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Formation and Propagation of Discontinuity for Boltzmann Equation in Non-Convex Domains

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Abstract: The formation and propagation of singularities for the Boltzmann equation in bounded domains has been an important question in numerical studies as well as in theoretical studies. In this paper, we consider the nonlinear Boltzmann solution near Maxwellians under in-flow, diffuse, or bounce-back boundary conditions. We demonstrate that discontinuity is created at the non-convex part of the grazing boundary, and then it propagates only along the forward characteristics inside the domain before it hits on the boundary again.

1. Introduction

A density of a dilute gas is governed by the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F) \quad , \quad F|_{t=0} = F_0, \tag{1}$$

where $F(t, x, v)$ is a distribution function for the gas particles at a time $t \geq 0$, a position $x \in \Omega \subset \mathbb{R}^3$ and a velocity $v \in \mathbb{R}^3$. Throughout this paper, the collision operator takes the form

$$\begin{aligned} Q(F_1, F_2) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega) F_1(u') F_2(v') d\omega du \\ &\quad - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega) F_1(u) F_2(v) d\omega du \\ &\equiv Q_+(F_1, F_2) - Q_-(F_1, F_2), \end{aligned} \tag{2}$$

where $u' = u + [(v-u)\cdot\omega]\omega$, $v' = v - [(v-u)\cdot\omega]\omega$ and $B(v-u,\omega) = |v-u|^\gamma q_0(\frac{v-u}{|v-u|}\cdot\omega)$, with $0 < \gamma \leq 1$ (hard potential) and $\int_{\mathbb{S}^2} q_0(\hat{u}\cdot\omega)d\omega < +\infty$, (angular cutoff) for all $\hat{u} \in \mathbb{S}^2$.

In terms of the standard perturbation f such that $F = \mu + \sqrt{\mu} f$, the Boltzmann equation can be rewritten as

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f) \quad , \quad f|_{t=0} = f_0, \tag{3}$$

where the standard linear Boltzmann operator([15,20]) is given by

$$Lf \equiv \nu f - Kf,$$

with the collision frequency $\nu(v) \equiv \int |v-u|^{\gamma} \mu(u) q_0(\frac{v-u}{|v-u|} \cdot \omega) d\omega du$ and $\frac{1}{C_\nu}(1+|v|)^{\gamma} \leq \nu(v) \leq C_\nu (1+|v|)^{\gamma}$,

$$Kf \equiv \int_{\mathbb{R}^3} \mathbf{k}(v, v') f(v') dv' \equiv \frac{1}{\sqrt{\mu}} Q_+(\mu, \sqrt{\mu} f) + \frac{1}{\sqrt{\mu}} Q_+(\sqrt{\mu} f, \mu) - \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \mu),$$

$$\Gamma(f, f) \equiv \frac{1}{\sqrt{\mu}} Q_+(\sqrt{\mu} f, \sqrt{\mu} f) - \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \sqrt{\mu} f) \equiv \Gamma_+(f, f) - \Gamma_-(f, f).$$

If the gas is contained in bounded regions or flows past solid bodies, the Boltzmann equation must be accompanied by boundary conditions describing the interactions of the gas molecules with the solid walls. Let the domain Ω be a smooth bounded domain. We consider three basic types of boundary conditions ([11,12,21,30,39]) for $f(t, x, v)$ at $(x, v) \in \partial\Omega \times \mathbb{R}^3$ with $v \cdot n(x) < 0$, where $n(x)$ is an outward unit normal vector at $x \in \partial\Omega$:

1. *In-flow injection boundary condition*. Incoming particles are prescribed:

$$f(t, x, v) = g(t, x, v). \tag{4}$$

2. *Diffuse reflection boundary condition*. Incoming particles are the probability average of the outgoing particles;

$$f(t, x, v) = c_\mu \sqrt{\mu(v)} \int_{v' \cdot n(x) > 0} f(t, x, v') \sqrt{\mu(v)} \{n(x) \cdot v'\} dv', \tag{5}$$

with a normalized Maxwellian $\mu = e^{-\frac{|v|^2}{2}}$, a normalized constant $c_\mu > 0$ such that

$$c_\mu \int_{v' \cdot n(x) > 0} \mu(v') |n(x) \cdot v'| dv' = 1, \tag{6}$$

which implies that mass is conserved at the boundary and the temperature of the wall to be constant and equals 1.

3. *Bounce-back reflection boundary condition*. Incoming particles bounce back at the reverse of the velocity:

$$f(t, x, v) = f(t, x, -v). \tag{7}$$

The purpose of this paper is to investigate a possible formation and propagation of discontinuity for the nonlinear Boltzmann equation under these boundary conditions. In order to state our results, we need the following definitions.

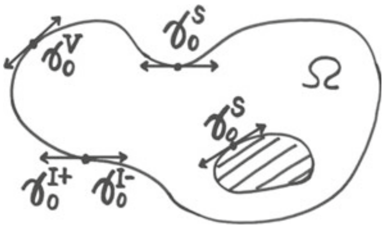

Fig. 1. Grazing Boundary γ_0

1.1. Domain. Throughout this paper, we assume the domain $\Omega \subset \mathbb{R}^3$ is open and bounded and connected. For simplicity, we assume that the boundary $\partial\Omega$ is smooth, i.e. for each point $x_0 \in \partial\Omega$, there exists $r > 0$ and a smooth function $\Phi_{x_0} : \mathbb{R}^2 \to \mathbb{R}$ such that - upon relabeling and reorienting the coordinates axes if necessary - we have

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_3 > \Phi_{x_0}(x_1, x_2)\}. \tag{8}$$

The outward normal vector at $\partial\Omega$ is given by

$$n(x_1, x_2) = \frac{1}{\sqrt{1 + |\nabla_x \Phi(x_1, x_2)|^2}} (\ \partial_{x_1}\Phi_{x_0}(x_1, x_2),\ \partial_{x_2}\Phi_{x_0}(x_1, x_2),\ -1\ ).$$

Given (t, x, v), let $[X(s), V(s)] = [X(s; t, x, v), V(s; t, x, v)] = [x - (t - s)v, v]$ be a trajectory (or a characteristics) for the Boltzmann equation (1):

$$\frac{dX(s)}{ds} = V(s), \qquad \frac{dV(s)}{ds} = 0,$$

with the initial condition: $[X(t; t, x, v), V(t; t, x, v)] = [x, v]$.

Definition 1 ([21]). *For $(x, v) \in \bar{\Omega} \times \mathbb{R}^3$, we define the* ***backward exit time****, $t_{\mathbf{b}}(x, v) \geq 0$ to be the last moment at which the back-time straight line $[X(s; 0, x, v), V(s; 0, x, v)]$ remains in the interior of Ω:*

$$t_{\mathbf{b}}(x, v) = \sup(\{0\} \cup \{\tau > 0 : x - sv \in \Omega \textit{ for all } \ 0 < s < \tau\}).$$

We also define the ***backward exit position*** *in $\partial\Omega$,*

$$x_{\mathbf{b}}(x, v) = x - t_{\mathbf{b}}(x, v)v \in \partial\Omega,$$

and we always have $v \cdot n(x_{\mathbf{b}}(x, v)) \leq 0$ (Fig. 1).

1.2. Discontinuity set and discontinuity jump. We denote the phase boundary in the phase space $\Omega \times \mathbb{R}^3$ as $\gamma = \partial\Omega \times \mathbb{R}^3$, and split it into outgoing boundary γ_+, the incoming boundary γ_-, and the grazing boundary γ_0 ([21]):

$$\gamma_+ = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : \ n(x) \cdot v > 0\},$$
$$\gamma_- = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : \ n(x) \cdot v < 0\},$$
$$\gamma_0 = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : \ n(x) \cdot v = 0\}.$$

We need to study the grazing boundary γ_0 more carefully.

Definition 2. *We define the* ***concave(singular) grazing boundary*** *in the grazing boundary* γ_0 *as*

$$\gamma_0^{\mathbf{S}} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) \neq 0 \text{ and } t_{\mathbf{b}}(x,-v) \neq 0\},$$

and the ***outward inflection grazing boundary*** *in the grazing boundary* γ_0 *as*

$$\gamma_0^{I+} = \Big\{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) \neq 0 \text{ and } t_{\mathbf{b}}(x,-v) = 0 \text{ and there is } \delta > 0 \text{ such that } x + \tau v \in \bar{\Omega}^c \text{ for } \tau \in (0,\delta)\Big\},$$

and the ***inward inflection grazing boundary*** *in the grazing boundary* γ_0 *as*

$$\gamma_0^{I-} = \Big\{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) = 0 \text{ and } t_{\mathbf{b}}(x,-v) \neq 0 \text{ and there is } \delta > 0 \text{ such that } x - \tau v \in \bar{\Omega}^c \text{ for } \tau \in (0,\delta)\Big\},$$

and the ***convex grazing boundary*** *in the grazing boundary* γ_0 *as*

$$\gamma_0^{V} = \{(x,v) \in \gamma_0 : t_{\mathbf{b}}(x,v) = 0 \text{ and } t_{\mathbf{b}}(x,-v) = 0\}.$$

We say an open subset Ω of $\mathbb{R}^3$ is **non-convex** if and only if $\gamma_0^{\mathbf{S}} \neq \emptyset$. It turns out that the concave (singular) grazing boundary $\gamma_0^{\mathbf{S}}$ is the only part at which discontinuity can be created and propagates into the interior of the phase space $\Omega \times \mathbb{R}^3$.

Definition 3. *Define the* ***discontinuity set*** *in* $[0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$ *as*

$$\mathfrak{D} = \Big\{(0,\infty) \times [\, \gamma_0^{\mathbf{S}} \cup \gamma_0^{V} \cup \gamma_0^{I+} \,]\Big\} \cup \Big\{ (t,x,v) \in (0,\infty) \times \{\Omega \times \mathbb{R}^3 \cup \gamma_+\} : \ t \geq t_{\mathbf{b}}(x,v) \text{ and } (x_{\mathbf{b}}(x,v), v) \in \gamma_0^{\mathbf{S}} \Big\}, \tag{9}$$

and the ***continuity set*** *in* $[0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$ *as*

$$\mathfrak{C} = \Big\{\{0\} \times \bar{\Omega} \times \mathbb{R}^3\Big\} \cup \Big\{(0,\infty) \times [\, \gamma_- \cup \gamma_0^{I-} \,]\Big\} \cup \Big\{ (t,x,v) \in (0,\infty) \times \{\Omega \times \mathbb{R}^3 \cup \gamma_+\} : \ t < t_{\mathbf{b}}(x,v) \text{ or } (x_{\mathbf{b}}(x,v), v) \in \gamma_- \cup \gamma_0^{I-} \Big\}. \tag{10}$$

For the bounce-back reflection boundary condition case [(7)](#), we need slightly different definitions: the ***bounce-back discontinuity set*** *and the* ***bounce-back continuity set*** *are*

$$\mathfrak{D}_{bb} = \mathfrak{D} \cup \Big\{(t,x,v) \in (0,\infty) \times \Omega \times \mathbb{R}^3 : t \geq 2t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x,-v) \text{ and } (x_{\mathbf{b}}(x,-v), -v) \in \gamma_0^{\mathbf{S}}\Big\},$$

$$\begin{aligned}\mathfrak{C}_{bb} = &\Big\{\{0\} \times \bar{\Omega} \times \mathbb{R}^3\Big\} \cup \Big\{(0,\infty) \times [\, \gamma_- \cup \gamma_0^{I-} \,]\Big\} \\ &\cup \Big\{(t,x,v) \in [0,\infty) \times \{\Omega \times \mathbb{R}^3 \cup \gamma_+\} : \ t < t_{\mathbf{b}}(x,v) \\ &\text{or } \Big[(x_{\mathbf{b}}(x,v), v) \in \gamma_- \cup \gamma_0^{I-} \ \text{ and } \ t < 2t_{\mathbf{b}}(x,v) + t_{\mathbf{b}}(x,-v)\Big] \\ &\text{or } \Big[(x_{\mathbf{b}}(x,-v), -v) \in \gamma_- \cup \gamma_0^{I-} \ \text{ and } \ (x_{\mathbf{b}}(x,v), v) \in \gamma_- \cup \gamma_0^{I-} \Big]\Big\},\end{aligned}$$

respectively.

The discontinuity set $\mathfrak{D}$ consists of two parts: The first set of (9) is the grazing boundary part γ_0 of $\mathfrak{D}$. This set mainly consists of the phase boundary where the backward exit time $t_{\mathbf{b}}(x,v)$ is not continuous (Lemma 2). The second set of (9) is mainly the interior phase space part of $\mathfrak{D}$, i.e. $\mathfrak{D} \cap \{[0,\infty) \times \Omega \times \mathbb{R}^3\}$, which is a subset of a union of all forward trajectories in the phase space emanating from $\gamma_0^{\mathbf{S}}$. Notice that $\mathfrak{D}$ does not include the forward trajectories emanating from $\gamma_0^V \cup \gamma_0^{I+}$ because those forward trajectories are not in the interior phase space $[0,\infty) \times \Omega \times \mathbb{R}^3$. We also exclude the case $t < t_{\mathbf{b}}(x,v)$ from $\mathfrak{D}$. In fact, considering the pure transport equation, $t < t_{\mathbf{b}}(x,v)$ implies the transport solution at (t,x,v) equals the initial data at $(x - tv, v)$ and if the initial data is continuous, we expect the transport solution is continuous around (t,x,v). Notice that we exclude the initial plane $\{0\} \times \bar{\Omega} \times \mathbb{R}^3$ from $\mathfrak{D}$ because we assume that the Boltzmann solution is continuous at $t = 0$. The continuity set $\mathfrak{C}$ consists of points either emanating from the initial plane or from $\gamma_- \cup \gamma_0^{I-}$, but not from $\gamma_0^{\mathbf{S}}$.

Furthermore we define a set including the grazing boundary γ_0 and all forward trajectories emanating from the whole grazing boundary γ_0.

Definition 4. *The **grazing set** is defined as*

$$\mathfrak{G} = \{(x,v) \in \bar{\Omega} \times \mathbb{R}^3 : n(x_{\mathbf{b}}(x,v)) \cdot v = n(x - t_{\mathbf{b}}(x,v)v) \cdot v = 0\}, \tag{11}$$

*and the **grazing section** is defined as*

$$\mathfrak{G}_x = \{v \in \mathbb{R}^3 : (x,v) \in \mathfrak{G}\} = \{v \in \mathbb{R}^3 : n(x_{\mathbf{b}}(x,v)) \cdot v = 0\}.$$

Obviously the grazing set $\mathfrak{G}$ includes the discontinuity set $\mathfrak{D}$. In order to study the continuity property of the Boltzmann solution, we define:

Definition 5. *For a function $\phi(t,x,v)$ defined on $[0,\infty) \times \{\bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}\}$, we define the **discontinuity jump in space and velocity***

$$[\phi(t)]_{x,v} = \lim_{\delta \downarrow 0} \sup_{(x',v'),(x'',v'') \in \{\bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}\} \cap \{B((x,v);\delta) \backslash (x,v)\}} |\phi(t,x',v') - \phi(t,x'',v'')|,$$

*and the **discontinuity jump in time and space and velocity***

$$[\phi]_{t,x,v} = \lim_{\delta \downarrow 0} \sup_{\substack{t',t'' \in B(t;\delta) \\ (x',v'),(x'',v'') \in \{\bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}\} \cap \{B((x,v);\delta) \backslash (x,v)\}}} |\phi(t',x',v') - \phi(t'',x'',v'')|,$$

where $\mathfrak{G}$ is defined in Definition 4. We say a function ϕ is discontinuous in space and velocity (in time and space and velocity) at (t,x,v) if and only if $[\phi(t)]_{x,v} \neq 0$ ($[\phi]_{t,x,v} \neq 0$) and continuous in space and velocity (in time and space and velocity) at (t,x,v) if and only if $[\phi(t)]_{x,v} = 0$ ($[\phi]_{t,x,v} = 0$).

Notice that the function ϕ is only defined away from the grazing set $\mathfrak{G}$. If the discontinuity jump of a given function ϕ is zero at (t,x,v) then the function ϕ can be extended to $[0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$ near (t,x,v). Because of these definitions we can consider a function which has a removable discontinuity as a continuous function. And a non-zero discontinuity jump $[\phi]_{t,x,v} \neq 0$ means ϕ has a 'real' discontinuity which is not removable.

1.3. Main result. The main result consists of two parts:

- continuity and discontinuity of the Boltzmann solution (Theorem 1, 2, 3),
- continuity of the gain term Q_+ (Theorem 4).

In the first part, we study such qualitative properties of the Boltzmann solution which has been established near Maxwellian regime in [21]. Recall the theorem of [21]. In order to state the theorem in a unified way for several boundary conditions, we introduce the weight function

$$w(v) = \{1 + \rho^2|v|^2\}^{\beta}. \tag{12}$$

Theorem of [21]. Assume $w^{-2}\{1+|v|\}^3 \in L^1$. Let Ω be an open subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. There exists $\delta > 0$ such that if $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$ and

$$||wf_0||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} + \sup_{t\in[0,\infty)} e^{\lambda_0 t}||wg(t)||_{L^\infty(\gamma_-)} < \delta, \tag{13}$$

for the in-flow injection boundary condition (4) and

$$||wf_0||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} < \delta, \tag{14}$$

for the diffuse reflection (5), bounce-back reflection (7) boundary conditions with $\lambda_0 > 0$, then there exists a unique Boltzmann solution $F(t,x,v) = \mu + \sqrt{\mu} f \geq 0$ to the in-flow injection (4), the diffuse reflection (5), the bounce-back reflection (7) boundary conditions respectively. Moreover, there exists $\lambda \in (0, \lambda_0)$ such that

$$\sup_{t\in[0,\infty)} e^{\lambda t}||wf(t)||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} \leq C\left\{||wf_0||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} + \sup_{t\in[0,\infty)} e^{\lambda_0 t}||wg(t)||_{L^\infty(\gamma_-)}\right\}, \tag{15}$$

for the in-flow injection boundary condition (4) and

$$\sup_{t\in[0,\infty)} e^{\lambda t}||wf(t)||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} \leq C||wf_0||_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)}, \tag{16}$$

for the diffuse reflection (5), bounce-back reflection (7) boundary conditions.

Now we are ready to state the main theorems of this paper.

Theorem 1 (Formation of Discontinuity). *Let Ω be an open subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Assume Ω is non-convex, i.e. $\gamma_0^{\mathbf{S}} \neq \emptyset$. Choose any non-convex point $(x_0, v_0) \in \gamma_0^{\mathbf{S}}$ with $v_0 \neq 0$.*

1. *For in-flow boundary condition(4), there exist an initial datum $F_0 = \mu + \sqrt{\mu} f_0 \in C^0(\Omega\times\mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\})$ and an in-flow boundary datum $G = \mu + \sqrt{\mu} g \in C^0([0,\infty)\times\{\gamma_- \cup \gamma_0^{\mathbf{S}}\})$ satisfying (13) and*

$$f_0(x,v) = g(0,x,v) \ \ for \ \ (x,v) \in \gamma_- \cup \gamma_0^{\mathbf{S}}, \tag{17}$$

such that the Boltzmann solution $F = \mu + \sqrt{\mu} f$ of (1) with the in-flow boundary condition (4) is discontinuous in space and velocity at (t_0, x_0, v_0), i.e. $[F(t_0)]_{x_0,v_0} \neq 0$ for some $t_0 \in (0, t_{\mathbf{b}}(x_0, -v_0))$.

2. *For diffuse boundary condition(5), there exists an initial datum* $F_0 = \mu + \sqrt{\mu} f_0 \in C^0(\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\})$ *satisfying (14) and*

$$f_0(x, v) = c_\mu \sqrt{\mu(v)} \int_{v' \cdot n(x) > 0} f_0(x, v') \sqrt{\mu(v')} \{n(x) \cdot v'\} dv' \text{ for } (x, v) \in \gamma_- \cup \gamma_0^{\mathbf{S}}, \tag{18}$$

such that the Boltzmann solution $F = \mu + \sqrt{\mu} f$ *of (1) with the diffuse boundary condition (5) is discontinuous in space and velocity at* (t_0, x_0, v_0)*, i.e.* $[F(t_0)]_{x_0, v_0} \neq 0$ *for some* $t_0 \in (0, t_{\mathbf{b}}(x_0, -v_0))$.
3. *For bounce-back boundary condition(7), there exists an initial datum* $F_0 = \mu + \sqrt{\mu} f_0 \in C^0(\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\})$ *satisfying (14) and*

$$f_0(x, v) = f_0(x, -v) \text{ for } (x, v) \in \gamma_- \cup \gamma_0^{\mathbf{S}}, \tag{19}$$

such that the Boltzmann solution $F = \mu + \sqrt{\mu} f$ *of (1) with the bounce-back boundary condition(7) is discontinuous in space and velocity at* (t_0, x_0, v_0)*, i.e.* $[F(t_0)]_{x_0, v_0} \neq 0$ *for some* $t_0 \in (0, \min\{t_{\mathbf{b}}(x_0, -v_0), t_{\mathbf{b}}(x_0, v_0)\})$.

Notice that in Theorem 1 we construct an initial datum $F_0 = \mu + \sqrt{\mu} f_0$ (and an in-flow boundary datum $G = \mu + \sqrt{\mu} g$ for the in-flow boundary condition case) satisfying the smallness condition for f_0 and g (13) or (14). Due to Theorem of [21], this condition (13) or (14) ensures that the Boltzmann solution $F = \mu + \sqrt{\mu} f$ in Theorem 1 exists globally in time. Moreover, since the initial datum F_0 (and G for the in-flow boundary condition case) is continuous and satisfies the compatibility conditions (17), (18) and (19), the Boltzmann solution is initially continuous. However the continuity breaks down after a time $t_0 > 0$ at the chosen point (x_0, v_0) of the concave (singular) grazing boundary $\gamma_0^{\mathbf{S}}$. Therefore, for any generic non-convex domain Ω, we are able to observe the formation of discontinuity. In particular if $t_0 < t_{\mathbf{b}}(x_0, -v_0)(t_0 < \min\{t_{\mathbf{b}}(x_0, -v_0), t_{\mathbf{b}}(x_0, v_0)\}$ for the bounce-back boundary condition case) we said the Boltzmann solution F has a **local-in-time formation of discontinuity** at (t_0, x_0, v_0).

Once we have the formation of discontinuity at $(t_0, x_0, v_0) \in \gamma_0^{\mathbf{S}}$, we further establish that the discontinuity propagates along the forward characteristics.

Theorem 2 (Propagation of Discontinuity). *Let* Ω *be an open bounded subset of* $\mathbb{R}^3$ *with a smooth boundary* $\partial\Omega$*. Let* $F = \mu + \sqrt{\mu} f$ *be the Boltzmann solution to the initial datum* $F_0 = \mu + \sqrt{\mu} f_0$ *which is continuous on* $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$*, and with one of the following boundary conditions:*

1. *For in-flow boundary condition (4), let (17) and (13) be valid and* $G(t, x, v) = \mu + \sqrt{\mu} g$ *be continuous on* $[0, \infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$.
2. *For diffuse boundary condition (5), assume (14) and (18).*
3. *For bounce-back boundary condition (7), assume (14) and (19).*

Then for all $t \in [t_0, t_0 + t_{\mathbf{b}}(x_0, -v_0))$ *we have*

$$[F]_{t, x_0 + (t - t_0) v_0, v_0} \leq e^{-C_1 (1 + |v_0|)^\gamma (t - t_0)} [F(t_0)]_{x_0, v_0}, \tag{20}$$

where $C_1 > 0$ *only depends on* $||wf||_{L^\infty([0,\infty) \times \bar{\Omega} \times \mathbb{R}^3)}$.

On the other hand, assume $[F(t_0)]_{x_0,v_0} \neq 0$, *and* $t_0 \in (0, t_{\mathbf{b}}(x_0, -v_0))$ *for in-flow and diffuse boundary conditions and* $t_0 \in (0, \min\{t_{\mathbf{b}}(x_0, -v_0), t_{\mathbf{b}}(x_0, v_0)\})$ *for bounce-back boundary condition, and a strict concavity of* $\partial\Omega$ *at* x_0 *along* v_0, *i.e.*

$$\sum_{i,j} (v_0)_i \partial_{x_i} \partial_{x_j} \Phi(x_0)(v_0)_j < -C_{x_0,v_0}. \tag{21}$$

Then for all $t \in [t_0, t_0 + t_{\mathbf{b}}(x_0, -v_0))$, *the Boltzmann solution* F *is discontinuous in time and space and velocity at* $(t, x_0 + (t - t_0)v_0, v_0)$, *i.e.* $[F]_{t,x_0+(t-t_0)v_0,v_0} \neq 0$ *and*

$$Ce^{-C_2(1+|v_0|)^{\gamma}(t-t_0)}[F(t_0)]_{x_0,v_0} \leq [F]_{t,x_0+(t-t_0)v_0,v_0}, \tag{22}$$

where $0 < C < 1$, *and* $C_2 = C_2(\|w\frac{F-\mu}{\sqrt{\mu}}\|_{L^\infty}) \in \mathbb{R}$ *which is positive for sufficiently small* $\|w\frac{F-\mu}{\sqrt{\mu}}\|_{L^\infty([0,\infty)\times\bar{\Omega}\times\mathbb{R}^3)}$.

The strict concavity condition (21) rules out some technical issue of the backward exit time $t_{\mathbf{b}}$. Our theorem characterize the propagation of discontinuity before the forward trajectory reaches the boundary. In the case that the forward trajectory reaches the boundary, i.e. $t \geq t_0 + t_{\mathbf{b}}(x_0, -v_0)$, the situation is much more complicated. Denote $x_1 = x_0 + t_{\mathbf{b}}(x_0, -v_0)v_0$, $t_1 = t_0 + t_{\mathbf{b}}(x_0, -v_0)$. If the trajectory hits on the boundary non-tangentially, i.e. $(x_1, v_0) \in \gamma_+$, for in-flow and diffuse boundary cases, the discontinuity disappears because of the continuity of the in-flow datum and the average property of diffuse boundary operator. For bounce-back case the discontinuity is reflected and continues to propagate along the trajectory. If the trajectory hits on the boundary tangentially, i.e. $(x_1, v_0) \in \gamma_0$, there are three possibilities. First, if $(x_1, v_0) \in \gamma_0^{I+}$, then the situation is the same as the case $(x_1, v_0) \in \gamma_+$ above. Second, if the trajectory is contained in the boundary for awhile, i.e. there exists $\delta > 0$ so that $x_1 + sv_0 \in \partial\Omega$ for $s \in (0, \delta)$ then it is difficult to predict the propagation of discontinuity in general. Assuming a certain condition on Ω, Definition 6 for example, we can rule out such an unlikely case.

The last case is that $(x_1, v_0) \in \gamma_0^{\mathbf{S}}$. Assume we have a sequence of $\{t_n = t_{n-1} + t_{\mathbf{b}}(x_{n-1}, -v_0)\}$ and $\{x_n = x_{n-1} + t_{\mathbf{b}}(x_{n-1}, -v_0)v_0\} \in \partial\Omega$ so that $(x_n, v_0) \in \gamma_0^{\mathbf{S}}$, and a directional strict concavity (21) is valid for each (x_n, v_0). We can show the propagation of discontinuity also between the first and the second intersections, i.e. $[F]_{t,x_0(t-t_0)v_0,v_0} \neq 0$ for $t \in [t_1, t_2)$ in general. For $t \geq t_2$, if we have a very simple geometry, for example the first picture of Fig. 2, we can show the propagation of discontinuity, i.e. $[F]_{t,x_0(t-t_0)v_0,v_0} \neq 0$ for $t \in [t_n, t_{n+1})$ even if $n = 2, 3$. But in general,

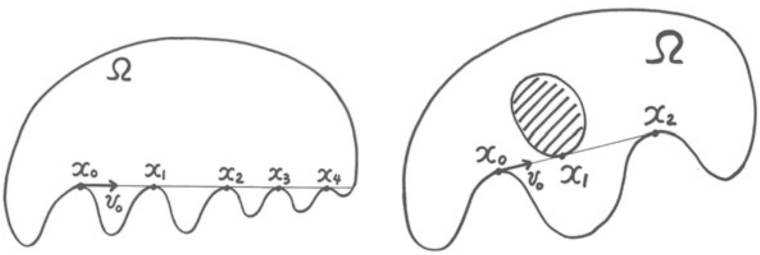

Fig. 2. Grazing Again

for example the second picture of Fig. 2, we cannot show $[F]_{t,x_0(t-t_0)v_0,v_0} \neq 0$ for $t \in [t_n, t_{n+1})$ for $n \geq 2$.

The next result states that Theorem 1 and Theorem 2 capture all the possible singularities (discontinuities) for the generic non-convex domain (Definition 6), despite the nonlinearity in the Boltzmann equation. In other words, the singularity of the Boltzmann solution is propagating as the linear Boltzmann equation and no new singularities created from the nonlinearity of the Boltzmann equation.

Definition 6. *Assume $\Omega \in \mathbb{R}^3$ is open and the boundary $\partial\Omega$ is smooth. We say the boundary $\partial\Omega$* ***does not include a line segment*** *if and only if for each $x_0 \in \partial\Omega$ and for all $(u_1, u_2) \in \mathbb{S}^1$ there is no $\delta > 0$ such that*

$$\Phi_{x_0}(\tau u_1, \tau u_2)$$

is a linear function for $\tau \in (-\delta, \delta)$ where Φ_{x_0} from (8).

Theorem 3 (Continuity away from $\mathfrak{D}$). *Let Ω be an open bounded subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Let $F(t, x, v)$ be a Boltzmann solution of (1) with the initial datum F_0 which is continuous on $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_+ \cup \gamma_0^{I-}\}$ and with one of*

1. *In-flow boundary condition (4). Assume (13) is valid and the compatibility condition*

$$F_0(x, v) = G(0, x, v) \ \ for \ \ (x, v) \in \gamma_- \cup \gamma_0^{I-}, \tag{23}$$

and $G(t, x, v)$ is continuous on $[0, \infty) \times \{\gamma_- \cup \gamma_0^{I-}\}$.

