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A New Light on Nets of C*-Algebras and Their Representations

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Dedicated to John E. Roberts on the occasion of his seventieth birthday

Abstract: The present paper deals with the question of representability of nets of C*-algebras whose underlying poset, indexing the net, is not upward directed. A particular class of nets, called C*-net bundles, is classified in terms of C*-dynamical systems having as group the fundamental group of the poset. Any net of C*-algebras has a canonical morphism into a C*-net bundle, the enveloping net bundle, which generalizes the notion of universal C*-algebra given by Fredenhagen to nonsimply connected posets. This allows a classification of nets; in particular, we call injective those nets such that the canonical morphism is faithful. Injectivity turns out to be equivalent to the existence of faithful representations. We further relate injectivity to a generalized Čech cocycle of the net, and this allows us to give examples of nets exhausting the above classification.

Using these results we have shown, in another paper, that any conformal net over S^1 is injective.

Contents

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1. Introduction

A net of C*-algebras is a covariant functor from a poset, considered as a category, to the category of unital C*-algebras having faithful *-morphisms as arrows. Actually, this structure is not a net unless the poset is upward directed; it is, rather, a precosheaf of C*-algebras, however we prefer to maintain the term net throughout this paper, according to the convention used in algebraic quantum field theory. The present paper addresses the analysis of nets over posets that are not upward directed and, in particular, the question of their representability on Hilbert spaces.

The basic idea of the algebraic approach to quantum fields over a spacetime is that the physical content of the theory is completely encoded in the relation linking observables measurable within a suitable region of the spacetime to that region ([1,4,8,30,32,38]). Mathematically this is expressed by a net of C*-algebras defined over a poset given as a suitable set of regions of the spacetime ordered under inclusion. Two physical inputs are imposed on this net: causality and covariance under spacetime symmetries, if there are any. This is what is called the *observable net*. Quantum systems, like for instance quantum fields with particle spectra, are described by certain representations of the observable net on a Hilbert space. Some remarkable results in this direction are [5,20,21].

The set of regions indexing the observable net must be chosen to best fit the topological and the causal properties of the spacetime and the global symmetries. In Minkowski space this is the set of double cones, which turns out to be upward directed under inclusion; so the net embeds in the inductive limit C*-algebra (the colimit). Symmetries on the net lift to the inductive limit, and (covariant) representations of the inductive limit yield (covariant) representations of the observable net. However, when one deals with theories over curved spacetimes, where nontrivial topologies are allowed, or over the circle S^1, the appropriate set of regions is not upward directed any more,[1] see [13,29,42]. So the question of the existence of representations arises.

A similar problem arises in geometric group theory ([9,27]). Any simple complex of groups is a net of groups over its set of simplices ordered under opposite inclusion, and the realizability of the complex, namely the existence of an embedding into the colimit, is equivalent to the existence of a faithful representation of the net (the analogue of a Hilbert space representation, see below). Neverthless the powerful results obtained in this context seem to be very far from applicable to the nets arising in quantum field theory, mainly because the posets involved are finite and, in many applications, the groups are finitely generated.

[1] An upward directed poset is simply connected; the first homotopy group of the spacetime is isomorphic to that of the poset indexing the net, see [42]. About the definitions of homotopy and cohomology (homology) of a poset, see the above-cited reference and [39,40].

The question of representability, initially approached by Blackadar in terms of generators and relations ([11,12]), has been considered in the context of quantum field theory by Fredenhagen, who generalized the inductive limit C*-algebra to posets that are not upward directed introducing the universal C*-algebra of a net, characterized by the property that representations of the net (those that we call, in the present paper, *Hilbert space representations*) lift to representations of the universal C*-algebra ([25, 28]). Symmetries of the net lift to the universal C*-algebra, which reduces to the inductive limit when the poset is upward directed. The advantage, apart from that of dealing with a single algebra rather than a net, is that some of the symmetries and representations of the observable net may be realized in the universal C*-algebra, see for instance [19,26,33].

Nevertheless we consider this approach unsatisfactory, for two main reasons. The first one is that the universal C*-algebra may be trivial, but there is no result connecting the nontriviality of the universal C*-algebra to intrinsic properties of the net itself. The second one is that the universal C*-algebra is too restrictive, since it does not take the fundamental group of the poset into account.[2] Two recent papers, [17] and [14] (see also [6]) illustrate the problems that arise. In particular, in [14] quantum effects due to the topology of the spacetime have been studied. Guided by Roberts' cohomology [37,38], the authors introduced a different notion of representation for a net of C*-algebras, that we use here and call *representation* in the sequel. These induce a representation of the fundamental group of the poset; generalize the above notion of Hilbert space representation in the sense that the two notions coincide when the poset is simply connected; describe, as representations of the observable net over a spacetime, charges affected by the topology of the spacetime. Nevertheless, these representations do not, in general, admit any extension to the universal C*-algebra.

The present paper answers the above two questions and is organized as follows. We begin by giving a combinatorial construction of nets of C*-algebras over arbitrary posets (§3.2). Afterwards we focus on a particular class of nets, the C*-net bundles, giving a classification in terms of C*-dynamical systems carrying an action of the fundamental group of the poset (§3.3). The importance of net bundles is that any net defines a C*-net bundle called *the enveloping net bundle*, which is the codomain of a *canonical morphism* from the given net. This gives a first answer to the above questions, since the enveloping net bundle takes into account the topology of the poset and reduces to the universal C*-algebra when the poset is simply connected. We distinguish nets between *nondegenerate*, whenever the enveloping net bundle is nonvanishing, and *injective*, whenever the canonical morphism is faithful (§3.4). Injectivity turns out to be equivalent to the existence of faithful representations (§4). We give an intrinsic description of injective nets in terms of a generalized Čech cocycle. This allows us to find examples of nets exhausting the above classification: degenerate nets, nondegenerate but noninjective nets and injective nets having no Hilbert space representations (so, the universal C*-algebra is trivial but we have an embedding into the enveloping net bundle, §5).

Finally, we stress that this paper is followed by a second one ([44]), in which the ideas of the present work are used to prove that any (covariant) net over the standard, nondirected base of S^1 has faithful (covariant) representations, a scenario of interest in conformal field theory.

[2] A geometric invariant appearing in the context of Yang-Mills theory is the 2-homology of the spacetime, see [3]. In the language of algebraic quantum field theory, we have 2-cycles yielding central elements of the universal *-algebra of the net of perturbative quantum fields (see [33, App. A]). We also mention the 2-cohomology defined by the electromagnetic field over the Minkowski spacetime ([34, §4]), which, in spite of simply-connectedness, is nontrivial since has coefficients in a net.

2. Preliminaries

We introduce basic notions of posets, some motivated by algebraic quantum field theory, and some related algebraic structures. In particular we discuss the fundamental group of a poset in terms of a related simplicial set.

2.1. Posets. A *poset* is a nonempty set K with an order relation $\leq$, that is, $\leq$ is a binary relation which is reflexive, antisymmetric and transitive. We shall denote the elements of K by Latin letters o, a. We shall write $a < o$ to indicate that $a \leq o$ and $a \neq o$. A poset K is said to be *upward directed* if for any pair $o_1, o_2 \in K$ there is $o \in K$ with $o_1, o_2 \leq o$. The *dual* poset of K is the set K° having the same elements as K and order relation $\leq^\circ$ defined by $a \leq^\circ o$ if, and only if, $o \leq a$. We say that K is *downward directed* if K° is upward directed. A subset $C \subseteq K$ is said to be *contained* in $a \in K$ whenever $o \leq a$ for all $o \in C$ and in this case we write $C \subseteq a$.

A *morphism* from K to a poset P is an order preserving map $\mathrm{f} : K \to P$: $\mathrm{f}(o) \leq \mathrm{f}(a)$ whenever $o \leq a$, and we say that it is an isomorphism if it is injective and surjective. We shall denote the set of automorphisms of a poset K by $\mathrm{Aut}(K)$. A group G is a *symmetry group* for K if there is an order preserving left action $G \times K \ni (g, o) \to go \in K$, that is, if there is an injective group morphism from G to $\mathrm{Aut}(K)$.

The next definition is a key notion for posets used in algebraic quantum field theory. A *causal disjointness relation* for a poset K is a symmetric binary relation $\perp$ such that

$$a \perp o \text{ and } \tilde{o} \leq o \;\Rightarrow\; a \perp \tilde{o}.$$

If K is endowed with a causal disjointness relation $\perp$ and G is a symmetry group for K, *we always assume that G preserves* $\perp$; this amounts to saying that

$$a \perp o \iff ga \perp go,$$

for any $g \in G$.

In algebraic quantum field theory, the main object of study is the observable net: an inclusion-preserving mapping from a set of regions K of a given spacetime manifold to the class of C*-algebras [30]. So the poset structure of the set K, ordered under inclusion, enters the theory. In general this set of regions K is a base for the topology of the spacetime manifold, consisting of open, connected and simply connected subsets of the spacetime. If the spacetime has a global symmetry group, then one considers only bases stable under its action.

In Minkowski space $\mathbb{M}^4$, K is the set of double cones, the symmetry group is the Poincaré group, and the causal disjointness relation is spacelike separation. This poset is upward directed under inclusion. For arbitrary 4-dimensional globally hyperbolic spacetimes $\mathcal{M}$, K is the set of diamonds [14,42], which is stable under isometries, and the causal disjointness relation is induced by the causal structure of the spacetime. This poset is not upward directed when $\mathcal{M}$ is not simply connected or when $\mathcal{M}$ has compact Cauchy surfaces. For theories on the circle S^1, K is the set of connected open intervals of S^1 having a proper closure; the symmetry group is $\mathrm{Diff}(S^1)$ or the Möbius subgroup; the usual notion of disjointness between sets is the causal disjointness relation. This poset is not upward directed either.

2.2. A simplicial set for posets. We introduce a simplicial set associated with a poset and, in particular, discuss the notion of the fundamental group of a poset in terms of this simplicial set. The standard symbols ∂_i and σ_i are used to denote the face and degeneracy maps. The symbols ∂_{ij} and σ_{ij} denote, respectively, the compositions $\partial_i \partial_j$ and $\sigma_i \sigma_j$. References for this section are [38,40].

We consider the simplicial set $\Sigma_*(K)$ of *singular simplices* associated with a poset K, introduced by Roberts in [38]. A brief description of the set $\Sigma_n(K)$ of n-simplices is the following. A 0-simplex is just an element of K. Inductively, for $n \geq 1$, and n-simplex x is formed by $n+1$, $(n-1)$-simplices $\partial_0 x, \ldots, \partial_n x$ and by an element of the poset $|x|$, called the *support* of x, such that $|\partial_i x| \leq |x|$ for $i = 0, \ldots, n$. We shall denote 0-simplices either by a or by o, 1-simplices by b, and 2-simplices by c. Given a 1-simplex b the *opposite* $\overline{b}$ is the 1-simplex having the same support as b and such that $\partial_0 \overline{b} = \partial_1 b$, $\partial_1 \overline{b} = \partial_0 b$.

Composing 1-simplices one gets paths. A *path* p is an expression of the form $b_n * \cdots * b_1$, where b_i are 1-simplices satisfying the relations $\partial_0 b_{i-1} = \partial_1 b_i$ for $i = 2, \ldots, n$. We define the 0-simplices $\partial_1 p := \partial_1 b_1$ and $\partial_0 p := \partial_0 b_n$ and call them, respectively, the starting and the ending point of p. The *support* of a path p is the subset $|p|$ of K whose elements are the supports of the 1-simplices which compose the path. By $p : a \to \tilde{a}$ we mean a path starting from a and ending at $\tilde{a}$. A path $p : o \to o$ is called a *loop* over o. The *opposite* of p is the path $\overline{p} : \tilde{a} \to a$ defined by $\overline{p} := \overline{b}_1 * \cdots * \overline{b}_n$. If q is a path from $\tilde{a}$ to $\hat{a}$, then we can define, in an obvious way, the composition $q * p : a \to \hat{a}$.

Any poset morphism $\mathrm{f} : K \to P$ induces a morphism between the corresponding simplicial sets. Given $o \in \Sigma_0(K)$, we let $\mathrm{f}(o) \in \Sigma_0(P)$ be the image of o by f. Inductively, for $n \geq 1$, given an n-simplex x of K we define $\mathrm{f}(x)$ as the n-simplex of P with faces $\partial_i \mathrm{f}(x) := \mathrm{f}(\partial_i x)$ for $i = 0, 1, \ldots n$, and support $|\mathrm{f}(x)| := \mathrm{f}(|x|)$. This, clearly, induces a mapping between the corresponding set of paths: $\mathrm{f}(p) := \mathrm{f}(b_n) * \cdots * \mathrm{f}(b_2) * \mathrm{f}(b_1)$ is a path of P for any path p of K of the form $p = b_n * \cdots * b_2 * b_1$.

A poset K is said to be connected whenever for any pair o, a of 0-simplices there is a path $p : o \to a$. In the present paper *we shall always consider pathwise connected posets*. In a pathwise connected poset we can define *path frames*: fix a 0-simplex o, the pole, a path frame P_o, with respect to o, is a choice for any 0-simplex a of a path $p_{(a,o)} : o \to a$ such that $p_{(o,o)}$ is homotopic (see below) to ι_o the trivial loop over o which i.e. the degenerate 1-simplex $\sigma_0 o$. We shall always denote the opposite $\overline{p_{(a,o)}}$ of the path $p_{(a,o)}$ in P_o by $p_{(o,a)}$.

A deformation of a path p is a path obtained either by replacing two subsequent 1-simplices $\partial_0 c * \partial_2 c$ of p by $\partial_1 c$, or by replacing a 1-simplex $\partial_1 c$ of p by $\partial_0 c * \partial_2 c$, where $c \in \Sigma_2(K)$. Two paths p and q are *homotopy equivalent*, written $\sim$, if one can be obtained from the other by a finite sequence of deformations. We shall denote the homotopy class of a path p by $[p]$. We then define the *first homotopy group* of K, with base point $o \in \Sigma_0(K)$, as

$$\pi_1^o(K) := \{p : o \to o\}/\sim, \quad [p] \cdot [q] := [p * q].$$

Note that, if $\mathrm{f} : K \to P$ is a poset morphism, then the induced mappings between the set of paths (see above) preserves homotopy equivalence. So, by setting

$$\mathrm{f}_*([p]) := [\mathrm{f}(p)], \qquad [p] \in \pi_1^o(K),$$

one has that $\mathrm{f}_* : \pi_1^o(K) \to \pi_1^{\mathrm{f}(o)}(P)$ is a group morphism. Now, since we consider only pathwise connected posets, the first homotopy group does not depend, up to isomorphism, on the choice of the base point; this isomorphism class, written $\pi_1(K)$, is the *fundamental group* of K. We shall say that K is simply connected whenever $\pi_1(K)$ is trivial.

It turns out that a poset K is simply connected if it is upward directed, because there is a contracting homotopy [38], but also if it is downward directed because the fundamental group of a poset and that of its dual are equivalent [40]. A useful result is the following (see [42]): when K is a base of neighbourhoods of a space X consisting of arcwise and simply connected subsets, $\pi_1(K)$ is isomorphic to the homotopy group $\pi_1(X)$. This has important consequences for the present paper.

In the following we shall also use the *nerve* $N_*(K)$ of K. Considering the poset K as a category (taking the elements of K as objects and the inclusions as arrows), a 0-simplex of the nerve is an object of this category; a 1-simplex is an arrow; an n-simplex is a composition of n arrows. The nerve can be realized as a subsimplicial set of $\Sigma_*(K)$. Clearly 0-simplices of the nerve are nothing but that 0-simplices of $\Sigma_0(K)$. For $n \geq 1$, the elements of $N_n(K)$ are those elements x of $\Sigma_n(K)$ whose vertices $x^i := \partial_{012\cdots(i-1)(i+1)\cdots(n-1)n}x$, for $i = 0, 1, \ldots, n$, satisfy the relation

$$x^0 \leq x^1 \leq x^2 \leq \cdots \leq x^n = |x|.$$

In the following, by $(ao) \in N_1(K)$ we shall mean the 1-simplex of the nerve with $\partial_1(ao) = o$ and $\partial_0(ao) = a$; clearly $a \geq o$ and $|(ao)| = a$.

Finally, we introduce a notion of continuity for symmetries of a poset, which is mainly used for posets arising as a base for the topology of a space as those introduced in §2.1. Some preliminary observations are in order. Let G be a symmetry group of the poset K. Extend the action of the group from the poset K to the simplicial set $\Sigma_*(K)$ and hence to paths, as done above for morphisms of posets. It is clear, from the definition of symmetry of a poset, that $p \sim q$ if, and only if, $gp \sim gq$. So $g_* : \pi_1^o(K) \to \pi_1^{go}(K)$ is a group isomorphism. Now, a *continuous symmetry group* is a topological symmetry group G of K such that, for any path p and $a_0, a_1 \in K$ with $\partial_0 p < a_0$, $\partial_1 p < a_1$, there is a neighbourhood U_e of the identity $e \in G$ such that $g\partial_0 p \leq a_0$, $g\partial_1 p \leq a_1$, and

$$(a, g\partial_0 p) * gp * \overline{(o, g\partial_1 p)} \sim (a, \partial_0 p) * p * \overline{(o, \partial_1 p)} \tag{2.1}$$

for any $g \in U_e$. In this equation $(o, g\partial_1 p)$ denotes the 1-simplex of the nerve having $g\partial_1 p$ as 1-face and o as 0-face, that is, $\partial_1(o, g\partial_1 p) = g\partial_1 p$ and $\partial_0(o, g\partial_1 p) = o$, and $\overline{(o, g\partial_1 p)}$ denotes the opposite of the 1-simplex $(o, g\partial_1 p)$.

The meaning of (2.1) is that, in the limit $g \to e$, the path gp becomes homotopic to p, up to rescaling the starting and the ending points. Examples of posets having a continuous symmetry group are those described in §2.1, arising as a base for the topology of a G-space.

3. Abstract Nets of C*-Algebras

We develop the abstract theory of nets of C*-algebras over posets, focusing on those aspects of the theory involving the question of representability of nets. We define some basic notions concerning nets of C*-algebras and give several examples motivating these definitions. Apart from the examples coming from the algebraic quantum field theory, we give new examples of nets of C*-algebras over any poset.

The first important result concerns a particular class of nets, those that we call C*-net bundles, which are classified in terms of C*-dynamical systems having as group the first homotopy group of the poset. The importance of this result relies on two related facts.

The first one is that, as we shall see in §4, this result implies that any C*-net bundle can be faithfully represented. The second one is that any net of C*-algebras has a canonical morphism into a C*-net bundle, the enveloping net bundle. So the existence of faithful representations turns out to be equivalent to the faithfulness of the canonical morphism. Nets satisfying the latter property are called injective.

3.1. Basic definitions. A *net* of C*-algebras $(\mathcal{A}, \jmath)_K$ is defined by a poset K, a correspondence $\mathcal{A} : o \to \mathcal{A}_o$ associating a unital C*-algebra $\mathcal{A}_o$ to any $o \in K$, *the fibre over* o, and a family $\jmath_{oa} : \mathcal{A}_a \to \mathcal{A}_o$, with $a \leq o$, of unital faithful *-morphisms, the *inclusion maps*, satisfying the *net relations*

$$\jmath_{oa} \circ \jmath_{ae} = \jmath_{oe}, \qquad e \leq a \leq o.$$

Whenever the inclusion maps are all *-isomorphisms we say that $(\mathcal{A}, \jmath)_K$ is a C*-*net bundle*. If $P \subset K$, then restricting $\mathcal{A}$ and $\jmath$ to elements of P yields a net called the *restriction of* $(\mathcal{A}, \jmath)_K$ *to* P, that we denote by $(\mathcal{A}, \jmath)_P$.

Remark 3.1. Some observations are in order.

1. According to the above definition, a net of C*-algebras is a pre-cosheaf of C*-algebras. We are adopting the practice in algebraic quantum field theory of calling these objects nets of C*-algebras even if it properly only applies when the poset is upward directed.
2. The term *net bundle* derives from the fact that, as indicated in [40], these objects are fibre bundles over posets where geometrical concepts like connections and their curvatures can be introduced and analyzed. Moreover, when K is a good base for the topology of a space X, the category of net bundles is, in essence, a (non-full) subcategory of the category of bundles on X in the usual sense [40].
3. Note that, when we have a C*-net bundle $(\mathcal{A}, \jmath)_K$, each $\jmath_{oa}, a \leq o$, is invertible, and it makes sense to consider the inverses $\jmath_{oa}{}^{-1}$. To be concise we will write $\jmath_{ao} := \jmath_{oa}{}^{-1}$.

