Commun. Math. Phys. 307, 713–759 (2011)
Digital Object Identifier (DOI) 10.1007/s00220-011-1350-6

Communications in **Mathematical Physics**

Global Solutions to the 3-D Incompressible Anisotropic Navier-Stokes System in the Critical Spaces

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Received: 13 April 2010 / Accepted: 29 May 2011
Published online: 24 September 2011 –

Abstract: In this paper, we consider the global wellposedness of the 3-D incompressible anisotropic Navier-Stokes equations with initial data in the critical Besov-Sobolev type spaces $\mathcal{B}$ and $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$ (see Definitions 1.1 and 1.2 below). In particular, we proved that there exists a positive constant C such that (ANS_ν) has a unique global solution with initial data $u_0 = (u_0^h, u_0^3)$ which satisfies $\|u_0^h\|_{\mathcal{B}} \exp\big(\frac{C}{\nu^4}\|u_0^3\|_{\mathcal{B}}^4\big) \le c_0\nu$ or $\|u_0^h\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \exp\big(\frac{C}{\nu^4}\|u_0^3\|^4_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\big) \le c_0\nu$ for some c_0 sufficiently small. To overcome the difficulty that Gronwall's inequality can not be applied in the framework of Chemin-Lerner type spaces, $\widetilde{L_t^p}(\mathcal{B})$, we introduced here sort of weighted Chemin-Lerner type spaces, $\widetilde{L^2_{t,f}}(\mathcal{B})$ for some apropriate L^1 function $f(t)$.

1. Introduction

We first recall the classical (isotropic) Navier-Stokes system for incompressible fluids in the whole space:

$$(NS_\nu)\quad \begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p, & (t,x)\in(0,\infty)\times\mathbb{R}^3,\\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $u(t,x)$ denote the fluid velocity and $p(t,x)$ the pressure. In the seminal paper [21], J. Leray proved the global existence of finite energy weak solutions to (NS_ν). This result used the structure of the nonlinear terms in (NS_ν) in order to obtain the energy inequality.

An approach due to T. Kato reduces the solving of (NS_ν) to the search of a fixed point for some quadratic functional. The first result in that direction is the theorem of H. Fujita and T. Kato (see [13]) in which the authors proved that the system (NS_ν) is globally wellposed for small initial data in the homogeneous Sobolev spaces $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ which is the space of tempered distributions u with Fourier transform of which satisfy

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi|\,|\widehat{u}(\xi)|^2 d\xi < \infty.$$

Cannone, Meyer and Planchon [3], Cannone [4], and Planchon [25] proved a similar result for (NS_ν) with initial data in the negative Besov spaces $B^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ for $3 < p < \infty$. The interest of Besov spaces with negative regularity indices concerns the global wellposedness of the Navier-Stokes equations with highly oscillatory initial data. A different important role of Besov spaces is to give a functional framework to construct global self-similar solutions for (NS_ν) with small data homogeneous of degree -1 (see [3]). This approach has reached its end point with the theorem of Koch and Tataru [18]. Their theorem implies in particular that, for a given function ϕ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, if we consider the family of initial data u_0^ε defined by

$$u_0^\varepsilon(x) \stackrel{\text{def}}{=} \frac{\lambda}{\varepsilon} \sin\left(\frac{x_1}{\varepsilon}\right) (0, -\partial_3\phi, \partial_2\phi)\,, \tag{1.1}$$

if λ is small enough, a positive ε_0 exists such that for any $\varepsilon \leq \varepsilon_0$ the initial data u_0^ε generates a unique global solution to (NS_ν). Those theorems are global existence results for a generalized Navier-Stokes system with small initial data and do not take into account any particular properties of the nonlinear structure in the Navier-Stokes equation. One may check [20] for complete references in this direction.

In this text, we are going to study a version of the system (NS_ν) where the usual Laplacian Δ is substituted by the Laplacian in the horizontal variables $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$, namely

$$(ANS_\nu) \begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta_h u = -\nabla p, & (t,x) \in (0,\infty)\times\mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

Systems of this type appear in geophysical fluids (see for instance [8]). In fact, instead of putting the classical viscosity $-\nu\Delta$ in (NS_ν), meteorologists often modelize turbulent diffusion by putting a viscosity of the form: $-\nu_h\Delta_h - \nu_3\partial_{x_3}^2$, where ν_h and ν_3 are empirical constants, and ν_3 is usually much smaller than ν_h. We refer to the book of J. Pedlovsky [22], Chap. 4 for a more complete discussion. We note also that in the particular case of the so-called Ekman layers (see [12,14]) for rotating fluids, $\nu_3 = \epsilon\nu_h$ and ϵ is a very small parameter. The system (ANS_ν) has been studied first by J. Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier in [7] and D. Iftimie in [17] where it is proved that the anisotropic Navier-Stokes system (ANS_ν) is locally wellposed for initial data in the anisotropic Sobolev space

$$H^{0,\frac{1}{2}+\varepsilon} \stackrel{\text{def}}{=} \left\{ u \in L^2(\mathbb{R}^3) \,/\, \|u\|^2_{\dot{H}^{0,\frac{1}{2}+\varepsilon}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi_3|^{1+2\varepsilon} |\widehat{u}(\xi_h,\xi_3)|^2 d\xi < +\infty \right\},$$

for some $\varepsilon > 0$. Moreover, it has also been proved that if the initial data u_0 is small enough in the sense that

$$\|u_0\|^\varepsilon_{L^2} \|u_0\|^{1-\varepsilon}_{\dot{H}^{0,\frac{1}{2}+\varepsilon}} \leq c\nu \tag{1.2}$$

for some sufficiently small constant c, then they have a global wellposedness result. Let us notice that the space in which uniqueness is proved **in** [7,17], is the space of continuous functions with value in $H^{0,\frac{1}{2}+\varepsilon}(\mathbb{R}^3)$ and the horizontal gradient of which belongs to $L^2_{loc}([0,T];H^{0,\frac{1}{2}+\varepsilon}(\mathbb{R}^3))$.

On the other hand, we notice that, as a classical Navier-Stokes system, the system (ANS_ν) has a scaling. Indeed, if u is a solution of (ANS_ν) on a time interval $[0,T]$ with initial data u_0, then the vector field u_λ defined by

$$u_\lambda(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x) \tag{1.3}$$

is also a solution of (ANS_ν) on the time interval $[0,\lambda^{-2}T]$ with the initial data $\lambda u_0(\lambda x)$. The smallness condition (1.2) is of course scaling invariant. But the norm $\|\cdot\|_{\dot{H}^{\frac{1}{2}+\varepsilon}}$ is not and this norm determines the level of regularity required to have wellposedness. M. Paicu proved in [23] a theorem of the same type for the system (ANS_ν) in the case when the initial data u_0 belongs to $\mathcal{B}$ (see Definition 1.1 below). This result can be looked upon as the equivalence of Fujita-Kato's theorem in the case of the (ANS_ν) system.

In [11], the authors proved a theorem which in particular implies the following global wellposedness result of (ANS_ν) for initial data with high oscillation in the horizontal variable: for a given function ϕ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, if we consider the family of initial data u_0^ε defined by

$$u_0^\varepsilon(x) \stackrel{\text{def}}{=} \frac{\lambda}{\varepsilon^{\frac{1}{2}}}\sin\Big(\frac{x_1}{\varepsilon}\Big)\,(0,-\partial_3\phi,\partial_2\phi)\,, \tag{1.4}$$

then, if λ is small enough, there exists a small positive constant ε_0 such that for any small enough $\varepsilon \le \varepsilon_0$, the system (ANS_ν) is globally wellposed with the initial data u_0^ε. This is analogous (with a smaller power of ε) to the example of initial data given in (1.1) for the case to the (NS_ν) system.

The purpose of this paper is to extend the wellposedness results in [11,23], in particular, our theorem here will provide examples of larger initial data such that (ANS_ν) has a unique global solution. In the first part of the paper, we shall prove the global wellposedness of the anisotropic Navier-Stokes system with initial data having a large vertical component provided that the horizontal component is small enough (compared with the vertical one and with the horizontal viscosity coefficient). The functional framework that we shall use is invariant by scaling and by vertical dilation and is given by the anisotropic Besov-Sobolev space $\mathcal{B}$ (see Definition 1.1 below). Moreover, we note that we are able to obtain the energy estimate in this anisotropic space without using an additional vertical derivative, and consequently our results are valid also in the case of a vanishing vertical viscosity. The main idea to handle this anisotropic model is that the velocity field verifies a 2-D Navier-Stokes type system in the horizontal variables while in the vertical variable we have to deal with a 1D hyperbolic type equation. We shall use energy estimates in the horizontal variables so that the divergence free condition allows to control the vertical derivatives to the vertical component of the velocity field.

We emphasize that our proof uses in a fundamental way the algebraic structure of the Navier-Stokes system. The first step is to obtain energy estimates on the horizontal components on the one hand and on the vertical component on the other hand. One of the difficulties with this strategy is that the pressure term does not disappear but has to be estimated. We remark that the equation on the vertical component is a linear equation with coefficients depending on the horizontal components. Therefore, the equation on the vertical component does not demand any smallness condition. While the equation on the horizontal component contains bilinear terms in the horizontal components and also terms taking into account the interactions between the horizontal components and the vertical one. In order to solve this equation, we need a smallness condition on the horizontal component (amplified by the vertical component) of the initial data. At this point, we need to use the Gronwall Lemma which can not be applied directly in the framework of Chemin-Lerner type spaces $\widetilde{L_t^p}(\mathcal{B})$. To overcome this difficulty we shall introduce here sort of weighted Chemin-Lerner type spaces, $\widetilde{L^2_{t,f}}(\mathcal{B})$ for some appropriate L^1 function $f(t)$. As we already explained, our result allows to give some examples of large data which are slowly varying in the vertical direction and which are larger than the "well prepared" case studied in [9] for the classical Navier-Stokes system.

In the second part of this paper, we shall give an analogous result in the case of initial data with very rough regularity, namely belonging to Besov-Sobolev spaces with negative regularity in the horizontal variables. The main ingredient of the proof is a combination between the strategy that we already explained above and the methods introduced in [11]. The idea is to decompose the initial data in a part where the horizontal frequencies are higher than the vertical frequencies and the rest which belong to the Besov-Sobolev spaces $\mathcal{B}$ for which we can use our previous methods. This new result allows us to present another new example of large data where we combine the high frequencies feature in the horizontal variables with a slow varying vertical variable.

As in [7,23] and [11] (or more recently the book [1]), the definition of the spaces we are going to work with requires anisotropic dyadic decomposition of the Fourier space. Let us recall

$$\begin{aligned}
&\Delta_k^h a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{a}), \quad \Delta_\ell^v a = \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), \quad \text{and}\\
&S_k^h a = \sum_{k'\leq k-1} \Delta_{k'}^h a, \quad S_\ell^v a = \sum_{\ell'\leq \ell-1} \Delta_{\ell'}^v a,
\end{aligned} \tag{1.5}$$

where $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of the distribution a, and $\varphi(\tau)$ is a smooth function such that

$$\operatorname{Supp}\,\varphi \subset \left\{\tau \in \mathbb{R} \,/\, \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\right\} \quad \text{and} \quad \forall \tau > 0,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1.$$

Before we present the spaces we are going to work with, let us first recall the Besov-Soblev type space $\mathcal{B}$ from [16,23] and $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$ from [11].

Definition 1.1. *We call $\mathcal{B}$ the space of tempered distributions, which is the completion of $\mathcal{S}(\mathbb{R}^3)$ by the following norm:*

$$\|a\|_{\mathcal{B}} \stackrel{def}{=} \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v a\|_{L^2(\mathbb{R}^3)}. \tag{1.6}$$

The space $\mathcal{B}(T)$ is the completion of $C^\infty([0,T];\mathcal{S}(\mathbb{R}^3))$ by the norm:

$$\begin{aligned}
\|a\|_{\mathcal{B}(T)} &\stackrel{def}{=} \|a\|_{\widetilde{L^\infty_T}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h a\|_{\widetilde{L^2_T}(\mathcal{B})} \quad \text{with}\\
\|a\|_{\widetilde{L^\infty_T}(\mathcal{B})} &\stackrel{def}{=} \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v a\|_{L^\infty_T(L^2(\mathbb{R}^3))} \quad \text{and} \\
\|\nabla_h a\|_{\widetilde{L^2_T}(\mathcal{B})} &\stackrel{def}{=} \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \|\nabla_h \Delta_\ell^v a\|_{L^2_T(L^2(\mathbb{R}^3))}.
\end{aligned} \tag{1.7}$$

In [23], the author proved the local wellposedness of (ANS_ν) with initial data in $\mathcal{B}$. Moreover, if the initial data is small enough compared to the horizontal viscosity, he also established the global wellposedness result. Note from the above definition that u_0^ε defined in (1.4) is not small in this space no matter how small ε is. The main motivation for the authors to introduce the space $\mathcal{B}_4^{-\frac12,\frac12}$ in [11] is to find a scaling invariant Besov-Soblev type space, which is very close to the classical Besov spaces of negative indices, such that in particular u_0^ε defined in (1.4) is small in this space for ε and λ sufficiently small. We emphasize that such a space has to be Besov type space with negative regularity indices in the horizontal variables in order to take into account strong oscillations in the horizontal variables. Meanwhile for the vertical variable, we have to use a space which is invariant by vertical dilation in order to consider "slowly varying" data in the vertical variable. This justifies the following definition:

Definition 1.2. *We denote by $\mathcal{B}_4^{-\frac12,\frac12}$ the space of distributions, which is the completion of $\mathcal{S}(\mathbb{R}^3)$ by the following norm:*

$$\|a\|_{\mathcal{B}_4^{-\frac12,\frac12}} \stackrel{def}{=} \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \Big(\sum_{k=\ell-1}^{\infty} 2^{-k} \|\Delta_k^h \Delta_\ell^v a\|^2_{L^4_h(L^2_v)}\Big)^{\frac12} + \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|S_{j-1}^h \Delta_j^v a\|_{L^2(\mathbb{R}^3)}. \tag{1.8}$$

The space $\mathcal{B}_4^{-\frac12,\frac12}(T)$ is the completion of $C^\infty([0,T];\mathcal{S}(\mathbb{R}^3))$ by the norm:

$$\begin{aligned}
\|a\|_{\mathcal{B}_4^{-\frac12,\frac12}(T)} &\stackrel{def}{=} \|a\|_{\widetilde{L^\infty_T}(\mathcal{B}_4^{-\frac12,\frac12})} + \sqrt{\nu}\|\nabla_h a\|_{\widetilde{L^2_T}(\mathcal{B}_4^{-\frac12,\frac12})} \quad \text{with}\\
\|a\|_{\widetilde{L^\infty_T}(\mathcal{B}_4^{-\frac12,\frac12})} &\stackrel{def}{=} \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \Big(\sum_{k=\ell-1}^{\infty} 2^{-k} \|\Delta_k^h \Delta_\ell^v a\|^2_{L^\infty_T(L^4_h(L^2_v))}\Big)^{\frac12} \\
&\quad + \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|S_{j-1}^h \Delta_j^v a\|_{L^\infty_T(L^2(\mathbb{R}^3))} \quad \text{and} \\
\|\nabla_h a\|_{\widetilde{L^2_T}(\mathcal{B}_4^{-\frac12,\frac12})} &\stackrel{def}{=} \sum_{\ell\in\mathbb{Z}} 2^{\frac{\ell}{2}} \Big(\sum_{k=\ell-1}^{\infty} 2^{k} \|\Delta_k^h \Delta_\ell^v a\|^2_{L^2_T(L^4_h(L^2_v))}\Big)^{\frac12} \\
&\quad + \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \|\nabla_h S_{j-1}^h \Delta_j^v a\|_{L^2_T(L^2(\mathbb{R}^3))}.
\end{aligned} \tag{1.9}$$

More recently, Zhang [26] claimed the following result:

Theorem 1.1. *A positive constant* C_1 *exists such that if* $u_0 \in \mathcal{B}$ *which satisfies* $\operatorname{div} u_0 = 0$ *and*

$$C_1 \nu^{-1} \|u_0^h\|_{\mathcal{B}} \exp\{C_1(\nu^{-1}\|u_0^3\|_{\mathcal{B}} + 1)^4\} \leq 1, \tag{1.10}$$

then (ANS_ν) *has a unique global solution.*

However, we found that there is a serious gap in the proof of Theorem 1.1 in [26], the main reason is that one can not use the Gronwall type inequality in the framework of spaces $\widetilde{L^p_t}(\mathcal{B})$. Indeed, the only gap in the proof of Theorem 1.1 lies in the proof of Proposition 3.2 in [26]. For instance, using the Hölder inequality and Lemma 3.3 in [26], what one can obtain is

$$G_j^\nu(T) \lesssim \int_0^T d_j(t) 2^{-\frac{j}{2}} \|u^h(t)\|_{\mathcal{B}}^{\frac12} \|\nabla_h u^h(t)\|_{\mathcal{B}}^{\frac12} \|u^3(t)\|_{\mathcal{B}}^{\frac12} \|\nabla_h u^3(t)\|_{\mathcal{B}}^{\frac12} \|\Delta_j^v \nabla_h w(t)\|_{L^2}\, dt,$$

which can not be dominated by

$$d_j^2 2^{-j} \int_0^T \|u^h(t)\|_{\mathcal{B}}^{\frac12} \|\nabla_h u^h(t)\|_{\mathcal{B}}^{\frac12} \|u^3(t)\|_{\mathcal{B}}^{\frac12} \|\nabla_h u^3(t)\|_{\mathcal{B}}^{\frac12} \|\nabla_h w(t)\|_{\mathcal{B}}\, dt,$$

as it is claimed in [26], where both $(d_j)_{j\in\mathbb{Z}}$ and $(d_j(t))_{j\in\mathbb{Z}}$ are generic elements in $\ell^1(\mathbb{Z})$ with the norm of which equal to 1.

To overcome the difficulty mentioned above, the authors [15] basically proved the global wellposedness of (ANS_ν) provided that

$$\|u_0^h\|_{H^{0,s_0}} \exp\Big(C_0(\nu^{-1}\|u_0^3\|_{H^{0,s_0}})^4\Big) \leq c\nu$$

for $s_0 > \frac12$ and some c sufficiently small. Moreover, a sort of global stability result was proved for the classical Navier-Stokes system (NS_ν) with anisotropic type perturbation of the initial data to any given global smooth solution of (NS_ν). However, the regularity level of the initial data in this result is not scaling invariant and this is our main motivation to obtain the following Theorem 1.2.

Now we present the main results in this paper:

Theorem 1.2. *Let* $u_0 = (u_0^h, u_0^3) \in \mathcal{B}$ *be a divergence free vector field, and there exists a positive constant* L *such that*

$$\eta \stackrel{def}{=} \|u_0^h\|_{\mathcal{B}} \exp\left(\frac{L}{\nu^4}\|u_0^3\|_{\mathcal{B}}^4\right) \leq c_0\nu \tag{1.11}$$

for some c_0 *sufficiently small. Then the system* (ANS_ν) *has a unique global solution* $u \in C([0,\infty);\mathcal{B})$ *with* $\nabla_h u \in \widetilde{L^2}(\mathbb{R}^+;\mathcal{B})$. *Moreover,*

$$\begin{aligned} &\|u^h\|_{\widetilde{L^\infty}(\mathbb{R}^+;\mathcal{B})} + \|\nabla_h u^h\|_{\widetilde{L^2}(\mathbb{R}^+;\mathcal{B})} \leq 2\exp\left(\frac{L}{16}\right)\eta \quad \text{and} \\ &\|u^3\|_{\widetilde{L^\infty}(\mathbb{R}^+;\mathcal{B})} + \|\nabla_h u^3\|_{\widetilde{L^2}(\mathbb{R}^+;\mathcal{B})} \leq 2\|u_0^3\|_{\mathcal{B}} + \nu \quad \text{holds.} \end{aligned} \tag{1.12}$$

Remark 1.1. (1) This theorem ensures the global wellposedness of (ANS_ν) with initial data of the form $u_0 = (u_0^h, u_0^3)$ with $\operatorname{div} u_0 = 0$ and $\|u_0^3\|_{\mathcal{B}} \leq C\nu$ for any positive constant C while $\|u_0^h\|_{\mathcal{B}} \leq c\nu$ for some sufficiently small c, which in particular implies the global wellposedness result in [23].

(2) Theorem 1.2 also ensures the global wellposedness of (ANS_ν) with initial data of the form

$$u_0^\epsilon = \big(\epsilon(-\ln\epsilon)^\delta v_0^h(x_h, \epsilon x_3), (-\ln\epsilon)^\delta u_0^3(x_h, \epsilon x_3)\big)$$

for $0 < \delta < 1/4$ and ε sufficiently small, as that claimed in [26].

(3) Very recently, Zhang [27] corrected his former result (1.10) in [26] to be

$$C_1\nu^{-1}\|u_0^h\|_{\mathcal{B}} \exp\big\{C_1(\nu^{-1}\|u_0^3\|_{\mathcal{B}} + 1)^8\big\} \leq 1$$

by using basically the same idea of [26].

The main tool that we shall use to overcome the difficulty that we can not use the Gronwall inequality in the framework of $\widetilde{L_t^p}(\mathcal{B})$, as we mentioned before, is to introduce the following weighted Chemin-Lerner [10] type norm:

Definition 1.3. *Let $f(t) \in L^1_{loc}(\mathbb{R}_+)$, $f(t) \geq 0$. We define*

$$\|u\|_{\widetilde{L^2_{T,f}}(\mathcal{B})} = \sum_q 2^{\frac{q}{2}} \Big(\int_0^T f(t)\|\Delta_q^v u(t)\|_{L^2}^2 dt\Big)^{\frac{1}{2}}.$$

Remark 1.2. In fact, Definition 1.3 is very much motivated by the following variant of Gronwall's Lemma. Let $X(t)$, $f(t)$, $h(t)$ be positive functions so that

$$\begin{cases} \frac{d}{dt}X(t) \leq Cf(t)X(t) + h(t), \\ X(0) = X_0. \end{cases} \tag{1.13}$$

Instead of directly applying Gronwall's Lemma to (1.13), we get by multiplying $g_\lambda(t) \stackrel{\text{def}}{=} \exp\Big\{-\lambda\int_0^t f(t')\,dt'\Big\}$ to (1.13) and integrating the resulting inequality over $[0, t]$ that

$$g_\lambda(t)X(t) + \lambda\int_0^t g_\lambda(t')f(t')X(t')\,dt' \leq X_0 + C\int_0^t g_\lambda(t')f(t')X(t')\,dt' + \int_0^t g_\lambda(t')h(t')\,dt',$$

in particular, if we take $\lambda \geq C$, this gives rise to

$$g_\lambda(t)X(t) \leq X_0 + \int_0^t g_\lambda(t')h(t')\,dt',$$

that is

$$X(t) \leq X_0 \exp\Big\{\lambda\int_0^t f(t')\,dt'\Big\} + \int_0^t \exp\Big\{\lambda\int_{t'}^t f(\tau)\,d\tau\Big\}h(t')\,dt', \tag{1.14}$$

for $\lambda \geq C$. The main motivation to introduce Definition 1.3 is to adapt the proof of (1.14) to the framework of Chemin-Lerner type spaces, namely, integrating any dyadic block with respect to time first and then making the summation with respect to q.