2. *Diffuse boundary condition (5). Assume (14) is valid and the compatibility condition*

$$F_0(x, v) = c_\mu \mu(v) \int_{v' \cdot n(x) > 0} F_0(x, v')\{n(x) \cdot v'\} dv' \ \ for \ \ (x, v) \in \gamma_- \cup \gamma_0^{I-}. \tag{24}$$

3. *Bounce-back boundary condition (7). Assume (14) is valid and the compatibility condition*

$$F_0(x, v) = F_0(x, -v) \ \ for \ \ (x, v) \in \gamma_- \cup \gamma_0^{I-}. \tag{25}$$

Then $F(t, x, v)$ is a continuous function on $\mathfrak{C}$ for 1,2 and a continuous function on $\mathfrak{C}_{bb}$ for 3. If the domain Ω does not include a line segment (Definition 6) then the continuity set $\mathfrak{C}$ and $\mathfrak{C}_{bb}$ are the complementary of $\mathfrak{D}$ and $\mathfrak{D}_{bb}$ respectively. Therefore $F(t, x, v)$ is continuous on $(\mathfrak{D})^c$ for 1,2 and continuous on $(\mathfrak{D}_{bb})^c$ for 3.

The last theorem is a qualitative property of the gain term in (2). This theorem is crucial to prove Theorem 2 and Theorem 3.

Theorem 4 (Continuity of Q_+). *Assume that $F(t, x, v)$ is a function defined on $(t, x, v) \in [0, T] \times \bar{\Omega} \times \mathbb{R}^3$ and is continuous away from the grazing set in (4), i.e.*

$$F \in C^0([0, T] \times (\Omega \times \mathbb{R}^3) \backslash \mathfrak{G}),$$

and

$$||\bar{w}^{-1} F||_{L^\infty([0,T]\times\bar{\Omega}\times\mathbf{R}^3)} < +\infty,$$

where $\bar{w} = \frac{e^{-\frac{|v|^2}{4}}}{(1+\rho^2|v|^2)^\beta}$ with $\rho \in \mathbb{R}$ and $\beta > 0$.

Then the gain term $Q_+(F, F)(t, x, v)$ *is continuous, i.e.*

$$Q_+(F, F) \in C^0([0, T] \times \Omega \times \mathbb{R}^3),$$

and

$$\sup_{[0,T]\times\bar{\Omega}\times\mathbb{R}^3} |\nu^{-1}\bar{w}^{-1}Q_+(F, F)(t, x, v)| < \infty. \tag{26}$$

Notice that the function F in Theorem 4 need not be a solution of the Boltzmann equation.

1.4. Previous works and significance of this work. There are many references for the mathematical study of different aspects of the boundary value problem of the Boltzmann equation such as [2,10,18,21,23,32] and the references therein. In [21], an unified $L^2 - L^\infty$ theory in the near Maxwellian regime is developed to establish the existence, uniqueness and exponential decay toward a Maxwellian, for all four basic types of the boundary conditions and rather general domains.

The qualitative study of the particle-boundary interaction in a bounded domain and its effects on the global dynamics is a fundamental problem in the Boltzmann theory. One of the challenging questions is the regularity theory of kinetic equations in a bounded domain. This problem is particularly difficult because even for the simplest kinetic equations with the differential operator $v \cdot \nabla_x$, the phase boundary $\partial\Omega \times \mathbb{R}^3$ is always characteristic but not uniformly characteristic at the grazing set $\gamma_0 = \{(x, v) : x \in \partial\Omega,\ \text{and } v \cdot n(x) = 0\}$. In a convex domain a continuity of the Boltzmann solution away from γ_0 is established in [21] for all four basic boundary conditions. In a convex domain, backward trajectories starting from the interior points of the phase space cannot reach points of the grazing boundary γ_0, due to the Velocity Lemma ([19,25]), where possible singularities may exist.

On the other hand, in a non-convex domain, generally the backward trajectories starting at the interior points of the phase space can reach the grazing boundary. Therefore, we expect singularities will be created at some part of grazing boundary γ_0 and propagate in the inside of the phase space. In general, the formation and propagation of singularity has been an important issue for the various partial differential equations ([24,31,37]). For the Boltzmann equation, this question has been attracting much attention since the early '90s (the references in pp. 91–92 in Sone's book [34]). For the Boltzmann equation, most of the works are numerical studies [34–36] and few mathematical studies.

There are several works about the propagation of 'given' singularities. For example, an initial datum or a boundary datum already has some singularities (mathematical works [4,7–9,14] as well as numerical works [5,34]). In [4], for the linear BGK model, a propagation of discontinuity, which exists already in the boundary data, is studied mathematically and also numerically. In [7], for the full Boltzmann equation in the near vacuum regime, a propagation of the Sobolev $H^{1/25}$ singularity, which exists already in the initial data, is studied and the same effect has been recently shown in the near Maxwellian regime ([8,14]).

In Vlasov theory, we refer to [3,16,40] for the boundary value problem. Singular solutions were studied in [19] extensively. In [19], the non-convexity condition of boundary is replaced by the inward electric field which has a similar effect with non-convexity of the boundary. In convex domains, Hölder estimates of the Vlasov solution with specular reflection boundary is solved recently ([25,26]), but the Sovolev-type estimate is still widely open.

Our results give a rather complete characterization of formation and propagation of singularity for the nonlinear Boltzmann equation near Maxwellian in general domain under in-flow, diffuse, bounce-back boundary conditions. There is no restriction of the time interval. More precisely we show that for any non-convex point x of the boundary and velocity tangent to $\partial\Omega$ at x, there exists an initial datum (and in-flow datum, for the in-flow boundary condition case) such that the Boltzmann solution has a jump discontinuity at (x, v) (Theorem 1: Formation of Discontinuity). Once the discontinuity occurs at the grazing boundary, this discontinuity propagates inside along the forward trajectory until it hits the boundary again (Theorem 2: Propagation of Discontinuity). And except for those points we can show that the Boltzmann solution is continuous (Theorem 3: Continuity away from $\mathfrak{D}$).

1.5. Main ingredients of the proofs.

1. *The equality induced by non-convex domain*. We consider the near Maxwellian regime and the linearized Boltzmann equation (3). The formation of discontinuity is a consequence of the following estimate. Assume $(x, v) \in \gamma_0^{\mathbf{S}}$ as below, pictured so that for sufficiently small $t > 0$ the backward trajectory $x - tv$ is in an interior of the phase space. For simplicity we impose the trivial in-flow boundary condition $G(t, x, v) \equiv \mu(v)$ which corresponds to $g(t, x, v) \equiv 0$ (93). Consider points (x_n', v_n') in γ_- and (x_n'', v_n'') missing the non-convex part near (x, v) and both sequences converge (x, v) as $n \to \infty$.

   Now suppose the solution f of the linearized Boltzmann equation is continuous around (x, v). Then the Boltzmann solution f at (x_n', v_n')

$$f(t, x_n', v_n') = g(t, x_n', v_n') = 0,$$

   and at (x_n'', v_n''),

$$f(t, x_n'', v_n'') = e^{-\nu(v_n'')t} f_0(x_n'' - tv_n'', v_n'') + \int_0^t e^{-\nu(v_n'')(t-s)}\{Kf + \Gamma(f, f)\}(s, x_n'' - (t-s)v_n'', v_n'')ds$$

   converges with each other as $n \to \infty$. Then we have the following equality (Fig. 3):

$$f_0(x - tv, v) = -\int_0^t e^{\nu(v)s}\{Kf + \Gamma(f, f)\}(s, x - (t-s)v, v)ds. \tag{27}$$

   Thanks to [21], the pointwise estimate of f, with some standard estimates of K, Γ, the right-hand side of the above equality has magnitude $O(t)||f_0||_\infty(1 + ||f_0||_\infty)$. If you choose $f_0(x - tv, v) = ||f_0||_\infty$, then the above equality (27) cannot be true for sufficiently small t unless the trivial case $f_0 \equiv 0 (F \equiv \mu)$. Therefore the Boltzmann solution f cannot be continuous at (x, v). For diffuse (5), bounce-back (7) boundary conditions we also obtain the equality induced by the non-convex domain similar to (27). This argument is based on the idea that the free transport effect is dominant to the collision effect if time $t > 0$ and the perturbation $f = \frac{F-\mu}{\sqrt{\mu}}$ is small.
2. *New proof of continuity of Boltzmann solution with diffuse boundary condition*. We consider the near Maxwellian regime and the linearized Boltzmann equation (4). In Sect. 5.2 we prove a continuity away from $\mathfrak{D}$ of the Boltzmann solution with

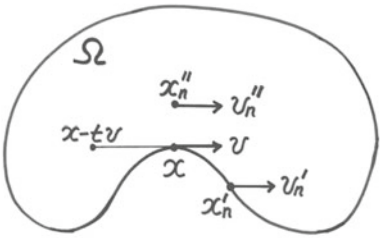

Fig. 3. Non-Convex Domain

diffuse boundary condition using a simple iteration scheme (102) with iteration diffuse boundary condition (131). This iteration scheme has several advantages. First it preserves a continuity away from $\mathfrak{D}$ as m increasing, that is, if h^m is continuous away from $\mathfrak{D}$ then h^{m+1} is also continuous away from $\mathfrak{D}$. Second, the sequence $\{h^m\}$ has uniform L^∞ bound and moreover it is Cauchy in L^∞ for the in-flow boundary condition $h^m|_{\gamma_-} = wg$. Therefore $h = \lim h^m$, a solution of the linear Boltzmann equation is continuous local in time. Combining with uniform-in-time boundedness of the Boltzmann solution ([21]), we achieve the continuity for all time. In order to apply this idea to diffuse boundary condition, we use Guo's idea [21]: A norm of the diffuse boundary operator is less than 1 effectively, if we trace back several bounces. This approach gives a simpler proof for the continuity of the Boltzmann equation with diffuse boundary condition with convex domain (see Lemma 23–25 of [21]).

3. *Continuity of the gain term* Q_+. In contrast to the previous two ingredients, in this part we consider the non-perturbation setting and the gain term Q_+ of the Boltzmann equation (1). The smoothing effect of the gain term Q_+ is one of the fundamental features of the Boltzmann theory. There are lots of results about the smoothing effect in Sobolev regularity, for example

$$||Q_+(\phi,\psi)||_{H^{\frac{N-1}{2}}} \leq C||\phi||_{L^1}||\psi||_{L^2},$$

with some assumption on various collision kernels [28,41,42]. To study the propagation of singularity and regularity, in the case of the angular cutoff kernel (1), it is standard to use Duhamel formulas and combine the Velocity Average Lemma and the regularity of Q_+ [7]. For details see [28,33] and the Villani review [39] especially pp. 77–79.

In order to study the propagation of discontinuity and continuity we need a totally different smooth effect of Q_+. For the discontinuity induced by the non-convex domain, we need the following: Recall the grazing set $\mathfrak{G}$ in Definition 4. A test function $\phi(t,x,v)$ is continuous on $[0,T]\times(\Omega\times\mathbb{R}^3)\backslash\mathfrak{G}$ and bounded on $[0,T]\times\Omega\times\mathbb{R}^3$. Then

$$Q_+(\phi,\phi)(t,x,v) \in C^0([0,T]\times\Omega\times\mathbb{R}^3). \tag{28}$$

Recall that the grazing set $\mathfrak{G} = \{(x,v)\in\bar{\Omega}\times\mathbb{R} : v\in\mathfrak{G}_x\}$. The grazing section $\mathfrak{G}_x = \{\tau u\in\mathbb{R}^3 : t\geq 0,\ u\in\mathfrak{G}_x\cap\mathbb{S}^2\}$ is a union of straight lines in velocity space $\mathbb{R}^3$ and two dimensional Lebesgue measure of $\mathfrak{G}_x\cap\mathbb{S}^2$ is zero (Hongjie Dong's Lemma, Lemma 17 of [21]). Moreover, using continuous behavior of $\mathfrak{G}_x$ in x, one can invent a very effective covering of $\mathfrak{G}_x$ (Guo's covering, Lemma 18 of [21]). Recall that the gain term Q_+ (2) is an integration operator in v alone and a local operator in x. Therefore, in order to show the continuity of $Q_+(\phi,\phi)$ for t, x, v(especially for x) we need to utilize both the geometric property and the smallness of $\mathfrak{G}$. Notice that the smoothing effect on $C^0_{t,x,v}$ has been believed to be true for long time without a mathematical proof in numerical communities [1], p1587 of [4], p502 of [35].

The main idea to prove the smoothing effect in $C^0_{t,x,v}$ is to use Carleman's representation for $Q_+(\phi,\phi)(t,x,v)$ which has been a very effective tool [20,41,42],

$$\int_{\mathbb{R}^3}\phi(t,x,v')\frac{1}{|v-v'|^2}\int_{E_{vv'}}\phi(t,x,v_1')B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|})dv_1'dv', \quad (29)$$

with the hyperplane $E_{vv'}=\{v_1'\in\mathbb{R}^3:(v_1'-v)\cdot(v'-v)=0\}$. We will show the smallness of

$$|Q_+(\phi,\phi)(\bar{t},\bar{x},\bar{v})-Q_+(\phi,\phi)(t,x,v)|,$$

for $|(t,x,v)-(\bar{t},\bar{x},\bar{v})|<\delta$. Assume we have sufficient decay of ϕ for large v. Replace the integrable kernel $\frac{1}{|v-v'|^2}$ by a smooth compactly supported function and cut off the singular part of $B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|})$ to control the above difference as

$$\begin{aligned}
&O(\delta)||\phi||^2_\infty + C\int_{|v'|<N}|\phi(t,x,v')-\phi(\bar{t},\bar{x},v'')|\int_{E_{\bar{v}v''}\cap\{|v_1''|<N\}}|\phi(\bar{t},\bar{x},v_1'')|dv_1''dv'\\
&+C\int_{|v'|<N}|\phi(t,x,v')|\left\{\int_{E_{vv'}\cap\{|v_1'|<N\}}\phi(t,x,v_1')dv_1'\right.\\
&\qquad\qquad\left.-\int_{E_{\bar{v}v''}\cap\{|v_1''|<N\}}\phi(\bar{t},\bar{x},v_1'')dv_1''\right\}dv',
\end{aligned}$$

where $v''(v')$ is chosen to be $v'-(v-\bar{v})$ for convenience.

One can easily control the integration at the first line. Because for the first term, integrating over v', we can cut off a small neighborhood of $\mathfrak{G}_x$ from $|v'|<N$. Away from that neighborhood, using the continuity of ϕ away from $\mathfrak{G}_x$ we can control the integrand pointwisely.

In order to control the second line integration we have to control the difference in big braces. To do that we choose a special change of variables for v_1'' ,(41). Under this change of variables the second line is bounded by

$$C\int_{|v'|<N}|\phi(t,x,v')|\underbrace{\int_{E_{vv'}\cap\{|v_1'|<N\}}|\,\phi(t,x,v_1')-\phi(\bar{t},\bar{x},v_1'')|\,dv_1'}\,dv'.$$

The underbraced integration above is a function of $t,x,\bar{t},\bar{x},v$ and v'. Unfortunately, for fixed $t,x,\bar{t},v$ one cannot expect a smallness of the underbraced integration for all v'. (Since ϕ might be discontinuous at $\mathfrak{G}_x$, the difference $|\,\phi(t,x,v_1')-\phi(\bar{t},\bar{x},v_1'')|$ could be large for $(t,x,v_1')\in\mathfrak{G}_x$. Moreover the 2-dimensional Lebesgue measure of the intersection of $\mathfrak{G}_x$ and the plane $E_{vv'}$ could be large or even infinite.) However, in Sect. 3.3, we can show that that bad situation happens for very rare v' in $\{v'\in\mathbb{R}^3:|v'|<N\}$ and use the integration over v' to control the above integration.

1.6. Structure of paper. In Sect. 2, we state some preliminary facts which are useful tools for this paper. In Sect. 3, we state and prove the continuity of $Q+$ (Theorem 4). In Sects. 4–6, we deal with in-flow boundary, diffuse boundary and bounce-back boundary, respectively. For each section, first we prove the formation of discontinuity (Theorem 1). Then we show the continuity away from $\mathfrak{D}$ (Theorem 3). Using this continuity, combining with continuity of Q_+, we show the propagation of discontinuity (Theorem 2).

2. Preliminary

In this section we study continuity properties of the backward exit time $t_{\mathbf{b}}(x,v)$ and, a measure theoretic property and geometric covering of the grazing set $\mathfrak{G}$, and estimates of Boltzmann operators and Carleman's representation.

We use Lemma 1 of [21], basic properties of the backward exist time $t_{\mathbf{b}}(x,v)$:

Lemma 1 [21]. *Let Ω be an open bounded subset of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Let (t,x,v) be connected with $(t-t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v)$ backward in time through a trajectory of (1.1).*

1. *The backward exit time $t_{\mathbf{b}}(x,v)$ is lower semicontinuous.*
2. *If*

$$v\cdot n(x_{\mathbf{b}}(x,v))<0, \tag{30}$$

then $(t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v))$ are smooth functions of (x,v) so that

$$\nabla_x t_{\mathbf{b}} = \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \quad \nabla_v t_{\mathbf{b}} = \frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})},$$
$$\nabla_x x_{\mathbf{b}} = I + \nabla_x t_{\mathbf{b}}\otimes v, \quad \nabla_v x_{\mathbf{b}} = t_{\mathbf{b}} I + \nabla_v t_{\mathbf{b}}\otimes v.$$

For a convex domain, if a point (x,v) is in the interior of the phase space, i.e. $(x,v)\in\Omega\times\mathbb{R}^3$, then the condition (30) is always satisfied and hence $t_{\mathbf{b}}(x,v)$ is smooth due to the second statement of Lemma 1. However for a non-convex domain, there is a point (x,v) in $\Omega\times\mathbb{R}^3$ but $(x_{\mathbf{b}}(x,v,),v)\in\gamma_0$, i.e. $v\cdot n(x_{\mathbf{b}}(x,v))=0$. Hence $t_{\mathbf{b}}(x,v)$ is not smooth at that point (x,v). We further investigate a continuity property of $t_{\mathbf{b}}$ for that case. Indeed, the discontinuity behavior of $t_{\mathbf{b}}(x,v)$ for $(x_{\mathbf{b}}(x,v),v)\in\gamma_0^{\mathbf{S}}$ is a main ingredient of the formation of discontinuity.

Lemma 2. *Let $\Omega\in\mathbb{R}^3$ be an open set with a smooth boundary $\partial\Omega$. Assume $(x_0,v_0)\in\Omega\times\mathbb{R}^3$ with $v_0\neq 0$ and $t_{\mathbf{b}}(x_0,v_0)<\infty$. Consider $(x_0,v_0)\in\mathfrak{G}$, i.e. $(x_{\mathbf{b}}(x_0,v_0),v_0)\in\gamma_0$,*

$$\textit{If } (x_{\mathbf{b}}(x_0,v_0),v_0)\in\gamma_0^{I-} \textit{ then } t_{\mathbf{b}}(x,v) \textit{ is continuous around } (x_0,v_0),$$
$$\textit{If } (x_{\mathbf{b}}(x_0,v_0),v_0)\in\gamma_0^{\mathbf{S}} \textit{ then } t_{\mathbf{b}}(x,v) \textit{ is not continuous around } (x_0,v_0).$$

Recall γ_0^{I-} and $\gamma_0^{\mathbf{S}}$ in Definition 2.

Proof. Throughout this proof, without loss of generality we assume that $\partial\Omega$ is a graph of Φ locally and $\Phi(0,0)=0$ and $(\partial_{x_1}\Phi,\partial_{x_2}\Phi)(0,0)=(0,0)$. Moreover assume $x_0=(|x_0|,0,0)$, $v_0=(|v_0|,0,0)$ and $t_{\mathbf{b}}(x_0,v_0)=\frac{|x_0|}{|v_0|}$ so that $x_{\mathbf{b}}(x_0,v_0)=(0,0,0)=(0,0,\Phi(0,0))$.

First, let $(x_{\mathbf{b}}(x_0,v_0),v_0)\in\gamma_0^{I-}$. By the definition of γ_0^{I-}, we have $\Phi(-\tau,0)>0$ and $\Phi(\tau,0)<0$ for $0<\tau<<1$. Using the continuity of Φ, choose sufficiently small

$\varepsilon > 0$, $\delta > 0$ such that $\Phi(-\delta, y) > \frac{\varepsilon}{2}$ and $\Phi(\delta, y) < -\frac{\varepsilon}{2}$ for $0 < |y| < \delta$. Fix $x = (x_1, x_2, x_3) \sim x_0$ and $v = (v_1, v_2, v_3) \sim v_0$. We define

$$\Psi(x, v, t) = x_3 - tv_3 - \Phi(x_1 - tv_1, x_2 - tv_2).$$

For $t' \equiv \frac{x_1-\delta}{v_1}$, $\Psi(x, v, t_0') = -\Phi(\delta, x_2 - \frac{x_1-\delta}{v_1}v_2) + x_3 - \frac{x_1-\delta}{v_1}v_3 > \frac{\varepsilon}{4}$ for $(x_1, x_2, x_3) \sim (|x_0|, 0, 0)$, $(v_1, v_2, v_3) \sim (|v_0|, 0, 0)$. For $t'' = \frac{x_1+\delta}{v_1}$, $\Psi(x, v, t'') = -\Phi(-\delta, x_2 - \frac{x_1+\delta}{v_1}v_2) + x_3 - \frac{x_1+\delta}{v_1}v_3 < -\frac{\varepsilon}{4}$ for $(x_1, x_2, x_3) \sim (|x_0|, 0, 0)$, $(v_1, v_2, v_3) \sim (|v_0|, 0, 0)$. Using the continuity of Φ and Ψ, there exists $t_* \in (\frac{x_1}{v_1} - \frac{\delta}{v_1}, \frac{x_1}{v_1} + \frac{\delta}{v_1})$ so that $\Psi(x, v, t_*) = 0$, i.e. $t_{\mathbf{b}}(x, v) = t_*$. If $x \sim x_0$ and $v \sim v_0$, then $\frac{x_1}{v_1} - \frac{\delta}{v_1} \sim \frac{|x_0|}{|v_0|} - \frac{\delta}{|v_0|} = t_{\mathbf{b}}(x_0, v_0) - \frac{\delta}{|v_0|}$ and $\frac{x_1}{v_1} + \frac{\delta}{v_1} \sim \frac{|x_0|}{|v_0|} + \frac{\delta}{|v_0|} \sim t_{\mathbf{b}}(x_0, v_0) + \frac{\delta}{|v_0|}$ so that $t_* \in (t_{\mathbf{b}}(x_0, v_0) - \frac{\delta}{|v_0|}, t_{\mathbf{b}}(x_0, v_0) + \frac{\delta}{|v_0|})$.

Next, let $(x_{\mathbf{b}}(x, v), v) \in \gamma_0^{\mathbf{S}}$. By the definition of the concave grazing boundary $\gamma_0^{\mathbf{S}}$, we have $\Phi(-\tau, 0) > 0$ and $\Phi(\tau, 0) < 0$ for $0 < \tau << 1$. Choose a sequence $x_n = (|x_0|, 0, \frac{1}{n})$. There exists $\varepsilon > 0$ such that $t_{\mathbf{b}}(x_n, v_0) > t_{\mathbf{b}}(x_0, v_0) + \varepsilon$ for sufficiently large n. This implies that $(x_n, v_0) \to (x_0, v_0)$ but $t_{\mathbf{b}}(x_n, v_0) \nrightarrow t_{\mathbf{b}}(x_0, v_0)$ as $n \to \infty$. □

In the next two lemmas, we consider the grazing set $\mathfrak{G}$ (Definition 4) including the discontinuity set $\mathfrak{D}$. Lemma 3, Lemma 17 of [21] due to Hongjie Dong, is important to control a size of $\mathfrak{G}$. We denote m_2 as a standard 2-dimensional Lebesgue measure and m_3 as a standard 3-dimensional Lebesgue measure. Recall the grazing section $\mathfrak{G}_x$ in Definition 4.

Lemma 3 [21]. *If $\partial\Omega$ is C^1 then the grazing section $\mathfrak{G}_x$ restricted to $\mathbb{S}^2$ has zero 2-dimensional Lebesgue measure for all $x \in \bar{\Omega}$,i.e.*

$$m_2(\mathfrak{G}_x \cap \mathbb{S}^2) = 0,$$

for all $x \in \bar{\Omega}$.

With condition $m_2(\mathfrak{G}_x \cap \mathbb{S}^2) = 0$, we can construct Guo's covering which is little bit stronger than the original one in Lemma 18 in [21].

Lemma 4 (Guo's covering) [21]. *Assume $m_2(\mathfrak{G}_x \cap \mathbb{S}^2) = 0$ is valid for all $x \in \bar{\Omega}$. Let $B_N = \{v \in \mathbb{R}^3 : |v| \leq N\}$. Then for any $\varepsilon > 0$ and $N_* > 0$ there exist $\delta_{\varepsilon,N,N_*} > 0$, and $l_{\varepsilon,N,N_*,\Omega}$ balls $B(x_1; r_1), B(x_2; r_2) \ldots, B(x_l; r_l) \subset \bar{\Omega}$, as well as open sets $O_{x_1}, O_{x_2}, \ldots O_{x_l}$ of B_N which are radial symmetric, i.e.*

$$O_{x_i} = \{t\hat{v} \in \mathbb{R}^3 : t \geq 0,\ \hat{v} \in O_{x_i} \cap \mathbb{S}^2\},$$

with $m_3(O_{x_i}) < \frac{\varepsilon}{N_*}$ and $m_2(O_{x_i} \cap \mathbb{S}^2) \leq \frac{\varepsilon}{N^2 N_*}$ for all $1 \leq i \leq l_{\varepsilon,N,N_*,\Omega}$, such that for any $x \in \bar{\Omega}$, there exists x_i so that $x \in B(x_i; r_i)$ and for $v \notin O_{x_i}$,

$$|v \cdot n(x_{\mathbf{b}}(x, v))| > \delta_{\varepsilon,N,N_*} > 0,$$

or equivalently

$$O_{x_i} \supset \bigcup_{x \in B(x_i; r_i)} \{v \in B_N : |v \cdot n(x_{\mathbf{b}}(x, v))| \leq \delta_{\varepsilon,N,N_*}\} \supset \bigcup_{x \in B(x_i; r_i)} \mathfrak{G}_x \cap B_N.$$

Combining Lemma 3 and Lemma 4, we have the following lemma. Later we will use this lemma to prove Theorem 4. Namely, a function which is continuous away from the grazing set $\mathfrak{G}$ is uniformly continuous except for an arbitrary small open set containing $\mathfrak{G}$.

Lemma 5. *Assume $\phi(t,x,v)$ is continuous on $[0,T]\times(\Omega\times\{v\in\mathbb{R}^3:\frac{1}{M}\le|v|\le N\})\backslash\mathfrak{G}$. For fixed $x\in\Omega$ and $\varepsilon>0$ and $N_*>0$, there exist*

$$\delta=\delta(\phi,\Omega,\varepsilon,N_*,x,\frac{1}{M},N)>0, \tag{31}$$

and an open set $U_x\subset\{v\in\mathbb{R}^3:\frac{1}{M}\le|v|\le N\}$ which is radial symmetric, i.e. $U_x=\{t\hat{v}\in\mathbb{R}^3:t\ge0\ ,\ \hat{v}\in U_x\cap\mathbb{S}^2\}$ with $m_3(U_x)<\frac{\varepsilon}{N_*}$ and $m_2(U_x\cap\mathbb{S}^2)<\frac{\varepsilon}{N_*N^2}$ such that

$$|\phi(t,x,v)-\phi(\bar{t},\bar{x},\bar{v})|<\frac{\varepsilon}{N_*},$$

for $v\in\{v\in\mathbb{R}^3:\frac{1}{M}\le|v|\le N\}\backslash U_x$ and $|(t,x,v)-(\bar{t},\bar{x},\bar{v})|<\delta$.

Proof. Let $x\sim\bar{x}$. Due to Guo's covering [21], Lemma 4, we can choose $B(x_i;r_i)$ including x and $\bar{x}$, as well as $O_{x_i}\subset\mathbb{R}^3$ so that

$$O_{x_i}\supset\bigcup_{y\in B(x_i;r_i)}\mathfrak{G}_y\cap B_N\supset\bigcup_{y\in B(x;\delta)}\mathfrak{G}_y\cap B_N,$$

with $m_3(O_{x_i})<\frac{\varepsilon}{N_*}$. Notice that $m_3(\bar{O}_{x_i})=m_3(O_{x_i})$. We can choose an open set U_{x_i} so that $m_3(U_{x_i})\le 2m_3(O_{x_i})$ and $\bar{O}_{x_i}\subset U_{x_i}$. Since both $\bar{O}_{x_i}$ and $B_N\backslash U_{x_i}$ are compact subsets of B_N, we have a positive distance between two sets, i.e.

$$0<\mathfrak{d}=\inf\{|\zeta-\xi|:\zeta\in\bar{O}_{x_i}\quad\text{and}\quad\xi\in B_N\backslash U_{x_i}\}.$$

Assume $\delta<\mathfrak{d}/2$. Fix $x\in\bar{\Omega}$ and $v\in\{v\in\mathbb{R}^3:\frac{1}{M}\le|v|\le N\}\backslash U_x$. Then $|(\bar{x},\bar{v})-(x,v)|<\delta$ implies that $\bar{v}\in\{v\in\mathbb{R}^3:\frac{1}{M}\le|v|\le N\}\backslash O_{x_i}$. For such x, v, $\bar{x}$ and $\bar{v}$, consider the function ϕ as its restriction on a compact set $[0,T]\times\bar{B}(x;\delta)\times B_N\backslash O_{x_i}$. Therefore $\phi|_{[0,T]\times\bar{B}(x;\delta)\times B_N\backslash O_{x_i}}$ is a uniformly continuous function. Hence $|\phi(t,x,v)-\phi(\bar{t},\bar{x},\bar{v})|$ can be controlled as small uniformly, if $\delta>0$ is chosen sufficiently small. □

We will use Carleman's representation ([20,41]) in the proof of Theorem 4 crucially. Let $Q_+(\phi,\psi)$ be defined by (2) and assume $Q_+(\psi,\phi)<\infty$ for $\psi=\psi(v)$ and $\phi=\phi(v)$. Then **Carleman's representation** is

$$Q_+(\psi,\phi)(v)=2\int_{\mathbb{R}^3}\psi(v')\frac{1}{|v-v'|^2}\underbrace{\int_{E_{vv'}}\phi(v_1')B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|})dv_1'dv'}, \tag{32}$$

where $E_{vv'}$ is a hyperplane containing $v\in\mathbb{R}^3$ and perpendicular to $\frac{v'-v}{|v'-v|}\in\mathbb{S}^2$, i.e.

$$E_{vv'}=\{v_1'\in\mathbb{R}^3:(v_1'-v)\cdot(v'-v)=0\}. \tag{33}$$

In the proof of Theorem 4, we need to control the underbraced integration over $E_{vv'}$ in (32) frequently:

Lemma 6. *For a rapidly decreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, we have*

$$\int_{E_{vv'}} \phi(|v_1{}'|) B\big(2v - v' - v_1{}', \frac{v' - v_1{}'}{|v' - v_1{}'|}\big) dv_1{}' \le C_\phi (1 + |v - v'|^\gamma), \tag{34}$$

where C_ϕ only depends on ϕ.