A *morphism* $(\phi, \mathrm{f}) : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_P$ of nets of C*-algebras is a pair (ϕ, f), where $\mathrm{f} : K \to P$ is a morphism of posets and ϕ is a family $\phi_o : \mathcal{A}_o \to \mathcal{B}_{\mathrm{f}(o)}$, $o \in K$, of *-morphisms fulfilling the relation

$$\phi_o \circ \jmath_{oa} = \imath_{\mathrm{f}(o)\mathrm{f}(a)} \circ \phi_a, \qquad a \leq o.$$

We say that (ϕ, f) is a *unital* morphism whenever all ϕ_o are unital, and *faithful on the fibres* whenever ϕ_o is faithful for any o. Moreover, we say that (ϕ, f) is a *monomorphism* if f is injective and ϕ_o is faithful for any o; it is an *isomorphism* if f and ϕ_o, for any o, are isomorphisms. There is an obvious composition rule between morphisms: given $(\psi, \mathrm{h}) : (\mathcal{B}, \imath)_P \to (\mathcal{C}, y)_S$ we define

$$(\psi, \mathrm{h}) \circ (\phi, \mathrm{f}) := (\psi \circ \phi, \mathrm{h} \circ \mathrm{f}) \; : \; (\mathcal{A}, \jmath)_K \to (\mathcal{C}, y)_S,$$

leading, in an obvious way, to the category of nets of C*-algebras. We shall mainly deal with nets over a fixed poset K where we shall denote morphisms of the form (ϕ, id_K) by ϕ (id_K is the identity morphism of K).

The *constant net bundle* with fibre the C*-algebra A is defined as the constant assignment $\mathcal{A}^t_o := \mathrm{A}$, $o \in K$, with inclusion maps $\jmath^t_{\tilde{o}o} := \mathrm{id}_{\mathrm{A}}$ for all $o \leq \tilde{o}$; we say that the net is *trivial* if it is isomorphic to a constant net bundle. Applying the reasoning of

[39,40] it can be proved that when K is simply connected every C*-net bundle $(\mathcal{A}, \jmath)_K$ is trivial; in §3.3 we shall give a proof using dynamical systems. Finally, we say that a net *vanishes* if it is isomorphic to the *null net bundle*, that is, the C*-net bundle having fibre $\{0\}$.

Remark 3.2. It is easily seen from the above definitions that a net is a functor from a poset, considered as a category, to the category of unital C*-algebras having unital faithful *-morphisms as arrows. Other types of nets can be obtained by changing the target category, as follows.

A *net of Hilbert spaces* $(\mathcal{H}, U)_K$ is given by the correspondence assigning Hilbert spaces $\mathcal{H}_o$, $o \in K$, and a family of isometries $U_{ao} : \mathcal{H}_o \to \mathcal{H}_a$, $o \leq a$, fulfilling $U_{eo} = U_{ea} U_{ao}$, $o \leq a \leq e$. When each U_{ao} is unitary we say that $(\mathcal{H}, U)_K$ is a *Hilbert net bundle*.

A *net of locally compact groups* $(\mathcal{G}, \jmath)_K$ is defined assigning locally compact groups $\mathcal{G}_o$, $o \in K$, and continuous group monomorphisms $\jmath_{ao} : \mathcal{G}_o \to \mathcal{G}_a$, $o \leq a$, fulfilling $\jmath_{eo} = \jmath_{ea} \circ \jmath_{ao}$, $o \leq a \leq e$. We say that $(\mathcal{G}, \jmath)_K$ is a group net bundle whenever each $\jmath_{ao}$ is an isomorphism.

Let G be symmetry group of K in the sense of §2.1. A net of C*-algebras $(\mathcal{A}, \jmath)_K$ is *G-covariant* if for any $g \in G$ there is a family $\alpha_o^g : \mathcal{A}_o \to \mathcal{A}_{go}$, $o \in K$, of *-isomorphisms such that

$$\alpha_{go}^h \circ \alpha_o^g = \alpha_o^{hg}, \qquad g, h \in G, \tag{3.1}$$

and

$$\alpha_o^g \circ \jmath_{oa} = \jmath_{go\,ga} \circ \alpha_a^g, \qquad a \leq o. \tag{3.2}$$

Let $(\mathcal{B}, \imath, \beta)$ be a G-covariant net. A morphism $\phi : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_K$ is said to be G-covariant whenever $\beta_o^g \circ \phi_o = \phi_{go} \circ \alpha_o^g$, $\forall o \in K$, $g \in G$. When G is a continuous symmetry group of K (see (2.1)) we assume that the action α on a covariant net $(\mathcal{A}, \jmath, \alpha)_K$ is *continuous*. This amounts to saying that if $\{g_\lambda\}_\Lambda$ is a net in G converging to the identity of the group, then for any $o \in K$, and for any $a \in K$ with $a > o$, there exists $\lambda_a \in \Lambda$ such that $g_\lambda o \leq a$ for any $\lambda \in \Lambda$ with $\lambda \geq \lambda_a$ and

$$\| \jmath_{a\, g_\lambda o} \circ \alpha_o^{g_\lambda}(A) - \jmath_{ao}(A) \| \to 0, \qquad \forall A \in \mathcal{A}_o. \tag{3.3}$$

We stress that this notion of continuity is intended for posets arising as the base for the topology of a topological space acted upon, continuously, by a group (see §2.1). In these cases, one can easily see that when the poset is upward directed this notion of continuity reduces to the usual notion in algebraic quantum field theory (see [30]).

Finally, when K is endowed with a causal disjointness relation $\perp$, we say that a net of C*-algebras $(\mathcal{A}, \jmath)_K$ is *causal* if, for any $o_1, o_2 \leq o$ with $o_1 \perp o_2$,

$$[\jmath_{oo_1}(A_1), \jmath_{oo_2}(A_2)] = 0 \text{ holds} \tag{3.4}$$

for any $A_1 \in \mathcal{A}_{o_1}$ and $A_2 \in \mathcal{A}_{o_2}$, where $[\cdot, \cdot]$ denotes the commutator.

3.2. Examples. Many of the examples of nets of C*-algebras arise as models of quantum fields over a spacetime, mainly *via* the Weyl quantization of the phase space of a classical field. For any spacetime mentioned in §2.1 (the Minkowski space, the circle S^1, or an arbitrary globally hyperbolic spacetime) there are causal nets, over a suited poset (see §2.1), which are covariant with respect to the symmetries of the spacetime. For these examples we refer the reader to [1,7,13,15,22,26,30,46]. We also quote [16] for generally locally covariant nets over globally hyperbolic spacetimes, and [2,15] for examples of nets over the lattice $\mathbb{Z}^n$.

In all the mentioned examples the representability problem does not arise: the nets are represented on the *physical* Hilbert space of the model.[3] However, in order to study model independent aspects of a theory one is led to consider abstract nets of C*-algebras. Apart from the cases, like the Minkowski space, where the poset indexing the net is upward directed (see §2.1), there is no result about the existence of representations for these nets; in these cases the existence of a representation is a working assumption. This motivates our investigation.

Important examples of nets outside the context of quantum field theory come from geometric group theory: as stated in the Introduction, any simple complex of groups is a net of groups ([9,27,31,45]). Examples are also the systems of C*-algebras and the systems of groups underlying the notions of the (generalized) amalgamated free product [10,11,35,36].

In the next subsections we provide new examples of nets. In particular, we give a combinatorial construction of nets of C*-algebras over arbitrary posets. These nets turn out to be covariant when the posets have a symmetry group. Aspects related to the causal structure will be discussed in a future work [18].

3.2.1. Nets of groups of loops. We introduce a way of deforming a path of a poset which turns out to be weaker than that underlying the homotopy equivalence relation. This new deformation allows us to construct examples of nets of discrete groups (in the present section) and of C*-algebras (later) over any poset.

Let p, q be two paths with the same endpoints. We say that q is a *w-deformation* of p if

(1) either q is obtained by inserting into p a degenerate 1-simplex;
(2) or by replacing two consecutive 1-simplices $\partial_0 c * \partial_2 c$ of the path p by $\partial_1 c$, where $c \in N_2(K)$ (a 2-simplex of the nerve);
(3) or q can be obtained by replacing two consecutive 1-simplices $\overline{b} * b$ of the path p by $\sigma_0 \partial_1 b$.

Two paths with the same endpoints are *w-equivalent* if one can be obtained from the other by a finite sequence of w-deformations.

This is an equivalence relation weaker than homotopy since any pair of w-equivalent paths are also homotopy equivalent. The following example points out the difference between these two relations. Consider a 1-simplex $b \in \Sigma_1(K)$, and the associated path

$$p_b := \overline{(|b|\partial_0 b)} * (|b|\partial_1 b),$$

where $(|b|\partial_1 b)$ is the 1-simplex of the nerve of K (see §2.2) having 1-face $\partial_1 b$ and 0-face $|b|$; while $\overline{(|b|\partial_0 b)}$ is the opposite in $\Sigma_1(K)$ of the 1-simplex of the nerve $(|b|\partial_0 b)$. Then

[3] For instance, models obtained by Weyl quantization are, in general, represented on a Fock space. More in general, in these models the net embeds into the C*-algebra associated with whole phase space. Faithful representations of this algebra induce faithful representations of the net.

b is homotopic to p_b: the relations

$$\partial_0 c_b := (\overline{|b|\partial_0 b}),\quad \partial_2 c_b := (|b|\partial_1 b),\quad \partial_1 c_b := b,\quad |c_b| := |b|,$$

define a 2-simplex c_b of $\Sigma_2(K)$. However

$$b \sim_w p_b \quad \Longleftrightarrow \quad b \in N_1(K).$$

In fact if b does not belong to the nerve of K, then c_b does not belong to the nerve of K, and this implies that $b \not\sim_w p_b$. Conversely if b belongs to the nerve, then $\partial_0 b = |b|$. So $(\overline{|b|\partial_0 b}) = \sigma_0\partial_0 b$ (the degenerate 1-simplex associated with the 0-face of b) and $b = (|b|\partial_1 b)$. Therefore $c_b \in N_2(K)$ and $b \sim_w p$ by property (2) of the above definition.

Now it is easily seen that w-equivalence is compatible with the composition of paths and with the operation of taking the opposite of a path. This allows us to associate two groups with any o of K. The first group is defined as the quotient set of the set of loops over o with respect to the w-equivalence:

$$\Lambda_o := \{p : o \to o\}/\sim_w . \tag{3.5}$$

The product is defined as $[p]_w \cdot [q]_w := [p * q]_w$. We call this group the *w-group of loops* over o. The second group is the subset of Λ_o defined by

$$\Lambda^l_o := \{[p]_w \in \Lambda_1(o) \mid \exists q \in [p]_w \text{ such that } |q| \subseteq o\}, \tag{3.6}$$

with the same product as Λ_o. In words Λ^l_o is the subset of those elements $[p]_w$ of Λ_o for which there is at least one path in the equivalence class of $[p]_w$ whose support is contained in o. This is a subgroup because composition of paths whose support is contained in o leads to a path supported in o. We call Λ^l_o the *w-group of loops supported in o*.

The next step is to prove that these groups form nets of discrete groups. Given an inclusion $o \le a$, define

$$\lambda_{ao}([p]_w) := [(ao) * p * (\overline{ao})]_w, \qquad [p]_w \in \Lambda_1(o). \tag{3.7}$$

Here (ao), as above, is the 1-simplex of the nerve associated with the inclusion $o \le a$; instead $(\overline{ao})$ is the opposite, in $\Sigma_1(K)$, of (ao) (see the preliminaries).

It is clear that $\lambda_{ao} : \Lambda_o \to \Lambda_a$. Moreover, it is easily seen from the definition that $\lambda_{ao} : \Lambda^l_o \to \Lambda^l_a$. Now, for any $[q]_w, [p]_w \in \Lambda_o$ we have

$$\begin{aligned}\lambda_{ao}([p]_w)\,\lambda_{ao}([q]_w) &= [(ao) * p * (\overline{ao})]_w \cdot [(ao) * q * (\overline{ao})]_w\\ &= [(ao) * p * (\overline{ao}) * (ao) * q * (\overline{ao})]_w\\ &= [(ao) * p * q * (\overline{ao})]_w = \lambda_{ao}([p * q]_w),\end{aligned}$$

because of property (3) of the definition of w-deformation. Moreover for any inclusion $e \le o \le a$,

$$\lambda_{ao}(\lambda_{oe}([p]_w)) = [(ao) * (oe) * p * (\overline{oe}) * (\overline{ao})]_w = [(ae) * p * (\overline{ae})]_w = \lambda_{ae}([p]_w),$$

for any $[p]_w \in \Lambda_e$, because of property (2) of the definition of w-deformation. Finally if $[p]_w \in \Lambda_a$, then $[(\overline{ao}) * p * (ao)]_w \in \Lambda_o$ for $o \le a$, and $\lambda_{ao}([(\overline{ao}) * p * (ao)]_w) = [p]_w$. This proves that $\lambda_{ao} : \Lambda_o \to \Lambda_a$ is a group isomorphism making $(\Lambda, \lambda)_K$ a net bundle of discrete groups and $(\Lambda^l, \lambda)_K$ a net of discrete groups.

Remark 3.3. In general the triple $(\Lambda^l, \lambda)_K$ is *is not a net bundle*. For, it is enough to consider inclusions $o, \tilde{o} < a$ such that o and $\tilde{o}$ are not related. In this case one can easily construct elements of Λ^l_a not belonging to the image of Λ^l_o by λ_{ao}.

Assume that K has a symmetry group G. Extend the action of the symmetry group from the poset K to the simplicial set $\Sigma_*(K)$, and hence to paths, as done in §2.1 for morphisms of posets. Note that $p \sim_w q$ if, and only if, $gp \sim_w gq$ for any $g \in G$. This is quite obvious from how g is defined on paths. Therefore,

$$g_*([p]_w) := [gp]_w, \qquad [p]_w \in \Lambda_o, \tag{3.8}$$

is well defined, and $g_* : \Lambda_o \to \Lambda_{go}$ and $g_* : \Lambda^l_o \to \Lambda^l_{go}$ are group isomorphisms. Moreover, given $o \leq a$ we have

$$g_*(\lambda_{ao}([p]_w)) = g_*([(ao) * p * (\overline{ao})]_w) = [g(ao) * gp * \overline{g(oa)}]_w = \lambda_{g(ao)}(g_*([p]_w)),$$

for any $[p]_w \in \Lambda_o$. Summing up, we have the following result.

Proposition 3.4. *Let K be a poset.*

- *(i) Then $(\Lambda, \lambda)_K$ and $(\Lambda^l, \lambda)_K$ are, respectively, a net bundle and a net of discrete groups with a monomorphism $i : (\Lambda^l, \lambda)_K \to (\Lambda, \lambda)_K$ defined by the inclusions $i_o : \Lambda^l_o \to \Lambda_o$.*
- *(ii) If K has a symmetry group G, then $i : (\Lambda^l, \lambda)_K \to (\Lambda, \lambda)_K$ is a monomorphism of G-covariant nets.*

Proof. It is clear that the mapping $i : \Lambda^l_o \to \Lambda_o$ defined by $i_o([p]_w) = [p]_w$, with $[p]_w \in \Lambda^l_o$, is a monomorphism. Moreover $g_* \circ i_o([p]_w) = [gp]_w = i_{go}([gp]_w) = i_{go} \circ g_*([p]_w)$ for any $[p]_w \in \Lambda^l_o$. This completes the proof. □

3.2.2. Nets of C**-algebras from nets of discrete groups.* In the present section we make use of the functor assigning the group C*-algebra to construct nets of C*-algebras starting from nets of discrete groups.

Let G be a discrete group and $C^*(G)$ denote the (full) group C*-algebra; this is a unital C*-algebra defined as the enveloping C*-algebra of the convolution algebra $\ell^1(G)$. The unital *-algebra $\mathbb{C}(G)$ of those functions $f : G \to \mathbb{C}$ which are a finite linear combination $f = \sum_g f(g)\,\delta_g$ of Kronecker delta functions is a dense subset of $\ell^1(G)$. In particular $\delta_g * \delta_h = \delta_{gh}$.

Given a group morphism $\sigma : G \to H$, for any $f \in \mathbb{C}(G)$ let

$$\tilde{\sigma}(f) := \sum f(g)\delta_{\rho(g)}.$$

This defines a morphism $\tilde{\sigma} : \mathbb{C}(G) \to \mathbb{C}(H)$ in the ℓ^1-norm. Moreover it is an isometry (isomorphism) when σ is injective (is an isomorphism). Therefore $\tilde{\sigma}$ lifts to a continuous morphism from $\ell^1(G)$ into $\ell^1(H)$. By the universality property of the enveloping C*-algebra there exists a unique morphism $C^*(\sigma) : C^*(G) \to C^*(H)$ such that

$$C^*(\sigma) \circ \iota_G = \iota_H \circ \tilde{\sigma},$$

where ι denotes the embedding of the ℓ^1 algebra into the enveloping C*-algebra. Therefore, the mapping

$$C^* : G \mapsto C^*(G), \quad \sigma \mapsto C^*(\sigma)$$

is a functor. When σ is a group isomorphism $C^*(\sigma)$ is, clearly, a *-isomorphism, nevertheless we are interested in the case where σ is simply a monomorphism, and for a locally compact group $C^*(\sigma)$ is not injective in general. On the other hand, assuming that G, H are discrete we find that for every unitary representation π of G there is an isometry $V \in (\pi, \mathrm{ind}(\pi) \circ \sigma)$, where $\mathrm{ind}(\pi)$ is the representation of H induced by π [4] (see for instance [24]). By the definition of the enveloping C*-algebra, and the relation between the unitary representations of a locally compact group and non-degenerate representations of the ℓ^1 algebra, $C^*(\sigma)$ is a faithful *-morphism.

The functor C^* allows one to construct in the obvious way a net of C*-algebras $(C^*(\mathcal{G}), C^*(\jmath))_K$, $C^*(\mathcal{G})_o := C^*(\mathcal{G}_o)$, $C^*(\jmath)_{o'o} := C^*(\jmath_{o'o})$, $o \leq o' \in K$, starting from the net of discrete groups $(\mathcal{G}, \jmath)_K$. Clearly, $(C^*(\mathcal{G}), C^*(\jmath))_K$ is a net bundle when $(\mathcal{G}, \jmath)_K$ is a net bundle, and this yields the following result.

Proposition 3.5. *Given a poset K, let $(\Lambda^l, \lambda)_K$ and $(\Lambda, \lambda)_K$ be the net of discrete groups defined in §3.2.1. Then $(C^*(\Lambda^l), C^*(\lambda))_K$ and $(C^*(\Lambda), C^*(\lambda))_K$ are respectively a net of C*-algebras and a C*-net bundle, and*

$$C^*(i) : (C^*(\Lambda^l), C^*(\lambda))_K \to (C^*(\Lambda), C^*(\lambda))_K$$

is a unital monomorphism. If K has a symmetry group G, then all the above nets and morphisms are G-covariant.

Proof. The first assertion follows as C^* is a functor. In a similar fashion, to prove covariance we note that by Prop.3.4 the two nets of groups are G-covariant, and, given $g \in G$, define

$$c_g := C^*(g^*) : C^*(\Lambda)_o \to C^*(\Lambda)_{go},$$

where $g^* : \Lambda_o \to \Lambda_{go}$ is defined in (3.8). It is easily verified that this makes our nets G-covariant, as desired. □

Remark 3.6. With the applications to algebraic quantum field theory in mind, this is a promising result: it allows us to construct nontrivial examples of covariant nets of C*-algebras over any spacetime, taking as poset a suitable base of neighbourhoods for the topology of the spacetime encoding the causal structure and the global symmetry of the spacetime, as indicated in the preliminaries. However, this result, as it stands, is not complete. In fact, the above nets are not causal. Furthermore, if G is a continuous symmetry for the poset (2.1), then the action of G on the net of C*-algebras defined in Prop.3.5 is not continuous according to the definition given in §3.1. This is so because w-equivalence, used to define the groups of loops, is weaker than homotopy equivalence. These gaps are filled in [18].

3.3. The holonomy dynamical system. We now focus on net bundles. We shall see that the fibres of a net bundle are acted upon by the homotopy group of the poset. This in turn will lead to an equivalence between the category of net bundles over a poset and that of $\pi_1^o(K)$-dynamical systems for any $o \in K$. We shall work using morphisms over different posets, a scenario that will appear in the forthcoming paper [44] and is useful in generally locally covariant field theory ([16]).

[4] We are grateful to A. Valette for drawing our attention to this fact, and to a counterexample when the groups involved are not discrete.

We start by making some preliminary definitions. Let (A, G, α) and (B, H, β) be C*-dynamical systems; a morphism from (A, G, α) to (B, H, β) is a pair (η, p) where $\eta : \mathrm{A} \to \mathrm{B}$ is a *-morphism and $\mathrm{p} : G \to H$ is a group morphism satisfying the relation

$$\eta \circ \alpha_g = \beta_{\mathrm{p}(g)} \circ \phi \ , \qquad g \in G. \tag{3.9}$$

We say that (η, f) is an *isomorphism* when η is a *-isomorphism and f is a group isomorphism. When $G = H$ we shall denote morphisms of the form (η, id_G) by η.

