Commun. Math. Phys. 309, 371–412 (2012)
Digital Object Identifier (DOI) 10.1007/s00220-011-1334-6

Communications in
Mathematical Physics

Lagrange Structure and Dynamics for Solutions to the Spherically Symmetric Compressible Navier-Stokes Equations

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Received: 15 December 2010 / Accepted: 3 April 2011
Published online: 1 October 2011 –

Abstract: The compressible Navier-Stokes system (CNS) with density-dependent viscosity coefficients is considered in multi-dimension, the prototype of the system is the viscous Saint-Venat model for the motion of shallow water. A spherically symmetric weak solution to the free boundary value problem for CNS with stress free boundary condition and arbitrarily large data is shown to exist globally in time with the free boundary separating fluids and vacuum and propagating at finite speed as particle path, which is continuous away from the symmetry center. Detailed regularity and Lagrangian structure of this solution have been obtained. In particular, it is shown that the particle path is uniquely defined starting from any non-vacuum region away from the symmetry center, along which vacuum states shall not form in any finite time and the initial regularities of the solution is preserved. Starting from any non-vacuum point at a later-on time, a particle path is also uniquely defined backward in time, which either reaches at some initial non-vacuum point, or stops at a small middle time and connects continuously with vacuum. In addition, the free boundary is shown to expand outward at an algebraic rate in time, and the fluid density decays and tends to zero almost everywhere away from the symmetry center as the time grows up. This finally leads to the formation of vacuum state almost everywhere as the time goes to infinity.

1. Introduction

The compressible isentropic Navier-Stokes equations (CNS) with density-dependent viscosity coefficients in $\mathbb{R}^N$, $N = 2, 3$, can be written for $t > 0$ as

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{U}) = 0, \\ (\rho \mathbf{U})_t + \mathrm{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \mathrm{div}(2\mu(\rho)\mathbb{D}(\mathbf{U})) - \nabla(\lambda(\rho)\mathrm{div}\mathbf{U}) + \nabla P(\rho) = 0, \end{cases} \tag{1.1}$$

where $\rho(\mathbf{x}, t)$, $\mathbf{U}(\mathbf{x}, t)$ and $P(\rho) = \rho^\gamma (\gamma > 1)$ stand for the fluid density, velocity and pressure, respectively, $\mathbb{D}(\mathbf{U}) = \frac{1}{2}(\nabla \mathbf{U} + \nabla \mathbf{U}^T)$ is the stress tensor, and $\mu(\rho)$ and $\lambda(\rho)$ are the Lamé viscosity coefficients satisfying $\mu(\rho) \geq 0$ and $\mu(\rho) + N\lambda(\rho) \geq 0$ for $\rho \geq 0$.

There is huge literature on the studies about global existence and behaviors of solutions to (1.1) in the case that the viscosity coefficients μ and ξ are both constants. The important progress on the global existence of strong or weak solutions in spatial one-dimension (1D) or multi-dimension (multi-D) has been made by many authors, refer to [7,9,13,27,28,31,35] and references therein. However, the regularity, uniqueness and dynamical behavior of the weak solutions for arbitrary initial data remain largely open for the compressible Navier-Stokes equations with constant viscosity coefficients, but there is new progress recently [19,21]. As emphasized in many related papers (refer to [6,15,16,18–20,32,42,53] for instance), the possible appearance of vacuum is one of the main difficulties, which indeed leads to the singular behaviors of solutions in the presence of vacuum, such as the failure of continuous dependence of weak solutions on initial data [15] and the finite time blow-up of smooth solutions [19,53]. To overcome the above singularities of solutions near the vacuum state, Liu-Xin-Yang [32] investigate the compressible Navier-Stokes equations with density-dependent viscosities, derived from the fluid-dynamical approximation to the Boltzmann equation and the isentropic reduction of temperature, and show the well-posedness of the local weak solution even in the appearance of vacuum. Moreover, it should be emphasized that the viscous Saint-Venant system in the description of the motion for shallow water was also derived recently [10,34], which is expressed exactly as (1.1) with $N = 2$, $\mu(\rho) = \rho$, $\lambda(\rho) = 0$ and $\gamma = 2$. Such viscous compressible models with density-dependent viscosity coefficients and its variants appear in geophysical flows [31,40].

The global existence of weak solution with large aptitude to (1.1) remains to be carried in terms of the Lions' compactness framework of renormalized solutions [31] due to the new mathematical challenges encountered below. Indeed, the system (1.1) is highly degenerate at vacuum because of the dependence of viscosity coefficients on the flow density. This makes it very difficult to obtain the uniformly a-priori estimates for the velocity and trace the motion of particle paths near vacuum regions, which is the essential difference of (1.1) from compressible Navier-Stokes equations with constant viscosity coefficients where the particle path can be defined a-priorily even near vacuum [14,15]. In particular, unlike the case for CNS with constant viscosity coefficients investigated in [16,54], it is not known yet whether the vacuum states shall form or not for global (weak) solutions to (1.1) even if initial data is far from vacuum. The significant progress on global existence of weak solutions has been established recently to (1.1) for the case with either a drag friction or a cold pressure term and for the case with spherical symmetry, refer to [1–3,11] and references therein. In general, however, it still seems to be a challenge to show the global existence of weak solutions for general multi-dimensional data.