Proof. For fixed v' and v, let us denote $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, with $\tilde{\mathbf{e}}_3 = \frac{v'-v}{|v'-v|}$, and let be the orthonormal basis of $\mathbb{R}^3$ such that any $v_1' \in E_{vv'}$ can be written as $v_1' = v + \eta_1 \tilde{\mathbf{e}}_1 + \eta_2 \tilde{\mathbf{e}}_2$. Since $v' - v \perp E_{vv'}$ from (33), there is η_3 such that $v' - v = \eta_3 \tilde{\mathbf{e}}_3$, where $|\eta_3| = |v - v'|$. Then we can write $2v - v' - v_1' = v - v' + v - v_1' = -\eta_1 \tilde{\mathbf{e}}_1 - \eta_2 \tilde{\mathbf{e}}_2 - \eta_3 \tilde{\mathbf{e}}_3$ and $|2v - v' - v_1'|^2 = \eta_1^2 + \eta_2^2 + |v' - v|^2$. Moreover $v' - v_1' = -\eta_1 \tilde{\mathbf{e}}_1 - \eta_2 \tilde{\mathbf{e}}_2 + \eta_3 \tilde{\mathbf{e}}_3$. We can write the left hand side of (34) as

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\eta_1^2 + \eta_2^2 + |v|^2) \left| \begin{pmatrix} -\eta_1 \\ -\eta_2 \\ -\eta_3 \end{pmatrix} \right|^\gamma \frac{1}{\eta_1^2 + \eta_2^2 + |v - v'|^2} \begin{pmatrix} -\eta_1 \\ -\eta_2 \\ -\eta_3 \end{pmatrix} \cdot \begin{pmatrix} -\eta_1 \\ -\eta_2 \\ \eta_3 \end{pmatrix} d\eta_1 d\eta_2 \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\eta_1^2 + \eta_2^2)(\eta_1^2 + \eta_2^2 + |v' - v|^2)^{\frac{\gamma}{2} - 1}(\eta_1^2 + \eta_2^2 - |v' - v|^2) d\eta_1 d\eta_2 \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\eta_1^2 + \eta_2^2)\big(\eta_1^2 + \eta_2^2 + |v' - v|^2\big)^{\frac{\gamma}{2}} d\eta_1 d\eta_2 \\
&\le C_\phi (1 + |v' - v|^\gamma).
\end{aligned}$$

□

We recall two estimates of linearized operators K and Γ from [21].

Lemma 7 ([21]). *The Grad estimate for hard potentials:*

$$|\mathbf{k}(v, v')| \le C_{\mathbf{k}} \{|v - v'| + |v - v'|^{-1}\} e^{-\frac{1}{8}|v - v'|^2 - \frac{1}{8}\frac{||v|^2 - |v'|^2|^2}{|v - v'|^2}}.$$

Recall w in (12). Let $0 \le \theta < \frac{1}{4}$. Then there exists $0 \le \varepsilon(\theta) < 1$ and $C_\theta > 0$ such that for $0 \le \varepsilon < \varepsilon(\theta)$,

$$\int \{|v - v'| + |v - v'|^{-1}\} e^{-\frac{1-\varepsilon}{8}|v - v'|^2 - \frac{1-\varepsilon}{8}\frac{||v|^2 - |v'|^2|^2}{|v - v'|^2}} \frac{w(v) e^{\theta |v|^2}}{w(v') e^{\theta |v'|^2}} dv' \le \frac{C_{\mathbf{k}}}{1 + |v|}. \tag{35}$$

For the nonlinear collision operator,

$$|w \Gamma(g_1, g_2)(v)| \le C_\Gamma (1 + |v|)^\gamma ||w g_1||_\infty ||w g_2||_\infty. \tag{36}$$

Also we recall a standard estimate

$$\int_{\mathbb{R}^3} \phi(v') |v - v'|^\gamma dv' \sim (1 + |v|)^\gamma, \tag{37}$$

for $\phi \in L^1(\mathbb{R}^3)$.

3. Continuity of the Collision Operators

In this section we mainly prove Theorem 4, a smooth effect of Q_+ in $C^0_{t,x,v}$, Theorem 4 is the crucial ingredient to prove Theorem 2 and Theorem 3. This smooth effect of the gain term ensures that there is no singularity created by the nonlinearity of the Boltzmann equation.

Proof of (26). It is easy to show the boundedness (26) from

$$\begin{aligned}
&\nu^{-1}\bar{w}^{-1}Q_+(F,F)(t,x,v)\\
&\leq \frac{1}{\nu(v)\bar{w}(v)}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)\bar{w}(u')\bar{w}(v')d\omega du \times ||\bar{w}^{-1}F||^2_\infty\\
&\leq \nu(v)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)\frac{e^{-\frac{|u|^2}{4}}}{(1+\rho^2|u|^2)^\beta}d\omega du \times ||\bar{w}^{-1}F||^2_\infty\\
&\leq C\ \nu(v)^{-1}\nu(v)||\bar{w}^{-1}F||^2_\infty \leq C||\bar{w}^{-1}F||^2_{L^\infty([0,T]\times(\bar{\Omega}\times\mathbf{R}^3))},
\end{aligned}$$

where we used (37) and $|u'|^2+|v'|^2=|u|^2+|v|^2$. $\square$

Next we will show the continuity part of Theorem 4. The goal of following three subsections is to show

$$\begin{aligned}
&\text{For fixed } \varepsilon>0 \text{ and } (t,x,v)\in[0,T]\times\Omega\times\mathbb{R}^3, \text{ there is } \delta>0 \text{ such that}\\
&|Q_+(\bar{w}h,\bar{w}h)(\bar{t},\bar{x},\bar{v})-Q_+(\bar{w}h,\bar{w}h)(t,x,v)|<\varepsilon \text{ for } |(\bar{t},\bar{x},\bar{v})-(t,x,v)|<\delta.
\end{aligned} \tag{38}$$

3.1. Decomposition and change of variables. In this section, we use Carleman's representation (32) to split $Q_+(\bar{w}h,\bar{w}h)(\bar{t},\bar{x},\bar{v})-Q_+(\bar{w}h,\bar{w}h)(t,x,v)$ in a natural way (39), and introduce two change of variables (40) and (41).

It is convenient to define

$$h\equiv\bar{w}^{-1}F,$$

where $||h||_\infty\equiv||h||_{L^\infty([0,T]\times(\bar{\Omega}\times\mathbb{R}^3))}=||\bar{w}^{-1}F||_{L^\infty([0,T]\times(\bar{\Omega}\times\mathbf{R}^3))}$. Choose $(\bar{t},\bar{x},\bar{v})\sim(t,x,v)$. Using Carleman's Representation (32) we have

$$\begin{aligned}
&Q_+(\bar{w}h,\bar{w}h)(\bar{t},\bar{x},\bar{v})-Q_+(\bar{w}h,\bar{w}h)(t,x,v)\\
&=2\int_{\mathbb{R}^3}\underbrace{\bar{w}(v'')h(\bar{t},\bar{x},v'')\frac{1}{|\bar{v}-v''|^2}}_{\mathcal{A}}\int_{E_{\bar{v}v''}}\underbrace{\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'')B\Big(2\bar{v}-v''-v_1'',\frac{v''-v_1''}{|v''-v_1''|}\Big)}_{\mathcal{B}}dv_1''dv''\\
&\quad-2\int_{\mathbb{R}^3}\underbrace{\bar{w}(v')h(t,x,v')\frac{1}{|v-v'|^2}}_{\mathcal{A}'}\int_{E_{vv'}}\underbrace{\bar{w}(v_1')h(t,x,v_1')B\Big(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|}\Big)}_{\mathcal{B}'}dv_1'dv'\\
&=2\int_{\mathbb{R}^3}\{\mathcal{A}-\mathcal{A}'\}\int_{E_{\bar{v}v''}}\mathcal{B}\,dv_1''dv''+2\int_{\mathbb{R}^3}\mathcal{A}'\int_{E_{\bar{v}v''}}\{\mathcal{B}-\mathcal{B}'\}\,dv_1''dv'.
\end{aligned} \tag{39}$$

In order to control the first term of (39), we need to compare the arguments $v'',\bar{t},\bar{x},v'$ of $\mathcal{A}$ and the arguments v',t,x,v of $\mathcal{A}'$. For that purpose, we introduce the following change of variables:

Lemma 8. *For fixed v and $\bar{v}$ in $\mathbb{R}^3$, we define*

$$v'' \equiv v''(v'; v, \bar{v}) = v' - (v - \bar{v}). \tag{40}$$

Then two planes $E_{\bar{v}v''}$ and $E_{vv'}$ have the same normal direction. The distance between to planes is $|(\bar{v} - v) \cdot \frac{v'-v}{|v'-v|}|$.

Proof. Assume (40). Clearly the Jacobian matrix $\frac{\partial v''(v')}{\partial v'} = I$, where I is 3×3 identity matrix. The normal direction of $E_{\bar{v}v''}$ is $\frac{v''-\bar{v}}{|v''-\bar{v}|} = \frac{v'-v}{|v'-v|}$ which is also the normal direction of $E_{vv'}$. To measure a distance between two planes $E_{vv'}$ and $E_{\bar{v}v''}$, we consider the line passing v and directing $\frac{v'-v}{|v'-v|}$, which is $v(s) = \frac{v'-v}{|v'-v|}s + v$. The solution of $v(s_*) \in E_{\bar{v}v''}$ is a solution of $0 = (v''-\bar{v})\cdot(v(s)-\bar{v}) = (v'-v)\cdot(v(s)-\bar{v}) = |v'-v|s+(v'-v)\cdot(v-\bar{v})$. Easily we have the solution $s_* = \frac{(v'-v)\cdot(\bar{v}-v)}{|v'-v|}$. Since $v(s)$ is the unit-speed line we know that $|v(s_*) - v(0)|$ is the distance between $E_{\bar{v}v''}$ and $E_{vv'}$. $\square$

An important property of (40) is that two planes $E_{\bar{v}v''}$ and $E_{vv'}$ have the same normal direction. In order to control the second term of (39), we need to compare the arguments v_1'', $\bar{t}$, $\bar{x}$, $\bar{v}$, v'' of $\mathcal{B}$ and the argument v_1', t, x, v, v' of $\mathcal{B}'$, especially $v_1' \in E_{vv'}$ and $v_1'' \in E_{\bar{v}v''}$. For that purpose, we introduce the following change of variables:

Lemma 9. *For fixed v, v' and $\bar{v}$ in $\mathbb{R}^3$, we define a unit Jacobian change of variables*

$$v_1'' \equiv v_1''(v_1'; v, v', \bar{v}) = v_1' + \frac{v'-v}{|v'-v|}\{(\bar{v} - v) \cdot \frac{v'-v}{|v'-v|}\}. \tag{41}$$

In this change of variables $v_1'' \in E_{\bar{v}v''}$ if and only if $v_1' \in E_{vv'}$.

Proof. Assume (40) and (41). Clearly $\frac{\partial v_1''(v_1')}{\partial v_1'} = I$. We can check the following equality:

$$\begin{aligned}
(v_1'' - \bar{v}) \cdot (v'' - \bar{v}) &= ({v_1}' - \bar{v} + \frac{v'-v}{|v'-v|}\{(\bar{v} - v) \cdot \frac{v'-v}{|v'-v|}\}) \cdot (v' - v) \\
&= ({v_1}' - \bar{v}) \cdot (v' - v) + |v' - v|\{(\bar{v} - v) \cdot \frac{v'-v}{|v'-v|}\} \\
&= (v_1' - v) \cdot (v' - v) + (v - \bar{v}) \cdot (v' - v) + (\bar{v} - v) \cdot (v' - v) \\
&= (v_1' - v) \cdot (v' - v).
\end{aligned}$$

By definition, $v_1' \in E_{vv'}$ is equivalent to $(v_1' - v) \cdot \frac{v'-v}{|v'-v|} = 0$. From the above equality, we conclude $(v_1'' - \bar{v}) \cdot \frac{v''-\bar{v}}{|v''-\bar{v}|} = 0$ which is equivalent to $v_1'' \in E_{\bar{v}v''}$. $\square$

Under the first change of variables (40), we can rewrite the first term of (39) as

$$2 \int_{\mathbb{R}^3} \underbrace{\frac{1}{|v-v'|^2}\Big\{\bar{w}(v'')h(\bar{t}, \bar{x}, v'') - \bar{w}(v')h(t, x, v')\Big\}}_{(\mathcal{C})}$$
$$\times \int_{E_{\bar{v}v''}} \underbrace{\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'')B\big(2\bar{v} - v'' - v_1'', \frac{v''-v_1''}{|v''-v_1''|}\big)}_{(\mathcal{D})} dv_1'' dv'. \tag{42}$$

Under the second change of variables (41), we can rewrite the second term of (39) as

$$2\int_{\mathbb{R}^3}\underbrace{\bar{w}(v')h(t,x,v')\frac{1}{|v-v'|^2}}_{(\mathcal{E})}$$
$$\times\int_{E_{vv'}}\underbrace{\Big\{\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'')B\big(2\bar{v}-v''-v_1'',\frac{v''-v_1''}{|v''-v_1''|}\big)-\bar{w}(v_1')h(t,x,v_1')B\big(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|}\big)\Big\}}_{(\mathcal{F})}dv_1'dv'. \tag{43}$$

We will estimate (42) and (43) separately in following two sections.

3.2. Estimate of (42). We divide into several cases:

Case 1. $|v|\geq N$. From Lemma 6, for $N>0$ we can estimate

$$\begin{aligned}Q_+(\bar{w}h,\bar{w}h)(t,x,v)\mathbf{1}_{|v|>N}&\leq C||h||_\infty^2\mathbf{1}_{|v|>N}\int_{\mathbf{R}^3}\bar{w}(v')\left(\frac{1}{|v-v'|^2}+\frac{1}{|v-v'|^{2-\gamma}}\right)dv'\\&\leq C||h||_\infty^2\left(\frac{1}{(1+|v|)^2}+\frac{1}{(1+|v|)^{2-\gamma}}\right)\mathbf{1}_{|v|>N}\leq\frac{C}{N}||h||_\infty^2.\end{aligned}$$

Hence we have

$$(42)\mathbf{1}_{|v|\geq N}\leq\frac{C}{N}||h||_\infty^2. \tag{44}$$

Case 2. $|v|\leq N$ and $|v'|\geq 2N$, or $|v|\leq N$ and $|v'|\leq\frac{1}{M}$. Also assume $0<\delta<<1$:

$$\begin{aligned}&2\times\mathbf{1}_{|v|\leq N}\int_{\{|v'|\geq 2N\text{ or }|v'|\leq\frac{1}{M}\}}(\mathcal{C})\int_{E_{\bar{v}v''}}(\mathcal{D})\,dv_1''dv'\\&\quad\leq C\mathbf{1}_{|v|\leq N}\int_{|v'|\geq 2N}\left\{\frac{1}{|v-v'|^2}+\frac{1}{|v-v'|^{2-\gamma}}\right\}e^{-\frac{|v'|^2}{8}}dv'e^{\frac{\delta^2}{4}}||h||_\infty^2\\&\qquad+\underbrace{C\int_{|v'|\leq\frac{1}{M}}\left\{\frac{1}{|v'|^2}+\frac{1}{|v'|^{2-\gamma}}\right\}e^{-\frac{|v'|^2}{8}}dv'\,e^{\frac{\delta^2}{4}}}_{o(\frac{1}{M})}||h||_\infty^2\\&\quad\leq C\left(\frac{1}{N^2}+\frac{1}{N^{2-\gamma}}\right)||h||_\infty^2+o(\frac{1}{M})||h||_\infty^2,\end{aligned} \tag{45}$$

where we have used $\bar{w}(v')\leq e^{-\frac{|v'|^2}{4}}$ and $\bar{w}(v'')\leq e^{-\frac{|v'|^2}{8}}e^{\frac{\delta^2}{4}}$ and Lemma 6.

Case 3. $|v|\leq N$ and $\frac{1}{M}\leq|v'|\leq 2N$.

$$\begin{aligned}&2\times\mathbf{1}_{|v|\leq N}\int_{\frac{1}{M}\leq|v'|\leq 2N}(\mathcal{C})\int_{E_{\bar{v}v''}}(\mathcal{D})dv_1''dv'\\&\quad\leq C||h||_\infty\int_{\frac{1}{M}\leq|v'|\leq 2N}\mathbf{1}_{|v|\leq N}\left(\frac{1}{|v-v'|^2}+\frac{1}{|v-v'|^{2-\gamma}}\right)|\bar{w}(v'')h(\bar{t},\bar{x},v'')\\&\qquad-\bar{w}(v')h(t,x,v')|dv'.\end{aligned} \tag{46}$$

Since $\left(\frac{1}{|v-v'|^2}+\frac{1}{|v-v'|^{2-\gamma}}\right)$ is integrable we can choose a smooth function $\mathbf{z}(v,v')$ with compact support such that

$$\sup_{|v|\le N}\int_{|v'|\le 2N}\left|\left(\frac{1}{|v-v'|^2}+\frac{1}{|v-v'|^{2-\gamma}}\right)-\mathbf{z}(v,v')\right|dv'\le\frac{1}{N}. \tag{47}$$

Therefore we can bound (46) by two parts

$$C||h||^2_{L^\infty}\int_{|v'|\le 2N}\mathbf{1}_{|v|\le N}\left|\left(\frac{1}{|v-v'|^2}+\frac{1}{|v-v'|^{2-\gamma}}\right)-\mathbf{z}(v,v')\right|e^{-\frac{|v'|^2}{8}}e^{\frac{\delta^2}{4}}dv' \tag{48}$$

$$+C\sup_{|v|\le N,|v'|\le 2N}|\mathbf{z}(v,v')|\times||h||_{L^\infty}\int_{\frac{1}{M}\le|v'|\le 2N}\mathbf{1}_{|v|\le N}|\bar{w}(v''(v'))h(\bar{t},\bar{x},v''(v'))$$
$$-\bar{w}(v')h(t,x,v')|dv'. \tag{49}$$

From (47), it is easy to control the first term

$$|(48)|\le\frac{C}{N}||h||^2_\infty. \tag{50}$$

Now we are going to estimate the second term (49). Applying Lemma 5 to $\bar{w}(v')h(t,x,v')$, we can choose $\delta=\delta(\bar{w}h,\Omega,\varepsilon,N_*,x,\frac{1}{M},2N)>0$ and an open set $U_x\subset\{\frac{1}{M}\le|v|\le 2N\}$ with $|U_x|<\frac{\varepsilon}{N_*}$ such that

$$|\bar{w}(v''(v'))h(\bar{t},\bar{x},v''(v'))-\bar{w}(v')h(t,x,v')|<\frac{\varepsilon}{N_*},$$

for $v'\in\{v\in\mathbb{R}^3:\frac{1}{M}\le|v|\le N\}\backslash U_x$ and $|(\bar{t},\bar{x},\bar{v})-(t,x,v)|<\delta$. Therefore we can split the second part (49) as an integration over U_x and U_x^c and control it as

$$C\sup_{|v|\le N,|v'|\le 2N}|\mathbf{z}(v,v')|\times||h||^2_\infty\times m_3(U_x)+C||h||_\infty$$
$$\times\int_{\{\frac{1}{M}\le|v'|\le 2N\}\cap U_x^c}|\bar{w}(v''(v'))h(\bar{t},\bar{x},v''(v'))-\bar{w}(v')h(t,x,v')|dv'$$
$$\le C\sup_{|v|\le N,|v'|\le 2N}|\mathbf{z}(v,v')|\times||h||^2_\infty\frac{\varepsilon}{N_*}+C||h||_\infty N^3\frac{\varepsilon}{N_*}. \tag{51}$$

In summary, combining (44), (45), (50) and (51), we have established

$$(42)\le C||h||^2_\infty\left\{\frac{1}{N}+o(\frac{1}{M})+\sup_{|v|\le N,|v'|\le 2N}|\mathbf{z}(v,v')|\frac{\varepsilon}{N_*}\right\}+C||h||_\infty N^3\frac{\varepsilon}{N_*}.$$

Choosing sufficiently large $N,M>0$ and $N_*>0$, then

$$(42)\le\frac{\varepsilon}{2}. \tag{52}$$

3.3. Estimate of (43). The estimate of (43) is much more delicate. The reason is that we cannot expect $\int_{E_{vv'}}(\mathcal{F})\,dv_1'$ in (43) is small for all $v'\in\mathbb{R}^3$. We know that $h(t,x,v_1')$ may not be continuous on $v_1'\in\mathfrak{G}_x$. Even $\mathfrak{G}_x$ is radial symmetric and has a small measure by Lemma 3, a bad situation the intersection of $\mathfrak{G}_x$ and $E_{vv'}$ could have large (even infinite) 2-dimensional Lebesgue measure, can happen. However we can show that such bad situations only happen for very rare v''s in $\mathbb{R}^3$. Using the integration over $v'\in\mathbb{R}^3$, we are able to control (43) small.

Recall $(\mathcal{E})$ and $(\mathcal{F})$ in (43). We divide into several cases:

Case 1. $|v|\geq N$. Follow exactly the same proof of Case 1 of the previous subsection, we conclude

$$(43)\mathbf{1}_{|v|\geq N}\leq \frac{C}{N}||h||_\infty^2. \tag{53}$$

Case 2. $|v|\leq N$ and $|v'|\geq 2N$. We go back to the original formula, the second term of (39), and use Lemma 6 to estimate

$$\begin{aligned}
&2\int_{|v'|\geq 2N}(\mathcal{E})\int_{E_{vv'}}(\mathcal{F})dv_1'dv'\mathbf{1}_{|v|\leq N}\\
&\leq 4||h||_\infty^2\int_{|v'|\geq 2N}\bar{w}(v')\frac{1}{|v-v'|^2}(1+|v-v'|)^\gamma dv'\mathbf{1}_{|v|\leq N}\\
&\leq 4||h||_\infty^2\left(\frac{1}{N^2}+\frac{1}{N^{2-\gamma}}\right).
\end{aligned}\tag{54}$$

Case 3. $|v|\leq N$, $|v'|\leq 2N$, and $|v_1'|\leq\frac{1}{N}$ or $|v_1'|\geq N$. In the case of $|v_1'|\leq\frac{1}{N}$, we have

$$\begin{aligned}
&2\times\mathbf{1}_{|v|\leq N}\int_{|v'|\geq 2N}(\mathcal{E})\int_{\{|v_1'|\leq\frac{1}{N}\}\cap E_{vv'}}(\mathcal{F})\,dv_1'dv'\\
&\leq 2||h||_\infty^2\int_{\mathbb{R}^3}\frac{\bar{w}(v')}{|v-v'|^2}dv'\int_{\{|v_1'|\leq\frac{1}{N}\}\cap E_{vv'}}\\
&\quad\times\left\{e^{-\frac{|v_1'|^2}{8}}e^{\frac{\delta^2}{4}}(4N+\frac{1}{N}+\delta)^\gamma+e^{-\frac{|v_1'|^2}{4}}(4N+\frac{1}{N})^\gamma\right\}dv_1'\\
&\leq C\frac{||h||_\infty^2}{N^{2-\gamma}}.
\end{aligned}\tag{55}$$

In the case of $|v_1'|\geq N$ we have

$$\begin{aligned}
&2\times\mathbf{1}_{|v|\leq N}\int_{|v'|\geq 2N}(\mathcal{E})\int_{\{|v_1'|\leq\frac{1}{N}\}\cap E_{vv'}}(\mathcal{F})\,dv_1'dv'\\
&\leq 2||h||_\infty^2\int_{\mathbb{R}^3}\frac{\bar{w}(v')}{|v-v'|^2}dv'\int_{\{|v_1'|\geq N\}\cap E_{vv'}}\\
&\quad\times\left\{e^{-\frac{|v_1'|^2}{8}}e^{\frac{\delta^2}{4}}(4N+\frac{1}{N}+\delta)^\gamma+e^{-\frac{|v_1'|^2}{4}}(4N+\frac{1}{N})^\gamma\right\}dv_1'\\
&\leq C||h||_\infty^2e^{-\frac{N^2}{16}}\int_{\mathbb{R}^3}e^{-\frac{|v_1'|^2}{16}}dv'\times N^\gamma e^{-\frac{N^2}{16}}\leq C||h||_\infty^2e^{-\frac{N^2}{16}}.
\end{aligned}\tag{56}$$

Case 4. $|v| \le N$, $|v'| \le 2N$, and $\frac{1}{N} \le |v_1'| \le N$. In order to remove the unboundedness of $\frac{1}{|v-v'|^2}$ in (43), we choose a positive smooth function $\mathbf{Z}(v, v')$ with compact support such that

$$\sup_{|v| \le N} \int_{|v'| \le 2N} \left| \frac{1}{|v-v'|^2} - \mathbf{Z}(v, v') \right| dv' < \frac{1}{N^{10}}. \tag{57}$$

Splitting $2 \times \mathbf{1}_{|v| \le N} \int_{|v'| \le 2N} (\mathcal{E}) \int_{\frac{1}{N} \le |v_1'| \le N} (\mathcal{F}) \, dv_1' dv'$ into two parts

$$2 \times \mathbf{1}_{|v| \le N} \int_{|v'| \le 2N} \bar{w}(v') |h(t, x, v')| \left| \frac{1}{|v-v'|^2} - \mathbf{Z}(v, v') \right| \times \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\}} (\mathcal{F}) \, dv_1 dv' \le C ||h||_\infty^2 \frac{1}{N^{10}} N^{\gamma+2}, \tag{58}$$

$$C \int_{|v'| \le 2N} ||h||_\infty \sup_{|v| \le N, |v'| \le 2N} |\mathbf{Z}(v, v')| \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\}} (\mathcal{F}) \, dv_1' dv', \tag{59}$$

where we used (57) for the first line. From now we will focus on estimate (59).

Case 5. $|v| \le N$, $|v'| \le 2N$, $\frac{1}{N} \le |v_1'| \le N$ and $|2v - v' - v_1'| < \frac{1}{M}$ or $|v' - v_1'| < \frac{1}{M}$. This region includes the part where the collision kernel $B(\cdot, \cdot)$ has a singular behavior.

$$\begin{aligned}
&C \int_{|v'| \le 2N} ||h||_\infty \sup_{|v| \le N, |v'| \le 2N} |\mathbf{Z}(v, v')| \\
&\quad \times \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\}} (\mathcal{F}) \, \mathbf{1}_{\{|(2v-v')-v_1'| < \frac{1}{M} \text{ or } |v'-v_1'| < \frac{1}{M}\}}(v', v_1') \, dv_1' dv' \\
&\le C \sup_{|v| \le N, |v'| \le 2N} |\mathbf{Z}(v, v')| \times ||h||_\infty^2 \\
&\quad \times \int_{|v'| \le 2N} dv' e^{-\frac{|v'|^2}{4}} \int_{E_{vv'}} dv_1' \left\{ \mathbf{1}_{\{|(2v-v')-v_1'| < \frac{1}{M}\}}(v_1') + \mathbf{1}_{\{|v'-v_1'| < \frac{1}{M}\}}(v_1') \right\} \times N^\gamma \\
&\le C \sup_{|v| \le N, |v'| \le 2N} |\mathbf{Z}(v, v')| \times ||h||_\infty^2 \frac{N^\gamma}{M^2}.
\end{aligned} \tag{60}$$

Case 6. $|v| \le N$, $|v'| \le 2N$, $\frac{1}{N} \le |v_1'| \le N$ and $|2v - v' - v_1'| > \frac{1}{M}$ and $|v' - v_1'| > \frac{1}{M}$ and $0 < \delta < \frac{1}{10M}$. We estimate

$$\begin{aligned}
&2 \times \mathbf{1}_{|v| \le N} \int_{|v'| \le 2N} dv' \bar{w}(v') h(t, x, v') \mathbf{Z}(v, v') \\
&\times \int_{E_{vv'} \cap \{\frac{1}{N} \le |v_1'| \le N\}} \{ \bar{w}(v_1'') h(\bar{t}, \bar{x}, v_1'') B(2\bar{v} - v'' - v_1'', \frac{v'' - v_1''}{|v'' - v_1''|}) \\
&\times - \bar{w}(v_1') h(t, x, v_1') B(2v - v' - v_1', \frac{v' - v_1'}{|v' - v_1'|}) \} \mathbf{1}_{\{|2v - v' - v_1'| > \frac{1}{M}\}} \mathbf{1}_{\{|v' - v_1'| > \frac{1}{M}\}} dv_1'.
\end{aligned} \tag{61}$$

We need this step because of the singular behavior of

$$B(u_1, u_2) = |u_1|^\gamma q_0 (\frac{u_1}{|u_1|} \cdot \frac{u_2}{|u_2|}) = |u_1|^\gamma (q_0 \circ \mathfrak{F})(u_1, u_2),$$

where $\mathfrak{F}:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ with $\mathfrak{F}(u_1,u_2)=\frac{u_1}{|u_1|}\cdot\frac{u_2}{|u_2|}$. The function $\mathfrak{F}(u_1,u_2)$ is not continuous at $(u_1,u_2)=(0,0)$ and continuous away from $(0,0)$, i.e. the restriction of $\mathfrak{F}$ on a compact set,

$$\mathfrak{F}_{M,N}:\{\frac{1}{2M}\le|u_1|\le 6N\}\times\{\frac{1}{2M}\le|u_2|\le 4N\}\to\mathbb{R}$$

is uniformly continuous. From $|2v-v'-v_1'|>\frac{1}{M}$ and $|v-\bar{v}|<\delta<\frac{1}{10M}$ we have a lower bound of

$$|2\bar{v}-v''-v_1''|\ge\left||2v-v'-v_1'|-|\bar{v}-v-\frac{v'-v}{|v'-v|}\{(\bar{v}-v)\cdot\frac{v'-v}{|v'-v|}\}|\right|\ge\frac{1}{2M}.$$

Similarly from $|v'-v_1'|>\frac{1}{M}$ and $|v-\bar{v}|<\delta<\frac{1}{10M}$ we have a lower bound of

$$|v''-v_1''|\ge\left||v'-v_1'|-|\bar{v}-v-\frac{v'-v}{|v'-v|}\{(\bar{v}-v)\cdot\frac{v'-v}{|v'-v|}\}|\right|\ge\frac{1}{2M}.$$

Therefore for any $\varepsilon>0$, we can choose $\delta>0$ so that

$$\begin{aligned}&\left|B(2\bar{v}-v''-v_1'',\frac{v''-v_1''}{|v''-v_1''|})-B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|})\right|\\&=\Big||2\bar{v}-v''-v_1''|^{\gamma}(q_0\circ\mathfrak{F})(2\bar{v}-v''-v_1'',v''-v_1'')\\&\quad-|2v-v'-v_1'|^{\gamma}(q_0\circ\mathfrak{F})(2v-v'-v_1',v'-v_1')\Big|<\frac{\varepsilon}{N_*},\end{aligned}\tag{62}$$

for $|2v-v'-v_1'|>\frac{1}{M}$ and $|v'-v_1'|>\frac{1}{M}$ and $0<\delta<\frac{1}{10M}$.