On the other hand, it follows from [11] that: given $u_0=(u_0^h,u_0^3)\in\mathcal{B}_4^{-\frac12,\frac12}$, (ANS_ν) has a unique solution of the form

$$u=u_F+w, \tag{1.15}$$

where $w\in\mathcal{B}(T)$ for some positive time T and u_F is given by

$$u_F\stackrel{\text{def}}{=}e^{\nu t\Delta_h}u_{hh}\qquad\text{and}\qquad u_{hh}\stackrel{\text{def}}{=}\sum_{k\geq\ell-1}\Delta_k^h\Delta_\ell^v u_0. \tag{1.16}$$

Substituting (1.15) into (ANS_ν) results in

$$\begin{cases}\partial_t w+w\cdot\nabla w-\nu\Delta_h w+w\cdot\nabla u_F+u_F\cdot\nabla w=-u_F\cdot\nabla u_F-\nabla p,\\ \operatorname{div} w=0,\\ w|_{t=0}=u_{\ell h}\stackrel{\text{def}}{=}u_0-u_{hh}.\end{cases} \tag{1.17}$$

Notice that

$$\|\Delta_j^v u_{\ell h}\|_{L^2}\lesssim\sum_{|j-j'|\leq1}\|S_{j'-1}^h\Delta_{j'}^v u_0\|_{L^2}, \tag{1.18}$$

which along with Definition 1.2 implies that if u_0 belongs to $\mathcal{B}_4^{-\frac12,\frac12}$, then $u_{\ell h}$ belongs to $\mathcal{B}$ and

$$\|u_{\ell h}\|_{\mathcal{B}}\lesssim\|u_0\|_{\mathcal{B}_4^{-\frac12,\frac12}}. \tag{1.19}$$

Combining the techniques used in the proof of Theorem 1.2 and the above observation, we can prove the following wellposedness result for (ANS_ν) :

Theorem 1.3. *Let $u_0=(u_0^h,u_0^3)\in\mathcal{B}_4^{-\frac12,\frac12}$ be a divergence free vector field, and there exists a positive constant M such that*

$$\eta_1\stackrel{\text{def}}{=}\|u_0^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\exp\left(\frac{M}{\nu^4}\|u_0^3\|^4_{\mathcal{B}_4^{-\frac12,\frac12}}\right)\leq c_1\nu \tag{1.20}$$

for some c_1 sufficiently small. Then the system (ANS_ν) has a unique global solution $u=u_F+w$ such that u_F is given by (1.16) and $w\in C([0,\infty);\mathcal{B})$ with $\nabla_h w\in\widetilde{L^2}(\mathbb{R}^+;\mathcal{B})$. Moreover, there exists a positive constant K such that

$$\begin{aligned}&\|w^h\|_{\widetilde{L^\infty}(\mathbb{R}^+;\mathcal{B})}+\|\nabla_h w^h\|_{\widetilde{L^2}(\mathbb{R}^+;\mathcal{B})}\leq K\eta_1\quad\textit{and}\\ &\|w^3\|_{\widetilde{L^\infty}(\mathbb{R}^+;\mathcal{B})}+\|\nabla_h w^3\|_{\widetilde{L^2}(\mathbb{R}^+;\mathcal{B})}\leq K\|u_0^3\|_{\mathcal{B}_{4,1}^{-\frac12,\frac12}}+\nu.\end{aligned} \tag{1.21}$$

Remark 1.3. (1) This theorem ensures the global wellposedness of (ANS_ν) with initial data of the form $u_0=(u_0^h,u_0^3)$ with $\operatorname{div}u_0=0$ and $\|u_0^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\leq C\nu$ for any positive constant C while $\|u_0^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\leq c\nu$ for some sufficiently small c. In particular, this theorem implies the global wellposedness result in [11].

(2) For a given function ϕ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, Theorem 1.3 along with Proposition 1.1 (which claims that $\|e^{ix_1/\varepsilon}\phi\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \leq c_\phi \varepsilon^{\frac{1}{2}}$) also ensures the global wellposedness of (ANS_ν) with initial data of the form

$$u_0^\epsilon = (-\ln\varepsilon)^\delta \sin\left(\frac{x_1}{\varepsilon}\right)(0, -\varepsilon^{\frac{1}{2}}\partial_3\phi(x_h, \varepsilon x_3), \varepsilon^{-\frac{1}{2}}\partial_2\phi(x_h, \varepsilon x_3))$$

for $0 < \delta < 1/4$ and ε sufficiently small.

(3) Both Theorem 1.2 and Theorem 1.3 holds for classical Navier-Stokes equations (NS_ν). We emphasize that our results strongly use the algebraic structure of the nonlinear terms and the divergence-free condition of the velocity field. In our subsequent paper [24], we obtained similar global wellposedness results for the 3-D inhomogeneous Navier-Stokes equations with initial data in the critical Besov spaces.

In the rest of the paper, we shall constantly use the anisotropic version of the isotropic para-differential decomposition due to J. M. Bony [2] that: for $a, b \in \mathcal{S}'(\mathbb{R}^3)$,

$$\begin{aligned}&ab = T_a^v b + R^v(a,b), \quad \text{or} \quad ab = T_a^v b + T_b^v a + \mathcal{R}^v(a,b), \quad \text{where}\\ &T_a^v b = \sum_{q\in\mathbb{Z}} S_{q-1}^v a \Delta_q^v b, \quad R^v(a,b) = \sum_{q\in\mathbb{Z}} \Delta_q^v a S_{q+2}^v b, \quad \text{and}\\ &\mathcal{R}^v(a,b) = \sum_{|q-q'|\leq 1} \Delta_q^v a \Delta_{q'}^v b.\end{aligned} \tag{1.22}$$

Similar decomposition for the horizontal variables will also be used frequently.

The organization of this paper follows:

Scheme of the proof and organization of the paper. In Sect. 2, we shall present the proof of Theorem 1.2. The main idea can be outlined as follows: we first work out the energy estimate for the horizonal components of the velocity field so that

$$\begin{aligned}&\|u_\lambda^h\|_{\widetilde{L}_t^\infty(\mathcal{B})} + \sqrt{\lambda}\|u_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u_\lambda\|_{\widetilde{L}_t^2(\mathcal{B})}\\ &\quad \leq \|u_0^h\|_{\mathcal{B}} + C_0\Big(\|u^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac{1}{2}}\|\nabla_h u_\lambda^h\|_{\widetilde{L}_t^2(\mathcal{B})} + \nu^{-\frac{3}{2}}\|u_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})}\Big),\end{aligned} \tag{1.23}$$

where

$$u_\lambda(t,x) \stackrel{\text{def}}{=} e^{-\lambda\int_0^t f(t')\,dt'}u(t,x), \quad \text{with} \quad f(t) \stackrel{\text{def}}{=} \|u^3(t)\|_{\mathcal{B}}^2\|\nabla_h u^3(t)\|_{\mathcal{B}}^2. \tag{1.24}$$

Along the same lines as Remark 1.2, if we take $\lambda = \frac{4C_0^2}{\nu^3}$ in (1.23) and take T^* so small that

$$T^* \stackrel{\text{def}}{=} \max\Big\{t : \|u^h\|_{\widetilde{L}_t^\infty(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u^h\|_{\widetilde{L}_t^2(\mathcal{B})} \leq \min(\frac{1}{4C_0^2}, \varepsilon_0)\nu\Big\}, \tag{1.25}$$

we infer from (1.23) that

$$\|u_\lambda^h\|_{\widetilde{L}_t^\infty(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u_\lambda^h\|_{\widetilde{L}_t^2(\mathcal{B})} \leq 2\|u_0^h\|_{\mathcal{B}} \qquad \text{for} \quad t < T^*.$$

Then thanks to (1.24), we arrive at

$$\begin{aligned}&\|u^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}+\sqrt{\nu}\|\nabla_h u^h\|_{\widetilde{L^2_t}(\mathcal{B})}\\&\quad\le 2\|u_0^h\|_{\mathcal{B}}\exp\Big\{\frac{4C_0^2}{\nu^3}\int_0^t\|u^3(t')\|_{\mathcal{B}}^2\|\nabla_h u^3(t')\|_{\mathcal{B}}^2\,dt'\Big\}\end{aligned}\tag{1.26}$$

for $t<T^*$.

The second step of the proof is to obtain the energy estimate for the vertical component of the velocity field. We use at this point, the important fact that u^3 verifies a linear equation with coefficients depending on u^h so that

$$\begin{aligned}\|u^3\|_{\widetilde{L^\infty_t}(\mathcal{B})}+\sqrt{2\nu}\|\nabla_h u^3\|_{\widetilde{L^2_t}(\mathcal{B})}\le\|u_0^3\|_{\mathcal{B}}+\widetilde{C}\Bigg(&\|u^h\|^{1/2}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h u^h\|_{\widetilde{L^2_t}(\mathcal{B})}\\&+\|u^3\|^{1/4}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h u^3\|^{1/4}_{\widetilde{L^2_t}(\mathcal{B})}\|u^h\|^{1/4}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla u^h\|^{3/4}_{\widetilde{L^2_t}(\mathcal{B})}\Bigg).\end{aligned}$$

Then thanks to (1.25), we obtain

$$\|u^3\|_{\widetilde{L^\infty_t}(\mathcal{B})}+\sqrt{2\nu}\|\nabla_h u^3\|_{\widetilde{L^2_t}(\mathcal{B})}\le\|u_0^3\|_{\mathcal{B}}+\widetilde{C}\left(\varepsilon_0^{\frac32}\nu+\varepsilon_0\nu^{\frac58}\|u^3\|^{1/4}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla u^3\|^{1/4}_{\widetilde{L^2_t}(\mathcal{B})}\right),$$

for $t<T^*$. Taking $\varepsilon_0=\varepsilon_0(\widetilde{C})$ small enough, we obtain

$$\|u^3\|_{\widetilde{L^\infty_t}(\mathcal{B})}+\sqrt{\nu}\|\nabla_h u^3\|_{\widetilde{L^2_t}(\mathcal{B})}\le 2\|u_0^3\|_{\mathcal{B}}+\nu\qquad\text{for}\quad t<T^*.\tag{1.27}$$

The last step is to prove that $T^*=\infty$, provided that c_0 in (1.11) is sufficiently small. Indeed if $T^*<\infty$, it follows from (1.27) that

$$\int_0^t\|u^3(t')\|_{\mathcal{B}}^2\|\nabla_h u^3(t')\|_{\mathcal{B}}^2\,dt'\le\|u^3\|^2_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h u^3\|^2_{\widetilde{L^2_t}(\mathcal{B})}\le\frac{16}{\nu}\left(16\|u_0^3\|^4_{\mathcal{B}_4^{-\frac12,\frac12}}+\nu^4\right).$$

Substituting the above inequality into (1.26) ensures that

$$\begin{aligned}\|u^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}+\sqrt{\nu}\|\nabla_h u^h\|_{\widetilde{L^2_t}(\mathcal{B})}&\le 2\exp(64C_0^2)\|u_0^h\|_{\mathcal{B}}\exp\left(\frac{1024C_0^2}{\nu^4}\|u_0^3\|_{\mathcal{B}}^4\right)\\&\le\frac12\min\Big(\frac{1}{4C_0^2},\varepsilon_0\Big)\nu\qquad\text{for}\quad t<T^*,\end{aligned}$$

provided that we take $L=1024C_0^2$ and $c_0\le\frac14\exp(-64C_0^2)\min(\frac{1}{4C_0^2},\varepsilon_0)$ in (1.11). This contradicts the definition of T^* defined in (1.25), and therefore $T^*=+\infty$.

In Sect. 2, we shall rigorously work out the above estimates for appropriate approximate solutions of (ANS_ν), and then prove the existence part via the compactness argument.

Exactly following the same line as the proof of Theorem 1.2 in Sect. 2, we shall present the proof of Theorem 1.3 in Sect. 3. However, comparing (1.17) with (ANS_ν), three additional new terms appear in (1.17). Therefore more a complicated argument has to be involved in the proof of the estimates like (1.26) and (1.27) for w. One may check (3.50) and (3.54) for more details.

Let us complete the introduction with the notations we are going to use in this context.

Notations. Let A, B be two operators; we denote $[A; B] = AB - BA$, the commutator between A and B, $a \lesssim b$. We mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We shall denote by $(a|b)$ (or $(a|b)_{L^2}$) the $L^2(\mathbb{R}^3)$ inner product of a and b. We denote $L^r_T(L^p_h(L^q_v))$ the space $L^r([0,T]; L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; L^q(\mathbb{R}_{x_3})))$. Finally, we denote by $(c_k)_{k\in\mathbb{Z}}$ (resp. $(d_j)_{j\in\mathbb{Z}}$) a generic element of the sphere of $\ell^2(\mathbb{Z})$ (resp. $\ell^1(\mathbb{Z})$), and $(d_{k,j})_{(k,j)\in\mathbb{Z}^2}$ a generic sequence such that

$$\sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} d_{k,j}^2 \Big)^{\frac{1}{2}} = 1.$$

2. The Proof of Theorem 1.2

The goal of this section is to present the proof of Theorem 1.2. For the convenience of the readers, we first recall the following Bernstein type lemma from [6,11]:

Lemma 2.1. *Let $\mathcal{B}_v$ be a ball of $\mathbb{R}_v$, and $\mathcal{C}_v$ a ring of $\mathbb{R}_v$; let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then the following hold:*

If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_h$, then

$$\|\partial_{x_h}^{\alpha} a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k\left(|\alpha|+2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)} \|a\|_{L_h^{p_2}(L_v^{q_1})}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{B}_v$, then

$$\|\partial_3^{\beta} a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell\left(\beta+\left(\frac{1}{q_2}-\frac{1}{q_1}\right)\right)} \|a\|_{L_h^{p_1}(L_v^{q_2})}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_h$, then

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_{x_h}^{\alpha} a\|_{L_h^{p_1}(L_v^{q_1})}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{C}_v$, then

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-\ell N} \|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})}.$$

Proof. Those inequalities are classical (see for instance [23] or [11]). For the reader's convenience, we shall prove the last one in the particular case when $N = 1$. Let us consider $\widetilde{\varphi}$ in $\mathcal{D}(\mathbb{R}^2 \setminus \{0\})$ such that $\widetilde{\varphi}$ has value 1 near $\mathcal{C}_v$. Then for any tempered distribution a such that the support of $\widehat{a}$ is included in $2^\ell\mathcal{C}_v$, we have

$$\widehat{a} = 2^{-\ell} i\xi_3 \overline{\varphi}(2^{-\ell}\xi_3)\widehat{a} \quad \text{with} \quad \overline{\varphi}(\xi_3) \stackrel{\text{def}}{=} -\frac{i\xi_3\widetilde{\varphi}(\xi_3)}{|\xi_3|^2}.$$

Then, we have

$$a = 2^{-\ell}\partial_3 \Delta_\ell^{v,3} a \quad \text{with} \quad \Delta_\ell^{v,3} a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\overline{\varphi}(2^{-\ell}\xi_3)\widehat{a}). \tag{2.1}$$

This formula will be useful later on and it proves the fourth inequality of the lemma in the particular case when $N = 1$. □

The proof of Theorem 1.2 will mainly be based on the following two propositions. The first proposition is to deal with the L^2 inner product of $a \cdot \nabla b$ and b for any dyadic block on vertical frequencies, without using any vertical derivative of a and b. Here, the divergence free condition of a plays an essential role. The idea is to use integration by parts for the term $\int_{\mathbb{R}^3} S^v_{q-1} a^3 \partial_3 \Delta^v_q a \cdot \Delta^v_q a \, dx$ and to use the divergence free condition of $a = (a^h, a^3)$ in the form $\partial_3 a^3 = -\operatorname{div}_h a^h$. We remark that in the horizontal variables, we shall use classical energy estimates for the 2D Navier-Stokes equation. The second proposition presents the estimate to the anisotropic L^2 inner product of the horizontal derivative to the pressure with the horizontal components of the velocity field, which does not disappear in our energy estimates. There again the divergence-free condition of the velocity field as well as the algebraic expression of the equation satisfied by the pressure plays a key role.

Proposition 2.1. *Let* $a = (a^h, a^3), b \in \mathcal{B}(t)$ *with* $\operatorname{div} a = 0$. *Let* $g \in L^\infty(0,t)$, *and we denote* $a_g(t,x) \overset{def}{=} g(t)a(t,x)$ *and* $b_g(t,x) \overset{def}{=} g(t)b(t,x)$. *Then there holds for all* $q \in \mathbb{Z}$,

$$\begin{aligned}
&\Big|\int_0^t \big(\Delta^v_q(a\cdot\nabla b_g) \mid \Delta^v_q b_g\big)_{L^2}\,dt\Big| \\
&\quad\lesssim d_q^2 2^{-q}\Big(\|a^h\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^h_g\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}\|b\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h b_g\|^{\frac32}_{\widetilde{L^2_t}(\mathcal{B})} \\
&\qquad\qquad + \|\nabla_h a^h_g\|_{\widetilde{L^2_t}(\mathcal{B})}\|b\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h b_g\|_{\widetilde{L^2_t}(\mathcal{B})}\Big). \qquad (2.2)
\end{aligned}$$

Proof. The main idea of the proof to this lemma essentially follows from that of Lemma 3 of [7] and proposition 3.3 of [11]. Noticing that the right-hand side of (2.2) does not contain the term with $\partial_3 b_g$, we distinguish the terms with the horizontal derivatives from the term with the vertical one so that

$$I_{q,g}(t) \overset{\text{def}}{=} \int_0^t \Big(\Delta^v_q(a\cdot\nabla b_g)|\Delta^v_q b_g\Big)_{L^2}\,dt' \overset{\text{def}}{=} I^h_{q,g}(t) + I^v_{q,g}(t),$$

with

$$I^h_{q,g}(t) \overset{\text{def}}{=} \int_0^t \big(\Delta^v_q(a^h\cdot\nabla_h b_g)|\Delta^v_q b_g\big)_{L^2}dt' \text{ and } I^v_{q,g}(t) \overset{\text{def}}{=} \int_0^t \Big(\Delta^v_q(a^3\partial_3 b_g)|\Delta^v_q b_g\Big)_{L^2}\,dt'.$$

Thanks to Bony's decomposition (1.22), we have

$$a^h\cdot\nabla_h b_g = T^v_{a^h}\nabla_h b_g + R^v(a^h, \nabla_h b_g).$$

Whereas a simple interpolation in 2-D gives

$$\begin{aligned}
\|\sqrt{g}\Delta^v_q a^h\|_{L^4_t(L^4_h(L^2_v))} &\lesssim \|\Delta^v_q a^h\|^{\frac12}_{L^\infty_t(L^2)}\|\nabla_h\Delta^v_q a^h_g\|^{\frac12}_{L^2_t(L^2)} \\
&\lesssim d_q 2^{-\frac{q}{2}}\|a^h\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^h_g\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}, \qquad (2.3)
\end{aligned}$$

which along with Lemma 2.1 implies

$$\begin{aligned}\|\sqrt{g}\Delta_q^v(R^v(a^h,\nabla_h b_g))\|_{L_t^{\frac43}(L_h^{\frac43}(L_v^2))} &\lesssim \sum_{q'\geq q-4}\|\sqrt{g}\Delta_{q'}^v a^h\|_{L_t^4(L_h^4(L_v^2))}\|S_{q'+2}^v\nabla_h b_g\|_{L_t^2(L_h^2(L_v^\infty))}\\ &\lesssim d_q 2^{-\frac{q}{2}}\|a^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h a_g^h\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12}\|\nabla_h b_g\|_{\widetilde{L}_t^2(\mathcal{B})},\end{aligned}$$

and similarly

$$\begin{aligned}\|\sqrt{g}\Delta_q^v(T_{a^h}\nabla_h b))\|_{L_t^{\frac43}(L_h^{\frac43}(L_v^2))} &\lesssim \sum_{|q'-q|\leq 5}\|\sqrt{g}S_{q'-1}^v a^h\|_{L_t^4(L_h^4(L_v^\infty))}\|\Delta_{q'}^v\nabla_h b_g\|_{L_t^2(L^2)}\\ &\lesssim d_q 2^{-\frac{q}{2}}\|a^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h a_g^h\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12}\|\nabla_h b_g\|_{\widetilde{L}_t^2(\mathcal{B})}.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}\left|I_{q,g}^h(t)\right| &\lesssim \|\sqrt{g}\Delta_q^v(a^h\cdot\nabla_h b_g)\|_{L_t^{\frac43}(L_h^{\frac43}(L_v^2))}\|\sqrt{g}\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))}\\ &\lesssim d_q^2 2^{-q}\|a^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h a_g^h\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12}\|b\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h b_g\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac32}.\end{aligned}\tag{2.4}$$

On the other hand, to deal with $I_{q,g}^v(t)$, we need to use the assumption that $\operatorname{div} a = 0$ and the trick from [7,11]. Toward this, we first use Bony's decomposition for $a^3\partial_3 b_g$ in the vertical variables and then a commutator process for $\Delta_q^v(T_{a^3}\partial_3 b_g)$ so that

$$\begin{aligned}I_{q,g}^v(t) &= \int_0^t \big(S_{q-1}^v a^3\partial_3\Delta_q^v b_g \,|\, \Delta_q^v b_g\big)_{L^2}\,dt\\ &+\sum_{|q'-q|\leq 5}\int_0^t\big([\Delta_q^v; S_{q'-1}^v a^3]\partial_3\Delta_{q'}^v b_g \,|\, \Delta_q^v b_g\big)_{L^2}\,dt'\\ &+\sum_{|q'-q|\leq 5}\int_0^t\big((S_{q'-1}^v a^3 - S_{q-1}^v a^3)\partial_3\Delta_q^v\Delta_{q'}^v b_g \,|\, \Delta_q^v b_g\big)_{L^2}\,dt'\\ &+\sum_{q'\geq q-4}\int_0^t\big(\Delta_q^v(\Delta_{q'}^v a^3 S_{q'+2}^v\partial_3 b_g) \,|\, \Delta_q^v b_g\big)_{L^2}\,dt'\\ &\overset{\text{def}}{=} I_{q,g}^{1,v}(t)+I_{q,g}^{2,v}(t)+I_{q,g}^{3,v}(t)+I_{q,g}^{4,v}(t).\end{aligned}\tag{2.5}$$

In what follows, we shall successively estimate all the terms above. Firstly as $\operatorname{div} a = 0$, we get by using integration by parts that

$$I_{q,g}^{1,v}(t) = \frac12\int_0^t\int_{\mathbb{R}^3} g^2 S_{q-1}^v \operatorname{div}_h a^h |\Delta_q^v b|^2\,dx\,dt',$$

from which and (2.3), we deduce that

$$\begin{aligned}|I_{q,g}^{1,v}(t)| &\leq \|S_{q-1}^v(\operatorname{div}_h a_g^h)\|_{L_t^2(L_h^2(L_v^\infty))}\|\sqrt{g}\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))}^2\\ &\lesssim d_q^2 2^{-q}\|\operatorname{div}_h a_g^h\|_{\widetilde{L}_t^2(\mathcal{B})}\|b\|_{\widetilde{L}_t^\infty(\mathcal{B})}\|\nabla_h b_g\|_{\widetilde{L}_t^2(\mathcal{B})}.\end{aligned}$$

To handle the commutator in $I_{q,g}^{2,v}(t)$, we first use Taylor's formula to get

$$I_q^{2,v}(t) = \sum_{|q-q'|\le 5} 2^{q'} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}} h(2^{q'}(x_3-y_3)) \int_0^1 S_{q'-1}^v \partial_3 a^3(x_h, \tau y_3 + (1-\tau)x_3)\, d\tau \\ \times (y_3 - x_3)\partial_3 \Delta_{q'}^v b_g(t', x_h, y_3)\, dy_3\ \Delta_q^v b_g(t', x)\, dx\, dt', \tag{2.6}$$

where $h(x_3) \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(|\xi_3|))(x_3)$. Applying Lemma 2.1, (2.3) and Young's inequality yields

$$|I_{q,g}^{2,v}(t)| \lesssim \sum_{|q-q'|\le 5} \|S_{q'-1}^v \partial_3 a_g^3\|_{L_t^2(L_h^2(L_v^\infty))} \|\sqrt{g}\Delta_{q'}^v b\|_{L_t^4(L_h^4(L_v^2))} \|\sqrt{g}\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))} \\ \lesssim d_q^2 2^{-q} \|\operatorname{div}_h a_g^h\|_{\widetilde{L_t^2}(\mathcal{B})} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}.$$