It is a natural and interesting problem to investigate the influence of the vacuum state on the existence and dynamics of global solutions to (1.1). One of the prototype problems is the time-evolution of the compressible viscous flow of finite mass expanding into infinite vacuum. This corresponds to free boundary value problems (FBVP) for the compressible Navier-Stokes equations (1.1) for general initial data and variant boundary conditions imposed on the free surface. The study is a fundamental issue of fluid mechanics and has attracted lots of research interests [38,45]. These free boundary problems have been studied with rather abundant results concerned with the existence and dynamics of global solution for CNS (1.1) in 1D, refer to [8,26,32,33,37,41,51,55] and references therein. Some important progress has been made about free boundary value problems for multi-dimensional compressible viscous Navier-Stokes equations with constant viscosity coefficients for either barotropic or heat-conducive fluids by many authors, refer to [4,44,46,47,49,56–61] and references therein. In particular, in the case that across the free surface stress tensor is balanced by a constant exterior pressure and/or the surface tension, classical solutions with strictly positive densities in the fluid regions to FBVP for CNS (1.1) with constant viscosity coefficients is shown locally in time for either heat-conductive flows [44,49,57] or barotropic flows [46,61,59]. In the case that across the free surface the stress tensor is balanced by surface tension [47], exterior pressure [59], or both surface tension and exterior pressure [60] respectively, global existence of classical solutions with small amplitude and positive densities in fluid region to the FBVP for CNS (1.1) with constant viscosity coefficients is established, where initial data is assumed to be near to non-vacuum equilibrium state. Global existence of classical solutions to FBVP for compressible viscous and heat-conductive fluids are also obtained with the stress tensor balanced by the surface tension and/or exterior pressure across the free surface, refer to [56,58] and references therein. There are also very interesting investigations about free boundary value problems for the compressible Navier-Stokes equations with the self-gravitation force taken for granted, refer to [22,39,43,48,62,63] and references therein.

It is not known in general about the existence and dynamics of global solutions to the free boundary value problem for (1.1) in multi-dimension with stress free boundary condition imposed on the free surface. This problem is rather interesting and hard to investigate. Indeed, it should be noted that the previous studies on FBVP for (1.1) in [47,59,60,63], subject to the boundary condition that stress tensor is balanced by an exterior pressure and/or surface tension, depend crucially on the facts that in fluid region the density is strictly positive everywhere and the momentum equation is uniformly parabolic. In our case, however, the fluid region expands outward at an algebraic rate in time due to the stress free boundary condition (already observed in 1D for (1.1) in [30,33,37]), which implies the decay of fluid density to zero almost everywhere time-asymptotically and the loss of uniform parabolicity of momentum equation. These make the analysis rather delicate and difficult.

It is also of great interest to study the dynamical behaviors and the Lagrangian properties (such as existence and uniqueness of particle paths, transportation of initial regularities, non-formation of vacuum, or finite time vanishing of vacuum, etc.) for global weak solutions to FBVP for CNS (1.1). It is noted that for CNS (1.1) with constant viscosity coefficients, Hoff-Smoller [16] prove that vacuum states shall not form for global weak solutions in 1D so long as there is no vacuum state initially, which is also generalized to multi-D spherically symmetric case [54], Hoff-Santos make important progress on the analysis of Lagrange structure and the propagation of jump discontinuities in (2D and 3D) whole space, Hoff-Tsyganov successfully show the time analyticity and backward uniqueness of global weak solutions with small (relative) energy in terms of Lagrangian formulation (fluid particle trajectory) [17]. Li-Li-Xin [29] discover and prove rigorously the phenomena of the finite time vanishing of vacuum states to (1.1) in 1D where a global entropy weak solution gains regularities to become a strong one after the vanishing of vacuum. There is also interesting recent progress on local well-posedness of free boundary value problems for the compressible Euler equations based on Lagrangian formulation, see [23,24] and references therein. It should be noted, however, that the dynamical behaviors and the Lagrangian properties are also unknown for CNS (1.1) in multi-dimension even if the global existence of weak solution is already shown for special cases as mentioned above.

In the present paper, we study the free boundary value problem for the compressible multi-dimensional Navier-Stokes equations (1.1) with stress free across the free surface and focus on the existence and dynamical behaviors of global solutions. For simplicity, we deal with the case $\mu(\rho) = \rho$ and $\lambda(\rho) = 0$ (namely, the FBVP (2.1)-(2.7) in Sect. 2 below). First, we show that a global spherically symmetric entropy weak solution to the FBVP (2.1)-(2.7) exists for general initial data with finite entropy, and the free surface moves as particle paths in radial direction. Away from the symmetry center the density is continuous in both space and time (up to the free surface), and the total mass is conserved for all time (refer to Theorem 2.1 in Sect. 2 for details).

Then, we investigate the Lagrangian structure of global weak solutions. It is shown that starting from any point at initial non-vacuum regions (away from the symmetry center), a particle path is uniquely defined globally in time, along which the flow density is strictly positive and bounded from upper and below in finite time. Any two particle paths starting from two initially separated points (including the initial boundary point) in non-vacuum regions shall be separated uniformly from each other for all time. Between the two particle paths, vacuum states shall not form in any finite time so long as there is no vacuum state initially, and the initial regularities of the solution are maintained, in particular, the solution gains enough regularities to become a classical one. In addition, the free surface (or the free boundary) is shown to expand outward in the radial direction at an algebraic rate in time and the density decays to zero time-asymptotically almost everywhere away from the symmetry center, which lead to the formation of vacuum states as the time goes to infinity (refer to Theorems 2.2–2.3 in Sect. 2 for details). This is a completely different phenomena compared with the initial boundary value problem for (2.1) investigated in [12,29,30] where it was shown that any finite vacuum shall vanish in finite time.