Now we split (61) into two parts

$$\begin{aligned}&2\times\mathbf{1}_{|v|\le N}\int_{|v'|\le 2N}dv'..\int_{E_{vv'}\cap\{\frac{1}{N}\le|v_1'|\le N\}}\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'')\\&\times\left\{B(2\bar{v}-v''-v_1'',\frac{v''-v_1''}{|v''-v_1''|})-B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|})\right\}\\&\times\mathbf{1}_{\{|2v-v'-v_1'|>\frac{1}{M}\}}\mathbf{1}_{\{|v'-v_1'|>\frac{1}{M}\}}dv_1'\\&+2\times\mathbf{1}_{|v|\le N}\int_{|v'|\le 2N}dv'..\int_{E_{vv'}\cap\{\frac{1}{N}\le|v_1'|\le N\}}\left\{\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'')-\bar{w}(v_1')h(t,x,v_1')\right\}\\&\times B(2v-v'-v_1',\frac{v'-v_1'}{|v'-v_1'|}).\end{aligned}\tag{63}$$

Using (62), the continuity of $B(\cdot,\cdot)$ away from $(0,0)$, the first line above is bounded by

$$C\sup_{v,v'}|\mathbf{Z}(v,v')|\times||h||_{\infty}^2\frac{\varepsilon}{N_*}.\tag{64}$$

In the remainder of this section we will focus on (63):

Estimate of (63).

$$(63) \leq CN^2||h||_\infty \sup_{v,v'} |\mathbf{Z}(v,v')| \int_{|v'|\leq 2N} \bar{w}(v') \times \underbrace{\int_{E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} |\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'') - \bar{w}(v_1')h(t,x,v_1')| dv_1'\, dv'}_{\odot}, \quad (65)$$

where we used $\sup_{|v|\leq N,|v'|\leq 2N,|v_1'|\leq N} B(2v - v' - v_1', \frac{v'-v_1'}{|v'-v_1'|}) < \infty$. Recall our choice of v'' and v_1'' from (40) and (41) to have

$$|v_1'' - v_1'| \leq \left| \frac{v'-v}{|v'-v|}\{(\bar{v}-v)\cdot\frac{v'-v}{|v'-v|}\}\right| \leq |\bar{v}-v| < \delta.$$

We will use the following strategy: separate $\int_{E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} \ldots dv_1'$ into two parts

$$\int_{U_x\cap E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} \ldots dv_1' + \int_{U_x^c\cap E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} \ldots dv_1'.$$

The first part is the integration over U_x, a neighborhood of $\mathfrak{G}_x$ that contains possible discontinuity of h. Moreover we expect the measure of the neighborhood U_x is small so we can control the first term. For the second term, we will use the continuity of the integrand $\bar{w}h$. However if $v = 0$ then $\mathfrak{G}_x$ could be a large measure set in $E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\}$. For example if $\mathfrak{G}_x \cap \mathbb{S}^2 = \{u \in \mathbb{S}^2 : u_3 = 0\}$ then $\mathfrak{G}_x$ is the xy−plane and $E_{0\mathbf{e}_3}$ is also the xy−plane. Therefore we have to divide the two cases $v \neq 0$ and $v = 0$ and study them separately.

Case of $v \neq 0$. In the case of $v \neq 0$, assume $\varrho < |v|^2/2$ for sufficiently small $\varrho > 0$. We will divide the velocity space $\mathbb{R}^3$ into

$$\mathfrak{B} = \left\{ v' \in \mathbb{R}^3 : |v| - \frac{\varrho}{|v|} \leq v' \cdot \frac{v}{|v|} \leq |v| + \frac{\varrho}{|v|} \right\} \text{ and}$$
$$\mathfrak{B}^c = \left\{ v' \in \mathbb{R}^3 : \left| v' \cdot \frac{v}{|v|} - |v| \right| > \frac{\varrho}{|v|} \right\}.$$

The important property of $\mathfrak{B}$ is that if $v \in \mathfrak{B}^c$ then $E_{vv'}$ does not contain zero. We can split the underbraced integration $\odot$ of (65) into

$$\int_{v'\in B_{2N}\cap\mathfrak{B}} \bar{w}(v') \int_{E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} |\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'') - \bar{w}(v_1')h(t,x,v_1')| dv_1' dv' \quad (66)$$

$$+ \int_{v'\in B_{2N}\setminus\mathfrak{B}} \bar{w}(v') \int_{E_{vv'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} |\bar{w}(v_1'')h(\bar{t},\bar{x},v_1'') - \bar{w}(v_1')h(t,x,v_1')| dv_1' dv'. \quad (67)$$

Notice that $\mathfrak{B} \cap B_{2N}$ has a small measure:

$$m_3(\mathfrak{B} \cap B_{2N}) \leq 2\pi(2N)^2 \times 2\frac{\varrho}{|v|} \leq 2\pi(2N)^2 \times 2\frac{\varrho}{\sqrt{2\varrho}} \leq 2\sqrt{2}\pi(2N)^2\sqrt{\varrho}.$$

Therefore we have

$$|(66)| \leq CN^4||h||_{L^\infty}\sqrt{\varrho}. \tag{68}$$

Now we are going to estimate (67). Here we use a property of $\mathfrak{B}^c$: for $v' \in \mathfrak{B}^c$ we have

$$\text{dist}(0, E_{vv'}) = \left| v \cdot \frac{v'-v}{|v'-v|} \right| = \frac{|v' \cdot v - |v|^2|}{|v'-v|} > \frac{\varrho}{|v'-v|} > \frac{\varrho}{2N+|v|} \geq \frac{\varrho}{3N},$$

where we also have used $|v'| \leq 2N$ and $|v| \leq N$. From Lemma 5 we use U_x, an open radial symmetric subset of $\{\frac{1}{N} \leq |v_1'| \leq N\}$ with a small measure and $\bar{w}h$ is uniformly continuous on U_x^c, to split (67) into

$$\int_{v' \in B_{2N}\backslash\mathfrak{B}} \bar{w}(v') \int_{E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x} |\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| dv_1' dv' \tag{69}$$

$$+ \int_{v' \in B_{2N}\backslash\mathfrak{B}} \bar{w}(v') \int_{E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x^c} |\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| dv_1' dv'. \tag{70}$$

For the last line, we use Lemma 5 to know estimate $|\bar{w}(v_1'')h(\bar{t}, \bar{x}, v_1'') - \bar{w}(v_1')h(t, x, v_1')| < \frac{\varepsilon}{N_*}$, for $v_1' \in E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\}\backslash U_x$ and $|v_1'' - v_1'| \leq |v - \bar{v}| < \delta$. Therefore

$$|(70)| \leq CN^2\frac{\varepsilon}{N_*}||h||_\infty. \tag{71}$$

In order to show that (69) is small, we introduce following projection:

Lemma 10. *Assume* $0 < \varrho < \frac{|v|^2}{2}$. *Let* $E_{vv'} = \{v_1' \in \mathbb{R}^3 : (v_1 - v) \cdot (v' - v) = 0\}$. *We define a projection*

$$\begin{aligned} \mathbb{P}\ \mathbb{S}^2 &\to E_{vv'}, \\ u \in \mathbb{S}^2 &\mapsto \left\{ \frac{v \cdot (v'-v)}{u \cdot (v'-v)} \right\} u \ \in\ E_{vv'}. \end{aligned}$$

For $v' \in \{v' \in \mathbb{R}^3 : |v'| \leq 2N\}\backslash\mathfrak{B}$, *define the restricted projection*

$$\begin{aligned} \mathbb{P}' \equiv \mathbb{P}|_{\mathbb{P}^{-1}(E_{vv'} \cap \{1/N \leq |v_1'| \leq N\})}\ &\mathbb{P}^{-1}(E_{vv'} \cap \{1/N \leq |v_1'| \leq N\}) \\ &\to E_{vv'} \cap \{1/N \leq |v_1'| \leq N\}. \end{aligned}$$

Then for $v' \in B_{2N}\backslash\mathfrak{B}$ *the Jacobian of* $\mathbb{P}'$ *is bounded:*

$$Jac(\mathbb{P}') = \left| \frac{\partial \mathbb{P}'}{\partial u} \right| = \left(v \cdot \frac{v'-v}{|v'-v|} \right)^2 |\sec^2\theta \tan\theta| \leq \frac{3N^4}{\varrho},$$

where θ *is defined by* $\cos\theta = u \cdot \frac{v'-v}{|v'-v|}$ *(Fig. 4).*

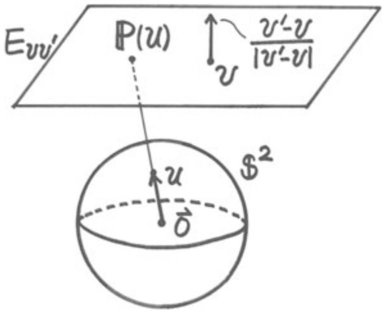

Fig. 4. Projection Map

Proof. Without loss of generality, we may assume $\frac{v'-v}{|v'-v|} = (0, 0, 1)^T$. Using the spherical coordinate,

$$\mathbb{P}'(u) = \frac{v \cdot (v'-v)}{u \cdot (v'-v)} u = \frac{v \cdot \frac{v'-v}{|v'-v|}}{u \cdot \frac{v'-v}{|v'-v|}} u = \frac{v \cdot \frac{v'-v}{|v'-v|}}{\cos\theta} \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$$= v \cdot \frac{v'-v}{|v'-v|} \begin{pmatrix} \tan\theta\cos\phi \\ \tan\theta\sin\phi \\ 1 \end{pmatrix},$$

and a Jacobian matrix of $\mathbb{P}'$,

$$\frac{\partial \mathbb{P}'}{\partial(\theta,\phi)} = v \cdot \frac{v'-v}{|v'-v|} \begin{pmatrix} \sec^2\theta\cos\phi & -\tan\theta\sin\phi \\ \sec^2\theta\sin\phi & \tan\theta\cos\phi \end{pmatrix}.$$

Therefore a Jacobian of $\mathbb{P}'$ is

$$Jac(\mathbb{P}') = \left| \frac{\partial \mathbb{P}'}{\partial(\theta,\phi)} \right| = \left(v \cdot \frac{v'-v}{|v'-v|} \right)^2 \sec^2\theta |\tan\theta| \leq \text{dist}(0, E_{vv'})^2 |\sec\theta|^3.$$

Notice that

$$|\sec\theta| = \frac{1}{|\cos\theta|} = \frac{1}{\left| u \cdot \frac{v'-v}{|v'-v|} \right|} = \left| \left\{ \frac{v \cdot (v'-v)}{u \cdot (v'-v)} \right\} u \right| \frac{1}{\left| v \cdot \frac{v'-v}{|v'-v|} \right|} = \frac{|\mathbb{P}'(u)|}{\text{dist}(0, E_{vv'})}.$$

Because $\mathbb{P}'(u) \in \{\frac{1}{N} \leq |v_1'| \leq N\}$ and $\text{dist}(0, E_{vv'}) \geq \frac{\varrho}{3N}$ we have

$$Jac(\mathbb{P}') \leq \frac{|\mathbb{P}'(u)|^3}{|\text{dist}(0, E_{vv'})|} \leq \frac{3N^4}{\varrho}.$$

□

Assume we choose $m_2(U_x \cap \mathbb{S}^2) \leq \frac{\varrho\varepsilon}{N_* N^2}$. By definition we know that $\mathbb{P}'(U_x \cap \mathbb{S}^2) = E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x$ and the 2-dimension Lebesgue measure of $E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x$ is bounded by

$$m_2(E_{vv'} \cap \{\frac{1}{N} \leq |v_1'| \leq N\} \cap U_x) = m_2(\mathbb{P}'(U_x \cap \mathbb{S}^2)) \leq Jac(\mathbb{P}') \times |U_x \cap \mathbb{S}^2|$$

$$\leq \frac{3N^4}{\varrho} \times \frac{\varepsilon}{N_* N^2} = \frac{3N^2}{\varrho N_*}\varepsilon.$$

Therefore we have an upper bound of (69):

$$|(69)| \leq CN^2\varepsilon ||h||_\infty, \tag{72}$$

where $C = \int_{\mathbb{R}^3} \bar{w}(v')dv'$. In case of $v \neq 0$, from (68), (71) and (72), we have

$$\begin{aligned}(63) &\leq CN^2 ||h||_\infty \sup_{v,v'} |\mathbf{Z}(v,v')| \times \odot \\ &\leq CN^4 ||h||_\infty^2 \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')| \{N^2\sqrt{\varrho} + (1+\frac{3N^2}{\varrho})\frac{\varepsilon}{N_*}\},\end{aligned} \tag{73}$$

where $\odot$ is the underlined integration in (65).

Case of $v = 0$. In this case, we do not have a upper bound of the Jacobian of $\mathbb{P}'$. Instead we will use the structure of $\mathfrak{G}_x$ of Lemma 4 crucially. In the case of $v = 0$, we split (65)

$$\begin{aligned}&\int_{|v'|\leq 2N} \bar{w}(v') \int_{E_{0v'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} |\bar{w}(v_1'')h(\bar{t},\bar{x},v_1') - \bar{w}(v_1')h(t,x,v_1')| dv_1' dv' \\ &= \int_{|v'|\leq 2N} \bar{w}(v') \int_{E_{0v'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} |\bar{w}(v_1'')h(\bar{t},\bar{x},v_1') - \bar{w}(v_1')h(t,x,v_1')| \times \mathbf{1}_{U_x}(v_1') dv_1' dv' \end{aligned} \tag{74}$$

$$\begin{aligned}&+ \int_{|v'|\leq 2N} \bar{w}(v') \int_{E_{0v'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}} |\bar{w}(v_1'')h(\bar{t},\bar{x},v_1') - \bar{w}(v_1')h(t,x,v_1')| \\ &\times \mathbf{1}_{E_{0v'}\cap\{\frac{1}{N}\leq|v_1'|\leq N\}\backslash U_x}(v_1') dv_1' dv'.\end{aligned} \tag{75}$$

For v', we use spherical polar coordinates (r', θ', ϕ') so that

$$v' = (r'\sin\theta'\cos\phi', r'\sin\theta'\sin\phi', r'\cos\theta'). \tag{76}$$

By definition, $E_{0v'}$ is a plane containing the origin and normal to v'. We know that $E_{0v'}$ is generated by two unit vectors

$$E_{0v'} = \left\langle \begin{pmatrix} \cos\theta'\cos\phi' \\ \cos\theta'\sin\phi' \\ -\sin\theta' \end{pmatrix}, \begin{pmatrix} -\sin\phi' \\ \cos\phi' \\ 0 \end{pmatrix} \right\rangle.$$

We will use a polar coordinate (r_1', θ_1') for $v_1' \in E_{0v'}$, i.e.

$$v_1' = \begin{pmatrix} (v_1')_1 \\ (v_1')_2 \\ (v_1')_3 \end{pmatrix} (r_1', \theta_1'; \theta', \phi') \equiv r_1' \begin{pmatrix} \cos\theta'\cos\phi' & -\sin\phi' & \sin\theta'\cos\phi' \\ \cos\theta'\sin\phi' & \cos\phi' & \sin\theta'\sin\phi' \\ -\sin\theta' & 0 & \cos\theta' \end{pmatrix} \begin{pmatrix} \cos\theta_1' \\ \sin\theta_1' \\ 0 \end{pmatrix}. \tag{77}$$

Direct computation gives $\det\left(\frac{\partial(v_1')}{\partial(r_1',\theta_1',\theta')}\right) =$

$$\begin{aligned}&(r_1')^2\cos\theta_1' \\ &\times \det\begin{pmatrix} \cos\theta'\cos\phi'\cos\theta_1' - \sin\phi'\sin\theta\theta_1' & -\cos\theta'\cos\phi'\sin\theta_1' - \sin\phi'\cos\theta_1' & \sin\theta'\cos\phi' \\ \cos\theta'\sin\phi'\cos\theta_1' + \cos\phi'\sin\theta_1' & -\cos\theta'\sin\phi'\sin\theta_1' + \cos\phi'\cos\theta_1' & \sin\theta'\sin\phi' \\ \sin\theta'\cos\theta_1' & \sin\theta'\sin\theta_1' & \cos\theta' \end{pmatrix} \\ &= (r_1')^2\cos\theta_1'.\end{aligned}$$

Therefore we have following identity:

$$\int_{\mathbb{R}^3} \cdots dv_1' = \int_0^\infty \int_0^{2\pi} \int_0^\pi \cdots (r_1')^2 \cos\theta_1' d\theta' d\theta_1' dr_1'. \tag{78}$$

Recall the standard 3-dimensional polar coordinates and 2-dimensional polar coordinates:

$$\int_{|v'|\le 2N} \cdots dv' = \int_0^{2N} \int_0^{2\pi} \int_0^\pi \cdots (r')^2 \sin\theta' d\theta' d\phi' dr',$$

$$\int_{E_{0v'}\cap\{\frac{1}{N}\le|v_1'|\le N\}} \cdots dv_1' = \int_{\frac{1}{N}}^{N} \int_0^{2\pi} \cdots r_1' d\theta_1' dr_1',$$

and use the above identities to control (74) by

$$\begin{aligned}&\int_0^{2N} dr'(r')^2 \bar{w}(r') \int_0^{2\pi} d\phi' \\ &\times \underbrace{\int_0^\pi d\theta' \sin\theta' \int_{\frac{1}{N}}^N dr_1' r_1' e^{-\frac{(r_1')^2}{8}} \int_0^{2\pi} d\theta_1' \mathbf{1}_{U_x}(v_1'(r_1',\theta_1';\theta',\phi'))}\, ||h||_\infty.\end{aligned} \tag{79}$$

We focus on the underbraced integration in (79) and divide it into

$$\int_0^\pi d\theta' \sin\theta' \int_{\frac{1}{N}}^N dr_1' r_1' e^{-\frac{(r_1')^2}{8}} \int_0^{2\pi} d\theta_1' \mathbf{1}_{U_x}(v_1') \mathbf{1}_{\theta_1'\in(\frac{\pi}{2}-\varrho,\frac{\pi}{2}+\rho)\cup(\frac{3\pi}{2}-\varrho,\frac{3\pi}{2}+\varrho)} \tag{80}$$

$$+\int_0^\pi d\theta' \sin\theta' \int_{\frac{1}{N}}^N dr_1' r_1' e^{-\frac{(r_1')^2}{8}} \int_0^{2\pi} d\theta_1' \mathbf{1}_{U_x}(v_1') \mathbf{1}_{\theta_1'\in[0,\frac{\pi}{2}-\varrho]\cup[\frac{\pi}{2}+\varrho,\frac{3\pi}{2}-\varrho]\cup[\frac{3\pi}{2}+\varrho,2\pi]}. \tag{81}$$

Easily (80) $\le 2\varrho(e^{-\frac{1}{8N^2}} - e^{-\frac{N^2}{8}}) \le 4\varrho$. For (81), we use $1 \le \frac{\cos\theta_1'}{\varrho}$ and $1 \le N r_1'$ on $\theta_1' \in [0, \frac{\pi}{2} - \varrho] \cup [\frac{\pi}{2} + \varrho, \frac{3\pi}{2} - \varrho] \cup [\frac{3\pi}{2} + \varrho, 2\pi]$ and $r_1' \in [\frac{1}{N}, N]$ to have

$$\begin{aligned}(81) &\le \varrho^{-1} N \int_0^\pi d\theta' \int_{\frac{1}{N}}^N dr_1' (r_1')^2 \int_0^{2\pi} d\theta_1' \cos\theta_1' \mathbf{1}_{U_x}(v_1'(r_1',\theta_1';\theta',\phi')) \\ &= \varrho^{-1} N \times m_3(U_x \cap \{\frac{1}{N} \le |v_1'| \le N\}),\end{aligned} \tag{82}$$

where we used (78). To sum we have

$$(74) \le (79) \le C||h||_\infty \left\{4\varrho + \varrho^{-1} N \times \frac{\varepsilon}{N_*}\right\}. \tag{83}$$

On the other hand for (75) we can use Lemma 5 to have

$$(75) \le C\frac{\varepsilon}{N_*}. \tag{84}$$

From (83) and (84) we have

$$(63) \leq CN^2||h||_\infty \sup_{v,v'} |\mathbf{Z}(v,v')| \times \bigodot = CN^2||h||_\infty \sup_{v,v'} |\mathbf{Z}(v,v')| \times \{(74)+(75)\}$$
$$\leq CN^2||h||_\infty \sup_{v,v'} |\mathbf{Z}(v,v')| \left\{ \frac{\varepsilon}{N_*} + ||h||_\infty \Big\{ 4\varrho + \varrho^{-1} N \times \frac{\varepsilon}{N_*} \Big\} \right\}, \tag{85}$$

where $\bigodot$ is the underbraced integration in (65).

To summarize, from (53), (54), (55), (56), (58), (60), (64), (73) and (85), we have established

$$(43) \leq C||h||_\infty^2 \{ \frac{1}{N} + e^{-\frac{N^2}{16}} \} + C||h||_\infty^2 \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')| \frac{N^\gamma}{M^2}$$
$$+ C||h||_\infty^2 \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')| (N^6\sqrt{\varrho} + 4N^2\varrho)$$
$$+ \frac{\varepsilon}{N_*} C||h||_\infty \sup_{|v|\leq N, |v'|\leq 2N} |\mathbf{Z}(v,v')| \left\{ N^2 + ||h||_\infty \Big(1 + \frac{3N^6}{\varrho} + N^4 + N^3 \varrho \Big) \right\}. \tag{86}$$

We choose $N, M, N_* > 0$ sufficiently large and $\varrho > 0$ sufficiently small so that we can control (43) $< \frac{\varepsilon}{2}$. Combining with the result of the previous subsection (52), we conclude (38) and and prove Theorem 4.

3.4. Continuity of collision operators Kf and $\Gamma(f,f)$. The following is a consequence of Theorem 4.

Corollary 5. *Assume $f(t,x,v)$ is continuous on $[0,T] \times (\bar{\Omega} \times \mathbb{R}^3) \backslash \mathfrak{G}$ and*

$$w(v) f(t,x,v) = (1+\rho^2|v|^2)^\beta f(t,x,v) \in L^\infty([0,T] \times (\bar{\Omega} \times \mathbb{R}^3)).$$

Then $Kf(t,x,v)$ and $\Gamma_+(f,f)(t,x,v)$ are continuous in $[0,T] \times \Omega \times \mathbb{R}^3$ and

$$\sup_{[0,T]\times\bar{\Omega}\times\mathbb{R}^3} |\nu^{-1}(v) w(v) K(f)| < \infty, \qquad \sup_{[0,T]\times\bar{\Omega}\times\mathbb{R}^3} |\nu^{-1}(v) w(v) \Gamma_+(f,f)| < \infty.$$

Proof. The above boundedness is a direct consequence of (35) and (36). Thanks to Theorem 4, we already established the continuity of Γ_+. Therefore we only need to show the continuity of

$$\frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \mu) = e^{-\frac{|v|^2}{4}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\omega) f(t,x,u) e^{-\frac{|u|^2}{4}} d\omega du.$$

Choose $(\bar{t}, \bar{x}, \bar{v}) \sim (t, x, v)$ so that $|(\bar{t}, \bar{x}, \bar{v}) - (t, x, v)| < \delta$. We will estimate

$$\frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \mu)(\bar{t}, \bar{x}, \bar{v}) - \frac{1}{\sqrt{\mu}} Q_-(\sqrt{\mu} f, \mu)(t, x, v)$$
$$= \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{4}} \{B(v-u,\omega) f(t,x,v) - \underline{B(\bar{v}-u,\omega) f(\bar{t},\bar{x},u)}\} d\omega du$$
$$= \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u,\omega) e^{-\frac{|u|^2}{4}} f(t,x,v) d\omega du$$
$$- \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v-u',\omega) e^{-\frac{|u'-(v-\bar{v})|^2}{4}} f(\bar{t},\bar{x},u'-(v-\bar{v})) d\omega du'$$
$$\leq \frac{1}{\sqrt{\mu}} \int_{u\in\mathbb{R}^3} \int_{\mathbb{S}^2} |B(v-u,\omega)| |e^{-\frac{|u|^2}{4}} - e^{-\frac{|u-(v-\bar{v})|^2}{4}}| w^{-1}(u-(v-\bar{v})) ||wf||_\infty d\omega du \tag{87}$$
$$+ \frac{1}{\sqrt{\mu}} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \underbrace{|B(v-u,\omega)| e^{-\frac{|u|^2}{4}} |f(t,x,u) - f(\bar{t},\bar{x},u-(v-\bar{v}))|}_{\bigoplus} d\omega du, \tag{88}$$

where we used a change of variables $u' = u + (v - \bar{v})$ for the underlined term. Using Taylor's expansion we control

$$e^{-\frac{|u-(v-\bar{v})|^2}{4}} = e^{-\frac{|u|^2}{4}} + \frac{1}{2} |u_*| e^{-\frac{|u_*|^2}{4}} |v-\bar{v}| \leq \frac{1}{2} (|u| + \delta) e^{\frac{\delta^2}{4}} e^{-\frac{|u|^2}{4}} \times |v-\bar{v}|,$$

where $u_* = s_*\{u - (v - \bar{v})\} + (1 - s_*)u$ for some $s_* \in (0, 1)$ and $|v - \bar{v}| < \delta$. Therefore we control

$$|(87)| \leq e^{\frac{|v|^2}{4}} \int_{\mathbb{R}^3} |v-u|^\gamma \frac{1}{2} (|u|+\delta) e^{\frac{\delta^2}{4}} e^{-\frac{|u|^2}{4}} du \times \sup_{v,u} \int_{\mathbb{S}^2} q_0\big(\frac{v-u}{|v-u|} \cdot \omega\big) d\omega |v-\bar{v}| ||wf||_\infty$$
$$\leq C(1+|v|)^\gamma e^{\frac{|v|^2}{4}} ||wf||_\infty, \tag{89}$$

where we have used the the angular cutoff assumption (1). Now we estimate (88) with following steps:

Case 1: $|u| \geq N$. Since $e^{-\frac{|u|^2}{4}} \leq e^{-\frac{N^2}{8}} e^{-\frac{|u|^2}{8}}$, we estimate

$$\int_{|u|\geq N} \int_{\mathbb{S}^2} \bigoplus d\omega du \leq C e^{-\frac{N^2}{8}} \int_{\mathbb{R}^3} e^{-\frac{|u|^2}{8}} |u-v|^\gamma du \times ||wf||_\infty$$
$$\leq C e^{-\frac{N^2}{8}} \nu(v) ||wf||_\infty, \tag{90}$$

where $\bigoplus$ is the underbraced term in (88).

Case 2: $|u| \leq N$. A function f is continuous on $[0, T] \times (\bar{\Omega} \times B(0; N)) \backslash \mathfrak{G}$. By Lemma 5, we can choose $U_x \subset B(0; N)$ with $|U_x| < \frac{\varepsilon}{N}$ with $|f(t, x, u) - f(\bar{t}, \bar{x}, u - (v - \bar{v}))| < \frac{\varepsilon}{N}$ for $|(t, x, u) - (\bar{t}, \bar{x}, u - (v - \bar{v}))| \leq \delta$ with $u \in B(0; N)\backslash U_x$. Therefore $\int_{|u|\leq N} \int_{\mathbb{S}^2} \bigoplus d\omega du$ is bounded by

$$\int_{u\in B(0;N)\cap U_x} \int_{\mathbb{S}^2} \bigoplus d\omega du + \int_{u\in B(0;N)\backslash U_x} \int_{\mathbb{S}^2} \bigoplus d\omega du \leq C \frac{\varepsilon}{N} \nu(v) ||wf||_\infty, \tag{91}$$

where $\bigoplus$ is the underbraced term in (88). From (89), (90) and (91), we summarize

$$\frac{1}{\sqrt{\mu}}|Q_{-}(\sqrt{\mu}f,\mu)(\bar{t},\bar{x},\bar{v})-Q_{-}(\sqrt{\mu}f,\mu)(t,x,v)|\leq(o(\delta)+e^{-\frac{N^2}{8}}+\frac{\varepsilon}{N})\frac{\nu(v)}{\sqrt{\mu}}||wf||_{\infty},$$

which is less than ε for sufficiently large N and sufficiently small δ. □

4. In-Flow Boundary Condition

In this section, we consider the weighted linearized Boltzmann equation

$$\{\partial_t + v\cdot\nabla_x + \nu - K_w\}h = w\Gamma(\frac{h}{w},\frac{h}{w}), \tag{92}$$

where $K_w h \equiv wK(\frac{h}{w})$ with the in-flow injection boundary condition:

$$h(t,x,v) = w(v)g(t,x,v) \quad \text{ for } (x,v)\in\gamma_{-}. \tag{93}$$

where the weight function $w(v) = \{1+\rho^2|v|^2\}^{\beta}$ in (12). First we will show the formation of discontinuity using a pointwise estimate of the Boltzmann solution [21]. Then we use the continuity of collision operators, Theorem 4, to show a continuity of solution on the continuity set $\mathfrak{C}$ and the propagation of discontinuity on the discontinuity set $\mathfrak{D}$.