Now, consider a net bundle $(\mathcal{A}, \jmath)_K$ and define

$$\jmath_b := \jmath_{\partial_0 b|b|} \circ \jmath_{|b|\partial_1 b}, \qquad b \in \Sigma_1(K), \tag{3.10}$$

where $\jmath_{\partial_0 b|b|} := \jmath^{-1}_{|b|\partial_0 b}$. Since the inclusion maps are isomorphisms, we have a field

$$\Sigma_1(K) \ni b \to \jmath_b \in \mathrm{Iso}(\mathcal{A}_{\partial_1 b}, \mathcal{A}_{\partial_0 b}),$$

satisfying the *1-cocycle* equation

$$\jmath_{\partial_0 c} \circ \jmath_{\partial_2 c} = \jmath_{\partial_1 c}, \qquad c \in \Sigma_2(K), \tag{3.11}$$

where $\mathrm{Iso}(\mathcal{A}_{\partial_1 b}, \mathcal{A}_{\partial_0 b})$ is the set of *-isomorphisms from $\mathcal{A}_{\partial_1 b}$ to $\mathcal{A}_{\partial_0 b}$. Extend this 1-cocycle from 1-simplices to paths by setting $\jmath_p := \jmath_{b_n} \circ \cdots \circ \jmath_{b_2} \circ \jmath_{b_1}$ for any path $p := b_n * \cdots * b_2 * b_1$. Then the 1-cocycle equation implies *homotopy invariance*, that is, $\jmath_p = \jmath_q$ whenever $p \sim q$ (see [42]). In this way, fixing an element o of K and defining $\mathcal{A}_* := \mathcal{A}_o$ yields the action

$$\jmath_* : \pi_1^o(K) \to \mathrm{Aut}\mathcal{A}_*, \quad \jmath_{*,[p]} := \jmath_p, \quad [p] \in \pi_1^o(K). \tag{3.12}$$

We call $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$ the *holonomy dynamical system* associated with the net bundle $(\mathcal{A}, \jmath)_K$. A different choice of the base element o leads to an isomorphic dynamical system. If $(\phi, \mathrm{f}) : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_S$ is a morphism, there is an induced morphism $\mathrm{f}_* : \pi_1^o(K) \to \pi_1^{\mathrm{f}(o)}(S)$, and defining

$$(\phi_*, \mathrm{f}_*) : (\mathcal{A}_*, \pi_1^o(K), \jmath_*) \to (\mathcal{B}_*, \pi_1^{\mathrm{f}(o)}(S), \imath_*), \quad \phi_* := \phi_o,$$

yields a morphism of dynamical systems, in fact

$$\phi_* \circ \jmath_{*,[p]} = \phi_o \circ \jmath_p = \imath_{\mathrm{f}_*(p)} \circ \phi_o = \imath_{*,\mathrm{f}_*[p]} \circ \phi_*.$$

Remark 3.7. Just a comment on the terminology: as said before, a net bundle can be seen as a fibre bundle over the underlying poset; the fibre over o is nothing but the algebra $\mathcal{A}_o$, and the inclusion maps $\jmath$ define a connection of the fibre bundle. So, for any loop p over o, the automorphism $\jmath_p$ is the parallel transport along p for the connection $\jmath$, i.e., the holonomy.

Now, we want to prove that the mapping

$$(\mathcal{A}, \jmath)_K \mapsto (\mathcal{A}_*, \pi_1^o(K), \jmath_*) \tag{3.13}$$

is bijective up to isomorphism. Thus we have to find an inverse (up to isomorphism). To this end, we fix a path frame $P_o := \{p_{(a,o)}, a \in K\}$ over $o \in K$ (see §2.2) and, given the dynamical system $(\mathrm{A}, \pi_1^o(K), \alpha)$, define

$$\alpha_{*,\tilde{a}a} := \alpha_{[p_{(o,\tilde{a})} * (a\tilde{a}) * p_{(a,o)}]} \in \mathrm{Aut}\mathrm{A} \ , \qquad a \le \tilde{a} \ , \tag{3.14}$$

where, by convention, $p_{(o,\tilde{a})}$ denotes the opposite $\overline{p_{(\tilde{a},o)}}$ of $p_{(\tilde{a},o)}$. Observe that

$$\begin{aligned}\alpha_{*,\tilde{a}a}\circ\alpha_{*,ae} &= \alpha_{[p_{(o,\tilde{a})}*(\tilde{a}a)*p_{(a,o)}]}\circ\alpha_{[p_{(o,a)}*(ae)*p_{(e,o)}]}\\ &= \alpha_{[p_{(o,\tilde{a})}*(\tilde{a}a)*p_{(a,o)}*p_{(o,a)}*(ae)*p_{(e,o)}]}\\ &= \alpha_{[p_{(o,\tilde{a})}*(\tilde{a}a)*(ae)*p_{(e,o)}]} = \alpha_{[p_{(o,\tilde{a})}*(\tilde{a}e)*p_{(e,o)}]}\\ &= \alpha_{*,\tilde{a}e},\end{aligned}$$

where we have used the homotopy equivalence of the paths involved. So $(\mathrm{A}_*, \alpha_*)_K$, where A_* is the constant assignment $\mathrm{A}_{*,a} := \mathrm{A}$ for any $a \in K$, is a C*-net bundle. We call $(\mathrm{A}_*, \alpha_*)_K$ the *net bundle associated* with the dynamical system $(\mathrm{A}, \pi_1^o(K), \alpha)$. It is easily seen that a different choice of path frame leads to isomorphic net bundles.

Passing to the level of morphisms requires more attention. Consider a poset morphism $\mathrm{f} : K \to S$. Take a path frame P_o in K, and choose, in S a path frame $P'_{\mathrm{f}(o)} := \{q_{(a',\mathrm{f}(o))},\ a' \in S\}$ whose elements satisfy the condition

$$q_{(\mathrm{f}(a),\mathrm{f}(o))} = \mathrm{f}(p_{(a,o)}), \qquad a \in K.$$

So, this path frame is an extension of the image $\mathrm{f}(P_o)$ of P_o to S. Such an extension exists since no other restriction is imposed on a path frame but that $q_{(\mathrm{f}(o),\mathrm{f}(o))}$ be homotopic to the degenerate 1-simplex $\sigma_0\mathrm{f}(o)$. Since $q_{(\mathrm{f}(o),\mathrm{f}(o))} = \mathrm{f}(p_{(o,o)})$, this condition is automatically fulfilled because poset morphisms preserve homotopy equivalence (see §2.2).

Given a morphism $(\eta, \mathrm{f}_*) : (\mathrm{A}, \pi_1^o(K), \alpha) \to (\mathrm{B}, \pi_1^{\mathrm{f}(o)}(S), \beta)$ of dynamical systems, we construct the net bundle $(\mathrm{B}_*, \beta_*)_S$ using $P'_{\mathrm{f}(o)}$ as above.[5] Define $\eta_{*,a} := \eta$, $\forall a \in K$, giving

$$\eta_{*,\tilde{a}}\circ\alpha_{*,\tilde{a}a} = \eta\circ\alpha_{[p_{(o,\tilde{a})}*(\tilde{a}a)*p_{(a,o)}]} = \beta_{\mathrm{f}_*[p_{(o,\tilde{a})}*(\tilde{a}a)*p_{(a,o)}]}\circ\eta = \beta_{*,\mathrm{f}(\tilde{a})\mathrm{f}(a)}\circ\eta_{*,a}.$$

Thus η_* is a morphism of net bundles. We are ready to prove that the mapping

$$(\mathrm{A}, \pi_1^o(K), \alpha) \mapsto (\mathrm{A}_*, \alpha_*)_K \qquad (3.15)$$

is, up to isomorphism, the inverse of (3.13). This amounts to showing that the dynamical system $(\mathrm{A}, \pi_1^o(K), \alpha)$ and the net bundle $(\mathcal{A}, \jmath)_K$ are, respectively, isomorphic to the dynamical system $(\mathrm{A}_{**}, \pi_1^o(K), \alpha_{**})$ and to the net bundle $(\mathcal{A}_{**}, \jmath_{**})_K$ defined with respect to a fixed pole o and to a fixed path frame P_o of K. In the first case, by construction we have $\mathrm{A}_{**} = \mathrm{A}_{*,a} \equiv \mathrm{A}$ for all $a \in K$, and, for any loop q over o of the form $q = b_n * \cdots * b_1$,

$$\alpha_{**,[q]} = \alpha_{*,q} = \alpha_{[p_{(o,\partial_0 b_n)}*b_n*p_{(\partial_1 b_n,o)}*\cdots*p_{(o,\partial_0 b_1)}*b_1*p_{(\partial_1 b_1,o)}]} = \alpha_{[p_{(o,o)}*q*p_{(o,o)}]} = \alpha_{[q]},$$

thus $(\mathrm{A}_{**}, \pi_1^o(K), \alpha_{**}) = (\mathrm{A}, \pi_1^o(K), \alpha)$. In the second case, we define the family of *-isomorphisms

$$\tau_a := \jmath_{p_{(o,a)}} : \mathcal{A}_a \to \mathcal{A}_o, \qquad a \in K; \qquad (3.16)$$

to prove that τ defines an isomorphism $\tau : (\mathcal{A}, \jmath)_K \to (\mathcal{A}_{**}, \jmath_{**})_K$, we compute

$$\begin{aligned}\tau_a\circ\jmath_{ae} &= \jmath_{p_{(o,a)}*(ae)} = \jmath_{p_{(o,a)}*(ae)*p_{(e,o)}}\circ\jmath_{p_{(o,e)}}\\ &= \jmath_{*,[p_{(o,a)}*(ae)*p_{(e,o)}]}\circ\jmath_{p_{(o,e)}} = \jmath_{**,ae}\circ\tau_e,\end{aligned}$$

and this shows that τ preserves the net structures, as desired. Thus we have proved:

[5] We consider only morphisms of the form (η, f_*) where f_* is the extension of a poset morphism $\mathrm{f} : K \to S$ to the corresponding homotopy groups. This suffices for our purpose.

Proposition 3.8. *There exists a correspondence, bijective up to isomorphism, between net bundles over a poset and dynamical systems having as group the first homotopy group of the poset. In particular, the category of net bundles over K with morphisms (ϕ, id_K) is equivalent to the category of* C*-*dynamical systems with group $\pi_1^o(K)$, for some $o \in K$, with morphisms $(\eta, \mathrm{id}_{\pi_1^o(K)})$.*

Proof. The first part of the statement has already been proved. The second part, the categorical equivalence, follows directly from the above calculations. □

The preceding analysis has a wider scope than that indicated in Prop. 3.8. In algebraic quantum field theory one deals with nets of C*-algebras over a poset associated with a fixed spacetime. Thus considering morphisms of nets between different posets means dealing with nets defined over different spacetimes. This is part of the more general framework of the generally locally covariant quantum field theories [16]. However, this topic would lead us too far from the mainline of the paper.

Corollary 3.9. *If K is simply connected then any net bundle $(\mathcal{A}, \jmath)_K$ is trivial.*

Proof. If K is simply connected then $\pi_1^o(K)$ is trivial and hence only trivial $\pi_1^o(K)$-actions occur. On the other hand, by construction, the net bundle associated with a dynamical system with trivial action is clearly trivial. □

The above result applies analogously to other categories than that of C*-algebras, for example, the category of Hilbert spaces or topological groups.

Example 3.10. Let $\mathrm{f} : \pi_1^o(K) \to G$ be a continuous group morphism (here $\pi_1^o(K)$ has the discrete topology). If (A, G, α) is a dynamical system, then there is an induced net bundle $(\mathrm{A}_*, (\alpha \circ \mathrm{f})_*)_K$. In particular:

(i) The unitary group $\mathbb{U}(d)$, $d \in \mathbb{N}$, acts on the Cuntz algebra $\mathcal{O}_d$, thus every unitary representation of $\pi_1^o(K)$ induces a net bundle with fibre $\mathcal{O}_d$.
(ii) Let (V, ω) be a symplectic space and $\mathrm{f} : \pi_1^o(K) \to \mathrm{Aut}(V, \omega)$ a group morphism. Since the symplectic group $\mathrm{Aut}(V, \omega)$ acts by automorphisms on the Weyl algebra $\mathrm{W}_{(V,\omega)}$, we conclude that f defines a net bundle with fibre $\mathrm{W}_{(V,\omega)}$.

3.4. The enveloping net bundle and injectivity. As we saw in the previous section net bundles can be efficiently classified in terms of dynamical systems and contain interesting geometric information, thus it is of interest to characterize those nets of C*-algebras admitting an embedding into a C*-net bundle. In a certain sense, this is a slight generalization of the problem of characterizing those nets admitting an embedding into a *single* C*-algebra, as for example the Fredenhagen universal algebra [25]. In the present section we shall show that any net of C*-algebras defines a C*-net bundle, the enveloping net bundle, carrying a canonical morphism lifting any representation of the initial net. We then define to be injective those nets whose canonical morphism is faithful.

Let $(\mathcal{A}, \jmath)_K$ denote a net of C*-algebras. Given $o \in K$, we let $\overline{\mathcal{A}}_o$ be the free unital algebra generated by the set of symbols

$$\{ (p, A) \mid \partial_0 p = o\,,\ A \in \mathcal{A}_{\partial_1 p} \}. \tag{3.17}$$

This is indeed a *-algebra: the adjoint is defined on generators by

$$(p, A)^* := (p, A^*), \tag{3.18}$$

and extended by anti-multiplicativity and anti-linearity to all of $\overline{\mathcal{A}}_o$.

We now add some additional relations. The first set of relations are of algebraic nature:

$$(p, A) \cdot (p, B) = (p, AB), \tag{3.19}$$
$$(p, \alpha A + \beta B) = \alpha (p, A) + \beta (p, B), \tag{3.20}$$
$$(p, \mathbb{1}) = \mathbb{1}, \tag{3.21}$$

which hold for any path p, for any $A, B \in \mathcal{A}_{\partial_1 p}$ and for any $\alpha, \beta \in \mathbb{C}$. The next two relations encode the net structure and the topology of the poset. The first one is *isotony*: given $\tilde{a} \geq a$,

$$(p, \jmath_{\tilde{a}a}(A)) = (p * (\tilde{a}a), A), \tag{3.22}$$

for any path $p : \tilde{a} \to o$ and $A \in \mathcal{A}_a$. The second is *homotopy invariance*: if $p \sim q$ then

$$(p, A) = (q, A). \tag{3.23}$$

With an abuse of notation, we again denote the *-algebra obtained by imposing these additional relations by $\overline{\mathcal{A}}_o$.[6]

Now, we want to make the family $\{\overline{\mathcal{A}}_o\}$ into a net. To this end, given $o \leq \tilde{o}$, we define

$$\overline{\jmath}_{\tilde{o}o}(p, A) := ((\tilde{o}o) * p, A) , \qquad (p, A) \in \overline{\mathcal{A}}_o, \tag{3.24}$$

and extend it by multiplicativity and linearity to all of $\overline{\mathcal{A}}_o$. It is easily seen that $\overline{\jmath}_{\tilde{o}o} : \overline{\mathcal{A}}_o \to \overline{\mathcal{A}}_{\tilde{o}}$ is a well defined *-morphism, which, by homotopy invariance, is invertible. Moreover, given $o \leq \tilde{o} \leq e$ and $(p, A) \in \overline{\mathcal{A}}_o$, by homotopy invariance we have

$$\overline{\jmath}_{e\tilde{o}} \circ \overline{\jmath}_{\tilde{o}o}(p, A) = \overline{\jmath}_{e\tilde{o}}((\tilde{o}o) * p, A) = ((e\tilde{o}) * (\tilde{o}o) * p, A) = ((eo) * p, A) = \overline{\jmath}_{eo}(p, A).$$

This proves that $(\overline{\mathcal{A}}, \overline{\jmath})_K$ is a net bundle of *-algebras.

We now introduce a norm making $(\overline{\mathcal{A}}, \overline{\jmath})_K$ a C*-net bundle. Given $o \in K$, for any $W \in \overline{\mathcal{A}}_o$, we define

$$\|W\| := \sup_{\pi} \|\pi_o(W)\| , \tag{3.25}$$

where the sup is taken over the set of morphisms $\pi : (\overline{\mathcal{A}}, \overline{\jmath})_K \to (\mathcal{B}, \imath)_K$ taking values in C*-net bundles. If $\|W\| = 0$ then for every π and $\tilde{o} \geq o$ we have $\|\pi_o(W)\| = \|\imath_{\tilde{o}o} \circ \pi_o(W)\| = \|\pi_{\tilde{o}} \circ \overline{\jmath}_{\tilde{o}o}(W)\|$; thus $\|\overline{\jmath}_{\tilde{o}o}(W)\| = 0$ and $\|\cdot\|$ is well-defined with respect to the inclusion maps $\overline{\jmath}_{\tilde{o}o}$. Clearly $\|\cdot\|$ is a seminorm for $\overline{\mathcal{A}}_o$. The completion of the quotient of $\overline{\mathcal{A}}_o$ by the ideal of null elements, is a C*-algebra that, with an abuse of notation, we again denote by $\overline{\mathcal{A}}_o$. This yields a C*-net bundle $(\overline{\mathcal{A}}, \overline{\jmath})_K$ that we call the *enveloping net bundle of* $(\mathcal{A}, \jmath)_K$.

Proposition 3.11. *Given a net* $(\mathcal{A}, \jmath)_K$ *of* C**-algebras there is a unital morphism*

$$\epsilon : (\mathcal{A}, \jmath)_K \to (\overline{\mathcal{A}}, \overline{\jmath})_K,$$

satisfying the following properties:

[6] We note the close relation between the algebra $\overline{\mathcal{A}}_o$ and the fundamental group of a complex of groups (cfr. [31])

(i) let $(\varphi, \mathrm{h}), (\theta, \mathrm{h})$ *be a pair of morphisms from the enveloping net bundle to a* C*-*net bundle, if* $(\varphi, \mathrm{h}) \circ \epsilon = (\theta, \mathrm{h}) \circ \epsilon$*, then* $\varphi = \theta$*;*

(ii) for any morphism (ψ, f) *from* $(\mathcal{A}, \jmath)_K$ *into a* C*-*net bundle* $(\mathcal{B}, \imath)_S$ *there is a unique morphism* $(\psi^{\uparrow}, \mathrm{f}) : (\overline{\mathcal{A}}, \overline{\jmath})_K \to (\mathcal{B}, \imath)_S$ *such that* $(\psi, \mathrm{f}) = (\psi^{\uparrow}, \mathrm{f}) \circ \epsilon$.

Proof. Given $o \in K$, let

$$\epsilon_o(A) := (\iota_o, A), \qquad A \in \mathcal{A}_o, \tag{3.26}$$

where ι_o is the trivial loop over o. Properties (3.19, 3.20, 3.21) imply that $\epsilon_o : \mathcal{A}_o \to \overline{\mathcal{A}}_o$ is a unital *-morphism. Moreover, for $\tilde{o} \leq o$ and $A \in \mathcal{A}_{\tilde{o}}$ we have

$$\begin{aligned}\epsilon_o \circ \jmath_{o\tilde{o}}(A) &= (\iota_o, \jmath_{o\tilde{o}}(A)) = (\iota_o * (o\tilde{o}), A)\\ &= ((o\tilde{o}), A) = \overline{\jmath}_{o\tilde{o}}(\iota_{\tilde{o}}, A) = \overline{\jmath}_{o\tilde{o}} \circ \epsilon_{\tilde{o}}(A),\end{aligned}$$

where isotony and homotopy invariance have been used. Thus the collection $\epsilon := \{\epsilon_o,\ o \in K\}$ is a unital morphism from $(\mathcal{A}, \jmath)_K$ into $(\overline{\mathcal{A}}, \overline{\jmath})_K$. Since, for any morphism $\overline{\pi}$ from $(\overline{\mathcal{A}}, \overline{\jmath})_K$ into a C*-net bundle, the composition $\overline{\pi} \circ \epsilon$ is a morphism from $(\mathcal{A}, \jmath)_K$ into a C*-net bundle, $\|\epsilon_o(A)\| = \sup_{\overline{\pi}} \|\overline{\pi}_o \circ \epsilon_o(A)\| \leq \sup_{\overline{\pi}} \|\overline{\pi}_o(A)\| = \|A\|$, so ϵ_o extends by continuity to all of $\mathcal{A}_o$ proving the first part of the statement.

(i) Let $(\varphi, \mathrm{h}), (\theta, \mathrm{h}) : (\overline{\mathcal{A}}, \overline{\jmath})_K \to (\mathcal{C}, y)_P$ be a pair of morphisms as in the statement, where $(\mathcal{C}, y)_P$ is a C*-net bundle. Given $o \in K$, let $p : a \to o$ and $A \in \mathcal{A}_a$. Using the definition of ϵ, and of the inclusion maps (3.24),

$$\begin{aligned}\varphi_o(p, A) &= (\varphi_o \circ \overline{\jmath}_p)(\iota_a, A) = (y_{\mathrm{h}(p)} \circ \varphi_a)(\iota_a, A)\\ &= (y_{\mathrm{h}(p)} \circ \varphi_a \circ \epsilon_a)(A) = (y_{\mathrm{h}(p)} \circ \theta_a \circ \epsilon_a)(A) = \theta_o(p, A).\end{aligned}$$

So $\varphi_o = \theta_o$ because they coincide on the generators of $\overline{\mathcal{A}}_o$.

