence by symmetry breaking. If one adds a term $-d_A d^* A$ to the right side of the Yang-Mills heat equation (2.6) the equation becomes strictly parabolic but is no longer gauge invariant. Remarkably, a solution to the modified equation can be transformed to a solution of the original equation by a time dependent gauge transformation. This was first observed by D. Zwanziger, [60], in the context of stochastic quantum field theory. Donaldson, [9], independently added such a term to the evolution equation of a classical Yang-Mills heat equation and similarly "gauged it away". L. Sadun, motivated by Zwanziger's work, used this technique in proving existence of solutions to (1.3) over $\mathbb{R}^3$ in his Ph. D. thesis, [46], as a step in carrying out stochastic quantization for Yang-Mills fields. Donaldson's work, which is carried out in the C^∞ category, is also summarized in the book [11, Sect. 6.3]. When the initial data is only in $W_1(M)$, as in our case, there is, unfortunately, a singularity in the time dependent gauge transformation at time zero. Our gauge invariant apriori estimates will play a key role in addressing this problem.

To distinguish the desired solution $A(\cdot)$ of the Yang-Mills heat equation, from the solution to the modified equation let us denote the latter by $C(t)$. The equation and boundary conditions for A then translate into the following initial value problem for $C(\cdot)$:

$$(\partial/\partial t)C = -(d_C^* B_C + d_C d^* C), t > 0, \quad C(0) = A_0, \tag{2.14}$$

along with one of the following two kinds of boundary conditions: (N) or (D).

$$(N) \quad C(t)_{norm} = 0 \ \text{ for } t \geq 0, \quad (B_{C(t)})_{norm} = 0 \quad \text{for } t > 0, \tag{2.15}$$

$$(D) \quad C(t)_{tan} = 0 \quad \text{for } t \geq 0, \quad (d^* C(t))|_{\partial M} = 0 \quad \text{for } t > 0, \tag{2.16}$$

where B_C denotes the curvature of a connection form C.

Equation (2.14) is a strictly parabolic differential equation, unlike (2.6). The boundary conditions (D) are relative boundary conditions in the sense of Ray and Singer, [45], while, in view of Remark 2.11, the boundary conditions (N) are equivalent to absolute boundary conditions. For recent systematic discussions of absolute and relative boundary conditions for real valued forms see for example the book [57, Chap. 5, Sect. 9] and [38], especially Chap. 5.

Concerning the parabolic system (2.14)–(2.16) we will prove the following short time existence and uniqueness theorem.

Theorem 2.13. *Let $A_0 \in W_1$. Assume that $(A_0)_{norm} = 0$, respectively $(A_0)_{tan} = 0$. Then there exists $T > 0$ and a continuous function $C : [0, T) \to W_1$ such that $C(0) = A_0$ and*

a) $B_{C(t)} \in W_1$ and $d^*C(t) \in W_1$ for each $t \in (0, T)$,
b) the strong $L^2(M)$ derivative $(d/dt)C(t)$ exists for each $t > 0$,
c) Eq. (2.14) holds for each $t > 0$ along with the boundary conditions (2.15), respectively (2.16),
f) $t^{3/4}\|B_{C(t)}\|_\infty$ is bounded on $(0, T)$.

The solution is unique under the preceding conditions. Moreover, $C(\cdot)$ lies in $C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$.

This will be proved in Sect. 7.1. The proof proceeds by a fairly standard reduction to an integral equation and a contraction mapping argument, followed then by a regularity theorem. However our form of the contraction argument will allow us to deduce some important regularity for these non-real valued functions that seems unavailable by more standard means. We will use a quadratic form version of the boundary conditions (2.15) and (2.16).

Here is an informal description of the gauging procedure which transforms a solution of the parabolic equation (2.14) to a solution of the Yang-Mills heat equation. A precise version will be given in Theorem 8.2.

Lemma 2.14 (Heuristic). *Let $C(t)$ be a solution to (2.14) with boundary conditions (2.15), respectively (2.16). Define a function $g : [0, T) \to C^\infty(M; K) \subset C^\infty(M; End\, \mathcal{V})$ as the solution to the initial value problem*

$$g'(t, x)g(t, x)^{-1} = d^*C(t, x), \quad g(0, x) = I_{\mathcal{V}} \tag{2.17}$$

for each $x \in M$. Let $A = C^g$. That is, $A(t, x) = g(t, x)^{-1}C(t, x)g(t, x) + g(t, x)^{-1}dg(t, x)$. Then A solves (2.6) with the Neumann type boundary conditions (2.9), (2.10), respectively the Dirichlet type boundary conditions (2.11), (2.12).

It is interesting that, for the solution $A = C^g$ produced in this way, the relative and absolute boundary conditions imposed on $C(\cdot)$ partly disappear, while for Marini boundary conditions, imposed on A, the relative and absolute boundary conditions disappear completely.

Because of the singular behavior of $d^*C(t, x)$ as $t \downarrow 0$, it is difficult to establish the regularity of $g(t, x)$ needed to ensure that $A(t) \in W_1(M)$ for $t \geq 0$. We will instead define $g_\epsilon(t)$ for $t \geq \epsilon$ using the same differential equation (2.17), but with initial condition $g_\epsilon(\epsilon) = I_{\mathcal{V}}$. Defining $A_\epsilon(t) = C(t)^{g_\epsilon(t)}$ for $t \geq \epsilon$, we will then show that the connection forms $A_\epsilon(\cdot)$ define smooth solutions which converge in a strong sense to the desired solution to (2.6) as $\epsilon \downarrow 0$. To carry out this transition from the parabolic equation to the weakly parabolic Yang-Mills heat equation we will need to use the gauge invariant Gaffney-Friedrichs inequality described below, along with the gauge invariant apriori estimates that follow from it. See Sect. 8 for precise statements and proof.

Remark 2.15 (Uniqueness and reverse gauge transformation). If one should wish to prove uniqueness for the weakly parabolic equation (2.6) by referring back to the strictly parabolic equation (2.14), for which uniqueness is well known, one must reverse the gauge transformation procedure described in Lemma 2.14. To this end, one must express the gauge function $g(t, x)$ in terms of $A(\cdot)$ rather than in terms of $C(\cdot)$. But, whereas the function $g(t, x)$ can be recovered from $C(\cdot)$ via the simple ordinary differential equation (2.17), its recovery from $A(\cdot)$ requires solving a nonlinear partial differential equation similar to the harmonic map equation. In our Sobolev class 1 category this is itself a difficult problem. Instead we are going to give a direct proof of uniqueness for (2.6) without passing back to (2.14).

2.3. Gauge invariant Gaffney-Friedrichs inequalities. Most of the estimates in this paper will depend on the use of Sobolev inequalities in which the energy form $\|\nabla^A\omega\|^2_{L^2(M)} + \|\omega\|_2^2$ is replaced by the Hodge version, $\|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + \lambda\|\omega\|_2^2$. Here we have written $(\nabla^A)_j\omega = \nabla_j\omega + [A_j, \omega]$ for the gauge covariant gradient of a $\mathfrak{k}$ valued form ω. It will be necessary to establish equivalences between these two energy forms because it is the former that controls L^p norms via Sobolev inequalities while it is the latter that relates well to the Yang-Mills heat equation. The constant λ depends on the curvature of the connection form A. The nature of this dependence is crucial for dealing with singular initial data. The equivalence of these two energy forms is dependent on the gauge-invariant Gaffney-Friedrichs inequality,

$$\|\nabla^A\omega\|^2_{L^2(M)} + \|\omega\|_2^2 \le const.(\|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + \lambda\|\omega\|_2^2). \tag{2.18}$$

This will be the key input to most of our results, including gauge invariant apriori estimates and regularity of solutions in the gauge covariant derivative sense. See Remark 2.9.

In the classical case, i.e., real valued forms, such equivalences go back to Gaffney, [13], and Friedrichs, [12]. See also Eells and Morrey, [41], for a very early work in this direction. The constants in these classical inequalities depend on the Riemannian curvature of M and the curvature of its boundary. M. Mitrea, [39], has shown that such inequalities can be established with no dependence on the Riemannian curvature of M and only mild dependence (convexity) on the curvature of the boundary in these classical cases. The benefit of using a convex domain for real valued forms was observed early on by Saranen, [47], for a convex domain in $\mathbb{R}^3$. A reader may consult the book by Taylor, [57, pp. 361–364] for a recent derivation of the Gaffney-Friedrichs inequality in the classical case and [38] for extensions to nonsmooth Riemannian manifolds in the classical case.

Our concern here is primarily with the dependence of the constant λ on the curvature of the connection form A.

Notation 2.16. Define the gauge invariant version of (2.1) by

$$\|\omega\|^2_{W_1^A(M)} = \|\nabla^A\omega\|^2_{L^2(M)} + \|\omega\|^2_{L^2(M)} \tag{2.19}$$

for any $\mathfrak{k}$ valued r-form ω on M. By Sobolev's inequality, there exists a constant κ, depending on the geometry of M but not on A, such that $\|\omega\|_6^2 \le (\kappa^2/2)(\int_M |grad|\omega|\,|^2 + \|\omega\|_2^2)$ for all $\omega \in W_1(M)$. (See e.g., [15, Thm. 7.26].) In view of Kato's inequality, $\int_M |grad|\omega|\,|^2 \le \|\nabla^A\omega\|_2^2$, it follows that

$$\|\omega\|_6^2 \le (\kappa^2/2)\|\omega\|^2_{W_1^A(M)} \quad \text{for } \omega \text{ and } A \in W_1(M). \tag{2.20}$$

Theorem 2.17 (Gauge invariant Gaffney-Friedrichs inequality). *Suppose that M is a compact Riemannian 3-manifold with smooth boundary and that A is a $\mathfrak{k}$ valued 1-form in $W_1(M)$ with curvature B. Let $p \in [2, \infty]$ and assume that $\|B\|_{L^p(M)} < \infty$. There* *are constants λ_M and γ_p, depending only on the geometry of M and not on A, such that, with*

$$\lambda_p(B) := \lambda_M + \gamma_p \|B\|_p^{2p/(2p-3)}, \tag{2.21}$$

there holds

$$(1/2)\|\omega\|^2_{W_1^A(M)} \leq \|d_A\omega\|^2_{L^2(M)} + \|d_A^*\omega\|^2_{L^2(M)} + \lambda_p(B)\|\omega\|_2^2 \tag{2.22}$$

for any $\mathfrak{k}$ valued r-form ω in $W_1(M)$ satisfying either

$$\omega_{tan} = 0 \quad or \quad \omega_{norm} = 0. \tag{2.23}$$

Here d_A is the covariant exterior derivative with domain matching the boundary condition on ω and d_A^* is its adjoint. If M is a convex subset of $\mathbb{R}^3$ one can take $\lambda_M = 1$.

Theorem 2.17 will be proven in Sect. 4. We want to emphasize that in this theorem, as well as all of its consequences, we aim to establish estimates that do not depend significantly on the size of the manifold M, because in our intended application we will allow $M \subset \mathbb{R}^3$ to expand to all of $\mathbb{R}^3$.

Remark 2.18 (Case $p = 2$). If $p = 2$ then (2.21) reduces to

$$\lambda_2(B) = \lambda_M + \gamma_2\|B\|_2^4, \tag{2.24}$$

with $\gamma_2 = (1/4)(3\kappa^2)^3c^4$. This is the case that will be needed in this paper. The constant $c \equiv \sup\{\|ad\, x\|_{\mathfrak{k}\to\mathfrak{k}} : |x|_{\mathfrak{k}} \leq 1\}$ measures the non-commutativity of K and is zero if K is commutative.

Remark 2.19 (Continuous dependence on initial data). The solution to (2.14) described in Theorem 2.13 is easily shown to depend continuously on the initial data A_0 in W_1 norm. So do the solutions in Theorems 2.5, 2.7 and 2.12. But the proofs for these three cases will be postponed to a later work in which the initial data space will be enlarged to include $H_{1/2}$ data.

3. Dirichlet and Neumann Boundary Conditions

In this section we will extend some of the machinery developed by Conner, [7], for real valued differential forms to forms in a product bundle over M with a connection. See [57, Chap. 5, Sect. 9] for a recent exposition of the real valued case. Our objective is to develop the mechanisms needed to make effective use of the gauge invariant Gaffney-Friedrichs inequality of Sect. 4.

3.1. The minimal and maximal exterior derivatives.

Notation 3.1. M^{int} will denote the interior of the compact Riemannian 3-manifold M. Denote by δ the coderivative on $C^\infty(M; \Lambda^{p+1} \otimes \mathfrak{k})$. Thus if ϕ is a $\mathfrak{k}$ valued p - form in $C_c^\infty(M^{int})$ and $\omega \in C^\infty(M; \Lambda^{p+1} \otimes \mathfrak{k})$ then

$$(\phi, \delta\omega)_{L^2(M;\Lambda^p\otimes\mathfrak{k})} = (d\phi, \omega)_{L^2(M;\Lambda^{p+1}\otimes\mathfrak{k})}. \tag{3.1}$$

If $u = \sum_{|I|=r} u_I dx^I$ and $v = \sum_{|J|=p} v_J dx^J$ are $End\,\mathscr{V}$ valued forms then their wedge product, $u \wedge v = \sum_{I,J} u_I v_J dx^I \wedge dx^J$, is another $End\,\mathscr{V}$ valued form. But when the appropriate action of u on v is via $ad\, u$ then we will write $[u \wedge v] = \sum_{I,J} [u_I, v_J] dx^I \wedge dx^J$. This will be the case when u is an $End\,\mathscr{V}$ valued connection form or its time derivative. If u and v take their values in $\mathfrak{k}$ then so does $[u \wedge v]$.

The interior product, $[u \lrcorner v]$, of an element $u \in \Lambda^p \otimes \mathfrak{k}$ with an element $v \in \Lambda^{p+r} \otimes \mathfrak{k}$ is defined, for $r \geq 0$, by

$$\langle w, [u \lrcorner v] \rangle_{\Lambda^r \otimes \mathfrak{k}} = \langle [u \wedge w], v \rangle_{\Lambda^{p+r} \otimes \mathfrak{k}} \text{ for all } w \in \Lambda^r \otimes \mathfrak{k}. \tag{3.2}$$

If u or v or both are real valued then we will write simply $u \lrcorner v$ since the commutator bracket should be omitted. If u and v are both in $\Lambda^1 \otimes \mathfrak{k}$ then (3.2) gives $\mathfrak{k} \ni [u \lrcorner v] = -[u \cdot v] = -\sum_j [u_j, v_j]$ in an orthonormal frame for Λ^1. And if $w \in \Lambda^2 \otimes \mathfrak{k}$ then $[w \lrcorner w] = 0$.

We wish to consider a connection on the product bundle $M \otimes \mathscr{V} \to M$. We may and will identify the connection with a $\mathfrak{k}$ valued 1-form A on M. The corresponding gauge covariant exterior derivative is then given by $d_A \omega = d\omega + [A \wedge \omega]$ on smooth $\mathfrak{k}$ valued forms. However we are going to use the symbols d and d_A for the closed versions of these differential operators as follows.

Notation 3.2. Denote by D the closure of the exterior derivative operator defined initially on $\mathfrak{k}$ valued p-forms in $C^\infty(M)$. Denote by d the closure of $D|C_c^\infty(M^{int})$. Then $d \subset D$. D and d are the maximal and minimal exterior derivative operators respectively.

For $A \in L^\infty(M; \Lambda^1 \otimes \mathfrak{k})$ define

$$D_A \omega = D\omega + [A \wedge \omega] \quad \text{for} \quad \omega \in \mathscr{D}(D), \tag{3.3}$$

$$d_A \omega = d\omega + [A \wedge \omega] \quad \text{for} \quad \omega \in \mathscr{D}(d). \tag{3.4}$$

The Hodge star operator $*$ on forms defines a unitary map from L^2 forms to itself with the following properties:

$$D_A^* = *^{-1}(d_A)*, \tag{3.5}$$

$$d_A^* = *^{-1}(D_A)*, \tag{3.6}$$

$$W_1 \subset \mathscr{D}(D_A) \cap \mathscr{D}(d_A^*), \tag{3.7}$$

$$d_A = (\delta_A)^*, \tag{3.8}$$

$$D_A = (\delta_A | C_c^\infty(M^{int}))^*, \tag{3.9}$$

where

$$\delta_A \omega = \delta\omega + [A \lrcorner \omega] \quad \text{for} \quad \omega \in C^\infty(M; \Lambda^p \otimes \mathfrak{k}), \quad p \geq 1. \tag{3.10}$$

Remark 3.3. D_A and d_A^* are maximal operators in the sense that their domains are restricted only by size and regularity and not by boundary conditions. However the domains of their adjoints, D_A^* and d_A are restricted also by boundary conditions as follows.

The symbol (D) in front of an equation will signify that the equation is relevant for Dirichlet boundary conditions. An (N) signifies that the equation is relevant for Neumann boundary conditions.

Lemma 3.4. *Suppose that* $\omega \in W_1(M; \Lambda^p \otimes \mathfrak{k})$ *and* $A \in L^\infty(M)$. *Then*

$$(D) \qquad \omega \in \mathscr{D}(d_A) \text{ if and only if } \omega_{tan} = 0, \tag{3.11}$$

$$(N) \qquad \omega \in \mathscr{D}(D_A^*) \text{ if and only if } \omega_{norm} = 0. \tag{3.12}$$

Proof. These boundary conditions are already known for the minimal and maximal operators when $A = 0$. See [7]. Since A is bounded the domains are the same as for $A = 0$. □

Proposition 3.5. *Assume that* ω *is a* $\mathfrak{k}$ *valued form and that* $A \in W_1 \cap L^\infty$. *Denote the curvature of* A *by* B, *as in* (2.2). *If* $[B \wedge \omega] \in L^2$ *then*

$$(N) \quad \omega \in \mathscr{D}(D_A) \text{ implies } \omega \in \mathscr{D}((D_A)^2) \text{ and } D_A^2\omega = [B \wedge \omega], \tag{3.13}$$

$$\text{and } (D) \quad \omega \in \mathscr{D}(d_A) \text{ implies } \omega \in \mathscr{D}((d_A)^2) \text{ and } d_A^2\omega = [B \wedge \omega]. \tag{3.14}$$

If $[B \lrcorner \omega] \in L^2$ *then*

$$(D) \quad \omega \in \mathscr{D}(d_A^*) \text{ implies } \omega \in \mathscr{D}((d_A^*)^2) \text{ and } (d_A^*)^2\omega = [B \lrcorner \omega], \tag{3.15}$$

$$\text{and } (N) \quad \omega \in \mathscr{D}(D_A^*) \text{ implies } \omega \in \mathscr{D}((D_A^*)^2) \text{ and } (D_A^*)^2\omega = [B \lrcorner \omega]. \tag{3.16}$$

Proof. It will be clarifying to distinguish the closed operators d_A and D_A from the pointwise defined differential operator $\{d_A\}$ acting on smooth forms, and which ignores boundary conditions. If A and ω are in $C^\infty(M)$, then the Bianchi identity $\{d_A\}^2\omega = [B \wedge \omega]$ holds and we need only address domain issues in the four assertions of the proposition. To this end observe that if ω and u are both in $C^\infty(M)$ and one has compact support in M^{int} then we may integrate by parts to find

$$(\{d_A\}\omega, \{\delta_A\}u) = (\{d_A\}^2\omega, u) = ([B \wedge \omega], u) = (\omega, [B \lrcorner u]). \tag{3.17}$$

Since the first, third and fourth terms are continuous in A in the W_1 norm the equality of these terms persists for $A \in W_1$.

Now since $C^\infty(M)$ is a core for D_A and the far right side is continuous in ω in the L^2 norm, it follows that

$$(D_A\omega, \{\delta_A\}u) = (\omega, [B \lrcorner u]) = ([B \wedge \omega], u) \tag{3.18}$$

for all $\omega \in \mathscr{D}(D_A)$ and $u \in C_c^\infty(M^{int})$. Since $[B \wedge \omega] \in L^2$ the right side is continuous in u in the L^2 norm and therefore so is $(D_A\omega, \{\delta_A\}u)$. Hence $D_A\omega \in \mathscr{D}(D_A)$, by (3.9) and $(D_A^2\omega, u) = ([B \wedge \omega], u)$. This proves (3.13).

To prove (3.14) take $\omega \in C_c^\infty(M^{int})$ and $u \in C^\infty(M)$ in (3.17). Then $\omega \in \mathscr{D}(d_A)$ and, since $C_c^\infty(M^{int})$ is a core for d_A and $[B \lrcorner u] \in L^2$, equality of the first and fourth terms in (3.17) implies that $(d_A\omega, \{\delta_A\}u) = (\omega, [B \lrcorner u]) = ([B \wedge \omega], u)$ for all $\omega \in \mathscr{D}(d_A)$ and $u \in C^\infty(M)$. Since $[B \wedge \omega] \in L^2(M)$ the equality of the first and third terms now shows that $(d_A\omega, \{\delta_A\}u)$ is continuous in u in L^2 norm and therefore $d_A\omega \in \mathscr{D}(d_A)$. Thus $((d_A)^2\omega, u) = ([B \wedge \omega], u)$ for all $u \in C^\infty(M)$. This proves (3.14).

The assertions (3.15) and (3.16) could be derived in the same way as (3.13) and (3.14). But they also follow directly from these by use of (3.5) and (3.6). Thus if $\omega \in \mathscr{D}(d_A^*)$, then (3.6) shows that $*\omega \in \mathscr{D}(D_A)$. By (3.13) $*\omega$ is therefore in $\mathscr{D}(D_A^2)$. It now follows from (3.6) again that $\omega \in \mathscr{D}((d_A^*)^2)$. Of course $(d_A^*)^2\omega = [B \lrcorner \omega]$ since the adjoint of $[B \wedge \cdot]$ is $[B \lrcorner \cdot]$ by (3.2). The proof of (3.15) is similar. □

Corollary 3.6. *Suppose that ω is a $\mathfrak{k}$ valued p-form in W_1, that $A \in W_1 \cap L^\infty$ and that the function $x \mapsto |B(x)||\omega(x)|$ is in $L^2(M)$.*

$$(D) \qquad \text{If } \omega_{tan} = 0 \text{ and } d_A\omega \in W_1 \text{ then } (d_A\omega)_{tan} = 0. \tag{3.19}$$

$$(N) \qquad \text{If } \omega_{norm} = 0 \text{ and } D_A^*\omega \in W_1 \text{ then } (D_A^*\omega)_{norm} = 0. \tag{3.20}$$

Proof. If $\omega \in W_1$ and $\omega_{tan} = 0$ then $\omega \in \mathscr{D}(d_A)$ by (3.11). From (3.14) we see that $d_A\omega \in \mathscr{D}(d_A)$. Therefore if $d_A\omega \in W_1$ then $(d_A\omega)_{tan} = 0$ by (3.11). This proves (3.19). The proof of (3.20) follows from (3.12) and (3.16) similarly. □

Corollary 3.7 (Functional Bianchi identity). *Assume that $A \in W_1 \cap L^\infty$. Then*

$$B \in \mathscr{D}(D_A) \quad \text{and} \quad D_A B = 0. \tag{3.21}$$

If, moreover, $A_{tan} = 0$ and $B \in W_1$, then

$$(D) \qquad B_{tan} = 0, \ B \in \mathscr{D}(d_A) \text{ and } d_A B = 0. \tag{3.22}$$

Proof. For $A \in C^\infty(M)$ and $u \in C_c^\infty(M^{int})$ an integration by parts and Bianchi's identity gives $(B, \delta_A u) = (\{d_A\}B, u) = 0$. The left side of this identity is continuous in $A \in W_1$ and therefore

$$(B, \delta_A u) = 0 \tag{3.23}$$

for all $A \in W_1$ and all $u \in C_c^\infty(M^{int})$. Since the right side of this identity, being zero, is continuous in u in the L^2 norm, it follows that $B \in \mathscr{D}((\delta_A|C_c^\infty(M^{int}))^*) = \mathscr{D}(D_A)$ and that $D_A B = 0$, proving (3.21).

Now suppose that $A_{tan} = 0$ and that $B \in W_1$. Take $\omega = A$ in (3.11) and take the form A of that lemma to be our present $A/2$. It follows that $A \in \mathscr{D}(d_{A/2})$. But $d_{A/2}A = B$. Further, we see that $[B \wedge \omega] = [B \wedge A] \in L^2$ because $B \in L^2$ and $A \in L^\infty$. Hence $B \in \mathscr{D}(d_{A/2})$ by (3.14). Reapplying (3.11) again we find that $B_{tan} = 0$. Equation (3.11) now shows that $B \in \mathscr{D}(d_A)$. But $d_A B = D_A B = 0$, by (3.21). □

Corollary 3.8. *Assume that $A \in W_1 \cap L^\infty$ and $B \in W_1$. Then $B \in \mathscr{D}((d_A^*)^2)$ and*

$$(d_A^*)^2 B = 0. \tag{3.24}$$

If, in addition, $B_{norm} = 0$ then $B \in \mathscr{D}((D_A^*)^2)$ and

$$(D_A^*)^2 B = 0. \tag{3.25}$$

Proof. We see that $B \in W_1 \subset \mathscr{D}(d_A^*)$, by (3.7), and, since $[B \lrcorner B] = 0$, we may choose $\omega = B$ in (3.15), from which it follows that $B \in \mathscr{D}((d_A^*)^2)$ and that (3.24) holds. Suppose, further, that $B_{norm} = 0$. Then (3.12) implies that $B \in \mathscr{D}(D_A^*)$. Therefore (3.16) now shows that $D_A^* B \in \mathscr{D}(D_A^*)$ and $(D_A^*)^2 B = [B \lrcorner B] = 0$, which proves (3.25). □

Remark 3.9. For real valued forms the use of the maximal and minimal operators D and d goes back to Conner, [7]. In particular, Proposition 3.5 in the real valued case, which simply reads $d^2 = 0, \ D^2 = 0, \ (d^*)^2 = 0$ and $(D^*)^2 = 0$, along with proper statements about the domains, was proved by Conner [7, page 9].