4.1. Formation of discontinuity. We prove Part 1 of Theorem 1. Without loss of generality we may assume $x_0 = (0,0,0)$ and $v_0 = (1,0,0)$ and $(x_0,v_0)\in\gamma_0^{\mathbf{S}}$. Locally the boundary is a graph, i.e. $\Omega\cap B(\mathbf{0};\delta) = \{(x_1,x_2,x_3)\in B(\mathbf{0};\delta) : x_3 > \Phi(x_1,x_2)\}$. The condition $(x_0,v_0)\in\gamma_0^{\mathbf{S}}$ implies $t_{\mathbf{b}}(x_0,v_0)\neq 0$ and $t_{\mathbf{b}}(x_0,-v_0)\neq 0$ which means $\Phi(\xi,0)<0$ for $\xi\in(-\delta,\delta)\backslash\{0\}$ (see Fig. 3).

For simplicity we assume a zero boundary datum, i.e. $g\equiv 0$. From Theorem 1 of [21], we have a global solution of the linearized Boltzmann equation (92) with zero in-flow boundary condition, satisfying the decay estimate (15). In the proof we do not use the decay estimate but just boundedness

$$\sup_{t\in[0,\infty)}||h(t)||_{\infty}\leq C'||h_0||_{\infty}. \tag{94}$$

Recall the constants $C_{\mathbf{k}}$ and C_{Γ} from (35) and (36). Choose $t_0\in(0,\min\{\frac{\delta}{2},\frac{t_{\mathbf{b}}(x_0,-v_0)}{2}\})$ sufficiently small so that

$$\frac{1}{2}\leq\left(e^{-\nu(1)t_0} - t_0C_{\mathbf{k}}C' - (1-e^{-\nu(1)t_0})C_{\Gamma}(C')^2\right), \tag{95}$$

where $\nu(1)\equiv\nu(v_0)$ for any $v_0\in\mathbb{R}^3$ with $|v_0|=1$. This choice is possible because the right-hand side of (95) is a continuous function of $t_0\in\mathbb{R}$ and it has a value 1 when $t_0=0$. Furthermore assume a condition for our initial datum h_0: there is sufficiently small $\delta'=\delta'(\Omega,t_0)>0$ such that $B((-t_0,0,0);\delta')\subset\Omega$ and

$$h_0(x_0,v_0)\equiv||h_0||_{\infty}>0 \quad \text{for } (x,v)\in B((-t_0,0,0);\delta')\times B((1,0,0);\delta'). \tag{96}$$

We claim the Boltzmann solution h with such an initial datum h_0 and zero in-flow boundary condition is not continuous at $(t_0, x_0, v_0) = (t_0, (0, 0, 0), (1, 0, 0))$. We will use a contradiction argument: Suppose

$$[h(t_0)]_{x_0, v_0} = 0. \tag{97}$$

Choose sequences of points $(x_n', v_n') = ((0, 0, \frac{1}{n}), (1, 0, 0))$ and $(x_n, v_n) = ((\frac{1}{n}, 0, \Phi(\frac{1}{n}, 0)), (1, 0, \frac{1}{n}))$. Because of our choice, for sufficiently large $n \in \mathbb{N}$, the characteristics $[X(0; t_0, x_0, v_0), V(0; t_0, x_0, v_0)]$ is near to $((-t_0, 0, 0), (1, 0, 0))$, i.e.

$$(x_n' - t_0 v_n', v_n') = ((-t_0, 0, \frac{1}{n}), (1, 0, 0)) \in B((-t_0, 0, 0); \delta') \times B((1, 0, 0); \delta').$$

Hence the Boltzmann solution at (t_0, x_n', v_n') is

$$\begin{aligned} h(t_0, x_n', v_n') &= h_0(x_n' - t_0 v_n', v_n') e^{-\nu(v_n')t_0} \\ &\quad + \int_0^{t_0} e^{-\nu(v_n')(t_0 - \tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} (\tau, x_n' - v(t_0 - \tau), v_n') d\tau \\ &= ||h_0||_\infty e^{-\nu(v_n')t_0} + \int_0^{t_0} e^{-\nu(v_n')(t_0 - \tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} \\ &\quad \times (\tau, x - v_n'(t_0 - \tau), v_n') d\tau. \end{aligned}$$

Combining $h(t_0, x_n, v_n) = w(v_n) g(t_0, x_n, v_n) = 0$ with (97), we conclude

$$h(t_0', x_n', v_n') \to 0 \quad \text{as } n \to 0. \tag{98}$$

On the other hand, using (94) we can estimate

$$\begin{aligned} &\liminf_{n\to\infty} |h(t_0, x_n', v_n')| \\ &\quad = \liminf_{n\to\infty} |h(t_0, x_n', v_n') - h(t_0, x_n, v_n)| \\ &\quad \geq \liminf_{n\to\infty} \Big| ||h_0||_\infty e^{-\nu(v_n')t_0} - \int_0^{t_0} C_{\mathbf{k}} C' ||h_0||_\infty d\tau \\ &\qquad + \int_0^{t_0} \nu(v_n') e^{-\nu(v_n')(t-\tau)} C_\Gamma (C')^2 ||h_0||_\infty^2 d\tau \Big| \\ &\quad \geq ||h_0||_\infty e^{-\nu(1)t_0} - t_0 C_{\mathbf{k}} C' ||h_0||_\infty - (1 - e^{-\nu(1)t_0}) C_\Gamma (C')^2 ||h_0||_\infty^2 \\ &\quad = ||h_0||_\infty \left(e^{-\nu(1)t_0} - t_0 C_{\mathbf{k}} C' - (1 - e^{-\nu(1)t_0}) C_\Gamma (C')^2 \right) \geq \frac{||h_0||_\infty}{2} \neq 0, \end{aligned}$$

which is contradiction to (98).

4.2. Continuity away from $\mathfrak{D}$. We aim to prove Part 1 of Theorem 3 in this section. First we recall Lemma 12 of [21], the representation for the solution operator $G(t, 0)$ for the homogeneous transport equation with in-flow boundary condition:

Lemma 11 [21]. *Let* $h_0 \in L^\infty$ *and* $wg \in L^\infty$. *Let* $\{G(t,0)h_0\}$ *be the solution to the transport equation*

$$\{\partial_t + v\cdot\nabla_x\}G(t,0)h_0 = 0, \quad G(0,0)h_0 = h_0, \quad \{G(t,0)h_0\}_{\gamma_-} = wg.$$

For $(x,v) \notin \gamma_0 \cap \gamma_-$,

$$\{G(t,0)h_0\}(t,x,v) = \mathbf{1}_{t-t_{\mathbf{b}}\le 0}h_0(x-tv,v) + \mathbf{1}_{t-t_{\mathbf{b}}>0}\{wg\}(t-t_{\mathbf{b}}, x-t_{\mathbf{b}}v, v).$$

Next we prove a generalized version of Lemma 13 in [21].

Lemma 12 (Continuity away from $\mathfrak{D}$: Transport Equation). *Let* Ω *be an open subset of* $\mathbb{R}^3$ *with a smooth boundary* $\partial\Omega$ *and an initial datum* $h_0(x,v)$ *be continuous in* $\Omega\times\mathbb{R}^3\cup\{\gamma_-\cup\gamma_+\cup\gamma_0^{I-}\}$ *and a boundary datum* g *be continuous in* $[0,T]\times\{\gamma_-\cup\gamma_0^{I-}\}$. *Also assume* $q(t,x,v)$ *and* $\phi(t,x,v)$ *are continuous in the interior of* $[0,T]\times\Omega\times\mathbb{R}^3$ *and satisfy* $\sup_{[0,T]\times\Omega\times\mathbb{R}^3}\left|q(t,x,v)\right| < \infty$ *and* $\sup_{[0,T]\times\Omega}\left|\phi(\cdot,\cdot,v)\right| < \infty$ *for all* $v\in\mathbb{R}^3$. *Let* $h(t,x,v)$ *be the solution of*

$$\{\partial_t + v\cdot\nabla_x + \phi\}h = q\ , \quad h(0,x,v) = h_0\ , \quad h|_{\gamma_-} = wg.$$

Assume the compatibility condition on $\gamma_-\cup\gamma_0^{I-}$,

$$h_0(x,v) = w(v)g(0,x,v). \tag{99}$$

Then the Boltzmann solution $h(t,x,v)$ *is continuous on the continuity set* $\mathfrak{C}$. *Furthermore, if the boundary* $\partial\Omega$ *does not include a line segment (Definition 6) then* $h(t,x,v)$ *is continuous on a complementary set of the discontinuity set, i.e.* $\{[0,T]\times\bar\Omega\times\mathbb{R}^3\}\backslash\mathfrak{D}$.

Proof. Continuity on $\{\{0\}\times\bar\Omega\times\mathbb{R}^3\}\cup\{(0,\infty)\times[\gamma_-\cup\gamma_0^{I-}]\}$ is obvious from the assumption. Fix $(t,x,v)\in\mathfrak{C}$. Notice that

$$\left\{\frac{d}{ds}\{h(s,X(s),V(s))e^{-\int_s^t \phi(\tau,X(\tau),V(\tau))d\tau}\} - q(s,X(s),V(s))e^{-\int_s^t \phi(\tau,X(\tau),V(\tau))d\tau}\right\}$$
$$\times\mathbf{1}_{[\max\{0,t-t_{\mathbf{b}}(x,v)\},t]}(s) = 0, \tag{100}$$

along the characteristics $X(s;t,x,v) = x - v(t-s)$, $V(s;t,x,v) = v$ until the characteristics hits on the boundary. Choose $(\bar t,\bar x,\bar v)\sim(t,x,v)$ and use a change of variables $\bar s = s-(\bar t - t)$ with $\bar s\in[t-\bar t, t]$ to have

$$\Big\{\frac{d}{d\bar s}\{h(\bar s+(\bar t-t),\bar X(\bar s),\bar V(\bar s))e^{-\int_{\bar s}^t \phi(\tau+(\bar t-t),\bar X(\tau),\bar V(\tau))d\tau}\}$$
$$-q(\bar s+(\bar t-t),\bar X(\bar s),\bar V(\bar s))e^{-\int_{\bar s}^t \phi(\tau+(\bar t-t),\bar X(\tau),\bar V(\tau))d\tau}\Big\}\mathbf{1}_{[-(\bar t-t)+\max\{0,\bar t-t_{\mathbf{b}}(\bar x,\bar v)\},t]}(s) = 0, \tag{101}$$

where $\bar X(\bar s) = X(\bar s+(\bar t-t);\bar t,\bar x,\bar v)$ and $\bar V(\bar s) = V(\bar s+(\bar t-t);\bar t,\bar x,\bar v)$.

By the definition $\mathfrak{C}$, we can separate two cases: $t < t_{\mathbf{b}}(x,v)$, $(x_{\mathbf{b}}(x,v),v)\in\gamma_-\cup\gamma_0^{I-}$.

Case of $t - t_{\mathbf{b}}(x,v) < 0$. From the assumption $t-t_{\mathbf{b}}(x,v)<0$, we know that (100) holds for $0\le s\le t$. Now we choose $(\bar t,\bar x,\bar v)$ near (t,x,v) so that $\bar t - t_{\mathbf{b}}(\bar x,\bar v) < 0$, and

$\bar{X}(\bar{s}) = X(\bar{s} + (\bar{t} - t); \bar{t}, \bar{x}, \bar{v})$ is in the interior of Ω for all $\bar{s} \in [t - \bar{t}, t]$. Taking the integration over $[\min\{0, t - \bar{t}\}, t]$ of (100)–(101) to have

$$
\begin{aligned}
&h(t, x, v) - h(\bar{t}, \bar{x}, \bar{v}) = h_0(X(0), V(0))e^{-\int_0^t \phi(\tau, X(\tau), V(\tau))d\tau} \\
&\quad -h_0(\bar{X}(t - \bar{t}), \bar{V}(t - \bar{t}))e^{-\int_{t-\bar{t}}^t \phi(\tau+(\bar{t}-t), \bar{X}(\tau), \bar{V}(\tau))d\tau} \\
&\quad + \int_{\min\{0,t-\bar{t}\}}^t \Big\{ \mathbf{1}_{[\max\{0,t-t_{\mathbf{b}}(x,v)\},t]}(s) q(s, X(s), V(s))e^{-\int_s^t \phi(\tau, X(\tau), V(\tau))d\tau} \\
&\quad -\mathbf{1}_{[t-\bar{t}+\max\{0,\bar{t}-t_{\mathbf{b}}(\bar{x},\bar{v})\},t]}(s) q(s + (\bar{t} - t), \bar{X}(s), \bar{V}(s))e^{-\int_s^t \phi(\tau+(\bar{t}-t), \bar{X}(\tau), \bar{V}(\tau))d\tau} \Big\} ds.
\end{aligned}
$$

Since h_0 and ϕ is continuous, it is easy to see that the first line above goes to zero when $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$. For the remainder we separate cases: $t - \bar{t} > 0$ and $t - \bar{t} \le 0$. If $t - \bar{t} > 0$ the remainder is bounded by

$$
\begin{aligned}
&\int_{t-\bar{t}}^t |q(s)e^{\int_s^t \phi(\tau)\tau} - q(s + (\tau t - t))e^{-\int_s^t \phi(\tau+(\bar{t}-t)}| \\
&\quad +|t - \bar{t}| \sup_{0\le s\le t} ||q(s)||_\infty e^{t \sup_{0\le s\le t} ||\phi(s)||_\infty},
\end{aligned}
$$

where the first term is small using continuity of q and ϕ, and the second term is small as $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$. The case $t - \bar{t} \le 0$ is similar.

Case of $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$. We only have to consider cases of $t > t_{\mathbf{b}}(x, v)$ and $t = t_{\mathbf{b}}(x, v)$. By definition $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$. From Lemma 2, we know that $t_{\mathbf{b}}(x, v)$ is a continuous function when $(x_{\mathbf{b}}(x, v), v) \notin \gamma_- \cup \gamma_0^{I-}$. In the case of $t > t_{\mathbf{b}}(x, v)$, for $(\bar{t}, \bar{x}, \bar{v}) \sim (t, x, v)$, we have $\bar{t} > t_{\mathbf{b}}(\bar{x}, \bar{v})$. Taking the integration over $[\min\{0, t - \bar{t}\}, t]$ of (100)–(101) to have

$$
\begin{aligned}
&h(t, x, v) - h(\bar{t}, \bar{x}, \bar{v}) \\
&\quad = wg(t - t_{\mathbf{b}}(x, v), X(t_{\mathbf{b}}(x, v)), V(t_{\mathbf{b}}(x, v)))e^{-\int_{t-t_{\mathbf{b}}(x,v)}^t \phi(\tau, X(\tau), V(\tau))d\tau} \\
&\quad\quad -wg(\bar{t} - t_{\mathbf{b}}(\bar{x}, \bar{v}), X(t_{\mathbf{b}}(\bar{x}, \bar{v})), V(t_{\mathbf{b}}(\bar{x}, \bar{v})))e^{-\int_{\bar{t}-t_{\mathbf{b}}(\bar{x},\bar{v})}^t \phi(\tau+(\bar{t}-t), \bar{X}(\tau), \bar{V}(\tau))d\tau} \\
&\quad\quad + \int_{t-t_{\mathbf{b}}(x,v)}^t q(s, X(s), V(s))e^{-\int_s^t \phi(\tau, X(\tau), V(\tau))d\tau} ds \\
&\quad\quad - \int_{t-t_{\mathbf{b}}(\bar{x},\bar{v})}^t q(s + (\bar{t} - t), \bar{X}(s), \bar{V}(s))e^{-\int_s^t \phi(\tau+(\bar{t}-t), \bar{X}(\tau), \bar{V}(\tau))d\tau} ds.
\end{aligned}
$$

Using the continuity of $t_{\mathbf{b}}$ and q and ϕ, it is easy to show that $|h(t, x, v) - h(\bar{t}, \bar{x}, \bar{v})| \to 0$ as $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$. In the case of $t = t_{\mathbf{b}}(x, v)$ we can choose $(\bar{t}, \bar{x}, \bar{v}) \sim (t, x, v)$ so that $t_{\mathbf{b}}(\bar{x}, \bar{v}) \in (t - \epsilon, t + \epsilon)$. Taking the integration over $[\min\{0, t - \bar{t}\}, t]$ of (100)–(101) to have

$$
\begin{aligned}
&|h(t, x, v) - h(\bar{t}, \bar{x}, \bar{v})| \le wg(t - t_{\mathbf{b}}(x, v), X(t_{\mathbf{b}}(x, v)), \\
&\quad \times V(t_{\mathbf{b}}(x, v)))e^{-\int_{t-t_{\mathbf{b}}(x,v)}^t \phi(\tau, X(\tau), V(\tau))d\tau} \\
&\quad -\mathbf{1}_{\bar{t}>t_{\mathbf{b}}(\bar{x},\bar{v})} wg(\bar{t} - t_{\mathbf{b}}(\bar{x}, \bar{v}), X(t_{\mathbf{b}}(\bar{x}, \bar{v}), V(t_{\mathbf{b}}(\bar{x}, \bar{v}))))e^{-\int_{\bar{t}-t_{\mathbf{b}}(\bar{x},\bar{v})}^t \phi(\tau+(\bar{t}-t), \bar{X}(\tau), \bar{V}(\tau))d\tau} \\
&\quad -\mathbf{1}_{\bar{t}\le t_{\mathbf{b}}(\bar{x},\bar{v})} h_0(\bar{X}(t - \bar{t}), \bar{V}(t - \bar{t}))e^{-\int_{t-\bar{t}}^t \phi(\tau+(\bar{t}-t), \bar{X}(\tau), \bar{V}(\tau))d\tau}
\end{aligned}
$$

$$+\int_{t-t_{\mathbf{b}}(x,v)+\varepsilon}^{t}\Big|q(s,X(s),V(s))e^{-\int_s^t\phi(\tau,X(\tau),V(\tau))d\tau}$$

$$-q(s+(\bar{t}-t),\bar{X}(s),\bar{V}(s))e^{-\int_s^t\phi(\tau+(\bar{t}-t),\bar{X}(\tau),\bar{V}(\tau))d\tau}\Big|ds$$

$$+2\varepsilon\sup_{0\le s\le t}||q(s)||_\infty e^{t\sup_{0\le s\le t}||\phi(s)||_\infty},$$

where the first three lines can be small using the compatibility condition and continuity of h_0 in $\Omega\times\mathbb{R}^3\cup\{\gamma_-\cup\gamma_+\cup\gamma_0^{I-}\}$ and a continuity of g on $[0,T]\times\{\gamma_-\cup\gamma_0^{I-}\}$ and continuity of ϕ. For the fourth line above, we use the continuity of q and ϕ.

If the boundary $\partial\Omega$ does not include a line segment (Definition 6) we have $\mathfrak{C}=\{[0,T]\times\bar{\Omega}\times\mathbb{R}^3\}\backslash\mathfrak{D}$. □

Proof of Part 1 of Theorem 3. We will use the following iteration scheme

$$\{\partial_t+v\cdot\nabla_x+\nu\}h^{m+1}=K_wh^m+w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right)-w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right),\quad(102)$$

with $h^{m+1}|_{t=0}=h_0$ and $h^{m+1}(t,x,v)=wg(t,x,v)$ with $(t,x,v)\in\gamma_-\cup\gamma_0^{I-}$. Notice that this sequence is used in (242), p. 803 of [21] and we use the smallness of the initial datum and in-flow datum crucially. For simplicity we define

$$q^m=K_wh^m+w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right)-w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right).\quad(103)$$

Step 1. We claim

$$h^i\text{ is a continuous function in }\mathfrak{C}_T\quad(104)$$

for all $i\in\mathbb{N}$ and for any $T>0$ where

$$\mathfrak{C}_T\equiv\mathfrak{C}\cap\{[0,T]\times\bar{\Omega}\times\mathbb{R}^3\},\quad(105)$$

where the continuity set $\mathfrak{C}$ is defined in (10). We will use mathematical induction to show (104). We choose $h^0=0$, then (104) is satisfied for $i=0$. Assume (104) for all $i=0,1,2,\ldots,m$. Rewrite $w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right)=\nu\left(\sqrt{\mu}\frac{h^m}{w}\right)h^{m+1}$, then the equation of h^{m+1} is

$$\{\partial_t+v\cdot\nabla_x+\nu(v)+\nu\left(\sqrt{\mu}\frac{h^m}{w}\right)\}h^{m+1}=K_wh^m+w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right).\quad(106)$$

From Theorem 4 and Corollary 5 we know that $\nu\left(\sqrt{\mu}\frac{h^m}{w}\right)$ and $w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right)$ is continuous in $[0,T]\times\Omega\times\mathbb{R}^3$. Apply Lemma 12 where $\phi(t,x,v)$ corresponds to $\nu(v)+\nu(\sqrt{\nu}\frac{h^m}{w})$ and $q(t,x,v)$ corresponds to the right hand side of (106). Then we check (104) for $i=m+1$.

Step 2. We claim that there exist $C>0$ and $\delta>0$ such that if $C\{||h_0||_\infty+\sup_{0\le s<\infty}||wg(s)||_\infty\}<\delta$ and $C||h_0||_\infty<\delta$ then there exists $T=T(C,\delta)>0$ so that

$$\sup_{0\le s\le T}||h^m(s)||_\infty\le C||h_0||_\infty,\quad(107)$$

for all $m\in\mathbb{N}$. Moreover $\{h^m\}_{m=0}^\infty$ is Cauchy in $L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}^3)$.

First we will show a boundedness (107) for all $m \in \mathbb{N}$. We use mathematical induction on m. Assume $\sup_{0\le s\le T} ||h^m(s)||_\infty \le C||h_0||_\infty$, where $T > 0$ will be determined later. Integrating (102) along the trajectory, we have

$$\begin{aligned}
h^{m+1}(t,x,v) &= \mathbf{1}_{t<t_{\mathbf{b}}(x,v)} e^{-\nu(v)t} h_0(x-tv,v) \\
&\quad + \mathbf{1}_{t\ge t_{\mathbf{b}}(x,v)} e^{-\nu(v)t_{\mathbf{b}}(x,v)} w(v) g(t-t_{\mathbf{b}}(x,v), x_{\mathbf{b}}(x,v), v) \\
&\quad + \int_{\max\{t-t_{\mathbf{b}}(x,v),0\}}^{t} e^{-\nu(v)(t-s)} \Big\{ K_w h^m + w\Gamma_+\left(\frac{h^m}{w},\frac{h^m}{w}\right) \\
&\quad - w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}}{w}\right)\Big\}(s, x-(t-s)v, v)\,ds \\
&\le ||h_0||_\infty + \sup_{0\le s\le t} ||wg(s)||_\infty + tC_{\mathbf{k}} \sup_{0\le s\le t} ||h^m(s)||_\infty \\
&\quad + C_\Gamma \sup_{0\le s\le t} ||h^m(s)||_\infty \sup_{0\le s\le t}\left(||h^m(s)||_\infty + ||h^{m+1}(s)||_\infty \right),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{0\le s\le t} ||h^{m+1}(s)||_\infty &\le \frac{1 + tC_{\mathbf{k}}C + C_\Gamma C\{||h_0||_\infty + \sup_s ||wg(s)||_\infty\}}{1 - C_\Gamma C\{||h_0||_\infty + \sup_s ||wg(s)||_\infty\}} \\
&\quad \times \left\{ ||h_0||_\infty + \sup_{0\le s\le t} ||wg(s)||_\infty \right\} \\
&\le C \left\{ ||h_0||_\infty + \sup_{0\le s\le t} ||wg(s)||_\infty \right\},
\end{aligned}$$

where we choose $C > 4$ and then $\{||h_0||_\infty + \sup_{0\le s\le t} ||wg(s)||_\infty\} \le \frac{1}{2C_\Gamma C}$ and then $T = \frac{C-3}{2C_{\mathbf{k}}C}$.

Next we will show the sequence $\{h^m\}$ is Cauchy in $L^\infty([0,T]\times\bar{\Omega}\times\mathbb{R}^3)$. The equation of $h^{m+1}-h^m$ is

$$\begin{aligned}
&\{\partial_t + v\cdot\nabla_x + \nu\}(h^{m+1}-h^m) = \tilde{q}^m, \\
&(h^{m+1}-h^m)|_{t=0} = 0, \quad (h^{m+1}-h^m)|_{\gamma_-} = 0,
\end{aligned} \tag{108}$$

where

$$\begin{aligned}
\tilde{q}^m &= K_w(h^m - h^{m-1}) + w\Gamma_+\left(\frac{h^m}{w},\frac{h^m - h^{m-1}}{w}\right) - w\Gamma_+\left(\frac{h^{m-1}-h^m}{w},\frac{h^{m-1}}{w}\right) \\
&\quad - w\Gamma_-\left(\frac{h^m}{w},\frac{h^{m+1}-h^m}{w}\right) + w\Gamma_-\left(\frac{h^{m-1}-h^m}{w},\frac{h^m}{w}\right).
\end{aligned} \tag{109}$$

From (35) and (36), we have a bound of $\tilde{q}^m$,

$$\begin{aligned}
\sup_{0\le s\le t} ||\tilde{q}^m(s)||_\infty &\le C_{\mathbf{k}} \sup_{0\le s\le t} ||\{h^m - h^{m-1}\}(s)||_\infty \\
&\quad + C_\Gamma \nu(v)\{ \sup_{0\le s\le t} ||\{h^m - h^{m-1}\}(s)||_\infty + \sup_{0\le s\le t} ||\{h^{m+1}-h^m\}(s)||_\infty \} \\
&\quad \times(\sup_{0\le s\le t} ||h^m(s)||_\infty + \sup_{0\le s\le t} ||h^{m+1}(s)||_\infty).
\end{aligned} \tag{110}$$

Integrating (108) along the trajectory, we have

$$
\begin{aligned}
&||\{h^{m+1} - h^m\}(t)||_\infty \\
&\quad \le \int_0^t e^{-\nu(v)(t-s)} ||\tilde{q}^m(s, x-(t-s)v, v)||_\infty ds \\
&\quad \le C_{\mathbf{k}} t \sup_{0\le s\le t} ||\{h^m - h^{m-1}\}(s)||_\infty \\
&\qquad + CC_\Gamma \left(||h_0||_\infty + \sup_{0\le s\le t} ||wg(s)||_\infty \right) \\
&\qquad \times \left\{ \sup_{0\le s\le t} ||\{h^m - h^{m-1}\}(s)||_\infty + \sup_{0\le s\le t} ||\{h^{m+1} - h^m\}(s)||_\infty \right\}.
\end{aligned}
$$

If we choose $CC_\Gamma ||h_0||_\infty \le \frac{1}{4}$ and $C_{\mathbf{k}} T \le \frac{1}{8}$ then

$$
\sup_{0\le s\le T} ||\{h^{m+1} - h^m\}(s)||_\infty \le \frac{1}{2} \sup_{0\le s\le T} ||\{h^m - h^{m-1}\}(s)||_\infty .
$$

Then we have

$$
\begin{aligned}
&\sup_{0\le s\le T} ||\{h^m - h^{m-1}\}(s)||_\infty \\
&\quad \le \sup_{0\le s\le T} ||\{h^m - h^{m-1}\}(s)||_\infty + \cdots + \sup_{0\le s\le T} ||\{h^{n+1} - h^n\}(s)||_\infty \\
&\quad \le \{\frac{1}{2^{m-n-1}} + \cdots + \frac{1}{2^0}\} \sup_{0\le s\le T} ||\{h^{n+1} - h^n\}(s)||_\infty \\
&\quad \le \frac{2}{2^n} \sup_{0\le s\le T} ||\{h^1 - h^0\}(s)||_\infty \\
&\quad \le \frac{4}{2^n} C\{||h_0||_\infty + \sup_{0\le s\le T} ||wg(s)||_\infty\},
\end{aligned}
$$

which means that the sequence $\{h^m\}$ is Cauchy in $L^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^3)$.

Step 3. From previous steps we obtain that h with $\lim_{n\to\infty} h^n$ is a continuous function on $\mathfrak{C}_T$. Now we claim that h is continuous in $\mathfrak{C}$. Notice that T only depends on $||h_0||_\infty$ and $\sup_{0\le s\le T} ||wg(s)||_\infty$. Using a uniform bound of $\sup_{0\le s<\infty} ||h(s)||_\infty$ (Theorem 1 of [21]) we can obtain the continuity for h for all time by repeating $[0, T], [T, 2T], \ldots$. If the boundary $\partial\Omega$ does not include a line segment (Definition 6) then every step is valid with $[0, \infty) \times \{\bar{\Omega} \times \mathbb{R}^3\} \backslash \mathfrak{D}$ instead of $\mathfrak{C}$ and $[0, T] \times \{\bar{\Omega} \times \mathbb{R}^3\} \backslash \mathfrak{D}$ instead of $\mathfrak{C}_T$.

4.3. Propagation of discontinuity.

Proof of 1 of Theorem 2.