Next, we study the dynamics of vacuum states for the global spherically symmetric entropy weak solution constructed in Theorem 2.1. It is proved that starting from any non-vacuum point (r_0', t_0') at any positive time $t_0' > 0$, a particle path is uniquely defined backward in time along which the density remains positive. It propagates either backward to some initial non-vacuum point r_0 as the time approaches zero, or terminates at a smaller time $t_1' \in [0, t_0')$ so that the density connects continuously to some vacuum state which is originated from the initial one and separated from any other vacuum states by non-vacuum fluid regions. This actually implies the finite time vanishing of initial vacuum and the blow-up phenomena (refer to Theorem 2.4 and Remark 2.4 in Sect. 2 for details).

Finally, the large time behavior of any global entropy weak solution and formation of vacuum state almost everywhere time-asymptotically are shown too. It is proved that the free surface (or the free boundary) moves outward in the radial direction at an algebraic rate in time from above and below, along which the density decays algebraically in time. These together with the uniform entropy estimates lead to the decay of fluid density to zero almost everywhere away from the symmetry center as the time approaches infinity (refer to Theorem 2.5 in Sect. 2 for details).

As stated above, the dynamics of vacuum states and the Lagrangian properties of the global spherically symmetric entropy weak solution (refer to Theorems 2.2–2.4) imply that any two separated vacuum states shall never meet in any finite time, and in particular, any initial vacuum state separated from the symmetry center (by some region with positive mass) shall not go into or be originated from the center in any finite time. On the other hand, any particle path along which the density is strictly positive in positive finite time can not go into either the symmetry center or the free surface backward or forward in time, unless it coincides identically with them for all time. It should be noted that all results stated above apply to the viscous Saint-Venant system.

It seems more involved to show all the results stated above for the FBVP (2.1)–(2.7). Indeed, besides the difficulties already mentioned above, additional difficulties also appear. For example, in contrast to the FBVP problems studied for (1.1) in one-dimension [26,32,41,51,55] and in multi-dimensional spherically symmetric domain between solid core and free boundary [4,5,52], the FBVP (2.1)-(2.7) investigated here contains the symmetry center and is a truly multi-dimensional problem. In particular, the spherically symmetric form of (2.1) becomes singular at the center $r = 0$ (see (2.3) in Sect. 2 and (3.11) in Sect. 3.2 respectively), which makes it hard to show the existence of global weak solutions. Furthermore, unlike CNS (1.1) with constant viscous coefficients where any particle path can be defined a-priorily even near vacuum, it is hard to define particle path a-priorily for CNS (2.1) due to possible degeneracy of viscous diffusion near vacuum states. Therefore, no information can be obtained a-priorily about Lagrange structures of solutions.

To overcome these difficulties and avoid the possible singularity at the origin, we first consider the FBVP for the spherically symmetric (2.3) on the domain excluding the ball $B_\varepsilon(0)$ centered at the origin with radius $\varepsilon > 0$, which then makes it possible to construct global smooth approximate solutions, trace the motion of particle paths, control lower and upper bounds for the density of the approximate solutions, and investigate their Lagrange properties. With these, one can extend the global approximate solutions to the whole domain including the ball $B_\varepsilon(0)$ as [11], obtain the Bresch-Desjardins (BD) entropy estimates as [1,2,11], and especially establish the desired uniform estimates on Lagrange properties for the approximate solutions with respect to $\varepsilon > 0$. These enable us to show the strong convergence of approximate solutions near the free boundary, while the convergence of approximate solutions on the whole domain follows from the uniform entropy estimates and the compactness framework founded by Mellet-Vasseur in [36]. And the expected Lagrange structure of solutions to the original problem can be justified also.

It should be noted that although the general strategy to construct the global approximate solution sequences for the FBVP (2.1)–(2.7) is similar to the initial boundary problem for (2.1) investigated in [11], however, the free boundary yields new phenomena and difficulties and new arguments are introduced in the present paper to obtain the regularities, dynamics of vacuum states and long time behaviors of global weak solutions to the FBVP (2.1)–(2.7). As already shown in [11,12,29,30] on compressible Navier-Stokes equations (2.1) in bounded domains, any possible existing vacuum state shall vanish in finite time and the density becomes uniformly positive in large time. This makes the equation of velocity strictly parabolic so that the weak solution can gain enough regularity. Therefore, it is possible to define the particle path and investigate the Lagrangian properties and dynamical behaviors of the weak solution. However, the situation is quite different for the FBVP (2.1)–(2.7). Indeed, as one can see in the present paper, the free boundary (the interface) moves outwards and the region of finite fluid mass expands continuously into the vacuum due to the free motion of viscous compressible fluid and the dispersion of total pressure. These in particular make the fluid density decay to zero (the formation of vacuum state) time-asymptotically and lead to the strong degeneracy of the system (loss of strict parabolicity and so on). Thus, it is nontrivial to investigate the regularity of the weak solution to the FBVP (2.1)–(2.7), define the particle path to analyze the Lagrangian structure and dynamical behaviors of global weak solutions (refer to Sect. 4–Sect. 5 for details). Moreover, these dynamical behaviors established in Theorems 2.2–2.5 for the first time are new and different from the phenomena observed in [12,29,30].