(ii) Given a morphism $(\psi, \mathrm{f}) : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_S$, where $(\mathcal{B}, \imath)_S$ is a C*-net bundle, define $(\psi^{\uparrow}, \mathrm{f})$ on the generators of $\overline{\mathcal{A}}_o$ as follows,

$$\psi^{\uparrow}_o(p, A) := \imath_{\mathrm{f}(p)} \circ \psi_a(A)\,, \qquad A \in \mathcal{A}_a. \tag{3.27}$$

It easily follows from this definition that $\psi^{\uparrow}_o$ preserves isotony and homotopy invariance, and that $\psi^{\uparrow}_o(p, A) \in \mathcal{B}_{\mathrm{f}(o)}$, since $\psi_a : \mathcal{A}_a \to \mathcal{B}_{\mathrm{f}(a)}$. Extend $\psi^{\uparrow}_o$ by multiplicativity and linearity to all of $\overline{\mathcal{A}}_o$. Note that

$$\begin{aligned}\psi^{\uparrow}_o \circ \overline{\jmath}_{o\tilde{o}}(q, A) &= \psi^{\uparrow}_o((o\tilde{o}) * q, A) = \imath_{\mathrm{f}((o\tilde{o})*q)} \circ \psi_a(A)\\ &= \imath_{\mathrm{f}(o)\mathrm{f}(\tilde{o})} \circ \imath_{\mathrm{f}(q)} \circ \psi_a(A) = \imath_{\mathrm{f}(o)\mathrm{f}(\tilde{o})} \circ \psi^{\uparrow}_{\tilde{o}}(q, A);\end{aligned}$$

moreover, $\psi^{\uparrow}_o \circ \epsilon_o(A) = \psi^{\uparrow}_o(\iota_o, A) = \psi_o(A)$. Uniqueness follows from (i), completing the proof. □

In the following we shall refer to the morphism $\epsilon : (\mathcal{A}, \jmath)_K \to (\overline{\mathcal{A}}, \overline{\jmath})_K$ defined by Eq. (3.16) as the *canonical morphism* of the net into its enveloping net bundle, and to Prop. 3.11 as the *universal property* of the enveloping net bundle. This property characterizes the enveloping net bundle up to isomorphism. In particular, it implies that any C*-net bundle is isomorphic to its enveloping net bundle.

It is clear from the definition that the enveloping net bundle of a net may vanish, i.e., the seminorm (3.25) may be zero for every W. On these grounds we introduce the following terminology.

Definition 3.12. *We say that a net of* C*-*algebras is* **degenerate** *if its enveloping net bundle vanishes, and is* **nondegenerate** *otherwise. A nondegenerate net of* C*-*algebras is* ***injective*** *if the canonical morphism is a monomorphism.*

Injectivity is the central notion of the present paper. As we shall see in §4.2, injectivity for a net of C*-algebras turns out to be equivalent to the existence of faithful representations. So from now on our main task shall be to understand what conditions on a net are necessary or sufficient for injectivity.

The next result shows that the enveloping net bundle is the object that we were looking for: an object uniquely associated with a net of C*-algebras which takes into account the topology of the poset and reduces, in the simply connected case, to the universal algebra defined by Fredenhagen.

Lemma 3.13. *Given a net* $(\mathcal{A}, \jmath)_K$*, if* K *is simply connected, then there is a canonical isomorphism*

$$\rho : (\overline{\mathcal{A}}, \overline{\jmath})_K \to (\mathcal{A}^t, \jmath^t)_K,$$

where $(\mathcal{A}^t, \jmath^t)_K$ *is the trivial net with fibre Fredenhagen's universal* C*-*algebra* $\mathcal{A}^{\mathrm{u}}$*. In particular, if* K *is upward directed* $\mathcal{A}^u$ *is isomorphic to the* C*-*inductive limit of* $(\mathcal{A}, \jmath)_K$*.*

Proof. K being simply connected, ρ is defined at the *-algebraic level observing that: (1) the generators of $\overline{\mathcal{A}}_o$ are the same as the generators of $\mathcal{A}^{\mathrm{u}}$, since $(p, A) = (q, A)$ for any pair of paths $p, q : a \to o$; (2) the relations (3.19)–(3.22) are the same as those defining $\mathcal{A}^{\mathrm{u}}$ (see [25]). □

Remark 3.14. Let G be a symmetry group for K and $(\mathcal{A}, \jmath)_K$ a G-covariant net (see §3.1). Define, for any $p : a \to o$ and $A \in \mathcal{A}_a$,

$$\overline{\alpha}^g_o(p, A) := (gp, \alpha^g_a(A)), \qquad g \in G. \tag{3.28}$$

Then it is easily seen that $\overline{\alpha}$ yields an action on $(\overline{\mathcal{A}}, \overline{\jmath})_K$, making it G-covariant, and that ϵ is a G-covariant morphism. Furthermore, assume that G is a continuous symmetry group of K (in the sense of (2.1)), and that the action α of G on $(\mathcal{A}, \jmath)_K$ is continuous (see §3.1). The action $\overline{\alpha}$ of G on the enveloping net bundle is continuous too. It is enough to prove this on the generators of the fibres of the enveloping net bundle. Consider a net $\{g_\lambda\}$ converging to the identity of the group G. Given o, let $\tilde{o} > o$. For any $p : a \to o$ and $A \in \mathcal{A}_a$ we have, using $(p, A) = \overline{\jmath}_p \circ \epsilon_a(A)$ and (3.28),

$$\begin{aligned}
&\|\overline{\jmath}_{\hat{o}\, g_\lambda o} \circ \overline{\alpha}^{g_\lambda}_o(p, A) - \overline{\jmath}_{\hat{o}\, o} \circ (p, A)\| \\
&\quad = \|\overline{\jmath}_{\hat{o}\, g_\lambda o} \circ \overline{\alpha}^{g_\lambda}_o \circ \overline{\jmath}_p \circ \epsilon_a(A) - \overline{\jmath}_{\hat{o}\, o} \circ \overline{\jmath}_p \circ \epsilon_a(A)\| \\
&\quad = \|\overline{\jmath}_{\hat{o}\, g_\lambda o} \circ \overline{\jmath}_{gp} \circ \overline{\alpha}^{g_\lambda}_a \circ \epsilon_a(A) - \overline{\jmath}_{\hat{o}\, o} \circ \overline{\jmath}_p \circ \epsilon_a(A)\| \\
&\quad = \|\overline{\jmath}_{\hat{o}\, g_\lambda o} \circ \overline{\jmath}_{gp} \circ \epsilon_{g_\lambda a} \circ \alpha^{g_\lambda}_a\,(A) - \overline{\jmath}_{\hat{o}\, o} \circ \overline{\jmath}_p \circ \epsilon_a(A)\|.
\end{aligned}$$

Now, take $\hat{a} > a$. Since $g_\lambda a$ are eventually smaller than $\hat{a}$, using the above relation we have

$$\begin{aligned}
&\|\overline{\jmath}_{\hat{o}\, g_\lambda o} \circ \overline{\alpha}^{g_\lambda}_o(p, A) - \overline{\jmath}_{\hat{o}\, o} \circ (p, A)\| \\
&\quad = \|\overline{\jmath}_{\hat{o}\, g_\lambda o} \circ \overline{\jmath}_{gp} \circ \epsilon_{g_\lambda a} \circ \alpha^{g_\lambda}_a\,(A) - \overline{\jmath}_{\hat{o}\, o} \circ \overline{\jmath}_p \circ \epsilon_a(A)\| \\
&\quad = \|\overline{\jmath}_{(\hat{o}, g_\lambda o)*gp*\overline{(\hat{a}, g_\lambda a)}} \circ \overline{\jmath}_{\hat{a}\, g_\lambda a} \circ \epsilon_{g_\lambda a} \circ \alpha^{g_\lambda}_a\,(A) - \overline{\jmath}_{(\hat{o}, o)*p*\overline{(\hat{a}, a)}} \circ \overline{\jmath}_{\hat{a}\, a} \circ \epsilon_a(A)\| \\
&\quad = \|\overline{\jmath}_{(\hat{o}, g_\lambda o)*gp*\overline{(\hat{a}, g_\lambda a)}} \circ \epsilon_{\hat{a}} \circ \jmath_{\hat{a}\, g_\lambda a} \circ \alpha^{g_\lambda}_a\,(A) - \overline{\jmath}_{(\hat{o}, o)*p*\overline{(\hat{a}, a)}} \circ \epsilon_{\hat{a}} \circ \jmath_{\hat{a}\, a}(A)\|.
\end{aligned}$$

By continuity of the G-action on K (see (2.1)), the paths $(\hat{o}, g_\lambda o) * gp * \overline{(\hat{a}, g_\lambda a)}$ and $(\hat{o}, o) * p * \overline{(\hat{a}, a)}$ are homotopic. So that

$$\begin{aligned}
&\|\overline{\jmath}_{\hat{o}\, g_\lambda o} \circ \overline{\alpha}_o^{g_\lambda}(p, A) - \overline{\jmath}_{\hat{o}\, o} \circ (p, A)\| \\
&\quad = \|\overline{\jmath}_{(\hat{o},o)*p*\overline{(\hat{a},a)}} \circ \epsilon_{\hat{a}} \circ \jmath_{\hat{a}\, g_\lambda a} \circ \alpha_a^{g_\lambda}(A) - \overline{\jmath}_{(\hat{o},o)*p*\overline{(\hat{a},a)}} \circ \epsilon_{\hat{a}} \circ \jmath_{\hat{a}\, a}(A)\| \\
&\quad \le \|\jmath_{\hat{a}\, g_\lambda a} \circ \alpha_a^{g_\lambda}(A) - \jmath_{\hat{a}\, a}(A)\| ,
\end{aligned}$$

which goes to zero as $g_\lambda \to e$ because of the continuity of α. This proves that $\overline{\alpha}$ is continuous.

We have just seen that if a net is covariant then the enveloping net bundle is covariant too. A property of a net not in general inherited by the enveloping net bundle is causality. This is clear from the definition, since all fibres of the net are involved in defining a single fibre of the enveloping net bundle.

We conclude by showing the stability of injectivity under morphisms faithful on the fibres (see §3.1), and the functoriality of the enveloping net bundle.

Proposition 3.15. *The following assertions hold:*

(i) *To any morphism* $(\pi, \mathrm{f}) : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_P$ *there corresponds a morphism* $(\overline{\pi}, \mathrm{f}) : (\overline{\mathcal{A}}, \overline{\jmath})_K \to (\overline{\mathcal{B}}, \overline{\imath})_P$ *satisfying*

$$(\overline{\pi}, \mathrm{f}) \circ \epsilon = \tilde{\epsilon} \circ (\pi, \mathrm{f}), \tag{3.29}$$

where ϵ *and* $\tilde{\epsilon}$ *are, respectively, the canonical morphisms of the nets* $(\mathcal{A}, \jmath)_K$ *and* $(\mathcal{B}, \imath)_P$ *into the corresponding enveloping net bundles. If* (π, f) *is faithful on the fibres and* $(\mathcal{B}, \imath)_P$ *is injective, then* $(\mathcal{A}, \jmath)_K$ *is injective too.*

(ii) *Assigning the enveloping net bundle yields a functor from the category of net of* C**-algebras to the category of* C**-net bundles.*

Proof. (i) Given $o \in K$, define

$$(\overline{\pi}, \mathrm{f})_o(p, A) := (\mathrm{f}(p), \pi_a(A)) , \qquad (p, A) \in \overline{\mathcal{A}}_o. \tag{3.30}$$

Clearly $(\overline{\pi}, \mathrm{f})_o(p, A) \in \overline{\mathcal{B}}_{\mathrm{f}(o)}$. To prove that (3.30) is well defined, we consider $a \le \tilde{a}$ and compute

$$\begin{aligned}
(\overline{\pi}, \mathrm{f})_o(p * (\tilde{a}a), A) &= (\mathrm{f}(p * (\tilde{a}a)), \pi_a(A)) = (\mathrm{f}(p) * \mathrm{f}(\tilde{a}a), \pi_a(A)) \\
&= (\mathrm{f}(p), \imath_{\mathrm{f}(\tilde{a})\mathrm{f}(a)} \circ \pi_a(A)) = (\mathrm{f}(p), \pi_{\tilde{a}} \circ \jmath_{\tilde{a}a}(A)) \\
&= (\overline{\pi}, \mathrm{f})_o(p, \jmath_{\tilde{a}a}(A)) ;
\end{aligned}$$

thus (3.30) is well-defined at the level of isotony. Passing to homotopy invariance, we note that if $p, q : a \to o$ are homotopic then $\mathrm{f}(p)$ is homotopic to $\mathrm{f}(q)$, and we have $(\mathrm{f}(p), \pi_a(A)) = (\mathrm{f}(q), \pi_a(A))$. This proves that (3.30) is well posed. If $o \le \tilde{o}$, then by homotopy invariance

$$\begin{aligned}
\overline{\imath}_{\mathrm{f}(\tilde{o})\mathrm{f}(o)} \circ (\overline{\pi}, \mathrm{f})_o(p, A) &= ((\mathrm{f}(\tilde{o})\mathrm{f}(o)) * \mathrm{f}(p), \pi_a(A)) = (\mathrm{f}_*(\tilde{o}o) * p, \pi_a(A)) \\
&= (\overline{\pi}, \mathrm{f})_{\tilde{o}}((\tilde{o}o) * p, A) = (\overline{\pi}, \mathrm{f})_{\tilde{o}} \circ \overline{\jmath}_{\tilde{o}o}(p, A).
\end{aligned}$$

Observing that

$$(\overline{\pi}, \mathrm{f})_o \circ \epsilon_o(A) = (\overline{\pi}, \mathrm{f})_o(\iota_o, A) = (\iota_{\mathrm{f}(o)}, \pi_o(A)) = \tilde{\epsilon}_{\mathrm{f}(o)}(\pi_o(A)) = \tilde{\epsilon}_{\mathrm{f}(o)} \circ (\pi, \mathrm{f})_o(A),$$

for any o and $A \in \mathcal{A}_o$, Eq. (3.29) follows. Finally, if (π, f) is faithful on the fibres and $(\mathcal{B}, \imath)_P$ is injective, the r.h.s. of the above equation is the composition of faithful morphisms for any o; so, ϵ is a monomorphism. (ii) Clearly $(\overline{\pi' \circ \pi}, \mathrm{f}' \circ \mathrm{f}) = (\overline{\pi'}, \mathrm{f}') \circ (\overline{\pi}, \mathrm{f})$, and this concludes the proof. □

This proposition and Prop. 3.5 imply that the net $(\mathrm{C}^*(\Lambda^l), \mathrm{C}^*(\lambda))_K$, given in §3.2.2, is injective.

4. States and Representations

We study states and representations of nets of C*-algebras providing sufficient conditions for the existence of states (representations), and of invariant states (covariant representations) when the poset is endowed with a symmetry group. Furthermore, we show that for a net of C*-algebras, injectivity is equivalent to the existence of faithful representations. Aspects of the decomposition of representations are studied in Appendix A.

4.1. States. We relate states of a C*-net bundle to those of the corresponding holonomy dynamical system. This will allow us to prove that, when the homotopy group of the underlying poset is amenable, any nondegenerate net has states, which are invariant whenever the net is covariant under an amenable symmetry group.

A *state* of a net of C*-algebras $(\mathcal{A}, \jmath)_K$ is a family $\omega := \{\omega_o\ , o \in K\}$, where ω_o is a state of the C*-algebra $\mathcal{A}_o$, fulfilling the relation

$$\omega_o = \omega_a \circ \jmath_{ao}, \quad o \leq a. \tag{4.1}$$

We shall denote the set of states of $(\mathcal{A}, \jmath)_K$ by $\mathcal{S}(\mathcal{A}, \jmath)_K$.

It is easily seen that if $(\phi, \mathrm{f}) : (\mathcal{B}, \imath)_P \to (\mathcal{A}, \jmath)_K$ is a unital morphism and ω is a state of $(\mathcal{A}, \jmath)_K$, then the composition $\omega \circ \phi$ defined by

$$(\omega \circ \phi)_o = \omega_{\mathrm{f}(o)} \circ \phi_o\ , \qquad o \in P, \tag{4.2}$$

yields a state of the net $(\mathcal{B}, \imath)_P$. Another property easy to verify is the following. When $(\mathcal{A}, \jmath)_K$ is a C*-net bundle then, by (4.1)

$$\omega_a = \omega_o \circ \jmath_p, \qquad p : a \to o. \tag{4.3}$$

Our first result relates states of a C*-net bundle to invariant states of the corresponding holonomy dynamical system.

Lemma 4.1. *The set of states of a* C**-net bundle is in one-to-one correspondence with the set of invariant states of the associated holonomy dynamical system.*

Proof. Consider the holonomy dynamical system $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$ defined with respect to $o \in K$ (3.12). Let ω be a state of a C*-net bundle $(\mathcal{A}, \jmath)_K$. Then $\omega_* := \omega_o$ is an invariant state of $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$. In fact by (4.3) we have that

$$\omega_* \circ \jmath_{*,[p]} = \omega_o \circ \jmath_{*,[p]} = \omega_o \circ \jmath_p = \omega_o = \omega_*,$$

for any $[p] \in \pi_1^o(K)$. Conversely, let φ be an invariant state of $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$. Take a path frame $P_o := \{p_{(a,o)},\ a \in K\}$ and define a state on the associated net bundle $(\mathcal{A}_{**}, \jmath_{**})_K$ (see §3.3) by

$$\varphi_{*,a} := \varphi \circ \jmath_{**,p} \overset{(3.14)}{=} \varphi \circ \jmath_{*,[p_{(o,o)} * p * p_{(a,o)}]} = \varphi \circ \jmath_{*,[p * p_{(a,o)}]}, \qquad a \in K,$$

for some path $p : a \to o$, where the fact that $p_{(o,o)}$ is homotopic to the trivial loop over o has been used. Since φ is $\jmath_*$-invariant for every $q : a \to o$ we have $\varphi_{*,a} = \varphi \circ \jmath_{**,p} = \varphi \circ \jmath_{**,p*\overline{q}} \circ \jmath_{**,q} = \varphi \circ \jmath_{**,q}$, so that the family $\varphi_* := \{\varphi_{*,a}\}$ is well-defined (note in particular that $\varphi_{*,o} = \varphi$). For the same reason we have

$$\varphi_{*,a} \circ \jmath_{**,a\tilde{a}} = \varphi \circ \jmath_{**,p} \circ \jmath_{**,a\tilde{a}} = \varphi \circ \jmath_{**,p*(a,\tilde{a})} = \varphi_{*,\tilde{a}}$$

for any $\tilde{a} \leq a$, thus $\varphi_* \in \mathcal{S}(\mathcal{A}_{**}, \jmath_{**})_K$. Composing φ_* with the isomorphism $\tau : (\mathcal{A}, \jmath)_K \to (\mathcal{A}_{**}, \jmath_{**})_K$ defined by Eq. (3.16) yields a state of $(\mathcal{A}, \jmath)_K$.

Finally, we prove that these mappings are the inverse of one another. Given a state ω of $(\mathcal{A}, \jmath)_K$, we have

$$(\omega_{**} \circ \tau)_a = \omega_{**,a} \circ \tau_a = \omega_* \circ \jmath_{**,p} \circ \tau_a = \omega_* \circ \tau_o \circ \jmath_p = \omega_o \circ \jmath_p = \omega_a \ ,$$

for some path $p : a \to o$, where we have used the fact that $\tau_o = \mathrm{id}_o$ (see Definition 3.16) and Eq. (4.3). Conversely, if φ is a state of the holonomy dynamical system, then $(\varphi_* \circ \tau)_* = (\varphi_* \circ \tau)_o = \varphi_{*,o} \circ \tau_o = \varphi_{*,o}$, because, as observed above, $\varphi_{*,o} = \varphi$ completing the proof. □

We now are ready to give the main result on the existence of states for nets of C*-algebras.

Proposition 4.2. *Let K be a poset with amenable homotopy group. Then any nondegenerate net of* C*-*algebras over K has states.*

Proof. Since a nondegenerate net has a nonvanishing enveloping net bundle it is enough, by (4.2), to prove the statement when $(\mathcal{A}, \jmath)_K$ is a C*-net bundle. This follows by the previous lemma, since any C*-dynamical system with an amenable group has invariant states. □

Let now $(\mathcal{A}, \jmath, \alpha)_K$ be a G-covariant net. A state $\varphi \in \mathcal{S}(\mathcal{A}, \jmath)_K$ is said to be *G-invariant* whenever

$$\varphi_{go} \circ \alpha_o^g := \varphi_o, \quad \forall o \in K, \ g \in G.$$

The next result gives conditions for the existence of G-invariant states.

Proposition 4.3. *Let G be an amenable group. Then the following assertions hold:*

(i) Any G-covariant C*-*net bundle having states has G-invariant states.*
(ii) If the fundamental group of K is amenable, then any nondegenerate G-covariant net over K has G-invariant states.