Whereas thanks to Lemma 2.1 and $\operatorname{div} a = 0$, we have

$$\|\Delta_q^v a_g^3\|_{L_t^2(L_h^2(L_v^\infty))} \lesssim 2^{-\frac{q}{2}} \|\Delta_q^v \partial_3 a_g^3\|_{L_t^2(L^2)} \lesssim d_q 2^{-q} \|\operatorname{div}_h a_g^h\|_{\widetilde{L_t^2}(\mathcal{B})}, \tag{2.7}$$

which together with (2.3) ensures that

$$|I_{q,g}^{3,v}(t)| \lesssim 2^q \sum_{|q-q'|\le 5} \|\Delta_{q'}^v a_g^3\|_{L_t^2(L_h^2(L_v^\infty))} \|\sqrt{g}\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))}^2 \\ \lesssim d_q^2 2^{-q} \|\operatorname{div}_h a_g^h\|_{\widetilde{L_t^2}(\mathcal{B})} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}.$$

Finally again thanks to Lemma 2.1 and (2.7), we obtain

$$|I_{q,g}^{4,v}(t)| \lesssim \sum_{q'\ge q-4} 2^{q'} \|\Delta_{q'}^v a_g^3\|_{L_t^2(L^2)} \|\sqrt{g} S_{q'+2}^v b\|_{L_t^4(L_h^4(L_v^\infty))} \|\sqrt{g}\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))} \\ \lesssim d_q^2 2^{-q} \|\operatorname{div}_h a_g^h\|_{\widetilde{L_t^2}(\mathcal{B})} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}.$$

Therefore, we obtain

$$|I_{q,g}^v(t)| \lesssim d_q^2 2^{-q} \|\nabla_h a_g^h\|_{\widetilde{L_t^2}(\mathcal{B})} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}.$$

This along with (2.4) completes the proof of Lemma 2.1. □

Taking $g(t) = 1$ in Proposition 2.1 immediately gives

Corollary 2.1. *Under the assumptions of Proposition 2.1, we have*

$$\Big|\int_0^t \big(\Delta_q^v(a\cdot\nabla b) \mid \Delta_q^v b\big)_{L^2}\, dt\Big| \lesssim d_q^2 2^{-q} \bigg(\|a^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12} \|\nabla_h a^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12} \|\nabla_h b\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32} \\ + \|\nabla_h a^h\|_{\widetilde{L_t^2}(\mathcal{B})} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b\|_{\widetilde{L_t^2}(\mathcal{B})} \bigg).$$

To handle the pressure term, we need the following proposition:

Proposition 2.2. *Let* $\lambda \geq 0$, $u = (u^h, u^3) \in \mathcal{B}(t)$ *with* $\operatorname{div} u = 0$. *We define*

$$u_\lambda(t,x) \stackrel{def}{=} g_\lambda(t)u(t,x) \quad \text{and} \quad p_\lambda \stackrel{def}{=} \sum_{\ell,k=1}^{3} (-\Delta)^{-1}\partial_\ell\partial_k(u^\ell u_\lambda^k),$$

for $f(t) \stackrel{def}{=} \|u^3(t)\|_{\mathcal{B}}^2 \|\nabla_h u^3(t)\|_{\mathcal{B}}^2$ *and* $g_\lambda(t) = \exp\bigl(-\lambda \int_0^t f(t')\,dt'\bigr)$. *Then, one has for all* $q \in \mathbb{Z}$,

$$\begin{aligned} &\Bigl|\int_0^t (\Delta_q^v \nabla_h p_\lambda \mid \Delta_q^v u_\lambda^h)_{L^2}\,dt'\Bigr| \\ &\qquad \lesssim d_q^2 2^{-q}\Bigl(\|u^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h u_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^2 + \|u_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})}^{\frac12}\|\nabla_h u_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}\Bigr). \end{aligned} \tag{2.8}$$

Proof. Motivated by [15,19,26], here we again distinguish the terms with horizontal derivatives from the terms with a vertical one so that

$$P_{q,\lambda}(t) \stackrel{\text{def}}{=} \sum_{\ell,k=1}^{3} \int_0^t \bigl(\Delta_q^v(-\Delta)^{-1}\partial_\ell\partial_k(u^\ell u_\lambda^k) \mid \Delta_q^v \operatorname{div}_h u_\lambda^h\bigr)_{L^2}\,dt' = P_{q,\lambda}^h(t) + P_{q,\lambda}^v(t), \tag{2.9}$$

where

$$P_{q,\lambda}^h(t) \stackrel{\text{def}}{=} \sum_{\ell,k=1}^{2} \int_0^t \bigl(\Delta_q^v(-\Delta)^{-1}\partial_\ell\partial_k(u^\ell u_\lambda^k) \mid \Delta_q^v \operatorname{div}_h u_\lambda^h\bigr)_{L^2}\,dt',$$

and

$$P_{q,\lambda}^v(t) \stackrel{\text{def}}{=} \int_0^t \left(\Delta_q^v(-\Delta)^{-1}[\partial_3^2(u^3u_\lambda^3) + 2\sum_{k=1}^{2}\partial_3\partial_k(u^3u_\lambda^k)] \mid \Delta_q^v \operatorname{div}_h u_\lambda^h\right)_{L^2} dt'.$$

We start with the estimate to $P_{q,\lambda}^h(t)$. Indeed thanks to Bony's decomposition (1.22), we have

$$\begin{aligned} \sum_{\ell,k=1}^{2} \|\Delta_q^v(u^\ell u_\lambda^k)\|_{L_t^2(L^2)} \lesssim \sum_{\ell,k=1}^{2} \Biggl(&\sum_{|q'-q|\leq 5} \|\sqrt{g_\lambda}S_{q'-1}^v u^\ell\|_{L_T^4(L_h^4(L_v^\infty))}\|\sqrt{g_\lambda}\Delta_{q'}^v u^k\|_{L_T^4(L_h^4(L_v^2))} \\ &+ \sum_{q'\geq q-4} \|\sqrt{g_\lambda}\Delta_{q'}^v u^\ell\|_{L_T^4(L_h^4(L_v^2))}\|\sqrt{g_\lambda}S_{q'+2}^v u^k\|_{L_T^4(L_h^4(L_v^\infty))}\Biggr), \end{aligned}$$

from which and (2.3), we deduce that

$$\sum_{\ell,k=1}^{2} \|\Delta_q^v(u^\ell u_\lambda^k)\|_{L_t^2(L^2)} \lesssim d_q 2^{-\frac{q}{2}}\|u^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h u_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}.$$

Whence we obtain

$$\begin{aligned}\left|P^h_{q,\lambda}(t)\right| &\lesssim \sum_{\ell,k=1}^{2} \|\Delta^v_q(u^\ell u^k_\lambda)\|_{L^2_t(L^2)}\|\Delta^v_q \operatorname{div}_h u^h_\lambda\|_{L^2_t(L^2)}\\ &\lesssim d_q^2 2^{-q}\|u^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h u^h_\lambda\|^2_{\widetilde{L^2_t}(\mathcal{B})}.\end{aligned} \tag{2.10}$$

While as $\operatorname{div} u = 0$, we get by using Bony's decomposition that

$$\begin{aligned}P^v_{q,\lambda}(t) &= 2\sum_{k=1}^{2}\int_0^t \left(\Delta^v_q(-\Delta)^{-1}\partial_3(u^k_\lambda \partial_k u^3) \mid \Delta^v_q \operatorname{div}_h u^h_\lambda\right)_{L^2} dt'\\ &= P^{v,1}_{q,\lambda}(t) + P^{v,2}_{q,\lambda}(t),\end{aligned} \tag{2.11}$$

with

$$P^{v,1}_{q,\lambda}(t) \stackrel{\text{def}}{=} 2\sum_{k=1}^{2}\sum_{q'\geq q-5}\int_0^t \left(\Delta^v_q(-\Delta)^{-1}\partial_3(\Delta^v_{q'}u^k_\lambda S^v_{q'+2}\partial_k u^3) \mid \Delta^v_q \operatorname{div}_h u^h_\lambda\right)_{L^2} dt',$$

and

$$P^{v,2}_{q,\lambda}(t) \stackrel{\text{def}}{=} 2\sum_{k=1}^{2}\sum_{|q'-q|\leq 5}\int_0^t \left(\Delta^v_q(-\Delta)^{-1}\partial_3(S^v_{q'-1}u^k_\lambda \Delta^v_{q'}\partial_k u^3) \mid \Delta^v_q \operatorname{div}_h u^h_\lambda\right)_{L^2} dt'.$$

Let us begin by the estimate of $P^{v,1}_{q,\lambda}(t)$. Indeed using integration by parts, one has

$$\begin{aligned}P^{v,1}_{q,\lambda} = 2\sum_{k=1}^{2}\sum_{q'\geq q-5}\Big(&\int_0^t \left(\Delta^v_q(\Delta^v_{q'}\partial_k u^k_\lambda S^v_{q'+1}u^3) \mid \Delta^v_q(-\Delta)^{-1}\partial_3 \operatorname{div}_h u^h_\lambda\right)_{L^2} dt'\\ &+\int_0^t (\Delta^v_q(\Delta^v_{q'}u^k_\lambda S^v_{q'+1}u^3) \mid \Delta^v_q(-\Delta)^{-1}\partial_3 \operatorname{div}_h \partial_k u^h_\lambda)_{L^2}dt'\Big) \stackrel{\text{def}}{=} \mathcal{A}_{1,\lambda}(t)+\mathcal{A}_{2,\lambda}(t).\end{aligned}$$

Then it follows from

$$\|a(D)g\|_{L^4_h(L^2_v)} \leq C\|a(D)g\|^{\frac12}_{L^2}\|a(D)\nabla_h g\|^{\frac12}_{L^2} \leq C\|g\|^{\frac12}_{L^2}\|\nabla_h g\|^{\frac12}_{L^2}$$

for any homogeneous function $a(\xi)$ of degree zero, that

$$\begin{aligned}\left|\mathcal{A}_{1,\lambda}(t)\right| &\lesssim \sum_{q'\geq q-5}\int_0^t \|S^v_{q'+2}u^3\|_{L^4_h(L^\infty_v)}\|\Delta^v_{q'}\nabla_h u^h_\lambda\|_{L^2}\|\Delta^v_q u^h_\lambda\|_{L^4_h(L^2_v)}\, dt'\\ &\lesssim \sum_{q'\geq q-5}\int_0^t \|S^v_{q'+2}u^3\|^{\frac12}_{L^2_h(L^\infty_v)}\|S^v_{q'+2}\nabla_h u^3\|^{\frac12}_{L^2_h(L^\infty_v)}\|\Delta^v_{q'}\nabla_h u^h_\lambda\|_{L^2}\\ &\quad\times\|\Delta^v_q u^h_\lambda\|^{\frac12}_{L^2}\|\Delta^v_q\nabla_h u^h_\lambda\|^{\frac12}_{L^2}\, dt',\end{aligned} \tag{2.12}$$

applying Hölder's inequality gives

$$\left|\mathcal{A}_{1,\lambda}(t)\right| \lesssim \sum_{q'\geq q-5}\left(\int_0^t \|u^3\|^2_{\mathcal{B}}\|\nabla_h u^3\|^2_{\mathcal{B}}\|\Delta^v_q(u^h_\lambda)\|^2_{L^2}\right)^{\frac14}\|\Delta^v_{q'}(\nabla_h u^h_\lambda)\|_{L^2_t(L^2)}\|\Delta^v_q(\nabla_h u^h_\lambda)\|^{\frac12}_{L^2_t(L^2)},$$

from which and Definition 1.3, we deduce that

$$\big|\mathcal{A}_{1,\lambda}(t)\big| \lesssim d_q^2 2^{-q} \|u_\lambda^h\|_{\widetilde{L}^2_{t,f}(\mathcal{B})}^{\frac12} \|\nabla_h u_\lambda^h\|_{\widetilde{L}^2_t(\mathcal{B})}^{\frac32}. \tag{2.13}$$

The same estimate holds for $\mathcal{A}_{2,\lambda}(t)$ and for $P^{v,1}_{q,\lambda}(t)$ as well.

Whereas again we get by using integration by parts and $\operatorname{div} u = 0$ that

$$\begin{aligned} P^{v,2}_{q,\lambda}(t) &= -2\sum_{k=1}^{2}\sum_{|q'-q|\le 5}\Big(\int_0^t \big(\Delta_q^v \partial_3 (S^v_{q'-1} u_\lambda^k \Delta^v_{q'} u^3) \mid \Delta_q^v (-\Delta)^{-1} \operatorname{div}_h \partial_k u_\lambda^h\big)_{L^2}\,dt' \\ &\quad -2\int_0^t (\Delta_q^v (S^v_{q'-1}\partial_3 u^3 \Delta^v_{q'} u^3_\lambda) \mid \Delta_q^v(-\Delta)^{-1}\operatorname{div}_h \partial_3 u^h_\lambda)_{L^2} dt' \\ &\overset{\text{def}}{=} \mathcal{D}_{1,\lambda}(t) + \mathcal{D}_{2,\lambda}(t). \end{aligned}$$

Thanks to Lemma 2.1, one has

$$\big|\mathcal{D}_{1,\lambda}(t)\big| \lesssim \sum_{|q'-q|\le 5} \|\sqrt{g_\lambda} S^v_{q'-1} u^h\|_{L^4_t(L^4_h(L^\infty_v))} \|\Delta^v_{q'}\partial_3 u^3_\lambda\|_{L^2_t(L^2)} \|\sqrt{g_\lambda}\Delta_q^v u^h\|_{L^4_t(L^4_h(L^2_v))},$$

which along with (2.3) and $\operatorname{div} u = 0$ ensures that

$$\begin{aligned} \big|\mathcal{D}_{1,\lambda}(t)\big| &\lesssim \Big(\sum_{|q'-q|\le 5} d_{q'} 2^{-\frac{q'}{2}}\Big) d_q 2^{-\frac q2} \|u^h\|_{\widetilde{L}^\infty_t(\mathcal{B})} \|\nabla_h u^h_\lambda\|^2_{\widetilde{L}^2_t(\mathcal{B})} \\ &\lesssim d_q^2 2^{-q} \|u^h\|_{\widetilde{L}^\infty_t(\mathcal{B})} \|\nabla_h u^h_\lambda\|^2_{\widetilde{L}^2_t(\mathcal{B})}. \end{aligned} \tag{2.14}$$

And a similar argument gives

$$\begin{aligned} \big|\mathcal{D}_{2,\lambda}(t)\big| &\lesssim \sum_{|q'-q|\le 5}\int_0^t \|S^v_{q'-1} u^3\|_{L^4_h(L^\infty_v)} \|\Delta^v_{q'}\partial_3 u^3_\lambda\|_{L^2} \|\Delta^v_q u^h_\lambda\|_{L^4_h(L^2_v)}\,dt' \\ &\lesssim \sum_{|q'-q|\le 5}\int_0^t \|u^3\|_{\mathcal{B}}^{\frac12} \|\nabla_h u^3\|_{\mathcal{B}}^{\frac12} \|\Delta^v_{q'}\nabla_h u^h_\lambda\|_{L^2} \|\Delta^v_q u^h_\lambda\|_{L^2}^{\frac12} \|\Delta^v_q \nabla_h u^h_\lambda\|_{L^2}^{\frac12}\,dt', \end{aligned}$$

from which and Definition 1.3, we deduce that

$$\begin{aligned} \big|\mathcal{D}_{2,\lambda}(t)\big| &\lesssim \sum_{|q'-q|\le 5}\Big(\int_0^t \|u^3\|^2_{\mathcal{B}} \|\nabla_h u^3\|^2_{\mathcal{B}} \|\Delta^v_q u^h_\lambda\|^2_{L^2}\,dt'\Big)^{\frac14} \|\Delta^v_{q'}\nabla_h u^h_\lambda\|_{L^2_t(L^2)} \|\Delta^v_q \nabla_h u^h_\lambda\|^{\frac12}_{L^2_t(L^2)} \\ &\lesssim d_q^2 2^{-q} \|u^h_\lambda\|^{\frac12}_{\widetilde{L}^2_{t,f}(\mathcal{B})} \|\nabla_h u^h_\lambda\|^{\frac32}_{\widetilde{L}^2_t(\mathcal{B})}. \end{aligned}$$

As a consequence, we obtain

$$\big|P^{v,2}_{q,\lambda}(t)\big| \lesssim d_q^2 2^{-q}\bigg(\|u^h\|_{\widetilde{L}^\infty_t(\mathcal{B})} \|\nabla_h u^h_\lambda\|^2_{\widetilde{L}^2_t(\mathcal{B})} + \|u^h_\lambda\|^{\frac12}_{\widetilde{L}^2_{t,f}(\mathcal{B})} \|\nabla_h u^h_\lambda\|^{\frac32}_{\widetilde{L}^2_t(\mathcal{B})}\bigg). \tag{2.15}$$

Summing up (2.9), (2.10), (2.13) and (2.15), we complete the proof of (2.8). □

Taking $\lambda = 0$ in the above proposition implies that

Corollary 2.2. *Under the assumptions of Proposition 2.2, we have for all $q \in \mathbb{Z}$,*

$$\Big|\int_0^t (\Delta_q^v \nabla_h p_0 \mid \Delta_q^v u^h)_{L^2}\, dt'\Big| \lesssim d_q^2 2^{-q} \Big(\|u^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h u^h\|^2_{\widetilde{L_t^2}(\mathcal{B})} + \|u^3\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h u^3\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B})} \|u^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h u^h\|^{\frac32}_{\widetilde{L_t^2}(\mathcal{B})}\Big).$$

Proof. Indeed taking $\lambda = 0$ in (2.12) gives

$$\begin{aligned}\big|\mathcal{A}_{1,0}(t)\big| &\lesssim \sum_{q' \geq q-5} \|S^v_{q'+2} u^3\|^{\frac12}_{L_t^\infty(L_h^2(L_v^\infty))} \|S^v_{q'+2}\nabla_h u^3\|^{\frac12}_{L_t^2(L_h^2(L_v^\infty))} \|\Delta^v_{q'}\nabla_h u^h\|_{L_t^2(L^2)} \\ &\quad \times \|\Delta_q^v u^h\|^{\frac12}_{L_t^\infty(L^2)} \|\Delta_q^v \nabla_h u^h\|^{\frac12}_{L_t^2(L^2)} \\ &\lesssim d_q^2 2^{-q} \|u^3\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h u^3\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B})} \|u^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h u^h\|^{\frac32}_{\widetilde{L_t^2}(\mathcal{B})}. \end{aligned} \tag{2.16}$$

The same estimate also holds for $\mathcal{A}_{2,0}(t)$.

On the other hand, notice that

$$\big|\mathcal{D}_{2,0}(t)\big| \lesssim \sum_{|q'-q|\leq 5} \|S^v_{q'-1} u^3\|_{L_t^4(L_h^4(L_v^\infty))} \|\Delta^v_{q'} \partial_3 u^3_\lambda\|_{L_t^2(L^2)} \|\Delta_q^v u^h_\lambda\|_{L_t^4(L_h^4(L_v^2))},$$

which along with the proof of (2.16) ensures that

$$\big|\mathcal{D}_{2,0}(t)\big| \lesssim d_q^2 2^{-q} \|u^3\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h u^3\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B})} \|u^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h u^h\|^{\frac32}_{\widetilde{L_t^2}(\mathcal{B})}.$$

This together with (2.10), (2.14) for $\lambda = 0$ and (2.16) completes the proof of the corollary. □

Now we are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We shall use the classical Friedrichs' regularization method to construct the approximate solutions to (ANS_ν). For simplicity, we just outline it here (for the details in this context, see [23] or [6]). In order to do so, let us first define the sequence of projection operators $(\mathcal{P}_n)_{n\in\mathbb{N}}$ by

$$\mathcal{P}_n a \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\mathbf{1}_{B(0,n)}\widehat{a}\big) \tag{2.17}$$

and we define (u_n^h, u_n^3) via

$$\begin{cases} \partial_t u_n^h - \nu\Delta_h u_n^h + \mathcal{P}_n(u_n\cdot\nabla u_n^h) + \sum_{\ell,k=1}^3 \mathcal{P}_n \nabla_h(-\Delta)^{-1}\partial_\ell\partial_k(u_n^\ell u_n^k) = 0, \\ \partial_t u_n^3 - \nu\Delta_h u_n^3 + \mathcal{P}_n(u_n\cdot\nabla u_n^3) + \sum_{\ell,k=1}^3 \mathcal{P}_n \partial_3(-\Delta)^{-1}\partial_\ell\partial_k(u_n^\ell u_n^k) = 0, \\ \operatorname{div}_h u_n^h + \partial_3 u_n^3 = 0, \\ (u_n^h, u_n^3)|_{t=0} = (\mathcal{P}_n u_0^h, \mathcal{P}_n u_0^3), \end{cases} \tag{2.18}$$

where $(-\Delta)^{-1}\partial_j\partial_k$ is defined precisely by

$$(-\Delta)^{-1}\partial_j\partial_k a \stackrel{\text{def}}{=} -\mathcal{F}^{-1}(|\xi|^{-2}\xi_j\xi_k\widehat{a}).$$

Because of properties of L^2 and L^1 functions the Fourier transform of which are supported in the ball $B(0,n)$, the system (2.18) appears to be an ordinary differential equation in the space

$$L^2_n \stackrel{\text{def}}{=} \Big\{a \in L^2(\mathbb{R}^3) : \text{Supp}\, \widehat{a} \subset B(0,n)\Big\}. \tag{2.19}$$

This ordinary differential equation is globally wellposed because

$$\|u_n(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla_h u_n(t')\|^2_{L^2}\, dt' = \|P_n u_0\|^2_{L^2} \leq \|u_0\|^2_{L^2}. \tag{2.20}$$

We refer to [6] and [23] for the details.

Next let us turn to the uniform estimates for the thus obtained approximate solution sequence $(u_n)_{n\in\mathbb{N}}$. To overcome the difficulty that we can not use Gronwall inequality in the framework of Chemin-Lerner type spaces $\widetilde{L^p_t}(\mathcal{B})$, for any positive $\lambda > 0$, we define:

$$u_{n,\lambda}(t,x) \stackrel{\text{def}}{=} e^{-\lambda \int_0^t f_n(t')\,dt'} u_n(t,x), \quad \text{with} \quad f_n(t) \stackrel{\text{def}}{=} \|u^3_n(t)\|^2_{\mathcal{B}} \|\nabla_h u^3_n(t)\|^2_{\mathcal{B}}. \tag{2.21}$$

Then u^h_λ solves

$$\begin{cases} \partial_t u^h_{n,\lambda} + \lambda f_n(t) u^h_{n,\lambda} - \nu \Delta_h u^h_{n,\lambda} + \mathcal{P}_n(u_n \cdot \nabla u^h_{n,\lambda}) \\ \quad = -\sum_{i,j=1}^3 \mathcal{P}_n \nabla_h (-\Delta)^{-1} \partial_i \partial_j (u^i_n u^j_{n,\lambda}), \\ \operatorname{div} u_{n,\lambda} = 0, \\ u^h_{n,\lambda}|_{t=0} = \mathcal{P}_n u^h_0. \end{cases} \tag{2.22}$$

The main idea will be now to use the sort of weighted Chemin-Lerner spaces $\widetilde{L^2_{t,f}}(\mathcal{B})$ introduced in Definition 1.3 which allows to avoid Gronwall-type lemma in the energy estimates. This kind of spaces will be useful also in the proof of Theorem 1.3 in Sect. 3.