The rest of this paper is as follows. In Sect. 2 we state the main results of this paper. In Sect. 3, we construct the global approximate solutions and derive the desired entropy estimates. The key uniform estimates away from the symmetry center are established in Sect. 4, and the convergence of the global approximate solutions and the main results of the paper are presented in Sect. 5. The long time behavior of global weak solutions is shown in Sect. 6.

2. Main Results

For simplicity, the viscosity terms are assumed to satisfy $\mu(\rho) = \rho, \lambda(\rho) = 0$ in (1.1). The pressure is assumed to be $P(\rho) = \rho^\gamma$. Then (1.1) become

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{U}) = 0, \\ (\rho \mathbf{U})_t + \operatorname{div}(\rho \mathbf{U} \otimes \mathbf{U}) - \operatorname{div}(2\rho \mathbb{D}(\mathbf{U})) + \nabla \rho^\gamma = 0. \end{cases} \tag{2.1}$$

Consider a spherically symmetric solution $(\rho, \mathbf{U})$ to (2.1) in $\mathbb{R}^3$ so that

$$\rho(\mathbf{x}, t) = \rho(r, t), \quad \rho \mathbf{U}(\mathbf{x}, t) = \rho u(r, t)\frac{\mathbf{x}}{r}, \quad r = |\mathbf{x}|, \quad \mathbf{x} \in \mathbb{R}^3, \tag{2.2}$$

and (2.1) are changed to

$$\begin{cases} (r^2\rho)_t + (r^2\rho u)_r = 0, \\ (r^2\rho u)_t + (r^2\rho u^2)_r + r^2(\rho^\gamma)_r - r^2\big(\rho(u_r + \frac{2}{r}u)\big)_r + 2r\rho_r u = 0, \end{cases} \tag{2.3}$$

for $(r, t) \in \Omega_T$ with

$$\Omega_T = \{(r, t) |\, 0 \le r \le a(t),\ 0 \le t \le T\}. \tag{2.4}$$

The initial data is taken as

$$(\rho, \rho u)(r, 0) = (\rho_0, m_0)(r) =: (\rho_0, \rho_0 u_0)(r), \quad r \in (0, a_0). \tag{2.5}$$

At the center of symmetry we impose the Dirichlet boundary condition

$$\rho u(0, t) = 0, \tag{2.6}$$

and the free surface $\partial\Omega_t$ moves in the radial direction along the "particle path" $r = a(t)$ with the stress-free boundary condition

$$(\rho^\gamma - \rho(u_r + \frac{2}{r}u))(a(t), t) = 0, \quad t > 0, \tag{2.7}$$

where $a'(t) = u(a(t), t), t > 0$, and $a(0) = a_0$.

First, we define a weak solution to the FBVP (2.1)–(2.7) as follows.

Definition 2.1. $(\rho, \mathbf{U}, a)$ *with* $\rho \ge 0$ *a.e. is said to be a weak solution to the free surface problem* (2.1)–(2.7) *on* $\Omega_t \times [0, T]$, *provided that it holds that*

$$\rho \in L^\infty(0,T;L^1(\Omega_t)\cap L^\gamma(\Omega_t))\cap C([0,T];L^{3/2}(\Omega_t)),\quad \sqrt{\rho}\in L^\infty(0,T;H^1(\Omega_t)),$$
$$\sqrt{\rho}\,\mathbf{U}\in L^\infty(0,T;L^2(\Omega_t)),\quad \sqrt{\rho}\,\nabla\mathbf{U}\in L^2(0,T;W^{-1,1}(\Omega_t)),$$
$$\rho(a(t),t)>0,\ t\in[0,T],\quad a(t)\in H^1([0,T])\cap C^0([0,T]),$$

and the equations are satisfied in the sense of distributions. Namely, it holds for any $t_2 > t_1 \geq 0$ *and* $\phi \in C^1(\bar{\Omega}_t \times [0,T])$ *that*

$$\int_{\Omega_t} \rho\phi d\mathbf{x}|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Omega_t} (\rho\phi_t + \rho\mathbf{U}\cdot\nabla\phi)d\mathbf{x}dt, \tag{2.8}$$

and for $\psi = (\psi^1,\psi^2,\psi^3) \in C^1(\bar{\Omega}_t\times[0,T])$ *satisfying* $\psi(\mathbf{x},t) = 0$ *on* $\partial\Omega_t$ *and* $\psi(\mathbf{x},T) = 0$ *that*

$$\int_{\Omega_t} \mathbf{m_0}\cdot\psi(\mathbf{x},0)d\mathbf{x} + \int_0^T\int_{\Omega_t}[\sqrt{\rho}\,(\sqrt{\rho}\,\mathbf{U})\cdot\partial_t\psi + \sqrt{\rho}\,\mathbf{U}\otimes\sqrt{\rho}\,\mathbf{U}:\nabla\psi]d\mathbf{x}dt$$
$$+\int_0^T\int_{\Omega_t}\rho^\gamma \operatorname{div}\psi\, d\mathbf{x}dt - <2\rho\mathbb{D}(\mathbf{U}),\nabla\psi> = 0, \tag{2.9}$$