Remark 3.10. A 1-form ω need not be even weakly differentiable in order to be in the domain of the minimal operators d or d_A. For example if $M \subset \mathbb{R}^3$ and f is a $\mathfrak{k}$ valued smooth function with compact support in M^{int} of the form $f(x_1, x_2, x_3) = h(x_1)g(x_2, x_3)$ then, defining $\omega = f dx_1$, one has $d\omega = h(x_1)\{(\partial_2 g)dx_2 \wedge dx_1 + (\partial_3 g)dx_3 \wedge dx_1\}$ so that h is not differentiated. Thus if we now allow $h \in L^2(\mathbb{R}^1; \mathfrak{k})$ and $g \in C_c^\infty(\mathbb{R}^2)$, still insisting that support $f \subset M^{int}$, then the resulting form ω can easily be approximated in the graph norm of d by functions in $C_c^\infty(M^{int})$ of the same form. So $\omega \in \mathscr{D}(d)$. This example also shows that if A is unbounded then d and d_A will not have the same domain. One need only take $A = A_2(x_1)dx_2$. Then $d_A\omega - d\omega = [A_2(x_1), h(x_1)]g(x_2, x_3)dx_2 \wedge dx_1$, which need not be in $L^2(M)$ if A is unbounded and $h \in L^2(\mathbb{R}; \mathfrak{k})$. We conjecture, however that all results in this section will remain valid if the condition $A \in L^\infty$ is replaced by $A \in L^3$.

4. Gauge Invariant Gaffney-Friedrichs-Sobolev Inequalities

In this section we will prove Theorem 2.17 and derive from it Sobolev inequalities in a form that will be needed for establishing gauge invariant apriori estimates. The exterior derivative operators d and d_A and their adjoints are to be interpreted in this section as acting on smooth forms or on W_1 forms, as indicated, without boundary conditions built in.

4.1. A gauge invariant Gaffney identity.

Theorem 4.1 (A gauge invariant Gaffney identity). *Let M be a compact Riemannian n-manifold with smooth boundary. Suppose that $A \in C^\infty(M; \Lambda^1 \otimes \mathfrak{k})$. Let α and β be smooth $\mathfrak{k}$ valued p-forms on M with either*

$$\alpha_{tan} = \beta_{tan} = 0 \quad on\, \partial M \quad or \quad \alpha_{norm} = \beta_{norm} = 0 \quad on\, \partial M. \tag{4.1}$$

Then

$$(d_A\alpha, d_A\beta) + (d_A^*\alpha, d_A^*\beta) - (\nabla^A\alpha, \nabla^A\beta) - ((W + B)\circ\alpha, \beta) = \int_{\partial M} \langle K(x)\alpha(x), \beta(x)\rangle, \tag{4.2}$$

where W denotes the Riemannian Bochner-Weitzenboch operator, B is the curvature of A, $\circ$ denotes a pointwise product operation,

$$(\nabla^A\alpha, \nabla^A\beta) = \sum_{i=1}^n (\nabla^A_{e_i}\alpha, \nabla^A_{e_i}\beta)$$

locally, for any orthonormal frame field $e_1, \ldots, e_n$ of $T(M)$ and

$$K(x) : \Lambda^p(T_x(\partial M)) \to \Lambda^p(T_x(\partial M))$$

is a symmetric operator, bounded uniformly in x, and dependent only on the second fundamental form of ∂M, on the value of p and on the choice of boundary condition in (4.1). Moreover, $K(x) \geq 0$ for all $x \in \partial M$ if M is convex in the sense that the second fundamental form is non-negative on ∂M.

In particular, if ω is a smooth $\mathfrak{k}$ valued p-form on M satisfying either

$$\omega_{tan} = 0 \quad or \quad \omega_{norm} = 0 \tag{4.3}$$

then

$$\|\nabla^A \omega\|_2^2 = \|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 - (B \circ \omega, \omega) - (W \circ \omega, \omega) - \int_{\partial M} \langle K(x)\omega(x), \omega(x)\rangle. \tag{4.4}$$

The proof depends on the following lemmas. It is important for our applications that the boundary terms in (4.2) above do not depend on the gauge connection form. For this reason we are going to carry out explicitly what is otherwise a standard kind of integration by parts computation.

Notation 4.2 (Adapted coordinates). We will make use in this section of an adapted coordinate system for a neighborhood U containing a part of the boundary of M. This is a coordinate system $x = (x^1, \dots, x^n)$ in U such that a) $|x^j| < 1$ for $j = 1, \dots, n-1$ and $-1 < x^n \le 0$, while $U \cap \partial M = \{x : x^n = 0\}$. $(x^1, \dots, x^{n-1})$ form coordinates on $U \cap \partial M$. b) The curve $(-1, 0] \ni t \mapsto x(t) = (a^1, \dots, a^{n-1}, t)$ is a geodesic normal to ∂M at $t = 0$, and $\langle \partial/\partial x^n, \partial/\partial x^j\rangle = 0$ on U for $j = 1, \dots, n-1$. See for example [45, page 167] for the existence of such a coordinate chart.

Lemma 4.3. *Assume that $A \in C^\infty(M; \Lambda^1 \otimes \mathfrak{k})$ and that α and β are smooth $\mathfrak{k}$ valued p-forms on M. Then*

$$\begin{aligned}(d_A\alpha, d_A\beta) + (d_A^*\alpha, d_A^*\beta) - (\nabla^A\alpha, \nabla^A\beta) - ((W + B) \circ \alpha, \beta)\\ = L^A(\alpha, \beta),\end{aligned} \tag{4.5}$$

where

$$L^A(\alpha, \beta) = \int_{\partial M} \{\langle \nu \wedge \beta, d_A\alpha\rangle - \langle \beta, \nu \wedge d_A^*\alpha\rangle - \langle \beta, \nabla_\nu^A \alpha\rangle\}. \tag{4.6}$$

Here ν is the outward drawn unit co-normal and ∇_ν^A is the covariant gradient in the normal direction.

Proof. The Bochner-Weitzenboch formula for a $\mathfrak{k}$ valued p-form on M is

$$\{d_A^* d_A + d_A d_A^*\}\alpha - (W + B) \circ \alpha = (\nabla^A)^* \nabla^A \alpha, \tag{4.7}$$

which may be found in [4]. We need only take the inner product of (4.7) with β and do three integrations by parts to deduce (4.5). Two of the integrations by parts will follow from Stokes' theorem,

$$(d_A\omega, u) - (\omega, \delta_A u) = (\nu \wedge \omega, u)_{\partial M}, \ \ \omega \in C^\infty(M), \ \ u \in C^\infty(M), \tag{4.8}$$

which itself can be derived from the standard Stokes theorem by observing first that the terms involving the connection form A cancel on the left, in view of (3.1), (3.2), (3.3) and (3.10), and second, that the resulting identity holds for forms $\omega \in C^\infty(M)$ and $u \in C^\infty(M)$ because it holds for the real valued components of these forms with respect to an orthonormal basis of $\mathfrak{k}$.

Now inserting first $\omega = \beta,\ u = d_A\alpha$ into (4.8) and then inserting $\omega = d_A^*\alpha,\ u = \beta$ into (4.8) we find, respectively,

$$\langle d_A\beta, d_A, \alpha) = (\beta, d_A^* d_A\alpha) + \int_{\partial M} \langle \nu \wedge \beta, d_A\alpha\rangle,$$
$$\langle d_A^*\alpha, d_A^*\beta\rangle = (\beta, d_A d_A^*\alpha) - \int_{\partial M} \langle \nu \wedge d_A^*\alpha, \beta\rangle.$$

Combining these with (4.7) we find that the left side of (4.5) is equal to $((\nabla^A)^*(\nabla^A)\alpha, \beta) - ((\nabla^A)\alpha, (\nabla^A)\beta) + (\nu \wedge \beta, d_A\alpha)_{\partial M} - (\nu \wedge d_A^*\alpha, \beta)_{\partial M}$.

To complete the proof of (4.5) it suffices to show that

$$((\nabla^A)^*(\nabla^A)\alpha, \beta) - ((\nabla^A)\alpha, (\nabla^A)\beta) = -\int_{\partial M} \langle \nabla_\nu^A \alpha, \beta\rangle. \tag{4.9}$$

For the needed integration by parts we may write, with the help of a partition of unity, $\alpha = \alpha_0 + \sum_{j=1}^r \alpha_j$, where α_0 is supported in M^{int} and each α_j is supported in an adapted coordinate patch U_j. For an arbitrary $\mathfrak{k}$ valued p-form β in $C^\infty(M)$ the identity (4.9) holds for α_0 and β by an integration by parts because there are no boundary terms. It suffices therefore to prove (4.9) for each α_j. To this end we will prove (4.9) in case $\beta \in C^\infty(M)$ while α is supported in an adapted coordinate patch $U \subset M$.

For any smooth vector field X on U and real valued function $f \in C_c^\infty(U)$ we may apply the identity $\int_U Xf + \int_U f(div\ X) = \int_{\partial U} f(\nu \cdot X)$, to the real valued function $f(x) = \langle \omega(x), \beta(x)\rangle_{\Lambda^p \otimes \mathfrak{k}}$ to find

$$\int_U (div\ X)\langle \omega, \beta\rangle + (\nabla_X^A \omega, \beta) + (\omega, \nabla_X^A \beta) = \int_{\partial M} \langle \omega, \beta\rangle(\nu \cdot X)$$

for any p-form $\omega \in C_c^\infty(U)$. We read off from this that the formal adjoint of ∇_X^A is given by $(\nabla_X^A)^*\omega = -\nabla_X^A\omega - (div\ X)\omega$ and that

$$((\nabla_X^A)^*\omega, \beta) = (\omega, \nabla_X^A \beta) - \int_{\partial M} \langle \omega, \beta\rangle(\nu \cdot X). \tag{4.10}$$

Choose an orthonormal frame field $e_1, \ldots, e_n$ in the coordinate patch U and apply (4.10) with $X = e_j$ and $\omega = \nabla_{e_j}^A \alpha$ to find

$$((\nabla_{e_j}^A)^* \nabla_{e_j}^A \alpha, \beta) = (\nabla_{e_j}^A \alpha, \nabla_{e_j}^A \beta) - \int_{\partial M} \langle \nabla_{e_j}^A \alpha, \beta\rangle \nu \cdot e_j. \tag{4.11}$$

Summing over j gives (4.9). □

Unlike Stokes' theorem, (4.8), the connection form A shows up in the boundary term $L^A(\alpha, \beta)$ of (4.6). We may disentangle the A dependence in $L^A(\alpha, \beta)$. We find

$$L^A(\alpha, \beta) = \int_{\partial M} \{\langle \nu \wedge \beta, d\alpha\rangle - \langle \beta, \nu \wedge d^*\alpha\rangle - \langle \beta, \nabla_\nu \alpha\rangle\} \tag{4.12}$$
$$+ \int_{\partial M} \{\langle \nu \wedge \beta, [A \wedge \alpha]\rangle - \langle \beta, \nu \wedge [A \lrcorner \alpha]\rangle - \langle \beta, [A_\nu, \alpha]\rangle\}, \tag{4.13}$$

where $A_\nu(x)\nu = A_{norm}(x)$ is the normal component of A at x. It will be important for us that the boundary term be independent of A when the p-forms α and β satisfy appropriate boundary conditions.

Lemma 4.4. *The integrand in line* (4.13) *is zero at a point $x \in \partial M$ if either*

$$\alpha_{tan} = \beta_{tan} = 0 \quad at\ x \tag{4.14}$$

or

$$\alpha_{norm} = \beta_{norm} = 0 \quad at\ x. \tag{4.15}$$

$A(x)$ need not satisfy any boundary condition in either case.

Proof. Fix $x \in \partial M$. Assume first that $\alpha_{tan} = \beta_{tan} = 0$ at x. Then $\nu \wedge \beta = 0$. So the first term in (4.13) is zero at x. We assert that the remaining two terms cancel. Indeed, since $\beta_{tan} = 0$ we may write $\beta = \nu \wedge \phi$ at x with $\phi_{norm} = 0$. Then $\langle \beta, \nu \wedge [A \lrcorner \alpha] \rangle = \langle \nu \wedge \phi, \nu \wedge [A \lrcorner \alpha] \rangle = \langle \phi, [A \lrcorner \alpha] \rangle = \langle [A \wedge \phi], \alpha \rangle = \langle [A_\nu, \nu \wedge \phi] + \text{a tangential term}, \alpha \rangle = -\langle \nu \wedge \phi, [A_\nu, \alpha] \rangle = -\langle \beta, [A_\nu, \alpha] \rangle$. Thus the second and third terms in (4.13) cancel.

Assume next that $\alpha_{norm} = \beta_{norm} = 0$ at x. The middle term is zero because $\beta_{norm} = 0$. We assert that the first and third terms cancel. Indeed $\langle \nu \wedge \beta, [A \wedge \alpha] \rangle = \langle \nu \wedge \beta, [A_\nu, \nu \wedge \alpha] + \text{a tangential term} \rangle = \langle \nu \wedge \beta, \nu \wedge [A_\nu, \alpha] \rangle = \langle \beta, [A_\nu, \alpha] \rangle$, which shows that the first and third terms in (4.13) cancel. □

Remark 4.5. It is illuminating to understand when the integrand in (4.13) is identically zero, independently of boundary conditions on α and β. It can be shown that

a) the integrand is zero at a point $x \in \partial M$ for all α and β if $A_{tan}(x) = 0$.
b) If $\mathfrak{k}$ is semisimple and $\alpha_{tan}(x) = 0$ then there exist A and β such that the integrand is not zero at x.

We omit the proofs.

Notation 4.6 (Extended shape operator). An adapted coordinate system (see Notation 4.2) will be useful for describing the shape operator and its extension to the exterior algebra. Writing $\partial_j = \partial/\partial x^j$, the outward drawn unit normal and co-normal are given by ∂_n and $\nu = dx^n$, respectively, on $U \cap \partial M$. The shape operator at a point $P \in U \cap \partial M$ is given by $S(X) = \nabla_X \partial_n$ for $X \in T_P(\partial M)$, [14, page 217], where ∇_X is the Riemannian covariant derivative. The adjoint $S^* \in End(T_P^*(\partial M))$ extends uniquely to a derivation Q of the exterior algebra $\Lambda(T_P^*(\partial M))$. We may identify $\Lambda(T_P^*(\partial M))$ with the algebra of exterior polynomials in the 1-forms $dx^1, \ldots, dx^{n-1}$ with constant coefficients. The action of Q on such an exterior polynomial ω is given by

$$-(\nabla_n \omega)|_{\partial M} = Q(\omega|_{\partial M}), \tag{4.16}$$

as one sees by observing first, that $\nabla_n \partial_n = 0$ because $t \mapsto (a_1, \ldots, a_{n-1}, t)$ is a geodesic, second, that ∇_n therefore leaves invariant the span of $dx^1, \ldots, dx^{n-1}$, third, that $S^* = \nabla_n^* = -\nabla_n$ on this span, and finally, that Q and $-\nabla_n$ are derivations of this algebra.

Proof of Theorem 4.1. In view of (4.5) and Lemma 4.4 we can ignore the connecton form A and just show that the integrand in (4.12) has the form assserted in (4.2). Explicitly, we will show that

$$\langle K(x)\alpha(x), \beta(x) \rangle = \begin{cases} \langle \{I_{\mathfrak{k}} \otimes Q(x)\}\alpha(x), \beta(x) \rangle & \text{if } \alpha_{norm} = \beta_{norm} = 0 \\ \langle \{I_{\mathfrak{k}} \otimes (*^{-1} Q(x) *)\}\alpha(x), \beta(x) \rangle & \text{if } \alpha_{tan} = \beta_{tan} = 0. \end{cases} \tag{4.17}$$

Assume first that $\alpha_{norm} = \beta_{norm} = 0$. Choose an orthonormal basis $e_1, \dots, e_d$ of $\mathfrak{k}$ and write $\alpha = \sum_{i=1}^d e_i \alpha^i$ and $\beta = \sum_{i=1}^d e_i \beta^i$, where α^i and β^i are real valued p-forms. Then the integrand in (4.12) is

$$\sum_{i=1}^{d} \{ \langle \nu \wedge \beta^i, d\alpha^i \rangle_{\Lambda^{p+1}} - \langle \beta^i, \nu \wedge d^* \alpha^i \rangle_{\Lambda^p} - \langle \beta^i, \nabla_\nu \alpha^i \rangle \}$$

at a point $P \in \partial M$. It suffices to show that this has the form $\sum_{i=1}^d \langle Q(x)\alpha^i, \beta^i \rangle$, for then one can take $K(x) = I_{\mathfrak{k}} \otimes Q(x)$ in (4.2).

Now $\alpha(x)_{norm} = 0$ if and only if $\alpha^i(x)_{norm} = 0$ for each i. Thus it suffices to prove that the integrand in (4.12) is equal to $\langle Q(x)\alpha(x), \beta(x) \rangle$ when α and β are real valued p-forms such that $\alpha_{norm} = \beta_{norm}$ on $U \cap \partial M$. In this case the middle term in (4.12), $\langle \beta, \nu \wedge d^* \alpha \rangle = 0$ and we are left with $\langle \beta, \nu \lrcorner d\alpha - \nabla_\nu \alpha \rangle$.

We will compute this in an adapted coordinate system. We may write

$$\alpha(x) = \sum_{J<n} a_J(x) dx^J + \sum_{I<n} b_I(x) dx^I \wedge dx^n. \tag{4.18}$$

Here and below $J = (j_1, \dots j_p)$ with $j_1 < \cdots < j_p < n$ and $I = (i_1, \dots, i_{p-1})$ with $i_1 < \cdots < i_{p-1} < n$. Moreover $b_I(x) = 0$ if $x \in U \cap \partial M$. Then

$$\begin{aligned} \nu \lrcorner (d\alpha(x)) &= \sum_{J<n} \{ \nu \lrcorner \sum_{k=1}^{n-1} \partial_k a_J(x) dx^k \wedge dx^J + \nu \lrcorner (\partial_n a_J(x)) dx^n \wedge dx^J \} \\ &\quad + \sum_{I<n} \{ \nu \lrcorner \sum_{k=1}^{n-1} \partial_k b_I(x) dx^k \wedge dx^I \wedge dx^n \} \\ &= \sum_{J<n} \partial_n a_J(x) dx^J \end{aligned}$$

because $\nu \lrcorner (dx^k \wedge dx^J) = 0$ and $\partial_k b^I(x) = 0$ on $U \cap \partial M$ for $k = 1, \cdots, n-1$.

On the other hand, on ∂M,

$$\begin{aligned} \nabla_\nu \alpha = \nabla_n \alpha &= \sum_{J<n} \{ (\partial_n a_J) dx^J + a_J \nabla_n (dx^J) \} \\ &\quad + \sum_{I<n} \{ (\partial_n b_I) dx^I \wedge dx^n + b_I \nabla_n (dx^I \wedge dx^n) \}. \end{aligned}$$

On ∂M, therefore, we find some cancellation in the following difference and, since $b_I = 0$ on ∂M, we arrive at

$$\nu \lrcorner (d\alpha(x)) - \nabla_\nu \alpha(x) = -\sum_{J<n} a_J(x) \nabla_n (dx^J) - \sum_{I<n} (\partial_n b_I(x)) dx^I \wedge dx^n.$$

Finally, since $\beta_{norm} = 0$ we find, at $x \in \partial M$, in view of (4.16),

$$\begin{aligned} \langle \beta, \nu \lrcorner d\alpha - \nabla_\nu \alpha \rangle &= -\sum_{J<n} \langle \beta, a_J \nabla_n (dx^J) \rangle \\ &= \sum_{J<n} \langle \beta, a_J Q dx^J \rangle \\ &= \langle \beta, Q(x)\alpha \rangle. \end{aligned} \tag{4.19}$$

This proves (4.17) and (4.2) if $\alpha_{norm} = \beta_{norm} = 0$.

In the case $\alpha_{tan} = \beta_{tan} = 0$ we may reduce to real valued forms in the same way as above. Denoting the Hodge star operator on $\Lambda(T^*(M))$ by $*$ we can reduce this case to the preceding by applying the preceding case to the $n-p$ forms $*\alpha$ and $*\beta$, which, as is well known, satisfy now $(*\alpha)_{norm} = (*\beta)_{norm} = 0$. Applying the identity (4.19) to these two forms we find

$$\langle *\beta, \nu \lrcorner d * \alpha - \nabla_\nu * \alpha\rangle_{\Lambda^{n-p}} = \langle *\beta, Q(x) * \alpha\rangle_{\Lambda^{n-p}},$$

and therefore

$$\langle \beta, *^{-1}(\nu \lrcorner (d * \alpha)) - *^{-1}\nabla_\nu * \alpha\rangle_{\Lambda^p} = \langle \beta, *^{-1} Q(x) * \alpha\rangle_{L^p}.$$

But $*^{-1}\nabla_\nu * = \nabla_\nu$, while

$$*^{-1}(\nu \lrcorner (d * \alpha)) = -\nu \wedge d^*\alpha$$

by [38, Lemma 4.1, items (1), (6) and (10)]. Hence

$$-\langle \beta, \nu \wedge d^*\alpha + \nabla_\nu\alpha\rangle = \langle \beta, *^{-1} Q(x) * \alpha\rangle. \tag{4.20}$$

Now the first term in the integrand in (4.12) is zero because $\nu \wedge \beta = 0$. Thus (4.20) shows that the integrand in (4.12) is $\langle \beta(x), *^{-1}Q(x)*\alpha(x)\rangle$. Hence we may take $K(x) = I_{\mathfrak{k}} \otimes (*^{-1}Q(x)*)$ in this case. This completes the proof of (4.17) and (4.2).

Finally, observe that if the second fundamental form is greater than or equal to zero, i.e. $S(x) \geq 0$ on ∂M, then $S^* \geq 0$ also, as is also $Q(x)$ and the unitary transform $*^{-1}Q(x)*$. The identity (4.17) therefore shows that $K(x) \geq 0$ in both cases. □

Example 4.7. If M is a closed ball of radius R in $\mathbb{R}^3$, then $W = 0$ and the principal curvatures of its boundary are both $1/R$. Hence

$$K(x) = \begin{cases} 1/R & \text{if } p = 1 \\ 2/R & \text{if } p = 2 \end{cases}$$

in the case $\alpha_{norm} = \beta_{norm} = 0$. In the case $\alpha_{tan} = \beta_{tan} = 0$ the two lines should be interchanged.

4.2. A Gaffney-Friedrichs inequality in 3 dimensions.

Proof of Theorem 2.17. Any number q in $(1, 3)$ is a convex sum, $q = \alpha \cdot 1 + \beta \cdot 3$ with $\alpha + \beta = 1$. Then $2\alpha = 3 - q$ and $2\beta = q - 1$. For non-negative functions f, g on a measure space, Hölder's inequality, with conjugate exponents α^{-1}, β^{-1}, gives $\int g^q = \int g^\alpha (g^3)^\beta \leq (\int g)^\alpha (\int g^3)^\beta$, which is valid for $q \in \{1, 3\}$ also. In case $g = f^2$ this asserts that $\|f^2\|_q^q \leq \|f\|_2^{2\alpha} \|f\|_6^{6\beta}$. Into this standard convexity inequality insert $q := p'$, the conjugate exponent to the exponent p of the theorem. Then $2\alpha/q = (3/q) - 1 = 2 - (3/p)$ and $6\beta/q = 3(q-1)/q = 3/p$. Hence

$$\|f^2\|_{p'} \leq \|f\|_2^{2-(3/p)} \|f\|_6^{3/p}, \quad 3/2 \leq p \leq \infty. \tag{4.21}$$

For any $\mathfrak{k}$ valued r-form ω over M it follows from (4.21) and from the inequality $|(B \circ \omega, \omega)| \leq c \int_M |B(x)||\omega(x)|^2 dx$ that

$$|(B \circ \omega, \omega)| \leq c\|B\|_p \|\omega\|_2^{2-(3/p)} \|\omega\|_6^{3/p}, \quad 3/2 \leq p \leq \infty. \tag{4.22}$$

Let $\gamma > 0$ and define $u = \gamma c\|B\|_p\|\omega\|_2^{2-(3/p)}$ and $v = \gamma^{-1}\|\omega\|_6^{3/p}$. The convexity inequality $uv \leq ru^{r^{-1}} + sv^{s^{-1}}$, $r + s = 1$, which follows from the convexity of the exponential function, and with the choice $r = 1 - (3/2p)$, $s = 3/2p$, yields

$$\begin{aligned}|(B \circ \omega, \omega)| &\leq r\{\gamma c\|B\|_p\|\omega\|_2^{2-(3/p)}\}^{r^{-1}} + s\{\gamma^{-1}\|\omega\|_6^{3/p}\}^{s^{-1}}\\ &= [1 - (3/2p)]\{\gamma c\|B\|_p\}^{2p/(2p-3)}\ \|\omega\|_2^2 + (3/2p)\{\gamma^{-1}\}^{(2p/3)}\ \|\omega\|_6^2. \end{aligned} \tag{4.23}$$

Use (2.20) to bound the last term of (4.23) and at the same time choose γ so that $(3/2p)\gamma^{-2p/3}(\kappa^2/2) = 1/(4p)$. That is, $\gamma = (3\kappa^2)^{3/2p}$. We find then

$$\begin{aligned}|(B \circ \omega, \omega)| \leq\ & [1 - (3/2p)](3\kappa^2)^{3/(2p-3)}(c\|B\|_p)^{2p/(2p-3)}\|\omega\|_2^2\\ &+\frac{1}{4p}\|\omega\|^2_{W_1^A(M)}. \end{aligned} \tag{4.24}$$

Adding $\|\omega\|_2^2$ to both sides of the Gaffney identity (4.4), it then follows that

$$\begin{aligned}\|\omega\|^2_{W_1^A(M)} \leq\ & \|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + \lambda_p\|\omega\|_2^2 + \frac{1}{4p}\|\omega\|^2_{W_1^A(M)}\\ &-(W \circ \omega, \omega) - \int_{\partial M}\langle K(x)\omega(x), \omega(x)\rangle, \end{aligned} \tag{4.25}$$

where

$$\lambda_p = 1 + \gamma_p\|B\|_p^{2p/(2p-3)} \quad \text{and} \quad \gamma_p = [1 - (3/2p)](3\kappa^6c^{2p})^{1/(2p-3)}. \tag{4.26}$$

If M is a convex open subset of $\mathbb{R}^3$ then $W = 0$ and the last term in (4.25) is negative. Since $1/(4p) \leq 1/4$ for all $p > 3/2$ the inequality (4.25) implies

$$(3/4)\|\omega\|^2_{W_1^A(M)} \leq \|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + \lambda_p\|\omega\|_2^2 \tag{4.27}$$

for all $p > 3/2$. This proves (2.22) with $\lambda_M = 1$. If M is convex, but $W \neq 0$ then the last term in (4.25) is negative and we can estimate the next to last term in (4.25) by $|(W\omega, \omega)| \leq \|W\|_\infty\|\omega\|_2^2$, which then gives

$$(3/4)\|\omega\|^2_{W_1^A(M)} \leq \|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + (\|W\|_\infty + \lambda_p)\|\omega\|_2^2. \tag{4.28}$$

This proves (2.22) with $\lambda_M = 1 + \|W\|_\infty$. Finally, for the general case we need only estimate the boundary term in (4.25). We will show that there is a constant τ_1, depending only on the geometry of M, such that

$$-\int_{\partial M}\langle K(x)\omega(x), w(x)\rangle \leq \tau_1\|\omega\|_2^2 + (1/4)\|\omega\|^2_{W_1^A(M)}. \tag{4.29}$$

The insertion of this estimate into (4.25) then yields

$$(1/2)\|\omega\|^2_{W_1^A(M)} \leq \|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + (\|W\|_\infty + \tau_1 + \lambda_p)\|\omega\|_2^2, \tag{4.30}$$

which is (2.22) with $\lambda_M = 1+\|W\|_\infty+\tau_1$. This will conclude the proof of Theorem 2.17.