Proof of (20). In order to show the upper bound of discontinuity jump (20), we will show

$$
[h(t)]_{x_0+(t-t_0)v_0, v_0} \le [h]_{t_0, x_0, v_0} e^{-(\frac{1}{C_v} + \frac{C'}{C_w}||h_0||_\infty)(1+|v_0|)^\gamma (t-t_0)}, \tag{111}
$$

when $(x_0, v_0) \in \gamma_0^{\mathbf{S}}$ and $t \in (t_0, t_0 + t_{\mathbf{b}}(x_0, -v_0))$. Choose two points $(x', v'), (x'', v'') \in \{\bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}\} \cap B((x, v); \delta) \backslash (x, v)$ and compare the representation

$$
\begin{aligned}
&|h(t, x', v') - h(t, x'', v'')| \\
&\le \Big| \mathbf{1}_{t-t_0 \ge t_{\mathbf{b}}(x', v')} h(t - t_{\mathbf{b}}(x', v'), x_{\mathbf{b}}(x', v'), v') \\
&\quad \times e^{-\nu(v') t_{\mathbf{b}}(x', v') - \int_{t - t_{\mathbf{b}}(x', v')}^{t} \nu(\sqrt{\mu} \frac{h}{w})(\tau, x' - (t-\tau) v', v') d\tau} \\
&\quad + \mathbf{1}_{t-t_0 < t_{\mathbf{b}}(x', v')} h(t_0, x' - (t - t_0) v', v') e^{-\nu(v')(t-t_0) - \int_{t_0}^{t} \nu(\sqrt{\mu} \frac{h}{w})(\tau, x' - (t-\tau) v', v') d\tau} \\
&\quad - \mathbf{1}_{t-t_0 \ge t_{\mathbf{b}}(x'', v'')} h(t - t_{\mathbf{b}}(x'', v''), x_{\mathbf{b}}(x'', v''), v'') \\
&\quad \times e^{-\nu(v'') t_{\mathbf{b}}(x'', v'') - \int_{t - t_{\mathbf{b}}(x'', v'')}^{t} \nu(\sqrt{\mu} \frac{h}{w})(\tau, x' - (t-\tau) v', v') d\tau} \\
&\quad - \mathbf{1}_{t-t_0 < t_{\mathbf{b}}(x'', v'')} h(t_0, x'' - (t - t_0) v'', v'') e^{-\nu(v'')(t-t_0) - \int_{t_0}^{t} \nu(\sqrt{\mu} \frac{h}{w})(\tau, x'' - (t-\tau) v'', v'') d\tau} \Big| \\
&\quad + \Big| \int_{\max\{0, t - t_0 - t_{\mathbf{b}}(x', v')\}}^{t} \{K_w h + w \Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, x' - (t-s) v', v') \\
&\quad \times e^{-\nu(v')(t-s) - \int_s^t \nu(\sqrt{\mu} \frac{h}{w})(\tau, x' - (t-\tau) v', v') d\tau} ds \\
&\quad - \int_{\max\{0, t - t_0 - t_{\mathbf{b}}(x'', v'')\}}^{t} \{K_w h + w \Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, x'' - (t-s) v'', v'') \\
&\quad \times e^{-\nu(v'')(t-s) - \int_s^t \nu(\sqrt{\mu} \frac{h}{w})(\tau, x'' - (t-\tau) v'', v'') d\tau} ds \Big| .
\end{aligned}
$$

It is easy to see that if $t - t_0 \ge t_{\mathbf{b}}(x', v')$ then as $\delta \to 0$ we have

$$
t - t_{\mathbf{b}}(x', v') \to t_0\ , \ x_{\mathbf{b}}(x', v') \to x_0\ ,
$$

and if $t - t_0 < t_{\mathbf{b}}(x', v')$ then as $\delta \to 0$ we have

$$
x' - (t - t_0) v' \to x_0 .
$$

Therefore the first four lines converge to
$[h]_{t_0, x_0, v_0} \times e^{-\nu(v_0)(t-t_0) - \int_{t_0}^{t} \nu(\sqrt{\mu} \frac{h}{w})(\tau, x_0 - (t_0 - \tau) v_0, v_0) d\tau}$. For the last two lines, using the continuity of $K_w h$, $\Gamma(\frac{h}{w}, \frac{h}{w})$, $\nu(\sqrt{\mu} \frac{h}{w})$ we conclude that it converges to zero. Therefore we have

$$
\begin{aligned}
[h(t)]_{x_0 + (t - t_0) v_0, v_0} &\le [h]_{t_0, x_0, v_0} e^{-\nu(v_0)(t-t_0) - \int_{t_0}^{t} \nu(\sqrt{\mu} \frac{h}{w})(\tau, x_0 - (t_0 - \tau) v_0, v_0) d\tau} \\
&\le [h]_{t_0, x_0, v_0} \times e^{-(\frac{1}{C_\nu} - C_w C' ||h_0||_\infty)(1 + |v_0|)^\gamma (t - t_0)},
\end{aligned}
$$

where we used

$$
\nu_w(v) \equiv \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \omega) e^{-\frac{|u|^2}{4}} w^{-1}(u) d\omega du \tag{112}
$$

$$
\text{with } \ \frac{1}{C_w} (1 + |v|)^\gamma \le \nu_w(v) \le C_w (1 + |v|)^\gamma . \tag{113}
$$

Remark that the **Proof of (20)** is valid for in-flow, diffuse and bounce-back cases.

Proof of (22). Assume $[h(t_0)]_{x_0, v_0} \neq 0$ and $t_0 \in (0, t_{\mathbf{b}}(x_0, -v_0))$ with $(x_0, v_0) \in \gamma_0^{\mathbf{S}}$. Further assume that the boundary $\partial\Omega$ is strictly concave at x_0 along the v_0 direction (21).

Step 1 Claim. We can choose sequences $(t'_n, x'_n, v'_n), (t''_n, x''_n, v''_n) \in [0,\infty) \times \bar{\Omega} \times \mathbb{R}^3 \cap B((t_0, x_0, v_0); \frac{1}{n})\backslash(t_0, x_0, v_0)$ such that $\lim_{n\to\infty} |h(t'_n, x'_n, v'_n) - h(t''_n, x''_n, v''_n)| \geq \frac{1}{2}[h(t_0)]_{x_0,v_0} \neq 0$. From $[h(t_0)]_{x_0,v_0} \neq 0$ we may assume

$$\sup_{(x'_0,v'_0),(x''_0,v''_0)\in B((x_0,v_0);\frac{1}{n})\backslash(x_0,v_0)} |h(t_0, x'_0, v'_0) - h(t_0, x''_0, v''_0)| \geq \frac{3}{4}[h(t_0)]_{x_0,v_0} \neq 0, \tag{114}$$

for all $n \in \mathbb{N}$. And for each $n \in \mathbb{N}$ we can choose the desired sequences.

Step 2 Claim. For given $\varepsilon > 0$, up to the subsequence we may assume that

$$\begin{aligned} &(x_{\mathbf{b}}(x'_n, v'_n), v'_n) \in B((x_0, v_0); \varepsilon)\backslash\mathfrak{G}\ , \\ &\ (x_{\mathbf{b}}(x''_n, v''_n), v''_n) \notin B((x_0, v_0); \varepsilon) \cup \mathfrak{G} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{115}$$

We remark that a continuity $G(t,x,v) = w(v)g(t,x,v)$ on $[0,\infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$, i.e.

$$\begin{aligned} &[\ G|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} = w(v_0)[\ g|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} = 0 \quad \text{for all} \\ &\ (t_0, x_0, v_0) \in [0,\infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\} \end{aligned} \tag{116}$$

is crucially used in this step. In order to show the final goal (115) of this step, we need to prove following statement.

$$\begin{aligned} &\text{Assume } (x_0, v_0) \in \gamma_0^{\mathbf{S}} \text{ and } t_{\mathbf{b}}(x_0, v_0) > t_0. \text{ Then for sufficiently small } \varepsilon > 0 \\ &\quad \text{there exists } N > 0 \text{ such that if } (x,v) \in B((x_0, v_0); \frac{1}{n}) \text{ for } n > \mathbf{N} \text{ and } x_{\mathbf{b}}(x,v) \\ &\ \notin B((x_0, v_0); \varepsilon) \text{ then we have } t_{\mathbf{b}}(x,v) > t_0. \end{aligned} \tag{117}$$

We will prove (117) later and show (115) using (117). It suffices to show that there are only finite $n \in \mathbb{N}$ such that

$$(x_{\mathbf{b}}(x'_n, v'_n), v'_n) \in B((x_0, v_0); \frac{1}{n})\backslash\mathfrak{G}\ ,\ (x_{\mathbf{b}}(x''_n, v''_n), v''_n) \in B((x_0, v_0); \frac{1}{n})\backslash\mathfrak{G}, \tag{118}$$

$$\textit{or}\ \ (x_{\mathbf{b}}(x'_n, v'_n), v'_n) \notin B((x_0, v_0); \frac{1}{n}) \cup \mathfrak{G}\ ,\ (x_{\mathbf{b}}(x''_n, v''_n), v''_n) \notin B((x_0, v_0); \frac{1}{n}) \cup \mathfrak{G}. \tag{119}$$

Suppose there are infinitely many $n' \in \mathbb{N}$ satisfying (118). If $\varepsilon > 0$ is sufficiently small then (118) implies that $t_0 > t_{\mathbf{b}}(x'_{n'}, v'_{n'})$ and $t_0 > t_{\mathbf{b}}(x''_{n'}, v''_{n'})$. The Boltzmann solution h at $(t_0, x'_{n'}, v'_{n'})$ is

$$\begin{aligned} h(t_0, x'_{n'}, v'_{n'}) &= h(t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'}), x_{\mathbf{b}}(x'_{n'}, v'_{n'}), v'_{n'}) \\ &\quad \times e^{-\nu(v'_{n'})(t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'})) - \int_{t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'})}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x'_{n'} - (t_0-\tau)v'_{n'}, v'_{n'})d\tau} \\ &\quad + \int_{t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'})}^{t_0} \{K_w h + \Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, x'_{n'} - (t_0 - s)v'_{n'}, v'_{n'}) \\ &\quad \times e^{-\nu(v'_{n'})(t_0 - s) - \int_s^{t_0} \nu(F)(\tau, x'_{n'} - (t_0-\tau)v'_{n'}, v'_{n'})d\tau} ds, \end{aligned}$$

and a similar representation for $h(t_0, x''_{n'}, v''_{n'})$. Compare representations of $h(t_0, x'_{n'}, v'_{n'})$ and $h(t_0, x''_{n'}, v''_{n'})$ to conclude

$$\begin{aligned}
&\lim_{n'\to\infty} |h(t_0, x'_{n'}, v'_{n'}) - h(t_0, x''_{n'}, v''_{n'})| \\
&= \lim_{n'\to\infty} |h(t_0 - t_{\mathbf{b}}(x'_{n'}, v'_{n'}), x_{\mathbf{b}}(x'_{n'}, v'_{n'}), v'_{n'}) \\
&\quad -h(t_0 - t_{\mathbf{b}}(x''_{n'}, v''_{n'}), x_{\mathbf{b}}(x''_{n'}, v''_{n'}), v''_{n'})| \\
&\quad \times e^{-\nu(v_0)(t_0-t_{\mathbf{b}}(x_0,v_0))-\int_{t_0-t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau,x_0-(t_0-\tau)v_0,v_0)d\tau} \\
&\le [h|_{[0,\infty)\times\gamma_-}]_{t_0-t_{\mathbf{b}}(x_0,v_0),x_{\mathbf{b}}(x_0,v_0),v_0} \\
&\quad \times e^{-\nu(v_0)(t_0-t_{\mathbf{b}}(x_0,v_0))-\int_{t_0-t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau,x_0-(t_0-\tau)v_0,v_0)d\tau},
\end{aligned}$$

where we used the continuity of $\nu(\sqrt{\mu}\frac{h}{w})$ and $\Gamma_+(\frac{h}{w}, \frac{h}{w})$. Further using the in-flow boundary condition $h|_{\gamma_-} = wg$, we have

$$\begin{aligned}
&\lim_{n'\to\infty} |h(t_0, x'_{n'}, v'_{n'}) - h(t_0, x''_{n'}, v''_{n'})| \le [\, g|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} w(v_0) \\
&\times e^{-\nu(v_0)(t_0-t_{\mathbf{b}}(x_0,v_0))-\int_{t_0-t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau,x_0-(t_0-\tau)v_0,v_0)d\tau} = 0,
\end{aligned}$$

where we used the continuity of g on $[0, \infty) \times \{\gamma_- \cup \gamma_0\}$, (116) at the last line. This is contradicted because we choose the sequences $(x'_{n'}, v'_{n'})$, $(x''_{n'}, v''_{n'})$ satisfying $\lim_{n\to\infty} |h(t_0, x'_{n'}, v'_{n'}) - h(t_0, x''_{n'}, v''_{n'})| \ge \frac{1}{2}[h(t_0)]_{x_0,v_0} \neq 0$ in **Step 1**.

Now suppose there are infinitely many $n'' \in \mathbb{N}$ satisfying (119). Because of (117) we have $t_0 < t_{\mathbf{b}}(x'_{n''}, v'_{n''})$ and $t_0 < t_{\mathbf{b}}(x''_{n''}, v''_{n''})$. The Boltzmann solution h at $(t_0, x'_{n''}, v'_{n''})$ is

$$\begin{aligned}
h(t_0, x'_{n''}, v'_{n''}) &= h_0(x'_{n''} - t_0 v'_{n''}, v'_{n''}), v'_{n''})e^{-\nu(v'_{n''})t_0-\int_0^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau,x'_{n''}-(t_0-\tau)v'_{n''},v'_{n''})d\tau} \\
&\quad + \int_0^{t_0} \{K_w h + \Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, x'_{n''} - (t_0 - s)v'_{n''}, v'_{n''}) \\
&\quad \times e^{-\nu(v'_{n''})(t_0-s)-\int_s^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau,x'_{n''}-(t_0-\tau)v'_{n''},v'_{n''})d\tau} ds,
\end{aligned}$$

and same representation for $h(t_0, x''_{n''}, v''_{n''})$. Using the continuity of h_0 we see that

$$\begin{aligned}
&\lim_{n\to\infty} |h(t_0, x'_{n''}, v'_{n''}) - h(t_0, x''_{n''}, v''_{n''})| \\
&= \lim_{n\to\infty} |h_0(x'_{n''} - t_0 v'_{n''}, v'_{n''}) - h_0(x''_{n''} - t_0 v''_{n''}, v''_{n''})| \\
&\quad \times w(v_0)e^{-\nu(v_0)(t_0-t_{\mathbf{b}}(x_0,v_0))-\int_{t_0-t_{\mathbf{b}}(x_0,v_0)}^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau,x_0-(t_0-\tau)v_0,v_0)d\tau} \\
&= 0,
\end{aligned}$$

which is also a contradiction.

Now we prove (117). We can choose $\varepsilon > 0$ sufficiently small so that $\partial\Omega \cap B(x_0; \varepsilon) = \{(x_1, x_2, \Phi(x_1, x_2)) \in B(x_0; \varepsilon)\}$. From $t_{\mathbf{b}}(x_0, v_0) > t_0$ we know that a line segment between x_0 and $x_0 - t_0 x_0$ has only one intersection point x_0 with $\partial\Omega$, i.e. $\overline{x_0, x_0 - t_0 v_0} \cap \partial\Omega = \{x_0\}$. Furthermore we can choose $\varrho > 0$ so large that $\bigcup_{s\in[0,t_0]} B(x_0 - s v_0; \varrho) \cap \partial\Omega \subset B(x_0; \varepsilon)$. Choose $N \in \mathbb{N}$ sufficiently large so that $\overline{x, x - t_0 v} \subset \bigcup_{s\in[0,t_0]} B(x_0 - s v_0; \varrho)$ for all $(x, v) \in B((x_0, v_0); \frac{1}{n})$. If $x_{\mathbf{b}}(x, v) \notin B((x_0, v_0); \varepsilon)$, then $\overline{x, x - t_0 v} \cap \partial\Omega = \emptyset$ and this implies $t_{\mathbf{b}}(x, v) > t_0$.

Step 3 Claim. Choose $t > 0$ so that $t - t_0 \in [0, t_{\mathbf{b}}(x_0, -v_0))$ and denote $x = x_0 + (t - t_0)v_0,\ v = v_0$. Then there exists $N \in \mathbb{N}$ so that $t - t_0 < t_{\mathbf{b}}(x_n', -v_n')$ for all $n > N$. Using (117) we only have to prove $x_{\mathbf{b}}(x_n', -v_n') \notin B((x_0, -v_0); \varepsilon)$. From (115) we know that $x_{\mathbf{b}}(x_n', v_n') \in B(x_0; \varepsilon)$. We assume that $\Omega \cap B(x_0; \varepsilon) = \{x \in B(x_0; \varepsilon) : x_3 > \Phi(x_1, x_2)\}$ and $n(x_0) = (0, 0, -1)$ and $v_0 = |v_0|(1, 0, 0)$. Let's define

$$\Psi(s) = \Phi((x_n')_1 + s(v_n')_1, (x_n')_2 + s(v_n')_2) - ((x_n')_3 + s(v_n')_2).$$

Since $x_n' \in \Omega$ we have $\Psi(0) < 0$ and $\Psi(t_{\mathbf{b}}(x_n', -v_n')) = 0 = \Psi(-t_{\mathbf{b}}(x_n', v_n'))$. Because of the strict concavity along the v_0 direction at x_0 (21), for sufficiently large n so that $(x_n', v_n') \sim (x_0, v_0)$ we have

$$\Psi''(s) = ((v_n')_1, (v_n')_2) \begin{pmatrix} \partial_{x_1}^2 \Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_2}\partial_{x_1}\Phi & \partial_{x_2}^2 \Phi \end{pmatrix} \begin{pmatrix} (v_n')_1 \\ (v_n')_2 \end{pmatrix} < -\frac{1}{2} C_{x_0, v_0},$$

where the Hessian of Φ is evaluated at $((x_n')_1 + s(v_n')_1, (x_n')_2 + s(v_n')_2)$. Since $\{x_n' + s v_n' : s \in (-t_{\mathbf{b}}(x_n', v_n'), t_{\mathbf{b}}(x_n', -v_n'))\} \subset \Omega$ we have $\Psi(s) < 0$ for $s \in (-t_{\mathbf{b}}(x_n', v_n'), t_{\mathbf{b}}(x_n', -v_n'))$. Therefore $\Phi'(-t_{\mathbf{b}}(x_n', v_n')) \le 0$ and $\Phi'(t_{\mathbf{b}}(x_n', -v_n')) \ge 0$. This is a contradiction because

$$\begin{aligned} 0 \le \Phi'(t_{\mathbf{b}}(x_n', -v_n')) = \Phi'(-t_{\mathbf{b}}(x_n', v_n')) + \int_{-t_{\mathbf{b}}(x_n', v_n')}^{t_{\mathbf{b}}(x_n', -v_n')} \Phi''(s) ds \le 0 \\ -\frac{1}{2} C_{x_0, v_0} \{t_{\mathbf{b}}(x_n', -v_n') + t_{\mathbf{b}}(x_n', v_n')\} < 0. \end{aligned}$$

The consequence of this step is that for $n > N$ we have a representation of h at (t, x, v),

$$\begin{aligned} h(t, x_n' + (t - t_0)v_n', v_n') &= h(t_0, x_n', v_n') e^{-\nu(v_n')(t-t_0) - \int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_n' + (\tau - t_0)v_n', v_n') d\tau} \\ &\quad + \int_{t_0}^t \{K_w + w\Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, x_n + (s - t_0)v_n', v_n') \\ &\quad \times e^{-\nu(v_n')(t-s) - \int_s^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_n' + (\tau - t_0)v_n', v_n') d\tau} ds. \end{aligned} \tag{120}$$

Step 4 Claim. For given $\varepsilon > 0$ there exists $\delta > 0$ so that if $|(y, u) - (x_0, v_0)| < \delta$ and $|(x, v) - (x_0, v_0)| < \delta$ and $t_0 < t_{\mathbf{b}}(y, u)$ and $t_0 < t_{\mathbf{b}}(x, v)$ then

$$|h(t_0, y, u) - h(t_0, x, v)| < \varepsilon. \tag{121}$$

We have $h(t_0, y, u) = h_0(y - t_0 u, u) e^{-\nu(u)t_0 - \int_0^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, y - (t_0 - \tau)u, u) d\tau}$

$$+ \int_0^{t_0} \{K_w h + \Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, y - (t_0 - s)u, u) e^{-\nu(u)(t_0 - s) - \int_s^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, y - (t_0 - \tau)u, u) d\tau} ds,$$

and similarly $h(t_0, x, v) = h_0(x - t_0 v, v) e^{-\nu(v)t_0 - \int_0^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x - (t_0 - \tau)v, v) d\tau}$

$$+ \int_0^{t_0} \{K_w h + \Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, x - (t_0 - s)v, v) e^{-\nu(v)(t_0 - s) - \int_s^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x - (t_0 - \tau)v, v) d\tau} ds.$$

Let's compare the arguments of two representations:

$$|(y - t_0 u, u) - (x - t_0 v, v)| < 2(1 + t_0)\delta \ \text{ for } \ h_0,$$

$$|(\tau, y - (t_0 - \tau)u, u) - (\tau, x - (t_0 - \tau)v, v)| < 2(1 + t_0)\delta \ \text{ for } \ \nu(\sqrt{\mu}\frac{h}{w}),$$

$$|(s, y - (t_0 - s)u, u) - (s, x - (t_0 - s)v, v)| < 2(1 + t_0)\delta \ \text{ for } \ K_w h + \Gamma_+(\frac{h}{w}, \frac{h}{w}).$$

Using the continuity of h_0, $\nu(\sqrt{\mu}\frac{h}{w})$, $K_w h$ and $\Gamma_+(\frac{h}{w}, \frac{h}{w})$ we can choose the desired $\varepsilon > 0$ to conclude (121).

Step 5 Claim. Choose $t > 0$ so that $t \in [t_0, t_0 + t_{\mathbf{b}}(x_0, -v_0))$ and denote $x = x_0 + (t - t_0)v_0$, $v = v_0$. Let $\varepsilon \leq \frac{1}{10}[h(t_0)]_{x_0, v_0}$ and $\delta > 0$ be chosen in **Step 4**. Then we can choose $u_n'' \in \Omega$ so that $|u_n'' - v_n''| < \delta$ and $t_0 < t_{\mathbf{b}}(x_n'', u_n'')$ and $t - t_0 < t_{\mathbf{b}}(x_n'', -u_n'')$.

If there are infinitely many u_n'' so that $t_0 < t_{\mathbf{b}}(x_n'', u_n'')$ and $t - t_0 < t_{\mathbf{b}}(x_n'', -u_n'')$, then up to subsequence we can define $u_n'' = v_n''$. Therefore we may assume $t - t_0 \geq t_{\mathbf{b}}(x_n'', -v_n'')$ for all $n \in \mathbb{N}$. We assume that $\Omega \cap B(x_0; \varepsilon) = \{x \in B(x_0; \varepsilon) : x_3 > \Phi(x_1, x_2)\}$ and $n(x_0) = (0, 0, -1)$ and $v_0 = |v_0|(1, 0, 0)$. Now we illustrate how to choose such a u_n''. Denote $x_n'' = x = (x_1, x_2, x_3)$ and $v_n'' = (v_1, v_2, v_3)$. First we will choose (u_1, u_2, u_3) and $s > 0$ so that

$$n(x_{\mathbf{b}}(x, -u)) \cdot u = 0, \tag{122}$$

and $x_{\mathbf{b}}(x, -u) = (x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}}, \Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}}))$. The condition (122) implies that

$$\frac{u_3}{\sqrt{u_1^2 + u_2^2}} = \frac{d}{ds}\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2 + u_2^2}}) = \frac{\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}) - x_3}{s}. \tag{123}$$

In order to use the implicit function theorem we define

$$\begin{aligned}\Psi(x_1, x_2, x_3; u_1, u_2; s) = \Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2 + u_2^2}}) - x_3 \\ -s\left\{\frac{u_1}{\sqrt{u_1^2 + u_2^2}}\partial_{x_1}\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2 + u_2^2}})\right. \\ \left. +\frac{u_2}{\sqrt{u_1^2 + u_2^2}}\partial_{x_2}\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2 + u_2^2}})\right\},\end{aligned}$$

and compute, using (21),

$$\partial_s\Psi = -s(\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, \frac{u_2}{\sqrt{u_1^2 + u_2^2}})\begin{pmatrix}\partial_{x_1}^2\Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_1}\partial_{x_2}\Phi & \partial_{x_2}^2\Phi\end{pmatrix}\begin{pmatrix}\frac{u_1}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_2}{\sqrt{u_1^2+u_2^2}}\end{pmatrix} < -\frac{1}{2}C_{x_0, v_0}, \tag{124}$$

for $x \sim x_0, v \sim v_0$ and the Hessian is evaluated at $(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}})$. Hence $s = s(x_1, x_2, x_3; w_1, w_2)$ is a smooth function near $x \sim x_0$ and $(u_1, u_2) \sim (v_1, v_2)$. In order to study the behavior of s we use Taylor's expansion: from $\Psi(x_1, x_2, x_3; u_1, u_2; s) = 0$ we have

$$\Phi(x_1, x_2) - x_3 = \frac{1}{u_1^2 + u_2^2}\Bigg\{(u_1, u_2)\underbrace{\begin{pmatrix} \partial_{x_1}^2\Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_1}\partial_{x_2}\Phi & \partial_{x_2}^2\Phi \end{pmatrix}}_{(*)}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$-\frac{1}{2}(u_1, u_2)\underbrace{\begin{pmatrix} \partial_{x_1}^2\Phi & \partial_{x_1}\partial_{x_2}\Phi \\ \partial_{x_1}\partial_{x_2}\Phi & \partial_{x_2}^2\Phi \end{pmatrix}}_{(**)}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\Bigg\}s^2,$$

where the Hessian $(*)$ is evaluated at $(x_1+s_*\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2+s_*\frac{u_2}{\sqrt{u_1^2+u_2^2}})$ and the Hessian $(**)$ is evaluated at $(x_1 + s_{**}\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s_{**}\frac{u_2}{\sqrt{u_1^2+u_2^2}})$ with $s_*, s_{**} \in (0, s)$. For $x \sim x_0$ and $(u_1, u_2) \sim (v_1, v_2)$ we know that the right hand side of the above equation converges to

$$-\frac{1}{2(v_1^2 + v_2^2)}(v_1, v_2)\begin{pmatrix} \partial_{x_1}^2\Phi((x_0)_1, (x_0)_2) & \partial_{x_1}\partial_{x_2}\Phi((x_0)_1, (x_0)_2) \\ \partial_{x_1}\partial_{x_2}\Phi((x_0)_1, (x_0)_2) & \partial_{x_2}^2\Phi((x_0)_1, (x_0)_2) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq 0.$$

Hence we have control of s, i.e

$$\frac{1}{C}|\Phi(x_1, x_2) - x_3|^{\frac{1}{2}} \le s \le C|\Phi(x_1, x_2) - x_3|^{\frac{1}{2}}. \tag{125}$$

From (123), $u_3 = \sqrt{u_1^2 + u_2^2}\frac{d}{ds}\Phi(x_1 + s\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2 + s\frac{u_2}{\sqrt{u_1^2+u_2^2}})$ equals

$$\sqrt{u_1^2 + u_2^2}\begin{pmatrix} \frac{u_1}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_2}{\sqrt{u_1^2+u_2^2}} \end{pmatrix}$$
$$\cdot\begin{pmatrix} \partial_{x_1}\Phi(x_1, x_2) + \frac{u_1}{\sqrt{u_1^2+u_2^2}}\partial_{x_1}^2\Phi(x_1', x_2')s + \frac{u_2}{\sqrt{u_1^2+u_2^2}}\partial_{x_1}\partial_{x_2}\Phi(x_1', x_2')s \\ \partial_{x_2}\Phi(x_1, x_2) + \frac{u_1}{\sqrt{u_1^2+u_2^2}}\partial_{x_1}\partial_{x_2}\Phi(x_1', x_2')s + \frac{u_2}{\sqrt{u_1^2+u_2^2}}\partial_{x_2}^2\Phi(x_1', x_2')s \end{pmatrix}, \tag{126}$$

where $x_1' = x_1 + s'\frac{u_1}{\sqrt{u_1^2+u_2^2}}, x_2' = x_2 + s'\frac{u_2}{\sqrt{u_1^2+u_2^2}}$ for some $0 < s' < s \le C|\Phi(x_1, x_2) - x_3|^{\frac{1}{2}}$. Using the smoothness of Φ we can bound (126) as

$$\frac{1}{C}|(u_1, u_2)|\left(|(x_1, x_2)| + |\Phi(x_1, x_2) - x_3|^{\frac{1}{2}}\right)$$
$$\le (126) \le C|(u_1, u_2)|\left(|(x_1, x_2)| + |\Phi(x_1, x_2) - x_3|^{\frac{1}{2}}\right). \tag{127}$$

To sum for fixed x and direction $\frac{1}{\sqrt{u_1^2+u_2^2}}(u_1, u_2)$ we can choose u_3 such that $n(x_{\mathbf{b}}(x, -(u_1, u_2, u_3))) \cdot (u_1, u_2, u_3) = 0$ and u_3 is controlled by (127). Finally we choose $(u_1, u_2) = \sqrt{\frac{u_1^2+u_2^2}{v_1^2+v_2^2}}(v_1, v_2)$ and find the corresponding u_3 so that $|v| = |(u_1, u_2, u_3)|$. Define $u_n'' = -v + 2(v \cdot \frac{(u_1,u_2,u_3)}{|(u_1,u_2,u_3)|})(u_1, u_2, u_3)$. Then we have the desired u_n'' for sufficiently large $n \in \mathbb{N}$.

Step 6. To sum for $(t, x_n''+(t-t_0)u_n'', u_n'')$ we have $t-t_0 < t_{\mathbf{b}}(x_n'', -u_n'')$ and $t_0 < t_{\mathbf{b}}(x_n'', u_n'')$ and $|h(t_0, x_n'', u_n'') - h(t_0, x_n'', v_n'')| < \frac{1}{10}[h(t_0)]_{x_0,v_0}$. Hence the representation of the Boltzmann solution h at $(t, x_n'' + (t - t_0)u_n'', u_n'')$ is given by

$$\begin{aligned} h(t, x_n'' + (t-t_0)v_n'', u_n'') &= h(t_0, x_n'', u_n'')e^{-\nu(u_n'')(t-t_0)-\int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_n''+(\tau-t_0)u_n'', u_n'')d\tau} \\ &\quad + \int_{t_0}^t \{K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w})\}(s, x_n'' + (s-t_0)u_n'', u_n'') \\ &\quad \times e^{-\nu(u_n'')(t-s)-\int_s^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_n''+(\tau-t_0)u_n'', u_n'')d\tau} ds. \end{aligned}$$

Using (120) we have

$$\begin{aligned} &\lim_{n\to\infty} |h(t, x_n' + (t-t_0)v_n', v_n') - h(t, x_n'' + (t-t_0)u_n'', u_n'')| \\ &= \lim_{n\to\infty} |h(t_0, x_n', v_n') - h(t_0, x_n'', u_n'')| e^{-\nu(v_0)(t-t_0)-\int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0+(\tau-t_0)v_0, v_0)d\tau} \\ &\geq \Big\{ \lim_{n\to\infty} |h(t_0, x_n', v_n') - h(t_0, x_n'', v_n'')| \\ &\quad - \lim_{n\to\infty} |h(t_0, x_n'', v_n'') - h(t_0, x_n'', u_n'')| \Big\} e^{-\nu(v_0)(t-t_0)-\int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0+(\tau-t_0)v_0, v_0)d\tau} \\ &\geq \frac{1}{4}[h(t_0)]_{x_0,v_0} e^{-\nu(v_0)(t-t_0)-\int_{t_0}^t \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0+(\tau-t_0)v_0, v_0)d\tau}, \end{aligned}$$

which implies that

$$[h(t)]_{x,v} \geq \frac{1}{4}[h(t_0)]_{x_0,v_0} \times e^{-(C_\mu + C'C_w\|h_0\|_\infty)(1+|v|)^\gamma (t-t_0)} \neq 0.$$

Remark. Through Step 1 to Step 6, we only used the in-flow boundary datum g explicitly in Step 2. All the other steps are valid for diffuse and bounce-back boundary condition cases. In Step 2, we only used (116), the continuity of $G = wg$ on $[0, \infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$. Therefore, if we can show the continuity of F on $[0, \infty) \times \{\gamma_- \cup \gamma_0^{\mathbf{S}}\}$ then we can prove (22). For diffuse and bounce-back boundary we will prove such a continuity to conclude (22).