where $\mathbf{m_0} = m_0\frac{\mathbf{x}}{r}$ *and the diffusion term is defined for any* $\phi\in C^1(\bar{\Omega}_t\times[0,T])$ *as*

$$<\rho\partial_j\mathbf{U}_i,\phi> = -\int_0^T\int_{\Omega_t}\sqrt{\rho}(\sqrt{\rho}\,\mathbf{U}_i)\partial_j\phi\, d\mathbf{x}dt - \int_0^T\int_{\Omega_t}(\sqrt{\rho}\,\mathbf{U}_i)\phi\partial_j\sqrt{\rho}\, d\mathbf{x}dt$$

for $i,j=1,2,3$. *The free boundary condition* (2.7) *is satisfied in the sense of trace.*

Remark 2.1. That a weak solution to compressible Navier-Stokes equations (2.1) admits additional regularity on the fluid density $\sqrt{\rho}\in L^\infty(0,T;H^1(\Omega_t))$ is due to the Bresch-Desjardins (BD) entropy estimate, as discovered by Bresch and Desjardins [1,3]. It should be mentioned that the BD entropy estimate is also important in establishing the compactness estimates and analyzing the qualitative behaviors of solutions, refer to [1,2,11,29,30] and the references therein.

Notations. Throughout this paper, C and c denote generic positive constants, $C_{f,g}>0$ denotes a generic constant which may depend on the sub-index f and g, and $C_T>0$ a generic constant depending on T.

Assume further for $\Omega_0=[0,a_0]$ that

$$\begin{cases} \rho_0(r)\geq 0,\ \ r\in\Omega_0,\ \ \rho_0\in W^{1,\infty}(\Omega_0),\ \ \nabla\sqrt{\rho_0}\in L^2(\Omega_0),\ \ \int_{\Omega_0} r^2\rho_0(r)dr=1, \\ m_0(r)=0 \text{ for } r\in\Omega_0^0=:\{r\in\Omega_0|\,\rho_0(r)=0\},\ m_0\in W^{1,\infty}(\Omega_0),\ \frac{m_0^{2+\eta}}{\rho_0^{1+\eta}}\in L^1(\Omega_0), \end{cases} \tag{2.10}$$

with $\eta\in(0,1)$ a constant small enough. Throughout this paper the initial data and boundary value are assumed to be consistent at the point $(r,t)=(a_0,0)$. Then, we have the following global existence result.

Theorem 2.1 (Global existence). *Let $N = 2, 3$, $\gamma \in (1, \frac{N}{N-2})$. Assume that (2.10) holds and the initial data and boundary values are consistent in the sense*

$$\left(\rho_0^\gamma - \rho_0\left(u_{0r} + \frac{2}{a_0}u_0\right)\right)(a_0) = 0, \quad \rho_0(a_0) > 0. \tag{2.11}$$

Then, the FBVP (2.1)–(2.7) has a global spherically symmetric weak solution

$$(\rho, \rho\mathbf{U}, a)(\mathbf{x}, t) = \left(\rho(r,t), \rho u(r,t)\frac{\mathbf{x}}{r}, a(t)\right), \quad r = |\mathbf{x}|,$$

in the sense of Definition 2.1 satisfying for any $T > 0$ that

$$\int_0^{a(t)} r^2\rho(r,t)dr = \int_0^{a_0} r^2\rho_0(r)dr, \tag{2.12}$$

$$c \le a(t) \le C_T, \quad t \in [0, T], \quad \|a\|_{H^1([0,T])} \le C_T, \tag{2.13}$$

$$0 \le \rho(r,t) \in C^0((0, a(t)] \times [0, T]), \quad \rho \in C^0([0, T]; L^{3/2}(\bar{\Omega}_t)), \tag{2.14}$$

$$\sup_{t\in[0,T]}\int_{\Omega_t}(\rho^\gamma + |\sqrt{\rho}\,\mathbf{U}|^2)(\mathbf{x}, t)d\mathbf{x} + \int_0^T\int_{\Omega_t}|\sqrt{\rho}\,\nabla\mathbf{U}|^2 d\mathbf{x}dt \le C, \tag{2.15}$$

$$\sup_{t\in[0,T]}\int_{\Omega_t}|\nabla\sqrt{\rho}\,|^2(\mathbf{x}, t)d\mathbf{x} + \int_0^T\int_{\Omega_t}|\nabla\rho^{\frac{\gamma}{2}}|^2 d\mathbf{x}dt + \rho^\gamma(a(t), t)a^N(t) \le C, \tag{2.16}$$

with $C > 0$ and $C_T > 0$ two constants. Furthermore, it holds that $\rho^\gamma - \rho(u_r + \frac{2}{r}u) \in L^2(0, T; H^1(\Omega_\delta))$ with $\Omega_\delta = (a(t) - \delta, a(t))$ for some small constant $\delta > 0$, and the free boundary condition (2.7) is satisfied in the sense of trace.