Denoting by Δ the self-adjoint Neumann Laplacian on real valued functions on M, the fractional Sobolev norms are defined by $\|f\|_{H_a(M)} = \|(1 - \Delta)^{a/2}f\|_{L^2(M)}$ for $0 < a \leq 1$. The function $x \mapsto K(x) \in End\,\Lambda^r(T_x(\partial M))$ is continuous and therefore bounded below. Let $\tau_0 \geq$ be such that $-K(x) \leq \tau_0 I_{\Lambda^r(T_x(\partial M))}$ for all $x \in \partial M$. If $1/2 < a \leq 1$ then, by a trace inequality, (see e.g. [57, Chap. 4, Prop. 4.5]), there is a constant τ_a such that $\|f|\partial M\|_{L^2(\partial M))} \leq \|f|\partial M\|_{H_{a-(1/2)}(\partial M)} \leq \tau_a \|f\|_{H_a(M)}$. Choose $a = 3/4$ and write $\tau = \tau_{3/4}$. Then, by the spectral theorem for $1 - \Delta$ and Hölder's inequality, one has

$$\tau^2 \|f\|^2_{H_{3/4}(M)} \leq \tau^2 \|f\|^{1/2}_{L^2(M)} \|f\|^{3/2}_{H_1(M)}$$
$$\leq (1/4)(\frac{\tau^2}{\epsilon})^4 \|f||^2_{L^2(M)} + (3/4)\epsilon^{4/3} \|f\|^2_{H_1(M)},$$

wherein we may choose ϵ so that $\tau_0(3/4)\epsilon^{4/3} = 1/4$. Then we find $\tau_0 \|f|\partial M\|^2_{L^2(\partial M)} \leq \tau_1 \|f\|^2_{L^2(M)} + (1/4)\|f\|^2_{H_1(M)}$ with $\tau_1 = (27/4)(\tau^2\tau_0)^4$. Insert $f = |\omega|$ into this inequality and keep in mind Kato's Inequality, as in Notation 2.16, to arrive at

$$-\int_{\partial M} \langle K(x)\omega(x), \omega(x)\rangle \leq \tau_0 \int_{\partial M} |\omega(x)|^2$$
$$\leq \tau_1 \|\omega\|^2_{L^2(M)} + (1/4)\|\omega\|^2_{W_1^A(M)},$$

which is (4.29). □

Corollary 4.8 (Gaffney-Friedrichs-Sobolev inequality). *Under the hypothesis of Theorem 2.17 there holds*

$$\|\omega\|^2_{L^6(M)} \leq \kappa^2(\|d_A\omega\|_2^2 + \|d_A^*\omega\|^2_{L^2(M)} + \lambda_p(B)\|\omega\|^2_{L^2(M)}). \tag{4.31}$$

Proof. Equation (4.31) follows from (2.22) and the Sobolev inequality (2.20). □

Example 4.9. If one takes M to be a cube in $\mathbb{R}^3$ then, although M does not have a smooth boundary, the identity (4.2) is easily verified directly and, since $K(x) = 0$ on the flat sides of ∂M, one finds no boundary terms. The inequality (2.22) holds, therefore, in this case also, and, indeed, with $\lambda_M = 1$.

In the following remark we resume the notation for minimal and maximal operators from Sect. 3.

Remark 4.10. For a $\mathfrak{k}$ valued r-form ω on M with r=1 or 2 define

$$Q_N(\omega) = \|D\omega\|_2^2 + \|D^*\omega\|_2^2 + \|\omega\|_2^2, \quad \omega \in \mathscr{D}(D) \cap \mathscr{D}(D^*), \tag{4.32}$$
$$Q_D(\omega) = \|d\omega\|_2^2 + \|d^*\omega\|_2^2 + \|\omega\|_2^2, \quad \omega \in \mathscr{D}(d) \cap \mathscr{D}(d^*). \tag{4.33}$$

Both of these quadratic forms are coercive in the sense that their domains are contained in W_1 and each controls the W_1 norm. This follows from (2.22) if one chooses $A = 0$, because then $B = 0$ and $\lambda_p(B) = \lambda_M$ for any $p \geq 2$. This coercivity is the content of [40, Lem. 4.5]. See also [39]. The Laplacians associated to these closed quadratic forms will be used in Sect. 7.

5. Sobolev Inequalities for Solutions

Throughout this section we will assume that $A \in C^\infty((0,T) \times M : \Lambda^1 \otimes \mathfrak{k})$, with $T \le \infty$, and satisfies

$$A'(s) = -\delta_{A(s)} B(s) \quad \text{on } (0,T), \tag{5.1}$$

where δ_A, defined in (3.1) and (3.10), is to be interpreted as a differential operator without boundary conditions. We will also assume that either

$$(D) \quad A(s)_{tan} = 0 \qquad \text{for } 0 < s < T, \tag{5.2}$$

$$\text{or } (M) \quad B(s)_{norm} = 0 \quad \text{for } 0 < s < T. \tag{5.3}$$

We are going to establish apriori estimates for solutions to the Yang-Mills heat equation (5.1) over $(0, T)$. It will be necessary to integrate by parts in Lemma 5.2 and the use of the maximal or minimal operators D_A or d_A and their Hilbert space adjoints will be a very useful bookkeeping tool for this. The gauge invariant Sobolev inequalities in Hodge format, established in Sect. 4, simplify when applied to a form ω which is annihilated by any one of these four operators. In particular, when $A(\cdot)$ is a solution to (5.1), all Sobolev estimates can be conveniently expressed in terms of the time derivatives $A^{(n)}$ or $B^{(n)}$.

5.1. Pointwise and integral identities.

Lemma 5.1 (Pointwise Identities). *Suppose that $A(\cdot)$ is a smooth solution to the differential equation (5.1) and satisfies either (5.2) or (5.3). Then the following identities hold, wherein the symbol d_A is the minimal operator in case the Dirichlet boundary condition (5.2) is assumed, or represents the maximal operator D_A in case the Marini boundary condition (5.3) is assumed.*

$$B' = d_A A', \tag{5.4}$$

$$A'' + d_A^* B' = -[A' \lrcorner B], \tag{5.5}$$

$$d_A^* A' = d_A^* A'' = 0. \tag{5.6}$$

Proof. Let us first compute the derivatives in all cases, ignoring boundary conditions, but recalling that $\delta_A = d_A^*$ in all cases, aside from boundary conditions. Equation (5.4) follows from the definition of B. Differentiate (5.1) with respect to s to derive (5.5). By (5.1) we have $d_A^* A' = -(d_A^*)^2 B = -[B \lrcorner B] = 0$, which is half of (5.6). Differentiate this identity with respect to s to find $0 = (\partial/\partial s)(d_A^* A') = d_A^* A'' + [A' \lrcorner A'] = d_A^* A'' - [A' \cdot A'] = d_A^* A''$, since $[A' \cdot A'] = 0$. This proves (5.6).

Concerning the boundary conditions, consider first the Dirichlet case, (5.2). Since $A(s)_{tan} = 0$ for all $s \in (0, T)$ we may differentiate this equation with respect to s at a point on ∂M and find $A'(s)_{tan} = 0$. Thus the application of the minimal operator d_A in (5.4) is justified. Since d_A^* is a maximal operator there is no boundary issue in (5.5) or (5.6).

In the Marini case d_A is now the maximal operator D_A. So there is no domain issue in (5.4). We may differentiate Eq. (5.3) with respect to time to find $B'(s)_{norm} = 0$. By (3.12) $B(s)$ and $B'(s)$ are therefore both in the domain of the minimal operator D_A^*. Thus all the terms in (5.5) are well defined. Moreover (3.16) shows that $D_A^* B$ is again in the domain of the minimal operator D_A^*. From this and (5.1) it follows that A' is in the domain of D_A^* and from (3.12) it now follows that $A'(s)_{norm} = 0$. Of course then $A''(s)_{norm} = 0$ also and so $A''(s) \in \mathscr{D}(D_A^*)$. This justifies the identities in (5.6). □

Lemma 5.2 (Integral Identities). *Suppose that $A(\cdot)$ is a smooth solution to the differential equation* (5.1) *and satisfies either* (5.2) *or* (5.3). *Then*

$$(d/ds)\|B(s)\|_2^2 = -2\|A'(s)\|_2^2, \tag{5.7}$$

$$(d/ds)\|A'(s)\|_2^2 = -2\|B'(s)\|_2^2 - 2([A'(s) \wedge A'(s)], B(s)). \tag{5.8}$$

Proof. It was emphasized in Lemma 5.1 that, whether one assumes Dirichlet or Marini boundary conditions, B and its time derivatives as well as A' and its time derivatives all lie in the domain of the corresponding minimal operators d_A or D_A^*, respectively, and of course in the domain of the corresponding maximal operators d_A^* or D_A. All of the integrations by parts implicit in the following computations are thereby justified under either boundary condition (5.2) or (5.3). We will write the proof for the Dirichlet boundary condition. This uses the minimal operator d_A. But the proof is identical for the Marini boundary condition (5.3). One need only replace d_A by the maximal operator D_A:

$$\begin{aligned}(1/2)(d/ds)\|B(s)\|_{L^2}^2 &= (B', B)\\ &= (d_A A', B)\\ &= (A', d_A^* B)\\ &= -\|A'(s)\|_{L^2}^2.\end{aligned}$$

This proves (5.7). In view of (5.5) and (5.4) we have

$$\begin{aligned}(1/2)(d/ds)\|A'(s)\|^2 &= (A''(s), A'(s))\\ &= (-d_A^* B' - [A' \lrcorner B], A')\\ &= -(B', d_A A') - ([A' \lrcorner B], A'),\end{aligned}$$

which proves (5.8). □

5.2. Sobolev inequalities for smooth solutions. The derivation of the Sobolev inequalities (5.9) and (5.10) relies on use of more differentiability than is available from the definition of strong solution. We will assume therefore that $A(\cdot)$ is a smooth solution. But it will be shown in Corollary 9.2, by an approximation procedure for strong solutions, that (5.9) holds for all strong solutions. It can also be shown that (5.10) holds for strong solutions. But the proof relies on higher order apriori estimates which will not be needed in this paper.

Lemma 5.3 (Sobolev inequalities for smooth solutions). *Suppose that $A(\cdot)$ is a smooth solution to* (5.1) *and satisfies either* (5.2) *or* (5.3). *Let $p \in [2, \infty]$. Then, suppressing s,*

$$\|B\|_6^2 \le \kappa^2(\|A'\|_2^2 + \lambda\|B\|_2^2), \tag{5.9}$$

$$\|A'\|_6^2 \le \kappa^2(\|B'\|_2^2 + \lambda\|A'\|_2^2), \tag{5.10}$$

where κ is the Sobolev constant defined in (2.20) *and $\lambda = \lambda_p(B(s))$, defined in* (2.21).

Proof. All of these inequalities follow from the inequality, (4.31),

$$\|\omega\|_6^2 \leq \kappa^2(\|d_A\omega\|_2^2 + \|d_A^*\omega\|_2^2 + \lambda\|\omega\|_2^2) \tag{5.11}$$

in the presence of an identity that simplifies one of the terms. Choose $A = A(s)$ in (5.11), $\lambda = \lambda_p(B(s))$ and $\omega = B(s)$. Observe that $d_A\omega = d_{A(s)}B(s) = 0$ by Bianchi's identity (3.22) (for the Dirichlet case) or (3.21) (for the Marini case), while $d_A^*\omega = d_{A(s)}^*B(s) = -A'(s)$. Thus (5.11) reduces to (5.9) with these choices.

Similarly, to derive (5.10) from (5.11), choose $\omega = A'(s)$ in (5.11) and observe that $d_A^*\omega = d_{A(s)}^*A'(s) = 0$ by (5.6), while $d_A\omega = d_{A(s)}A'(s) = B'(s)$. Thus (5.11) reduces to (5.10) with these choices.

As in the preceding subsection, the symbol d_A represents the minimal operator in the case of Dirichlet boundary conditions, (5.2), or the maximal operator D_A in the case of Marini boundary conditions, (5.3). □

6. Apriori Estimates

We want to understand the nature of the singularities of the various gauge covariant spatial derivatives of $B(t)$ as $t \downarrow 0$, under the sole assumption of finite initial energy. The word "order", below, refers to the number of spatial derivatives of A involved in the inequalities. For example B involves one spatial derivative of A while A' involves two spatial derivatives by virtue of the equation $A' = -d_A^*B$.

The key ingredients for our gauge invariant estimates are the gauge invariant Sobolev inequalities (5.9) and (5.10), which were derived from the Gaffney-Friedrichs inequality of Theorem 2.17. These will be the basis for the gauge invariant estimates of Sect. 6.1. In Sect. 6.2 we will need to make estimates of $\|A(t)\|_{W^1(M)}$. These cannot be gauge invariant.

6.1. Gauge invariant apriori estimates.

Notation 6.1. Recall that $\lambda_2(B) = \lambda_M + \gamma_2\|B\|_2^4$, as defined in (2.24). Define

$$\psi(t) = 2\int_0^t \lambda_2(B(\sigma))d\sigma, \quad \text{and} \quad \psi_s^t = 2\int_s^t \lambda_2(B(\sigma))d\sigma. \tag{6.1}$$

Define also

$$\lambda_0 = \lambda_2(B_0) = \lambda_M + \gamma_2\|B_0\|_2^4. \tag{6.2}$$

Since $\|B(s)\|_2$ is a non-increasing function of s, it follows that $\lambda_2(B(s)) \leq \lambda_0$ for all s, and consequently $\psi(t) \leq 2t\lambda_0$. We are going to use the bound $\lambda_2(B(s)) \leq \lambda_0$ in the proofs of (6.6) and (6.8) because it simplifies these proofs considerably. But we will avoid using it in the proof of Lemma 6.5 because if the initial data A_0 should be in Sobolev class 1/2, which will be of later interest to us, then it can happen that $\|B_0\|_2 = \infty$. In fact B_0 need not be a function. Yet even in this case the integral defining $\psi(t)$ is finite. The techniques in this section will be applied in a later work to this larger class of initial data.

In the remainder of this paper we will use the Sobolev inequalities (5.9) and (5.10) with the choice $\lambda = \lambda_2(B(s))$. The bound $\lambda_2(B(s)) \le \lambda_0$ then yields

$$\|B(s)\|_6^2 \le \kappa^2(\|A'(s)\|_2^2 + \lambda_0\|B_0\|_2^2), \tag{6.3}$$

$$\|A'(s)\|_6^2 \le \kappa^2(\|B'(s)\|_2^2 + \lambda_0\|A'(s)\|_2^2) \tag{6.4}$$

for any smooth solution over $(0, T)$.

Theorem 6.2. *Let $0 < T \le \infty$. Suppose that $A(\cdot)$ is a smooth solution to (5.1) and satisfies either (5.2) or (5.3). Assume further that $\|B_0\|_2 < \infty$. Then $\|B(t)\|_2$ is non-increasing and there exist continuous non-decreasing functions $C_j : [0, \infty)^2 \to [0, \infty)$, for $j = 1, 2, 3$, such that*

$$\|B(t)\|_2^2 + 2\int_0^t \|A'(s)\|_2^2 ds = \|B_0\|_2^2, \quad \textit{Order}\, 1, \tag{6.5}$$

$$\int_0^t \|B(s)\|_6^2 ds \le C_2(t, \|B_0\|_2), \quad \textit{Order}\, 1, \tag{6.6}$$

and

$$t\|A'(t)\|_2^2 + \int_0^t e^{\psi_s^t} s\|B'(s)\|_2^2 ds \le C_1(t, \|B_0\|_2), \quad \textit{Order}\, 2, \tag{6.7}$$

$$t\|B(t)\|_6^2 + \int_0^t e^{\psi_s^t} s\|A'(s)\|_6^2 ds \le C_3(t, \|B_0\|_2), \quad \textit{Order}\, 2, \tag{6.8}$$

where ψ_s^t is defined in (6.1).

Corollary 6.3. *Let $\tau > 0$. Then*

$$2\tau\|A'(t)\|_2^2 \le e^{2\tau\lambda_0}\Big(\|B(t-\tau)\|_2^2 - \|B(t)\|_2^2\Big) \ \ \textit{for } t \ge \tau. \tag{6.9}$$

In particular, if $T = \infty$ then $\|A'(t)\|_2 \to 0$ as $t \to \infty$. Moreover

$$\sup_{t\ge 1} \|\nabla^{A(t)} B(t)\|_2 < \infty. \tag{6.10}$$

Corollary 6.4. *For each $p \in [2, 6)$, there is a continuous, non-decreasing function $C_4 : [0, \infty)^2 \to [0, \infty)$ such that*

$$\int_0^t \|A'(s)\|_p ds \le C_4(t, \|B_0\|_2), \quad \textit{Order}\, 2. \tag{6.11}$$

Proof of Order 1 inequalities.

Proof of (6.5) and (6.6). Integrate the identity (5.7) over $(0, t)$ to find $\int_0^t \|A'(s)\|_2^2 ds = (1/2)(\|B_0\|_2^2 - \|B(t)\|_2^2)$, which is (6.5). Taking the integral of the inequality (6.3) over $(0, t)$ and using (6.5) we find

$$\int_0^t \|B(s)\|_6^2 ds \le \kappa^2\|B_0\|_2^2\{1/2 + t\lambda_0\}, \tag{6.12}$$

which proves (6.6). □

Proof of Order 2 inequalities.

Lemma 6.5. *Let* $0 < T \le \infty$. *Suppose that* $A(\cdot)$ *is a smooth solution to* (5.1) *and satisfies either* (5.2) *or* (5.3). *Then*

$$(d/ds)\Big(e^{-\psi(s)}\|A'(s)\|_2^2\Big) \le -e^{-\psi(s)}\|B'(s)\|_2^2. \tag{6.13}$$

Proof. Hölder's inequality and (5.10) with $p = 2$ give, for any number $\gamma > 0$,

$$\begin{aligned}
2|([A'(s)\wedge A'(s)], B(s))| &\le 2c\|B(s)\|_2\|A'(s)\|_4^2\\
&\le 2c\|B(s)\|_2\|A'(s)\|_2^{1/2}\|A'(s)\|_6^{3/2}\\
&\le (1/4)\Big(\gamma 2c\|B(s)\|_2\|A'(s)\|_2^{1/2}\Big)^4 + (3/4)\Big(\gamma^{-1}\|A'(s)\|_6^{3/2}\Big)^{4/3}\\
&\le (1/4)\Big(\gamma 2c\|B(s)\|_2\Big)^4\|A'(s)\|_2^2 + (3/4)\gamma^{-4/3}\|A'(s)\|_6^2\\
&\le (1/4)\Big(\gamma 2c\|B(s)\|_2\Big)^4\|A'(s)\|_2^2\\
&\quad + (3/4)\gamma^{-4/3}\kappa^2\Big(\|B'(s)\|_2^2 + \lambda_2(B(s))\|A'(s)\|_2^2\Big).
\end{aligned}$$

Choose γ such that $(3/4)\gamma^{-4/3}\kappa^2 = 1$. So $\gamma = (3\kappa^2/4)^{3/4}$ and $(2\gamma)^4 = (1/4)(3\kappa^2)^3$. Then we arrive at

$$\begin{aligned}
&2|([A'(s)\wedge A'(s)], B(s))|\\
&\quad\le \Big\{(1/4)^2(3\kappa^2)^3(c\|B(s)\|_2)^4 + \lambda_2(B(s))\Big\}\|A'(s)\|_2^2 + \|B'(s)\|_2^2\\
&\quad\le 2\lambda_2(B(s))\|A'(s)\|_2^2 + \|B'(s)\|_2^2\\
&\quad= \psi'(s)\|A'(s)\|_2^2 + \|B'(s)\|_2^2.
\end{aligned}$$

From (5.8) we now find $(d/ds)\|A'(s)\|_2^2 \le -\|B'(s)\|_2^2 + \psi'(s)\|A'(s)\|_2^2$, which implies (6.13). □

Proof of (6.7). Integrate (6.13) over (σ, t) and multiply by $e^{\psi(t)}$ to find

$$\|A'(t)\|_2^2 + \int_\sigma^t e^{\psi_s^t}\|B'(s)\|_2^2 ds \le e^{\psi_\sigma^t}\|A'(\sigma)\|_2^2 \le e^{\psi(t)}\|A'(\sigma)\|_2^2. \tag{6.14}$$

Integrate this inequality with respect to σ over $(0, t)$, reverse the order of integration in the double integral and then use (6.5) to arrive at

$$\begin{aligned}
t\|A'(t)\|_2^2 + \int_0^t e^{\psi_s^t} s\|B'(s)\|_2^2 ds &\le e^{\psi(t)}\Big(\|B_0\|_2^2 - \|B(t)\|_2^2\Big)/2\\
&\le e^{2t\lambda_0}\|B_0\|_2^2/2,
\end{aligned} \tag{6.15}$$

which gives (6.7). □

Proof of (6.8). Add $\lambda_0\|B_0\|_2^2+\int_\sigma^t e^{\psi_s^t}\lambda_0\|A'(s)\|_2^2 ds$ to both sides of (6.14) and use (6.3) and (6.4) to find

$$\begin{aligned}
&\kappa^{-2}\Big\{\|B(t)\|_6^2+\int_\sigma^t e^{\psi_s^t}\|A'(s)\|_6^2 ds\Big\}\\
&\quad\le e^{\psi(t)}\|A'(\sigma)\|_2^2+\lambda_0\|B_0\|_2^2+\int_\sigma^t e^{\psi_s^t}\lambda_0\|A'(s)\|_2^2 ds\\
&\quad\le e^{\psi(t)}\|A'(\sigma)\|_2^2+\lambda_0\|B_0\|_2^2+e^{\psi(t)}\lambda_0\|B_0\|_2^2/2, \qquad (6.16)
\end{aligned}$$

wherein we have used $e^{\psi_s^t}\le e^{\psi(t)}$ and (6.5) for the last term. Integrate with respect to σ over $(0,t)$, reverse the σ and s integrals on the left and apply (6.5) to the first term on the right, to arrive at

$$\begin{aligned}
&t\|B(t)\|_6^2+\int_0^t e^{\psi_s^t} s\|A'(s)\|_6^2 ds\\
&\quad\le \kappa^2\Big\{e^{\psi(t)}\|B_0\|_2^2/2+\lambda_0 t\|B_0\|_2^2+e^{\psi(t)}\lambda_0(t/2)\|B_0\|_2^2\Big\},
\end{aligned}$$

which proves (6.8). □

Proof of Corollary 6.3. Let $0\le\sigma<t$ and apply (6.15) over the interval $[\sigma,t]$ to find

$$2(t-\sigma)\|A'(t)\|_2^2+2\int_\sigma^t e^{\psi_s^t}(s-\sigma)\|B'(s)\|_2^2 ds\le e^{\psi_\sigma^t}\Big(\|B(\sigma)\|_2^2-\|B(t)\|_2^2\Big). \qquad (6.17)$$

If $t\ge\tau$ then we can put $\sigma=t-\tau$ in this inequality and observe that $e^{\psi_\sigma^t}\le e^{2(t-\sigma)\lambda_2(B(\sigma))}\le e^{2\tau\lambda_0}$ to deduce (6.9), after dropping the positive integral on the left of (6.17). If $T=\infty$ and $\tau>0$ then (6.5) shows that $\int_0^\infty\|A'(s)\|_2^2 ds<\infty$ and therefore, that $\int_{t-\tau}^t\|A'(s)\|_2^2 ds\to 0$ as $t\to\infty$. Hence $\lim_{t\to\infty}\|A'(t)\|_2^2=0$. Since $d_{A(t)}B(t)=0$, the Gaffney-Friedrichs inequality (2.22) gives

$$\begin{aligned}
(1/2)\Big(\|\nabla^{A(t)}B(t)\|_2^2+\|B(t)\|_2^2\Big)&\le\|d_{A(t)}^*B(t)\|_2^2+\lambda_2(B(t))\|B(t)\|_2^2\\
&\le\|A'(t)\|_2^2+\lambda_2(B(1))\|B(1)\|_2^2,
\end{aligned}$$

which is bounded on $[1,\infty)$ because $\lim_{t\to\infty}\|A'(t)\|_2^2=0$. □

The proof of Corollary 6.4 depends on the following interpolation lemma.