We first apply Δ^v_q to (2.22) and then take the L^2 inner product of the resulting equation with $\Delta^v_q u^h_{n,\lambda}$ to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta^v_q u^h_{n,\lambda}(t)\|^2_{L^2} + \lambda f_n(t)\|\Delta^v_q u^h_{n,\lambda}(t)\|^2_{L^2} + \nu\|\Delta^v_q \nabla_h u^h_{n,\lambda}(t)\|^2_{L^2} \\ &\quad = -\big(\Delta^v_q(u_n \cdot \nabla u^h_{n,\lambda}) \mid \Delta^v_q u^h_n\big) - \sum_{i,j=1}^3 \big(\nabla_h(-\Delta)^{-1}\partial_i\partial_j \Delta^v_q(u^i_n u^j_{n,\lambda}) \mid \Delta^v_q u^h_{n,\lambda}\big). \end{aligned}$$

Applying Proposition 2.1 and Proposition 2.2, we obtain

$$\begin{aligned} &\|\Delta^v_q u^h_{n,\lambda}(t)\|^2_{L^2} + 2\lambda \int_0^t f_n(t')\|\Delta^v_q u^h_{n,\lambda}(t')\|^2_{L^2}\, dt' + 2\nu \int_0^t \|\Delta^v_q \nabla_h u^h_{n,\lambda}(t')\|^2_{L^2}\, dt' \\ &\quad \leq \|\Delta^v_q u^h_0\|^2_{L^2} + C d_q^2 2^{-q}\Big(\|u^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h u^h_{n,\lambda}\|^2_{\widetilde{L^2_t}(\mathcal{B})} + \|u^h_{n,\lambda}\|^{\frac{1}{2}}_{\widetilde{L^2_{t,f_n}}(\mathcal{B})}\|\nabla_h u^h_{n,\lambda}\|^{\frac{3}{2}}_{\widetilde{L^2_t}(\mathcal{B})}\Big), \end{aligned}$$

which along with (1.7) and Definition 1.3 ensures

$$\begin{aligned} &\|u^h_{n,\lambda}\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \sqrt{2\lambda}\|u^h_{n,\lambda}\|_{\widetilde{L^2_{t,f_n}}(\mathcal{B})} + \sqrt{2\nu}\|\nabla_h u^h_{n,\lambda}\|_{\widetilde{L^2_t}(\mathcal{B})} \\ &\quad \leq \|u^h_0\|_{\mathcal{B}} + C\Big(\|u^h_n\|^{\frac{1}{2}}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h u^h_{n,\lambda}\|_{\widetilde{L^2_t}(\mathcal{B})} + \|u^h_{n,\lambda}\|^{\frac{1}{4}}_{\widetilde{L^2_{t,f_n}}(\mathcal{B})}\|\nabla_h u^h_{n,\lambda}\|^{\frac{3}{4}}_{\widetilde{L^2_t}(\mathcal{B})}\Big). \end{aligned}$$

Note that Young's inequality ensures that

$$C\|u^h_{n,\lambda}\|^{1/4}_{\widetilde{L^2_{t,f_n}}(\mathcal{B})}\|\nabla_h u^h_{n,\lambda}\|^{3/4}_{\widetilde{L^2_t}(\mathcal{B})} \leq C_0\nu^{-\frac{3}{2}}\|u^h_{n,\lambda}\|_{\widetilde{L^2_{t,f_n}}(\mathcal{B})} + (\sqrt{2}-1)\sqrt{\nu}\|\nabla_h u^h_{n,\lambda}\|_{\widetilde{L^2_t}(\mathcal{B})},$$

whence we obtain

$$\begin{aligned} &\|u^h_{n,\lambda}\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \sqrt{\lambda}\|u^h_{n,\lambda}\|_{\widetilde{L^2_{t,f_n}}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u_{n,\lambda}\|_{\widetilde{L^2_t}(\mathcal{B})} \\ &\quad \leq \|u^h_0\|_{\mathcal{B}} + C_0\Big(\|u^h_n\|^{1/2}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h u^h_{n,\lambda}\|_{\widetilde{L^2_t}(\mathcal{B})} + \nu^{-\frac{3}{2}}\|u^h_{n,\lambda}\|_{\widetilde{L^2_{t,f_n}}(\mathcal{B})}\Big). \end{aligned} \tag{2.23}$$

Let ε_0 be a small positive constant, which will be determined later on, we denote

$$T^*_n \stackrel{\text{def}}{=} \max\Big\{t:\ \|u^h_n\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u^h_n\|_{\widetilde{L^2_t}(\mathcal{B})} \leq \min(\frac{1}{4C_0^2}, \varepsilon_0)\nu\Big\}. \tag{2.24}$$

Then taking $\lambda = \frac{4C_0^2}{\nu^3}$ in (2.23), we obtain

$$\|u^h_{n,\lambda}\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u^h_{n,\lambda}\|_{\widetilde{L^2_t}(\mathcal{B})} \leq 2\|u^h_0\|_{\mathcal{B}} \qquad \text{for} \quad t < T^*_n. \tag{2.25}$$

While thanks to (2.21), we have

$$e^{-\lambda\int_0^t f_n(t')dt'}\big(\|u^h_n\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \|\nabla_h u^h_n\|_{\widetilde{L^2_t}(\mathcal{B})}\big) \leq \|u^h_{n,\lambda}\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \|\nabla_h u^h_{n,\lambda}\|_{\widetilde{L^2_t}(\mathcal{B})},$$

which together with (2.25) implies that

$$\begin{aligned} &\|u^h_n\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u^h_n\|_{\widetilde{L^2_t}(\mathcal{B})} \\ &\quad \leq 2\|u^h_0\|_{\mathcal{B}}\exp\Big\{\frac{4C_0^2}{\nu^3}\int_0^t \|u^3_n(t')\|^2_{\mathcal{B}}\|\nabla_h u^3_n(t')\|^2_{\mathcal{B}}\,dt'\Big\}. \end{aligned} \tag{2.26}$$

On the other hand, we get by applying Δ^v_q to the vertical equation in (2.18) and then taking the L^2 inner product of the resulting equation with $\Delta^v_q u^3_n$ that

$$\begin{aligned} &\frac{1}{2}\|\Delta^v_q u^3_n(t)\|^2_{L^2} + \nu\int_0^t \|\Delta^v_q\nabla_h u^3_n(t')\|^2_{L^2}\,dt' - \frac{1}{2}\|\Delta^v_q\mathcal{P}_n u^3_0\|^2_{L^2} \\ &\quad = -\int_0^t \big(\Delta^v_q(u_n\cdot\nabla u^3_n) \mid \Delta^v_q u^3_n\big)_{L^2}\,dt' \\ &\qquad - \sum_{\ell,k=1}^3\int_0^t \big(\partial_3(-\Delta)^{-1}\partial_\ell\partial_k\Delta^v_q(u^\ell_n u^k_n) \mid \Delta^v_q u^3_n\big)_{L^2}\,dt'. \end{aligned}$$

Notice that $\operatorname{div} u_n = 0$, and one gets by using integration by parts

$$\begin{aligned} &\sum_{\ell,k=1}^3\int_0^t \big(\partial_3(-\Delta)^{-1}\partial_\ell\partial_k\Delta^v_q(u^\ell_n u^k_n) \mid \Delta^v_q u^3_{n,}\big)_{L^2}\,dt' \\ &\quad = -\sum_{\ell,k=1}^3\int_0^t \big(\nabla_h(-\Delta)^{-1}\partial_\ell\partial_k\Delta^v_q(u^i_n u^j_n) \mid \Delta^v_q u^h_n\big)_{L^2}\,dt', \end{aligned}$$

which along with Corollary 2.1 and Corollary 2.2 gives

$$\|u_n^3\|_{\widetilde{L}_t^\infty(\mathcal{B})} + \sqrt{2\nu}\|\nabla_h u_n^3\|_{\widetilde{L}_t^2(\mathcal{B})} \le \|u_0^3\|_{\mathcal{B}} + \widetilde{C}\Big(\|u_n^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{1/2}\|\nabla_h u_n^h\|_{\widetilde{L}_t^2(\mathcal{B})} + \|u_n^3\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{1/4}\|\nabla_h u_n^3\|_{\widetilde{L}_t^2(\mathcal{B})}^{1/4}\|u_n^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{1/4}\|\nabla u_n^h\|_{\widetilde{L}_t^2(\mathcal{B})}^{3/4}\Big).$$

Then thanks to (2.24), we obtain

$$\|u_n^3\|_{\widetilde{L}_t^\infty(\mathcal{B})} + \sqrt{2\nu}\|\nabla_h u_n^3\|_{\widetilde{L}_t^2(\mathcal{B})} \le \|u_0^3\|_{\mathcal{B}} + \widetilde{C}\left(\varepsilon_0^{\frac{3}{2}}\nu + \varepsilon_0\nu^{\frac{5}{8}}\|u_n^3\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{1/4}\|\nabla u_n^3\|_{\widetilde{L}_t^2(\mathcal{B})}^{1/4}\right),$$

for $t < T_n^*$. Taking $\varepsilon_0 = \varepsilon_0(\widetilde{C})$ small enough, we obtain

$$\|u_n^3\|_{\widetilde{L}_t^\infty(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u_n^3\|_{\widetilde{L}_t^2(\mathcal{B})} \le 2\|u_0^3\|_{\mathcal{B}} + \nu \quad \text{for} \quad t < T_n^*. \tag{2.27}$$

Now we claim that $T_n^* = \infty$ provided that c_0 in (1.11) is sufficiently small. Indeed if $T_n^* < \infty$, it follows from (2.27) that

$$\int_0^t \|u_n^3(t')\|_{\mathcal{B}}^2\|\nabla_h u_n^3(t')\|_{\mathcal{B}}^2\,dt' \le \|u_n^3\|_{\widetilde{L}_t^\infty(\mathcal{B})}^2\|\nabla_h u_n^3\|_{\widetilde{L}_t^2(\mathcal{B})}^2 \le \frac{16}{\nu}\left(16\|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}^4 + \nu^4\right).$$

Substituting the above inequality into (2.26) claims that

$$\begin{aligned}\|u_n^h\|_{\widetilde{L}_t^\infty(\mathcal{B})} + \sqrt{\nu}\|\nabla_h u_n^h\|_{\widetilde{L}_t^2(\mathcal{B})} &\le 2\exp(64C_0^2)\|u_0^h\|_{\mathcal{B}}\exp\left(\frac{1024C_0^2}{\nu^4}\|u_0^3\|_{\mathcal{B}}^4\right)\\ &\le \frac{1}{2}\min(\frac{1}{4C_0^2}, \varepsilon_0)\nu,\end{aligned} \tag{2.28}$$

provided that we take $L = 1024C_0^2$ and $c_0 \le \frac{1}{4}\exp(-64C_0^2)\min(\frac{1}{4C_0^2}, \varepsilon_0)$ in (1.11). This contradicts the definition of T_n^* defined in (2.24), and therefore $T_n^* = +\infty$.

With (2.27) and (2.28) being obtained for $T_n^* = \infty$, one can prove the existence part of Theorem 1.2 via a standard compactness argument. And the uniqueness of the solution to (ANS_ν) in $\mathcal{B}(T)$ has already been proved in [23]. One may check [23] or [6] for the details. This completes the proof of the theorem. □

3. The Proof of Theorem 1.3

The goal of this section is to prove Theorem 1.3. As a convention in what follows, we shall always assume that $u_0 \in \mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}$, then according to Definition 1.2, we split u_0 as $u_{hh} + u_{\ell h}$ with

$$u_{hh} \overset{\text{def}}{=} \sum_{k\ge \ell-1}\Delta_k^h\Delta_\ell^v u_0, \quad u_{\ell h} \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} S_{j-1}^h\Delta_j^v u_0. \tag{3.1}$$

Correspondingly, we shall seek the solution of (ANS_ν) of the form $u = u_F + w$ given by (1.15). We notice that the low horizontal frequencies part $u_{\ell h}$ belongs to $\mathcal{B}$ so that we can use part of the techniques used in the previous section to build the solution of (1.17) for w. For u_F, we shall use the fact that any vertical derivative of u_{hh} can be controlled by its horizontal derivative. Finally, we have to use once again the weighted Chemin-Lerner spaces given by Definition 1.3 in order to avoid using the Gronwall type lemma in the energy estimates.

The following lemma from [11] will be very useful in this section.

Lemma 3.1. *Let u_F be given by (1.16). Then there hold*

(1)

$$\|\Delta_k^h\Delta_\ell^v u_F\|_{L^p_T(L^4_h(L^2_v))} \lesssim \begin{cases} \dfrac{d_{k,\ell}}{\nu^{\frac{1}{p}}} 2^{k\left(\frac{1}{2}-\frac{2}{p}\right)} 2^{-\frac{\ell}{2}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}, & \textit{for} \quad k \geq \ell - 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{3.2}$$

for any $1 \leq p \leq \infty$;

(2) *For any* (p, q) *in* $[1, \infty] \times [4, \infty]$*, we have*

$$\|\Delta_k^h u_F\|_{L^p(\mathbb{R}^+; L^q_h(L^\infty_v))} \lesssim \frac{1}{\nu^{\frac{1}{p}}} c_k 2^{-k\left(2\left(\frac{1}{p}+\frac{1}{q}\right)-1\right)} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}. \tag{3.3}$$

If in addition $\dfrac{1}{p} + \dfrac{1}{q} > \dfrac{1}{2}$*, we have*

$$\|\Delta_j^v u_F\|_{L^p(\mathbb{R}^+; L^q_h(L^2_v))} \lesssim \frac{1}{\nu^{\frac{1}{p}}} d_j 2^{-j\left(2\left(\frac{1}{p}+\frac{1}{q}\right)-\frac{1}{2}\right)} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}; \tag{3.4}$$

(3)

$$\begin{aligned} \|\Delta_j^v u_F\|_{L^2(\mathbb{R}^+; L^\infty_h(L^2_v))} &\lesssim \frac{d_j}{\sqrt{\nu}} 2^{-\frac{j}{2}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \quad \textit{and} \\ \|u_F\|_{L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))} &\lesssim \frac{1}{\sqrt{\nu}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}. \end{aligned} \tag{3.5}$$

Outline of the proof. Indeed thanks to Lemma 2.1 of [5], there exists a positive constant c such that

$$\begin{aligned} \|\Delta_k^h\Delta_\ell^v u_F(t)\|_{L^4_h(L^2_v)} &\lesssim e^{-c\nu t 2^{2k}} \|\Delta_k^h\Delta_\ell^v u_{hh}\|_{L^4_h(L^2_v)} \\ &\lesssim e^{-c\nu t 2^{2k}} d_{k,\ell} 2^{\frac{k}{2}} 2^{-\frac{\ell}{2}} \|u_0\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}, \end{aligned} \tag{3.6}$$

from which and Lemma 2.1, we deduce (3.2–3.4).

To prove (3.5), we write

$$\|\Delta_j^v u_F\|^2_{L^2(\mathbb{R}^+;L_h^\infty(L_v^2))} = \|(\Delta_j^v u_F)^2\|_{L^1(\mathbb{R}^+;L_h^\infty(L_v^1))},$$

and using Bony's paradifferential decomposition (1.22) in the horizonal variables, one has

$$(\Delta_j^v u_F)^2 = \sum_{k\in\mathbb{Z}} S_{k-1}^h \Delta_j^v u_F \Delta_k^h \Delta_j^v u_F + \sum_{k\in\mathbb{Z}} \Delta_k^h \Delta_j^v u_F S_{k+2}^h \Delta_j^v u_F,$$

which along with (3.2) gives (3.5). One may check Lemma 2.5, Corollary 2.3 and Lemma 2.6 of [11] for the detailed proof. □

In what follows, everything will be different for terms involving the horizontal derivatives and for terms involving the vertical derivative. For terms involving horizontal derivatives, the following Lemma 3.2 and Corollary 3.1 will be very useful.

Lemma 3.2. *Let $a \in \mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(t)$ and $b \in \mathcal{B}(t)$. Let $0 \leq g \in L^\infty(0,t)$; we denote $a_g(t,x) \overset{def}{=} g(t)a(t,x)$. Then there holds for all $q \in \mathbb{Z}$,*

$$\|\sqrt{g}\Delta_q^v(a\cdot\nabla_h b_g)\|_{L_t^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} \lesssim d_q 2^{-\frac{q}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}.$$

Proof. We first get by using Bony's decomposition (1.22) in the vertical variable that

$$\Delta_q^v(a\cdot\nabla_h b_g) = \sum_{|q'-q|\leq 5} \Delta_q^v(S_{q'-1}^v a \Delta_{q'}^v \nabla_h b_g) + \sum_{q'\geq q-4} \Delta_q^v(\Delta_{q'}^v a S_{q'+2}^v \nabla_h b_g). \quad (3.7)$$

While according to Definition 1.2, we can split a into a part where the horizontal frequencies are greater than the vertical ones and a part where the horizontal frequencies are smaller than the vertical ones, more precisely,

$$a = a_{\mathfrak{h}} + a_{\mathfrak{l}} \quad \text{with} \quad a_{\mathfrak{h}} \overset{\text{def}}{=} \sum_{k\geq \ell-1} \Delta_k^h \Delta_\ell^v a \quad \text{and} \quad a_{\mathfrak{l}} \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} S_{j-1}^h \Delta_j^v a. \quad (3.8)$$

Notice that

$$\|\sqrt{g}\Delta_{q'}^v a_{\mathfrak{h}}\|^2_{L_t^4(L_h^4(L_v^2))} = \|g(\Delta_{q'}^v a_{\mathfrak{h}})^2\|_{L_t^2(L_h^2(L_v^1))},$$

and using Bony's decomposition in the horizontal variables gives

$$(\Delta_{q'}^v a_{\mathfrak{h}})^2 = \sum_{k\in\mathbb{Z}} S_{k-1}^h \Delta_{q'}^v a_{\mathfrak{h}} \Delta_k^h \Delta_{q'}^v a_{\mathfrak{h}} + \sum_{k\in\mathbb{Z}} S_{k+2}^h \Delta_{q'}^v a_{\mathfrak{h}} \Delta_k^h \Delta_{q'}^v a_{\mathfrak{h}}.$$

However it follows from Lemma 2.1 and Definition 1.2 that

$$\begin{aligned}
&\|g\sum_{k\in\mathbb{Z}} S_{k+2}^h \Delta_{q'}^v a_{\mathfrak{h}} \Delta_k^h \Delta_{q'}^v a_{\mathfrak{h}}\|_{L_t^2(L_h^2(L_v^1))} \\
&\quad\lesssim \sum_{k\in\mathbb{Z}} \|S_{k+2}^h \Delta_{q'}^v a_{\mathfrak{h}}\|_{L_t^\infty(L_h^4(L_v^2))} \|\Delta_k^h \Delta_{q'}^v (g a_{\mathfrak{h}})\|_{L_t^2(L_h^4(L_v^2))} \\
&\quad\lesssim \sum_{k\in\mathbb{Z}} d_{k,q'}^2 2^{-q'} \|a\|_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \\
&\quad\lesssim d_{q'}^2 2^{-q'} \|a\|_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})},
\end{aligned}$$

which gives

$$\|\sqrt{g}\Delta_{q'}^v a_{\mathfrak{h}}\|_{L_t^4(L_h^4(L_v^2))} \lesssim d_{q'} 2^{-\frac{q'}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})}. \tag{3.9}$$

On the other hand, notice that

$$\Delta_{q'}^v a_{\mathfrak{l}} = \sum_{|j-q'|\leq 1} \Delta_{q'}^v (S_{j-1}^h \Delta_j^v a),$$

which along with the simple interpolation in 2-D gives

$$\begin{aligned}\|\sqrt{g}\Delta_{q'}^v a_{\mathfrak{l}}\|_{L_t^4(L_h^4(L_v^2))} &\lesssim \|\Delta_{q'}^v a_{\mathfrak{l}}\|^{\frac{1}{2}}_{L_t^\infty(L^2)} \|g\Delta_{q'}^v \nabla_h a_{\mathfrak{l}}\|^{\frac{1}{2}}_{L_t^2(L^2)} \\ &\lesssim d_{q'} 2^{-\frac{q'}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})}. \end{aligned} \tag{3.10}$$

Thanks to Lemma 2.1 and (3.9), (3.10), we obtain

$$\|\sqrt{g}S_{q'-1}^v a\|_{L_t^4(L_h^4(L_v^\infty))} \lesssim \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})}, \tag{3.11}$$

from which we deduce that

$$\begin{aligned}&\|\sqrt{g}\sum_{|q'-q|\leq 5} \Delta_q^v (S_{q'-1}^v a \Delta_{q'}^v \nabla_h b_g)\|_{L_t^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} \\ &\lesssim \sum_{|q'-q|\leq 5} \|\sqrt{g}S_{q'-1}^v a\|_{L_t^4(L_h^4(L_v^\infty))} \|\Delta_{q'}^v \nabla_h b_g\|_{L_t^2(L^2)} \\ &\lesssim \Big(\sum_{|q'-q|\leq 5} d_{q'} 2^{\frac{q-q'}{2}}\Big) 2^{-\frac{q}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})} \\ &\lesssim d_q 2^{-\frac{q}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}. \end{aligned} \tag{3.12}$$

Similar argument shows that

$$\begin{aligned}&\|\sqrt{g}\sum_{q'\geq q-4} \Delta_q^v (\Delta_{q'}^v a S_{q'+2}^v \nabla_h b_g)\|_{L_t^{\frac{4}{3}}(L_h^{\frac{4}{3}}(L_v^2))} \\ &\lesssim \sum_{q'\geq q-4} \|\sqrt{g}\Delta_{q'}^v a\|_{L_t^4(L_h^4(L_v^2))} \|S_{q'+2}^v \nabla_h b_g\|_{L_t^2(L_h^2(L_v^\infty))} \\ &\lesssim d_q 2^{-\frac{q}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}. \end{aligned}$$

This along with (3.7) and (3.12) concludes the proof of the lemma. □

An immediate corollary implied by the proof of the above lemma is:

Corollary 3.1. (1) *Under the assumptions of Lemma 3.1, we have*

$$\|\Delta_q^v(ab_g)\|_{L_t^2(L^2)} \lesssim d_q 2^{-\frac{q}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|b\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}.$$

(2) *Let a, b be in $\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}(t)$. We have*

$$\|\Delta_q^v(ab_g)\|_{L_t^2(L^2)} \lesssim d_q 2^{-\frac{q}{2}} \|a\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h a_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|b\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})} \|\nabla_h b_g\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}})}.$$

Proof. Indeed thanks to (3.7), one has

$$\begin{aligned}\|\Delta_q^v(ab_g)\|_{L_t^2(L^2)} \lesssim & \sum_{|q'-q|\le 5} \|\sqrt{g} S_{q'-1}^v a\|_{L_t^4(L_h^4(L_v^\infty))} \|\sqrt{g}\Delta_{q'}^v b\|_{L_t^4(L_h^4(L_v^2))} \\ & + \sum_{q'\ge q-4} \|\sqrt{g}\Delta_{q'}^v a\|_{L_t^4(L_h^4(L_v^2))} \|\sqrt{g} S_{q'+2}^v b\|_{L_t^4(L_h^4(L_v^\infty))},\end{aligned}$$

which along with (2.3) and (3.9–3.11) gives the corollary. □

The following propositions are the key ingredients in the proof of Theorem 1.3.