Remark 2.2. (i) The assumption $(\rho_0, u_0) \in W^{1,\infty}(\Omega_0)$ in (2.10) can be relaxed. For instance, it can be replaced by $(\rho_0, u_0) \in W^{1,\infty}(\Omega_0\backslash\Omega_0^0)$ as follows from the proofs in Sect. 5.

(ii) Theorem 2.1 yields the global existence of spherically symmetric weak solutions for two/three dimensional compressible Navier-Stokes equation with free surface separating fluid and vacuum states. In particular, it applies to the viscous Saint-Venant model for shallow water (which is (2.1) with $N = 2$, $\mu(\rho) = \rho$, $\lambda(\rho) = 0$, and $\gamma = 2$). As it will follow from our analysis, the same existence result holds for general viscosity coefficients in (1.1), for instance, $\mu(\rho) = \rho^\alpha$ and $\lambda(\rho) = (\alpha - 1)\rho^\alpha$ with $\frac{N}{N+1} < \alpha \le 1$ and $N \ge 2$.

Next, we investigate the Lagrangian properties of global weak solutions to the FBVP (2.1)–(2.7), such as the motion of particle paths, transportation of fluid mass, non-formation of vacuum states, maintenance of initial regularities, etc. For simplicity, we treat only the case $N = 3$, the case $N = 2$ can be dealt with similarly. We have the following results

Theorem 2.2 (Lagrangian structure). *Let $N = 3$ and $\gamma \in [2, 3)$. Assume that (2.10) and (2.11) hold. Then, the global weak solution $(\rho, \rho\mathbf{U}, a) = (\rho(r,t), \rho u(r,t)\frac{\mathbf{x}}{r}, a(t))$ to the FBVP (2.1)–(2.7) constructed in Theorem 2.1 satisfies the following properties:*

(i) (**Non-concentration of mass**) *There is no mass concentration at the symmetry center, namely,*

$$\int_0^{\eta} r^2\rho(r,t)dr \to 0, \quad as\ \eta \to 0_+ \text{ holds.} \tag{2.17}$$

(ii) (**Non-formation of vacuum state in finite time**) *For any* $r_0 \in (0, a_0)$ *with* $\rho_0(r_0) > 0$*, there exists a particle path* $r = r_x(t)$ *for* $t \in [0, T]$ *uniquely defined by* $\frac{d}{dt}r_x(t) = u(r_x(t), t)$ *with* $r_{x_0}(0) = r_0$ *and* $x_0 = 1 - \int_{r_0}^{a_0} r^2\rho_0(r)dr \in (0, 1)$*, so that*

$$\begin{cases} 0 < cx_0^{\frac{\gamma}{3(\gamma-1)}} \le r_{x_0}(t) < a(t) \le C_T \quad t \in [0, T], \\ 0 < c_{x_0,T} \le \rho(r_{x_0}(t), t) \le Cx_0^{-\frac{2\gamma}{3(\gamma-1)}}, \quad t \in [0, T] \text{ holds,} \end{cases} \tag{2.18}$$

where C, C_T, c *and* $c_{x_0,T}$ *are positive constants, and* $c_{x_0,T} \to 0$ *as* $x_0 \to 0_+$*. Furthermore, for any* $0 < r_1 < r_2 \le a_0$ *with* $\rho_0(r_i) > 0$*, there exist two particle paths* $r = r_{x_i}(t)$ *uniquely defined by*

$$\frac{d}{dt}r_{x_i}(t) = u(r_{x_i}(t), t), \tag{2.19}$$

with $r_{x_i}(0) = r_i$ *and* $x_i = 1 - \int_{r_i}^{a_0} r^2\rho_0(r)dr \in (0, 1]$, $i = 1, 2$*, such that*

$$\begin{cases} c(x_2 - x_1)^{\frac{\gamma}{\gamma-1}} \le r_{x_2}^3(t) - r_{x_1}^3(t), \quad t \in [0, T], \\ 0 < c_{x_i,T} \le \rho(r_{x_i}(t), t) \le Cx_i^{-\frac{2\gamma}{3(\gamma-1)}}, \quad i = 1, 2,\ t \in [0, T], \end{cases} \tag{2.20}$$

where the constant $c_{x_i,T} > 0$ *satisfies* $c_{x_i,T} \to 0$ *as* $x_i \to 0_+$.

(iii) (**Long time dynamics**) *The free surface expands at the following rates*

$$C(1+t)^{\frac{\gamma}{3(\gamma-1)}} \ge a(t) \ge c(1+t)^{\frac{1}{3\gamma}}, \quad as\ t \to +\infty, \tag{2.21}$$

and the fluid density ρ *decays almost everywhere as follows:*

$$\rho(a(t), t) = \mathcal{O}(1)(1+t)^{-\frac{1}{\gamma-1}}, \quad \gamma > 1, \tag{2.22}$$

$$\frac{1}{a^3(t)}\int_0^{a(t)} \rho(r,t)r^2dr \le C(1+t)^{-\frac{1}{(\gamma-1)}}, \tag{2.23}$$

where $C > 0$ *and* $c > 0$ *are constants independent of time, and*

$$\rho(r,t) \longrightarrow 0, \quad t \to \infty, \quad r \in [r_0, a(t)], \tag{2.24}$$

for any fixed $r_0 \in (0, a(t))$ *with* $t > 0$.