Proof. (i) Let $(\mathcal{A}, \jmath, \alpha)_K$ be a G-covariant C*-net bundle and $\omega \in \mathcal{S}(\mathcal{A}, \jmath)_K$. For any $o \in K$ and $A \in \mathcal{A}_o$, define

$$f_o^A(g) := \omega_{go} \circ \alpha_o^g(A), \qquad g \in G.$$

It is clear that $f_o^A \in L^\infty(G)$ for any $o \in K$ and $A \in \mathcal{A}_o$. Moreover the mapping $\mathcal{A}_o \ni A \to \mathrm{f}_o^A \in L^\infty(G)$ is linear and positive. We also note that the following two relations hold:

$$f_a^{\jmath_{ao}(A)} = f_o^A, \qquad o \leq a, \qquad (*)$$
$$f_{ho}^{\alpha_o^h(A)} = (f_o^A)_h, \quad h \in G, \qquad (**)$$

where $(f_o^A)_h$ means the right translation of f_o^A by h. In fact, for any $A \in \mathcal{A}_o$ and $g \in G$ we have

$$f_a^{\jmath_{ao}(A)}(g) = \omega_{ga}(\alpha_a^g(\jmath_{ao}(A)) = \omega_{ga}(\jmath_{ga\,go}(\alpha_o^g(A)) = \omega_{go}(\alpha_o^g(A)) = f_o^A(g),$$

proving $(*)$. Moreover, for any $A \in \mathcal{A}_o$ and $g, h \in G$ we have

$$f_{ho}^{\alpha_o^h(A)}(g) = \omega_{gho}(\alpha_{ho}^g(\alpha_o^h(A)) = \omega_{gho}(\alpha_o^{gh}(A)) = f_o^A(gh) = (f_o^A)_h(g),$$

proving $(**)$. Now, let μ be a right invariant mean over $L^\infty(G)$. Define

$$\varphi_o(A) := \mu(f_o^A), \qquad A \in \mathcal{A}_o;$$

this is a state over $\mathcal{A}_o$. Moreover, the collection $\varphi := \{\varphi_o\ ,\ o \in K\}$ is a G-invariant state of $(\mathcal{A}, \jmath)_K$, in fact by the relation $(*)$ $\varphi_o \circ \jmath_{oa}(A) = \mu(f_o^{\jmath_{oa}(A)}) = \mu(f_o^A) = \varphi_o(A)$. On the other hand, by the relation $(**)$ it follows that

$$\varphi_{ho} \circ \alpha_o^h(A) = \mu(f_{ho}^{\alpha_o^h(A)}) = \mu((f_o^A)_h) = \mu(f_o^A) = \varphi_o(A),$$

where the invariance of μ has been used.

(ii) Since $\pi_1^o(K)$ is amenable, by Proposition 4.2 the enveloping net bundle $(\overline{\mathcal{A}}, \overline{\jmath})_K$ has states. Applying Rem.3.14 we conclude that $(\overline{\mathcal{A}}, \overline{\jmath})_K$ is G-covariant with an action $\overline{\alpha}$ satisfying

$$\overline{\alpha}_o^g \circ \epsilon_o = \epsilon_{go} \circ \alpha_o^g, \ \ \forall o \in K,\ g \in G.$$

Applying (i), we conclude that $(\overline{\mathcal{A}}, \overline{\jmath}, \overline{\alpha})_K$ has an invariant state $\overline{\varphi}$, thus defining $\varphi_o := \overline{\varphi}_o \circ \epsilon_o$, for any $o \in K$, yields an invariant state of $(\mathcal{A}, \jmath)_K$.

4.2. Representations. We now study representations of nets of C*-algebras. We relate representations of a C*-net bundle to those of the corresponding holonomy dynamical system, and representations of a net to those of the enveloping net bundle. This leads to the equivalence between injectivity and existence of faithful representations. Injective nets defined over a poset with amenable fundamental group, and an amenable symmetry group, have covariant representations. We also characterize those nets having Hilbert space representations and those nets having a trivial enveloping net bundle.

Let $(\mathcal{A}, \jmath)_K$ be a net of C*-algebras. A *representation* of $(\mathcal{A}, \jmath)_K$ is a pair (π, U), where π is a family of Hilbert space representations $\pi_o : \mathcal{A}_o \to \mathfrak{B}(\mathcal{H}_o)$, $o \in K$ and U is a family of unitaries $U_{ao} : \mathcal{B}(\mathcal{H}_o) \to \mathcal{B}(\mathcal{H}_a)$, $o \le a$, called *inclusion operators*, such that

$$U_{ao} \in (\pi_o, \pi_a \circ \jmath_{ao}), \qquad o \le a, \tag{4.4}$$

and

$$U_{ea}\, U_{ao} = U_{eo}, \qquad o \le a \le o. \tag{4.5}$$

The representation (π, U) is said to be *faithful* if π_o is a faithful representation of $\mathcal{A}_o$ for any $o \in K$. A *Hilbert space representation* of $(\mathcal{A}, \jmath)_K$ is a representation of the form $(\pi, 1)$ (here we assume that every Hilbert space $\mathcal{H}_o$, $o \in K$, coincides with a fixed Hilbert space $\mathcal{H}$ whose identity is denoted by 1).

Remark 4.4. In the context of the algebraic quantum field theory a representation of a net of C*-algebras usually means what we call a Hilbert space representation. The first time a representation, in the sense of the present paper, appeared was in [23]; the reconstruction of a state of an algebra associated to a region of a spacetime from a family of states of its subregions yielded a collection of representations and unitary operators satisfying the above relations. This structure has been promoted to the rôle of a representation of a net of C*-algebras in [14] (and called a unitary net representation) where its topological content was analyzed and where, in particular, its rôle in the description of charges induced by the topology of a spacetime was pointed out.

With the above notation, it is easily seen that $U_{oo} = 1_o$ for any $o \in K$. Since unitaries U are invertible, we define $U_{oa} := U^*_{ao}$ for any $o \leq a$. Note that the pair $(\mathcal{H}, U)_K$, $\mathcal{H} := \{\mathcal{H}_o\}$, defines a Hilbert net bundle, and (4.4), (4.5) imply that (π, U) yields a morphism

$$\pi : (\mathcal{A}, \jmath)_K \to (\mathcal{BH}, \mathrm{ad}U)_K, \tag{4.6}$$

where $(\mathcal{BH}, \mathrm{ad}U)_K$ is the net bundle with fibres $\mathcal{B}(\mathcal{H}_o)$, $o \in K$, with inclusion maps $\mathrm{ad}U_{ao}$, $o \leq a$, defined by the adjoint action. So a representation can be seen as a morphism from the given net to the C*-net bundle defined by a Hilbert net bundle. Both the concrete C*-net bundle and the Hilbert net bundle are trivial in the case of Hilbert space representations.

An *intertwiner* between two representations (π, U) and (π', U') of $(\mathcal{A}, \jmath)_K$ is a family of bounded linear operators $T := \{T_o : \mathcal{H}_o \to \mathcal{H}'_o\, ,\ o \in K\}$ such that $T_o \in (\pi_o, \pi'_o)$, and

$$T_o\, U_{oa} = U'_{oa}\, T_a\ , \qquad a \leq o. \tag{4.7}$$

When all T_o are unitaries we shall say that T is a *unitary* intertwiner. Two representations (π, U) and (π', U') are *equivalent* if they have a unitary intertwiner. A representation is said to be *topologically trivial* whenever it is equivalent to a Hilbert space representation (the motivation of this terminology will soon be clear). Finally, a representation (π, U) is said to *vanish* whenever $\pi_a(A) = 0$ for any $a \in K$ and $A \in \mathcal{A}_a$.

We can always assume that a representation (π, U) is defined on a fixed Hilbert space (see [14] for details). For any 1-simplex b define $U_b := U_{\partial_0 b|b|}\, U_{|b|\partial_1 b}$, so we have a unitary $U_b : \mathcal{H}_{\partial_1 b} \to \mathcal{H}_{\partial_0 b}$. Extend U from 1-simplices to paths in the usual way,

$$U_p := U_{b_n} \cdots U_{b_2}\, U_{b_1}, \qquad p = b_n * \cdots b_2 * b_1. \tag{4.8}$$

Afterwards, fix a path frame P_o and define $\pi'_a(\cdot) := U_{p_{(o,a)}}\, \pi_a(\cdot)\, U_{p_{(a,o)}}$ with $a \in K$ (note that $\pi'_o = \pi_o$) and $U'_{ae} := U_{p_{(o,a)}}\, U_{ae}\, U_{p_{(e,o)}}$ with $a \leq e$. Then the pair (π', U') defines a representation of the net $(\mathcal{A}, \jmath)$ into a fixed Hilbert space; the family $T_a := U_{p_{(o,a)}}$, with $a \in K$, is a unitary intertwiner from (π, U) to (π', U'). On these grounds, unless otherwise stated, we assume from now on that *all representations are defined on a fixed Hilbert space*.

The topological content of a representation (π, U) can be easily seen by considering the associated Hilbert net bundle $(\mathcal{H}, U)$. The same reasoning used in defining the holonomy dynamical system (§3.3), yields a group morphism

$$U_* : \pi_1^o(K) \to \mathcal{U}(\mathcal{H}), \tag{4.9}$$

where $\mathcal{U}(\mathcal{H})$ is the group of unitary operators of the Hilbert space $\mathcal{H}$. Explicitly, one extends U to simplices as above and observes that the mapping $U : \Sigma_1(K) \to U_b \in \mathcal{B}(\mathcal{H})$ satisfies the 1-cocycle relation $U_{\partial_0 c} U_{\partial_2 c} = U_{\partial_1 c}$ for any 2-simplex c. So U, in turn, defines a representation U_* of the fundamental group of K. We shall call U_* *the holonomy representation* associated with (π, U).

We list the following results from [14], which can be proved as in §3.3:

(i) If two representations are equivalent then the corresponding representations of $\pi_1(K)$ are equivalent.
(ii) If K is simply connected then any representation is equivalent to a Hilbert space representation.

These two points explain the term 'topologically trivial representation'.

The first task is to relate representations of a net to those of the enveloping net bundle.

Lemma 4.5. *Any representation of the net $(\mathcal{A}, \jmath)_K$ extends uniquely to a representation of $(\overline{\mathcal{A}}, \overline{\jmath})_K$, and this yields a bijective correspondence between representations of $(\mathcal{A}, \jmath)_K$ and representations of $(\overline{\mathcal{A}}, \overline{\jmath})_K$.*

Proof. Let $\epsilon : (\mathcal{A}, \jmath)_K \to (\overline{\mathcal{A}}, \overline{\jmath})_K$ be the canonical morphism. For any representation (σ, V) of the enveloping net bundle, the pair $(\sigma \circ \epsilon, V)$ defines a representation of the net. Conversely, by (4.6), any representation (π, U) of $(\mathcal{A}, \jmath)_K$ defines a morphism $\pi : (\mathcal{A}, \jmath)_K \to (\mathcal{BH}, \mathrm{ad}U)$. By Prop. 3.11 there is a unique morphism $\pi^{\uparrow} : (\overline{\mathcal{A}}, \overline{\jmath})_K \to (\mathcal{BH}, \mathrm{ad}U)$, defined by Eq. (3.27), such that $\pi^{\uparrow} \circ \epsilon = \pi$. This completes the proof. □

We note that the extension of a faithful representation of a net to the enveloping net bundle need not be faithful.

The next result relates representations of C*-net bundles to covariant representations of the associated dynamical system.

Lemma 4.6. *Representations of a* C**-net bundle are, up to equivalence, in bijective correspondence with covariant representations of the corresponding holonomy dynamical system.*

Proof. We give a sketch of the proof since the reasoning is similar to that of the proof of Lemma 4.1. By (4.6), we have that every representation (π, U) of $(\mathcal{A}, \jmath)_K$ defines a morphism $\pi : (\mathcal{A}, \jmath)_K \to (\mathcal{BH}, \mathrm{ad}U)$. Thus, by Prop.3.8, we have the morphism of dynamical systems

$$\pi_* : \mathcal{A}_* \to \mathcal{B}(\mathcal{H}), \ \pi_* \circ \jmath_{*,[p]} = \mathrm{ad}U_{*,[p]} \circ \pi_*, \quad \forall [p] \in \pi_1^o(K), \tag{4.10}$$

i.e., π_* is a covariant representation of $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$. Conversely, given a covariant representation (η, V) of $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$ on the Hilbert space $\mathcal{H}$ then, again by Prop.3.8, we can define the net bundle $(\mathcal{B}(\mathcal{H})_*, \mathrm{ad}V_*)_K$ and the morphism

$$\eta_* : (\mathcal{A}_{**}, \jmath_{**})_K \to (\mathcal{B}(\mathcal{H})_*, \mathrm{ad}V_*)_K .$$

Since $(\mathcal{A}_{**}, \jmath_{**})_K$ is isomorphic to $(\mathcal{A}, \jmath)_K$, composing with η_* gives the desired representation. □

After this the relation between injectivity and the existence of faithful representations follows easily.

Theorem 4.7. *The following assertions hold.*

(i) A net of C*-*algebras is injective if, and only if, it has faithful representations.*
(ii) A net of C*-*algebras is nondegenerate if, and only if, it has nonvanishing representations.*

Proof. (i) ($\Leftarrow$) follows by Prop.3.15.i. ($\Rightarrow$) By the previous lemma it suffices to show the existence of faithful covariant representations for the dynamical system associated with the enveloping net bundle. But this is true for any dynamical system, completing the proof of (i). A similar reasoning leads to the proof of (ii). □

We discuss the existence of Hilbert space representations. Let (A, G, α) be a C*-dynamical system. A *G-invariant representation* is a representation π of A such that $\pi \circ \alpha_g = \pi$ for any $g \in G$. Let J_{A}^G be the G-invariant ideal of A generated by the elements $A - \alpha_g(A)$ for $A \in \mathrm{A}$ and $g \in G$; then we have the following result:

Lemma 4.8. *A* C*-*dynamical system* (A, G, α) *has nontrivial G-invariant representations if, and only if, the ideal* J_{A}^G *is proper.*

Proof. ($\Rightarrow$) If π is an invariant representation, then J_{A}^G lays within the $Ker(\pi)$. ($\Leftarrow$) If J_{A}^G is proper, then the quotient $\widehat{\mathrm{A}} := \mathrm{A}/J_{\mathrm{A}}^G$ is not trivial, and since J_{A}^G is G-invariant, the action α lifts to an action $\widehat{\alpha}$ on $\widehat{\mathrm{A}}$. However this action is trivial according to the definition of J_{A}^G. So take a faithful representation σ of $\widehat{\mathrm{A}}$ and define $\pi(A) := \sigma(\widehat{A})$ for any $A \in \mathrm{A}$. Then $\pi \circ \alpha_g(A) = \sigma(\widehat{\alpha_g(A)}) = \sigma(\widehat{A}) = \pi(A)$ for any $A \in \mathrm{A}$ and $g \in G$, completing the proof. □

We are ready to provide a necessary and sufficient condition for the existence of Hilbert space representations.

Proposition 4.9. *An injective net of* C*-*algebras* $(\mathcal{A}, \jmath)_K$ *has nontrivial Hilbert space representations if, and only if, the ideal* $J_{\overline{\mathcal{A}}_*}^{\pi_1^o(K)}$ *is proper, where* $\overline{\mathcal{A}}_*$ *is the holonomy dynamical system of the enveloping net bundle.*

Proof. Follows straighforwardly from the previous lemma and from Lemma 4.6. □

Examples of injective nets where this ideal fails to be proper will be given in §5.1.

We now want to characterize those nets having a trivial enveloping net bundle. To this end, let us introduce the following notion. A representation (π, U) of a net $(\mathcal{A}, \jmath)_K$ is said to be *quasi (topologically) trivial* whenever for any $o \in K$ the relation

$$U_p \in (\pi_o, \pi_o), \qquad p : o \to o, \tag{4.11}$$

holds. This means that the coupling between the analytical and the topological content of a quasi-trivial representation is 'artificial', and can be completely removed from this representation. More precisely, let (π, U) be a quasi-trivial representation. Take a path frame $P_o = \{p_{(a,o)},\ a \in K\}$, and define

$$\sigma_a := \mathrm{ad}U_{p_{(o,a)}} \circ \pi_a, \qquad a \in K.$$

Then $(\sigma, \mathbb{1})$ is a Hilbert space representation of the net. In fact given $a \le \tilde{a}$, by (4.11) we have

$$\begin{aligned}\sigma_{\tilde{a}} \circ \jmath_{\tilde{a}a} &= \mathrm{ad}U_{p_{(o,\tilde{a})}} \circ \pi_{\tilde{a}} \circ \jmath_{\tilde{a}a} = \mathrm{ad}U_{p_{(o,\tilde{a})}} \circ \mathrm{ad}U_{\tilde{a}a} \circ \pi_a \\ &= \mathrm{ad}U_{p_{(o,\tilde{a})} * (\tilde{a}a)} \circ \pi_a = \mathrm{ad}U_{p_{(o,a)}} \circ \mathrm{ad}U_{p_{(a,o)} * p_{(o,\tilde{a})} * (\tilde{a}a)} \circ \pi_a \\ &= \mathrm{ad}U_{p_{(o,a)}} \circ \pi_a = \sigma_a.\end{aligned}$$

So, any quasi-trivial representation defines in a natural way a Hilbert space representation. However these two representations are not, in general, equivalent. In fact consider the unitary T defined by $T_a := U_{p_{(a,o)}}$, $a \in K$. Then $T_a \in (\sigma_a, \pi_a)$ for any $a \in K$, but T does not intertwine the inclusion operators since $U_{\tilde{a}a}\, T_a = U_{(\tilde{a},a)*p_{(a,o)}} = T_{\tilde{a}}\, U_{p_{o,\tilde{a})}*(\tilde{a},a)*p_{(a,o)}}$, which is different from T_a unless the holonomy representation defined by U is trivial. It is not surprising that to any Hilbert space representation $(\sigma, \mathbb{1})$ of a net one can associate a quasi trivial representation carrying a nontrivial representation of the fundamental group. In fact, extend σ to the representation $(\sigma^{\uparrow}, \mathbb{1})$ of the enveloping net bundle and consider the covariant representation $(\sigma^{\uparrow}_{*}, \mathbb{1})$ of the holonomy dynamical system $(\overline{\mathcal{A}}_*, \pi_1^o(K), \overline{\jmath}_*)$. If the homotopy group has representations V taking values in the commutant $(\sigma^{\uparrow}_{*}, \sigma^{\uparrow}_{*})$,[7] then the pair $(\sigma^{\uparrow}_{*}, V)$ is still a covariant representation of the dynamical system. Turning back to the net, this yields a quasi trivial representation of the net not equivalent to $(\sigma, \mathbb{1})$.

We now characterize those nets whose enveloping net bundle is trivial.

Proposition 4.10. *The enveloping net bundle of a nondegenerate net is trivial if, and only if, the net has only quasi trivial representations.*

Proof. ($\Rightarrow$) Let (π, U) be a representation of a net $(\mathcal{A}, \jmath)_K$ with trivial enveloping net bundle. Let $(\pi^{\uparrow}, U)$ be the extension of this representation to $(\overline{\mathcal{A}}, \overline{\jmath})_K$. Because of triviality $\overline{\jmath}_p = \mathrm{id}_{\overline{\mathcal{A}}_o}$ for any $o \in K$ and for any loop p over o. Therefore $\mathrm{ad}_{U_p} \circ \pi^{\uparrow}_o = \pi^{\uparrow}_o \circ \overline{\jmath}_p = \pi^{\uparrow}_o$. By the definition of $\pi^{\uparrow}$, the quasi-triviality of (π, U) follows.

($\Leftarrow$) Consider the holonomy dynamical system $(\overline{\mathcal{A}}_*, \pi_1^o(K), \overline{\jmath}_*)$. Let ρ be a faithful representation of the crossed product $\overline{\mathcal{A}}_* \rtimes \pi_1^o(K)$. ρ defines a covariant representation (ρ, U^{ρ}) of the holonomy dynamical system and this, in turn, defines a representation of the enveloping net bundle, Lemma 4.6, and so a representation of the net $(\mathcal{A}, \jmath)_K$, Lemma 4.5. Since only quasi trivial representations of the net are allowed, $U^{\rho}_{[p]} \in (\rho, \rho)$ for any $[p] \in \pi_1^o(K)$. So $\rho \circ \overline{\jmath}_{*,[p]} = \mathrm{ad}U^{\rho}_{[p]} \circ \rho = \rho$ for any $[p] \in \pi_1^o(K)$. Since ρ is faithful $\overline{\jmath}_{*,[p]} = \mathrm{id}_{\overline{\mathcal{A}}_*}$ for any $[p] \in \pi_1^o(K)$, and the proof follows. □

The next result relates representations and states.

Lemma 4.11. *For any* C*-*net bundle* $(\mathcal{A}, \jmath)_K$, *the following properties are equivalent:*

(i) $(\mathcal{A}, \jmath)_K$ *has states.*
(ii) *There is a representation* (π, U) *of* $(\mathcal{A}, \jmath)_K$ *on a Hilbert space* $\mathcal{H}$ *and a family of nonzero vectors* $\Omega = \{\Omega_a \in \mathcal{H}\}$ *such that* $U_{oa}\Omega_a = \Omega_o$ *for all* $o \leq a$.
(iii) *The holonomy dynamical system* $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$ *admits a covariant representation with an invariant vector.*

Proof. $(i) \Rightarrow (ii)$ Let ω be a state of $(\mathcal{A}, \jmath)_K$. Given $o \in K$, let $(\pi_o, \mathcal{H}_o, \Omega_o)$ be the GNS representation associated with the state ω_o of the algebra $\mathcal{A}_o$. For any inclusion $o \leq a$ define

$$U_{ao}\pi_o(A)\Omega_o := \pi_a(\jmath_{ao}(A))\Omega_a, \qquad A \in \mathcal{A}_o.$$

[7] If for instance $\sigma^{\uparrow}_{*}$ is irreducible (see Appendix A), then $\pi_1^o(K)$ must have 1-dimensional representations.