Proposition 3.1. *Let u_F be given by (1.16). Let $a \in \mathcal{B}(t)$ and $0 \le m \in L^\infty(0,t)$. We denote*

$$\begin{aligned} g_\lambda(t) &\stackrel{def}{=} \exp\big(-\lambda \|u_{hh}^3\|^4_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} - m(t)\big), \\ u_{F,\lambda}(t,x) &\stackrel{def}{=} g_\lambda(t) u_F(t,x), \quad a_\lambda(t,x) \stackrel{def}{=} g_\lambda(t) a(t,x), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I_{\lambda,q}^h(t) &\stackrel{def}{=} \int_0^t \big(\Delta_q^v(u_F\cdot\nabla u_{F,\lambda}^h) \mid \Delta_q^v a_\lambda\big)_{L^2}\,dt \qquad \textit{and} \\ I_q^v(t) &\stackrel{def}{=} \int_0^t \big(\Delta_q^v(u_F\cdot\nabla u_F^3) \mid \Delta_q^v a\big)_{L^2}\,dt. \end{aligned}$$

Then we have for all $q \in \mathbb{Z}$,

$$\begin{aligned} \big|I_{\lambda,q}^h(t)\big| \lesssim d_q^2 2^{-q} \Bigg(& \frac{1}{\sqrt{\nu}} \|u_{hh}^h\|^2_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \|\nabla_h a_\lambda\|_{\widetilde{L_t^2}(\mathcal{B})} \\ & + \frac{1}{\lambda^{\frac{1}{4}}\nu} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \|a_\lambda\|_{\widetilde{L_t^\infty}(\mathcal{B})} \Bigg), \end{aligned} \tag{3.14}$$

$$\big|I_q^v(t)\big| \lesssim d_q^2 2^{-q} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \|u_{hh}^3\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \left(\frac{1}{\sqrt{\nu}} \|\nabla_h a\|_{\widetilde{L_t^2}(\mathcal{B})} + \frac{1}{\nu} \|a\|_{\widetilde{L_t^\infty}(\mathcal{B})} \right). \tag{3.15}$$

Proof. As $\operatorname{div} u_F = 0$, we first get by using integration by parts that

$$I^h_{\lambda,q}(t) = -\int_0^t \big(\Delta^v_q(u^h_F \otimes u^h_{F,\lambda}) \mid \nabla_h \Delta^v_q a_\lambda\big)_{L^2}\, dt + \int_0^t \big(\partial_3 \Delta^v_q(u^3_F u^h_{F,\lambda}) \mid \Delta^v_q a_\lambda\big)_{L^2}\, dt$$
$$\stackrel{\text{def}}{=} I^{h,1}_{\lambda,q}(t) + I^{h,2}_{\lambda,q}(t). \tag{3.16}$$

Applying Corollary 3.1 gives

$$\big|I^{h,1}_{\lambda,q}(t)\big| \leq \|\Delta^v_q\left(u^h_F \otimes u^h_{F,\lambda}\right)\|_{L^2_t(L^2)}\|\nabla_h \Delta^v_q a_\lambda\|_{L^2_t(L^2)}$$
$$\lesssim d_q^2 2^{-q}\|u^h_F\|_{\widetilde{L^\infty_t}(\mathcal{B}_4^{-\frac12,\frac12})}\|\nabla_h u^h_F\|_{\widetilde{L^2_t}(\mathcal{B}_4^{-\frac12,\frac12})}\|\nabla_h a_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}.$$

However, thanks to Definition 1.2 and (3.2), we have

$$\|u^h_F\|_{\widetilde{L^\infty_t}(\mathcal{B}_4^{-\frac12,\frac12})} \lesssim \|u^h_{hh}\|_{B_4^{-\frac12,\frac12}} \quad \text{and} \quad \|\nabla_h u^h_F\|_{\widetilde{L^2_t}(\mathcal{B}_4^{-\frac12,\frac12})} \lesssim \frac{1}{\sqrt{\nu}}\|u^h_{hh}\|_{B_4^{-\frac12,\frac12}}, \tag{3.17}$$

from which we deduce that

$$\big|I^{h,1}_{\lambda,q}(t)\big| \lesssim \frac{d_q^2}{\sqrt{\nu}} 2^{-q}\|u^h_{hh}\|^2_{B_4^{-\frac12,\frac12}}\|\nabla_h a_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}. \tag{3.18}$$

For the term with the vertical derivative, let us write using Lemma 2.1 that

$$\big|I^{h,2}_{\lambda,q}(t)\big| \lesssim 2^q\|\Delta^v_q(u^3_F u^h_{F,\lambda})\|_{L^1_t(L^2)}\|\Delta^v_q a_\lambda\|_{L^\infty_t(L^2)}.$$

Using Bony's decomposition in the vertical variable, we infer

$$\Delta^v_q(u^3_F u^h_{F,\lambda}) = \sum_{|q'-q|\leq 5}\Delta^v_q(S^v_{q'-1}u^3_F \Delta^v_{q'}u^h_{F,\lambda}) + \sum_{q'\geq q-4}\Delta^v_q(\Delta^v_{q'}u^3_F S^v_{q'+2}u^h_{F,\lambda}), \tag{3.19}$$

whereas using Bony's decomposition (1.22) in the horizontal variables and the definition of u_{hh} gives

$$S^v_{q'-1}u^3_F \Delta^v_{q'}u^h_{F,\lambda} = \sum_{k\geq q'-4}\Big(S^h_{k-1}S^v_{q'-1}u^3_F \Delta^h_k \Delta^v_{q'}u^h_{F,\lambda}$$
$$+ \Delta^h_k S^v_{q'-1}u^3_F S^h_{k+2}\Delta^v_{q'}u^h_{F,\lambda}\Big). \tag{3.20}$$

The two terms of the above sum can be estimated along the same lines. Whereas thanks to (3.2) and (3.6), we have

$$\|\sum_{k\geq q'-4} S^h_{k-1}S^v_{q'-1}u^3_F \Delta^h_k \Delta^v_{q'}u^h_{F,\lambda}\|_{L^1_t(L^2)}$$
$$\lesssim \sum_{k\geq q'-4}\|S^h_{k-1}S^v_{q'-1}u^3_F\|_{L^\infty_t(L^4_h(L^\infty_v))}\|\Delta^h_k \Delta^v_{q'}u^h_{F,\lambda}\|_{L^1_t(L^4_h(L^2_v))}$$
$$\lesssim \Big(\sum_{k\geq q'-4} c_k d_{k,q'}2^k\int_0^t e^{-c\nu t' 2^{2k}}\,dt'\Big)2^{-\frac{q'}{2}}\|u^3_{hh}\|_{B_4^{-\frac12,\frac12}} e^{-\lambda\|u^3_{hh}\|^4_{B_4^{-\frac12,\frac12}}}\|u^h_{hh}\|_{B_4^{-\frac12,\frac12}},$$

which gives

$$\|\sum_{k\geq q'-4} S^h_{k-1}S^v_{q'-1}u^3_F\Delta^h_k\Delta^v_{q'}u^h_{F,\lambda}\|_{L^1_t(L^2)}\lesssim\frac{d_q}{\lambda^{\frac14}\nu}2^{-\frac{3q'}{2}}\|u^h_{hh}\|_{B_4^{-\frac12,\frac12}},$$

and consequently

$$\left|I^{h,2}_{\lambda,q}(t)\right|\lesssim\frac{d_q^2}{\lambda^{\frac14}\nu}2^{-q}\|u^h_{hh}\|_{B_4^{-\frac12,\frac12}}\|a_\lambda\|_{\widetilde{L^\infty_t}(\mathcal{B})}.$$

This along with (3.18) proves (3.14).

On the other hand, again as $\operatorname{div} u_F=0$, we write by using integration by parts

$$I^v_q(t)=-\int_0^t\left(\Delta^v_q(u^h_Fu^3_F)\mid\Delta^v_q\nabla_h a\right)_{L^2}dt'-\int_0^t\left(\Delta^v_q((u^3_F)^2)\mid\partial_3\Delta^v_q a\right)_{L^2}dt',$$

which along with Lemma 2.1 ensures

$$\begin{aligned}\left|I^v_q(t)\right|\lesssim&\|\Delta^v_q(u^h_Fu^3_F)\|_{L^2_t(L^2)}\|\Delta^v_q\nabla_h a\|_{L^2_t(L^2)}\\&+2^q\|\Delta^v_q((u^3_F)^2)\|_{L^1_t(L^2)}\|\Delta^v_q a\|_{L^\infty_t(L^2)}.\end{aligned}\tag{3.21}$$

Applying Corollary 3.1 and (3.17) gives

$$\|\Delta^v_q(u^h_Fu^3_F)\|_{L^2_t(L^2)}\lesssim\frac{d_q}{\sqrt{\nu}}2^{-\frac q2}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|u^3_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}.\tag{3.22}$$

On the other hand, applying Lemma 2.1 and $\operatorname{div} u_F=0$ yields

$$\begin{aligned}&\|\sum_{k\geq q'-4}S^h_{k-1}S^v_{q'-1}u^3_F\Delta^h_k\Delta^v_{q'}u^3_F\|_{L^1_t(L^2)}\\&\lesssim 2^{-q'}\sum_{k\geq q'-4}\|S^h_{k-1}S^v_{q'-1}u^3_F\|_{L^\infty_t(L^4_h(L^\infty_v))}\|\Delta^h_k\Delta^v_{q'}\partial_3u^3_F\|_{L^1_t(L^4_h(L^2_v))}\\&\lesssim\frac1\nu\Big(\sum_{k\geq q'-4}c_kd_{k,q'}\Big)2^{-\frac{3q'}{2}}\|u^3_{hh}\|_{B_4^{-\frac12,\frac12}}\|u^h_{hh}\|_{B_4^{-\frac12,\frac12}}\\&\lesssim\frac{d_{q'}}{\nu}2^{-\frac{3q'}{2}}\|u^3_{hh}\|_{B_4^{-\frac12,\frac12}}\|u^h_{hh}\|_{B_4^{-\frac12,\frac12}},\end{aligned}$$

from which we deduce by a similar version of (3.19) and (3.20) for $\Delta_q((u^3_F)^2)$ that

$$\|\Delta^v_q((u^3_F)^2)\|_{L^1_t(L^2)}\lesssim\frac{d_q}{\nu}2^{-\frac{3q}{2}}\|u^3_{hh}\|_{B_4^{-\frac12,\frac12}}\|u^h_{hh}\|_{B_4^{-\frac12,\frac12}},$$

which along with (3.21) and (3.22) ensures (3.15). This completes the proof of the proposition. □

Proposition 3.2. *Let u_F be given by (1.16). Let $a = (a^h, a^3)$ and b be in $\mathcal{B}(t)$ with* $\operatorname{div} a = 0$. *For a given nonnegative function $m \in L^\infty(0,t)$ and $\lambda > 0$, we denote*

$$f(t) \stackrel{def}{=} \|a^3(t)\|_{\mathcal{B}}^2 \|\nabla_h a^3(t)\|_{\mathcal{B}}^2 \qquad and \qquad g_\lambda(t) \stackrel{def}{=} \exp\big(-\lambda \int_0^t f(t')\,dt' - m(t)\big), \tag{3.23}$$

and

$$J_{\lambda,q}^h(t) \stackrel{def}{=} \int_0^t \big(\Delta_q^v(a\cdot\nabla u_{F,\lambda}^h) \mid \Delta_q^v b_\lambda\big)_{L^2}\,dt \qquad and$$
$$J_q^v(t) \stackrel{def}{=} \int_0^t \big(\Delta_q^v(a\cdot\nabla u_F^3) \mid \Delta_q^v b\big)_{L^2}\,dt,$$

with $u_{F,\lambda}$, a_λ and b_λ being given by (3.13). Then there holds for $q \in \mathbb{Z}$,

$$\begin{aligned}\big|J_{\lambda,q}^h(t)\big| \lesssim d_q^2 2^{-q}\Big(&\frac{1}{\nu^{\frac14}}\|\nabla_h a_\lambda^h\|_{\widetilde{L}_t^2(\mathcal{B})}\|b\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h b_\lambda\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12} + \frac{1}{\nu^{\frac34}}\|b_\lambda\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|b_\lambda\|_{\widetilde{L}_{t,f}^2(\mathcal{B})}^{\frac12}\\ &+\frac{1}{\nu^{\frac14}}\|a^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h a_\lambda^h\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12}\|\nabla_h b_\lambda\|_{\widetilde{L}_t^2(\mathcal{B})} + \frac{1}{\nu^{\frac12}}\|\nabla_h a_\lambda^h\|_{\widetilde{L}_t^2(\mathcal{B})}\|b\|_{\widetilde{L}_t^\infty(\mathcal{B})}\Big)\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}},\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\big|J_q^v(t)\big| \lesssim \frac{d_q^2}{\nu^{\frac14}}2^{-q}\Big\{&\big(\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h a^3\|_{\widetilde{L}_t^2(\mathcal{B})} + \|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h a^h\|_{\widetilde{L}_t^2(\mathcal{B})}\big)\\ &\times\|b\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h b\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12} + \big(\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^3\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h a^3\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12}\\ &+\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h a^h\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12}\big)\|\nabla_h b\|_{\widetilde{L}_t^2(\mathcal{B})}\Big\}.\end{aligned} \tag{3.25}$$

Proof. Again we first distinguish the terms with horizontal derivatives from terms with the vertical one to write

$$\begin{aligned}J_{\lambda,q}^h(t) &= \int_0^t \big(\Delta_q^v(a^h\cdot\nabla_h u_{F,\lambda}^h) \mid \Delta_q^v b_\lambda\big)_{L^2}\,dt' + \int_0^t \big(\Delta_q^v(a^3\partial_3 u_{F,\lambda}^h) \mid \Delta_q^v b_\lambda\big)_{L^2}\,dt'\\ &\stackrel{\text{def}}{=} J_{\lambda,q}^{h,1}(t) + J_{\lambda,q}^{h,2}(t).\end{aligned} \tag{3.26}$$

Note that one gets by using integration by parts

$$J_{\lambda,q}^{h,1}(t) = -\int_0^t \big(\Delta_q^v(\operatorname{div}_h a^h u_{F,\lambda}^h) \mid \Delta_q^v b_\lambda\big)_{L^2}\,dt' - \int_0^t \big(\Delta_q^v(a^h u_{F,\lambda}^h) \mid \nabla_h\Delta_q^v b_\lambda\big)_{L^2}\,dt',$$

so that applying (2.3), Lemma 3.2 and Corollary 3.1 gives

$$\begin{aligned}\big|J_{\lambda,q}^{h,1}(t)\big| &\lesssim \|\sqrt{g_\lambda}\Delta_q^v(\operatorname{div}_h a_\lambda^h u_F^h)\|_{L_t^{\frac43}(L_h^{\frac43}(L_v^2))}\|\sqrt{g_\lambda}\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))}\\ &\quad+\|\Delta_q^v(a_\lambda^h u_F^h)\|_{L_t^2(L^2)}\|\Delta_q^v\nabla_h b_\lambda\|_{L_t^2(L^2)}\\ &\lesssim d_q^2 2^{-q}\|u_F^h\|_{\widetilde{L}_t^\infty(\mathcal{B}_4^{-\frac12,\frac12})}^{\frac12}\|\nabla_h u_{F,\lambda}^h\|_{\widetilde{L}_t^2(\mathcal{B}_4^{-\frac12,\frac12})}^{\frac12}\big(\|\nabla_h a_\lambda^h\|_{\widetilde{L}_t^2(\mathcal{B})}\|b\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\\ &\quad\times\|\nabla_h b_\lambda\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12} + \|a^h\|_{\widetilde{L}_t^\infty(\mathcal{B})}^{\frac12}\|\nabla_h a_\lambda^h\|_{\widetilde{L}_t^2(\mathcal{B})}^{\frac12}\|\nabla_h b_\lambda\|_{\widetilde{L}_t^2(\mathcal{B})}\big),\end{aligned}$$

from which and (3.17), we deduce that

$$\left|J^{h,1}_{\lambda,q}(t)\right| \lesssim \frac{d_q^2}{\nu^{\frac14}} 2^{-q} \|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}} \big(\|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})} \|b\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})} \|\nabla_h b_\lambda\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})} + \|a^h\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})} \|\nabla_h a^h_\lambda\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})} \|\nabla_h b_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}\big). \tag{3.27}$$

While again we get by using Bony's decomposition (1.22) in the vertical variable, we write

$$J^{h,2}_{\lambda,q}(t) = \sum_{|q'-q|\le 5} \int_0^t \big(\Delta_q^v(S^v_{q'-1} a^3 \partial_3 \Delta^v_{q'} u^h_{F,\lambda}) \mid \Delta^v_q b_\lambda\big)_{L^2}\, dt' + \sum_{q'\ge q-4} \int_0^t \big(\Delta_q^v(\Delta^v_{q'} a^3 \partial_3 S^v_{q'+2} u^h_{F,\lambda}) \mid \Delta^v_q b_\lambda\big)_{L^2}\, dt' \stackrel{\text{def}}{=} \mathcal{H}_{1,q}(t) + \mathcal{H}_{2,q}(t).$$

Then it follows from Lemma 2.1 that

$$\begin{aligned}|\mathcal{H}_{1,q}(t)| &\lesssim \sum_{|q'-q|\le 5} 2^{q'} \int_0^t \|S^v_{q'-1} a^3\|_{L^4_h(L^2_v)} \|\Delta^v_{q'} u^h_F\|_{L^4_h(L^2_v)} \|\Delta^v_q b_\lambda\|_{L^2}\, dt' \\ &\lesssim \sum_{|q'-q|\le 5} 2^q \Big(\int_0^t \|a^3(t')\|^2_{\mathcal{B}} \|\nabla_h a^3(t')\|^2_{\mathcal{B}} \|\Delta^v_q b_\lambda(t')\|^2_{L^2}\, dt'\Big)^{\frac14} \\ &\quad \times \|\Delta^v_{q'} u^h_F\|_{L^{\frac43}_t(L^4_h(L^2_v))} \|\Delta^v_q b_\lambda\|^{\frac12}_{L^\infty_t(L^2)},\end{aligned}$$

which along with Definition 1.2 and (3.4) gives

$$|\mathcal{H}_{1,q}(t)| \lesssim \frac{d_q^2}{\nu^{\frac34}} 2^{-q} \|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|b_\lambda\|^{\frac12}_{\widetilde{L^2_{t,f}}(\mathcal{B})} \|b_\lambda\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})}.$$

And again thanks to Lemma 2.1, we write

$$|\mathcal{H}_{2,q}(t)| \lesssim \sum_{q'\ge q-4} \|\Delta^v_{q'} \partial_3 a^3_\lambda\|_{L^2_t(L^2)} \|S^v_{q'+2} u^h_F\|_{L^2_t(L^\infty)} \|\Delta^v_q b\|_{L^\infty_t(L^2)},$$

which along with $\partial_3 a^3 = -\operatorname{div}_h a^h$ and (3.5) gives

$$\begin{aligned}|\mathcal{H}_{2,q}(t)| &\lesssim \frac{1}{\nu^{\frac12}} \Big(\sum_{q'\ge q-4} d_{q'} 2^{\frac{q-q'}{2}}\Big) d_q 2^{-q} \|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})} \|b\|_{\widetilde{L^\infty_t}(\mathcal{B})} \\ &\lesssim \frac{d_q^2}{\nu^{\frac12}} 2^{-q} \|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})} \|b\|_{\widetilde{L^\infty_t}(\mathcal{B})}.\end{aligned}$$

As a consequence, we obtain

$$\left|J^{h,2}_{q,\lambda}(t)\right| \lesssim d_q^2 2^{-q} \|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}} \Big(\frac{1}{\nu^{\frac34}} \|b_\lambda\|^{\frac12}_{\widetilde{L^2_{t,f}}(\mathcal{B})} \|b_\lambda\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})} + \frac{1}{\nu^{\frac12}} \|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})} \|b\|_{\widetilde{L^\infty_t}(\mathcal{B})}\Big).$$

This together with (3.26) and (3.27) proves (3.24).

To deal with $J_q^v(t)$, we first use $\operatorname{div} u_F = 0$ and integration by parts to get

$$\begin{aligned} J_q^v(t) = &-\int_0^t \big(\Delta_q^v(\operatorname{div}_h a^h u_F^3 - \nabla_h a^3 u_F^h) \mid \Delta_q^v b\big)_{L^2}\, dt' \\ &-\int_0^t \big(\Delta_q^v(a^h u_F^3 - a^3 u_F^h) \mid \nabla_h \Delta_q^v b\big)_{L^2}\, dt' \stackrel{\text{def}}{=} J_q^{v,1}(t) + J_q^{v,2}(t). \end{aligned} \tag{3.28}$$

Thanks to (2.3) and Corollary 3.1, we write

$$\begin{aligned} \big|J_q^{v,1}(t)\big| \leq\ & \|\Delta_q^v(\operatorname{div}_h a^h u_F^3 - \nabla_h a^3 u_F^h)\|_{L_t^{\frac43}(L_h^{\frac43}(L_v^2))} \|\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))} \\ \lesssim\ & d_q^2 2^{-q}\Big(\|u_F^3\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac12,\frac12})} \|\nabla_h u_F^3\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac12,\frac12})} \|\nabla_h a^h\|_{\widetilde{L_t^2}(\mathcal{B})} \\ &+\|u_F^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B}_4^{-\frac12,\frac12})} \|\nabla_h u_F^h\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B}_4^{-\frac12,\frac12})} \|\nabla_h a^3\|_{\widetilde{L_t^2}(\mathcal{B})}\Big) \|b\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B})}, \end{aligned}$$

which along with (3.17) gives

$$\begin{aligned} \big|J_q^{v,1}(t)\big| \lesssim\ & \frac{d_q^2}{\nu^{\frac14}} 2^{-q}\Big(\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|\nabla_h a^h\|_{\widetilde{L_t^2}(\mathcal{B})} + \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|\nabla_h a^3\|_{\widetilde{L_t^2}(\mathcal{B})}\Big) \\ & \times \|b\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B})}. \end{aligned} \tag{3.29}$$

Exactly following the same line, we obtain

$$\begin{aligned} \big|J_q^{v,2}(t)\big| \leq\ & \|\Delta_q^v(a^h u_F^3 - a^3 u_F^h)\|_{L_t^2(L^2)} \|\Delta_q^v \nabla_h b\|_{L_t^2(L^2)} \\ \lesssim\ & \frac{d_q^2}{\nu^{\frac14}} 2^{-q}\Big(\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h a^h\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B})} \\ &+\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a^3\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h a^3\|^{\frac12}_{\widetilde{L_t^2}(\mathcal{B})}\Big) \|\nabla_h b\|_{\widetilde{L_t^2}(\mathcal{B})}, \end{aligned}$$

which together with (3.28) and (3.29) shows (3.25). This completes the proof of the proposition. □

Proposition 3.3. *Let $0 \leq g(t) \leq 1$ and $b \in \mathcal{B}(t)$. We denote*

$$b_g(t,x) \stackrel{def}{=} g(t)b(t,x) \quad and \quad F_{q,g}(t) \stackrel{def}{=} \int_0^t \big(\Delta_q^v(u_F\cdot\nabla b_g) \mid \Delta_q^v b_g\big)_{L^2}\, dt'.$$

Then there holds for all $q \in \mathbb{Z}$,

$$\big|F_{q,g}(t)\big| \lesssim d_q^2 2^{-q} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \Big(\frac{1}{\nu^{\frac14}} \|b\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|^{\frac32}_{\widetilde{L_t^2}(\mathcal{B})} + \frac{1}{\nu^{\frac12}} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}\Big).$$

Proof. The proof of this proposition basically follows from the proof of Proposition 2.1. Indeed, we first write $F_{q,g}(t)$ as

$$\begin{aligned} F_{q,g}(t) &= \int_0^t \big(\Delta_q^v(u_F^h \cdot \nabla_h b_g) \mid \Delta_q^v b_g\big)_{L^2}\, dt' + \int_0^t \big(\Delta_q^v(u_F^3 \partial_3 b_g) \mid \Delta_q^v b_g\big)_{L^2}\, dt' \\ &\overset{\text{def}}{=} F_{q,g}^h(t) + F_{q,g}^v(t). \end{aligned} \tag{3.30}$$

Thanks to (2.3), (3.17) and Lemma 3.2, we have

$$\begin{aligned} \big|F_{q,g}^h(t)\big| &\lesssim \|\sqrt{g}\Delta_q^v(u_F^h \cdot \nabla_h b_g)\|_{L_t^{\frac43}(L_h^{\frac43}(L_v^2))} \|\sqrt{g}\Delta_q^v b\|_{L_t^4(L_h^4(L_v^2))} \\ &\lesssim \frac{d_q^2}{\nu^{\frac14}} 2^{-q} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}. \end{aligned} \tag{3.31}$$

Whereas similar to (2.5), we write

$$\begin{aligned} F_{q,g}^v(t) &= \int_0^t \big(S_{q-1}^v u_F^3 \partial_3 \Delta_q^v b_g | \Delta_q^v b_g\big)_{L^2}\, dt \\ &\quad + \sum_{|q'-q|\le 5} \int_0^t \big([\Delta_q^v; S_{q'-1}^v u_F^3]\partial_3 \Delta_{q'}^v b_g | \Delta_q^v b_g\big)_{L^2}\, dt' \\ &\quad + \sum_{|q'-q|\le 5} \int_0^t \big((S_{q'-1}^v u_F^3 - S_{q-1}^v u_F^3)\partial_3 \Delta_q^v \Delta_{q'}^v b_g | \Delta_q^v b_g\big)_{L^2}\, dt' \\ &\quad + \sum_{q'\ge q-4} \int_0^t \big(\Delta_q^v(\Delta_{q'}^v u_F^3 S_{q'+2}^v \partial_3 b_g) | \Delta_q^v b_g\big)_{L^2}\, dt' \\ &\overset{\text{def}}{=} F_{q,g}^{1,v}(t) + F_{q,g}^{2,v}(t) + F_{q,g}^{3,v}(t) + F_{q,g}^{4,v}(t). \end{aligned}$$

Then one gets by using $\operatorname{div} u_F = 0$ and integration by parts that

$$\begin{aligned} \big|F_{q,g}^{1,v}(t)\big| &= \Big|\int_0^t \int_{\mathbb{R}^3} S_{q-1}^v u_F^h \Delta_q^v b_g \Delta_q^v \nabla_h b_g\, dx\, dt'\Big| \\ &\lesssim \|S_{q-1}^v u_F^h\|_{L_t^2(L^\infty)} \|\Delta_q^v b\|_{L_t^\infty(L^2)} \|\Delta_q^v \nabla_h b_g\|_{L_t^2(L^2)} \\ &\lesssim \frac{d_q^2}{\sqrt{\nu}} 2^{-q} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|b\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h b_g\|_{\widetilde{L_t^2}(\mathcal{B})}, \end{aligned}$$

where we used (3.5) in the last step.