Lemma 6.6 (Interpolation). *Let $0\le a<b<\infty$ and let $2\le p<6$. Suppose that $f:(a,b)\to L^2(M)\cap L^6(M)$ is continuous. Then*

$$\int_a^b\|f(s)\|_p ds\le\Big(\int_a^b s^{\frac{3}{p}-\frac{3}{2}}ds\Big)^{1/2}\Big(\int_a^b\|f(s)\|_2^2 ds\Big)^{\alpha/p}\Big(\int_a^b s\|f(s)\|_6^2\Big)^{3\beta/p}, \qquad (6.18)$$

where $p=2\alpha+6\beta$, $\alpha+\beta=1$ and $0\le\beta<1$.

Proof. By interpolation $\|f(s)\|_p \le \|f(s)\|_2^{2\alpha/p}\|f(s)\|_6^{6\beta/p}$. Hence

$$\int_a^b \|f(s)\|_p ds \le \int_a^b \{s^{-3\beta/p}\}\{\|f(s)\|_2^{2\alpha/p}\}\{(s^{1/2}\|f(s)\|_6)^{6\beta/p}\}ds.$$

Apply Hölder's inequality to the product of the three functions in braces to find

$$\int_a^b \|f(s)\|_p ds$$
$$\le \Big(\int_a^b \{s^{-3\beta/p}\}^q ds\Big)^{1/q}\Big(\int_a^b \|f(s)\|_2^{2\alpha r/p} ds\Big)^{1/r}\Big(\int_a^b \{s^{1/2}\|f(s)\|_6\}^{6\beta m/p} ds\Big)^{1/m},$$

provided q, r, m are nonnegative and $q^{-1} + r^{-1} + m^{-1} = 1$. Choose $q = 2$, $r = p/\alpha$ and $m = p/(3\beta)$ and observe that $6\beta/p = (3/2) - (3/p)$ to arrive at (6.18). □

Proof of Corollary 6.4. Choose $(a, b) = (0, t)$ and $f(s, x) = |A'(s, x)|_{\Lambda^1\otimes\mathfrak{k}}$ in Lemma 6.6. Since $p < 6$ the exponent in the first factor is $(3/p) - (3/2) > -1$. Therefore the first factor on the right in (6.18) is finite. The second and third factors on the right are also finite, by (6.5) and (6.8) respectively. □

6.2. Growth of $\|A(t)\|_{W_1(M)}$. In the previous sections all apriori estimates were gauge invariant. However for our proof of long time existence of solutions we will need estimates that depend on A_0 itself, not just on its gauge equivalence class. Correspondingly, we will have to replace Marini boundary conditions by the stronger Neumann boundary conditions in order to get estimates on $A(t)$ itself, not just on certain of its derivatives.

The smoothness hypothesis in the following theorem, that $A(\cdot) \in C^\infty((0, T) \times M)$, will be removed in Sect. 9, Cor. 9.3.

Theorem 6.7. *There is a continuous increasing function* $C_5 : [0, \infty)^2 \to [0, \infty)$, *depending only on the geometry of* M, *such that, for any strong solution to the Yang-Mills heat equation satisfying Neumann,* (2.9)*,* (2.10)*, or Dirichlet,* (2.11)*, boundary conditions on an interval* $[0, T)$*, with* $0 < T \le \infty$*,*

$$\|A(t)\|_{W_1(M)} \le C_5(t, \|A_0\|_{W_1(M)}),\ \ 0 \le t < T \text{ holds,} \tag{6.19}$$

under the additional hypothesis that $A(\cdot) \in C^\infty((0, T) \times M)$.

The proof depends on the following estimates, which will be derived for smooth $A(\cdot)$. But the smoothness requirement will be removed in Corollary 9.2, thereby proving the following four inequalities for any strong solution satisfying Neumann or Dirichlet boundary conditions.

Lemma 6.8. *Suppose that* $A(\cdot)$ *is a strong solution to the Yang-Mills heat equation satisfying Neumann,* (2.9)*,* (2.10)*, or Dirichlet,* (2.11)*, boundary conditions on an interval* $[0, T)$*, with* $0 < T \le \infty$*. Assume also that* $A \in C^\infty((0, T))$*. Then*

$$\|A(t)\|_2 \le \|A_0\|_2 + t^{1/2}\|B_0\|_2, \tag{6.20}$$

$$\|A(s)\|_4 \le \|A_0\|_4 + C_4(t, \|B_0\|_2), \quad 0 < s \le t, \tag{6.21}$$

$$\|dA(t)\|_2 \le \|B_0\|_2 + (c/2)\Big(\|A_0\|_4 + C_4(t, \|B_0\|_2)\Big)^2, \tag{6.22}$$

$$\text{and} \quad \|d^*A(t)\|_2 \le \|d^*A_0\|_2 + c\Big(\|A_0\|_4 + C_4(t, \|B_0\|_2)\Big)C_4(t, \|B_0\|_2), \tag{6.23}$$

where $C_4(\cdot, \cdot)$ *is defined by* (6.11) *for* $p = 4$.

Proof. The identity

$$A(s) = A_0 + \int_0^s A'(\sigma)d\sigma, \tag{6.24}$$

is valid for any strong solution, even if not smooth on $(0, T)$. We may take the L^2 norm in (6.24) (with $s = t$) to find $\|A(t)\|_2 \le \|A_0\|_2 + \int_0^t \|A'(\sigma)\|_2 d\sigma \le \|A_0\|_2 + t^{1/2}(\int_0^t \|A'(\sigma)\|_2^2)^{1/2}$. Equation (6.20) now follows from (6.5).

The rest of the proof hinges on the estimate (6.11) for $p = 4$, which asserts

$$\int_0^t \|A'(\sigma)\|_4 d\sigma \le C_4(t, \|B_0\|_2) \tag{6.25}$$

for some non-decreasing continuous function $C_4 : [0, \infty)^2 \to [0, \infty)$. Now (6.24) implies that $\|A(s)\|_4 \le \|A_0\|_4 + \int_0^s \|A'(\sigma)\|_4 d\sigma$, which proves (6.21) in view of (6.25). Observe next the identities

$$dA(t) = B(t) - (1/2)[A(t) \wedge A(t)], \tag{6.26}$$

$$d^*A'(s) = [A(s) \cdot A'(s)]. \tag{6.27}$$

The first just rewrites the definition of curvature (2.2), while the second just rewrites the first identity in (5.6). It follows that

$$\|dA(t)\|_2 \le \|B(t)\|_2 + (c/2)\|A(t)\|_4^2, \tag{6.28}$$

which yields (6.22) upon insertion of (6.21), given that $\|B(t)\|_2$ is non-increasing. Finally, the identity (6.27) gives

$$d^*A(t) = d^*A_0 + \int_0^t [A(s) \cdot A'(s)]ds, \tag{6.29}$$

and therefore

$$\begin{aligned} \|d^*A(t)\|_2 &\le \|d^*A_0\|_2 + c\int_0^t \|A(s)\|_4 \|A'(s)\|_4 ds \\ &\le \|d^*A_0\|_2 + c \sup_{0<s\le t} \|A(s)\|_4 \int_0^t \|A'(s)\|_4 ds. \end{aligned} \tag{6.30}$$

Equation (6.23) now follows from (6.21) and (6.25). Notice that $\|d^*A_0\|_2 < \infty$ because, by assumption, $A(\cdot)$ maps $[0, T)$ into W_1. □

Note. The proof of (6.25) relies on use of third spatial derivatives of A in the identity (5.8), and therefore is not immediately applicable to a strong solution. Moreover the identity (6.27) and its consequences, (6.29) and (6.30), also uses the third spatial derivatives of A. However we will construct in Sect. 9 an approximation method that allows us to prove (6.11), and in particular (6.25), as well as (6.30), for all strong solutions and indeed with the same function $C_4(\cdot, \cdot)$. This entire proof will then apply to all strong solutions without the additional hypothesis that $A(\cdot) \in C^\infty((0, T)) \times M)$.

Proof of Theorem 6.7. We are going to make use of the Gaffney-Friedrichs inequality (2.22) with $A = 0$ in that inequality and ω chosen to be the form $A(t)$ of the present theorem, with $t > 0$. In this case (2.22) asserts that

$$(1/2)\|A(t)\|_{W_1}^2 \leq \|dA(t)\|_2^2 + \|d^*A(t)\|_2^2 + \lambda_M\|A(t)\|_2^2. \tag{6.31}$$

This is applicable because $A(t) \in W_1(M)$ and either $A(t)_{norm} = 0$ or $A(t)_{tan} = 0$. The three terms on the right may be estimated by (6.22), (6.23) and (6.20) respectively. The theorem now follows if one takes into account that $\|A_0\|_2$, $\|A_0\|_4$, $\|B_0\|_2$ and $\|d^*A_0\|_2$ are all dominated by a linear or quadratic polynomial in $\|A_0\|_{W_1}$, given the definition (2.1) of the W_1 norm. □

7. Short Time Existence and Uniqueness for the Parabolic Equation

In this section we will prove Theorem 2.13 for both sets of boundary conditions (2.15) and (2.16) simultaneously by encoding the boundary conditions into appropriate Sobolev spaces and then using a common approach.The Sobolev spaces will be the quadratic form domains of the absolute and relative Laplacians, [7,45], as described in Remark 4.10.

Notation 7.1. Define

$$(N) \quad \Delta_N = -(D^*D + DD^*), \tag{7.1}$$

$$\text{or } (D) \quad \Delta_D = -(d^*d + dd^*). \tag{7.2}$$

Here D and d are the maximal and minimal exterior derivative operators, respectively, discussed in Sect. 3. They act on p-forms.

For both kinds of boundary conditions we are going to write simply $H_1(M)$ (or $H_1(M; \Lambda^1 \otimes \mathfrak{k})$ when clarity demands) for the form domain of Δ_N or Δ_D, namely the domains, respectively, of the quadratic forms Q_N or Q_D in (4.32) and (4.33). This defines two distinct notions of H_1. Thus, writing Δ for either the absolute or relative Laplacian Δ_N or Δ_D, Remark 4.10 allows us to write the Sobolev norm as

$$\|\omega\|_{H_1} = \|(1 - \Delta)^{1/2}\omega\|_{L^2(M)} \tag{7.3}$$

in both cases. We remind the reader that Remark 4.10 shows that a form $\omega \in W_1(M)$ is in the Neumann version of $H_1(M)$ if and only if $\omega_{norm} = 0$ and is in the Dirichlet version of $H_1(M)$ if and only if $\omega_{tan} = 0$.

Throughout this section we will write d for the exterior derivative with the understanding that this represents the maximal or minimal version, in agreement with the boundary conditions.

Recall that we are dealing with a product bundle and may therefore apply this definition to A itself.

In the next section we will separate out the nonlinear terms in the parabolic equation (2.14) and reformulate it as an integral equation in a more or less standard way. A natural abstract setting for producing solutions to the integral equation may be found, for example, in [58, Chap. 15]. But we are going to use the following modified path space within which to seek solutions in order to get some precise regularity at the same time.

Notation 7.2 (Path space). Let $0 < T < \infty$. Denote by $\mathscr{P}_T$ the set of continuous functions

$$C : [0, T] \to H_1(M)$$

such that $\|C(t)\|_\infty$, $\|dC(t)\|_\infty$ and $\|d^*C(t)\|_\infty$ are finite for each $t > 0$ and

$$\infty > \|C\|_{\mathscr{P}_T} \equiv \sup_{0<t\le T} \Big\{ \|C(t)\|_{H_1(M)} + t^{1/4}\|C(t)\|_{L^\infty(M)} + t^{3/4}\Big(\|dC(t)\|_\infty + \|d^*C(t)\|_\infty\Big)\Big\}. \tag{7.4}$$

Notice that the last two terms are well defined because the boundary conditions on $C(t)$ agree with the choice of d as a minimal or maximal operator.

Theorem 7.3. *Let $A_0 \in H_1(M)$ and suppose that $\beta \ge \|A_0\|_{H_1(M)}$. Then there exists $T > 0$ depending only on β such that the integral equation (7.9) has a solution in $\mathscr{P}_T$. The solution is unique in $\mathscr{P}_T$. Moreover the solution is strongly differentiable for $t > 0$ as a function into $L^2(M)$. For $t > 0$, $C(t)$ is in $\mathscr{D}(\Delta)$ and (2.14) holds. Further, the solution lies in $C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$.*

7.1. The integral equation and locally bounded strong solutions. We will prove Theorems 7.3 and 2.13 in this section.

We are going to operate mostly with the integral form of Eq. (2.14) as follows. Writing $B \equiv B_C = dC + (1/2)[C \wedge C]$, we can compute that

$$d_C^* B + d_C d^* C = (d^*d + dd^*)C - X(C), \tag{7.5}$$

where X is the first order nonlinear differential operator on $\mathfrak{k}$ valued 1-forms defined by

$$-X(C) = -[C \lrcorner B] + (1/2)d^*[C \wedge C] + [C, d^*C], \quad C : M \to \Lambda^1 \otimes \mathfrak{k}. \tag{7.6}$$

The term $d_C d^* C$ in (7.5) contributes the term dd^* to the second order operator on the right, thereby making the operator on the right elliptic. Without this term the equation (2.14) would be only weakly parabolic.

The terms in $X(C)$ which are cubic in C involve no derivatives of C while the terms which are quadratic all involve a factor of one spatial derivative of C. We may write this symbolically as

$$X(C) = C^3 + C \cdot \partial C. \tag{7.7}$$

$X(C)$ contains all the nonlinear terms in Eq. (2.14), which can now be rewritten as

$$C'(t) = \Delta C(t) + X(C(t)), \quad C(0) = A_0, \tag{7.8}$$

wherein Δ is given by (7.1) or (7.2).

Informally, Eq. (7.8) is equivalent to the integral equation

$$C(t) = e^{t\Delta}A_0 + \int_0^t e^{(t-\sigma)\Delta}X(C(\sigma))d\sigma. \tag{7.9}$$

The regularity lemma, Lemma 7.9, will show that (7.9) implies (7.8).

Lemma 7.4. *Let $C(\cdot) \in \mathscr{P}_T$. Define*

$$F(\sigma) = C(\sigma)^3 + C(\sigma) \cdot \partial C(\sigma), \tag{7.10}$$

and define $F_j(\sigma)$ similarly for paths C_j, $j = 1, 2$. Let $2 \le q \le \infty$. Suppose that $\|C\|_{\mathscr{P}_T} \le R$ and $\|C_j\|_{\mathscr{P}_T} \le R$. Then there are constants $a_1, \ldots, a_4$ independent of $C(\cdot)$, q, R and T such that, for $0 < \sigma < T$,

$$\|F(\sigma)\|_q \le \sigma^{-(3/2)(\frac{1}{2}-\frac{1}{q})}\{(R^3 a_1) + \sigma^{-1/4}(R^2 a_2)\}, \tag{7.11}$$

$$\|F_1(\sigma) - F_2(\sigma)\|_q \le \sigma^{-(3/2)(\frac{1}{2}-\frac{1}{q})}\|C_1 - C_2\|_{\mathscr{P}_T}\{(R^2 a_3) + \sigma^{-1/4}(R a_4)\}. \tag{7.12}$$

Proof. In the interpolation inequality $\|f\|_b \le \|f\|_a^{a/b}\|f\|_\infty^{1-(a/b)}$ for $1 \le a \le b$ choose $a = 2, b = q$ to find $\|f\|_q \le \|f\|_2^{2/q}\|f\|_\infty^{1-(2/q)}$. Take $f = |\partial C(\sigma)|$ to deduce

$$\begin{aligned}\|\partial C(\sigma)\|_q &\le \|\partial C(\sigma)\|_2^{(2/q)}\|\partial C(\sigma)\|_\infty^{1-(2/q)}\\ &\le \|C(\sigma)\|_{H_1}^{(2/q)}\left(\sigma^{-3/4}\|C\|_{\mathscr{P}_T}\right)^{1-(2/q)}\\ &\le \sigma^{-(3/4)(1-(2/q))}\|C\|_{\mathscr{P}_T},\end{aligned}$$

from which follows

$$\|C(\sigma) \cdot \partial C(\sigma)\|_q \le c\|C(\sigma)\|_\infty\|\partial C(\sigma)\|_q \le \sigma^{-1/4}\sigma^{-(3/4)(1-(2/q))}c\|C\|_{\mathscr{P}_T}^2.$$

Thus the term $C(\sigma) \cdot \partial C(\sigma)$ in $F(\sigma)$ is correctly estimated by the second term on the right of (7.11). Now choose $a = 6, b = 3q$ to find $\|f\|_{3q} \le \|f\|_6^{2/q}\|f\|_\infty^{1-(2/q)}$ and take $f = |C(\sigma)|$ to deduce

$$\begin{aligned}\|C(\sigma)^3\|_q &\le c^2\{\|C(\sigma)\|_{3q}\}^3\\ &\le c^2\{\|C(\sigma)\|_6^{(2/q)}\|C(\sigma)\|_\infty^{1-(2/q)}\}^3\\ &\le c^2\{(\kappa\|C(\sigma)\|_{H_1})^{(2/q)}(\sigma^{-1/4}\|C\|_{\mathscr{P}_T})^{1-(2/q)}\}^3\\ &= c^2\{\kappa^{(2/q)}\sigma^{-(1/4)(1-(2/q))}\|C\|_{\mathscr{P}_T}\}^3,\end{aligned}$$

which completes the verification of (7.11) with $a_1 = c^2 \max(\kappa^3, 1)$. The proof of (7.12) proceeds the same way but for differences in this cubic polynomial. □

Remark 7.5. We will need to use some heat kernel estimates for the absolute and relative Laplacians on a compact n-dimensional Riemannian manifold with smooth boundary. If Δ denotes either of these Laplacians then $e^{t\Delta}$ is given by an integral kernel $K_t(x, y)$, and $t^{n/2}|K_t(x, y)| + t^{(n+1)/2}|\mathrm{grad}_x K_t(x, y)|$ is bounded on any finite interval $0 < t \le T$. See [45, Prop. 5.3] for a proof.

It follows by interpolation that, in three dimensions, given $T_0 \in (0, \infty)$, there is a constant c_1 depending only on T_0 such that, for $1 \le q \le p \le \infty$ and $0 < t \le T_0$,

$$\|e^{t\Delta}\|_{q\to p} \le c_1 t^{-(3/2)((1/q)-(1/p))}, \tag{7.13}$$

$$\|\partial e^{t\Delta}\|_{q\to p} \le c_1 t^{-1/2} t^{-(3/2)((1/q)-(1/p))}, \quad \text{with } \partial = d \text{ or } \partial = d^*, \tag{7.14}$$

$$\|e^{t\Delta}\|_{L^2\to H_1} \le c_1 t^{-1/2}. \tag{7.15}$$

Equation ((7.15) actually follows directly from the spectral theorem.) In particular, each of the following are bounded by $c_1 = c_1(T_0)$ for $0 < t \le T_0$.

$$t^{3/4}\|e^{t\Delta}\|_{2\to\infty}, \qquad t^{1/4}\|e^{t\Delta}\|_{6\to\infty}, \qquad t^{1/4}\|e^{t\Delta}\|_{3/2\to 2}. \tag{7.16}$$

Lemma 7.6. *Let $0 < T_0 < \infty$. There is a constant c_0 depending on T_0 such that, for any $T \in (0, T_0]$ and any connection form $A_0 \in H_1(M)$, the path $[0, T] \ni t \mapsto C_0(t) \equiv e^{t\Delta} A_0$ lies in $\mathscr{P}_T$ and*

$$\|C_0(\cdot)\|_{\mathscr{P}_T} \le c_0 \|A_0\|_{H_1}. \tag{7.17}$$

Proof. Since $e^{t\Delta}$ is a contraction in the H_1 norm (7.3), we have $\|C_0(t)\|_{H_1} \le \|A_0\|_{H_1}$. Furthermore, by (7.16),

$$t^{1/4}\|C_0(t)\|_\infty = t^{1/4}\|e^{t\Delta}A_0\|_\infty \le t^{1/4}\|e^{t\Delta}\|_{6\to\infty}\|A_0\|_6 \le c_1\kappa\|A_0\|_{H_1}.$$

Writing $\partial = d$ or d^*, the last two terms in (7.4) are dominated for the path $C_0(\cdot)$ in accordance with the inequalities

$$\|\partial C_0(t)\|_\infty = \|\partial e^{t\Delta}A_0\|_\infty \le \|e^{t\Delta}\|_{2\to\infty}\|A_0\|_{H_1} \le c_1 t^{-3/4}\|A_0\|_{H_1}.$$

Multiply by $t^{3/4}$ and add to the previous two inequalities to arrive at (7.17). □

Lemma 7.7. *Let $0 < T < \infty$ and $0 < \alpha < 1$. There is a constant $c_{T,\alpha}$ such that, for all $\epsilon > 0$,*

$$\|(e^{\varepsilon\Delta} - 1)e^{s\Delta}\|_{L^2\to H_1} \le \varepsilon^\alpha s^{-\frac{1}{2}-\alpha} c_{T,\alpha} \quad \textit{for } 0 < s \le T, \tag{7.18}$$

$$\|(e^{\varepsilon\Delta} - 1)e^{s\Delta}\|_{L^2\to L^\infty} \le \varepsilon^\alpha s^{-\frac{3}{4}-\alpha} c_{T,\alpha} \quad \textit{for } 0 < s \le T. \tag{7.19}$$

Proof. Let $E = (1-\Delta)^{1/2}$ and let $b > 0$. We assert that there are constants c_α and $\hat{c}_{b,T}$ such that

$$\|E^{-2\alpha}(1 - e^{\varepsilon\Delta})\|_{2\to2} \le \varepsilon^\alpha c_\alpha \quad \text{and} \quad \|E^{2b}e^{s\Delta}\|_{2\to2} \le s^{-b}\hat{c}_{b,T} \tag{7.20}$$

for all $\varepsilon > 0$ and the specified ranges of α and s. The first follows from the spectral theorem for $-\Delta$ and the inequalities $(1+x)^{-\alpha}(1-e^{-\varepsilon x}) = (1+\varepsilon^{-1}y)^{-\alpha}(1-e^{-y}) \le \varepsilon^\alpha c_\alpha$, which hold for all $x > 0$, wherein we have put $y = \varepsilon x$. The second follows similarly from the inequalities $(1+x)^b e^{-sx} = (1+s^{-1}y)^b e^{-y} \le s^{-b}\hat{c}_{b,T}$, wherein we have put $y = sx$.

Defining $2b = 1 + 2\alpha$ in the second line below, we see that

$$\begin{aligned}\|(e^{\varepsilon\Delta} - 1)e^{s\Delta}\|_{L^2\to H_1} &= \|E(e^{\varepsilon\Delta} - 1)e^{s\Delta}\|_{L^2\to L^2} \\ &\le \|E^{-2\alpha}(e^{\varepsilon\Delta} - 1)\|_{2\to2}\|E^{1+2\alpha}e^{s\Delta}\|_{2\to2} \\ &\le \{\varepsilon^\alpha c_\alpha\}\{s^{-\frac{1}{2}-\alpha}\hat{c}_{b,T}\},\end{aligned}$$

which proves (7.18). Choosing next $b = \alpha$, we see that

$$\begin{aligned}\|(e^{\varepsilon\Delta} - 1)e^{s\Delta}\|_{L^2\to L^\infty} &\le \|e^{(s/2)\Delta}\|_{2\to\infty}\|E^{-2\alpha}(e^{\varepsilon\Delta} - 1)\|_{2\to2}\|E^{2\alpha}e^{(s/2)\Delta}\|_{2\to2} \\ &\le \{c_T s^{-3/4}\}\{\varepsilon^\alpha c_\alpha\}\{(s/2)^{-\alpha}\hat{c}_{\alpha,T}\},\end{aligned}$$

where $c_T = \sup_{0<s\le T} s^{3/4}\|e^{(s/2)\Delta}\|_{2\to\infty} < \infty$ in three dimensions by (7.16). □

Lemma 7.8 (Hölder continuity). *Suppose that $C(\cdot) \in \mathscr{P}_T$ and $\|C\|_{\mathscr{P}_T} \le R$. Let $0 < \alpha < 1/4$ and let $0 < a < T$. Define*

$$\rho(t) = \int_0^t e^{(t-\sigma)\Delta} F(\sigma) d\sigma. \tag{7.21}$$

Then there is a constant c_2 depending only on a and α and on R and T such that

$$\|\rho(t) - \rho(r)\|_\infty + \|\rho(t) - \rho(r)\|_{H_1} \le c_2 (t-r)^\alpha \quad \text{for } a \le r < t < T. \tag{7.22}$$

If, moreover, $C(\cdot)$ is a solution to the integral equation (7.9), then $[a, T) \ni \sigma \mapsto F(\sigma) \in L^2(M)$ is Hölder continuous of order α.