A routine calculation shows that the $U_{ao} : \mathcal{H}_o \to \mathcal{H}_a$ are unitary operators with $U_{ao}\,\pi_o(\cdot) = \pi_a \circ \jmath_{ao}(\cdot)\,U_{ao}$. Moreover

$$U_{ea}\,U_{ao}\pi_o(\cdot)\Omega_o = U_{ea}\pi_a(\jmath_{ao}(\cdot))\Omega_a = \pi_a(\jmath_{ea}\jmath_{ao}(\cdot))\Omega_e = U_{eo}\pi_o(\cdot)\Omega_o.$$

Clearly, by definition we have $U_{ao}\Omega_o = \Omega_a$. $(ii) \Rightarrow (iii)$ Define $\pi_* : \mathcal{A}_* \to \mathcal{B}(\mathcal{H})$ as in (4.10) and consider $\Omega_o \in \mathcal{H}$. Then the condition $U_{ao}\Omega_o = \Omega_a$, $o \le a$, implies that $U_p\Omega_o = \Omega_o$ for any loop p over o, where U_p is defined in (4.8). $(iii) \Rightarrow (i)$ Let (η, V) be a covariant representation of $(\mathcal{A}_*, \pi_1^o(K), \jmath_*)$ on the Hilbert space $\mathcal{H}$ and $\zeta \in \mathcal{H}$ be a V-invariant vector. We consider the representation (η_*, V_*) of $(\mathcal{A}_{**}, \jmath_{**})_K$ constructed as in Lemma 4.6 and define $\varphi_a(A) := (\zeta, \eta_{*,a}(A)\zeta)$, $A \in \mathcal{A}_{**,a} = \mathcal{A}_o$, $a \in K$. By V-invariance of ζ, and covariance of (η_*, V_*), we find

$$\varphi_{\tilde a} \circ \jmath_{**,\tilde a a}(A) = (\zeta, (\eta_{*,a} \circ \jmath_{**,\tilde a a}(A))\zeta) = (V_{*,\tilde a a}\zeta, \eta_{*,a}(A)V_{*,\tilde a a}\zeta) = \varphi_a(A).$$

Thus φ is a state of $(\mathcal{A}_{**}, \jmath_{**})_K$, and composing with the isomorphism with $(\mathcal{A}, \jmath)_K$ yields the desired state. □

We conclude the section giving an easy consequence of the above results in the setting of G-actions. Let G be a symmetry group for K and $(\mathcal{A}, \jmath, \alpha)_K$ a G-covariant net. A *G-covariant representation* of $(\mathcal{A}, \jmath, \alpha)_K$ is a representation (π, U) of $(\mathcal{A}, \jmath)_K$ such that the underlying family of Hilbert spaces $\mathcal{H} := \{\mathcal{H}_o\}$ is endowed with unitary operators $\Gamma_o^g : \mathcal{H}_o \to \mathcal{H}_{go}$, $g \in G$, $o \in K$, satisfying the relations

$$\begin{aligned} \Gamma^h_{go}\,\Gamma^g_o &= \Gamma^{hg}_o, && g, h \in G,\ \ o \in K,\\ \mathrm{ad}\Gamma^g_o \circ \pi_o &= \pi_{go} \circ \alpha^g_o, && g \in G,\ \ o \in K,\\ \Gamma^g_{\tilde o}\,U_{\tilde o o} &= U_{\tilde g o\, go}\,\Gamma^g_o, && o \le \tilde o,\ \ g \in G. \end{aligned} \tag{4.12}$$

Notice that when (π, U) is topologically trivial, Γ induces a unitary representation of the symmetry group, since the inclusion operators U_{ao} are constant. Furthermore, recall that, when G is a continuous symmetry group of the poset, we assume, by convention, that the G-action on the net is continuous (see §3.1). Similarly, under these circumstances we assume that for any G-covariant representation (π, U, Γ), Γ is *strongly continuous*. This amounts to saying that if $\{g_\lambda\}_\Lambda$ is a net in G converging to the identity of the group then, for any $o \in K$ and $a \in K$ with $a > o$, there exists $\lambda_a \in \Lambda$ such that $g_\lambda o \le a$ for any $\lambda \in \Lambda$ with $\lambda \ge \lambda_a$ and

$$\|U_{a\, g_\lambda o}\,\Gamma^{g_\lambda}_o\Omega - \Omega\| \to 0, \qquad \forall \Omega \in \mathcal{H}_a. \tag{4.13}$$

We have the following result.

Proposition 4.12. *Let K be a poset with amenable fundamental group and G an amenable symmetry group of K. Then every injective, G-covariant net of* C**-algebras over K has a G-covariant representation (π, U, Γ). In particular, if G is a continuous symmetry group of K, then Γ is strongly continuous.*

Proof. Let $(\mathcal{A}, \jmath, \alpha)_K$ be injective. Then by Rem. 3.14 it suffices to look for G-covariant representations of the enveloping net bundle $(\overline{\mathcal{A}}, \overline{\jmath}, \overline{\alpha})_K$. Now, by Prop. 4.3.ii $(\overline{\mathcal{A}}, \overline{\jmath}, \overline{\alpha})_K$ has a G-invariant state φ, which, by Lemma 4.11, induces a GNS representation π. Since φ is G-invariant, setting

$$\Gamma^g_o(\pi_o(T)\Omega_o) := (\pi_{go} \circ \overline{\alpha}^g_o(T))\Omega_{go}, \qquad T \in \overline{\mathcal{A}}_o,$$

for $o \in K$ and $g \in G$, we easily find that π is G-covariant. Finally note that if G is a continuous symmetry group of K, then the action α is continuous by assumption (see §3.1). As observed in Remark 3.14, the action $\overline{\alpha}$ of G on the enveloping net bundle is continuous as well. Using this, a routine calculation shows that Γ is strongly continuous.

5. Injectivity

Injectivity has been related to "outer properties" of the net: the existence of embeddings into C*-net bundles or, in particular, the existence of faithful representations. In the present section we analyze how injectivity relates to 'inner properties' of the net. More precisely, our aim is to find conditions on the net itself, which are either necessary or sufficient for injectivity. We shall discover that injectivity imposes a cohomological condition on the net. This will allow us to provide examples of degenerate nets, noninjective nets, and of injective nets having no nontrivial Hilbert space representation (for a further example of this last case see Ex. A.9). A remarkable result is that any C*-net bundle over S^1 whose Čech cocycle is globally defined is trivial.

Our aim is to introduce a simplicial set which will serve for defining the Čech cocycle mentioned above. To begin with, let K be a poset, and S, F nonempty subsets of K. We shall write $F \leq S$ whenever any element of F is smaller than any elements of S. We now are ready to define the simplicial set $\Sigma^\circ_*(K)$: for $n \geq 0$, an *n-simplex* is a string $(F; o_{n+1}, o_n, \ldots, o_1)$ where $o_{n+1}, o_n, \ldots, o_1$, are elements of K, called the *vertices* of the n-simplex, and F is a nonempty subset of K, called the *support* of the n-simplex, satisfying the relation $F \leq \{o_{n+1}, o_n, \ldots, o_1\}$. This simplicial set is symmetric since the string whose vertices are a permutation of the vertices of an n-simplex $(F; o_{n+1}, o_n, \ldots, o_1)$ and whose support is F is an n-simplex as well. As usual $\Sigma^\circ_n(K)$ will denote the set of n-simplices.

We denote the set of symmetric subsimplicial sets D_* of $\Sigma^\circ_*(K)$, with $D_i \subseteq \Sigma^\circ_i(K)$ for any i, by $\mathrm{Sub}(\Sigma^\circ_*(K))$. Note that $\mathrm{Sub}(\Sigma^\circ_*(K))$ is closed under finite or infinite union, and finite or infinite intersection, if not empty. Elements of $\mathrm{Sub}(\Sigma^\circ_*(K))$ are, for instance, $\Sigma^\circ_*(S)$ for any nonempty subset $S \subseteq K$.

For any 0-simplex (F, o) of D_0, we define the C*-algebra

$$\mathcal{A}^F_o := \mathrm{C}^*\{\jmath_{oa}(\mathcal{A}_a) \mid a \in F\} \subseteq \mathcal{A}_o.$$

Definition 5.1. *Let* $(\mathcal{A}, \jmath)_K$ *be a net of* C*-*algebras and* $D_* \in \mathrm{Sub}(\Sigma^\circ_*(K))$. *A* **(generalized) Čech cocycle** *of* $(\mathcal{A}, \jmath)_K$ *defined over* D_* *is a family* $\zeta := \{\zeta^F_{\tilde{o}o} \mid (F; \tilde{o}, o) \in D_1\}$ *of* *-*isomorphisms*

$$\zeta^F_{\tilde{o}o} : \mathcal{A}^F_o \to \mathcal{A}^F_{\tilde{o}}, \qquad \forall (F; o, \tilde{o}) \in D_1,$$

satisfying, for any 1*-simplex* $(F; \tilde{o}, o)$*, the relation*

$$\zeta^F_{\tilde{o}o} \circ \jmath_{oa} = \jmath_{\tilde{o}a}, \qquad \forall a \in F. \tag{5.1}$$

Before showing the relation to injectivity, it is convenient to draw some interesting consequences of the above definition. *First*, the defining relation implies that ζ satisfies

$$\zeta^F_{o_3 o_2} \circ \zeta^F_{o_2 o_1} = \zeta^F_{o_3 o_1}, \qquad (F; o_1, o_2, o_3) \in D_2, \tag{5.2}$$

and

$$\zeta^F_{o_2 o_1} = {\zeta^F_{o_1 o_2}}^{-1}, \qquad (F; o_1, o_2) \in D_1. \tag{5.3}$$

The first relation, the *cocycle equation*,[8] follows observing that

$$\zeta^F_{o_3o_2} \circ \zeta^F_{o_2o_1} \circ \jmath_{o_1o} = \zeta^F_{o_3o_2} \circ \jmath_{o_2o} = \jmath_{o_3o} = \zeta^F_{o_3o_1} \circ \jmath_{o_1o},$$

for any $o \in F$, where we have applied (5.1). A similar reasoning leads to the second relation. *Second*, the Čech cocycle reduces to the inclusion maps when defined on 1-simplices $(F; a, o)$ such that $o \leq a$, that is

$$\zeta^F_{ao} = \jmath_{ao} \restriction \mathcal{A}^F_o,$$

as an easy consequence of Eq. (5.1).

Lemma 5.2. *Any net* $(\mathcal{A}, \jmath)_K$ *has a unique Čech cocycle.*

Proof. Existence. Take a pair $o_1, o_2 \in K$ having a common minorant o. Consider $\Sigma^\circ_*(\{o_1, o_2, o\})$. Any 1-simplex of $\Sigma^\circ_1(\{o_1, o_2, o\})$ has the form $(o; \tilde{a}, a)$ with $a, \tilde{a} \in \{o_1, o_2\}$. Then, it is easily seen that the collection $\zeta := \{\zeta^o_{\tilde{a}a},\ a, \tilde{a} \in \{o_1, o_2\}\}$, where

$$\zeta^o_{\tilde{a}a} : \jmath_{ao}(\mathcal{A}_o) \to \jmath_{\tilde{a}o}(\mathcal{A}_o), \quad \zeta^o_{\tilde{a}a} \circ \jmath_{ao}(A) := \jmath_{\tilde{a}o}(A), \quad \forall A \in \mathcal{A}_o,$$

is a Čech cocycle defined over $\Sigma^\circ_*(\{o_1, o_2, o\})$.

Uniqueness. Let ζ^α, $\alpha \in \Lambda$, be the collection of all Čech cocycles of the net defined, respectively, over D^α_*, $\alpha \in \Lambda$. Set

$$\zeta^F_{\tilde{o}o} := \zeta^{\alpha,F}_{\tilde{o}o} \qquad \text{if } (F; \tilde{o}, o) \in D^\alpha_1.$$

The definition is well posed: if $(F; \tilde{o}, o) \in D^\alpha_1 \cap D^\beta_1$ for $\alpha, \beta \in \Lambda$, then by (5.1), $\zeta^{\alpha,F}_{\tilde{o}o} \circ \jmath_{oa} = \jmath_{\tilde{o}a} = \zeta^{\beta,F}_{\tilde{o}o} \circ \jmath_{oa}$, for any $a \in F$. So $\zeta^{\alpha,F}_{\tilde{o}o} = \zeta^{\beta,F}_{\tilde{o}o}$, because they coincide on the generators of the algebra $\mathcal{A}^F_o$. Therefore ζ is a Čech cocycle defined over $\cup_{\alpha\in\Lambda} D^\alpha_*$. □

We have established the existence and the uniqueness of the Čech cocycle of a net. Moreover, it is clear by the above proof that $\cup_{\alpha\in\Lambda} D^\alpha_*$ is the largest element of $\mathrm{Sub}(\Sigma^\circ_*(K))$ where the Čech cocycle is defined.

Definition 5.3. *Let* $(\mathcal{A}, \jmath)_K$ *be* C*-*net. We shall refer to the largest element of* $\mathrm{Sub}(\Sigma^\circ_*(K))$ *where the Čech cocycle of the net is defined as the* **the domain** *of the cocycle; we shall say that the cocycle is* **globally defined** *if the domain equals* $\Sigma^\circ_*(K)$.

From now on our purpose will be to characterize the domain of the Čech cocycle of the net and to establish the relation to injectivity. Part of the information about the domain of the Čech cocycle is transmitted under an embedding. This is the content of the next lemma.

Lemma 5.4. *Let* $\psi : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_K$ *be a monomorphism. Then the domain* D_* *of the Čech cocycle* ξ *of* $(\mathcal{B}, \imath)_K$ *is contained in the domain the Čech cocycle of* $(\mathcal{A}, \jmath)_K$.

[8] This cohomology generalizes the Čech cohomology for the dual poset introduced in [40].

Proof. Given $(F; \tilde{o}, o) \in D_1$ we have

$$\xi^F_{\tilde{o}o} \circ \psi_o \circ \jmath_{oa} = \xi^F_{\tilde{o}o} \circ \imath_{oa} \circ \psi_a = \imath_{\tilde{o}a} \circ \psi_a = \psi_{\tilde{o}} \circ \jmath_{\tilde{o}a},$$

for any $a \in F$, where we have used Eq. (5.1). So $(\xi^F_{\tilde{o}o} \circ \psi_o)(\mathcal{A}^F_o) = \psi_{\tilde{o}}(\mathcal{A}^F_{\tilde{o}})$. This and the uniqueness of the Čech cocycle, implies that the composition

$$\psi^{-1}_{\tilde{o}} \circ \xi^F_{\tilde{o}o} \circ \psi_o, \qquad (F; \tilde{o}, o) \in D_1,$$

is the Čech cocycle of the net $(\mathcal{A}, \jmath)_K$, and the proof follows. □

We now make a first step toward the understanding of the rôle played by the Čech cocycle in the theory.

Lemma 5.5. *Let $(\mathcal{A}, \jmath)_K$ be a* C*-*net bundle and let D_* be the domain of the Čech cocycle. Then given $(F; \tilde{o}, o) \in D_1$ we have*

$$\zeta^F_{\tilde{o}o} = \jmath_{\tilde{o}a} \circ \jmath_{ao}, \qquad \forall a \in F, \tag{5.4}$$

and

$$\jmath_{a_2o} \circ \jmath_{oa_1} = \jmath_{a_2\tilde{o}} \circ \jmath_{\tilde{o}a_1}, \qquad \forall a_1, a_2 \in F. \tag{5.5}$$

Proof. First of all note that $\mathcal{A}^F_o = \mathcal{A}_o$, for any $F \leq o$, since we are considering a C*-net bundle. Then the first relation derives directly from (5.1). Using the first relation we have $\jmath_{\tilde{o}a_1} \circ \jmath_{a_1o} = \jmath_{\tilde{o}a_2} \circ \jmath_{a_2o}$, for any $a_1, a_2 \in F$, and the second relation follows.

The second equation looks like a triviality result for the C*-net bundle. In particular, when the Čech cocycle is globally defined, this result asserts that the 1-cocycle defined by $\jmath$, giving the action of the homotopy group of the holonomy dynamical system (see (3.11)), does not depend on the support of the 1-simplex: $\jmath_b = \jmath_{\tilde{b}}$ for any pair of 1-simplices such that $\partial_i b = \partial_i \tilde{b}$ for $i = 0, 1$. We shall see soon that any C*-net bundle over S^1 having a globally defined Čech cocycle is indeed trivial.

We now start analyzing the relation between injectivity and the Čech cocycle.

Lemma 5.6. *Let $(\mathcal{A}, \jmath)_K$ be a net of* C*-*algebras and $P \subseteq K$ pathwise connected and such that $(\mathcal{A}, \jmath)_P$ has a faithful Hilbert space representation. Then $\Sigma^\circ_*(P)$ is contained in the domain of the Čech cocycle of the net $(\mathcal{A}, \jmath)_K$.*

Proof. By hypothesis there exists a C*-algebra $\mathcal{A}(P)$ (the universal one) and unital faithful *-morphisms $\psi_o : \mathcal{A}_o \to \mathcal{A}(P)$, with $o \in P$, such that $\psi_{o'} \circ \jmath_{o'o} = \psi_o$ for any $o \leq o'$. Now, given $(F; o_2, o_1) \in \Sigma^\circ_1(P)$, for any $a_1, \ldots, a_n \in F$, define

$$\zeta^F_{o_2o_1}\big(\jmath_{o_1a_1}(A_1) \cdots \jmath_{o_1a_n}(A_n)\big) := \jmath_{o_2a_1}(A_1) \cdots \jmath_{o_2a_n}(A_n), \tag{5.6}$$

where $A_i \in \mathcal{A}_{a_i}$ for $i = 1, \ldots, n$, and extend $\zeta^F_{o_2o_1}$ by linearity to all the *-algebra generated by $\jmath_{o_1a}(A_a)$ as a varies in F. To prove that $\zeta^F_{o_2o_1}$ is isometric we make use of the ψ_o's as follows:

$$\begin{aligned}
&\|\jmath_{o_2a_1}(A_1)\jmath_{o_2a_2}(A_2)\cdots\jmath_{o_2a_n}(A_n)\|\\
&\quad=\|\psi_{o_2}\big(\jmath_{o_2a_1}(A_1)\jmath_{o_2a_2}(A_2)\cdots\jmath_{o_2a_n}(A_n)\big)\|\\
&\quad=\|(\psi_{o_2}\circ\jmath_{o_2a_1})(A_1)\,(\psi_{o_2}\circ\jmath_{o_2a_2})(A_2)\cdots(\psi_{o_2}\circ\jmath_{o_2a_n})(A_n)\|\\
&\quad=\|\psi_{a_1}(A_1)\psi_{a_2}(A_2)\cdots\psi_{a_n}(A_n)\|\\
&\quad=\|(\psi_{o_1}\circ\jmath_{o_1a_1})(A_1)\,(\psi_{o_1}\circ\jmath_{o_1a_2})(A_2)\cdots(\psi_{o_1}\circ\jmath_{o_1a_n})(A_n)\|\\
&\quad=\|\psi_{o_1}\big(\jmath_{o_1a_1}(A_1)\jmath_{o_1a_2}(A_2)\cdots\jmath_{o_1a_n}(A_n)\big)\|\\
&\quad=\|\jmath_{o_1a_1}(A_1)\jmath_{o_1a_2}(A_2)\cdots\jmath_{o_1a_n}(A_n)\|.
\end{aligned}$$

The same reasoning applies to finite linear combinations, hence $\zeta^F_{o_2o_1}$ extends by continuity to a *-isomorphism from $\mathcal{A}^F_{o_1}$ to $\mathcal{A}^F_{o_2}$ that, by definition, fulfills (5.1). □

The following consequence of Lemma 5.6 will be useful to determine whether a given net is injective (see Example 5.8).

Corollary 5.7. *Let $(\mathcal{A},\jmath)_K$ be an injective net. Then $\Sigma^\circ_*(P)$ is contained in the domain of the Čech cocycle for any connected and simply connected subset P of K.*

Two observations are in order. *First*, the above result applies to C*-net bundles since any C*-net bundle is injective by definition. *Second*, even if any poset K is covered by the set of its connected and simply connected subsets, the Čech cocycle of an injective net may not be globally defined since $\Sigma^\circ_*(S)\cup\Sigma^\circ_*(P)$ may be smaller than $\Sigma^\circ_*(S\cup P)$, for any pair S, P of nonempty subsets of K. In fact, if there are $o\in S\setminus P$, $a\in P\setminus S$ and a nonempty subset F of $S\cup P$ with $F\leq\{o,a\}$, then the 1-simplex $(F;a,o)\in\Sigma^\circ_1(S\cup P)$ but $(F;a,o)\notin\Sigma^\circ_1(S)\cup\Sigma^\circ_1(P)$.

5.1. Examples. To illustrate the results of the previous section we give examples of injective, nondegenerate and degenerate nets. Afterwards we make some conjectures concerning the domain of the Čech cocycle and injectivity of the nets, and, finally, give interesting examples of nets coming from complexes of groups.

Example 5.8. We consider the poset B with elements $\{m,o,a,x,y\}$ and order relation $m\leq o,a\leq x,y$, so there is a minimum m and two maximal elements x,y. This poset is simply connected since it is downward directed (see preliminaries), so a net $(\mathcal{C},\jmath)_B$ can only have Hilbert space representations.