5. Diffuse Reflection Boundary Condition

In this section, we consider the weighted linearized Boltzmann equation (92) with the diffuse boundary condition

$$\begin{aligned} h(t, x, v) = w(v)\sqrt{\mu(v)} \int_{v'\cdot n(x)>0} h(t, x, v') \\ \times \frac{1}{w(v')\sqrt{\mu(v')}} c_\mu \mu(v')\{n_x \cdot v'\} dv' \quad \text{for } (x, v) \in \gamma_-. \qquad (128) \end{aligned}$$

In spite of the averaging effect of the diffuse reflection operator, we can observe the formation and propagation of discontinuity. Continuity away from $\mathfrak{D}$ is also established.

5.1. Formation of discontinuity. We prove Part 2 of Theorem 1. The idea of proof is similar to the in-flow case but we also use $|v_0|$ not only t_0 as a parameter. Without loss of generality we may assume $x_0 = (0,0,0)$ and $v_0 = (|v_0|,0,0)$ and $(x_0,v_0) \in \gamma_0^{\mathbf{S}}$. Locally the boundary is a graph, i.e. $\Omega \cap B(\mathbf{0};\delta) = \{(x_1,x_2,x_3) \in B(\mathbf{0};\delta) : x_3 > \Phi(x_1,x_2)\}$ and $\Phi(\xi,0) < 0$ for $\xi \in (-\delta,\delta)\backslash\{0\}$ (see Fig. 3).

Assume that $||h_0||_\infty < \delta$ is sufficiently small so that the global solution h of (92) with diffuse boundary (128) has a uniform bound (94), from Theorem 4 of [21]. Choose $t_0 \in (0, \min\{\delta, t_{\mathbf{b}}(x_0,-v_0)\})$ sufficiently small and $|v_0| > 0$ sufficiently large so that

$$\frac{1}{2} \leq \Bigg(e^{-\nu(|v_0|)t_0} - t_0 C_{\mathbf{k}} C' - (1 - e^{-\nu(|v_0|)t_0}) C_\Gamma (C')^2 ||h_0||_\infty \\ - C' \frac{1}{\tilde{w}(v_0)} \int_{\{v_1'>0\}} \tilde{w}(v') d\sigma(v') \Bigg), \tag{129}$$

where $\nu(|v|) = \nu(v)$ and $C_{\mathbf{k}}$ and C_Γ from (35) and (36). More precisely, first choose $|v_0| > 0$ large enough to have

$$\frac{1}{\tilde{w}(v_0)} = \frac{(1+\rho^2|v_0|^2)^\beta}{e^{\frac{|v_0|^2}{4}}} \leq \frac{1}{10C'},$$

then choose $t_0 > 0$ as

$$0 < t_0 = \min\Bigg\{ \frac{\delta}{2}, \frac{t_{\mathbf{b}}(x_0,-v_0)}{2}, \frac{\delta}{|v_0|}, \frac{1}{10\nu(|v_0|)}, \frac{1}{10C_{\mathbf{k}}C'}, \\ \frac{1}{\nu(|v_0|)} \log\left(\frac{10C_\Gamma (C')^2}{10C_\Gamma(C')^2 - 1} \right) \Bigg\}.$$

Assume the condition for initial datum h_0: there is sufficiently small $\delta' = \delta'(\Omega, t_0|v_0|) > 0$ such that $B((-t_0|v_0|,0,0),\delta') \subset \Omega$ and

$$h_0(x_0,v_0) \equiv ||h_0||_\infty > 0 \quad \text{for } (x,v) \in B((-t_0|v_0|,0,0);\delta') \times B((|v_0|,0,0);\delta'). \tag{130}$$

We claim that the Boltzmann solution h with such initial datum h_0 is not continuous at $(t_0,x_0,v_0) = (t_0,(0,0,0),(|v_0|,0,0))$. We will use a contradiction argument: Assume the Boltzmann solution h is continuous at (t_0,x_0,v_0), i.e (97) is valid. Choose sequences of points $(x_n', v_n') = ((0,0,\frac{1}{n}),(|v_0|,0,0))$ and $(x_n,v_n) = ((\frac{1}{n},0,\Phi(\frac{1}{n},0)), \frac{1}{\sqrt{1+\frac{1}{n^2}}}(|v_0|,0,\frac{|v_0|}{n}))$. Because of our choice, for sufficiently large $n \in \mathbb{N}$, we have

$$(x_n' - t_0 v_n', v_n') = ((-t_0|v_0|,0,\frac{1}{n}),(|v_0|,0,0)) \\ \in B((-t_0|v_0|,0,0);\delta') \times B((|v_0|,0,0);\delta').$$

Hence the Boltzmann solution at (t_0, x'_n, v'_n) is

$$\begin{aligned} h(t_0, x'_n, v'_n) &= h_0(x'_n - t_0 v'_n, v'_n)e^{-\nu(v'_n)t_0} \\ &\quad + \int_0^{t_0} e^{-\nu(v'_n)(t_0-\tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} (\tau, x'_n - v(t_0 - \tau), v'_n) d\tau \\ &= ||h_0||_\infty e^{-\nu(|v'_n|)t_0} \\ &\quad + \int_0^{t_0} e^{-\nu(|v'_n|)(t_0-\tau)} \left\{ K_w h + w\Gamma(\frac{h}{w}, \frac{h}{w}) \right\} (\tau, x - v'_n(t_0 - \tau), v'_n) d\tau. \end{aligned}$$

Using the diffuse boundary condition (128), the Boltzmann solution at $(t_0, x_n, v_n) \in [0, \infty) \times \gamma_-$ is

$$h(t_0, x_n, v_n) = \frac{1}{\tilde{w}(|v_0|)} \int_{\mathcal{V}(x_n)} h(t_0, x_n, v')\tilde{w}(v')d\sigma(v').$$

Using a pointwise boundedness (94) of h, and $||h_0||_\infty \le 1$, we can estimate

$$\begin{aligned} &|h(t_0, x'_n, v'_n) - h(t_0, x_n, v_n)| \\ &\ge \Big| \, ||h_0||_\infty e^{-\nu(|v_0|)t_0} - \int_0^{t_0} \{C_{\mathbf{k}} C' ||h_0||_\infty + \nu(v'_n) e^{-\nu(v'_n)(t_0-\tau)} C_\Gamma (C')^2 ||h_0||_\infty^2 \} d\tau \\ &\quad - C' ||h_0||_\infty \frac{1}{\tilde{w}(|v_0|)} \int_{\mathcal{V}} \tilde{w}(v')d\sigma(v') \Big| \\ &\ge ||h_0||_\infty e^{-\nu(|v_0|)t_0} - t_0 C_{\mathbf{k}} C' ||h_0||_\infty - (1 - e^{-\nu(|v_0|)t_0}) C_\Gamma (C')^2 ||h_0||_\infty^2 \\ &\quad - C' ||h_0||_\infty \frac{1}{\tilde{w}(|v_0|)} \int_{\mathcal{V}} \tilde{w}(v')d\sigma(v') \\ &= ||h_0||_\infty \Bigg(e^{-\nu(|v_0|)t_0} - t_0 C_{\mathbf{k}} C' - (1 - e^{-\nu(|v_0|)t_0}) C_\Gamma (C')^2 ||h_0||_\infty \\ &\qquad - C' \frac{1}{\tilde{w}(|v_0|)} \int_{\mathcal{V}} \tilde{w}(v')d\sigma(v') \Bigg) \\ &\ge \frac{||h_0||_\infty}{2} \neq 0, \end{aligned}$$

which is contradiction to (97).

5.2. Continuity away from $\mathfrak{D}$. Instead of using the argument of [21] to show continuity in the case of the diffuse reflection boundary condition we will use the sequence (102) with the boundary condition (131) and Lemma 12. Notice that this sequence is used in (242), p. 803 of [21]. This argument also gives a new proof of the continuity of the Boltzmann solution in a strictly convex domain in a simpler way than [21].

Proof of 2 of Theorem 3. We will use the sequence (102) with $h^{m+1}|_{t=0} = h_0$ with the following boundary condition:

$$h^{m+1}(t, x, v) = \frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} h^m(t, x, v')\tilde{w}(v')d\sigma, \tag{131}$$

with $(t, x, v) \in \gamma_-$. Notice that the smallness of the initial datum is used crucially in the following steps.

Step 1. We claim that

$$\frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} h^m(t, x, v')\tilde{w}(v')d\sigma(v'), \tag{132}$$

is a continuous function on $[0, T] \times \gamma$ even if $h^m \in L^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^3)$ is only continuous on $[0, T] \times \bar{\Omega} \times \mathbb{R}^3 \backslash \mathfrak{G}$. We will show as $(\bar{t}, \bar{x}, \bar{v}) \to (t, x, v)$,

$$\frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} h^m(t, x, v')\tilde{w}(v')d\sigma(v') \to \frac{1}{\tilde{w}(\bar{v})} \int_{\mathcal{V}(\bar{x})} h^m(\bar{t}, \bar{x}, v')\tilde{w}(v')d\sigma(v'). \tag{133}$$

Using the fact $|\mathcal{V}(x)\backslash\mathcal{V}(\bar{x})|, |\mathcal{V}(x)\backslash\mathcal{V}(\bar{x})| \to 0$ as $\bar{x} \to x$ and the exponentially decay weight function of $\tilde{w}d\sigma$ it suffices to show that

$$\int_{\mathcal{V}(x)\cap\mathcal{V}(\bar{x})\cap\{|v'|\le M\}} \{\tilde{w}(v)^{-1}h^m(t, x, v')\tilde{w}(v') - \tilde{w}(\bar{v})^{-1}h^m(\bar{t}, \bar{x}, v')\tilde{w}(v')\}d\sigma(v'), \tag{134}$$

for sufficiently large $M > 0$. Using Lemma 5 we can choose the open set $U_x \subset \{v' \in \mathbb{R}^3 : |v'| \le M\}$ so that $|U_x|$ is small and h^m is uniformly continuous on $\{|v'| \le M\}\backslash U_x$. Therefore we can make $\int_{\mathcal{V}(x)\cap\mathcal{V}(\bar{x})\cap\{|v'|\le M\}\cap U_x}$ small using the smallness of U_x and make $\int_{\mathcal{V}(x)\cap\mathcal{V}(\bar{x})\cap\{|v'|\le M\}\backslash U_x}$ small using the uniform continuity of h^m on $\{|v'| \le M\}\backslash U_x$. Hence (133) is valid.

Step 2. We claim

$$h^i \ \text{ is a continuous function in } \ \mathfrak{C}_T \tag{135}$$

for all $i \in \mathbb{N}$ where $\mathfrak{C}_T$ is defined in (105). By induction choose $h^0 = 0$ and (135) is satisfied for $i = 0$. Assume (135) for all $i = 0, 1, 2, \ldots, m$. Let $w\Gamma_-\left(\frac{h^m}{w}, \frac{h^{m+1}}{w}\right) = \nu\left(\frac{h^m}{w}\right)h^{m+1}$. Then the equation of h^{m+1} is

$$\{\partial_t + v \cdot \nabla_x + \nu(v) + \nu\left(\frac{h^m}{w}\right)\}h^{m+1} = K_w h^m + w\Gamma_+\left(\frac{h^m}{w}, \frac{h^m}{w}\right).$$

From Theorem 4 and Corollary 5 we know that $\nu\left(\frac{h^m}{w}\right)$ and $w\Gamma_+\left(\frac{h^m}{w}, \frac{h^m}{w}\right)$ are both continuous in $[0, T] \times \Omega \times \mathbb{R}^3$. Because of Step 1 we know that $\frac{1}{\tilde{w}(v)} \int_{\mathcal{V}(x)} h^m(t, x, v')\tilde{w}(v')d\sigma(v')$ is also a continuous function on $[0, T] \times \gamma$. So we can apply Lemma 12 to conclude (135) is valid for $i = m + 1$.

Step 3. We claim h^m is a Cauchy sequence in $\mathfrak{C}_T$ for some small $T > 0$. First we will compute some constants explicitly. From (6) the normalized constant c_μ is

$\left[\int_{n(x)\cdot v'>0} e^{-\frac{|v'|^2}{2}}\{n(x)\cdot v'\}dv'\right]^{-1}$. Choose $n(x) = (1, 0, 0)$ and then we can compute the right hand side of the above term:

$$\int_0^\infty dv_1\; v_1 e^{-\frac{v_1^2}{2}} \int_{-\infty}^\infty dv_2\; e^{-\frac{v_2^2}{2}} \int_{-\infty}^\infty dv_3\; e^{-\frac{v_3^2}{2}} = \int_0^\infty \frac{d}{dv_1}\left(-e^{-\frac{v_1^2}{2}}\right) dv_1 \times (\sqrt{2\pi})^2$$
$$= 2\pi \left[-e^{-\frac{v_1^2}{2}}\right]_0^\infty = 2\pi.$$

Therefore we have $c_\mu = \frac{1}{2\pi}$. Next we will show

$$\frac{1}{\tilde{w}(v)} \underbrace{\int_{v'\cdot n(x)>0} \tilde{w}(v')d\sigma(v')} \le \tilde{C}_\beta \rho^{2\beta-4}, \tag{136}$$

where $\tilde{w}(v)^{-1} = (1+\rho^2|v|^2)^\beta e^{-\frac{|v|^2}{4}}$. We follow the computation of Lemma 25 in [21]. For $\frac{1}{\tilde{w}(v)}$, in the case of $\beta\rho^2 > \frac{1}{4}$ we can see that $\tilde{w}(v)^{-1}$ has a maximum value at $|v| = \sqrt{\frac{4\beta\rho^2-1}{\rho^2}}$ which is

$$(1+\rho^2|v|^2)^\beta e^{-\frac{|v|^2}{4}}\Big|_{|v|=\sqrt{\frac{4\beta\rho^2-1}{\rho^2}}} = 4^\beta \beta^\beta e^{-\beta} e^{\frac{1}{4\rho^2}} \rho^{2\beta}, \tag{137}$$

and the underbraced integration in (136) is bounded above by

$$\int_{v'\cdot n(x)>0} \tilde{w}(v')d\sigma(v') = \frac{1}{2\pi}\int_{v_1'>0} (1+\rho^2|v'|^2)^{-\beta} e^{\frac{|v'|^2}{4}} e^{-\frac{|v'|^2}{2}} v_1' dv'$$
$$= \frac{1}{2\pi}\int_{u_1>0} (1+|u|^2)^{-\beta} e^{\frac{-2|u|^2}{4\rho}} \rho^{-4} u_1 du \le \rho^{-4} \times \frac{1}{2\pi}\int_{u_1>0} \frac{1}{(1+|u|^2)^{\beta-\frac{1}{2}}} du$$
$$= C_\beta \rho^{-4},$$

where $\beta \ge 2$ and combining with (137) we conclude (136).

First we will show a boundedness (107).

Lemma 13. *Let h^m be a solution of (102) with $h^{m+1}_{t=0} = h_0$ and the boundary condition (131). Then there exist $T_*, C, \delta > 0$ such that if $||h_0||_\infty < \delta$ then*

$$\sup_{0\le s\le T_*} ||h^m(s)||_\infty < C||h_0||_\infty \;\; \textit{for all} \;\; m \in \mathbb{N}.$$

Proof. We will use mathematical induction. Choose $h^0 = h_0$ and assume $||h_0||_\infty < \delta$ and

$$\sup_{0\le s\le T_*} ||h^i(s)||_\infty \le C||h_0||_\infty, \tag{138}$$

for $i = 0, 1, 2, \ldots, m$, where $\delta, C, T_* > 0$ will be determined later. From Lemma 24 of [21] the representation of h^{m+1} which is a solution of (102) with the boundary condition (131) is given by

$$
\begin{aligned}
&h^{m+1}(t,x,v)\\
&\quad = \mathbf{1}_{t_1\le 0}(t,x,v)\Big\{\underbrace{h_0(x-tv,v)e^{-\nu(v)t}}_{\text{[initial data]}}+\underbrace{\int_0^t e^{-\nu(v)(t-s)}q^m(s,x-(t-s)v,v)ds}_{\mathbf{I}}\Big\}
\end{aligned}
\tag{139}
$$

$$
+\mathbf{1}_{0<t_1}(t,x,v)\Big\{\underbrace{\int_{t_1}^t e^{-\nu(v)(t-s)}q^m(s,x-(t-s)v,v)ds}_{\mathbf{II}}+\frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}(v)}\int_{\prod_{j=1}^k\mathcal{V}_j}H\Big\},
\tag{140}
$$

where q^m was defined (103) and

$$
\begin{aligned}
H &= \sum_{l=1}^k \underbrace{\mathbf{1}_{t_{l+1}\le 0<t_l}h_0(x_l-t_lv_l,v_l)}_{\text{[initial data]}}d\Sigma_l(0)\\
&\quad+\sum_{l=1}^k\underbrace{\int_0^{t_l}\mathbf{1}_{t_{l+1}\le 0<t_l}q^{m-l}(s,x_l-(t_l-s)v_l,v_l)d\Sigma_l(s)ds}_{\mathbf{III}} \qquad (141)\\
&\quad+\sum_{l=1}^k\underbrace{\int_{t_{l+1}}^{t_l}\mathbf{1}_{0<t_{l+1}}q^{m-l}(s,x_l-(t_l-s)v_l,v_l)d\Sigma_l(s)ds}_{\mathbf{IV}}\\
&\quad+\underbrace{\mathbf{1}_{0<t_{k+1}}h^{m-k+1}(t_{k+1},x_{k+1},v_k)d\Sigma_k(t_{k+1})}_{\text{[many bounces]}}. \qquad (142)
\end{aligned}
$$

Here $d\Sigma_k(t_{k+1})$ is evaluated at $s=t_{k+1}$ of

$$
d\Sigma_l(s)=\{\Pi_{j=l+1}^k d\sigma_j\}\{e^{-\nu(v_l)(t_l-s)}\tilde{w}(v_l)d\sigma_l\}\Pi_{j=1}^{l-1}\{e^{-\nu(v_j)(t_j-t_{j+1})}d\sigma_j\}.
$$

First we can estimate [initial data] in (139) and (141),

$$
\begin{aligned}
&\int_{\prod_{j=1}^k\mathcal{V}_j}\left\{\mathbf{1}_{t_1\le 0}|h_0(x-tv,v)|+\frac{1}{\tilde{w}(v)}\sum_{l=1}^k\mathbf{1}_{t_{l+1}\le 0<t_l}|h_0(x_l-t_lv_l,v_l)|\tilde{w}(v_l)\right\}d\sigma_1\ldots d\sigma_k\\
&\le \max\left\{1,\frac{1}{\tilde{w}(v)}\max_{1\le l\le k}\int_{\prod_{j=1}^k\mathcal{V}_j}\tilde{w}(v_l)d\sigma_1\ldots d\sigma_k\right\}||h_0||_\infty\\
&\le\left\{1+\tilde{C}_\beta\rho^{2\beta-4}\right\}||h_0||_\infty,
\end{aligned}
$$

where we used (136).

Next we estimate the [many bounces] term in (142) which is a crucial estimate in this proof. We use Lemma 23 in [21] to bound a contribution of the [many bounces] term in

(142) in the last term of (140) by

$$\begin{aligned}
&\frac{1}{\tilde{w}(v)}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}(t,x,v,v_1,v_2,\ldots,v_k)>0\}}\tilde{w}(v_k)d\sigma_k d\sigma_{k-1}\ldots d\sigma_1\times\sup_{0\le s\le t}||h^{m-k+1}(s)||_\infty\\
&\le\frac{1}{\tilde{w}(v)}\int_{\mathcal{V}_k}\tilde{w}(v_k)d\sigma_k\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k(t,x,v,v_1,\ldots,v_{k-1})>0\}}d\sigma_{k-1}\ldots d\sigma_1\\
&\quad\times\sup_{0\le s\le t}||h^{m-k+1}(s)||_\infty\\
&\le\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}\sup_{0\le s\le t}||h^{m-k+1}(s)||_\infty\le\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}C||h_0||_\infty,
\end{aligned}$$

where we used (136) at the last step. The remainders **I**, **II**, **III** and **IV** are contributions of $q^m,\ldots,q^{m-k}$. We introduce a notation

$$\begin{aligned}
\mathcal{H}_i\equiv\ &tC_{\mathbf{k}}\sup_{0\le s\le t}||h^i(s)||_\infty+C_\Gamma\sup_{0\le s\le t}||h^i(s)||_\infty\\
&\times\left(\sup_{0\le s\le t}||h^i(s)||_\infty+\sup_{0\le s\le t}||h^{i+1}(s)||_\infty\right) &(143)\\
\le\ &C||h_0||_\infty(C_{\mathbf{k}}T_*+2CC_\Gamma||h_0||_\infty), &(144)
\end{aligned}$$

where the above inequality holds for $0\le t\le T_*$ and $i=0,1,2,\ldots,m-1$ and

$$\mathcal{H}_m\le(T_*C_{\mathbf{k}}+C_\Gamma C||h_0||_\infty)C||h_0||_\infty+C_\Gamma C||h_0||_\infty\sup_{0\le s\le T_*}||h^{m+1}(s)||_\infty,\quad(145)$$

where we used the induction hypothesis (138) for (144) and (145). Easily we have

$$\begin{aligned}
\mathbf{I},\mathbf{II}&\le\mathcal{H}_m,\\
\mathbf{III},\mathbf{IV}&\le\sum_{l=1}^{k}\frac{1}{\tilde{w}(v)}\int_{\mathcal{V}_1}d\sigma_1..\int_{\mathcal{V}_{l-1}}d\sigma_{l-1}\int_{\mathcal{V}_{l+1}}d\sigma_{l+1}..\int_{\mathcal{V}_k}d\sigma_k\\
&\quad\times\int_{\mathcal{V}_l}\int_0^{t_l}\mathcal{H}_{m-l}e^{-\nu(v_l)(t_l-s)}\tilde{w}(v_l)dsd\sigma_l\\
&\le\sum_{l=1}^{k}\mathcal{H}_{m-l}\frac{1}{\tilde{w}(v)}\int_{\mathcal{V}_l}\tilde{w}(v_l)d\sigma_l\le\tilde{C}_\beta\rho^{2\beta-4}\sum_{l=1}^{k}\mathcal{H}_{m-l}.
\end{aligned}$$

To summarize, we can estimate all terms of representation of $h^{m+1}(t,x,v)$ in (139) to obtain

$$\begin{aligned}
|h^{m+1}(t,x,v)|\le||h_0||_\infty\Big\{C\Big[&2T_*C_{\mathbf{k}}+2C_\Gamma C||h_0||_\infty\\
&+k\tilde{C}_\beta\rho^{2\beta-4}(C_{\mathbf{k}}T_*+2CC_\Gamma||h_0||_\infty)+\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}\Big]\\
&+1+\tilde{C}_\beta\rho^{2\beta-4}\Big\}+C_\Gamma C||h_0||_\infty\sup_{0\le s\le T_*}||h^{m+1}(s)||_\infty.
\end{aligned}$$

Choose $k = \rho^{5/4}$. Choose $\rho > 0$ sufficiently large so that $\tilde{C}_\beta \rho^{2\beta-4} \{\frac{1}{2}\}^{C_2\rho^{5/4}} \le \frac{1}{30}$ and then choose $T_* > 0$ sufficiently small so that $T_* \times C_\Gamma(1 + \tilde{C}_\beta \rho^{5/4}\rho^{2\beta-4}) \le \frac{1}{30}$ and then choose $C > 0$ sufficiently large $C > 10(1 + \tilde{C}_\beta \rho^{2\beta-4})$ and choose $\delta = \min\left\{\frac{1}{20C_\Gamma C}, \frac{1}{30C_\Gamma}\left(C\tilde{C}_\beta\rho^{5/4}\rho^{2\beta-4}\right)^{-1}\right\}$. Finally assume $||h_0||_\infty \le \delta$. Then we have

$$\begin{aligned}\sup_{0\le s\le T_*} ||h^{m+1}(s)||_\infty \le& \frac{1}{1 - C_\Gamma C||h_0||_\infty}||h_0||_\infty\Big\{1+\tilde{C}_\beta\rho^{2\beta-4}+C\Big[\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}\\ &+tC_\Gamma(1+\tilde{C}_\beta\rho^{5/4}\rho^{2\beta-4})\\ &+C_\Gamma C||h_0||_\infty + 2C_\Gamma\tilde{C}_\beta\rho^{5/4}\rho^{2\beta-4}C||h_0||_\infty\Big]\Big\}\\ \le& \frac{20}{19}||h_0||_\infty\left\{\frac{C}{10}+C\Big[\frac{1}{30}+\frac{1}{30}+\frac{1}{20}+\frac{1}{15}\Big]\right\} \le C||h_0||_\infty.\end{aligned}$$

□

Next we will show that h^m is a Cauchy sequence in L^∞.

Lemma 14. *Let h^m be a solution of (102) with $h^{m+1}|_{t=0} = h_0$ and the boundary condition (131). Then there exist $T_*, C, \delta > 0$ so that if $||h_0||_\infty < \delta$ then h^m is Cauchy in $L^\infty([0, T_*] \times \bar{\Omega} \times \mathbb{R}^3)$.*

Proof. The equation of $h^{m+1} - h^m$ is

$$\begin{aligned}&\{\partial_t + v\cdot\nabla_x + \nu\}(h^{m+1} - h^m) = \tilde{q}^m \quad \text{with} \quad \{h^{m+1} - h^m\}|_{t=0} = 0,\\ &\{h^{m+1} - h^m\}|_{\gamma_-} = \frac{1}{\tilde{w}(v)}\int_{(x)}\{h^m(t,x,v') - h^{m-1}(t,x,v')\}\tilde{w}(v')d\sigma(v'),\end{aligned}$$

where $\tilde{q}^m$ is defined at (109). From Lemma 24 of [21] we have the representation

$$\begin{aligned}\{h^{m+1} - h^m\}(t,x,v) =& \mathbf{1}_{t_1\le 0}(t,x,v)\underbrace{\int_0^t e^{-\nu(v)(t-s)}\tilde{q}^m(s, x-(t-s)v, v)ds}_{\tilde{\mathbf{I}}}\\ &+\mathbf{1}_{0<t_1}(t,x,v)\Big\{\underbrace{\int_{t_1}^t e^{-\nu(v)(t-s)}\tilde{q}^m(s, x-(t-s)v, v)ds}_{\tilde{\mathbf{II}}} + \frac{e^{-\nu(v)(t-t_1)}}{\tilde{w}(v)}\int_{\prod_{j=1}^k\mathcal{V}_j}\tilde{H}\Big\},\end{aligned} \tag{146}$$

where

$$\begin{aligned}\tilde{H} =& \sum_{l=1}^k \underbrace{\int_0^{t_l}\mathbf{1}_{t_{l+1}\le 0<t_l}\tilde{q}^{m-l}(s, x_l-(t_l-s)v_l, v_l)d\Sigma_l(s)ds}_{\tilde{\mathbf{III}}}\\ &+\sum_{l=1}^k\underbrace{\int_{t_{l+1}}^{t_l}\mathbf{1}_{0<t_{l+1}}\tilde{q}^{m-l}(s, x_l-(t_l-s)v_l, v_l)d\Sigma_l(s)ds}_{\tilde{\mathbf{IV}}}\\ &+\underbrace{\mathbf{1}_{0<t_{k+1}}\{h^{m-k+1} - h^{m-k}\}(t_{k+1}, x_{k+1}, v_k)d\Sigma_k(t_{k+1})}_{\text{[[many bounces]]}}.\end{aligned}$$

First using Lemma 24 of [21], we estimate the [[many bounces]] term for sufficiently large $k > 0$ by

$$\begin{aligned}
&\frac{1}{\tilde{w}(v)}\int_{\prod_{j=1}^{k}\mathcal{V}_j}\mathbf{1}_{\{t_{k+1}(t,x,v,v_1,v_2,\ldots,v_k)>0\}}\tilde{w}(v_k)d\sigma_k d\sigma_{k-1}\ldots d\sigma_1\\
&\quad\times\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty\\
&\le\frac{1}{\tilde{w}(v)}\int_{\mathcal{V}_k}\tilde{w}(v_k)d\sigma_k\int_{\prod_{j=1}^{k-1}\mathcal{V}_j}\mathbf{1}_{\{t_k(t,x,v,v_1,\ldots,v_{k-1})>0\}}d\sigma_{k-1}\ldots d\sigma_1\\
&\quad\times\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty\\
&\le\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty.
\end{aligned}$$

Easily we have $\tilde{\mathbf{I}},\ \tilde{\mathbf{II}}\le\delta\mathcal{H}_m,\quad\tilde{\mathbf{III}},\ \tilde{\mathbf{IV}}\le\tilde{C}_\beta\rho^{2\beta-4}\delta\mathcal{H}_{m-l}$, where

$$\begin{aligned}
\delta\mathcal{H}_i\equiv\ &tC_{\mathbf{k}}\sup_{0\le s\le t}||\{h^i-h^{i-1}\}(s)||_\infty\\
&+C||h_0||_\infty C_\Gamma\Big(\sup_{0\le s\le t}||\{h^i-h^{i-1}\}(s)||_\infty+\sup_{0\le s\le t}||\{h^{i+1}-h^i\}(s)||_\infty\Big)\\
\le\ &\frac{\tau}{4}\left\{\sup_{0\le s\le t}||\{h^i-h^{i-1}\}(s)||_\infty+\sup_{0\le s\le t}||\{h^{i+1}-h^i\}(s)||_\infty\right\},
\end{aligned}$$

with $\tau=4\max\{tC_{\mathbf{k}},C||h_0||_\infty C_\Gamma\}$.