Proof. Taking $0 < a \le r < t < T$, we may write

$$\rho(t) - \rho(r) = \int_0^r \left(e^{(t-r)\Delta} - 1\right) e^{(r-\sigma)\Delta} F(\sigma) d\sigma + \int_r^t e^{(t-\sigma)\Delta} F(\sigma) d\sigma. \tag{7.23}$$

We need to estimate the H_1 norm and L^∞ norm of each of these two integrals. In all four integrals we will use (7.11) with $q = 2$, namely

$$\|F(\sigma)\|_2 \le (R^3 a_1) + \sigma^{-1/4} (R^2 a_2), \quad 0 < \sigma < T, \tag{7.24}$$

from which follows, $c_{a,T} \equiv \sup_{a \le \sigma < T} \|F(\sigma)\|_2 < \infty$. Using (7.18) and (7.19) with $\varepsilon = t - r$ and $s = r - \sigma$, as well as (7.15), we find

$$\begin{aligned}
&\|\rho(t) - \rho(r)\|_{H_1} \\
&\quad \le \int_0^r \|(e^{(t-r)\Delta} - 1) e^{(r-\sigma)\Delta}\|_{L^2 \to H_1} \|F(\sigma)\|_2 d\sigma \\
&\qquad + \int_r^t \|e^{(t-\sigma)\Delta}\|_{L^2 \to H_1} d\sigma \sup_{a \le \sigma < T} \|F(\sigma)\|_2 \\
&\quad \le (t-r)^\alpha c_{T,\alpha} \int_0^r (r-\sigma)^{-\frac{1}{2} - \alpha} \|F(\sigma)\|_2 d\sigma + \int_r^t (t-\sigma)^{-1/2} d\sigma \; c_{a,T} \\
&\quad \le (t-r)^\alpha c_3 + (t-r)^{1/2} c_4.
\end{aligned}$$

Equation (7.24) shows that $c_3 < \infty$ if $\alpha < 1/2$. Similarly, by (7.19) and (7.16),

$$\begin{aligned}
&\|\rho(t) - \rho(r)\|_\infty \\
&\quad \le \int_0^r \|(e^{(t-r)\Delta} - 1) e^{(r-\sigma)\Delta}\|_{2 \to \infty} \|F(\sigma)\|_2 d\sigma \\
&\qquad + \int_r^t \|e^{(t-\sigma)\Delta}\|_{2 \to \infty} \sup_{a \le \sigma < T} \|F(\sigma)\|_2 \\
&\quad \le (t-r)^\alpha c_{T,\alpha} \int_0^r (r-\sigma)^{-\frac{3}{4} - \alpha} \|F(\sigma)\|_2 d\sigma + \int_r^t (t-\sigma)^{-3/4} c d\sigma \; c_{a,T} \\
&\quad \le (t-r)^\alpha c_5 + (t-r)^{1/4} c_6.
\end{aligned}$$

In view of (7.24), the constant $c_5 < \infty$ if $\alpha < 1/4$. This proves (7.22).

Now (7.22) shows that the integral term in (7.9) is Hölder continuous on $[a, T)$ into $L^\infty \cap H_1$ in the sum norm. So is the term $e^{t\Delta}A_0$, as one sees from the inequalities $\|(e^{t\Delta} - e^{r\Delta})A_0\|_{H_1} \le (t-r)^\alpha r^{-\frac{1}{2}-\alpha} C_{T,\alpha}\|A_0\|_2$ and $\|(e^{t\Delta} - e^{r\Delta})A_0\|_\infty \le (t-r)^\alpha r^{-\frac{3}{4}-\alpha} C_{T,\alpha}\|A_0\|_2$, which follow from (7.18) and (7.19), respectively. Hence $[a, T) \ni \sigma \mapsto C(\sigma) \in L^\infty \cap H_1(M)$ is bounded and Hölder continuous of order α. Therefore the term $C(\sigma)\cdot\partial C(\sigma)$ in $F(\sigma)$ is Hölder continuous into $L^2(M)$ while the term $C(\sigma)^3$ is Holder continuous into $L^\infty(M)$ and therefore into $L^2(M)$. $\square$

Lemma 7.9 (Strong solution). *Suppose that $C(\cdot)$ is a solution to the integral equation (7.9) lying in $\mathscr{P}_T$. Define $\rho(t)$ by (7.21). Then, for $t > 0$, $\rho(t) \in \mathscr{D}(\Delta)$ and is strongly differentiable as a function into $L^2(M)$. Moreover*

$$\rho'(t) = \Delta\rho(t) + F(t). \tag{7.25}$$

In particular $C(t) \in \mathscr{D}(\Delta)$ for $t > 0$. $C(\cdot)$ is strongly differentiable on $(0, T)$ into $L^2(M)$, and the differential equations (7.8) and (2.14) both hold.

Proof. For $a \le s < t$ define

$$\rho_s(t) = \int_0^s e^{(t-\sigma)\Delta}F(\sigma)d\sigma.$$

Since $t - \sigma \ge t - s > 0$ for all σ in the integrand, $\rho(t)$ is in $\mathscr{D}(\Delta)$ and

$$\Delta\rho_s(t) = \int_0^s \Delta e^{(t-\sigma)\Delta}F(\sigma)d\sigma \quad \text{for } a \le s < t.$$

We are going to show that $\Delta\rho_s(t)$ converges in $L^2(M)$ as $\varepsilon \equiv t - s \downarrow 0$ and in fact uniformly for $t \in [a, b] \subset (0, T)$. Observe first that if $a \le s_1 < s_2 < t$ then, for $0 < \alpha < 1/4$, and with c_7 denoting the Hölder constant for $F(\sigma)$ on $[a, T)$ into $L^2(M)$,

$$\begin{aligned}\|\Delta\int_{s_1}^{s_2} e^{(t-\sigma)\Delta}(F(\sigma)-F(t))d\sigma\|_2 &\le \int_{s_1}^{s_2}\|(t-\sigma)\Delta e^{(t-\sigma)\Delta}\frac{F(\sigma)-F(t)}{t-\sigma}\|d\sigma\\ &\le \int_{s_1}^{s_2}\|(t-\sigma)\Delta e^{(t-\sigma)\Delta}\|_{2\to2}\frac{\|F(\sigma)-F(t)\|_2}{t-\sigma}d\sigma\\ &\le \int_{s_1}^{s_2} c_7(t-\sigma)^{\alpha-1}d\sigma \to 0\end{aligned}$$

as $s_1 < s_2$ both increase to t, and uniformly for $t \in [a, b] \subset (0, T)$. Therefore,

$$\begin{aligned}&\|\Delta(\rho_{s_2}(t) - \rho_{s_1}(t))\|_2\\ &\quad= \|\int_{s_1}^{s_2}\Delta e^{(t-\sigma)\Delta}F(t)d\sigma + \int_{s_1}^{s_2}\Delta e^{(t-\sigma)\Delta}(F(\sigma)-F(t))d\sigma\|_2\\ &\quad\le \|e^{(t-\sigma)\Delta}|_{s_1}^{s_2}F(t)\|_2 + o(1) \to 0\end{aligned}$$

as $s_1 < s_2 \uparrow t$. Moreover, since $F(t)$ is continuous on $[a, T)$ into $L^2(M)$, we may conclude that $e^{\varepsilon\Delta}F(t) - F(t) \to 0$ uniformly for $t \in [a, b]$. Clearly $\rho_s(t) \to \rho(t)$ in $L^2(M)$ as $s \uparrow t$ and uniformly for $t \in [a, b]$. Since Δ is a closed operator it now follows that $\rho(t) \in \mathscr{D}(\Delta)$ and $t \mapsto \Delta\rho(t)$ is continuous into L^2.

To prove (7.25) observe that for $0 < r \le t_0 \le t$ we have

$$\rho(t) - \rho(r) = \int_r^t e^{(t-\sigma)\Delta} F(\sigma)d\sigma + \int_0^r (e^{(t-\sigma)\Delta} - e^{(r-\sigma)\Delta})F(\sigma)d\sigma$$
$$= \int_r^t e^{(t-\sigma)\Delta} F(t_0)d\sigma + \int_r^t e^{(t-\sigma)\Delta}(F(\sigma) - F(t_0))d\sigma + (e^{(t-r)\Delta} - 1)\rho(r)$$
$$= \Delta^{-1}(e^{(t-r)\Delta} - 1)F(t_0) + \int_r^t e^{(t-\sigma)\Delta}(F(\sigma) - F(t_0))d\sigma + (e^{(t-r)\Delta} - 1)\rho(r).$$

Divide by $t - r$ and note that as $t - r \downarrow 0$ one has

$$(t-r)^{-1}\Delta^{-1}(e^{(t-r)\Delta} - 1)F(t_0) \to F(t_0),$$

while

$$(t-r)^{-1}\| \int_r^t e^{(t-\sigma)\Delta}(F(\sigma)-F(t_0))d\sigma\|_2 \le (t-r)^{-1} \int_r^t \|F(\sigma)-F(t_0)\|_2 d\sigma \to 0.$$

Moreover

$$(t-r)^{-1}(e^{(t-r)\Delta} - 1)\rho(r) = (t-r)^{-1}\Delta^{-1}(e^{(t-r)\Delta} - 1)\Delta\rho(r) \to \Delta\rho(t_0)$$

because $r \mapsto \Delta\rho(r)$ is continuous into L^2. This proves (7.25).

Now $C(t) = e^{t\Delta}A_0 + \rho(t)$ by (7.9) and (7.21). Both terms are in the domain of Δ for $t > 0$ and both are differentiable on $(0, T)$ into $L^2(M)$. The equation $C'(t) = \Delta C(t) + F(t)$ now follows from (7.25). We may rearrange the terms in (7.8) to deduce that the differential equation (2.14) holds. We will show explicitly in the next corollary that $B_{C(t)} \in W_1(M)$, which is implicit in (2.14), the rearranged version of (7.8). □

Corollary 7.10 (Boundary conditions). *Under the hypotheses of Lemma 7.9, $DC(t)$ and $D^*C(t)$, resp. $dC(t)$ and $d^*C(t)$, are in $W_1(M)$ for $t > 0$ in the Neumann, resp. Dirichlet cases, as is also $B_{C(t)}$. Moreover, C satisfies the following respective boundary conditions for $t > 0$,*

$$(N)\ C(t)_{norm} = 0, \quad (DC(t))_{norm} = 0, \quad (B_{C(t)})_{norm} = 0, \tag{7.26}$$
$$(D)\ C(t)_{tan} = 0, \quad (dC(t))_{tan} = 0,\ (B_{C(t)})_{tan} = 0, \quad (d^*C(t))_{tan} = 0. \tag{7.27}$$

Proof. Writing d for both the minimal and maximal operators, we see that in both cases $C(t) \in \mathscr{D}(d^*d) \cap \mathscr{D}(dd^*)$ for $t > 0$ by Lemma 7.9. We may apply Proposition 3.5 with $A = 0$ and therefore $B = 0$. Take $\omega = C(t)$ in that proposition. Since $C(t) \in \mathscr{D}(d^*d)$ we have $C(t) \in \mathscr{D}(d)$ while $dC(t) \in \mathscr{D}(d^*)$. But also $dC(t) \in \mathscr{D}(d)$ by (3.13) in case (N) or by (3.14) in case (D). Therefore $dC(t) \in \mathscr{D}(d^*) \cap \mathscr{D}(d)$. By the Gaffney-Friedrichs inequality (2.22) it now follows that $dC(t) \in W_1(M)$. The same argument applies to $d^*C(t)$, upon use of (3.15) and (3.16) since $C(t) \in \mathscr{D}(dd^*)$. Thus $d^*C(t) \in W_1$ also. Further, since $C(t)$ is bounded for each $t > 0$ and in W_1, it follows that $[C(t) \wedge C(t)]$ is in W_1 and so, therefore, is $B_{C(t)}$. This proves the first assertion of the corollary.

Concerning the boundary conditions (7.26) and (7.27), there is a slight difference in the two cases and we will therefore distinguish between the minimal and maximal operators d and D in a repeated application of Lemma 3.4.

In case (N), since $C(t) \in \mathscr{D}(D^*) \cap W_1$, (3.12) shows that $C(t)_{norm} = 0$. Since also $DC(t) \in \mathscr{D}(D^*) \cap W_1$, (3.12) also shows that $(DC(t))_{norm} = 0$. But $(B_{C(t)})_{norm} =$

$(DC(t))_{norm} + (1/2)[C(t) \wedge C(t)]_{norm} = 0 + [C(t)_{norm} \wedge C(t)] = 0$. This establishes (7.26).

In case (D), since $C(t) \in \mathscr{D}(d) \cap W_1$, (3.11) shows that $C(t)_{tan} = 0$. And, since $d^*C(t) \in \mathscr{D}(d) \cap W_1$, (3.11) also shows that $(d^*C(t))_{tan} = 0$. This proves two of the equalities in (7.27). Taking now $\omega = C(t)$ in Proposition 3.5 we see that (3.14) implies $dC(t) \in \mathscr{D}(d)$ and, since $dC(t) \in W_1$, (3.11) shows that $(dC(t))_{tan} = 0$. Finally, $(B_{C(t)})_{tan} = (dC(t))_{tan} + (1/2)[C(t)_{tan} \wedge C(t)_{tan}] = 0$. □

Proof of Theorem 7.3. Let $A_0 \in H_1(M)$, choose $T_0 = 1$ in Lemma 7.6, and let c_0 be as described in that lemma. Choose $R > 2c_0\|A_0\|_{H_1}$. For $C(\cdot) \in \mathscr{P}_T$ define

$$W(C)(t) = e^{t\Delta}A_0 + \int_0^t e^{(t-\sigma)\Delta}F(\sigma)d\sigma, \quad 0 \le t \le T. \tag{7.28}$$

We will show that for T sufficiently small, W takes

$$\mathscr{P}_{T,R} \equiv \{C \in \mathscr{P}_T : \|C\|_{\mathscr{P}_T} \le R\} \tag{7.29}$$

into itself and is a strict contraction on this set. $\mathscr{P}_{T,R}$ is non-empty by Lemma 7.6 for any $T \le 1$. Observe first that, by (7.11) with $q = 2$, we have, for $0 \le t \le T \le 1$,

$$\begin{aligned}\int_0^t \|e^{(t-\sigma)\Delta}F(\sigma)\|_{H_1}d\sigma &\le \int_0^t \|e^{(t-\sigma)\Delta}\|_{L^2\to H_1}\|F(\sigma)\|_2 d\sigma \\ &\le \int_0^t (t-\sigma)^{-1/2}c_1\{(R^3a_1) + \sigma^{-1/4}(R^2a_2)\}d\sigma, \end{aligned}\tag{7.30}$$

while, for any $q \in [2, \infty]$,

$$\begin{aligned}\int_0^t \|e^{(t-\sigma)\Delta}F(\sigma)\|_\infty d\sigma &\le \int_0^t \|e^{(t-\sigma)\Delta}\|_{q\to\infty}\|F(\sigma)\|_q d\sigma \\ &\le \int_0^t (t-\sigma)^{-(3/2q)}c_1\sigma^{-(3/2)(\frac{1}{2}-\frac{1}{q})}\{(R^3a_1) + \sigma^{-1/4}(R^2a_2)\}d\sigma \end{aligned}\tag{7.31}$$

by (7.13) and

$$\begin{aligned}\int_0^t \|\partial e^{(t-\sigma)\Delta}F(\sigma)\|_\infty d\sigma &\le \int_0^t \|\partial e^{(t-\sigma)\Delta}\|_{q\to\infty}\|F(\sigma)\|_q d\sigma \\ &\le c_{q,\infty}\int_0^t (t-\sigma)^{-(3/2q)-(1/2)}\sigma^{-(3/2)(\frac{1}{2}-\frac{1}{q})}\{(R^3a_1) + \sigma^{-1/4}(R^2a_2)\} \end{aligned}\tag{7.32}$$

by (7.14). Although these inequalities are valid for any $q \in [2, \infty]$, nevertheless, for $q = 3$, the last integrand has a non-integrable singularity, $(t-\sigma)^{-1}$, and for $q \le 3$ it is even worse. Moreover for $q = \infty$ two of the four integrands in (7.31) and (7.32) contain the non-integrable singularity σ^{-1}. But use of any $q \in (3, \infty)$ will yield usable estimates and in fact will yield the same t dependence of the integrals. For simplicity we will use $q = 6$ in these estimates.

The six explicit σ integrals in (7.30)–(7.32) may all be done by substituting $\sigma = tr$. Choosing $q = 6$ in (7.31) and (7.32), so that $\sigma^{-(3/2)(\frac{1}{2}-\frac{1}{q})} = \sigma^{-1/2}$, and keeping in mind the three different powers of t dictated by the definition (7.4), one arrives at six integrals $t^\delta \int_0^t (t-\sigma)^{-\beta}\sigma^{-\gamma}d\sigma = c_{\delta,\beta,\gamma}t^{1+\delta-\beta-\gamma}$ which are all finite with the choice

$q = 6$. Choosing $\delta = 0$ for (7.30), $\delta = 1/4$ for (7.31) and $\delta = 3/4$ for (7.32) and adding, we find

$$\int_0^t \|e^{(t-\sigma)\Delta} F(\sigma)\|_{H_1} d\sigma + t^{1/4} \int_0^t \|e^{(t-\sigma)\Delta} F(\sigma)\|_\infty d\sigma$$
$$+ t^{3/4} \int_0^t \|\partial e^{(t-\sigma)\Delta} F(\sigma)\|_\infty d\sigma \le t^{1/2}\{c_8 a_1 R^3\} + t^{1/4}\{c_9 a_2 R^2\}.$$

Hence, in view of (7.17), and taking the supremum over $t \in [0, T]$, we find

$$\|W(C)\|_{\mathscr{P}_T} \le c_0\|A_0\|_{H_1} + T^{1/2}(c_{10}R^3) + T^{1/4}(c_{11}R^2). \tag{7.33}$$

Thus for T sufficiently small, depending on R, the second and third terms on the right add to at most $R - c_0\|A_0\|_{H_1}$. Therefore W takes $\mathscr{P}_{T,R}$ into itself.

For two elements C_1 and C_2 in $\mathscr{P}_{T,R}$ the estimate (7.12) yields, just as in the preceding estimates,

$$\|W(C_1) - W(C_2)\|_{P_T} \le \|C_1 - C_2\|_{\mathscr{P}_T}\{T^{1/2}(c_{16}R^2) + T^{1/4}(c_{17}R)\}, \tag{7.34}$$

since the term $e^{t\Delta}A_0$ cancels in the difference. The coefficient of $\|C_1 - C_2\|_{\mathscr{P}_t}$ may be made less than $1/2$ by choosing T sufficiently small, depending on R. The map W has therefore a unique fixed point in $\mathscr{P}_{T,R}$.

Suppose now that $\hat{C}$ is another solution to (7.9) in $\mathscr{P}_T$. Let $R_1 = \|\hat{C}\|_{\mathscr{P}_T}$. Then $R_1 > R$. Choose $T_1 \le T$ corresponding to R_1 as in the argument following (7.34) with R replaced by R_1. By what has just been proven we have uniqueness of solutions to (7.9) in $\mathscr{P}_{T_1,R_1}$. Since C, restricted to $[0, T_1]$, is in $\mathscr{P}_{T_1,R_1}$ it follows that $\hat{C}$ and C coincide on $[0, T_1]$. We may now apply the same argument on the interval $[T_1, 2T_1]$ (using the same R_1) to conclude that $\hat{C}$ coincides with C on the entire interval $[0, 2T_1]$. And so on. This proves uniqueness of solutions to (7.9) in $\mathscr{P}_T$.

Now Lemma 7.9 shows that the solution $C(t)$ to the integral equation (7.9) is actually a solution to the differential equation (7.8). We may therefore apply [58, Prop. 3.2, page 289] to conclude that the solution $C(\cdot)$ is in $C^\infty((0, T) \times M; \Lambda^1 \otimes \mathfrak{k})$. Rearranging the terms gives (2.14). □

Proof of Theorem 2.13. Choose $T \in (0, \infty)$ as in Theorem 7.3 and denote by $C(\cdot)$ the solution to the integral equation (7.9). Then $C(\cdot)$ lies in $\mathscr{P}_T$ and is therefore a continuous function from $[0, T)$ into W_1. Equation (7.9) shows that $C(0) = A_0$. Corollary 7.10 proves that $B_{C(t)}$ and $d^*C(t)$ are in W_1 for $t > 0$, which is the claim a) in Theorem 2.13, and proves as well that $C(\cdot)$ satisfies all the required boundary conditions, (2.15), resp. (2.16). For $t \in (0, T)$ Lemma 7.9 shows that $C(t)$ is strongly differentiable into $L^2(M)$ and that the differential equation (2.14) holds. The smoothness of $C(\cdot)$ is proved in Theorem 7.3. The boundedness of $t^{3/4}\|B_{C(t)}\|_\infty$, required in condition f) of Theorem 2.13, follows from the fact that $C(\cdot)$ lies in $\mathscr{P}_T$. Indeed, the norm definition (7.4) shows that, for $t \in (0, T)$,

$$\begin{aligned} t^{3/4}\|B_{C(t)}\|_\infty &\le t^{3/4}\{\|dC(t)\|_\infty + (c/2)\|C(t)\|_\infty^2\} \\ &= t^{3/4}\|dC(t)\|_\infty + t^{1/4}(c/2)(t^{1/4}\|C(t)\|_\infty)^2 \\ &\le \|C\|_{\mathscr{P}_T} + t^{1/4}(c/2)\|C\|_{\mathscr{P}_T}^2. \end{aligned} \tag{7.35}$$

Concerning uniqueness of solutions for the parabolic equation (2.14), a standard proof of existence and uniqueness for a semilinear parabolic equation may be found in [58, Chap. 15, Sect. 1]. It is based on a simpler path space than the space $\mathscr{P}_T$, defined in (7.4), that we have been using and relies on simpler estimates: Let $\hat{\mathscr{P}}_T = \{C(\cdot) \in C([0,T); H_1(M)) : \sup_{0\le t<T} \|C(t)\|_{H_1(M)} < \infty\}$. If $C(\cdot) \in \hat{\mathscr{P}}_T$ then $F(\sigma)$ (see (7.10)) is a continuous function into $L^{3/2}(M)$ because $C(\sigma)^3 \in L^6 \cdot L^6 \cdot L^6 \subset L^2(M)$ while $C(\sigma)\cdot\partial C(\sigma) \in L^6 \cdot L^2 \subset L^{3/2}(M)$. Moreover $\|e^{t\Delta}\|_{L^{3/2}\to H_1} \le \|e^{(t/2)\Delta}\|_{L^2\to H_1} \times \|e^{(t/2)\Delta}\|_{L^{3/2}\to L^2} = O(t^{-1/2}t^{-1/4})$ by (7.15) and (7.16). Since $3/4 < 1$ the integral equation (7.9) has a unique solution in $\hat{\mathscr{P}}_T$ for a given $A_0 \in H_1$ and small enough T.

Thus if $C(\cdot) \in \hat{\mathscr{P}}_T$ and is in addition strongly differentiable into $L^2(M)$ and satisfies (7.8) then the identity

$$C(t) - e^{t\Delta}C(0) = \int_0^t (d/d\sigma)\Big(e^{(t-\sigma)\Delta}C(\sigma)\Big)d\sigma = \int_0^t e^{(t-\sigma)\Delta}F(\sigma)d\sigma$$

shows that $C(\cdot)$ satisfies the integral equation (7.9) and uniqueness then follows. The last integrand is an integrable function into H_1 because $F : [0,t] \to L^{3/2}(M)$ is continuous, as we have seen above. The solution to (7.8) is unique, therefore, under the hypothesis that it is continuous and bounded on $[0,T)$ into H_1 and strongly differentiable on $(0,T)$ into L^2. $\square$

Remark 7.11. At the price of a more complicated proof we have used the smaller space $\mathscr{P}_T$ in our existence proof instead of the simpler space $\hat{\mathscr{P}}_T$. In order to derive the local boundedness condition (2.8), our use of the smaller space $\mathscr{P}_T$ seems unavoidable. The local boundedness will be an essential ingredient in our uniqueness proof for the weakly parabolic Yang-Mills heat equation.

7.2. An apriori estimate for the parabolic equation. The apriori estimates in Sect. 6 have parallels for the parabolic equation. But they get rapidly more complicated for the parabolic equation as the order of the inequality increases. We will need the following lowest order estimate. It will not artificially decompose the nonlinear terms in (2.14), as does the method of the previous subsection.

Lemma 7.12. *Assume that $C(\cdot)$ satisfies the conclusions of Theorem 7.3. Then $\|B_{C(t)}\|_2$ is non-increasing on $[0,T)$ and in fact*

$$\|B_{C(t)}\|_2^2 + 2\int_0^t \|d^*_{C(s)}B_{C(s)}\|_2^2 ds = \|B_0\|_2^2. \tag{7.36}$$

In particular,

$$\|B_{C(t)}\|_2 \le \|B_0\|_2. \tag{7.37}$$

Proof. For ease in reading define $\beta(t) = B_{C(t)}$. For $t > 0$, $\beta(t)$ is in the domain of $d^*_{C(t)}$ and therefore in the domain of the square of this operator, by (3.15) and (3.16), since $[B \lrcorner B] = 0$. In fact these identities show that $(d^*_{C(t)})^2\beta(t) = \beta(t)\cdot\beta(t) = 0$ for both (N) and (D) boundary conditions. Moreover $\beta(\cdot)$ is smooth on $(0, T) \times M$ by Theorem 7.3. The following computation is therefore justified for $t > 0$:

$$\begin{aligned}(1/2)(d/dt)\|\beta(t)\|_2^2 &= (\beta'(t), \beta(t)) \\ &= (d_{C(t)}C'(t), \beta(t)) \\ &= (C'(t), d^*_{C(t)}\beta(t)) \\ &= -(d^*_{C(t)}\beta(t) + d_{C(t)}d^*C(t), d^*_{C(t)}\beta(t)) \\ &= -\|d^*_{C(t)}\beta(t)\|_2^2 - (d^*C(t), (d^*_{C(t)})^2\beta(t)) \\ &= -\|d^*_{C(t)}\beta(t)\|_2^2.\end{aligned}$$

Since $C(\cdot)$ is continuous on $[0, T)$ into $W_1 \cap L^4(M)$, $B_{C(t)}$ is continuous into $L^2(M)$ on $[0, T)$. We may therefore integrate the last equality over $[0, t]$ to deduce (7.36) and (7.37). □

8. Short Time Existence and Uniqueness for the Yang-Mills Heat Equation

In this section we will prove the short time existence portions of Theorems 2.5 and 2.7 along with uniqueness. The space H_1 refers to either of the quadratic form domains defined in Remark 4.10 and used in Sect. 7, with the H_1 norm given by (7.3). d_A represents the minimal or maximal operator, in agreement with the boundary conditions.