A class of nets over B can be defined in the following way. Take a proper inclusion $\mathcal{A}\subset\mathcal{B}$ of unital C*-algebras and $\gamma\in\mathbf{aut}\mathcal{B}$ leaving $\mathcal{A}$ pointwise fixed; then we have the net $(\mathcal{C},\jmath)_B$,

$$\begin{cases}\mathcal{C}_m:=\mathcal{A},\ \ \mathcal{C}_a=\mathcal{C}_o=\mathcal{C}_x=\mathcal{C}_y=\mathcal{B},\\ \jmath_{om}=\jmath_{am}=\text{the inclusion},\\ \jmath_{yo}=\gamma,\quad \jmath_{xo}=\jmath_{xa}=\jmath_{ya}=id_{\mathcal{B}}.\end{cases}\tag{5.7}$$

Let us consider the 1-simplex $(F;x,y)$, $F:=\{a,o\}$; then we have $\mathcal{B}=\mathcal{C}^F_y=\mathcal{C}_y=\mathcal{C}_x=\mathcal{C}^F_x$, and any *-morphism $\zeta:\mathcal{C}^F_y\to\mathcal{C}^F_x$ such that $\zeta\circ\jmath_{yo}=\jmath_{xo}$, $\zeta\circ\jmath_{ya}=\jmath_{xa}$, can be regarded as a *-endomorphism of $\mathcal{B}$ fulfilling

$$\zeta\circ\gamma=id_{\mathcal{B}},\quad \zeta=id_{\mathcal{B}}.$$

Clearly, the previous equations are coherent if, and only if, $\gamma = id_{\mathcal{B}}$. In this case $(F; x, y)$ is in the domain of the Čech cocycle of $(\mathcal{C}, \jmath)_B$ and we have faithful Hilbert space representations, which, as can be easily seen, are Hilbert space representations of the amalgamated free product $\mathcal{B} *_{\mathcal{A}} \mathcal{B}$, that turns out to be the universal C*-algebra of the net.

We now assume $\gamma \neq id_{\mathcal{B}}$ and consider a Hilbert space representation $\pi : (\mathcal{C}, \jmath)_B \to \mathfrak{B}(\mathcal{H})$, which is necessarily not faithful because the 1-simplex $(F; x, y)$ does not belong to the domain of the Čech cocycle. Then, requiring coherence with (5.7) is equivalent to ask that[9] there is a representation $\Pi : \mathcal{B} \to \mathfrak{B}(\mathcal{H})$ such that $\Pi = \pi_a = \pi_x = \pi_y = \pi_o = \pi_y \circ \jmath_{yo} = \Pi \circ \gamma$, i.e. that any element of the type

$$\gamma(T) - T, \quad T \in \mathcal{B}, \tag{5.8}$$

is in ker Π. This characterizes Hilbert space representations of $(\mathcal{C}, \jmath)_B$, and in the following lines we discuss two particular cases. **(i)** Take a unital C*-algebra $\mathcal{M}$ and define

$$\mathcal{A} := \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \ A \in \mathcal{M} \right\}, \ \ \mathcal{B} := \mathbb{M}_2(\mathcal{M}), \ \ \gamma := \mathrm{ad} V, \ \ V := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{B}.$$

Then we have

$$\gamma(E) - E \ = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \ E := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so the identity $1 = (\gamma(E) - E)^2$ is in the kernel of any Hilbert space representation of $(\mathcal{C}, \jmath)_B$, which therefore is degenerate. **(ii)** Take a compact Hausdorff space X, a proper closed subset $W \subset X$ and a C*-algebra $\mathcal{M}$ with unit 1. Define

$$\mathcal{A} := C(X), \ \ \mathcal{B} := C(X, \mathcal{M}), \ \ \gamma := \mathrm{ad} V,$$

where $V : X \to \mathcal{M}$, $V \in \mathcal{B}$, is a continuous map taking values in the unitary group and such that $V(x) = 1$ for all $x \in W$ and $V(x') \notin \mathcal{M} \cap \mathcal{M}'$ for some $x' \in X - W$. We have

$$\{\gamma(T) - T\}(x) = 0, \ \ \forall T \in \mathcal{B}, \ x \in W,$$

so (5.8) generates a proper ideal $\mathcal{J}_\gamma$ of $\mathcal{B}$ contained in the one of $\mathcal{M}$-valued continuous maps vanishing on W. Thus there are non-trivial Hilbert space representations of $(\mathcal{C}, \jmath)_B$, labeled by representations of $\mathcal{B}/\mathcal{J}_\gamma$. We conclude that $(\mathcal{C}, \jmath)_B$ is nondegenerate, but also not injective.

The next is an example of an injective net having no Hilbert space representations (see Ex. A.9 for a further example in a geometrical context).

Example 5.9. The poset we consider is obtained from the one of the previous example by removing the minimum, so it is given by $C_2 := \{a, o, x, y\}$ with order relation $a, o \leq x, y$. We call C_2 a *2-cylinder*, a terminology that will be clarified in [44]. One can easily see that this poset is not simply connected and that the homotopy group is $\mathbb{Z}$. We consider the net $(\mathcal{C}, \jmath)_{C_2}$ defined by restricting on C_2 the net considered in the case **(i)** of Example 5.8. Clearly, $(\mathcal{C}, \jmath)_{C_2}$ is a C*-net bundle (hence it is injective), because there are no compositions of inclusions in C_2 and $\jmath_{xo}, \jmath_{xa}, \jmath_{yo}, \jmath_{ya}$ are one-to-one. Nevertheless, for the same reason as in case **(i)** of Example 5.8, $(\mathcal{C}, \jmath)_{C_2}$ has no Hilbert space representations and this implies that its universal C*-algebra is trivial, i.e. $\mathcal{C}^u = \{0\}$.

9 The authors thank an anonymous referee for illustrating the following argument.

We now analyse the relation between injectivity and the domain of the Čech cocycle in the opposite sense, and ask whether a net is injective on subsets of the poset associated with the domain of the Čech cocycle. We have no general result, however it is our opinion that the following assertions should hold. *First*, given a nondegenerate net, if the Čech cocycle is defined on $\Sigma_*^\circ(P)$ for any CONnected and simply connected subset P of the poset, then the net should be injective. If the Čech cocycle is globally defined, then the net should have faithful Hilbert space representations. *Second*, a C*-net bundle is trivial if its Čech cocycle is globally defined.

The next is an example supporting the first assertion.

Example 5.10. We consider the 2-cylinder $C_2 := \{o, a, x, y\}$ of the previous example. Let $(\mathcal{A}, \jmath)_{C_2}$ be a net of C*-algebras. Assume that this net has a globally defined Čech cocycle. Then in particular we have a *-isomorphism $\zeta_{yx}^{\{o,a\}} : \mathcal{A}_x^{\{o,a\}} \to \mathcal{A}_y^{\{o,a\}}$ such that

$$\zeta_{yx}^{\{o,a\}} \circ \jmath_{xe} = \jmath_{ye}, \quad \zeta_{xy}^{\{o,a\}} \circ \jmath_{ye} = \jmath_{xe}, \quad e = o, a. \tag{5.9}$$

Denoting a copy of $\mathcal{A}_x^{\{o,a\}}$ by $\mathcal{C}$ yields the diagram

$$\begin{array}{ccccc} \mathcal{A}_x & & & & \mathcal{A}_y \\ & \nwarrow^{\subseteq} & & \nearrow^{\zeta_{yx}^{\{o,a\}}} & \\ & & \mathcal{C} & & \\ & \nearrow^{\jmath_{xo}} & & \nwarrow^{\jmath_{xa}} & \\ \mathcal{A}_o & & & & \mathcal{A}_a \end{array}$$

By (5.9), the desired Hilbert space representation is obtained as a representation of the amalgamated free product $\mathcal{A}_x *_{\mathcal{C}} \mathcal{A}_y$.

A proof of the first conjecture seems being far from being reached. The above example suggests that it may be related to the "realizability" of the generalized amalgamated free product of C*-algebras, [10]. The next result supports the second assertion.

Lemma 5.11. *Any* C*-*net bundle over* S^1 *having a globally defined Čech cocycle is trivial.*

Proof. Let $(\mathcal{A}, \jmath)_{\mathcal{I}}$ be a C*-net bundle over S^1 (i.e., $\mathcal{I}$ is the set of open connected intervals of S^1 whose closure is properly contained in S^1). The homotopy group of $\mathcal{I}$ is the same as that of S^1, i.e. $\mathbb{Z}$. We now consider a path in $\mathcal{I}$, defined as follows. Let $x, y \in \mathcal{I}$ be such that $x \cup y = S^1$ and $x \cap y$ has two connected components a and o respectively. Clearly $a, o \in \mathcal{I}$. Consider the 1-simplices b_1 and b_2 defined by

$$|b_1| := x, \ \partial_1 b_1 := a, \ \partial_0 b_1 := o, \ \ |b_2| := y, \partial_1 b_2 := o, \ \partial_0 b_2 := a.$$

The path $b_2 * b_1$ is a loop over a which is not homotopically trivial and whose image under the isomorphism $\pi_1(\mathcal{I}) \simeq \pi_1(S^1)$ is the generator of $\pi_1(S^1)$. As a consequence, $(\mathcal{A}, \jmath)_{\mathcal{I}}$ is trivial if the isomorphism $\jmath_{b_2 * b_1}$ generating the holonomy of $(\mathcal{A}, \jmath)_{\mathcal{I}}$ is the identity. Now, since the Čech cocycle is globally defined, it is defined, in particular, on the 1-simplex $(\{x, y\}; a, o)$. By using Eq. (5.5) (within Lemma 5.5) we have that $\jmath_{ox} \circ \jmath_{xa} = \jmath_{oy} \circ \jmath_{ya}$, so $\jmath_{b_2 * b_1} = \underline{\jmath_{ay}} \circ \jmath_{yo} \circ \underline{\jmath_{ox} \circ \jmath_{xa}} = \mathrm{id}_{\mathcal{A}_a}$, where the homotopy equivalence of the paths $b_2 * b_1$ and $\overline{(ya)} * (yo) * \overline{(xo)} * (xa)$ has been used.

Note that also when a net has a globally defined Čech cocycle, the Čech cocycle of the enveloping net bundle may not be globally defined. In fact, in algebraic quantum field theory there are examples of nets having both faithful Hilbert space representations and faithful representations carrying a nontrivial representation of the homotopy group of the poset which are not quasi trivial (see [14,6]). So, by Prop. 4.10, the enveloping net bundle is not trivial.

In conclusion we present interesting examples of injective and degenerate nets coming from geometric group theory.

Example 5.12. The *triangle* poset is a set T of seven elements $\{m, A, B, C, X, Y, Z\}$ ordered as follows: $m \leq A, B, C$ and $A, B \leq X$, $B, C \leq Y$ and $A, C \leq Z$. T is downward directed, so it is simply connected. Any net of groups $(G, y)_T$ over T is a triangle of groups and, by the functor C^* (§3.2.2), a net of group C^*-algebras $(C^*(G), C^*(y))_K$ is associated with $(G, y)_T$. By the properties of functor C^*, the net $(G, y)_T$ embeds in the colimit group if, and only if, the net $(C^*(G), C^*(y))_K$ embeds in the universal C^*-algebra $C^*(G)^u$. Gersten and Stallings ([45]) gave sufficient conditions for the existence of this embedding in terms of angles associated to a triangle of groups. However, they also gave an example of a triangle of groups which does not satisfy these conditions and has a trivial colimit: take $H_m := \mathbb{1}$ the trivial group, $H_A := \langle a \rangle$, $H_B := \langle b \rangle$ and $H_C := \langle c \rangle$ the free groups of one generator, $H_X := \left\langle a, b \mid b^2 = aba^{-1} \right\rangle$, $H_Y := \left\langle b, c \mid c^2 = bcb^{-1} \right\rangle$ and $H_Z := \left\langle a, c \mid a^2 = cac^{-1} \right\rangle$. Taking the inclusion of groups as inclusion maps we get a net of groups $(H, i)_K$. The colimit of this net is the Mennicke group $M(2, 2, 2) = \left\langle a, b, c \mid b^2 = aba^{-1}, c^2 = bcb^{-1}, a^2 = cac^{-1} \right\rangle$ which is known to be trivial. So, the corresponding net of group C^*-algebras is degenerate.

Two observations about this example are in order. *First*, note that $(C^*(H), C^*(i))_T$ is a degenerate net of C^*-algebras having a globally defined Čech cocycle (this can be easily seen). *Second*, it is interesting to analyze the rôle of the minimum m. Consider the poset $T \setminus \{m\}$ obtained by removing m from T. Any net of C^*-algebras over $T \setminus \{m\}$ is injective since any such a net $(\mathcal{A}, \jmath)_{T \setminus \{m\}}$ admits faithful representations. To prove this we use an idea due to Blackadar ([11]). Let κ be a cardinal greater than the cardinality of the algebra $\mathcal{A}_a$ for any element a of $T \setminus \{m\}$. Then we set π_a as the tensor product of the universal representation of $\mathcal{A}_a$ times 1_κ. The trick of the cardinality implies that $\pi_a \circ \jmath_{ao}$ is unitarily equivalent to π_o for any inclusion $a \leq o$. So, choose such a unitary operator V_{ao} for any inclusion $o \leq a$. Then, since in $T \setminus \{m\}$ compositions of inclusions are not possible the pair (π, V) is a faithful representation of $(\mathcal{A}, \jmath)_{T \setminus \{m\}}$. Applying this result to the example of Gersten and Stallings, we have that the net $(C^*(H), C^*(i))_{T \setminus \{m\}}$ is injective and that the net $(H, i)_{T \setminus \{m\}}$ has faithful representations. However, $(C^*(H), C^*(i))_{T \setminus \{m\}}$ has no Hilbert space representations, because any Hilbert space representation of this net can be easily extended to a Hilbert space representation of $(C^*(H), C^*(y))_T$ and this, as observed before, is not possible.

6. Comments and Outlook

We list some topics and questions arising from the present paper.

1. The examples of nets of C^*-algebras defined by groups of loops, §3.2, might have interesting applications in algebraic quantum field theory. They are model independent and are constructed using only the principles of the theory. These examples are incomplete (see Remark 3.6); neverthless, generalizing the ideas of the present

paper, a model-independent construction of causal and covariant nets of C*- algebras over spacetimes is given in [18]. These nets admit, in the cases of Minkowski spacetime and of S^1, representations which are continuously covariant with respect to the symmetry group of the underlying spacetime (the Poincaré group and the conformal group respectively). It seems that there should not be obstruction to generalize these results to other spacetimes.

2. The present paper poses some new questions, in particular whether an injective net with a globally defined Čech cocycle has faithful Hilbert space representations. The existence of Hilbert space representations is equivalent to proving that a suitable ideal of the fibre of the enveloping net bundle is proper (Prop. 4.9). So, one should find a relation between the Čech cocycle of the net to this ideal. Examples where this ideal is not proper have been given in §5.1.
3. An application of the results of the present work is given in [44], where it is shown that any nontrivial net over S^1, the spacetime of chiral conformal quantum field theories, is injective.
4. Finally, we stress that nets of C*-algebras defined over a (base for the topology of a) space X carry topological information even when they are not net bundles. In fact, it can be proved that every net is a precosheaf of local sections of a canonical C*-bundle (in the topological sense). This result, and its consequences, is the object of a forthcoming paper ([43]).

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A. Elementary Properties of Representations

In this appendix we give some basic properties of representations of nets of C*-algebras. We will give particular emphasis to the fact that these can be conveniently described in terms of the covariant representations of the dynamical system associated with the enveloping net bundle of the given net.

More precisely, let $(\mathcal{A}, \jmath)_K$ be a net of C*-algebras and $(\overline{\mathcal{A}}, \overline{\jmath})_K$ denote the enveloping net bundle (see §3.4); then by Prop. 3.8 we have a dynamical system $(\overline{\mathcal{A}}_*, \pi_1^o(K), \overline{\jmath}_*)$, where $\overline{\mathcal{A}}_* := \overline{\mathcal{A}}_o$ for some fixed $o \in K$. A representation $(\pi, U)_K$ of $(\mathcal{A}, \jmath)_K$ extends, by Lemma 4.5, to a representation $(\pi^\uparrow, U)$ of $(\overline{\mathcal{A}}, \overline{\jmath})_K$ which, in turn, yields the $\pi_1^o(K)$-covariant representation

$$\pi_*^\uparrow : \overline{\mathcal{A}}_* \to \mathcal{B}(\mathcal{H}_o), \quad U_* : \pi_1^o(K) \to \mathcal{U}(\mathcal{H}) \tag{A.1}$$

(see Lemma 4.6). We shall keep the above notation throughout the appendix.

A.1. Decomposition of representations. Let $(\mathcal{H}, U)_K$ denote a Hilbert net bundle. Then the C*-net bundle $(\mathcal{BH}, \mathrm{ad}U)_K$ is defined, and we say that a family $T = \{T_a \in \mathcal{B}(\mathcal{H}_a)\}$ is a *section* of $(\mathcal{BH}, \mathrm{ad}U)_K$ whenever the following relations are fulfilled:

$$\mathrm{ad}U_{\tilde{a}a}(T_a) = T_{\tilde{a}}, \quad \forall a \leq \tilde{a}.$$

We denote the set of sections of $(\mathcal{BH}, \mathrm{ad}U)_K$ by $\Sigma(\mathcal{BH}, \mathrm{ad}U)_K$; this is a unital C*-algebra under the natural $*$-algebraic structure

$$T + T' := \{T_a + T'_a\}, \quad T^* := \{T^*_a\}, \quad TT' := \{T_a T'_a\},$$

and C*-norm defined by $\|T\| := \|T_o\|$, where $o \in K$ is fixed. (Since K is pathwise connected, for every $a \in K$ there is a path $p : a \to o$, so that $\|T_o\| = \|\mathrm{ad}U_{*,[p]}(T_a)\| = \|T_a\|$.)

Lemma A.1. *$\Sigma(\mathcal{BH}, \mathrm{ad}U)_K$ is isomorphic to the* C*-*algebra of operators in $\mathcal{B}(\mathcal{H}_o)$ that are invariant under the holonomy representation*

$$\mathrm{ad}U_* : \pi_1^o(K) \to \mathrm{Aut}\mathcal{B}(\mathcal{H}_o). \tag{A.2}$$

Proof. If $T \in \Sigma(\mathcal{BH}, \mathrm{ad}U)_K$, then it easily follows from the definition of the holonomy representation (4.9) that T_o is $\mathrm{ad}U_*$-invariant. On the converse, fix a path frame $P_o = \{p_{(a,o)},\ a \in K\}$. If $T_o \in \mathcal{B}(\mathcal{H}_o)$ is $\mathrm{ad}U_*$-invariant, define

$$T_a := \mathrm{ad}U_{p_{(a,o)}}(T_o), \qquad a \in K.$$

For any inclusion $a \le \tilde{a}$ we have that $\mathrm{ad}U_{\tilde{a}a}(T_a) = \mathrm{ad}U_{p_{(\tilde{a},o)}} \circ \mathrm{ad}\{U_{p_{(o,\tilde{a})}}\, U_{\tilde{a}o}\, U_{p_{(a,o)}}\}(T_o) = \mathrm{ad}U_{p_{(\tilde{a},o)}}(T_o) = T_{\tilde{a}}$, by $\mathrm{ad}U_*$-invariance. Thus T is a section and the lemma is proved. □

Now, let (π, U) be a representation of $(\mathcal{A}, \jmath)_K$ over the family of Hilbert spaces $\mathcal{H} := \{\mathcal{H}_a\}$, so that $(\mathcal{H}, U)_K$ is a Hilbert net bundle. We define

$$\Sigma(\pi, U) := \{T \in \Sigma(\mathcal{BH}, \mathrm{ad}U)_K : T_a \in (\pi_a, \pi_a), \forall a \in K\}, \tag{A.3}$$

this is clearly a C*-algebra. Given the covariant representation (A.1), we also define the C*-algebra

$$(\pi_*^\uparrow, \pi_*^\uparrow)_U := (\pi_*^\uparrow, \pi_*^\uparrow) \cap \{T \in \mathcal{B}(\mathcal{H}_o) : \mathrm{ad}U_{*,[p]}(T) = T\ ,\ \forall p \in \pi_1^o(K)\}.$$

It is clear that changing $o \in K$ we get a C*-algebra isomorphic to $(\pi_*^\uparrow, \pi_*^\uparrow)_U$.