While thanks to (2.6) and using integration by parts, we write

$$\begin{aligned} F_{q,g}^{2,v}(t) = \sum_{|q-q'|\le 5} & 2^{q'} \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}} h(2^{q'}(x_3 - y_3)) \int_0^1 S_{q'-1}^v u_F^h(x_h, \tau y_3 + (1-\tau)x_3)\, d\tau \\ & \times (y_3 - x_3)\Big[\partial_3 \Delta_{q'}^v \nabla_h b_g(t', x_h, y_3)\, dy_3\ \Delta_q^v b_g(t', x) \\ & + \partial_3 \Delta_{q'}^v b_g(t', x_h, y_3)\, dy_3\ \Delta_q^v \nabla_h b_g(t', x)\Big]\, dx\, dt', \end{aligned}$$

from which and (3.5), we deduce that

$$\begin{aligned}\big|F^{2,v}_{q,g}(t)\big| &\lesssim \sum_{|q'-q|\le 5}\|S^v_{q'-1}u^h_F\|_{L^2_t(L^\infty)}\big(\|\Delta^v_{q'}\nabla_h b_g\|_{L^2_t(L^2)}\|\Delta^v_q b\|_{L^\infty_t(L^2)}\\&\quad+\|\Delta^v_{q'}b_g\|_{L^\infty_t(L^2)}\|\Delta^v_q\nabla_h b_g\|_{L^2_t(L^2)}\big)\\&\lesssim \frac{d_q^2}{\nu^{\frac12}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|b\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h b_g\|_{\widetilde{L^2_t}(\mathcal{B})}.\end{aligned}$$

On the other hand, let $\Delta^{v,3}_q$ be defined by (2.1). Then it is easy to observe that

$$\begin{aligned}F^{4,v}_{q,g}(t) &= \sum_{q'\ge q-4}2^{-q'}\int_0^t\big(\Delta^v_q(\Delta^{v,3}_{q'}\Delta^v_{q'};\,\partial_3 u^3_F S^v_{q'+2}\partial_3 b_g)|\Delta^v_q b_g\big)_{L^2}\,dt'\\&=\sum_{q'\ge q-4}2^{-q'}\Big\{\int_0^t\big(\Delta^v_q(\Delta^{v,3}_{q'}\Delta^v_{q'};\,u^h_F S^v_{q'+2}\partial_3\nabla_h b_g)|\Delta^v_q b_g\big)_{L^2}\,dt'\\&\quad+\int_0^t\big(\Delta^v_q(\Delta^{v,3}_{q'}\Delta^v_{q'};\,u^h_F S^v_{q'+2}\partial_3 b_g)|\Delta^v_q\nabla_h b_g\big)_{L^2}\,dt'\Big\},\end{aligned}$$

which along with Lemma 2.1 and (3.5) ensures that

$$\begin{aligned}\big|F^{4,v}_{q,g}(t)\big| &\lesssim \sum_{q'\ge q-4}\|\Delta^v_{q'}u^h_F\|_{L^2_t(L^\infty_h(L^2_v))}\big(\|S^v_{q'+2}\nabla_h b_g\|_{L^2_t(L^2_h(L^\infty_v))}\|\Delta^v_q b_g\|_{L^\infty_t(L^2)}\\&\quad+\|S^v_{q'+2}b_g\|_{L^\infty_t(L^2_h(L^\infty_v))}\|\Delta^v_q\nabla_h b_g\|_{L^2_t(L^2)}\big)\\&\lesssim \frac{1}{\nu^{\frac12}}\Big(\sum_{q'\ge q-4}d_{q'}2^{\frac{q-q'}{2}}\Big)d_q 2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|b\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h b_g\|_{\widetilde{L^2_t}(\mathcal{B})}\\&\lesssim \frac{d_q^2}{\nu^{\frac12}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|b\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h b_g\|_{\widetilde{L^2_t}(\mathcal{B})}.\end{aligned}$$

A similar argument shows the same estimate for $F^{3,v}_{q,g}(t)$. As a consequence, we obtain

$$\big|F^v_{q,g}(t)\big|\lesssim \frac{d_q^2}{\nu^{\frac12}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|b\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h b_g\|_{\widetilde{L^2_t}(\mathcal{B})}.$$

This along with (3.30) and (3.31) concludes the proof of the proposition. □

To deal with the pressure term, we need the following two propositions:

Proposition 3.4. *Let $a=(a^h,a^3)$ be in $\mathcal{B}(t)$ and $g_\lambda, a_\lambda, u_{F,\lambda}$ be given by (3.13). We denote*

$$\bar P_{q,\lambda}(t)\overset{def}{=}\sum_{\ell,k=1}^3\int_0^t\big(\nabla_h\Delta^v_q(-\Delta)^{-1}\partial_\ell\partial_k(u^\ell_F u^k_{F,\lambda})\mid\Delta^v_q a^h_\lambda\big)_{L^2}\,dt'.$$

Then there holds for all $q\in\mathbb{Z}$,

$$\big|\bar P_{q,\lambda}(t)\big|\lesssim d_q^2 2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\Big(\frac{1}{\sqrt{\nu}}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}+\frac{1}{\nu\lambda^{\frac14}}\|a^h_\lambda\|_{\widetilde{L^\infty_t}(\mathcal{B})}\Big).$$

Proof. Similar to the proof of Proposition 2.2, we again distinguish the terms with horizontal derivatives from the terms with the vertical one so that

$$\bar{P}_{q,\lambda}(t)=\bar{P}^h_{q,\lambda}(t)+\bar{P}^v_{q,\lambda}(t), \tag{3.32}$$

with

$$\bar{P}^h_{q,\lambda}(t)\stackrel{\text{def}}{=}\sum_{\ell,k=1}^{2}\int_0^t\big(\nabla_h\Delta_q^v(-\Delta)^{-1}\partial_\ell\partial_k(u_F^\ell u_{F,\lambda}^k)\mid\Delta_q^v a_\lambda^h\big)_{L^2}\,dt',\qquad \textit{and}$$

$$\bar{P}^v_{q,\lambda}(t)\stackrel{\text{def}}{=}\int_0^t\Big(\Delta_q^v(-\Delta)^{-1}[\partial_3^2(u_F^3u_{F,\lambda}^3)+2\sum_{k=1}^{2}\partial_3\partial_k(u_F^3u_{F,\lambda}^k)]\mid\Delta_q^v a_\lambda^h\Big)_{L^2}\,dt'.$$

Then the proof of (3.18) ensures

$$\big|\bar{P}^h_{q,\lambda}(t)\big|\lesssim\frac{d_q^2}{\sqrt{\nu}}2^{-q}\|u_{hh}^h\|^2_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h a_\lambda^h\|_{\widetilde{L}^2_t(\mathcal{B})}. \tag{3.33}$$

Whereas a similar proof of (2.11) gives

$$\bar{P}^v_{q,\lambda}(t)=2\sum_{k=1}^{2}\int_0^t\big(\Delta_q^v(u_F^k\partial_k u_{F,\lambda}^3)\mid\Delta_q^v(-\Delta)^{-1}\partial_3\,\mathrm{div}_h\,a_\lambda^h\big)_{L^2}\,dt'=\bar{P}^{v,1}_{q,\lambda}(t)+\bar{P}^{v,2}_{q,\lambda}(t),$$

where

$$\bar{P}^{v,1}_{q,\lambda}(t)\stackrel{\text{def}}{=}2\sum_{k=1}^{2}\sum_{q'\geq q-4}\int_0^t\big(\Delta_q^v(\Delta_{q'}^v u_F^k S_{q'+2}^v\partial_k u_{F,\lambda}^3)\mid\Delta_q^v(-\Delta)^{-1}\partial_3\,\mathrm{div}_h\,a_\lambda^h\big)_{L^2}\,dt',$$

$$\bar{P}^{v,2}_{q,\lambda}(t)\stackrel{\text{def}}{=}2\sum_{k=1}^{2}\sum_{|q'-q|\leq 5}\int_0^t\big(\Delta_q^v(S_{q'-1}^v u_F^k\Delta_{q'}^v\partial_k u_{F,\lambda}^3)\mid\Delta_q^v(-\Delta)^{-1}\partial_3\,\mathrm{div}_h\,a_\lambda^h\big)_{L^2}\,dt'.$$

Using Bony's decomposition (1.22) in the horizontal variables, we write

$$\begin{aligned}\|\Delta_{q'}^v u_F^k S_{q'+2}^v\partial_k u_{F,\lambda}^3\|_{L^1_t(L^2)}\lesssim\sum_{j\geq q'-4}\Big(&\|\Delta_j^h\Delta_{q'}^v u_F^h\|_{L^2_t(L^4_h(L^2_v))}\|S_{j-1}^h S_{q'+2}^v\nabla_h u_{F,\lambda}^3\|_{L^2_t(L^4_h(L^\infty_v))}\\&+\|S_{j+2}^h\Delta_{q'}^v u_F^h\|_{L^\infty_t(L^4_h(L^2_v))}\|\Delta_j^h S_{q'+2}^v\nabla_h u_{F,\lambda}^3\|_{L^1_t(L^4_h(L^\infty_v))}\Big).\end{aligned}$$

Notice that (3.6) implies that there holds for any $1\leq p<\infty$,

$$\begin{aligned}\|\Delta_j^h\Delta_\ell^v u_{F,\lambda}^3\|_{L^p_t(L^4_h(L^2_v))}&\lesssim\Big(\int_0^t e^{-c\nu t'2^{2j}}\,dt'\Big)^{\frac1p}d_{j,\ell}2^{\frac j2}2^{-\frac\ell2}\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}e^{-\lambda\|u_{hh}^3\|^4_{\mathcal{B}_4^{-\frac12,\frac12}}}\\&\lesssim\frac{d_{j,\ell}}{\nu^{\frac1p}\lambda^{\frac14}}2^{(\frac12-\frac2p)j}2^{-\frac\ell2}.\end{aligned} \tag{3.34}$$

Whence applying Lemma 2.1 gives

$$\|S_j^h S_{q'+2}^v\nabla_h u_{F,\lambda}^3\|_{L^2_t(L^4_h(L^\infty_v))}\lesssim\frac{c_j}{\nu^{\frac12}\lambda^{\frac14}}2^{\frac j2}\qquad\text{and}$$

$$\|\Delta_j^h S_{q'+2}^v\nabla_h u_{F,\lambda}^3\|_{L^1_t(L^4_h(L^\infty_v))}\lesssim\frac{c_j}{\nu\lambda^{\frac14}}2^{-\frac j2},$$

as a consequence, we obtain

$$\begin{aligned}\|\Delta_{q'}^v u_F^k S_{q'+2}^v \nabla_h u_{F,\lambda}^3\|_{L_t^1(L^2)} &\lesssim \frac{1}{\nu\lambda^{\frac14}}\Big(\sum_{j\geq q'-4} c_j d_{j,q'}\Big)2^{-\frac{q'}{2}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\\ &\lesssim \frac{d_{q'}}{\nu\lambda^{\frac14}}2^{-\frac{q'}{2}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}},\end{aligned}$$

from which, we deduce that

$$\begin{aligned}\big|\bar{P}_{q,\lambda}^{v,1}(t)\big| &\lesssim \Big(\sum_{q'\geq q-4}\|\Delta_{q'}^v u_F^k S_{q'+2}^v \partial_k u_{F,\lambda}^3\|_{L_t^1(L^2)}\Big)\|\Delta_q^v a_\lambda^h\|_{L_t^\infty(L^2)}\\ &\lesssim \frac{d_q^2}{\nu\lambda^{\frac14}}2^{-q}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}.\end{aligned} \tag{3.35}$$

Whereas again using Bony's decomposition (1.22) in the horizontal variables, we obtain

$$\begin{aligned}\|S_{q'-1}^v u_F^h \Delta_{q'}^v \nabla_h u_{F,\lambda}^3\|_{L_t^1(L^2)} \lesssim \sum_{j\geq q'-4}\Big(&\|\Delta_j^h S_{q'-1}^v u_F^h\|_{L_t^2(L_h^4(L_v^\infty))}\|S_{j-1}^h \Delta_{q'}^v \nabla_h u_{F,\lambda}^3\|_{L_t^2(L_h^4(L_v^2))}\\ &+\|S_{j+2}^h S_{q'-1}^v u_F^h\|_{L_t^\infty(L_h^4(L_v^\infty))}\|\Delta_j^h \Delta_{q'}^v \nabla_h u_{F,\lambda}^3\|_{L_t^1(L_h^4(L_v^2))}\Big),\end{aligned}$$

which along with (3.34) yields

$$\begin{aligned}\|S_{q'-1}^v u_F^h \Delta_{q'}^v \nabla_h u_{F,\lambda}^3\|_{L_t^1(L^2)} &\lesssim \frac{1}{\nu\lambda^{\frac14}}\Big(\sum_{j\geq q'-4} c_j d_{j,q'}\Big)2^{-\frac{q'}{2}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\\ &\lesssim \frac{d_{q'}}{\nu\lambda^{\frac14}}2^{-\frac{q'}{2}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}},\end{aligned}$$

so we obtain

$$\begin{aligned}\big|\bar{P}_{q,\lambda}^{v,2}(t)\big| &\lesssim \sum_{|q'-q|\leq 5}\|S_{q'-1}^v u_F^h \Delta_{q'}^v \nabla_h u_{F,\lambda}^3\|_{L_t^1(L^2)}\|\Delta_q^v a_\lambda^h\|_{L_t^\infty(L^2)}\\ &\lesssim \frac{d_q^2}{\nu\lambda^{\frac14}}2^{-q}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}.\end{aligned}$$

This along with (3.32), (3.33) and (3.35) concludes the proof of the proposition. □

Corollary 3.2. *Under the assumptions of Proposition 3.4, we have for all* $q\in\mathbb{Z}$,

$$\begin{aligned}&\Big|\sum_{\ell,k=1}^3\int_0^t\big(\nabla_h\Delta_q^v(-\Delta)^{-1}\partial_\ell\partial_k(u_F^\ell u_F^k)\mid\Delta_q^v a^h\big)_{L^2}\,dt'\Big|\\ &\quad\lesssim d_q^2 2^{-q}\Big(\frac{1}{\sqrt{\nu}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}^2\|\nabla_h a^h\|_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu}\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\Big).\end{aligned}$$

Proof. Indeed notice that

$$\|\Delta_{q'}^v u_F^h S_{q'+2}^v \nabla_h u_F^3\|_{L_t^1(L^2)} \lesssim \sum_{j\geq q'-4} \Big(\|\Delta_j^h \Delta_{q'}^v u_F^h\|_{L_t^2(L_h^4(L_v^2))} \|S_{j-1}^h S_{q'+2}^v \nabla_h u_F^3\|_{L_t^2(L_h^4(L_v^\infty))}$$
$$+\|S_{j+2}^h \Delta_{q'}^v u_F^h\|_{L_t^\infty(L_h^4(L_v^2))} \|\Delta_j^h S_{q'+2}^v \nabla_h u_F^3\|_{L_t^1(L_h^4(L_v^\infty))}\Big),$$

which along with (3.2) ensures that

$$\|\Delta_{q'}^v u_F^k S_{q'+2}^v \nabla_h u_F^3\|_{L_t^1(L^2)} \lesssim \frac{1}{\nu}\Big(\sum_{j\geq q'-4} c_j d_{j,q'}\Big) 2^{-\frac{q'}{2}} \|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}$$
$$\lesssim \frac{d_{q'}}{\nu} 2^{-\frac{q'}{2}} \|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}},$$

from which, we deduce that

$$\big|\bar{P}_{q,0}^{v,1}(t)\big| \lesssim \sum_{q'\geq q-4} \|\Delta_{q'}^v u_F^k S_{q'+2}^v \nabla_h u_F^3\|_{L_t^1(L^2)} \|\Delta_q^v a^h\|_{L_t^\infty(L^2)}$$
$$\lesssim \frac{d_q^2}{\nu} 2^{-q} \|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}.$$

Following the same line, we obtain the same estimate for $\bar{P}_{q,0}^{v,2}(t)$. This along with (3.33) completes the proof of the corollary. □

Proposition 3.5. *Let $a = (a^h, a^3) \in \mathcal{B}(t)$ with $\operatorname{div} a = 0$. Let u_F be given by (1.16). For any positive number λ and $0 \leq m \in L^\infty(0,t)$, we denote*

$$f(t) \stackrel{def}{=} \|u_F^3(t)\|_{L^\infty}^2 + \|\nabla_h a^3(t)\|_{\mathcal{B}}^2, \quad g_\lambda(t) \stackrel{def}{=} \exp\Big(-\lambda \int_0^t f(t')\,dt' - m(t)\Big),$$
$$a_\lambda(t,x) \stackrel{def}{=} g_\lambda(t) a(t,x), \qquad u_{F,\lambda}(t,x) \stackrel{def}{=} g_\lambda(t) u_F(t,x), \tag{3.36}$$

and

$$\widetilde{P}_{q,\lambda}(t) \stackrel{def}{=} \sum_{\ell,k=1}^{3} \int_0^t \big(\nabla_h \Delta_q^v (-\Delta)^{-1} \partial_\ell \partial_k (u_F^\ell a_\lambda^k) \mid \Delta_q^v a_\lambda^h\big)_{L^2}\,dt'.$$

Then we have for all $q \in \mathbb{Z}$,

$$\big|\widetilde{P}_{q,\lambda}(t)\big| \lesssim d_q^2 2^{-q}\Big(\frac{1}{\sqrt{\nu}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h a_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})} + \|a_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})} \|\nabla_h a_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}$$
$$+\frac{1}{(\nu\lambda)^{\frac14}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})}^{\frac12} \|\nabla_h a_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12} + \frac{1}{\nu^{\frac12}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})}\Big).$$

Proof. Again as in the proof of Proposition 2.2 and Proposition 3.4 , we distinguish the terms with horizontal derivatives from the terms with a vertical one so that

$$\widetilde{P}_{q,\lambda}(t) = \widetilde{P}_{q,\lambda}^h(t) + \widetilde{P}_{q,\lambda}^v(t), \tag{3.37}$$

with

$$\widetilde{P}^h_{q,\lambda}(t) \stackrel{\text{def}}{=} \sum_{\ell,k=1}^{2}\int_0^t \big(\nabla_h\Delta_q^v(-\Delta)^{-1}\partial_\ell\partial_k(u_F^\ell a_\lambda^k) \mid \Delta_q^v a_\lambda^h\big)_{L^2}\,dt', \qquad \textit{and}$$

$$\widetilde{P}^v_{q,\lambda}(t) \stackrel{\text{def}}{=} \int_0^t \big(\nabla_h\Delta_q^v(-\Delta)^{-1}[\partial_3^2(u_F^3 a_\lambda^3) + \sum_{k=1}^{2}\partial_3\partial_k(u^3_{F,\lambda}a^k + a^3_\lambda u^k_F)] \mid \Delta_q^v a_\lambda^h\big)_{L^2}\,dt'.$$

Applying Bony's decomposition (1.22) and (3.5) gives

$$\begin{aligned}\|\Delta_q^v(u_F^h a_\lambda^h)\|_{L^2_t(L^2)} &\lesssim \sum_{|q'-q|\le 5}\|S^v_{q'-1}u_F^h\|_{L^2_t(L^\infty)}\|\Delta^v_{q'}a^h_\lambda\|_{L^\infty_t(L^2)}\\ &\quad+\sum_{q'\ge q-4}\|\Delta^v_{q'}u^h_F\|_{L^2_t(L^\infty_h(L^2_v))}\|S^v_{q'+2}a^h_\lambda\|_{L^\infty_t(L^2_h(L^\infty_v))}\\ &\lesssim \frac{d_q}{\sqrt{\nu}}2^{-\frac{q}{2}}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})},\end{aligned}$$

from which, we deduce that

$$\begin{aligned}\big|\widetilde{P}^h_{q,\lambda}(t)\big| &\lesssim \|\Delta_q^v(u_F^h a_\lambda^h)\|_{L^2_t(L^2)}\|\Delta_q^v\nabla_h a^h_\lambda\|_{L^2_t(L^2)}\\ &\lesssim \frac{d_q^2}{\sqrt{\nu}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}.\end{aligned} \tag{3.38}$$

On the other hand, using the fact that $\operatorname{div} a = 0$ and $\operatorname{div} u_F = 0$, we write

$$\begin{aligned}\widetilde{P}^v_{q,\lambda}(t) &= \sum_{k=1}^{2}\int_0^t \big(\nabla_h\Delta_q^v(-\Delta)^{-1}\partial_3(\partial_k u^3_{F,\lambda}a^k + \partial_k a^3_\lambda u^k_F) \mid \Delta_q^v a^h_\lambda\big)_{L^2}\,dt'\\ &\stackrel{\text{def}}{=} \widetilde{P}^{v,1}_{q,\lambda}(t) + \widetilde{P}^{v,2}_{q,\lambda}(t),\end{aligned} \tag{3.39}$$