Theorem 8.1. *Let $A_0 \in H_1(M)$ and suppose that $\beta \geq \|A_0\|_{H_1(M)}$. Then there exists $T > 0$, depending only on β, and a continuous function*

$$A(\cdot) : [0, T) \to H_1(M) \quad \textit{with } A(0) = A_0 \tag{8.1}$$

such that

$$a)\ B(t) \in H_1(M) \textit{ for each } t \in (0, T), \tag{8.2}$$

$$b)\ A(t) \textit{ is a strongly differentiable function into } L^2(M) \textit{ on } (0, T), \tag{8.3}$$

$$c)\ A'(t) = -d^*_{A(t)}B(t), \tag{8.4}$$

$$f)\ t^{3/4}\|B(t)\|_\infty \textit{ is bounded on } (0, T). \tag{8.5}$$

The previous theorem will be deduced from the following theorem, which makes precise the informal procedure described in Lemma 2.14.

Theorem 8.2. *Suppose that $C(\cdot)$ is a solution to (2.14) satisfying conditions a), b), c) and f) of Theorem 2.13 with $T < \infty$. Let $0 < \varepsilon < T$ and, for each $x \in M$, denote by $g_\varepsilon(t, x)$ the solution to the ordinary differential equation*

$$(d/dt)g_\varepsilon(t, x) = (d^*C(t, x))g_\varepsilon(t, x), \quad \varepsilon \leq t < T, \quad g_\varepsilon(\varepsilon) = I_{\mathcal{V}}. \tag{8.6}$$

Then $g_\varepsilon \in C^\infty([\varepsilon, T) \times M; K)$. Define

$$A_\varepsilon(t) = C(t)^{g_\varepsilon(t)} = g_\varepsilon(t)^{-1}C(t)g_\varepsilon(t) + g_\varepsilon(t)^{-1}dg_\varepsilon(t), \quad \varepsilon \leq t < T. \tag{8.7}$$

Then $A_\varepsilon \in C^\infty([\varepsilon, T) \times M; \Lambda^1 \otimes \mathfrak{k}) \cap H_1(M)$. There exists a continuous function

$$A(\cdot) : [0, T) \to H_1(M; \Lambda^1 \otimes \mathfrak{k}) \tag{8.8}$$

such that the curvature $B(t)$ of $A(t)$ is in H_1 for $t > 0$ and the strong L^2 derivative $A'(t)$ exists for all $t > 0$. Furthermore

$$\sup_{\varepsilon \le t < T} \|A(t) - A_\varepsilon(t)\|_{H_1} \to 0 \quad \text{as } \varepsilon \downarrow 0, \tag{8.9}$$

$$\sup_{\varepsilon \le t < T} t^{1/2} \|A'(t) - A'_\varepsilon(t)\|_{L^2} \to 0 \quad \text{as } \varepsilon \downarrow 0, \tag{8.10}$$

$$\sup_{\varepsilon \le t < T} t^{1/2} \|B(t) - B_\varepsilon(t)\|_{H_1} \to 0 \quad \text{as } \varepsilon \downarrow 0, \tag{8.11}$$

$$\text{and} \quad \sup_{\varepsilon \le t < T} t^{3/4} \|B(t) - B_\varepsilon(t)\|_\infty \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{8.12}$$

Moreover $A(\cdot)$ satisfies all the conditions of Theorem 8.1.

Notation 8.3. If $u(x) \in End\, \mathcal{V}$ for each $x \in M$ we will write $\|u\|_\infty = \sup_{x \in M} \|u(x)\|_{op}$, where the subscript op denotes the operator norm. In case u is a function into $\mathfrak{k} \subset End\, \mathcal{V}$ the operator norm and the $\mathfrak{k}$ norm are equivalent and we will not distinguish between them. Compare Notation 2.1. Although all products of $\mathfrak{k}$ valued forms have been, until now, commutator products, as e.g. in (7.6), we will need to estimate more general operators on $\mathcal{V}$ in the following.

Corollary 8.4. *The functions g_ε converge to a continuous function $g : [0, T) \times M \to K \subset End\, \mathcal{V}$ in the sense that*

$$\sup_{\varepsilon \le t < T} \|g(t) - g_\varepsilon(t)\|_\infty \to 0 \quad \text{as } \varepsilon \downarrow 0, \tag{8.13}$$

$$\text{and} \quad \sup_{\varepsilon \le t < T} \|h(t) - g_\varepsilon(t)^{-1} d g_\varepsilon(t)\|_{W_1(M)} \to 0 \quad \text{as } \varepsilon \downarrow 0, \tag{8.14}$$

for some continuous function $h : [0, T) \to W_1(M; \Lambda^1 \otimes \mathfrak{k})$. Here $g(0) = I_{\mathcal{V}}$ and $h(0) = 0$. A is given by

$$A(t) = g(t)^{-1} C(t) g(t) + h(t). \tag{8.15}$$

The proofs of these two theorems and corollary will be given at the end of this section.

Remark 8.5. Since $g_\varepsilon(t)$ is given fairly explicitly by (8.6) in terms of the solution $C(\cdot)$ to the parabolic equation (2.14), it would seem natural to prove (8.13) and (8.14) first, from which (8.9) would follow easily. But we have not been able to find a direct proof of the estimates on $g_\varepsilon(t)^{-1} d g_\varepsilon(t)$ needed for proving (8.14). Instead we will prove (8.13) and (8.9) first, using apriori estimates from Sect. 6.

8.1. g estimates. The following computations, observed by Zwanziger, [60], Donaldson [9] and Sadun [46], underlie the procedure described in Lemma 2.14. Throughout this subsection d and d^* act on all smooth forms on M. Boundary conditions on forms will be described explicitly when appropriate.

Lemma 8.6. *Let $C \in C^\infty((a, b) \times M; \Lambda^1 \otimes \mathfrak{k})$. For each $x \in M$, let $g(t, x)$ be a solution to the ordinary differential equation*

$$g'(t, x) g(t, x)^{-1} = d^* C(t, x), \quad t \in (a, b). \tag{8.16}$$

Define

$$C^g(t,x) = g(t,x)^{-1}C(t,x)g(t,x) + g(t,x)^{-1}dg(t,x). \tag{8.17}$$

Then

$$(g^{-1}dg)' = g^{-1}(dd^*C)g \qquad and \tag{8.18}$$

$$(C^g)' = g^{-1}(C' + d_C d^*C)g. \tag{8.19}$$

Let $A(t,x) = C^g(t,x)$ *and assume that* C *satisfies* (2.14) *over the interval* (a,b)*. Then*

$$A'(t) + d^*_{A(t)}B_{A(t)} = 0 \quad on\ (a,b), \tag{8.20}$$

and further,

$$A'(t) = -g(t)^{-1}\{d^*_{C(t)}B_{C(t)}\}g(t). \tag{8.21}$$

Proof. The easily verifiable identity $(g^{-1}dg)' = g^{-1}\{d(g'g^{-1})\}g$ proves (8.18), given (8.16). Writing $V = g'g^{-1}$ we can compute

$$\begin{aligned}(C^g)' &= g^{-1}\{C' + [C, g'g^{-1}]\}g + (g^{-1}dg)' \\ &= g^{-1}\{C' + [C, V] + dV\}g,\end{aligned}$$

which is (8.19) when (8.16) holds. In particular, if $C' + d_C d^*C = -d^*_C B_C$ over (a,b) then (8.19) shows that $A' = g^{-1}(-d^*_C B_C)g = -d^*_A B_A$. Here we have used the usual gauge transformation identities, $B_A = g^{-1}(B_C)g$ and $d^*_A B_A = g^{-1}(d^*_C B_C)g$ when $A = C^g$. □

Lemma 8.7 (Boundary conditions for g). *Suppose that* $C(\cdot) \in C^\infty((0,T))$ *and satisfies the differential equation* $(d/dt)C = -(d^*_C B_C + d_C d^*C)$ *on* $(0,T)$ *along with one of the two boundary conditions* (2.15) *or* (2.16)*. Define* g_ε *by* (8.6) *and let*

$$h_\varepsilon(t) = g_\varepsilon(t)^{-1}dg_\varepsilon(t), \quad \varepsilon \le t < T. \tag{8.22}$$

If C *satisfies the Neumann boundary contition* (2.15)*, then*

$$(N) \qquad h_\varepsilon(t)_{norm} = 0, \qquad \varepsilon \le t < T, \tag{8.23}$$

and if C *satisfies the Dirichlet boundary condition* (2.16)*, then*

$$(D) \qquad h_\varepsilon(t)_{tan} = 0, \qquad \varepsilon \le t < T. \tag{8.24}$$

In particular, $h_\varepsilon(t) \in H_1$ *in both cases.*

Proof. Since $g_\varepsilon(\varepsilon) = I_{\mathcal{V}}$ it follows that $h_\varepsilon(\varepsilon) = 0$. It suffices, therefore, to show that the normal, respectively tangential, component of $h'_\varepsilon(t)$ is zero on $[\varepsilon, T)$. The identity (8.18) shows that $h'_\varepsilon(t) = g_\varepsilon(t)^{-1}\{dd^*C(t)\}g_\varepsilon(t)$ and therefore it suffices to show that the normal, respectively tangential, component of $dd^*C(t)$ is zero for $\varepsilon \le t < T$.

In case (N) we have, by (2.15), $C(t)_{norm} = 0$ and $(B_{C(t)})_{norm} = 0$. From the first equality it follows that $C'(t)_{norm} = 0$ and from the second equality it follows, with the help of (3.20), that $(d^*_{C(t)}B_{C(t)})_{norm} = 0$. Therefore (2.14) shows that $(d_{C(t)}d^*C(t))_{norm} = 0$. Hence $(dd^*C(t))_{norm} = -[C(t), d^*C(t)]_{norm} = -[C(t)_{norm}, d^*C(t)] = 0$. This proves case (N).

In case (D), we have $(d^*C(t))_{tan} \equiv (d^*C(t))|_{\partial M} = 0$ by (2.16). Therefore $(dd^*C(t))_{tan} = 0$ by (3.19) (with $A = 0$). This proves case (D).

Since $h_\varepsilon(t) \in C^\infty(M)$ and satisfies the right boundary conditions it is in H_1 in both cases.

Although this proves the lemma, it may be worth noting that in case (D) the defining equation (8.6) already shows directly that $g_\varepsilon'(t)|_{\partial M} = 0$ because $d^*C(t)|_{\partial M} = 0$. Hence $g_\varepsilon(t) = I_{\mathcal{V}}$ on ∂M and therefore its tangential derivative, $h_\varepsilon(t)_{\tan}$ is zero. □

Corollary 8.8 (Boundary conditions for A_ε). *Define $A_\varepsilon(t)$ by (8.7). Then $A_\varepsilon(t) \in H_1(M)$ in both Neumann and Dirichlet cases, for $\varepsilon \le t < T$.*

Proof. Since $A_\varepsilon(t)$ is in $C^\infty(M)$ and $C(t)$ satisfies the right boundary conditions, the definition (8.7) shows that we need only prove that $g_\varepsilon(t)^{-1}dg_\varepsilon(t)$ satisfies the correct boundary conditions. But this is the assertion of Lemma 8.7. □

Lemma 8.9. *Define $g_\varepsilon : [\varepsilon, T) \to K$ as in (8.6). Then*

$$\sup_{\varepsilon \le t < T} \|g_\delta(t) - g_\varepsilon(t)\|_\infty \to 0 \quad \text{as } 0 < \delta < \varepsilon \downarrow 0. \tag{8.25}$$

Moreover there is a unique function $g \in C([0, T) \times M; K)$ such that $g(0) = I_{\mathcal{V}}$ and such that, for each $a \in (0, T)$, g_ε converges to g uniformly on $[a, T) \times M$.

Proof. For ease in reading let $V(t, x) = d^*C(t, x)$. All the estimates that need to be made are pointwise in x. For each $x \in M$, $V(t, x)$ is a continuous function on $(0, T)$ into $\mathfrak{k}$ and $\int_0^T \|V(s)\|_\infty ds < \infty$ by (7.4). We will suppress the x dependence in the following. If $0 < \delta < \varepsilon$ then the function $[\varepsilon, T) \ni t \mapsto g_\delta(t)g_\delta(\varepsilon)^{-1}$ satisfies the initial value problem (8.6), and consequently,

$$g_\delta(t) = g_\varepsilon(t)g_\delta(\varepsilon), \quad \varepsilon \le t < T. \tag{8.26}$$

Since $g_\varepsilon(t)$ is unitary it follows that

$$\|g_\delta(t) - g_\varepsilon(t)\|_{op} = \|g_\delta(\varepsilon) - I_{\mathcal{V}}\|_{op}, \quad \varepsilon \le t < T. \tag{8.27}$$

But

$$\|g_\delta(\varepsilon) - I_{\mathcal{V}}\|_{op} = \|\int_\delta^\varepsilon g_\delta'(s)ds\|_{op} \le \int_\delta^\varepsilon \|V(s)\|_\infty ds \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{8.28}$$

This proves (8.25). The existence of a uniform limit g over each set $[a, T) \times M$ now follows and the limit is clearly independent of a. Moreover letting $\delta \downarrow 0$ in (8.28) shows that $\|g(\varepsilon) - I_{\mathcal{V}}\|_\infty \le \int_0^\varepsilon \|V(s)\|_\infty ds$, and therefore g is continuous on all of $[0, T) \times M$ if defined to be $I_{\mathcal{V}}$ at $t = 0$. □

8.2. A estimates. Our goal in this section is to show that the smooth forms $A_\varepsilon(t)$ and $B_\varepsilon(t)$ converge in strong senses as $\varepsilon \downarrow 0$. Since $A_\varepsilon(\cdot)$ is in $C^\infty((\varepsilon, T)) \times M)$, all of the apriori estimates derived in Sects. 5 and 6 are applicable in this subsection.

With a view toward applying the Gaffney-Friedrichs inequality (2.22) (with $A = 0$ in that inequality), we are going to make estimates in the next few lemmas of $\|d\omega\|_2$ and $\|d^*\omega\|_2$ for several different choices of ω.

All four lemmas in this section depend on the apriori estimates of order two in Sect. 6.

Lemma 8.10. *Define* $C_4(\cdot,\cdot)$ *as in Corollary 6.4. Then*

$$\int_\varepsilon^t \|A'_\varepsilon(s)\|_4 ds \le C_4(t, \|B_0\|_2), \quad \varepsilon \le t < T, \tag{8.29}$$

$$\|A_\varepsilon(t)\|_4 \le \|C(\varepsilon)\|_4 + C_4(t, \|B_0\|_2), \quad \varepsilon \le t < T, \tag{8.30}$$

and

$$\|B_\varepsilon(t)\|_2 \le \|B_0\|_2, \quad \varepsilon \le t < T. \tag{8.31}$$

Proof. Since A_ε is a solution to (2.6) over the interval (ε, T), we may apply (6.25) over the interval $[\varepsilon, T)$ to find $\int_\varepsilon^t \|A'_\varepsilon(s)\|_4 ds \le C_4(t-\varepsilon, \|B_{A_\varepsilon(\varepsilon)}\|_2)$ for $\varepsilon \le t < T$. Since C_4 is monotone in both arguments and $\|B_{A_\varepsilon(\varepsilon)}\|_2 = \|B_{C(\varepsilon)}\|_2 \le \|B_0\|_2$, (8.29) follows. The derivation of (8.30) from (8.29) is similar to the derivation of (6.21), considering that $A_\varepsilon(\varepsilon) = C(\varepsilon)$. Further, $\|B_\varepsilon(t)\|_2 = \|g_\varepsilon(t)^{-1} B_{C(t)} g_\varepsilon(t)\|_2 = \|B_{C(t)}\|_2 \le \|B_0\|_2$ by (7.37), proving (8.31). □

Lemma 8.11. *As* $0 < \delta \le \varepsilon \downarrow 0$ *the following limits hold:*

$$\int_\varepsilon^T \|A'_\delta(s) - A'_\varepsilon(s)\|_4 ds \to 0, \tag{8.32}$$

$$\sup_{\varepsilon \le t < T} \|A_\delta(t) - A_\varepsilon(t)\|_4 \to 0, \tag{8.33}$$

$$\sup_{\varepsilon \le t < T} \|A_\delta(t) - A_\varepsilon(t)\|_2 \to 0, \tag{8.34}$$

$$\sup_{\varepsilon \le t < T} \|B_\delta(t) - B_\varepsilon(t)\|_2 \to 0. \tag{8.35}$$

Proof. To prove (8.32) observe that gauge transformations relate well to the forms A' in that

$$A'_\delta(s) = (Ad\ g_\delta(\varepsilon)^{-1}) A'_\varepsilon(s), \quad s \ge \varepsilon \tag{8.36}$$

because $-A'_\delta(s) = d^*_{A_\delta(s)} B_{A_\delta(s)} = (Ad\ g_\delta(\varepsilon)^{-1}) d^*_{A_\varepsilon(s)} B_{A_\varepsilon(s)}$. Therefore

$$\begin{aligned}\int_\varepsilon^t \|A'_\delta(s) - A'_\varepsilon(s)\|_4 ds &= \int_\varepsilon^t \|(Ad\ g_\delta(\varepsilon)^{-1} - I_{\mathfrak{k}}) A'_\varepsilon(s)\|_4 ds \\ &\le \|g_\delta(\varepsilon) - I_{\mathcal{V}}\|_\infty C_4(t, \|B_0\|_2),\end{aligned}$$

from which (8.32) follows.

To prove (8.33) we may again use the identity $A_\varepsilon(t) = C(\varepsilon) + \int_\varepsilon^t A'_\varepsilon(s) ds$ to find

$$\begin{aligned}\|A_\delta(t) - A_\varepsilon(t)\|_4 &= \|C(\delta) - C(\varepsilon) + \int_\delta^t A'_\delta(s) ds - \int_\varepsilon^t A'_\varepsilon(s) ds\|_4 \\ &\le \|C(\delta) - C(\varepsilon)\|_4 + \int_\delta^\varepsilon \|A'_\delta(s)\|_4 ds + \int_\varepsilon^t \|A'_\delta(s) - A'_\varepsilon(s)\|_4 ds.\end{aligned}$$

The first term goes to zero as $\delta < \varepsilon \downarrow 0$ because $C(\cdot)$ is continuous into H_1 and therefore into $L^4(M)$. The third term goes to zero uniformly for $t \in [\varepsilon, T)$ by (8.32). The middle term is equal to $\int_\delta^\varepsilon \|d^*_{C(s)} B_{C(s)}\|_4 ds$ by (8.21) and goes to zero because the integrand is integrable over $[0, T)$ by (8.29). Replace L^4 by L^2 in this proof to arrive at (8.34).

Now

$$\|B_\delta(t) - B_\varepsilon(t)\|_2 = \|(Ad\ g_\delta(t)^{-1} - Ad g_\varepsilon(t)^{-1}) B_{C(t)}\|_2 \le \|Ad\ g_\delta(\varepsilon) - I\|_\infty \|B_0\|_2$$

by (8.27). Thus (8.35) now follows from (8.28). □

Lemma 8.12. *As* $0 < \delta \le \varepsilon \downarrow 0$ *the following limits hold.*

$$\sup_{\varepsilon \le t < T} t^{1/2} \|A'_\delta(t) - A'_\varepsilon(t)\|_2 \to 0, \tag{8.37}$$

$$\sup_{\varepsilon \le t < T} t^{3/8} \|B_\delta(t) - B_\varepsilon(t)\|_4 \to 0, \tag{8.38}$$

$$\sup_{\varepsilon \le t < T} \|d^*(A_\delta(t) - A_\varepsilon(t))\|_2 \to 0, \tag{8.39}$$

$$\sup_{\varepsilon \le t < T} \|d(A_\delta(t) - A_\varepsilon(t))\|_2 \to 0. \tag{8.40}$$

Proof. Since $A_\delta(\delta) = C(\delta)$ we may apply the apriori estimate (6.7) to $A_\delta(t)$ on the interval $[\delta, T)$ to find $(t-\delta)\|A'_\delta(t)\|_2^2 \le C_1(t-\delta, \|B_{C(\delta)}\|_2)$. By (8.36) $\|A'_\delta(t)\|_2 = \|A'_\varepsilon(t)\|_2$ for $0 < \delta \le \varepsilon \le t$ while $\|B_{C(\delta)}\|_2 \le \|B_0\|_2$ by (7.37). Since $C_1(\cdot, \cdot)$ is nondecreasing in both arguments we find $(t - \delta)\|A'_\varepsilon(t)\|_2^2 \le C_1(t, \|B_0\|_2)$. We may now let $\delta \downarrow 0$ to find

$$t^{1/2}\|A'_\varepsilon(t)\|_2 \le C_1(t, \|B_0\|_2)^{1/2}, \quad \varepsilon \le t < T. \tag{8.41}$$

The assertion (8.37) now follows from the inequality $t^{1/2}\|A'_\delta(t) - A'_\varepsilon(t)\|_2 \le \|Ad\ g_\delta(\varepsilon)^{-1} - I_\mathfrak{k}\|_\infty t^{1/2}\|A'_\varepsilon(t)\|_2 \le \|Ad\ g_\delta(\varepsilon)^{-1} - I_\mathfrak{k}\|_\infty C_1(T, \|B_0\|_2)^{1/2}$.

To prove (8.38) observe that $(t-\delta)\|B_\delta(t)\|_6^2 \le C_3(t-\delta, \|B_{C(\delta)}\|_2) \le C_3(t, \|B_0\|_2)$ by (6.8) applied over the interval $[\delta, T)$. Since $\|B_\delta(t)\|_6 = \|Ad\ g_\delta(\varepsilon)^{-1})B_\varepsilon(t)\|_6 = \|B_\varepsilon(t)\|_6$, we can let $\delta \downarrow 0$ to find $t\|B_\varepsilon(t)\|_6^2 \le C_3(t, \|B_0\|_2)$. Interpolation between L^2 and L^4 now gives, in view of (8.31),

$$t^{3/8}\|B_\varepsilon(t)\|_4 \le \|B_\varepsilon(t)\|_2^{1/4}(t^{3/8})\|B_\varepsilon(t)\|_6^{3/4} \le \|B_0\|_2^{1/4} C_3(t, \|B_0\|_2)^{3/8}. \tag{8.42}$$

Hence

$$t^{3/8}\|B_\delta(t) - B_\varepsilon(t)\|_4 \le \|Ad\ g_\delta(\varepsilon)^{-1} - I_\mathfrak{k}\|_\infty \|B_0\|_2^{1/4} C_3(t, \|B_0\|_2)^{3/8}, \tag{8.43}$$

which proves (8.38).

To prove (8.39) observe that, since $A_\varepsilon(s)$ is a C^∞ solution to the Yang-Mills heat equation (5.1), the argument giving the identity (6.29) gives

$$d^* A_\varepsilon(t) = d^* C(\varepsilon) + \int_\varepsilon^t [A_\varepsilon(s) \cdot A'_\varepsilon(s)] ds, \tag{8.44}$$

because $A_\varepsilon(\varepsilon) = C(\varepsilon)$. Using (8.44) for both ε and δ we find

$$\|d^*\{A_\delta(t) - A_\varepsilon(t)\}\|_2 \le \|d^*\{C(\delta) - C(\varepsilon)\}\|_2 + \int_\delta^\varepsilon \|[A_\delta(s) \cdot A'_\delta(s)]\|_2 + \int_\varepsilon^t \|[A_\delta(s) \cdot A'_\delta(s)] - [A_\varepsilon(s) \cdot A'_\varepsilon(s)]\|_2 ds.$$

The first term on the right goes to zero as $0 < \delta < \varepsilon \downarrow 0$ because $C(\cdot)$ is continuous into H_1. The second term goes to zero because $\|A_\delta(s)\|_4$ is bounded, by (8.30), while $\|A'_\delta(s)\|_4 = \|d^*_{C(s)} B_{C(s)}\|_4$, which is integrable over $(0, T)$ by (8.29). The third term goes to zero as $0 < \delta < \varepsilon \downarrow 0$ in view of (8.30), (8.33), (8.29) and (8.32), which show that $\|A_\delta(s)\|_4$ is bounded, that $\|A_\delta(s) - A_\varepsilon(s)\|_4$ goes to zero uniformly in s over $[\varepsilon, T)$, while $\|A'_\varepsilon(s)\|_4$ is bounded in $L^1(\varepsilon, T)$ and $\|A'_\delta(s) - A'_\varepsilon(s)\|_4$ goes to zero in $L^1(\varepsilon, T)$.