Proposition A.2. *Let $(\mathcal{A}, \jmath)_K$ be a net of* C*-*algebras. Then for any representation (π, U) of $(\mathcal{A}, \jmath)_K$ there is an isomorphism $(\pi_*^\uparrow, \pi_*^\uparrow)_U \simeq \Sigma(\pi, U)$.*

Proof. We pick a path frame $P_o = \{p_{(a,o)},\ a \in K\}$ and define a map

$$\beta : (\pi_*^\uparrow, \pi_*^\uparrow)_U \to \Sigma(\pi, U), \quad \beta(w)_a := \mathrm{ad}U_{p_{(a,o)}}(w), \quad a \in K.$$

The above map has image in $\Sigma(\pi, U)$ since, for all $a \le \tilde{a}$,

$$\begin{aligned}
\beta(w)_{\tilde{a}}\, U_{\tilde{a}a} &= U_{p_{(\tilde{a},o)}}\, w\, U_{p_{(o,\tilde{a})}}\, U_{\tilde{a}a} = U_{\tilde{a}a}\, \mathrm{ad}U_{\overline{(\tilde{a}a)}*p_{(\tilde{a},o)}}(w) \\
&= U_{\tilde{a}a} \left\{ \mathrm{ad}U_{p_{(\tilde{a},o)}} \circ \mathrm{ad}U_{*,[p_{(o,a)}*\overline{(\tilde{a}a)}*p_{(\tilde{a},o)}]} \right\}(w) = U_{\tilde{a}a}\, \mathrm{ad}U_{p_{(\tilde{a},o)}}(w) \\
&= U_{\tilde{a}a} \beta(w)_a,
\end{aligned}$$

and, for all $T \in \mathcal{A}_a$, $a \in K$,

$$\begin{aligned}
\beta(w)_a\, \pi_a(T) &= \mathrm{ad}U_{p_{(a,o)}}(w)\, \pi_a(T) = \mathrm{ad}U_{p_{(a,o)}}(w)\, \pi_a^\uparrow(\iota_a, T) \\
&= \mathrm{ad}U_{p_{(a,o)}}(w\, \pi_a^\uparrow(p_{(o,a)}, T)) = \mathrm{ad}U_{p_{(a,o)}}(\pi_a^\uparrow(p_{(o,a)}, T)\, w) \\
&= \pi_a^\uparrow(\iota_a, T)\, \mathrm{ad}U_{p_{(a,o)}}(w) = \pi_a(T)\, \beta(w)_a.
\end{aligned}$$

Finally, β is obviously an isometric *-morphism, and defining $\beta'(T) := T_o$, $T \in \Sigma(\pi, U)$, yields an inverse of β.

Definition A.3. *A subrepresentation of* (π, U) *is given by a family* $\mathcal{H}' := \{\mathcal{H}'_a \subseteq \mathcal{H}_a\}_a$ *of* π_a*-stable Hilbert subspaces such that* $U_{\tilde{a}a}\mathcal{H}'_a = \mathcal{H}'_{\tilde{a}}$, $\forall a \leq \tilde{a}$.

Corollary A.4. *Subrepresentations of* (π, U) *are in one-to-one correspondence with projections of* $(\pi_*^\uparrow, \pi_*^\uparrow)_U$.

Proof. By the previous proposition it suffices to check projections of $\Sigma(\pi, U)$. If $\mathcal{H}'$ is a subrepresentation of (π, U) then for each $a \in K$ we define $E_a \in B(\mathcal{H}_a)$ to be the projection of $\mathcal{H}_a$ on $\mathcal{H}'_a$. Clearly $E_a \in (\pi_a, \pi_a)$, and since $U_{\tilde{a}a}E_a v = U_{\tilde{a}a}v = E_{\tilde{a}}\, U_{\tilde{a}a}v$ for all $v \in \mathcal{H}'_a$, we conclude that $\mathrm{ad}U_{\tilde{a}a}E_a = E_{\tilde{a}}, a \leq \tilde{a}$. Conversely, if $E = E^2 = E^* \in \Sigma(\pi, U)$ then we define $\mathcal{H}'_a := E_a\mathcal{H}_a$, $a \in K$, and check that $\mathcal{H}'$ is a subrepresentation of (π, U). □

Let $1_o \in \overline{\mathcal{A}}_*$ denote the identity; clearly 1_o is invariant under the $\pi_1^o(K)$-action, thus, by covariance, $\pi_*^\uparrow(1_o) \in (\pi_*^\uparrow, \pi_*^\uparrow)_U$. This yields a subrepresentation of (π, U) which, by construction, is unital and whose complement is a null representation. We say that (π, U) is *non-degenerate* whenever $\pi_*^\uparrow(1_o)$ is the identity of $\mathcal{B}(\mathcal{H}_o)$; note that in this case each π_o, $o \in K$, is a non-degenerate Hilbert space representation.

We say that (π, U) is *irreducible* whenever it does not admit non-vanishing subrepresentations. By the previous corollary, the irreducibility of (π, U) is equivalent to the condition

$$(\pi_*^\uparrow, \pi_*^\uparrow)_U \simeq \mathbb{C}.$$

Note that irreducibility of (π, U) does not necessarily imply irreducibility of the representations π_a, $a \in K$.

Cyclic vectors. Let us consider, for each $a \in K$, the concrete C*-algebra generated by

$$\{\mathrm{ad}U_p \circ \pi_{\tilde{a}}(\mathcal{A}_{\tilde{a}}), \quad p : \tilde{a} \to a\} \subseteq \mathcal{B}(\mathcal{H}_a)$$

coinciding, by Lemmas 4.5 and 4.6, with $\pi_a^\uparrow(\overline{\mathcal{A}}_a)$. We fix $o \in K$ as usual, consider $v \in \mathcal{H}_o$ and, given the path frame $P_o := \{p_{(a,o)}\ , a \in K\}$, define the closed vector spaces

$$U^a v := \text{closed span}\{U_p U_{p_{(a,o)}} v, \ p : a \to a\}, \quad a \in K.$$

The space $U^a v$ is independent of the choice of path frame, since a different path frame $\tilde{P}_o = \{\tilde{p}_{(a,o)}\ , a \in K\}$ yields $U_p U_{\tilde{p}_{(a,o)}} v = U_{p*\tilde{p}_{(a,o)}*\overline{p_{(a,o)}}}\, U_{p_{(a,o)}} v$. We then define the family $\mathcal{H}_v$ of Hilbert subspaces

$$\mathcal{H}_{v,a} := \pi_a^\uparrow(\overline{\mathcal{A}}_a)U^a v, \quad a \in K.$$

Elementary computations show that

$$\pi_a(\mathcal{A}_a)\mathcal{H}_{v,a} \subseteq \mathcal{H}_{v,a}, \quad U_{\tilde{a}a}\mathcal{H}_{v,a} = \mathcal{H}_{v,\tilde{a}}, \quad \forall a \leq \tilde{a},$$

proving the following result:

Lemma A.5. *Let* (π, U) *be a representation of* $(\mathcal{A}, \jmath)_K$. *Given* $o \in K$ *and* $v \in \mathcal{H}_o$, *the pair* $(\pi|_{\mathcal{H}_v}, U|_{\mathcal{H}_v})$ *yields a subrepresentation of* (π, U).

Definition A.6. *Let* (π, U) *be a representation over the family of Hilbert spaces* $\mathcal{H}$ *and* $o \in K$. *Then* $v \in \mathcal{H}_o$ *is said to be* **cyclic for** (π, U) *whenever* $\mathcal{H}_{v,a} = \mathcal{H}_a$ *for any* $a \in K$.

Proposition A.7. *Let (π, U) be a representation of the net $(\mathcal{A}, \jmath)_K$. Then cyclic vectors for (π, U) are in one-to-one correspondence with cyclic vectors for the induced crossed product representation $\pi_*^{\uparrow} \rtimes U_* : \overline{\mathcal{A}}_* \rtimes \pi_1^o(K) \to \mathcal{B}(\mathcal{H}_o)$.*

Proof. Let $v \in \mathcal{H}_o$ be cyclic for (π, U). Since p_{oo} is homotopic to the trivial loop we conclude that $U^o v = \{U_p v,\ p : o \to o\}$, thus $\mathcal{H}_{v,o} = \{\pi_o^{\uparrow}(\overline{\mathcal{A}}_o)\, U_p\, v\ ,\ p : o \to o\}$, and the condition $\mathcal{H}_{v,o} = \mathcal{H}_o$ is clearly equivalent to the desired cyclicity condition for $\pi_*^{\uparrow} \rtimes U_*$. Conversely, assume that $v \in \mathcal{H}_o$ is cyclic for $\pi_*^{\uparrow} \rtimes U_*$. Then the above argument implies that $\mathcal{H}_o = \mathcal{H}_{v,o}$, and since each $U_p : \mathcal{H}_{v,o} \to \mathcal{H}_a$, $p : o \to a$, is unitary, we conclude that $\mathcal{H}_{v,o} = \mathcal{H}_{v,a}$ for any $a \in K$, as desired. □

Corollary A.8. *Any non-degenerate representation (π, U) is a direct sum of cyclic representations.*

Proof. It suffices to perform the direct sum decomposition of $\pi_*^{\uparrow} \rtimes U_*$. □

A.2. Vector states. We start with a preliminary remark on representations. Let $(\mathcal{A}, \jmath)_K$ be a net of C*-algebras and $\pi := \{\pi_a : \mathcal{A}_a \to B(\mathcal{H}_a)\}$ a family of unital representations on the family of Hilbert spaces $\mathcal{H} = \{\mathcal{H}_a\}$. We say that (π, U) is a *quasi-representation* whenever U is a family of *isometries* fulfilling

$$U_{oa} \in (\pi_a, \pi_o \circ \jmath_{oa}), \quad U_{eo} = U_{ea} \circ U_{ao}, \quad \forall a \le o \le e.$$

It is easily verified that the pair $(\mathcal{H}, U)_K$ defines a net of Hilbert spaces.
The GNS construction. Let ω be a state of the net of C*-algebras $(\mathcal{A}, \jmath)_K$. Then the GNS construction yields a family $\mathcal{H} := \{\mathcal{H}_a\}$ of Hilbert spaces with maps $v_a : \mathcal{A}_a \to \mathcal{H}_a$, $a \in K$, and a family of representations

$$\pi_a : \mathcal{A}_a \to \mathcal{B}(\mathcal{H}_a), \quad \{\pi_a(T)\}\{v_a(T')\} := v_a(TT'), \quad \forall T, T' \in \mathcal{A}_a,$$

having cyclic vectors $v_a := v(1_a) \in \mathcal{H}_a$. Each $\jmath_{\tilde{a}a}$ defines the isometry

$$U_{\tilde{a}a} : \mathcal{H}_a \to \mathcal{H}_{\tilde{a}}, \quad U_{\tilde{a}a} v_a(T) := v_{\tilde{a}}(\jmath_{\tilde{a}a} T), \ T \in \mathcal{A}_o, \quad a \le \tilde{a},$$

and hence a quasi-representation (π, U) of $(\mathcal{A}, \jmath)_K$. Note that $v := \{v_a\}$ is a section of $(\mathcal{H}, U)_K$, i.e. $U_{\tilde{a}a} v_a = v_{\tilde{a}}$ for all $a \le \tilde{a}$.

Vector states. Let $(\mathcal{A}, \jmath)_K$ be a net of C*-algebras and (π, U) a representation. A *vector state* is given by a family $v := \{v_a \in \mathcal{H}, \|v\| = 1\}$ such that $v_{\tilde{a}} = U_{\tilde{a}a} v_a$, $a \le \tilde{a}$. This induces the state $\omega_a(\cdot) := (v_a, \cdot v_a)$, $a \in K$.

A representation does not necessarily yield vector states, indeed this happens if and only if the underlying Hilbert net bundle $\mathcal{H}$ has sections. In the following we give an example of net of C*-algebras having representations but no (vector) states.

Example A.9. Let Γ_n denote the free group with n generators and $\mathcal{A}_* := C_b(\Gamma_n)$ the C*-algebra of bounded continuous functions w.r.t. the Haar measure. We denote the left translation by

$$\lambda : \Gamma_n \times \mathcal{A}_* \to \mathcal{A}_*, \quad g, A \mapsto \lambda_g A =: A_g. \tag{A.4}$$

Since Γ_n is not amenable, there are no Γ_n-invariants states of $\mathcal{A}_*$. Now let M be a space with $\pi_1(M) = \Gamma_n$ (for example, the *n-bouquet*) and K a good base for the topology of M. Fix $o \in K$, so that $\pi_1^o(K) \simeq \Gamma_n$, and define $(\mathcal{A}_{**}, \lambda_*)_K$ as the C*-net bundle associated to the dynamical system $(\mathcal{A}_*, \pi_1^o(K), \lambda)$ (3.15). By Lemma 4.1 we conclude that $(\mathcal{A}_{**}, \lambda_*)_K$ does not have states.

Projective states. Let $(\mathcal{A}, \jmath)_K$ be a net of C*-algebras and (π, U) a representation over the Hilbert net bundle $(\mathcal{H}, U)_K$. Assume that there is a subnet of Hilbert spaces $(\mathcal{L}, \lambda)$ of $(\mathcal{H}, U)_K$ with rank 1. This means that there is a family $v = \{v_a \in \mathcal{H}_a\}$ of normalized vectors generating the subspaces $\mathcal{L}_a \subseteq \mathcal{H}_a$, $a \in K$, such that, with $\lambda = \{\lambda_{\tilde{a}a} \in \mathbb{T}, a \leq \tilde{a}\}$, we have

$$U_{\tilde{a}a} v_a = \lambda_{\tilde{a}a} v_{\tilde{a}}, \quad a \leq \tilde{a}.$$

In this scenario, the state $\omega \in \mathcal{S}(\mathcal{A}, \jmath)_K$, $\omega_a := (v_a, \cdot v_a)$, $a \in K$, is well defined, and we say ω is a *projective state of* $(\mathcal{A}, \jmath)_K$ *defined in* $(\mathcal{H}, U)_K$. The following result is an immediate consequence of the remarks at the beginning of this appendix.

Proposition A.10. *Let* $(\mathcal{A}, \jmath)_K$ *be a net of* C*-*algebras and* (π, U) *a representation over the family of Hilbert spaces* $\mathcal{H}$. *Then projective states defined in* $(\mathcal{H}, U)_K$ *are in one-to-one correspondence with rank one projections of* $(U_*, U_*) \subseteq \mathcal{B}(\mathcal{H}_o)$.

References

1. Araki, H.: *Mathematical theory of quantum fields.* Oxford: Oxford University Press, 2009
2. Araki, H., Moriya, H.: Equilibrium statistical mechanics of fermion lattice systems. Rev. Math. Phys. **15**(2), 93–198 (2003)
3. Ashtekar, A., Sen, A.: On the Role of Space-Time Topology in Quantum Phenomena: Superselection of Charge and Emergence of Nontrivial Vacua. J. Math. Phys. **21**, 526–533 (1980)
4. Brunetti, R., Fredenhagen. K.: Algebraic approach to Quantum Field Theory. In: *Elsevier Encyclopedia of Mathematical Physics*, Amsterdam: Elsevier, 2006, available a http://arxiv.org/abs/math-ph/0411072v1, 2004
5. Buchholz, D., Fredenhagen, K.: Locality and the structure of particle states. Commun. Math. Phys. **84**, 1–54 (1982)
6. Brunetti, R., Franecschini, L., Moretti, V.: Topological features of massive bosons on two dimensional Einstein space-time. Ann. Henri Poincaré **10**(6), 1027–1073 (2009)
7. Bär, C., Ginoux, N., Pfäffle, F.: *Wave equations on Lorentzian manifolds and quantization.* ESI Lectures in Mathematics and Physics. Zürich: European Mathematical Society (EMS), 2007
8. Buchholz, D., Haag, R.: The Quest for Understanding in Relativistic Quantum Physics J. Math. Phys. **41**(6), 3674–3697 (2000)
9. Bridson, M.R., Haefliger, A.: *Metric Spaces of Non-Positive Curvature.* Berlin: Springer-Verlag, 1999
10. Bildea, T.S.: Generalized free amalgamated product of C*-algebras. J. Operator Theory **56**(1), 143–165 (2006)
11. Blackadar, B.: Weak expectations and nuclear C*-algebras. Indiana Univ. Math. J. **27**(6), 1021–1026 (1978)
12. Blackadar, B.: Shape theory for C*-algebras. Math. Scand. **56**(2), 249–275 (1985)
13. Buchholz, D., Mack, G., Todorov, I.: The current algebra on the circle as a germ of local field theories. Conformal field theories and related topics. Nuclear Phys. B Proc. Suppl. **5B**, 20–56 (1988)
14. Brunetti, R., Ruzzi, G.: Quantum charges and spacetime topology: The emergence of new superselection sectors. Commun. Math. Phys. **287**(2), 523–563 (2009)
15. Bratteli, O., Robinson, D.W.: *Operator algebras and quantum statistical mechanics. 2. Equilibrium states. Models in quantum statistical mechanics.* Second edition. Texts and Monographs in Physics. Berlin: Springer-Verlag, 1997
16. Brunetti, R., Fredenhagen, K., Verch, R.: The generally covariant locality principle - A new paradigm for local quantum physics. Commun. Math. Phys. **237**, 31–68 (2003)
17. Carpi, S., Kawahigashi, Y., Longo, R.: Structure and classification of superconformal nets. Ann. Henri Poincaré **9**(6), 1069–1121 (2008)
18. Ciolli, F., Ruzzi, G., Vasselli, E.: *Causal posets, loops and the construction of nets of local algebras for QFT.* Preprint http://arXiv.org/abs/1109.4824v3 [math-ph], 2011
19. D'Antoni, C., Fredenhagen, K., Köster, S.: Implementation of conformal covariance by diffeomorphism symmetry. Lett. Math. Phys. **67**(3), 239–247 (2004)
20. Doplicher, S., Haag, R., Roberts, J.E.: Local observables and particle statistics I. Commun. Math Phys. **23**, 199–230 (1971)

21. Doplicher, S., Haag, R., Roberts, J.E.: Local observables and particle statistics II. Commun. Math Phys. **35**, 49–85 (1974)
22. Dimock, J.: Algebras of local observables on a manifold. Commun. Math. Phys. **77**(3), 219–228 (1980)
23. Fredenhagen, K., Haag, R.: Generally covariant quantum field theory and scaling limits. Commun. Math. Phys. **108**(1), 91–115 (1987)
24. Folland, G.: *A course in abstract harmonic analysis.* Studies in Advanced Mathematics. Boca Raton, FL: CRC Press, 1995
25. Fredenhagen, K.: *Generalizations of the theory of superselection sectors.* The algebraic theory of superselection sectors (Palermo, 1989), 379–387, River Edge, NJ: World Sci. Publ., 1990, pp. 379–387
26. Fredenhagen, K., Rehren, K.-H., Schroer, B.: *Superselection sectors with braid group statistics and exchange algebras. II.* Geometric aspects and conformal covariance. Special issue dedicated to R. Haag on the occasion of his 70th birthday. Rev. Math. Phys. 1992, Special Issue, no. 1, 113–157 (1992)
27. Group theory from a geometrical viewpoint. In: Proceedings of the workshop held in Trieste, March 26–April 6, 1990. Editors É. Ghys, A. Haefliger, A. Verjovsky. River Edge, NJ: World Scientific Publishing Co., Inc., 1991
28. Guido, D., Longo, R.: Relativistic invariance and charge conjugation in quantum field theory. Commun. Math. Phys. **148**(3), 521–551 (1992)
29. Guido, D., Longo, R., Roberts, J.E., Verch, R.: Charged sectors, spin and statistics in quantum field theory on curved spacetimes. Rev. Math. Phys. **13**(2), 125–198 (2001)
30. Haag, R.: *Local Quantum Physics*. Springer Texts and Monographs in Physics, 2nd edn. Berlin-Heidelberg New York: Springer, 1996
31. Haefliger, A.: *Complexes of groups and orbihedra.* Group theory from a geometrical viewpoint. (Trieste, 1990). River Edge, NJ: World Sci. Publ., 1991, pp. 504–540
32. Haag, R., Kastler, D.: An algebraic approach to quantum field theory. J. Math. Phys. **43**, 848–861 (1964)
33. Hollands, S.: Renormalized quantum Yang-Mills fields in curved spacetime. Rev. Math. Phys. **20**(9), 1033–1172 (2008)
34. Leyland, P., Roberts, J.E.: The Cohomology of Nets over Minkowski Space. Commun. Math. Phys. **62**, 173–189 (1978)
35. Neumann, H.: Generalized free products with amalgamated subgroups. Amer. J. Math. **70**, 590–625 (1948)
36. Neumann, H.: Generalized free products with amalgamated subgroups. II. Amer. J. Math. **71**, 491–540 (1949)
37. Roberts, J.E.: Local cohomology and superselection structure. Commun. Math. Phys. **51**(2), 107–119 (1976)
38. Roberts, J.E.: *Lectures on algebraic quantum field theory.* The algebraic theory of superselection sectors. (Palermo, 1989), River Edge, NJ: World Sci. Publ., 1990, pp. 1–112
39. Roberts, J.E., Ruzzi, G.: A cohomological description of connections and curvature over posets. Theo. App. Cat. **16**(30), 855–895 (2006)
40. Roberts, J.E., Ruzzi, G., Vasselli, E.: A theory of bundles over posets. Adv. Math. **220**(1), 125–153 (2009)
41. Rehren, K.-H., Schroer, B.: Quasiprimary fields: an approach to positivity of 2D conformal quantum field theory. Nucl. Phys. B **295**(2 FS21), 229–242 (1988)
42. Ruzzi, G.: Homotopy of posets, net-cohomology, and theory of superselection sectors in globally hyperbolic spacetimes. Rev. Math. Phys. **17**(9), 1021–1070 (2005)
43. Ruzzi, G., Vasselli, E.: The $C_0(X)$-algebra of a net and index theory. In preparation
44. Ruzzi, G., Vasselli, E.: *Representations of nets of C^*-algebras over S^1.* Preprint http://arXiv.org/abs/1104.3527v1 [math-ph], 2011
45. Stallings, J.R.: *Non-positively curved triangles of groups.* Group theory from a geometrical viewpoint (Trieste, 1990), River Edge, NJ: World Sci. Publ., 1991, pp. 491–503
46. Verch R.: Continuity of symplectically adjoint maps and the algebraic structure of Hadamard vacuum representations for quantum fields in curved spacetime. Rev. Math. Phys. **9**(5), 635–674 (1997)

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