Using integration by parts, we obtain

$$\begin{aligned}\widetilde{P}^{v,1}_{q,\lambda}(t) &= -\int_0^t \big(\Delta_q^v(u^3_{F,\lambda}\operatorname{div}_h a^h) \mid \Delta_q^v(-\Delta)^{-1}\partial_3\operatorname{div}_h a^h_\lambda\big)_{L^2}\,dt'\\ &\quad-\sum_{k=1}^{2}\int_0^t \big(\Delta_q^v(u^3_{F,\lambda}a^k) \mid \Delta_q^v\partial_k(-\Delta)^{-1}\partial_3\operatorname{div}_h a^h_\lambda\big)_{L^2}\,dt'\\ &\stackrel{\text{def}}{=} \mathcal{W}^a_{q,\lambda}(t) + \mathcal{W}^b_{q,\lambda}(t).\end{aligned} \tag{3.40}$$

Applying Bony's decomposition (1.22) in the vertical variables gives

$$\begin{aligned}\mathcal{W}^a_{q,\lambda}(t) &= \sum_{|q'-q|\le 5}\int_0^t \big(\Delta_q^v(S^v_{q'-1}u^3_F\Delta^v_{q'}\operatorname{div}_h a^h_\lambda) \mid \Delta_q^v(-\Delta)^{-1}\partial_3\operatorname{div}_h a^h_\lambda\big)_{L^2}\,dt'\\ &\quad+\sum_{q'\ge q-4}\int_0^t \big(\Delta_q^v(\Delta^v_{q'}u^3_{F,\lambda}S^v_{q'+2}\operatorname{div}_h a^h) \mid \Delta_q^v(-\Delta)^{-1}\partial_3\operatorname{div}_h a^h_\lambda\big)_{L^2}\,dt'\\ &\stackrel{\text{def}}{=} \widetilde{\mathcal{W}}^1_{q,\lambda}(t) + \widetilde{\mathcal{W}}^2_{q,\lambda}(t).\end{aligned}$$

Thanks to Definition 1.3, one has

$$\big|\widetilde{\mathcal{W}}^1_{q,\lambda}(t)\big| \lesssim \sum_{|q'-q|\le 5}\Big(\int_0^t \|u_F^3\|_{L^\infty}^2\|\Delta_q^v a_\lambda\|_{L^2}^2\,dt'\Big)^{\frac12}\|\Delta_{q'}^v\nabla_h a_\lambda^h\|_{L^2_t(L^2)}$$
$$\lesssim d_q^2 2^{-q}\|a_\lambda^h\|_{\widetilde{L^2_{t,f}}(\mathcal{B})}\|\nabla_h a_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})}.$$

Whereas thanks to (2.1), $\operatorname{div} a = 0$ and $\operatorname{div} u_F = 0$, we write by using integration by parts

$$\widetilde{\mathcal{W}}^2_{q,\lambda}(t) = \sum_{q'\ge q-4} 2^{-q'}\Big(-\int_0^t \big(\Delta_q^v(\Delta_{q'}^{v,3}\Delta_{q'}^v u^h_{F,\lambda} S^v_{q'+2}\partial_3\nabla_h a^3) | \Delta_q^v(-\Delta)^{-1}\partial_3\operatorname{div}_h a_\lambda^h\big)_{L^2}\,dt'$$
$$+\int_0^t \big(\Delta_q^v(\Delta_{q'}^{v,3}\Delta_{q'}^v u^h_F S^v_{q'+2}\operatorname{div}_h a_\lambda^h) \mid \Delta_q^v(-\Delta)^{-1}\partial_3\operatorname{div}_h \nabla_h a_\lambda^h\big)_{L^2}\,dt'\Big),$$

from which and Lemma 2.1, we deduce that

$$\big|\widetilde{\mathcal{W}}^2_{q,\lambda}(t)\big| \lesssim \sum_{q'\ge q-4}\|\Delta_{q'}^v u_F^h\|_{L^2_t(L^\infty)}\Big\{\Big(\int_0^t \|\nabla_h a^3\|_{\mathcal{B}}^2\|\Delta_q^v a_\lambda^h\|_{L^2}^2\,dt'\Big)^{\frac12}$$
$$+\|S^v_{q'+2}\nabla_h a_\lambda^h\|_{L^2_t(L^2)}\|\Delta_q^v a_\lambda^h\|_{L^\infty_t(L^2)}\Big\}$$
$$\lesssim \frac{d_q^2}{\sqrt{\nu}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\Big(\|a_\lambda^h\|_{\widetilde{L^2_{t,f}}(\mathcal{B})} + \|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})}\Big).$$

To deal with $\mathcal{W}^b_{q,\lambda}(t)$, again we use Bony's decomposition (1.22) in the vertical variable to write

$$\mathcal{W}^b_{q,\lambda}(t) = \sum_{k=1}^2\Big(\sum_{|q'-q|\le 5}\int_0^t \big(\Delta_q^v(S^v_{q'-1}u_F^3\Delta_{q'}^v a_\lambda^k) \mid \Delta_q^v\partial_k(-\Delta)^{-1}\partial_3\operatorname{div}_h a_\lambda^h\big)_{L^2}\,dt'$$
$$+\sum_{q'\ge q-4}\int_0^t \big(\Delta_q^v(\Delta_{q'}^v u^3_{F,\lambda} S^v_{q'+2}a^k) \mid \Delta_q^v\partial_k(-\Delta)^{-1}\partial_3\operatorname{div}_h a_\lambda^h\big)_{L^2}\,dt'\Big)$$
$$\overset{\text{def}}{=} \widetilde{\mathcal{W}}^3_{q,\lambda}(t) + \widetilde{\mathcal{W}}^4_{q,\lambda}(t).$$

It is easy to observe that

$$\big|\widetilde{\mathcal{W}}^3_{q,\lambda}(t)\big| \lesssim \sum_{|q'-q|\le 5}\Big(\int_0^t \|u_F^3\|_{L^\infty}^2\|\Delta_{q'}^v a_\lambda^h\|_{L^2}^2\,dt'\Big)^{\frac12}\|\Delta_q^v\nabla_h a_\lambda^h\|_{L^2_t(L^2)}$$
$$\lesssim d_q^2 2^{-q}\|a_\lambda^h\|_{\widetilde{L^2_{t,f}}(\mathcal{B})}\|\nabla_h a_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})}.$$

Whereas again thanks to (2.1) and $\operatorname{div} u_F = 0$, we get by using integration by parts that

$$\widetilde{\mathcal{W}}^4_{q,\lambda}(t) = \sum_{k=1}^2\sum_{q'\ge q-4} 2^{-q'}\Big\{\int_0^t \big(\Delta_q^v(\Delta_{q'}^{v,3}\Delta_{q'}^v u^h_{F,\lambda} S^v_{q'+2}\nabla_h a^k) \mid \Delta_q^v\partial_k(-\Delta)^{-1}\partial_3\operatorname{div}_h a_\lambda^h\big)_{L^2}\,dt'$$
$$+\int_0^t \big(\Delta_q^v(\Delta_{q'}^{v,3}\Delta_{q'}^v u^h_{F,\lambda} S^v_{q'+2}a^k) \mid \Delta_q^v\partial_k(-\Delta)^{-1}\partial_3\operatorname{div}_h \nabla_h a_\lambda^h\big)_{L^2}\,dt'\Big\},$$

from which and Lemma 2.1, we deduce

$$\begin{aligned}\big|\widetilde{\mathcal{W}}^4_{q,\lambda}(t)\big| &\lesssim \sum_{q'\geq q-4}\|\Delta^v_{q'}u^h_F\|_{L^2_t(L^\infty_h(L^2_v))}\Big(\|S^v_{q'+2}\nabla_h a^h_\lambda\|_{L^2_t(L^2_h(L^\infty_v))}\|\Delta^v_q a^h\|_{L^\infty_t(L^2)}\\ &\quad+\|S^v_{q'+2}a^h\|_{L^\infty_t(L^2_h(L^\infty_v))}\|\Delta^v_q\nabla_h a^h_\lambda\|_{L^2_t(L^2)}\Big)\\ &\lesssim \frac{d_q^2}{\sqrt{\nu}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L^2_t}(\mathcal{B})}\|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\big|\widetilde{P}^{v,1}_{q,\lambda}(t)\big| &\lesssim d_q^2 2^{-q}\Big(\|a^h_\lambda\|_{\widetilde{L^2_{t,f}}(\mathcal{B})}\|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}\\ &\quad+\frac{1}{\sqrt{\nu}}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\big(\|a^h_\lambda\|_{\widetilde{L^2_{t,f}}(\mathcal{B})}+\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}\big)\Big). \qquad (3.41)\end{aligned}$$

It remains to estimate of $\widetilde{P}^{v,2}_{q,\lambda}(t)$. Indeed using Bony's decomposition (1.22) in the vertical variable, we write

$$\begin{aligned}\widetilde{P}^{v,2}_{q,\lambda}(t) &= \sum_{k=1}^2\sum_{q'\geq q-4}\int_0^t\big(\nabla_h\Delta^v_q(-\Delta)^{-1}\partial_3(S^v_{q'+2}\partial_k a^3\Delta^v_{q'}u^k_{F,\lambda})\mid\Delta^v_q a^h_\lambda\big)_{L^2}\,dt'\\ &\quad+\sum_{k=1}^2\sum_{|q'-q|\leq 5}\int_0^t\big(\nabla_h\Delta^v_q(-\Delta)^{-1}\partial_3(\Delta^v_{q'}\partial_k a^3_\lambda S^v_{q'-1}u^k_F)\mid\Delta^v_q a^h_\lambda\big)_{L^2}\,dt'\\ &\overset{\text{def}}{=}\mathcal{G}^1_{q,\lambda}(t)+\mathcal{G}^2_{q,\lambda}(t). \qquad (3.42)\end{aligned}$$

It is easy to observe that

$$\begin{aligned}\big|\mathcal{G}^1_{q,\lambda}(t)\big| &\lesssim \sum_{q'\geq q-4}\int_0^t\|S^v_{q'+2}\nabla_h a^3\|_{L^2_h(L^\infty_v)}\|\Delta^v_{q'}u^h_{F,\lambda}\|_{L^4_h(L^2_v)}\|\Delta^v_q a^h_\lambda\|_{L^4_h(L^2_v)}\,dt'\\ &\lesssim \sum_{q'\geq q-4}\int_0^t\|\nabla_h a^3\|_{\mathcal{B}}\|\Delta^v_{q'}u^h_{F,\lambda}\|_{L^4_h(L^2_v)}\|\Delta^v_q a^h_\lambda\|^{\frac12}_{L^2}\|\Delta^v_q\nabla_h a^h_\lambda\|^{\frac12}_{L^2}\,dt'\\ &\lesssim \sum_{q'\geq q-4}\Big(\int_0^t\|\nabla_h a^3\|_{\mathcal{B}}\|\Delta^v_{q'}u^h_{F,\lambda}\|^2_{L^4_h(L^2_v)}\,dt'\Big)^{\frac12}\\ &\quad\times\Big(\int_0^t\|\nabla_h a^3\|^2_{\mathcal{B}}\|\Delta^v_q a^h_\lambda\|^2_{L^2}\,dt'\Big)^{\frac14}\|\Delta^v_q\nabla_h a^h_\lambda\|^{\frac12}_{L^2_t(L^2)}.\end{aligned}$$

However, applying Bony's decomposition (1.22) in the horizontal variables, we obtain

$$\begin{aligned}\int_0^t\|\nabla_h a^3\|_{\mathcal{B}}\|\Delta^v_{q'}u^h_{F,\lambda}\|^2_{L^4_h(L^2_v)}\,dt' &= \int_0^t\|\nabla_h a^3\|_{\mathcal{B}}\|(\Delta^v_{q'}u^h_{F,\lambda})^2\|_{L^2_h(L^1_v)}\,dt'\\ &\lesssim \sum_{k\geq q'-4}\big(\|S^h_{k-1}\Delta^v_{q'}u^h_F\|_{L^\infty_t(L^4_h(L^2_v))}+\|S^h_{k+2}\Delta^v_{q'}u^h_F\|_{L^\infty_t(L^4_h(L^2_v))}\big)\\ &\quad\times\int_0^t\|\nabla_h a^3\|_{\mathcal{B}}\|\Delta^h_k\Delta^v_{q'}u^h_{F,\lambda}\|_{L^4_h(L^2_v)}\,dt',\end{aligned}$$

whereas thanks to (3.6) and (3.36), we have

$$\begin{aligned}
&\int_0^t \|\nabla_h a^3\|_{\mathcal{B}} \|\Delta_k^h \Delta_{q'}^v u_{F,\lambda}^h\|_{L_h^4(L_v^2)}\,dt' \\
&\lesssim d_{k,q'} 2^{\frac{k}{2}} 2^{-\frac{q'}{2}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \int_0^t \|\nabla_h a^3\|_{\mathcal{B}} e^{-\lambda \int_0^{t'} \|\nabla_h a^3\|_{\mathcal{B}}^2\,d\tau} e^{-c\nu t' 2^{2k}}\,dt' \\
&\lesssim \frac{d_{k,q'}}{\sqrt{\lambda\nu}} 2^{-\frac{k}{2}} 2^{-\frac{q'}{2}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}},
\end{aligned}$$

which gives

$$\begin{aligned}
\int_0^t \|\nabla_h a^3\|_{\mathcal{B}} \|\Delta_{q'}^v u_{F,\lambda}^h\|_{L_h^4(L_v^2)}^2\,dt' &\lesssim \frac{1}{\sqrt{\lambda\nu}} \Big(\sum_{k\geq q'-4} d_{k,q'}^2\Big) 2^{-q'} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}^2 \\
&\lesssim \frac{d_{q'}^2}{\sqrt{\lambda\nu}} 2^{-q'} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}^2.
\end{aligned}$$

Whence thanks to Definition 1.3, we infer

$$\big|\mathcal{G}_{q,\lambda}^1(t)\big| \lesssim \frac{d_q^2}{(\lambda\nu)^{\frac14}} 2^{-q} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})}^{\frac12} \|\nabla_h a_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12}. \tag{3.43}$$

To deal with $\mathcal{G}_{q,\lambda}^2(t)$, we write, using integration by parts and $\operatorname{div} u_F = 0$,

$$\begin{aligned}
\mathcal{G}_{q,\lambda}^2(t) &= \sum_{|q'-q|\leq 5} \Big\{ \int_0^t \big(\Delta_q^v \partial_3 (\Delta_{q'}^v a_\lambda^3 S_{q'-1}^v u_F^h) \mid \Delta_q^v (-\Delta)^{-1} \nabla_h \operatorname{div}_h a_\lambda^h\big)_{L^2}\,dt' \\
&\quad + \int_0^t \big(\Delta_q^v (\Delta_{q'}^v a_\lambda^3 S_{q'-1}^v \partial_3 u_F^3) \mid \Delta_q^v (-\Delta)^{-1} \nabla_h \partial_3 a_\lambda^h\big)_{L^2}\,dt' \Big\} \\
&\overset{\text{def}}{=} \widetilde{\mathcal{G}}_{q,\lambda}^a(t) + \widetilde{\mathcal{G}}_{q,\lambda}^b(t).
\end{aligned} \tag{3.44}$$

Applying Lemma 2.1, (3.5) and $\operatorname{div} a = 0$, we obtain

$$\begin{aligned}
\big|\widetilde{\mathcal{G}}_{q,\lambda}^a(t)\big| &\lesssim \sum_{|q'-q|\leq 5} \|\Delta_{q'}^v \partial_3 a_\lambda^3\|_{L_t^2(L^2)} \|S_{q'-1}^v u_F^h\|_{L_t^2(L^\infty)} \|\Delta_q^v a^h\|_{L_t^\infty(L^2)} \\
&\lesssim \frac{d_q^2}{\sqrt{\nu}} 2^{-q} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} \|\nabla_h a_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})},
\end{aligned}$$

and

$$\begin{aligned}
\big|\widetilde{\mathcal{G}}_{q,\lambda}^b(t)\big| &\lesssim \sum_{|q'-q|\leq 5} \Big(\int_0^t \|u_F^3\|_{L^\infty}^2 \|\Delta_q^v a_\lambda^h\|_{L^2}^2\,dt'\Big)^{\frac12} \|\Delta_{q'}^v \partial_3 a_\lambda^3\|_{L_t^2(L^2)} \\
&\lesssim d_q^2 2^{-q} \|a_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})} \|\nabla_h a_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})},
\end{aligned}$$

from which, we deduce that

$$\big|\mathcal{G}_{q,\lambda}^2(t)\big| \lesssim d_q^2 2^{-q} \Big(\frac{1}{\sqrt{\nu}} \|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}} \|a^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} + \|a_\lambda^h\|_{\widetilde{L_{t,f}^2}(\mathcal{B})}\Big) \|\nabla_h a_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}. \tag{3.45}$$

Thanks to (3.42), (3.43) and (3.45), we arrive at

$$\big|\widetilde{P}^{v,2}_{q,\lambda}(t)\big| \lesssim d_q^2 2^{-q}\Big\{\frac{1}{(\lambda\nu)^{\frac14}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h_\lambda\|^{\frac12}_{\widetilde{L^2_{t,f}}(\mathcal{B})}\|\nabla_h a^h_\lambda\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})} + \Big(\frac{1}{\sqrt{\nu}}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})} + \|a^h_\lambda\|_{\widetilde{L^2_{t,f}}(\mathcal{B})}\Big)\|\nabla_h a^h_\lambda\|_{\widetilde{L^2_t}(\mathcal{B})}\Big\}.$$

This along with (3.37), (3.38) and (3.41) completes the proof of the proposition. □

Corollary 3.3. *Under the assumptions of Proposition 3.5, we have for all $q\in\mathbb{Z}$,*

$$\begin{aligned}&\Big|\sum_{\ell,k=1}^{3}\int_0^t\big(\nabla_h\Delta^v_q(-\Delta)^{-1}\partial_\ell\partial_k(u^\ell_F a^k)\mid\Delta^v_q a^h\big)_{L^2}\,dt'\Big|\\ &\lesssim d_q^2 2^{-q}\Big\{\frac{1}{\sqrt{\nu}}\big(\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h a\|_{\widetilde{L^2_t}(\mathcal{B})} + \|u^3_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h a^h\|_{\widetilde{L^2_t}(\mathcal{B})}\big)\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\\ &\quad+\frac{1}{\nu^{\frac14}}\|u^3_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^h\|^{\frac32}_{\widetilde{L^2_t}(\mathcal{B})}\Big\}.\end{aligned}$$

Proof. The proof of this corollary essentially follows from the proof of Proposition 3.5. Firstly thanks to (3.40), we have

$$\begin{aligned}\big|\widetilde{P}^{v,1}_{q,0}(t)\big| &\lesssim \|\Delta^v_q(u^3_F\nabla_h a^h)\|_{L_t^{\frac43}(L_h^{\frac43}(L^2_v))}\|\Delta^v_q a^h\|_{L^4_t(L^4_h(L^2_v))}\\ &\quad+\|\Delta^v_q(u^3_F a^h)\|_{L^2_t(L^2)}\|\Delta^v_q\nabla_h a^h\|_{L^2_t(L^2)},\end{aligned}$$

which along with (2.3), Lemma 3.2 and Corollary 3.1 ensures that

$$\big|\widetilde{P}^{v,1}_{q,0}(t)\big| \lesssim \frac{d_q^2}{\nu^{\frac14}}2^{-q}\|u^3_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|^{\frac12}_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^h\|^{\frac32}_{\widetilde{L^2_t}(\mathcal{B})}. \tag{3.46}$$

Whereas thanks to (3.42), we obtain

$$\begin{aligned}\big|\mathcal{G}^1_{q,0}(t)\big| &\lesssim \sum_{q'\ge q-4}\|S^v_{q'+2}\nabla_h a^3\|_{L^2_t(L^2_h(L^\infty_v))}\|\Delta^v_{q'}u^h_F\|_{L^2_t(L^\infty_h(L^2_v))}\|\Delta^v_q a^h\|_{L^\infty_t(L^2)}\\ &\lesssim \frac{1}{\sqrt{\nu}}\Big(\sum_{q'\ge q-4}d_{q'}2^{\frac{q-q'}{2}}\Big)d_q 2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^3\|_{\widetilde{L^2_t}(\mathcal{B})}\\ &\lesssim \frac{d_q^2}{\sqrt{\nu}}2^{-q}\|u^h_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^3\|_{\widetilde{L^2_t}(\mathcal{B})},\end{aligned} \tag{3.47}$$

and it follows from (3.44) that

$$\begin{aligned}\big|\mathcal{G}^2_{q,0}(t)\big| &\lesssim \sum_{|q'-q|\le5}\|\Delta^v_{q'}\partial_3 a^3\|_{L^2_t(L^2)}\big(\|S^v_{q'-1}u^h_F\|_{L^2_t(L^\infty)}\\ &\quad+\|S^v_{q'-1}u^3_F\|_{L^2_t(L^\infty)}\big)\|\Delta^v_q a^h\|_{L^\infty_t(L^2)}\\ &\lesssim \frac{d_q^2}{\sqrt{\nu}}2^{-q}\|u_{hh}\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|a^h\|_{\widetilde{L^\infty_t}(\mathcal{B})}\|\nabla_h a^h\|_{\widetilde{L^2_t}(\mathcal{B})}.\end{aligned}$$

This together (3.38), (3.46) and (3.47) completes the proof of the corollary. □

With the above preparations, we are in a position to compete the proof of Theorem 1.3.

Proof of Theorem 1.3. Motivated by [11], we shall look for a solution of (ANS_ν) with the form (1.15). Then w satisfies (1.17). Again we shall use the classical Friedrichs' regularization method to construct the approximate solutions of (1.17). For simplicity, we just outline it here. In order to do so, let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be the projection operator given by (2.17), we define w_n via

$$(\widetilde{ANS}_{\nu,n})\begin{cases}\partial_t w_n-\nu\Delta_h w_n+\mathcal{P}_n(w_n\cdot\nabla w_n)+\mathcal{P}_n(w_n\cdot\nabla u_{F,n})+\mathcal{P}_n(u_{F,n}\cdot\nabla w_n)\\ =-\mathcal{P}_n(u_{F,n}\cdot\nabla u_{F,n})-\mathcal{P}_n\nabla(-\Delta)^{-1}\partial_\ell\partial_k\left((u_{F,n}^\ell+w_n^\ell)(u_{F,n}^k+w_n^k)\right)\\ \qquad\operatorname{div}w_n=0,\\ \qquad w_n|_{t=0}=\mathcal{P}_n(u_{\ell h})\stackrel{\text{def}}{=}\mathcal{P}_n(u_0-u_{hh}),\end{cases}$$

where $u_{F,n}\stackrel{\text{def}}{=}(Id-S^v_{j_n})u_F$ with $j_n=-\log_2^n$. Because of properties of the L^2 and L^1 functions, the Fourier transforms of which are supported in the ball $B(0,n)$, the system $(\widetilde{ANS}_{\nu,n})$ appears to be an ordinary differential equation in the space L^2_n defined by (2.19). This ordinary differential equation is globally wellposed (one may check [11] for the details).