Theorem 2.3 (Regularity). *Let* $N = 3$ *and* $\gamma \in [2, 3)$*. Assume that* (2.10) *and* (2.11) *hold. Then, the global weak solution* $(\rho, \rho\mathbf{U}, a) = (\rho(r,t), \rho u(r,t)\frac{\mathbf{x}}{r}, a(t))$ *to the FBVP* (2.1)–(2.7) *constructed in Theorem 2.1 satisfies the following regularities:*

(i) (**Interior regularity**) *Assume further that there exist* $0 < r_1^- < r_1 < r_2 < r_2^+ \leq a_0$ *and a constant* $\rho_* > 0$ *such that*

$$\inf_{r\in[r_1,r_2]} \rho_0(r) \geq \rho_* > 0, \quad u_0 \in H^2([r_1^-, r_2^+]), \tag{2.25}$$

then

$$\begin{cases} 0 < c_{x_1,T} \leq \rho(r,t) \leq C x_1^{-\frac{2\gamma}{3(\gamma-1)}}, \quad r \in [r_{x_1}(t), r_{x_2}(t)],\ t \in [0,T], \\ \|(\rho,u)\|_{L^\infty(0,T;H^1([r_{x_1}(t),r_{x_2}(t)]))} + \|u\|_{L^2(0,T;H^2([r_{x_1}(t),r_{x_2}(t)]))} \leq C_{x_1,T}, \end{cases} \tag{2.26}$$

and the following interior regularities hold:

$$\begin{cases} (\rho,u) \in C^0([r_{x_1}(t), r_{x_2}(t)] \times [0,T]), \\ \rho \in L^\infty(0,T; H^1([r_{x_1}(t), r_{x_2}(t)])), \quad \rho_t \in L^\infty(0,T; L^2([r_{x_1}(t), r_{x_2}(t)])), \\ u \in L^\infty(0,T; H^2([r_{x_1}(t), r_{x_2}(t)])) \cap L^2(0,T; H^3([r_{x_1}(t), r_{x_2}(t)])), \\ u_t \in L^\infty(0,T; L^2([r_{x_1}(t), r_{x_2}(t)])) \cap L^2(0,T; H^1([r_{x_1}(t), r_{x_2}(t)])), \end{cases} \tag{2.27}$$

where $r = r_{x_i}(t)$ *is the particle path defined by* (2.19)*, and the constant* $C_{x_i,T} > 0$ *satisfies* $C_{x_1,T} \to \infty$ *as* $x_1 \to 0_+$.

(ii) (**Boundary regularity**) *Assume, in addition, that there exist* $0 < r_3^- < r_3 < a_0$ *and a constant* $\rho_* > 0$ *such that*

$$\inf_{r\in[r_1,a_0]} \rho_0(r) \geq \rho_* > 0, \quad u_0 \in H^2([r_3^-, a_0]), \tag{2.28}$$

then

$$\begin{cases} 0 < c_{x_3,T} \leq \rho(r,t) \leq C x_3^{-\frac{2\gamma}{3(\gamma-1)}}, \quad r \in [r_{x_3}(t), a(t)],\ t \in [0,T], \\ \|(\rho,u)\|_{L^\infty(0,T;H^1([r_{x_3}(t),a(t)]))} + \|u\|_{L^2(0,T;H^2([r_{x_3}(t),a(t)]))} \leq C_{x_3,T}, \end{cases} \tag{2.29}$$

and the following boundary regularities hold

$$\begin{cases} (\rho,u) \in C^0([r_{x_3}(t), a(t)] \times [0,T]), \quad a(t) \in H^2([0,T]) \cap C^1([0,T]), \\ \rho \in L^\infty(0,T; H^1([r_{x_3}(t), a(t)])), \quad \rho_t \in L^\infty(0,T; L^2([r_{x_3}(t), a(t)])), \\ u \in L^\infty(0,T; H^2([r_{x_3}(t), a(t)])) \cap L^2(0,T; H^3([r_{x_3}(t), a(t)])), \\ u_t \in L^\infty(0,T; L^2([r_{x_3}(t), a(t)])) \cap L^2(0,T; H^1([r_{x_3}(t), a(t)])), \end{cases} \tag{2.30}$$

where $r = r_{x_3}(t)$ *is the particle path with* $r_{x_3}(0) = r_3$ *and* $x_3 = 1 - \int_{r_3}^{a_0} r^2 \rho_0(r)dr \in (0,1)$. *The free boundary condition* (2.7) *is also satisfied pointwisely since*

$$\rho^\gamma - \rho(u_r + \frac{2}{r}u) \in L^\infty(0,T; H^1([r_{x_3}(t), a(t)])) \cap C^0([0,T] \times [r_{x_3}(t), a(t)]) \quad \text{holds.} \tag{2.31}$$

Remark 2.3. (i) It follows from Theorems 2.2–2.3 that for the FBVP (2.1)–(2.7) any initial non-vacuum point is transported along the particle path, vacuum states shall not appear away from the symmetry center in any finite time so long as there is no vacuum state initially, the initial regularities of solution are maintained on (non-vacuum) fluid regions. In fact, (2.27) and (2.30) imply that the solution gains regularities in the interior non-vacuum regions and becomes a classical one. The free surface expands outward in the radial direction along "particle path" at an algebraic rate in time. In particular, there is no mass concentration at the symmetry center, and the flow density shall tend to zero everywhere time-asymptotically away from the symmetry center, which leads to the formation of vacuum states as time goes to infinity.