To summarize, we can estimate all terms of representation of $h^{m+1}(t,x,v)-h^m(t,x,v)$ in (146) for any $m > k$ to obtain

$$\begin{aligned}
\sup_{0\le s\le t}||\{h^{m+1}-h^m\}(s)||_\infty\le\ &\frac{1}{1-2\tau}\left\{\frac{\tau}{2}\tilde{C}_\beta\rho^{2\beta-4}\sum_{l=1}^{k}\Big(\sup_{0\le s\le t}||\{h^{m-l}-h^{m-l-1}\}(s)||_\infty\right.\\
&+\sup_{0\le s\le t}||\{h^{m-l+1}-h^{m-l}\}(s)||_\infty\Big)\\
&+\frac{\tau}{2}\sup_{0\le s\le t}||\{h^m-h^{m-1}\}(s)||_\infty\\
&\left.+\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}}\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty\right\},
\end{aligned}$$

which is our starting point. Fix a small number $\tilde{\tau} > 0$ chosen later. Choose $\rho > 0$ sufficiently large so that $2\tilde{C}_\beta\rho^{2\beta-4}\left\{\frac{1}{2}\right\}^{C_2\rho^{5/4}} < \frac{\tilde{\tau}}{4}$ and then choose $\tau > 0$ so small that $\frac{\tau/2}{1-2\tau}\tilde{C}_\beta\rho^{2\beta-4} < \frac{\tilde{\tau}}{4}$ and $\frac{\tau/2}{1-2\tau} < \frac{\tilde{\tau}}{4}$. Then we have

$$\begin{aligned}
&\sup_{0\le s\le t}||\{h^{m+1}-h^m\}(s)||_\infty\\
&\quad\le\tilde{\tau}\left\{\sup_{0\le s\le t}||\{h^m-h^{m-1}\}(s)||_\infty+\cdots+\sup_{0\le s\le t}||\{h^{m-k+1}-h^{m-k}\}(s)||_\infty\right\}.
\end{aligned}\tag{147}$$

Using (147) for $m, j \in \mathbb{N}$ so that $m-(i+1)k>0$ and $j=0,1,\ldots,m-1$ it is easy to show

$$\sup_{0\le s\le t} ||\{h^{m-ik+1+j}-h^{m-ik+j}\}(s)||_\infty \le \tilde\tau(1+\tilde\tau)^j$$
$$\times\left\{\sup_{0\le s\le t}||\{h^{m-ik}-h^{m-ik-1}\}(s)||_\infty+\cdots+\sup_{0\le s\le t}||\{h^{m-(i+1)k+1}-h^{m-(i+1)k}\}(s)||_\infty\right\}.$$

We apply the above inequality term by term in (147) to have

$$\begin{aligned}\sup_{0\le s\le t}||\{h^{m+1}-h^m\}(s)||_\infty &\le \tilde\tau\{(1+\tilde\tau)^k-1\}\{\sup_{0\le s\le t}||\{h^{m-k}-h^{m-k-1}\}(s)||_\infty+\cdots\\ &\quad+\sup_{0\le s\le t}||\{h^{m-2k+1}-h^{m-2k}\}(s)||_\infty\}\\ &\le \tilde\tau\{(1+\tilde\tau)^k-1\}^i\{\sup_{0\le s\le t}||\{h^{m-ik}-h^{m-ik-1}\}(s)||_\infty+\cdots\\ &\quad+\sup_{0\le s\le t}||\{h^{m-(i+1)k+1}-h^{m-(i+1)k}\}(s)||_\infty\}.\end{aligned}$$

Now we estimate

$$\begin{aligned}\sup_{0\le s\le t}||\{h^m-h^n\}(s)||_\infty &\le \sum_{l=0}^{m-n-1}\sup_{0\le s\le t}||\{h^{m-l}-h^{m-l-1}\}(s)||_\infty\\
&\le \sum_{l=0}^{m-n-1}\tilde\tau\{(1+\tilde\tau)^k-1\}^i\{\sup_{0\le s\le t}||h^{m-ik-l-1}-h^{m-ik-l-2}||_\infty+\ldots\\
&\quad+\sup_{0\le s\le t}||h^{m-(i+1)k-l}-h^{m-(i+1)k-l-1}||_\infty\}\\
&\le \sum_{l=0}^{m-n-1}\tilde\tau\{(1+\tilde\tau)^k-1\}^{\left[\frac{m-l-1}{k}\right]-1}\{\sup_{0\le s\le t}||h^{2k}-h^{2k-1}||_\infty+\ldots+\sup_{0\le s\le t}||h^1-h^0||_\infty\}\\
&\le \tilde\tau\{(1+\tilde\tau)^k-1\}^{[\frac{n}{k}]-1}\sum_{l=0}^{m-n-1}\{(1+\tilde\tau)^k-1\}^{\left[\frac{m-l-1}{k}\right]-[\frac{n}{k}]}\{\sup_{0\le s\le t}||h^{2k}-h^{2k-1}||_\infty+\ldots\\
&\quad+\sup_{0\le s\le t}||h^1-h^0||_\infty\}\\
&\le \tilde\tau\{(1+\tilde\tau)^k-1\}^{[\frac{n}{k}]-1}\frac{1}{2-(1+\tilde\tau)^k}\{\sup_{0\le s\le t}||h^{2k}-h^{2k-1}||_\infty+\ldots+\sup_{0\le s\le t}||h^1-h^0||_\infty\},\end{aligned}$$

where we choose $i=\left[\frac{m-l-1}{k}\right]-1$ so that $m-(i+1)k-l-1\in[0,k)$. If $\tilde\tau>0$ is chosen sufficiently small so that $(1+\tilde\tau)^k-1\le\frac{1}{2}$, then $\{(1+\tilde\tau)^k-1\}^{[\frac{n}{k}]-1}\to 0$ as $n\to\infty$ which implies that

$$\sup_{0\le s\le t}||\{h^m-h^n\}(s)||_\infty\to 0, \tag{148}$$

as $m,n\to\infty$. Thus h^m is Cauchy in L^∞. □

Step 4. We claim that h is continuous in $\mathfrak{C}$. Notice that T only depends on $||h_0||_\infty$ and $\sup_{0\leq s\leq T}||wg(s)||_\infty$ (Theorem 1 of [21]). Using a unform bound of $\sup_{0\leq s<\infty}||h(s)||_\infty$, we can obtain the continuity of h for all time by repeating $[0,T]$, $[T,2T]$, ... If the boundary $\partial\Omega$ does not include a line segment (6) then every step is valid with $[0,\infty)\times\{\bar{\Omega}\times\mathbb{R}^3\}\backslash\mathfrak{D}$ instead of $\mathfrak{C}$ and $[0,T]\times\{\bar{\Omega}\times\mathbb{R}^3\}\backslash\mathfrak{D}$ instead of $\mathfrak{C}_T$.

5.3. Propagation of discontinuity.

Proof of 2 of Theorem 2.

Proof of (20). The proof is exactly same as the in-flow case in Sect. 4.3.

Proof of (22). The proof is exactly same as the proof of the in-flow case in Sect. 4.3 except for **Step 2**. As we mentioned in the Remark of **Step 2**, we need to show a continuity of a boundary datum on $\gamma_-\cup\gamma_0^{\mathbf{S}}$. In the diffuse reflection boundary condition case, we need

$$
\begin{aligned}
0 = [\, h|_{[0,\infty)\times\gamma_-}]_{t,y,v} &= \lim_{\delta\downarrow 0} \sup_{\substack{t',t''\in B(t;\delta)\\ (y',v'),(y'',v'')\in\gamma_-\cap B((y,v);\delta)\backslash(y,v)}} |h(t',y',v')-h(t'',y'',v'')| \\
&= \lim_{\delta\downarrow 0} \sup_{\substack{t',t''\in B(t;\delta)\\ (y',v'),(y'',v'')\in\gamma_-\cap B((y,v);\delta)\backslash(y,v)}} \left| \frac{1}{\tilde{w}(v')}\int_{\mathcal{V}(y')} h(t',y',\mathfrak{v})\tilde{w}(\mathfrak{v})d\sigma(\mathfrak{v}) \right. \\
&\quad \left. - \frac{1}{\tilde{w}(v'')}\int_{\mathcal{V}(y'')} h(t'',y'',\mathfrak{v})\tilde{w}(\mathfrak{v})d\sigma(\mathfrak{v}) \right|
\end{aligned}
$$

for $(y,v)\in\gamma_-\cup\gamma_0^{\mathbf{S}}$. This is already proven in Sect. 5.2, Continuity away from $\mathfrak{D}$.

6. Bounce-Back Boundary Condition

In this section, we consider the linear Boltzmann equation (92) with the bounce-back boundary condition

$$
h(t,x,v) = h(t,x,-v) \quad \text{for } (x,v)\in\gamma_-. \tag{149}
$$

6.1. Formation of discontinuity. We prove part 3 of Theorem 1. Without loss of generality we may assume $x_0=(0,0,0)$ and $v_0=(1,0,0)$ and $(x_0,v_0)\in\gamma_0^{\mathbf{S}}$. Locally the boundary is a graph, i.e. $\Omega\cap B(\mathbf{0};\delta)=\{(x_1,x_2,x_3)\in B(\mathbf{0};\delta): x_3>\Phi(x_1,x_2)\}$. The condition $(x_0,v_0)\in\gamma_0^{\mathbf{S}}$ implies $t_{\mathbf{b}}(x_0,v_0)\neq 0$ and $t_{\mathbf{b}}(x_0,-v_0)\neq 0$ which means $\Phi(\xi,0)<0$ for $\xi\in(-\delta,\delta)\backslash\{0\}$ (see Fig. 3).

Assume that $||h_0||_\infty<\delta$ is sufficiently small so that the global solution h of (92) with bounce-back boundary (149) has a uniform bound (94), from Theorem 2 of [21].

Recall the constants $C_{\mathbf{k}}$ and C_Γ from (35) and (36). Choose $t_0\in(0,\min\{\frac{\delta}{2},\frac{t_{\mathbf{b}}(x_0,-v_0)}{2},\frac{t_{\mathbf{b}}(x_0,v_0)}{2}\})$ sufficiently small so that

$$
\frac{1}{2}\leq\left(e^{-\nu(1)t_0}-t_0C_{\mathbf{k}}C'-(1-e^{-\nu(1)t_0})C_\Gamma(C')^2\right). \tag{150}
$$

Assume a condition for the initial datum h_0: there is sufficiently small $\delta'=\delta'(\Omega,t_0)>0$ such that

$B((-t_0,0,0),\delta'),\ B((t_0,0,0),\delta')\subset\Omega$ and

$$h_0(x,v)\equiv ||h_0||_\infty>0 \quad \text{for } (x,v)\in B((-t_0,0,0);\delta')\times B((1,0,0);\delta'),$$
$$h_0(x,v)\equiv -||h_0||_\infty>0 \quad \text{for } (x,v)\in B((t_0,0,0);\delta')\times B((-1,0,0);\delta').$$

We will use a contradiction argument: Assume the Boltzmann solution h is continuous at (t_0,x_0,v_0), i.e. (97) is valid. Choose sequences of points $(x_n',v_n')=((0,0,\frac{1}{n}),(1,0,0))$ and $(x_n,v_n)=((\frac{1}{n},0,\Phi(\frac{1}{n},0)),\frac{1}{\sqrt{1+\frac{1}{n^2}}}(1,0,\frac{1}{n}))$. Because of our choice, for sufficiently large $n\in\mathbb{N}$, we have

$$\begin{aligned}(x_n'-t_0v_n',v_n')&=((-t_0,0,\frac{1}{n}),(1,0,0))\in B((-t_0,0,0);\delta')\times B((1,0,0);\delta'),\\ (x_n-t_0(-v_n),-v_n)&=((\frac{1}{n}+\frac{t_0}{\sqrt{1+1/n^2}},0,\Phi(\frac{1}{n},0)+\frac{t_0}{n\sqrt{1+1/n^2}}),\\ &\qquad\frac{1}{\sqrt{1+1/n^2}}(-1,0,-\frac{1}{n}))\\ &\qquad\in B((t_0,0,0);\delta')\times B((-1,0,0);\delta').\end{aligned}$$

Hence the Boltzmann solution at (t_0,x_n',v_n') and (t_0,x_n,v_n) is

$$\begin{aligned}h(t_0,x_n',v_n')&=||h_0||_\infty e^{-\nu(v_n')t_0}\\ &\quad+\int_0^{t_0}e^{-\nu(-v_n')(t_0-\tau)}\{K_wh+w\Gamma\left(\frac{h}{w},\frac{h}{w}\right)\}(\tau,x_n'-(-v_n')(t_0-\tau),-v_n')d\tau,\\ h(t_0,x_n,v_n)&=h(t_0,x_n,-v_n)\\ &=-||h_0||_\infty e^{-\nu(-v_n)t_0}\\ &\quad+\int_0^{t_0}e^{-\nu(-v_n)(t_0-\tau)}\{K_wh+w\Gamma\left(\frac{h}{w},\frac{h}{w}\right)\}(\tau,x_n-(-v_n)(t_0-\tau),-v_n)d\tau.\end{aligned}$$

Using a pointwise boundedness (94) of h with (35) and (36), we have

$$h(t_0,x_n',v_n')\geq ||h_0||_\infty e^{-\nu(1)t_0}-t_0C_{\mathbf{k}}C'||h_0||_\infty-(1-e^{-\nu(1)t_0})C_\Gamma(C')^2||h_0||_\infty^2,$$
$$h(t_0,x_n,v_n)\leq -||h_0||_\infty e^{-\nu(1)t_0}+t_0C_{\mathbf{k}}C'||h_0||_\infty+(1-e^{-\nu(1)t_0})C_\Gamma(C')^2||h_0||_\infty^2.$$

Therefore using (150),

$$\begin{aligned}h(t_0,x_n',v_n')-h(t_0,x_n,v_n)&\geq 2||h_0||_\infty\left(e^{-\nu(1)t_0}-t_0C_{\mathbf{k}}C'-(1-e^{-\nu(1)t_0})C_\Gamma(C')^2\right)\\ &\geq ||h_0||_\infty\neq 0,\end{aligned}$$

which is a contradiction to (97).

6.2. Continuity away from $\mathfrak{D}_{bb}$. We recall some basic facts to study the bounce-back boundary condition from [21].

Definition 7 [21] (Bounce-Back Cycles). *Let* $(t, x, v) \notin \gamma_0 \cup \gamma_-$. *Let* $(t_0, x_0, v_0) = (t, x, v)$ *and inductively define for* $k \geq 1$*:*

$$(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_{\mathbf{b}}(x_k, v_k), x_{\mathbf{b}}(x_k, v_k), -v_k).$$

We define the back-time cycles as:

$$X_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)\{x_k + (s - t_k)v_k\}, \quad V_{\mathbf{cl}}(s; t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(s)v_k. \tag{151}$$

Clearly, we have $v_{k+1} \equiv (-1)^{k+1}v$, for $k \geq 1$,

$$x_k = \frac{1 - (-1)^k}{2}x_1 + \frac{1 + (-1)^k}{2}x_2, \tag{152}$$

where $x_1 = x - t_{\mathbf{b}}(x, v)v$ and $x_2 = x - [2t_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x, -v)](-v)$ and let $d = t_1 - t_2$, then $t_k - t_{k+1} = d \geq t_{\mathbf{b}}(t, x, v) > 0$ for $k \geq 1$, and

$$\begin{aligned} t_1(t, x, v) &= t - t_{\mathbf{b}}(x, v)\ , \\ t_2(t, x, v) &= t_1 - t_{\mathbf{b}}(x_1, v_1) = t_1 - (t_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x_1, v_1)) \\ &= t_1 - (2t_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x, -v)), \\ &\ \vdots \\ t_{k+1}(t, x, v) &= t_1 - k(2t_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x, -v)). \end{aligned} \tag{153}$$

Lemma 15 [21]. *Let* $h_0 \in L^\infty(\Omega \times \mathbf{R}^3)$ *and* $\phi(t, x, v)$ *with* $\sup_{[0,T]\times\Omega} |\phi(\cdot, \cdot, v)| < \infty$. *There exists a unique solution* $G(t)h_0$ *of*

$$\{\partial_t + v \cdot \nabla_x + \phi\}\{G(t)h_0\} = 0, \quad \{G(0)h_0\} = h_0,$$

with the bounce-back reflection $\{G(t)h_0\}(t, x, v) = \{G(t)h_0\}(t, x, -v)$ *for* $x \in \partial\Omega$. *For almost any* $(x, v) \in \bar{\Omega} \times \mathbf{R}^3 \setminus \gamma_0$,

$$\{G(t)h_0\}(t, x, v) = \sum_k \mathbf{1}_{[t_{k+1}, t_k)}(0)h_0\left(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)\right) e^{-\int_0^t \phi(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))d\tau}, \tag{154}$$

where $X_{\mathbf{cl}}(\tau) = X_{\mathbf{cl}}(\tau; t, x, v)$ *and* $V_{\mathbf{cl}}(\tau) = V_{\mathbf{cl}}(\tau; t, x, v)$ *in (151).*

Next we prove a generalized version of Lemma 16 in [21].

Lemma 16 (Continuity away from $\mathfrak{D}_{bb}$: Transport Equation). *Let* Ω *be an open subset of* $\mathbb{R}^3$ *with a smooth boundary* $\partial\Omega$ *and an initial datum* $h_0(x, v)$ *be continuous in* $\Omega \times \mathbb{R}^3 \cup \{\gamma_- \cup \gamma_+ \cup \gamma_0^I\}$. *Also assume* $q(t, x, v)$ *and* $\phi(t, x, v)$ *is continuous in the interior of* $[0, T] \times \Omega \times \mathbb{R}^3$ *and* $\sup_{[0,T]\times\Omega\times\mathbb{R}^3} |q(t, x, v)| < \infty$ *and* $\sup_{[0,T]\times\Omega} |\phi(\cdot, \cdot, v)| < \infty$ *for all* $v \in \mathbb{R}^3$. *Let* $h(t, x, v)$ *be the solution of*

$$\{\partial_t + v \cdot \nabla_x + \phi\}h = q\ , \quad h(0, x, v) = h_0\ , \quad h|_{\gamma_-}(t, x, v) = h(t, x, -v).$$

Assume the compatibility condition on $\gamma_- \cup \gamma_0^{I-}$,

$$h_0(x, v) = h_0(x, -v).$$

Then the Boltzmann solution $h(t, x, v)$ *is continuous on* $\mathfrak{C}_{bb}$. *Further, if the boundary* $\partial\Omega$ *does not include a line segment (6) then* $h(t, x, v)$ *is continuous on a complementary set of the discontinuity set, i.e.* $[0, T] \times \{\bar{\Omega} \times \mathbb{R}^3\}\backslash\mathfrak{D}_{bb}$.

Proof. The proof is similar to the proof of Lemma 16 of [21]. Take any point $(t, x, v) \in [0, T] \times \bar{\Omega} \times \mathbb{R}^3$ and recall its back-time cycle and (154). Assume $t_{m+1} \le 0 < t_m$. Using (154), $h(t, x, v)$ takes the form

$$\begin{aligned} &h_0(x_m - t_m v_m, v_m) e^{-\sum_{k=0}^{m-1} \int_{t_{k+1}}^{t_k} \phi(\tau, x_k - (t_k - \tau) v_k, v_k) d\tau - \int_0^{t_m} \phi(\tau, x_m - (t_m - \tau) v_m, v_m) d\tau} \\ &\quad + \sum_{k=0}^{m-1} \int_{t_{k+1}}^{t_k} q(s, x_k - (t_k - s) v_k, v_k) \\ &\qquad \times e^{-\sum_{i=0}^{k-1} \int_{t_{i+1}}^{t_i} \phi(\tau, x_i - (t_i - \tau) v_i, v_i) d\tau - \int_s^{t_k} \phi(\tau, x_k - (t_k - \tau) v_k, v_k) d\tau} \\ &\quad + \int_0^{t_m} q(s, x_m - (t_m - s) v_m, v_m) \\ &\qquad \times e^{-\sum_{i=0}^{m-1} \int_{t_{i+1}}^{t_i} \phi(\tau, x_i - (t_i - \tau) v_i, v_i) d\tau - \int_s^{t_m} \phi(\tau, x_m - (t_m - \tau) v_m, v_m) d\tau}. \end{aligned} \tag{155}$$

Take any point $(t, x, v) \in \mathfrak{C}_{bb}$. By the definition of $\mathfrak{C}_{bb}$ we assume that $(x, v) \in \Omega \times \mathbb{R}^3$ or $(x, v) \in \gamma_- \cup \gamma_0^{I-}$ and we can separate three cases: $t - t_{\mathbf{b}}(x, v) < 0$, $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$ with $t < 2t_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x, -v)$, and $(x_{\mathbf{b}}(x, -v), -v) \in \gamma_- \cup \gamma_0^{I-}$ with $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$.

Case of $t < t_{\mathbf{b}}(x, v)$. Simply we have $h(t, x, v) = h_0(x - tv, v) e^{-\int_0^t \phi(\tau, x-(t-\tau)v, v) d\tau} + \int_0^t q(s, x - (t - s)v, v) e^{\int_s^t \phi(\tau, x-(t-\tau)v, v) d\tau} ds$ and use the continuity of $q(t, x, v)$ and $\phi(t, x, v)$ to conclude the continuity of $h(t, x, v)$.

Case of $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$ *with* $t < 2t_{\mathbf{b}}(x, v) + t_{\mathbf{b}}(x, -v)$. A representation of $h(t, x, v)$ takes the form

$$\begin{aligned} &h_0(x_1 - t_1 v_1, v_1) e^{-\int_{t_1}^t \phi(\tau, x-(t-\tau)v, v) d\tau - \int_0^{t_1} \phi(\tau, x_1 - (t_1 - \tau) v_1, v_1) d\tau} \\ &\quad + \int_{t_1}^t q(s, x - (t - s)v, v) e^{-\int_s^t \phi(\tau, x-(t-\tau)v, v) d\tau} ds \\ &\quad + \int_0^{t_1} q(s, x_1 - (t_1 - s) v_1, v_1) e^{-\int_{t_1}^t \phi(\tau, x-(t-\tau)v, v) d\tau - \int_s^{t_1} \phi(\tau, x_1 - (t_1 - \tau) v_1, v_1) d\tau} ds. \end{aligned}$$

Thanks to Lemma 1 and Lemma 2, the condition $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$ implies continuity of $x_1(x, v) = x - x_{\mathbf{b}}(x, v)$, $t_1(t, x, v) = t - t_{\mathbf{b}}(x, v)$. Therefore we can show the continuity of $h(t, x, v)$.

Case of $(x_{\mathbf{b}}(x, -v), -v) \in \gamma_- \cup \gamma_0^{I-}$ *with* $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$. We have (155) for $h(t, x, v)$. Thanks to (152) and (153) and Lemma 1 and Lemma 2, the conditions $(x_{\mathbf{b}}(x, -v), -v) \in \gamma_- \cup \gamma_0^{I-}$ and $(x_{\mathbf{b}}(x, v), v) \in \gamma_- \cup \gamma_0^{I-}$ imply continuity of $x_k(x, v)$, $v_k(x, v)$, $t_k(t, x, v)$. Therefore we can show the continuity of $h(t, x, v)$. □

Proof of Part 1 of Theorem 3. Following the in-flow and diffuse cases, we use the iteration scheme (102) which is equivalent to (106) with bounce-back boundary condition $h^{m+1}|_{\gamma_-}(t, x, v) = h^{m+1}(t, x, -v)$ and an initial condition $h^{m+1}|_{t=0} = h_0$.

Step 1. We claim that h^i is a continuous function in $\mathfrak{C}_{bb,T}$ for all $i \in \mathbb{N}$ and for any $T > 0$, where $\mathfrak{C}_{bb,T} = \mathfrak{C}_{bb} \cap \{[0, T] \times \bar{\Omega} \times \mathbb{R}^3\}$. Choose $h^0 \equiv 0$ and use mathematical induction. Assume h^i is continuous $\mathfrak{C}_{bb,T}$ for $i = 0, 1, 2, \ldots, m$. Apply Lemma 16 to conclude that h^{m+1} is continuous in $\mathfrak{C}_{bb,T}$.

Step 2. We claim that there exist $C > 0$ and $\delta > 0$ such that if $C||h_0||_\infty < \delta$ then there exists $T = T(C, \delta) > 0$ so that $\sup_{0\le s\le T} ||h^m(s)||_\infty \le C||h_0||_\infty$ and $\{h^m\}_{m=0}^\infty$ is Cauchy in $L^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^3)$. First we will show the boundedness using mathematical induction. Assume $\sup_{0\le s\le T} ||h^m(s)||_\infty \le C||h_0||_\infty$, where $T > 0$ will be chosen later. Applying Lemma 15, ϕ and q correspond with ν and the right hand side of (102) respectively to have a representation of $h^{m+1}(t, x, v)$,

$$h_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))e^{-\nu(v)t} + \int_0^t e^{-\nu(v)(t-s)}\{K_w h^m + w\Gamma_+\left(\frac{h^m}{w}, \frac{h^m}{w}\right)$$
$$-w\Gamma_-\left(\frac{h^m}{w}, \frac{h^{m+1}}{w}\right)\}(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))ds,$$

where $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)] = [X_{\mathbf{cl}}(s; t, x, v), V_{\mathbf{cl}}(s; t, x, v)]$ is in (151). The above term is bounded by

$$||h_0||_\infty + tC_{\mathbf{k}} \sup_{0\le s\le t} ||h^m(s)||_\infty + C_\Gamma \sup_{0\le s\le t} ||h^m(s)||_\infty \sup_{0\le s\le t} (||h^m(s)||_\infty + ||h^{m+1}(s)||_\infty),$$

where the constants are coming from basic estimates, (35) and (36). Choose $C > 4$ and $\delta < \frac{1}{2C_\Gamma}$ and $T = \frac{C-3}{2C_{\mathbf{k}}C}$. Then we have $\sup_{0\le s\le T} ||h^{m+1}(s)||_\infty \le C||h_0||_\infty$.

Next we will show $\{h^m\}_{m=0}^\infty$ is Cauchy in $L^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^3)$. Recall $\tilde{q}^m(t, x, v)$ from (109). The equation of $h^{m+1} - h^m$ is (108) with a zero initial condition $(h^{m+1} - h^m)|_{t=0} = 0$ and a bounce-back boundary condition $(h^{m+1} - h^m)|_{\gamma_-}(t, x, v) = (h^{m+1} - h^m)(t, x, -v)$. Applying Lemma 15 to (108) we have

$$(h^{m+1} - h^m)(t, x, v) = \int_0^t e^{-\nu(v)(t-s)}\tilde{q}^m(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))ds,$$

where $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)] = [X_{\mathbf{cl}}(s; t, x, v), V_{\mathbf{cl}}(s; t, x, v)]$ is in (151). Then we have exactly the same estimates of the in-flow case to conclude $\{h^m\}$ is Cauchy.

Step 3. Same argument as the in-flow case but substitute $\mathfrak{C}_{bb,T}$, $\mathfrak{C}_{bb}$, $\mathfrak{D}_{bb,T}$, $\mathfrak{D}_{bb}$ for $\mathfrak{C}_T$, $\mathfrak{C}$, $\mathfrak{D}_T$, $\mathfrak{D}$ respectively.

6.3. Propagation of discontinuity.

Proof of 2 of Theorem 2.

Proof of (20). The proof is exactly same as the in-flow case in Sect. 4.3.

Proof of (22). Recall that we have $[h(t_0)]_{x_0,v_0} \neq 0$ for $(x_0, v_0) \in \gamma_0^{\mathbf{S}}$ and $t_0 \in (0, \min\{t_{\mathbf{b}}(x_0, -v_0), t_{\mathbf{b}}(x_0, v_0)\})$. The proof is exactly same as the proof of in-flow case in Sect. 4.3 except for **Step 2**. We need to show a continuity of a boundary datum on $\gamma_- \cup \gamma_0^{\mathbf{S}}$. In the bounce-back reflection boundary condition case, we need to show

$$\begin{aligned} 0 &= [\, h|_{[0,\infty)\times\gamma_-}]_{t_0,x_0,v_0} \\ &= \lim_{\delta\downarrow 0} \sup_{\substack{t',t''\in B(t;\delta) \\ (y',v'),(y'',v'')\in\gamma_-\cap B((x_0,v_0);\delta)\backslash(x_0,v_0)}} |h(t', y', v') - h(t'', y'', v'')|. \end{aligned}$$

Because (y', v') is in the incoming boundary γ_-, using the bounce-back boundary condition, we have $h(t', y', v') = h(t', y', -v')$. Further due to the condition $0 < t_0 < t_{\mathbf{b}}(x_0, -v_0)$ we have $0 < t' < t_{\mathbf{b}}(y', -v')$ and

$$\begin{aligned} h(t', y', v') = h(t', y', -v') &= h_0(y' + t'v', v')e^{-\nu(v')t' - \int_0^{t'} \nu(\sqrt{\mu}\frac{h}{w})(\tau, y'+(t'-\tau)v', v')d\tau} \\ &\quad + \int_0^{t'} \{K_w h + w\Gamma_+(\frac{h}{w}, \frac{h}{w})\}(s, y' + (t'-s)v', v') \\ &\quad \times e^{-\nu(v')(t'-s) - \int_s^{t'} \nu(\sqrt{\mu}\frac{h}{w})(\tau, y'+(t'-\tau)v', v')d\tau} ds, \end{aligned}$$

and a similar representation for $h(t', y', v')$. Using the continuity of $\nu(\sqrt{\mu}\frac{h}{w})$, $K_w h$ and $w\Gamma_+(\frac{h}{w}, \frac{h}{w})$ we have

$$\begin{aligned} 0 &= [\, h|_{[0,\infty)\times\gamma_-}]_{t_0, x_0, v_0} \\ &= \lim_{\delta \downarrow 0} \sup_{\substack{t', t'' \in B(t;\delta) \\ (y', v'), (y'', v'') \in \gamma_- \cap B((x_0, v_0);\delta) \backslash (x_0, v_0)}} |h_0(y' + t'v', v') - h_0(y'' + t''v'', v'')| \\ &\quad \times e^{-\nu(v_0)t_0 - \int_0^{t_0} \nu(\sqrt{\mu}\frac{h}{w})(\tau, x_0 + (t_0-\tau)v_0, v_0)d\tau}, \end{aligned}$$

where we used the continuity of the initial datum h_0 in the last equality.

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