To prove (8.40) we use again the identity $dA_\varepsilon = B_\varepsilon - (1/2)[A_\varepsilon \wedge A_\varepsilon]$ to arrive at

$$\|d\{A_\delta(t)-A_\varepsilon(t)\}\|_2 \le \|B_\delta(t)-B_\varepsilon(t)\|_2+(1/2)\|[A_\delta(t)\wedge A_\delta(t)]-[A_\varepsilon(t)\wedge A_\varepsilon(t)]\|_2.$$

The first term on the right goes to zero uniformly for $t \in [\varepsilon, T]$ by (8.35) while the second term goes similarly to zero by virtue of (8.30) and (8.33). □

Lemma 8.13. *There is a non-decreasing continuous function $C_6 : [0,\infty)^3 \to [0,\infty)$, depending only on the geometry of M, such that*

$$t^{1/2}\big(\|dB_\varepsilon(t)\|_2 + \|d^*B_\varepsilon(t)\|_2\big) \le C_6(t, \|B_0\|_2, \|C(\varepsilon)\|_{W_1})\ \varepsilon \le t < T. \tag{8.45}$$

Moreover, as $0 < \delta \le \varepsilon \downarrow 0$ the following limits hold:

$$\sup_{\varepsilon\le t<T} t^{1/2}\|d(B_\delta(t) - B_\varepsilon(t))\|_2 \to 0, \tag{8.46}$$

$$\sup_{\varepsilon\le t<T} t^{1/2}\|d^*(B_\delta(t) - B_\varepsilon(t))\|_2 \to 0. \tag{8.47}$$

Proof. The Bianchi identity and (8.20) yield, respectively,

$$dB_\varepsilon(t) = -[A_\varepsilon(t) \wedge B_\varepsilon(t)], \tag{8.48}$$

$$d^*B_\varepsilon(t) = -A'_\varepsilon(t) - [A_\varepsilon(t)\lrcorner B_\varepsilon(t)]. \tag{8.49}$$

Therefore,

$$\begin{aligned} &t^{1/2}\{\|dB_\varepsilon(t)\|_2 + \|d^*B_\varepsilon(t)\|_2\} \\ &\le t^{1/2}\{\|[A_\varepsilon(t)\wedge B_\varepsilon(t)]\|_2 + \|A'_\varepsilon(t)\|_2 + \|[A_\varepsilon(t)\lrcorner B_\varepsilon(t)]\|_2\} \\ &\le t^{1/2}\{\|A'_\varepsilon(t)\|_2 + 2c\|A_\varepsilon(t)\|_4\|B_\varepsilon(t)\|_4\} \\ &\le C_1(t, \|B_0\|_2)^{1/2} + 2c\big(\|A_\varepsilon(t)\|_4\big)t^{1/2}\|B_\varepsilon(t)\|_4 \\ &\le C_6(t, \|B_0\|_2, \|C(\varepsilon)\|_{H_1}) \end{aligned} \tag{8.50}$$

for some continuous function C_6, by virtue of (8.41), (8.30) and(8.42).

Using the identity (8.49) for ε and δ we may write

$$\begin{aligned} &t^{1/2}\|d^*(B_\delta(t) - B_\varepsilon(t))\|_2 \\ &\le t^{1/2}\|A'_\delta(t) - A'_\varepsilon(t)\|_2 + t^{1/2}\|[A_\delta(t)\lrcorner B_\delta(t)] - [A_\varepsilon(t)\lrcorner B_\varepsilon(t)]\|_2. \end{aligned}$$

The first term goes to zero uniformly for $\varepsilon \le t < T$ by (8.37). The second term goes similarly to zero by combining (8.30), (8.33) with (8.38) and (8.42). A similar argument applies to $t^{1/2}\|d(B_\delta(t) - B_\varepsilon(t))\|_2$ by using (8.48). □

Theorem 8.14. *There exist non-decreasing continuous functions C_7, C_8, C_9 from $[0,\infty)^2 \to [0,\infty)$ such that*

$$\|A_\varepsilon(t)\|_{H_1} \le C_7(t, \|C(\varepsilon)\|_{H_1}), \quad \varepsilon \le t < T, \tag{8.51}$$

$$t^{1/2}\|B_\varepsilon(t)\|_{H_1} \le C_8(t, \|C(\varepsilon)\|_{H_1}), \quad \varepsilon \le t < T, \tag{8.52}$$

$$t^{3/4}\|B_\varepsilon(t)\|_\infty \le C_9(T, \|C\|_{\mathscr{P}_T}), \quad \varepsilon \le t < T. \tag{8.53}$$

Moreover, as $0 < \delta \le \varepsilon \downarrow 0$ *the following limits hold.*

$$\sup_{\varepsilon \le t < T} \|A_\delta(t) - A_\varepsilon(t)\|_{H_1} \to 0, \tag{8.54}$$

$$\sup_{\varepsilon \le t < T} t^{1/2}\|B_\delta(t) - B_\varepsilon(t)\|_{H_1} \to 0, \tag{8.55}$$

$$\sup_{\varepsilon \le t < T} t^{3/4}\|B_\delta(t) - B_\varepsilon(t)\|_\infty \to 0. \tag{8.56}$$

Proof. Apply (6.19) to the smooth solution A_ε over $[\varepsilon, T)$ and recall that $A_\varepsilon(\varepsilon) = C(\varepsilon)$. We find that $\|A_\varepsilon(t)\|_{H_1(M)} \le C_5(t - \varepsilon, \|C(\varepsilon)\|_{H_1(M)}),\ \varepsilon \le t < T$. The monotonicity of C_5 in its first argument now yields (8.51) with $C_7 = C_5$.

To prove (8.54) apply the Gaffney-Friedrichs inequality (2.22) with $A = 0$ and $\omega = A_\delta - A_\varepsilon$. The inequality (8.54) then follows from (8.34), (8.39) and (8.40).

The Gaffney-Friedrichs inequality (2.22), with $A = 0$ and with $\omega = B_\varepsilon(t)$ gives

$$(1/2)t\|B_\varepsilon(t)\|_{H_1}^2 \le t\|dB_\varepsilon(t)\|_2^2 + t\|d^*B_\varepsilon(t)\|_2^2 + \lambda_M t\|B_\varepsilon(t)\|_2^2.$$

This, along with the inequality (8.45) and $\|B_\varepsilon(t)\|_2 \le \|B_0\|_2$, proves (8.52).

Similarly, the Gaffney-Friedrichs inequality (2.22), with $A = 0$ and with $\omega = B_\delta - B_\varepsilon$, proves (8.55) in view of (8.46), (8.47) and (8.35).

Since $t^{3/4}\|B_\varepsilon(t)\|_\infty = t^{3/4}\|B_{C(t)}\|_\infty$ the inequality (7.35) proves (8.53) with $C_9(T, \|C\|_{\mathscr{P}_T}) = \|C\|_{\mathscr{P}_T} + T^{1/4}(c/2)\|C\|_{\mathscr{P}_T}^2$.

Finally, for $\varepsilon \le t < T$, we have $t^{3/4}\|B_\delta(t) - B_\varepsilon(t)\|_\infty \le \|Ad\ g_\delta(\varepsilon) - I\|_\infty C_9(T, \|C\|_{\mathscr{P}_T})$, which goes to zero uniformly for $t \in [\varepsilon, T)$ by (8.28). □

8.3. Proof of Theorems 8.1 and 8.2. If $0 < a < T$ then (8.54) shows that $A_\varepsilon|_{[a,T)}$ is uniformly Cauchy in H_1 norm as $\varepsilon \downarrow 0$. The limit is clearly independent of $a > 0$ and defines a continuous function $A : (0, t) \to H_1$, being a uniform limit of continuous (in fact C^∞) functions on each interval $[a, T)$. Define $A(0) = A_0$. We need to show that the so extended function is continuous at $t = 0$. Since $C(\cdot)$ is continuous on $[0, T)$ into H_1, given $\alpha > 0$, there exists $\gamma > 0$, such that a) $\sup_{0\le t\le\gamma} \|A_0 - C(t)\|_{H_1} < \alpha$ and, by (8.54), b) $\sup_{\varepsilon\le t<T} \|A_\delta(t) - A_\varepsilon(t)\|_{H_1} < \alpha$ if $0 < \delta \le \varepsilon \le \gamma$. Suppose that $0 < t_0 \le \gamma$. Then $\|A_0 - C(t_0)\|_{H_1} < \alpha$ by a). Letting $\delta \downarrow 0$ in b) shows that $\|A(t_0) - A_\varepsilon(t_0)\|_{H_1} \le \alpha$ if $\varepsilon \le t_0$. Take $\varepsilon = t_0$. Then $\|A(t_0) - A_0\|_{H_1} \le \|A(t_0) - A_{t_0}(t_0)\|_{H_1} + \|C(t_0) - A_0\|_{H_1} < 2\alpha$. This proves the existence of a continuous function $A : [0, T) \to H_1(M)$ taking the correct initial value, A_0, and defined as the limit, in the sense of (8.9), of the C^∞ functions $A_\varepsilon : [\varepsilon, T) \to H_1(M)$. In particular (8.1) holds.

Now (8.37) shows that, for each $a > 0$, the derivatives $A'_\varepsilon(t)$ converge uniformly on $[a, T)$, as functions into $L^2(M)$. It follows that $A(t)$ is a strongly differentiable function on $(0, T)$ into $L^2(M)$, as required in (8.3), and that $A'(t) = L^2$ limit of $A'_\varepsilon(t)$ for each $t > 0$. In fact, letting $\delta \downarrow 0$ in (8.37) proves (8.10).

The curvature $B(t)$ of $A(t)$ is well defined because $A(t) \in H_1(M)$. Since, for each $t > 0$, $A_\varepsilon(t)$ converges to $A(t)$ in H_1 by (8.54) it follows that $B_\varepsilon(t)$ converges in L^2 to $B(t)$. But (8.55) shows that, for each $t > 0$, $B_\varepsilon(t)$ is Cauchy in H_1 norm as $\varepsilon \downarrow 0$. Hence $B_\varepsilon(t)$ converges in H_1 norm to an element in H_1, which is also the L^2 limit, $B(t)$. Thus $B(t)$ is in H_1 for each $t > 0$, as required in (8.2), and $\|B_\varepsilon(t) - B(t)\|_{H_1} \to 0$ for each $t > 0$. Therefore $d^*B_\varepsilon(t)$ converges to $d^*B(t)$ in L^2 while also $B_\varepsilon(t)$ converges to $B(t)$ in L^4. Hence $[A_\varepsilon(t) \lrcorner B_\varepsilon(t)]$ converges to $[A(t) \lrcorner B(t)]$ in L^2 in view of (8.33).

Therefore $d^*_{A_\varepsilon(t)} B_\varepsilon(t)$ converges in L^2 to $d^*_{A(t)} B(t)$ for each $t > 0$. It now follows that $A'(t) = -d^*_{A(t)} B(t)$ for $0 < t < T$, as required in (8.4). Furthermore, taking the limit in (8.55) as $\delta \downarrow 0$ proves (8.11).

A similar argument, based on (8.56), shows that, for each $t > 0$, one has $\|B_\varepsilon(t) - B(t)\|_\infty \to 0$ as $\varepsilon \downarrow 0$. In particular, (8.5) follows from (8.53). Moreover, (8.12) follows from (8.56) by letting $\delta \downarrow 0$. This proves Theorems 8.1 and 8.2. $\square$

8.4. Proof of Corollary 8.4. For $\varepsilon \le t < T$ define

$$h_\varepsilon(t) = g_\varepsilon(t)^{-1} dg_\varepsilon(t) \quad \text{and} \quad C_\varepsilon(t) = g_\varepsilon(t)^{-1} C(t) g_\varepsilon(t).$$

Since

$$h_\varepsilon(t) = A_\varepsilon(t) - C_\varepsilon(t), \tag{8.57}$$

we have $\|h_\varepsilon(t)\|_6 \le \|A_\varepsilon(t)\|_6 + \|C(t)\|_6$. Hence

$$\begin{aligned}\|h_\varepsilon(t)\|_6 &\le \kappa(\|A_\varepsilon(t)\|_{H_1} + \|C(t)\|_{H_1})\\ &\le \kappa\{C_7(t, \|C(\varepsilon)\|_{H_1}) + \|C\|_{\mathscr{P}_T}\}\end{aligned}$$

by (8.51) and (7.4). Moreover, from (8.54) and (8.25) we find

$$\begin{aligned}\sup_{\varepsilon \le t < T} \|h_\delta(t) - h_\varepsilon(t)\|_6 &\le \kappa \sup_{\varepsilon \le t < T} \|A_\delta(t) - A_\varepsilon(t)\|_{H_1}\\ &\quad + \sup_{\varepsilon \le t < T} \|C_\delta(t) - C_\varepsilon(t)\|_6 \to 0,\end{aligned} \tag{8.58}$$

as $0 < \delta \le \varepsilon \downarrow 0$. We assert that

$$\sup_{\varepsilon \le t < T} \|C_\delta(t) - C_\varepsilon(t)\|_{H_1} \to 0 \quad \text{as } 0 < \delta \le \varepsilon \downarrow 0. \tag{8.59}$$

It suffices to compute derivatives for some local orthonormal frame field e_1, e_2, e_3. We have $\nabla_j C_\delta(t) = (Ad\, g_\delta(t)^{-1}) \nabla_j C(t) + [C_\delta(t), \langle h_\delta(t), e_j\rangle]$ and therefore, denoting by $\|\cdot\|_2$ an L^2 norm over a coordinate patch, we find

$$\begin{aligned}\|\nabla_j\big(C_\delta(t) - C_\varepsilon(t)\big)\|_2 &\le \|Ad\, g_\delta(t) - Ad\, g_\varepsilon(t)\|_\infty \|\nabla_j C(t)\|_2\\ &\quad + \|[\{C_\delta(t) - C_\varepsilon(t)\}, h_\delta(t)\langle e_j\rangle]\|_2\\ &\quad + \|[C_\varepsilon(t), \{h_\delta(t) - h_\varepsilon(t)\}\langle e_j\rangle]\|_2.\end{aligned}$$

As $0 < \delta \le \varepsilon \downarrow 0$, the first term goes to zero, uniformly for $\varepsilon \le t < T$, by (8.25), since $\|\nabla_j C(t)\|_2 \le \|C\|_{\mathscr{P}_T}$. Since $\|h_\delta(t)\langle e_j\rangle\|_6 \le \|h_\delta(t)\|_6$ remains bounded as $\delta \downarrow 0$ and uniformly so over $t \in [\varepsilon, T)$, while $\|C_\delta(t) - C_\varepsilon(t)\|_3 \le \|Ad\, g_\delta(\varepsilon) - I\|_\infty \|C(t)\|_3 \to 0$ uniformly over $[\varepsilon, T)$ because $\|C(t)\|_3$ is dominated by $\|C(t)\|_{H_1} \le \|C\|_{\mathscr{P}_T}$, the second term also goes to zero uniformly over $[\varepsilon, T)$. The third term is dominated by $\|C(t)\|_3 \|h_\delta(t) - h_\varepsilon(t)\|_6$, which goes to zero uniformly over $[\varepsilon, T)$ by (8.58).

Upon adding the contributions to $\|C_\delta(t) - C_\varepsilon(t)\|^2_{H_1}$ from finitely many coordinate patches that cover M the assertion (8.59) follows. From (8.59) and (8.54) we deduce that

$$\begin{aligned}\sup_{\varepsilon \le t < T} \|h_\delta(t) - h_\varepsilon(t)\|_{H_1} &\le \sup_{\varepsilon \le t < T} \|A_\delta(t) - A_\varepsilon(t)\|_{H_1} + \sup_{\varepsilon \le t < T} \|C_\delta(t) - C_\varepsilon(t)\|_{H_1}\\ &\to 0 \quad \text{as } 0 < \delta \le \varepsilon \downarrow 0.\end{aligned} \tag{8.60}$$

Thus, for each $a \in (0, T)$, the h_ε converge uniformly over $[a, T)$ in H_1 to a function h which is clearly independent of a and defines a continuous function on $(0, T)$ into H_1.
Now (8.57) shows that, for $0 < \delta < \varepsilon \le t$, we have

$$\|h_\delta(t) - h_\varepsilon(t)\|_{H_1} \le \|A_\delta(t) - A_\varepsilon(t)\|_{H_1} + \|C_\delta(t) - C_\varepsilon(t)\|_{H_1}.$$

We have shown that all three differences converge as $\delta \downarrow 0$ and we may conclude that

$$\|h(t) - h_\varepsilon(t)\|_{H_1} \le \|A(t) - A_\varepsilon(t)\|_{H_1} + \|C(t) - C_\varepsilon(t)\|_{H_1}.$$

Take $t = \varepsilon$. Since $g_\varepsilon(\varepsilon) = I_{\mathscr{V}}$ on the fiber $\mathscr{V}$ we have $h_\varepsilon(\varepsilon) = 0$ and $C_\varepsilon(\varepsilon) = C(\varepsilon)$ and therefore $\|h(\varepsilon)\|_{H_1} \le \|A(\varepsilon) - C(\varepsilon)\|_{H_1}$. But $A(\varepsilon)$ and $C(\varepsilon)$ both converge to A_0 in H_1. Hence $\|h(\varepsilon)\|_{H_1} \to 0$ as $\varepsilon \downarrow 0$. Thus h is continuous on $[0, T)$ into H_1 if one defines $h(0) = 0$. The identity (8.15) now follows for each $t > 0$ by taking the $L^2(M)$ limit in (8.7) as $\varepsilon \downarrow 0$. At $t = 0$ Eq. (8.15) just asserts that $A_0 = A_0$ because $h(0) = 0$ and, by Lemma 8.9, $g(0) = I_{\mathscr{V}}$. This completes the proof of Corollary 8.4.

8.5. Uniqueness of solutions.

Theorem 8.15. *Let $T \le \infty$. Let $A_1(\cdot)$ and $A_2(\cdot)$ be two locally bounded strong solutions to* (2.6) *on the interval* $[0, T)$ *and having the same initial data in $W_1(M)$. Assume that either*

$$(N) \quad B_j(t)_{norm} = 0 \quad for\ j = 1, 2 \quad and \quad t > 0 \tag{8.61}$$

$$or \quad (D) \quad A_j(t)_{tan} = 0 \quad for\ j = 1, 2 \quad and \quad t > 0. \tag{8.62}$$

Then $A_1(t) = A_2(t)$ on $[0, T)$.

The proof depends on the next lemma.

Lemma 8.16 (An identity). *Suppose that A_1 and A_2 are two strong solutions satisfying either* (8.61) *or* (8.62)*. Then, for $t > 0$,*

$$\begin{aligned}(d/dt)\|A_1(t) - A_2(t)\|_2^2 &= -2\|B_1(t) - B_2(t)\|_2^2 \\ &\quad - (B_1(t) + B_2(t), [(A_1(t) - A_2(t)) \wedge (A_1(t) - A_2(t))]).\end{aligned} \tag{8.63}$$

Proof. Consider first the Neumann boundary condition (8.61). In this case the heat equation is $A'(s) = -D^*_{A(s)}B(s)$ wherein D denotes the maximal operator defined in Sect. 3 and $D^*_{A(s)}B(s) = D^*B(s) + [A(s) \lrcorner B(s)]$, which is in $L^2(M)$ because $B(s)$ and $A(s)$ are both in W_1 and $B(s)_{norm} = 0$. Since $\mathscr{D}(D) \supset W_1$ we may integrate by parts in the third line below.

$$\begin{aligned}(1/2)(d/dt)\|A_1(t) - A_2(t)\|_2^2 &= (A_1' - A_2', A_1 - A_2) \\ &= (-D^*_{A_1}B_1 + D^*_{A_2}B_2, A_1 - A_2) \\ &= -(B_1, D_{A_1}(A_1 - A_2)) + (B_2, D_{A_2}(A_1 - A_2)).\end{aligned} \tag{8.64}$$

But

$$\begin{aligned}D_{A_1}(A_1 - A_2) &= D(A_1 - A_2) + [A_1 \wedge (A_1 - A_2)] \\ &= B_1 - B_2 - (1/2)[A_1 \wedge A_1] + (1/2)[A_2 \wedge A_2] + [A_1 \wedge (A_1 - A_2)] \\ &= B_1 - B_2 + (1/2)[(A_1 - A_2) \wedge (A_1 - A_2)].\end{aligned}$$

Defining $\alpha = (1/2)[(A_1 - A_2) \wedge (A_1 - A_2)]$, we find, similarly, that $D_{A_2}(A_1 - A_2) = B_1 - B_2 - \alpha$. Hence

$$(1/2)(d/dt)\|A_1(t) - A_2(t)\|_2^2 = -(B_1, B_1 - B_2 + \alpha) + (B_2, B_1 - B_2 - \alpha) = -\|B_1 - B_2\|_2^2 - (B_1 + B_2, \alpha),$$

which is (8.63).

Next, consider the Dirichlet boundary condition (8.62). In this case the heat equation is $A'(s) = -d^*_{A(s)}B(s)$, wherein d is the minimal covariant exterior derivative operator. By (8.62) we have $(A_1(t) - A_2(t))_{tan} = 0$. Since $A_1(t) - A_2(t)$ is also in W_1, it is in the domain of d by (3.11). We may therefore integrate by parts, as in (8.64), to find $(1/2)(d/dt)\|A_1(t) - A_2(t)\|_2^2 = -(B_1, d_{A_1}(A_1 - A_2)) + (B_2, d_{A_2}(A_1 - A_2))$. The rest of the proof is the same as the Neumann case, with D replaced by d. □

Proof of Theorem 8.15. By (8.63) we have

$$(d/dt)\|A_1(t) - A_2(t)\|_2^2 \le |(B_1 + B_2, (A_1 - A_2) \wedge (A_1 - A_2))| \le (\|B_1(t)\|_\infty + \|B_2(t)\|_\infty)c\|A_1(t) - A_2(t)\|_2^2.$$

Here, for the first time, we need to use the assumption that the two solutions are locally bounded. By condition e) in Definition 2.2 we have, for some $b \in (0, T)$ and $a_5 < \infty$,

$$t^{3/4}\|B_j(t)\|_\infty \le a_5/2 \quad \text{for } 0 < t \le b,\ j = 1, 2.$$

Hence

$$(d/dt)\|A_1(t) - A_2(t)\|_2^2 \le a_5 t^{-3/4}\|A_1(t) - A_2(t)\|_2^2. \tag{8.65}$$

Since $\int_0^b t^{-3/4}dt < \infty$ and $\|A_1(0) - A_2(0)\|_2 = 0$, Gronwall's Lemma now shows that $\|A_1(t) - A_2(t)\|_2 = 0$ for $0 < t \le b$. (For example (8.65) shows that $(d/dt)\{e^{-4a_5 t^{1/4}}\|A_1(t) - A_2(t)\|_2^2\} \le 0$.) Now if $[0, a]$ is a maximal interval of equality and $a < T$ then, taking the origin now at $t = a$, condition d) in Definition 2.2 shows that $\|B_j(t)\|_\infty$ are both bounded on any finite interval $[a, b] \subset [a, T)$ and therefore $(t - a)^{3/4}\|B_j(t)\|_\infty$ is bounded on $[a, b]$. The preceding step in the proof now shows that $A_1 = A_2$ on $[a, b]$ and therefore $a = T$. □

Remark 8.17. Hong and Tian [24] have considered the Cauchy problem for the Yang-Mills heat equation interacting with a $\mathfrak{k}$ valued scalar field ϕ over a complete open three dimensional manifold. The flow equation is the gradient flow for the functional $\|B\|_2^2 + \|d_A\phi\|_2^2$. In the absence of the second term the flow equation reduces to the pure Yang-Mills equation (2.6). The flow equation for the combined system is weakly parabolic, just as in the pure Yang-Mills case. The method we used to prove uniqueness can be used for this combined system also. The seemingly fortuitous identity (8.63) extends to a similar identity for the combined system, namely

$$\begin{aligned}(d/dt)&\Big(\|A_1(t) - A_2(t)\|_2^2 + \|\phi_1(t) - \phi_2(t)\|_2^2\Big)\\ &= -2\|B_1 - B_2\|_2^2 - ((B_1 + B_2), [(A_1 - A_2) \wedge (A_1 - A_2)])\\ &\quad - 2\|d_{A_1}\phi_1 - d_{A_2}\phi_2\|_2^2 - 2((d_{A_1}\phi_1 + d_{A_2}\phi_2), [A_1 - A_2, \phi_1 - \phi_2]).\end{aligned} \tag{8.66}$$

Since Hong and Tian operate in the C^∞ category, their solutions are automatically locally bounded. The uniqueness proof given above applies, therefore, to their case as well, in the C^∞ category.

Remark 8.18. The weak parabolicity of the Yang-Mills heat equation is responsible for uniqueness under imposition of only two boundary conditions on the three component form $A(t)$, as already noted in Remark 2.10. It is potentially illuminating to see how the previous proof of uniqueness for the weakly parabolic equation translates to the parabolic case and why it should require three boundary conditions on the three component form $C(t)$. It is indeed possible to carry out the preceding proof for the parabolic case, although it is a little more complicated. While it sheds some light on the comparison of uniqueness proofs we will not pursue it further.

9. Long Time Existence

Here we complete the proof of Theorems 2.5, 2.7 and 2.12.

We will need the growth estimate (6.19) for strong solutions. But the proof of (6.19) given in Sect. 6.2 relies on existence of derivatives, e.g., $B'(t)$, which have not been proven to exist for a strong solution. We are therefore going to construct approximations of a given strong solution by a sequence of smooth solutions, locally in time, using the parabolic equation (2.14) and its partial gauge transforms A_ε, described in Sect. 8.

9.1. Covariant regularization of locally bounded strong solutions.

Lemma 9.1 (Local regularization). *Suppose that A is a locally bounded strong solution over $[0, T)$ for some $T \le \infty$. Let $0 < t < T$ and define $\beta = \sup_{0\le s\le t} \|A(s)\|_{W_1}$. Then there exists $\tau > 0$, depending only on β, such that, for any interval $[a, b] \subset (0, t]$ of length $b - a < \tau$, there exists a sequence A_n of smooth solutions over $[a, b]$ such that*

$$\sup_{a\le s\le b}\Big\{\|A_n(s) - A(s)\|_{W_1} + \|A_n'(s) - A'(s)\|_{L^2} + \|B_n(s) - B(s)\|_{W_1} + \|B_n(s) - B(s)\|_\infty\Big\} \to 0 \tag{9.1}$$

as $n \to \infty$.