Now let us turn to the uniform estimate of w_n. For a clear presentation, we shall neglect the subscript n. Toward this, as in the proof of Theorem 1.2, for arbitrary positive numbers λ_0, λ_1 and λ_2, which we shall choose later on, we denote

$$\begin{aligned}&f_1(t)\stackrel{\text{def}}{=}\|w^3(t)\|^2_{\mathcal{B}}\|\nabla_h w^3(t)\|^2_{\mathcal{B}},\quad f_2(t)\stackrel{\text{def}}{=}\|u_F^3(t)\|^2_{L^\infty}+\|\nabla_h w^3(t)\|^2_{\mathcal{B}},\\ &g_\lambda(t)\stackrel{\text{def}}{=}\exp\bigl(-\lambda_0\|u^3_{hh}\|^4_{\mathcal{B}_4^{-\frac12,\frac12}}-\int_0^t(\lambda_1f_1(t')+\lambda_2f_2(t'))\,dt'\bigr),\\ &w_\lambda(t,x)\stackrel{\text{def}}{=}g_\lambda(t)w(t,x)\quad\text{and}\quad u_{F,\lambda}(t,x)\stackrel{\text{def}}{=}g_\lambda(t)u_F(t,x).\end{aligned}\tag{3.48}$$

Then w_λ^h solves

$$\begin{cases}\partial_t w^h_\lambda+(\lambda_1f_1(t)+\lambda_2f_2(t))w^h_\lambda-\nu\Delta_hw^h_\lambda+\mathcal{P}_n(w\cdot\nabla w^h_\lambda)+\mathcal{P}_n(w\cdot\nabla u^h_{F,\lambda})\\ \quad+\mathcal{P}_n(u_F\cdot\nabla w^h_\lambda)=-\mathcal{P}_n(u_F\cdot\nabla u^h_{F,\lambda})-\mathcal{P}_n\nabla_h(-\Delta)^{-1}\partial_\ell\partial_k\left((u_F^\ell+w^\ell)(u^k_{F,\lambda}+w^k_\lambda)\right),\\ \qquad\operatorname{div}w_\lambda=0,\\ \qquad w_\lambda|_{t=0}=\mathcal{P}_n(u_{\ell h})\exp\bigl(-\lambda_0\|u^3_{hh}\|^4_{\mathcal{B}_4^{-\frac12,\frac12}}\bigr),\end{cases}$$

from which, we get by a standard energy estimate that

$$\begin{aligned}&\frac12\|\Delta^v_qw^h_\lambda(t)\|^2_{L^2}+\int_0^t(\lambda_1f_1(t')+\lambda_2f_2(t'))\|\Delta_q^vw^h_\lambda\|^2_{L^2}\,dt'+\nu\|\nabla_hw^h_\lambda\|^2_{L^2_t(L^2)}\\ &=\frac12\|\mathcal{P}_n(u_{\ell h})\|^2_{L^2}\exp\bigl(-2\lambda_0\|u^3_{hh}\|^4_{\mathcal{B}_4^{-\frac12,\frac12}}\bigr)\\ &\quad-\int_0^t\bigl(\Delta^v_q(w\cdot\nabla w^h_\lambda)\mid\Delta^v_qw^h_\lambda\bigr)_{L^2}\,dt'-\int_0^t\bigl(\Delta^v_q(w\cdot\nabla u^h_{F,\lambda})\mid\Delta^v_qw^h_\lambda\bigr)_{L^2}\,dt'\\ &\quad-\int_0^t\bigl(\Delta^v_q(u_F\cdot\nabla w^h_\lambda)\mid\Delta^v_qw^h_\lambda\bigr)_{L^2}\,dt'-\int_0^t\bigl(\Delta^v_q(u_F\cdot\nabla u^h_{F,\lambda})\mid\Delta^v_qw^h_\lambda\bigr)_{L^2}\,dt'\\ &\quad-\int_0^t\bigl(\Delta^v_q\nabla_h(-\Delta)^{-1}\partial_\ell\partial_k\left((u^\ell_F+w^\ell)(u^k_{F,\lambda}+w^k_\lambda)\right)\mid\Delta^v_qw^h_\lambda\bigr)_{L^2}\,dt'.\end{aligned}\tag{3.49}$$

Applying Proposition 2.1 for $a = w$ and $b = w^h$ gives

$$\Big|\int_0^t \big(\Delta_q^v(w\cdot\nabla w_\lambda^h) \mid \Delta_q^v w_\lambda^h\big)_{L^2}\,dt'\Big| \lesssim d_q^2 2^{-q}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^2.$$

Equation (3.24) applied with $a = w$ and $b = w^h$ yields

$$\Big|\int_0^t \big(\Delta_q^v(w\cdot\nabla u_{F,\lambda}^h) \mid \Delta_q^v w_\lambda^h\big)_{L^2}\,dt'\Big| \lesssim d_q^2 2^{-q}\Big(\frac{1}{\nu^{\frac14}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}$$
$$+\frac{1}{\nu^{\frac34}}\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|w_\lambda^h\|_{\widetilde{L_{t,f_1}^2}(\mathcal{B})}^{\frac12} + \frac{1}{\nu^{\frac12}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}\Big)\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}.$$

Proposition 3.3 applied with $b = w^h$ gives

$$\Big|\int_0^t \big(\Delta_q^v(u_F\cdot\nabla w_\lambda^h) \mid \Delta_q^v w_\lambda^h\big)_{L^2}\,dt'\Big| \lesssim d_q^2 2^{-q}\Big(\frac{1}{\nu^{\frac14}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}$$
$$+\frac{1}{\nu^{\frac12}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}\Big)\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}},$$

while (3.14) applied with $a = w^h$ gives

$$\Big|\int_0^t \big(\Delta_q^v(u_F\cdot\nabla u_{F,\lambda}^h) \mid \Delta_q^v w_\lambda^h\big)_{L^2}\,dt'\Big|$$
$$\lesssim d_q^2 2^{-q}\Big(\frac{1}{\nu^{\frac12}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}^2\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})} + \frac{1}{\lambda_0^{\frac14}\nu}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\Big).$$

Finally Proposition 2.2 applied with $u = w$, and Proposition 3.4, Proposition 3.5 applied with $a = w$ claims that

$$\Big|\int_0^t \Big(\Delta_q^v\nabla_h(-\Delta)^{-1}\partial_\ell\partial_k\Big((u_F^\ell + w^\ell)(u_{F,\lambda}^k + w_\lambda^k)\Big) \mid \Delta_q^v w_\lambda^h\Big)_{L^2}\,dt'\Big|$$
$$\lesssim d_q^2 2^{-q}\Big\{\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^2 + \|w_\lambda^h\|_{\widetilde{L_{t,f_1}^2}(\mathcal{B})}^{\frac12}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}$$
$$+\frac{1}{\nu^{\frac12}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}^2\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})} + \|w_\lambda^h\|_{\widetilde{L_{t,f_2}^2}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}$$
$$+\frac{1}{\nu^{\frac12}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})} + \frac{1}{\lambda_0^{\frac14}\nu}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}$$
$$+\frac{1}{(\lambda_2\nu)^{\frac14}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w_\lambda^h\|_{\widetilde{L_{t,f_2}^2}(\mathcal{B})}^{\frac12}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12} + \frac{1}{\nu^{\frac12}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w_\lambda^h\|_{\widetilde{L_{t,f_2}^2}(\mathcal{B})}\Big\}.$$

Plugging all the above estimates into (3.49), we arrive at

$$
\begin{aligned}
&\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}+\sqrt{2\lambda_1}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}\sqrt{2\lambda_2}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}+\sqrt{2\nu}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})}\\
&\leq \|u_{\ell h}^h\|_{\mathcal{B}}+C\Big\{\|w^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})}+\|w_\lambda^h\|^{\frac14}_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac34}_{\widetilde{L^2_t}(\mathcal{B})}\\
&\quad+\|w_\lambda^h\|^{\frac12}_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}+\|u_{hh}^h\|^{\frac12}_{\mathcal{B}_4^{-\frac12,\frac12}}\Big(\frac{1}{\nu^{\frac14}}\|w_\lambda^h\|^{\frac12}_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}\\
&\quad+\frac{1}{\nu^{\frac18}}\|w^h\|^{\frac14}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac34}_{\widetilde{L^2_t}(\mathcal{B})}+\frac{1}{\nu^{\frac38}}\|w_\lambda^h\|^{\frac14}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac14}_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}\\
&\quad+\frac{1}{\nu^{\frac14}}\|w^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}+\frac{1}{(\lambda_2\nu)^{\frac18}}\|w_\lambda^h\|^{\frac14}_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac14}_{\widetilde{L^2_t}(\mathcal{B})}\\
&\quad+\frac{1}{\lambda_0^{\frac18}\nu^{\frac12}}\|w_\lambda^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})}\Big)+\frac{1}{\nu^{\frac14}}\|u_{hh}^h\|^{\frac12}_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h w_\lambda^h\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}\Big\}. \qquad (3.50)
\end{aligned}
$$

For some small enough positive constant ε_1, which will be determined later on, we define

$$
T^*\stackrel{\text{def}}{=}\max\Big\{t:\ \|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}+\sqrt{\nu}\|\nabla_h w^h\|_{\widetilde{L^2_t}(\mathcal{B})}\leq \min\big(\varepsilon_1,\frac{1}{100C^2}\big)\nu\Big\}. \qquad (3.51)
$$

Applying (1.20) and Young's inequality, $a^{\frac1p}b^{\frac1q}\leq\frac ap+\frac bq$ for any positive numbers a,b,p,q satisfying $\frac1p+\frac1q=1$, gives

$$
\begin{aligned}
C\|w^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})}&\leq C\varepsilon_1^{\frac12}\sqrt{\nu}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})},\\
C\|w_\lambda^h\|^{\frac14}_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac34}_{\widetilde{L^2_t}(\mathcal{B})}&\leq C_1\nu^{-\frac32}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}+\frac{\sqrt{\nu}}{10}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})},\\
C\|w_\lambda^h\|^{\frac12}_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}&\leq C_1\nu^{-\frac12}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}+\frac{\sqrt{\nu}}{10}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})},\\
\frac{C}{\nu^{\frac14}}\|u_{hh}^h\|^{\frac12}_{\mathcal{B}_4^{-\frac12,\frac12}}\|\nabla_h w_\lambda^h\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}&\leq C_0c_1\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}+\frac{\sqrt{\nu}}{10}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})},
\end{aligned}
$$

and

$$
\begin{aligned}
&C\|u_{hh}^h\|^{\frac12}_{\mathcal{B}_4^{-\frac12,\frac12}}\Big(\frac{1}{\nu^{\frac18}}\|w^h\|^{\frac14}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac34}_{\widetilde{L^2_t}(\mathcal{B})}+\frac{1}{\nu^{\frac14}}\|w_\lambda^h\|^{\frac12}_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}\\
&\quad+\frac{1}{\nu^{\frac38}}\|w_\lambda^h\|^{\frac14}_{\widetilde{L_t^\infty}(\mathcal{B})}\|w_\lambda^h\|^{\frac14}_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}+\frac{1}{\nu^{\frac14}}\|w^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac12}_{\widetilde{L^2_t}(\mathcal{B})}\\
&\quad+\frac{1}{(\lambda_2\nu)^{\frac18}}\|w_\lambda^h\|^{\frac14}_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}\|\nabla_h w_\lambda^h\|^{\frac14}_{\widetilde{L^2_t}(\mathcal{B})}+\frac{1}{\lambda_0^{\frac18}\nu^{\frac12}}\|w_\lambda^h\|^{\frac12}_{\widetilde{L_t^\infty}(\mathcal{B})}\Big)\\
&\leq C_1(1+\varepsilon_1+c_1\varepsilon_1+(\lambda_0^{\frac14}\nu)^{-1})\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}+\frac{1}{10}\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}+\nu^{-\frac32}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}\\
&\quad+\nu^{-\frac12}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_2}}(\mathcal{B})}(1+(\lambda_2\nu)^{-\frac12})+\frac{\sqrt{\nu}}{10}\|\nabla_h w_\lambda^h\|_{\widetilde{L^2_t}(\mathcal{B})},
\end{aligned}
$$

for $t < T^*$. Without loss of generality, we may assume that $c_1, \varepsilon_1 \leq \frac{1}{2}$. Then it follows from (1.19) and (3.50) that

$$\begin{aligned}
&\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} + \sqrt{2\lambda_1}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_1}}(\mathcal{B})}\sqrt{2\lambda_2}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_2}}(\mathcal{B})} + \sqrt{2\nu}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})} \\
&\quad \leq C_1(3+(\lambda_0^{\frac{1}{4}}\nu)^{-1})\|u_0^h\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} + \frac{1}{10}\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} + 2C_1\nu^{-\frac{3}{2}}\|w_\lambda^h\|_{\widetilde{L^2_{t,f_1}}(\mathcal{B})} \\
&\quad\quad + C_1\nu^{-\frac{1}{2}}(2+(\lambda_2\nu)^{-\frac{1}{2}})\|w_\lambda^h\|_{\widetilde{L^2_{t,f_2}}(\mathcal{B})} + \frac{3\sqrt{\nu}}{5}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})},
\end{aligned}$$

for $t < T^*$. Taking $\lambda_0 = \nu^{-4}$, $\lambda_1 = 4C_1^2\nu^{-3}$ and $\lambda_2 = 9C_1^2\nu^{-1}$ in the above inequality, we infer that

$$\|w_\lambda^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h w_\lambda^h\|_{\widetilde{L_t^2}(\mathcal{B})} \leq 5C_1\|u_0^h\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \quad \text{for} \quad t < T^*,$$

which along with a similar derivation of (2.26) ensures that

$$\begin{aligned}
&\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})} + \sqrt{\nu}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})} \leq 5C_1\|u_0^h\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \exp\Big\{9C_1^2\big(\nu^{-4}\|u_{hh}^3\|^4_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} \\
&\quad + \nu^{-2}\|u_{hh}^3\|^2_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}} + \int_0^t (\nu^{-3}\|w^3(t')\|^2_{\mathcal{B}} + \nu^{-1})\|\nabla_h w^3(t')\|^2_{\mathcal{B}}\,dt'\big)\Big\}, \qquad (3.52)
\end{aligned}$$

where we used (3.5) so that $\int_0^t \|u_F^3(t')\|^2_{L^\infty}\,dt' \lesssim \nu^{-1}\|u_{hh}^3\|^2_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}$.

Now let us turn to the uniform estimate of w^3. Indeed we get by first applying Δ_q^v to the w^3 equation of $(\widetilde{ANS}_{\nu,n})$ and then taking the L^2 inner product of the resulting equation with $\Delta_q^v w^3$ that

$$\begin{aligned}
&\frac{1}{2}\|w^3(t)\|^2_{L^2} + \nu\|\nabla_h w^3\|^2_{L_t^2(L^2)} + \int_0^t \big(\Delta_q^v(w\cdot\nabla w^3) \mid \Delta_q^v w^3\big)_{L^2}\,dt' \\
&\quad + \int_0^t \big(\Delta_q^v(w\cdot\nabla u_F^3) \mid \Delta_q^v w^3\big)_{L^2}\,dt' + \int_0^t \big(\Delta_q^v(u_F\cdot\nabla w^3) \mid \Delta_q^v w^3\big)_{L^2}\,dt' \\
&\quad = \frac{1}{2}\|\Delta_q^v\mathcal{P}_n(u_{\ell h}^3)\|^2_{L^2} - \int_0^t \big(\Delta_q^v(u_F\cdot\nabla u_F^3) \mid \Delta_q^v w^3\big)_{L^2}\,dt' \\
&\quad\quad + \int_0^t \big(\nabla_h\partial_\ell\partial_k(-\Delta)^{-1}\Delta_q^v\big((u_F^\ell + w^\ell)(u_F^k + w^k)\big) \mid \Delta_q^v w^h\big)_{L^2}\,dt', \qquad (3.53)
\end{aligned}$$

where in the last step, we used div $w = 0$ and integration by parts.

Applying Proposition 2.1 for $a = w$, $b = w^3$ and $g = 1$ gives

$$\begin{aligned}
&\Big|\int_0^t \big(\Delta_q^v(w\cdot\nabla w^3) \mid \Delta_q^v w^3\big)_{L^2}\,dt'\Big| \\
&\quad \lesssim d_q^2 2^{-q}\Big(\|w^h\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^h\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}\|w^3\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^3\|^{\frac{3}{2}}_{\widetilde{L_t^2}(\mathcal{B})} \\
&\quad\quad + \|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}\Big).
\end{aligned}$$

Applying (3.25) for $a = w$ and $b = w^3$ implies

$$\Big|\int_0^t \big(\Delta_q^v(w\cdot\nabla u_F^3)\mid\Delta_q^v w^3\big)_{L^2}\,dt'\Big| \lesssim \frac{d_q^2}{\nu^{\frac14}}2^{-q}\Big\{\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}$$
$$+\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\big(\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12}$$
$$+\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}\big)\Big\},$$

whereas applying Proposition 3.3 for $b = w^3$ and $g = 1$ ensures

$$\Big|\int_0^t \big(\Delta_q^v(u_F\cdot\nabla w^3)\mid\Delta_q^v w^3\big)_{L^2}\,dt'\Big| \lesssim d_q^2 2^{-q}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\Big(\frac{1}{\nu^{\frac14}}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}$$
$$+\frac{1}{\nu^{\frac12}}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}\Big).$$

Applying (3.15) for $a = w^3$ gives

$$\Big|\int_0^t \big(\Delta_q^v(u_F\cdot\nabla u_F^3)\mid\Delta_q^v w^3\big)_{L^2}\,dt'\Big|$$
$$\lesssim d_q^2 2^{-q}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\Big(\frac{1}{\nu^{\frac12}}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}\Big).$$

Finally applying Corollary 2.2 for $u = w$, Corollary 3.2 and Corollary 3.3 for $a = w$ claims that

$$\Big|\int_0^t \big(\nabla_h\partial_\ell\partial_k(-\Delta)^{-1}\Delta_q^v\big((u_F^\ell+w^\ell)(u_F^k+w^k)\big)\mid\Delta_q^v w^h\big)_{L^2}\,dt'\Big|$$
$$\lesssim d_q^2 2^{-q}\Big\{\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^2+\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12}$$
$$+\frac{1}{\nu^{\frac12}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}^2\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}$$
$$+\frac{1}{\nu^{\frac12}}\|u_{hh}^h\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w\|_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu^{\frac14}}\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac32}$$
$$+\frac{1}{\nu^{\frac12}}\|u_{hh}^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}\Big\}.$$

Substituting all the above estimates into (3.53) and using (1.19), we obtain

$$\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}+\sqrt{2\nu}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}$$
$$\le C_2\|u_0^3\|_{\mathcal{B}_4^{-\frac12,\frac12}}+C_2\Big\{\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac14}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac14}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac14}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac34}$$
$$+\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac12}+\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac12}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}$$
$$+\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac14}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac34}\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}^{\frac14}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}^{\frac14}$$

$$
\begin{aligned}
&+\|u_{hh}^h\|^{\frac{1}{2}}_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\Big(\frac{1}{\nu^{\frac{1}{8}}}\|w^3\|^{\frac{1}{4}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^3\|^{\frac{3}{4}}_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu^{\frac{1}{4}}}\|w^3\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^3\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}\\
&\quad+\frac{1}{\nu^{\frac{1}{4}}}\|w^h\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})}(\|\nabla_h w^h\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}+\|\nabla_h w^3\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})})\Big)\\
&+\|u_{hh}^3\|^{\frac{1}{2}}_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\Big(\frac{1}{\nu^{\frac{1}{8}}}\|\nabla_h w^h\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}\|w^3\|^{\frac{1}{4}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^3\|^{\frac{1}{4}}_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu^{\frac{1}{8}}}\|w^h\|^{\frac{1}{4}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^h\|^{\frac{3}{4}}_{\widetilde{L_t^2}(\mathcal{B})}\\
&\quad+\frac{1}{\nu^{\frac{1}{8}}}\|w^h\|^{\frac{1}{4}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^h\|^{\frac{1}{4}}_{\widetilde{L_t^2}(\mathcal{B})}\|\nabla_h w^3\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu^{\frac{1}{4}}}\|w^h\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})}\|\nabla_h w^h\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}\Big)\\
&+\|u_{hh}^h\|^{\frac{1}{2}}_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\|u_{hh}^3\|^{\frac{1}{2}}_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\Big(\frac{1}{\nu^{\frac{1}{4}}}\|\nabla_h w^3\|^{\frac{1}{2}}_{\widetilde{L_t^2}(\mathcal{B})}+\frac{1}{\nu^{\frac{1}{2}}}\|w^h\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})}+\frac{1}{\nu^{\frac{1}{2}}}\|w^3\|^{\frac{1}{2}}_{\widetilde{L_t^\infty}(\mathcal{B})}\Big)\Big\}.
\end{aligned}
$$

Thanks to (1.20) and (3.51), we infer that for $t < T^*$,

$$
\begin{aligned}
&\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}+\sqrt{2\nu}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}\\
&\quad\le C_2\Big\{(1+\varepsilon_1+\sqrt{c_1\varepsilon_1}+\sqrt{c_1})\|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}+(\varepsilon_1+3\sqrt{c_1}+3\sqrt{\varepsilon_1})\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}\\
&\qquad+(\varepsilon_1+\sqrt{c_1\varepsilon_1}+\sqrt{c_1}\varepsilon_1)\nu+(\varepsilon_1+4\sqrt{\varepsilon_1}+2\sqrt{c_1}+\sqrt{c_1\varepsilon_1})\sqrt{\nu}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}\Big\}\quad\text{holds.}
\end{aligned}
\tag{3.54}
$$

Choosing c_1 in (1.20) and ε_1 in (3.51) small enough so that

$$
\begin{aligned}
&\varepsilon_1+\sqrt{c_1\varepsilon_1}+\sqrt{c_1}\le 1,\qquad \varepsilon_1+3\sqrt{c_1}+3\sqrt{\varepsilon_1}\le\frac{1}{2C_2}\quad\text{and}\\
&\varepsilon_1+\sqrt{c_1\varepsilon_1}+\sqrt{c_1}\varepsilon_1\le\frac{1}{2C_2},\qquad \varepsilon_1+4\sqrt{\varepsilon_1}+2\sqrt{c_1}+\sqrt{c_1\varepsilon_1}\le\frac{1}{2C_2},
\end{aligned}
$$

then we deduce from (3.54) that

$$
\|w^3\|_{\widetilde{L_t^\infty}(\mathcal{B})}+\sqrt{\nu}\|\nabla_h w^3\|_{\widetilde{L_t^2}(\mathcal{B})}\le 4C_2\|u_0^3\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}+\nu,\tag{3.55}
$$

for $t < T^*$. Thanks to (3.52), (3.55), and a similar proof of (2.28), there exist positive constants K, M which depend on C_1, C_2 such that

$$
\begin{aligned}
\|w^h\|_{\widetilde{L_t^\infty}(\mathcal{B})}+\sqrt{\nu}\|\nabla_h w^h\|_{\widetilde{L_t^2}(\mathcal{B})}&\le K\|u_0^h\|_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\exp\Big(M\nu^{-4}\|u_0^3\|^4_{\mathcal{B}_4^{-\frac{1}{2},\frac{1}{2}}}\Big)\\
&\le\frac{1}{2}\min\left(\varepsilon_1,\frac{1}{100C^2}\right)\nu\quad\text{for}\quad t<T^*,
\end{aligned}
\tag{3.56}
$$

provided that we take $c_1=\frac{1}{2K}\min\big(\varepsilon_1,\frac{1}{100C^2}\big)$ in (1.20). Equation (3.56) contradicts (3.52) if $T^*<\infty$. This shows that the solution sequence defined by $(\widetilde{ANS}_{\nu,n})$ satisfies (3.55) and (3.56) for $t=\infty$. With (3.55) and (3.56), one can follow the compactness argument in [11] to prove the global existence of solutions to $(\widetilde{ANS}_\nu)$ in the function space $\mathcal{B}(\infty)$. Moreover, the (1.21) holds. And the uniqueness part has been proved in [11]. This completes the proof of Theorem 1.3. □

Acknowledgements. The authors would like to thank the anonymous referee for many profitable suggestions. P. Zhang is partially supported by NSF of China under Grant 10421101 and 10931007, and one-hundred talents' plan from the Chinese Academy of Sciences under Grant GJHZ200829. M. Paicu gratefully acknowledges the hospitality of Morningside Center of Mathematics, CAS, from Beijing.

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Communicated by P. Constantin