Proof. The constant β is finite because $A : [0, T) \to W_1$ is continuous. By Theorem 2.13 there exists $\tau > 0$ such that, for any $t_0 \in [0, T)$, a solution $C(\cdot)$ to (2.14) with initial value $A(t_0)$, exists over $[t_0, t_0 + \tau)$. Suppose then that $[a, b] \subset (0, t]$ and that $b < a + \tau$. Choose $t_0 \in (0, a)$ with $b < t_0 + \tau$. Then $[a, b] \subset (t_0, t_0 + \tau)$ and the solution $C(\cdot)$ to (2.14) over $[t_0, t_0 + \tau)$, with $C(t_0) = A(t_0)$, exists over $[t_0, b]$, at least. Define the usual gauge transforms A_ε of C over $[t_0 + \varepsilon, b]$ as in (8.7). By Theorem 8.2 the smooth solutions A_ε converge as $\varepsilon \downarrow 0$ to a locally bounded strong solution on $[t_0, b]$ with initial data $A(t_0)$. Therefore, by the uniqueness theorem of Sect. 8.5, the solutions A_ε converge to A itself. The sense of convergence is specified in Theorem 8.2 in (8.9), (8.10), (8.11) and (8.12). In particular, choosing $\varepsilon = 1/n$, it follows from these that (9.1) holds because $a - t_0 > 0$. □

Corollary 9.2. *For any locally bounded strong solution $A(\cdot)$ on $[0, T)$,*

a) $\|B(\cdot)\|_2$ is non-increasing on $[0, T)$.
b) For $0 < s < T$ the Sobolev inequality (5.9) holds for $2 \le p \le \infty$ and the Sobolev inequality (6.3) holds.
c) The gauge invariant apriori estimates (6.5), (6.6), (6.8) and (6.11) hold.
d) The apriori estimates (6.20) – (6.23) hold.

Proof. Assume that $A(\cdot)$ is a locally bounded strong solution on $[0, T)$.

For the proofs of a) and b), if $0 < s < T$ pick $t \in (s, T)$ and choose $\tau > 0$ as in Lemma 9.1. Choose an interval $[a, b] \subset (0, t]$ of length $b - a < \tau$ with $a < s < b$, and choose a sequence A_n of smooth solutions over $[a, b]$ as in the lemma. By (5.9) there holds, for the given s,

$$\|B_n(s)\|_6^2 \le \kappa^2\Big(\|A_n'(s)\|_2^2 + \lambda_p(B_n(s))\|B_n(s)\|_2^2\Big), \tag{9.2}$$

where $\lambda_p(B)$ is given in (2.21). By Lemma 9.1, $A_n'(s)$ converges to $A'(s)$ in $L^2(M)$, while $B_n(s)$ converges to $B(s)$ in $W_1(M)$ and in L^∞ and therefore in $L^p(M)$ for $2 \le p \le \infty$. Hence $\lambda_p(B_n(s)) \to \lambda_p(B(s))$ for all $p \ge 2$. Letting $n \to \infty$ in (9.2) proves (5.9) for strong solutions for $2 \le p \le \infty$. Now $\|B_n(\cdot)\|_2$ is non-increasing on $[a, b]$ by Theorem 6.2 and therefore $\|B(\cdot)\|_2$ is non-increasing also on this interval. Thus $\|B(\sigma)\|_2$ is non-increasing on any interval $[a, b] \subset (0, t]$ of length less than τ, and, being continuous at $\sigma = 0$, is therefore non-increasing on $[0, t]$ for any $t < T$. Now (6.3) follows from (5.9) and the monotonicity of $\lambda_2(B(s))$ as in the original proof of (6.3). This proves the assertions of Parts a) and b).

For the proof of c) note first that, unlike the Sobolev inequality just proven for fixed s, all four of the inequalities in Theorem 6.2 are global, in the sense that they involve integrals over large intervals. To use Lemma 9.1 it will be necessary to partition the large intervals into small intervals of length less than τ and establish inequalities in each interval which can be added up with appropriate cancellation of boundary terms. We will illustrate the method by deriving the most complicated estimate, (6.8). Given a locally bounded strong solution A over $[0, T)$ and, given $t \in (0, T)$, pick τ as in Lemma 9.1. Suppose that $[a, b] \subset (0, t]$ with $b - a < \tau$. Denote by A_n a sequence of smooth solutions as prescribed in Lemma 9.1. We may apply the inequality (6.13) to A_n over the interval $[a, b]$ by taking the origin to be at a. Integrating (6.13) over $[a, b]$ we find

$$e^{-\psi_n(s)}\|A_n'(s)\|_2^2\Big|_a^b + \int_a^b e^{-\psi_n(s)}\|B_n'(s)\|_2^2 ds \le 0. \tag{9.3}$$

Here $\psi_n(s) = 2\int_a^s \lambda_2(B_n(\sigma))d\sigma$ as in (6.1). Before letting $n \to \infty$ we need to eliminate $\|B_n'(s)\|_2^2$, which we have no control over (at the present time.) To this end multiply (9.3) by κ^2 and use (6.4) in the integrand to find

$$\begin{aligned}\kappa^2 e^{-\psi_n(s)}\|A_n'(s)\|_2^2\Big|_a^b &+ \int_a^b e^{-\psi_n(s)}\|A_n'(s)\|_6^2 ds \\ &\le \kappa^2\int_a^b e^{-\psi_n(s)}\lambda_2(B_n(a))\|A_n'(s)\|_2^2 ds.\end{aligned} \tag{9.4}$$

By Lemma 9.1 $A_n'(s) \to A'(s)$ in $L^2(M)$ uniformly in s over $[a, b]$ and $\lambda_2(B_n(s)) \to \lambda_2(B(s))$ uniformly also. It now follows from Fatou's Lemma that $\|A'(s)\|_6^2 \le \liminf_{n\to\infty}\|A_n'(s)\|_6^2$ and the same argument applies to the entire integral on the left of (9.4), considering that $\psi_n(s)$ converges uniformly on $[a, b]$ to $\psi_a^s := 2\int_a^s \lambda_2(B(\sigma))d\sigma$. Since, by Part a), $\lambda_2(B(a)) \le \lambda_0$, we arrive at

$$\kappa^2 e^{-\psi_a^s}\|A'(s)\|_2^2\Big|_a^b + \int_a^b e^{-\psi_a^s}\|A'(s)\|_6^2 ds \le \kappa^2\lambda_0\int_a^b e^{-\psi_a^s}\|A'(s)\|_2^2 ds.$$

Now let $0 < \sigma < t$. If $[a, b] \subset [\sigma, t]$, then, using $\psi_\sigma^a + \psi_a^s = \psi_\sigma^s$ for $a \le s$, we can multiply the last inequality by $e^{-\psi_\sigma^a}$ to deduce that

$$\kappa^2 e^{-\psi_\sigma^s} \|A'(s)\|_2^2 \big|_a^b + \int_a^b e^{-\psi_\sigma^s} \|A'(s)\|_6^2 ds \le \kappa^2 \lambda_0 \int_a^b e^{-\psi_\sigma^s} \|A'(s)\|_2^2 ds. \quad (9.5)$$

Since the exponential factors no longer depend on a, (9.5) allows for cancellation of the boundary terms thus: Partition the interval $[\sigma, t]$ into small intervals, choosing $\sigma = a_0 < a_1 < \cdots < a_n = t$ with each interval of length less than τ. Taking $a = a_{j-1}$ and $b = a_j$ in (9.5) and summing from $j = 1$ to n we get cancellation of differences on the left and arrive at

$$\kappa^2 \Big\{ e^{-\psi_\sigma^t} \|A'(t)\|_2^2 - \|A'(\sigma)\|_2^2 \Big\} + \int_\sigma^t e^{-\psi_\sigma^s} \|A'(s)\|_6^2 ds \le \kappa^2 \lambda_0 \int_\sigma^t e^{-\psi_\sigma^s} \|A'(s)\|_2^2 ds,$$

which, upon multiplying by $e^{\psi_\sigma^t}$, gives

$$\kappa^2 \|A'(t)\|_2^2 + \int_\sigma^t e^{\psi_s^t} \|A'(s)\|_6^2 ds \le \kappa^2 \Big\{ e^{\psi_\sigma^t} \|A'(\sigma)\|_2^2 + \lambda_0 \int_\sigma^t e^{\psi_s^t} \|A'(s)\|_2^2 ds \Big\}.$$

By (6.3), which we already know holds for strong solutions by Part b) of this corollary, we have $\|B(t)\|_6^2 \le \kappa^2 \|A'(t)\|_2^2 + \kappa^2 \lambda_0 \|B_0\|_2^2$. Adding this to the last displayed inequality gives exactly (6.16), which, as before, implies (6.8).

For the proof of Part d) observe that, among the inequalities (6.20)–(6.23), the only one relying on more smoothness than is available from the definition of strong solutions is (6.23), because of its dependence on (6.27), which contains third spatial derivatives of A on the left side. But the integrated identity (6.29) is clearly derivable from Lemma 9.1 by adding finitely many identities of the form $d^*A(\sigma)|_a^b = \int_a^b [A(s) \cdot A'(s)] ds$ to arrive at $d^*A(t) - d^*A(r) = \int_r^t [A(s) \cdot A'(s)] ds$, and then letting $r \downarrow 0$. The rest of the proof is the same as the earlier derivation of (6.23). □

Corollary 9.3. *For any locally bounded strong solution* $A(\cdot)$ *over an interval* $[0, T)$ *the growth estimate* (6.19) *holds.*

Proof. The proof of the inequality (6.19) depends on the validity of the inequalities (6.20)–(6.23). These have been proven for locally bounded strong solutions in Corollary 9.2, wherein the restriction that $A \in C^\infty((0, T))$ was removed. The proof given of Theorem 6.7 is now applicable to any locally bounded strong solution. □

9.2. Dirichlet and Neumann boundary conditions.

Proof of Theorems 2.5 and 2.7. Suppose that $A(\cdot)$ is a locally bounded strong solution to (2.6) over $[0, T)$ satisfying either Neumann boundary conditions, (2.9) and (2.10) or Dirichlet boundary conditions, (2.11) and (2.12). If $T < \infty$ then by Theorem 6.7 there is a number $\beta < \infty$ such that $\|A(t)\|_{H_1(M)} \le \beta$ for $0 \le t < T$. By Theorem 8.1 there exists $\delta > 0$ such that short time solutions exist over $[0, 2\delta)$ if $\|A_0\|_{H_1} \le \beta$. Apply this theorem with $A_0 = A(T - \delta)$. Then we may conclude that there is a locally bounded strong solution $\hat{A}(t)$ over $[T - \delta, T + \delta)$ such that $\hat{A}(T - \delta) = A(T - \delta)$. From uniqueness, Theorem 8.15, we know that $\hat{A}(t) = A(t)$ on $[T - \delta, T)$. Hence $\hat{A}$ extends A to the entire interval $[0, T + \delta)$. We show now that the extended solution is locally bounded.

For $0 < a < T$, the condition (2.7) shows that $\|B_{A(t)}\|_\infty$ is bounded on $[a, T)$, and, since $\|B_{\hat{A}(T-\delta+s)}\|_\infty s^{3/4}$ is bounded for $0 < s < 2\delta$ by (8.5), it follows that, for the extension $A(\cdot)$ to $[0, T+\delta)$, one has $\sup_{a\le t<T+\delta}\|B_{A(t)}\|_\infty < \infty$. Therefore $A(\cdot)$ is a locally bounded strong solution on $[0, T+\delta)$. Hence the maximal interval of existence of a locally bounded strong solution is $[0, \infty)$.

The uniqueness portions of Theorems 2.5 and 2.7 follow from Theorem 8.15. □

9.3. Marini boundary conditions. The following lemma will be used to deduce Theorem 2.12 from Theorem 2.5.

Lemma 9.4. *Suppose that* $A \in C^2(M; \Lambda^1 \otimes \mathfrak{k})$. *Then there exists a function* $g \in C^2(M; K)$ *such that*

$$(A^g)_{norm} = 0. \tag{9.6}$$

Proof. For a point $P \in \partial M$ let $x_P(s), 0 \le s < \varepsilon$ be the geodesic in M starting at P and normal to ∂M at P. Thus $x_P'(0) = -\mathbf{n}$, where $\mathbf{n}$ is the outward drawn unit normal at P. We may choose $\varepsilon > 0$ so small that the map $\partial M \times [0, 2\varepsilon) \ni P, s \to x_P(s)$ is a diffeomorphism onto a collar neighborhood U of ∂M in M. Choose a function $h \in C_c^\infty([0, 2\varepsilon))$ such that $h(s) = s$ on $[0, \varepsilon)$. Define

$$g(y) = \begin{cases} e^{h(s)\langle A,\mathbf{n}\rangle_P} & \text{if } y = x_P(s) \in U \\ I_{\mathcal{V}} & \text{if } y \in M - U. \end{cases}$$

Then g is C^2 in U and, since $g(y) = e_K \equiv I_{\mathcal{V}}$ in a neighborhood of the inner boundary of U, it follows that $g \in C^2(M; K)$. Moreover $dg(x_P(s))/ds|_{s=0} = h'(s)|_{s=0}\langle A, \mathbf{n}\rangle_P = \langle A, \mathbf{n}\rangle_P$. Therefore

$$\begin{aligned}(A^g)_{norm}(P) &= g(P)^{-1}A_{norm}g(P) + g(P)^{-1}\langle dg(P), \mathbf{n}\rangle \\ &= A_{norm}(P) - dg(x_P(s))/ds|_{s=0} \\ &= 0.\end{aligned}$$

□

Remark 9.5. The preceding lemma has an imprecise analog for Dirichlet boundary conditions. Suppose that A is in $C^\infty(M)$ and that $B_{tan} = 0$. Then, given a point $P \in \partial M$, there is a smooth function $g : M \to K$ such that $(A^g)_{tan} = 0$ in some neighborhood of P in ∂M. Indeed, the connection form A_{tan} on ∂M has curvature form B_{tan}, which is zero. So A_{tan} is locally, on ∂M, a pure gauge. That is, there exists a smooth function ϕ on a neighborhood of $P \in \partial M$ such that $A_{tan} = \phi^{-1}d\phi$ on this neighborhood. Extend ϕ smoothly to a neighborhood U in M for which $P \in U \cap \partial M \subset$ domain ϕ and define $g = \phi^{-1}$ there. It is straightforward to verify then that $(A^g)_{tan} = 0$ on $U \cap \partial M$ as asserted. Moreover, choosing $\phi(P) = e_K$ and U small, one can ensure that ϕ takes its values in a contractible neighborhood of e_K in K and therefore g can be extended to all of M.

For a nontrivial bundle over M the boundary conditions $B_{norm} = 0$ and $B_{tan} = 0$ are both independent of gauge choices, as opposed to $A_{norm} = 0$ and $A_{tan} = 0$. This has been observed and used by W. Gryc, [20], in his work extending the no-section theorem of Narasimhan and Ramadas, [42], to manifolds with boundary.

Proof of Theorem 2.12. Suppose that $A_0 \in C^2(M; \Lambda^1 \otimes \mathfrak{k})$. By Lemma 9.4 there exists a function $g \in C^2(M; K)$ such that $\hat{A}_0 \equiv A_0^g$ has normal component zero. Clearly

$\hat{A}_0 \in C^1(M) \subset W_1(M)$. By Theorem 2.5 there exists a unique locally bounded strong solution $\hat{A}(\cdot)$ to (2.6) on $[0, \infty)$ such that $\hat{A}(s)_{norm} = 0$ for $s \geq 0$ and $\hat{B}(s)_{norm} = 0$ for $s > 0$. Define $A(s) = \hat{A}(s)^{g^{-1}}$ for $s \geq 0$. Since $g \in C^2(M; K)$, $A(s)$ is again a strong solution and $B(s)_{norm} = (\hat{B}(s)^{g^{-1}})_{norm} = 0$. Of course $A(s)_{norm}$ need not be zero for $s \geq 0$. However, the uniqueness portion of Theorem 2.5 applies, showing that $A(\cdot)$ is the unique locally bounded strong solution with $A(0) = A_0$ and $B(s)_{norm} = 0$ for $s > 0$.

□

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References

1. Alvarez, J., Eydenberg, M.S., Obiedat, H.: The action of operator semigroups on the topological dual of the Beurling-Björck space. J. Math. Anal. Appl. **339**(1), 405–418 (2008)
2. Arnaudon, M., Bauer, R.O., Thalmaier, A.: A probabilistic approach to the Yang-Mills heat equation. J. Math. Pures Appl. (9) **81**(2), 143–166 (2002)
3. Atiyah, M. F., Bott, R.: The Yang-Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A **308**(1505), 523–615 (1983)
4. Bourguignon, J.-P., Lawson, H.B. Jr.: Stability and isolation phenomena for Yang-Mills fields. Commun. Math. Phys. **79**(2), 189–230 (1981)
5. Butzer, P.L., Berens, H.: *Semi-groups of operators and approximation*. Die Grundlehren der mathematischen Wissenschaften, Band **145**, New York: Springer-Verlag New York Inc., 1967
6. Chen, Y.M., Shen, C.L.: *Evolution of Yang-Mills connections*. In: Differential geometry (Shanghai, 1991), River Edge, NJ: World Sci. Publ., 1993, pp. 33–41
7. Conner, P.E.: The Neumann's problem for differential forms on Riemannian manifolds. Mem. Amer. Math. Soc. **1956**(20), 56 (1956)
8. DeTurck, D.M.: Deforming metrics in the direction of their Ricci tensors. J. Diff. Geom. **18**(1), 157–162 (1983)
9. Donaldson, S.K.: Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. Proc. London Math. Soc. (3) **50**(1), 1–26 (1985)
10. Donaldson, S.K.: Boundary value problems for Yang-Mills fields. J. Geom. Phys. **8**(1–4), 89–122 (1992)
11. Donaldson, S.K., Kronheimer, P.B.: *The geometry of four-manifolds*. Oxford Mathematical Monographs, New York: The Clarendon Press/Oxford University Press, 1990
12. Friedrichs, K.O.: Differential forms on Riemannian manifolds. Comm. Pure Appl. Math. **8**, 551–590 (1955)
13. Gaffney, M.P.: The harmonic operator for exterior differential forms. Proc. Nat. Acad. Sci. U. S. A. **37**, 48–50 (1951)
14. Gallot, S., Hulin, D., Lafontaine, J.: *Riemannian geometry*. Third ed., Universitext, Berlin: Springer-Verlag, 2004
15. Gilbarg, D., Trudinger, N.S.: *Elliptic partial differential equations of second order*, Classics in Mathematics. Berlin: Springer-Verlag, 2001, reprint of the 1998 edition
16. Ginibre, J., Velo, G.: Global existence of coupled Yang-Mills and scalar fields in (2 + 1)-dimensional space-time. Phys. Lett. B **99**(5), 405–410 (1981)
17. Ginibre, J., Velo, G.: The Cauchy problem for coupled Yang-Mills and scalar fields in the temporal gauge. Commun. Math. Phys. **82**(1), 1–28 (1981/82)
18. Glimm, J., Jaffe, A.: *Quantum physics*. Second ed., New York: Springer-Verlag, 1987
19. Gross, L.: Convergence of $U(1)_3$ lattice gauge theory to its continuum limit. Commun. Math. Phys. **92**(2), 137–162 (1983)
20. Gryc, W.E.: On the holonomy of the Coulomb connection over manifolds with boundary. J. Math. Phys. **49**(6), 062904 (2008)
21. Hassell, A.: The Yang-Mills-Higgs heat flow on $\mathbf{R}^3$. J. Funct. Anal. **111**(2), 431–448 (1993)
22. Hong, M.-C.: Heat flow for the Yang-Mills-Higgs field and the Hermitian Yang-Mills-Higgs metric. Ann. Global Anal. Geom. **20**(1), 23–46 (2001)
23. Hong, M.-C., Tian, G.: Asymptotical behaviour of the Yang-Mills flow and singular Yang-Mills connections. Math. Ann. **330**(3), 441–472 (2004)

24. Hong, M.-C., Tian, G.: Global existence of the m-equivariant Yang-Mills flow in four dimensional spaces. Commun. Anal. Geom. **12**(1–2), 183–211 (2004)
25. Kogut, J.B., Suskind, L.: Hamiltonian formulation of Wilson's lattice gauge theories. Phys. Rev. D **11**, 395–408 (1975)
26. LePage, G.P., et al.: Accurate determinations of α_s from realistic lattice qcd. Phys. Rev. Lett. **95**, 052002-1–052002-4 (2005)
27. Lions, J.L.: Sur les espaces d'interpolation; dualité. Math. Scand. **9**, 147–177 (1961)
28. Lüscher, M.: *Properties and uses of the Wilson flow in lattice QCD*. J. High Energy Phys. no. 8, 071, 18. (2010)
29. Lüscher, M.: Trivializing maps, the Wilson flow and the HMC algorithm. Commun. Math. Phys. **293**(3), 899–919 (2010)
30. Lüscher, M., Weisz, P.: *Perturbative analysis of the gradient flow in non-abelian gauge theories*. J. High Energy Phys. no. 2, 051, i, 22 (2011)
31. Marini, A.: Dirichlet and Neumann boundary value problems for Yang-Mills connections. Comm. Pure Appl. Math. **45**(8), 1015–1050 (1992)
32. Marini, A.: Elliptic boundary value problems for connections: a non-linear Hodge theory. Mat. Contemp. **2**, 195–205 (1992), Workshop on the Geometry and Topology of Gauge Fields (Campinas, 1991)
33. Marini, A.: The generalized Neumann problem for Yang-Mills connections. Comm. Part. Diff. Eqs. **24**(3-4), 665–681 (1999)
34. Matsuzawa, T.: A calculus approach to hyperfunctions. I. Nagoya Math. J. **108**, 53–66 (1987)
35. Matsuzawa, T.: A calculus approach to hyperfunctions II. Trans. Amer. Math. Soc. **313**(2), 619–654 (1989)
36. Matsuzawa, T.: A calculus approach to hyperfunctions. III. Nagoya Math. J. **118**, 133–153 (1990)
37. Mendez-Hernandez, P.J., Murata, M.: Semismall perturbations, semi-intrinsic ultracontractivity, and integral representations of nonnegative solutions for parabolic equations. J. Funct. Anal. **257**(6), 1799–1827 (2009)
38. Mitrea, D., Mitrea, M., Taylor, M.: Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds. Mem. Amer. Math. Soc. **150**(713), x+120 (2001)
39. Mitrea, M.: Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains. Forum Math. **13**(4), 531–567 (2001)
40. Morrey, C.B. Jr.: A variational method in the theory of harmonic integrals. II. Amer. J. Math. **78**, 137–170 (1956)
41. Morrey, C.B. Jr., Eells, J. Jr.: A variational method in the theory of harmonic integrals. I. Ann. of Math. (2) **63**, 91–128 (1956)
42. Narasimhan, M.S., Ramadas, T.R.: Geometry of SU(2) gauge fields. Commun. Math. Phys. **67**(2), 121–136 (1979)
43. Pulemotov, A.: The Li-Yau-Hamilton estimate and the Yang-Mills heat equation on manifolds with boundary. J. Funct. Anal. **255**(10), 2933–2965 (2008)
44. Råde, J.: On the Yang-Mills heat equation in two and three dimensions. J. Reine Angew. Math. **431**, 123–163 (1992)
45. Ray, D.B., Singer, I.M.: R-torsion and the Laplacian on Riemannian manifolds. Adv. in Math. **7**, 145–210 (1971)
46. Sadun, L.A.: *Continuum regularized Yang-Mills theory*. Ph. D. Thesis, Univ. of California, Berkeley, 1987, 67+ pages
47. Saranen, J.: On an inequality of Friedrichs. Math. Scand. **51**(2), 310–322 (1982)
48. Seiler, E.: *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*. Lecture Notes in Physics, Vol. **159**, Berlin: Springer-Verlag, 1982
49. Sengupta, A.: Gauge theory on compact surfaces. Mem. Amer. Math. Soc. **126**(600) (1997)
50. Sengupta, A.N.: *Gauge theory in two dimensions: topological, geometric and probabilistic aspects*. In: Stochastic analysis in mathematical physics, Hackensack, NJ: World Sci. Publ., 2008, pp. 109–129
51. Singer, I.M.: Some remarks on the Gribov ambiguity. Commun. Math. Phys. **60**(1), 7–12 (1978)
52. Singer, I.M.: The geometry of the orbit space for nonabelian gauge theories. Phys. Scripta **24**(5), 817–820 (1981)
53. Streater, R.F., Wightman, A.S.: *PCT, spin and statistics, and all that*. Princeton Landmarks in Physics, Princeton, NJ: Princeton University Press, 2000, Corrected third printing of the 1978 edition
54. Taibleson, M.H.: On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties. J. Math. Mech. **13**, 407–479 (1964)
55. Taibleson, M.H.: On the theory of Lipschitz spaces of distributions on Euclidean n-space. II. Translation invariant operators, duality, and interpolation. J. Math. Mech. **14**, 821–839 (1965)
56. Taibleson, M.H.: On the theory of Lipschitz spaces of distributions on Euclidean n-space. III. Smoothness and integrability of Fourier tansforms, smoothness of convolution kernels. J. Math. Mech. **15**, 973–981 (1966)

57. Taylor, M.E.: *Partial differential equations*, Texts in Applied Mathematics, Vol. **23**, New York: Springer-Verlag, 1996
58. Taylor, M.E.: *Partial differential equations. III*. Applied Mathematical Sciences, Vol. **117**, New York: Springer-Verlag, 1997, Corrected reprint of the 1996 original
59. Wilson, K.G.: Confinement of quarks. Phys. Rev. D **10**, 2445–2459 (1974)
60. Zwanziger, D.: Covariant quantization of gauge fields without Gribov ambiguity. Nucl. Phys. B **192**(1), 259–269 (1981)

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