(ii) It should be emphasized that all Lagrange properties in Theorems 2.2–2.3 hold also for the FBVP (2.1)–(2.7) in 2D for $\gamma \in [2, \infty)$, in particular, it applies to the viscous Saint-Venat model.

(iii) Theorems 2.2 present the time-asymptotical formation of vacuum state for the compressible viscous fluid with free boundary. This is a completely different phenomena compared with the initial boundary value problem for CNS (2.1) in bounded domain investigated in [12,29,30], where it is shown that any finite vacuum shall vanish in finite time. Moreover, the analysis on Lagrangian properties of global weak solutions to the FBVP (2.1)–(2.7) can be used to study the dynamical behaviors of the global weak solution to the Dirichlet problem for CNS (2.1) in multi-dimension, where the regularity and finite vanishing of the vacuum state are shown, refer to [12] for details.

(iv) It should be mentioned that the Lagrangian analyticity and backward uniqueness of global weak solutions away from vacuum for CNS with constant viscosity coefficients have been obtained in [17].

Further properties on the dynamical behaviors of vacuum states away from the symmetry center are stated as follows.

Theorem 2.4 (Dynamics of vacuum states). *Let the assumptions in Theorem 2.2 hold. Assume that the global weak solution $(\rho, \rho\mathbf{U}, a)$ to the FBVP (2.1)-(2.7) constructed in Theorem 2.1 satisfies $\rho(r_0', t_0') > 0$ for some $(r_0', t_0') \in (0, a(t_0')) \times (0, T]$. Then, either case (a) or case (b) holds:*

(a) *There exist an initial point $r_0 \in (0, a_0)$ and a particle path $r = r_{x_0}(t)$ with $r_{x_0}(0) = r_0$ and $x_0 = 1 - \int_{r_0'}^{a(t_0')} r^2\rho(r, t')dr \in (0, 1)$ so that*

$$\begin{cases} r_{x_0}(0) = r_0, \quad r_{x_0}(t_0') = r_0', \quad \rho_0(r_0) > 0, \\ 0 < c x_0^{\frac{\gamma}{3(\gamma-1)}} \le r_{x_0}(t) < a(t) \le C_T, \quad t \in [0, t_0'], \\ 0 < c_{x_0,T} \le \rho(r_{x_0}(t), t) \le C_T x_0^{-\frac{2\gamma}{3(\gamma-1)}}, \quad t \in [0, t_0']. \end{cases} \tag{2.32}$$

(b) *There exist a time $t_1' \in [0, t_0')$ and a subset $\mathcal{V}_{x_0}^t \subset (0, a(t))$ defined for $t \in [0, t_0')$ as*

$$\mathcal{V}_{x_0}^t =: \{(r, t) | \rho(r, t) = 0, \ r \in (0, a(t)); \ \int_r^{a(t)} y^2\rho(y, t)dy = 1 - x_0\} \tag{2.33}$$

with $x_0 = 1 - \int_{r_0'}^{a(t_0')} r^2\rho(r, t_0')dr \in (0, 1)$, so that

$$\mathcal{V}_{x_0}^t \neq \phi_0 \ \ \text{for} \ \ t \in [0, t_1'], \quad \mathcal{V}_{x_0}^t = \phi_0 \ \ \text{for} \ \ t \in (t_1', t_0'], \tag{2.34}$$

with ϕ_0 being the empty set. In particular, there exist a point $(r_1', t_1') \in \mathcal{V}_{x_0}^{t_1'}$ and a particle path $r = r_{x_0}(t)$ on $(t_1', t_0']$ with $r_{x_0}(t_0') = r_0'$, so that

$$\rho(r_1', t_1') = 0, \quad 1 - x_0 = \int_{r_1'}^{a(t_1')} r^2 \rho(r, t_1') dr = \int_{r_{x_0}(t)}^{a(t)} r^2 \rho(r, t) dr, \quad t \in (t_1', t_0']. \tag{2.35}$$

Meanwhile, the solution blows up in the sense for any small but fixed $\eta_0 > 0$ that

$$\lim_{t \to t_1'^+} \int_t^{t_1'+\eta_0} \|u_r(s)\|_{L^\infty([r_{x_1}(s), r_{x_2}(s)])} ds = +\infty, \tag{2.36}$$

where $r = r_{x_i}(t)$ is the particle path on $[0, t_0']$ with $r_{x_i}(t_1') = r_i' \in (0, a_0)$ and $x_i = 1 - \int_{r_i'}^{a_0} r^2 \rho_0(r) dr$, $i = 1, 2$, for any $r_1' < r_2'$ chosen appropriately so that $\rho_0(r) > 0$ for $r \in [r_1', r_2'] \backslash \mathcal{V}_{x_0}^0$ and $\rho(r, t) > 0$ for $(r, t) \in [r_{x_1}(t), r_{x_2}(t)] \backslash \mathcal{V}_{x_0}^t$.

Remark 2.4. (i) It follows from Theorem 2.4 that starting from any non-vacuum point (r_0', t_0') with $t_0' > 0$, there is a unique particle path defined backward in time along which the density remains positive. It propagates either back to some initial point with positive density as the time approaches zero, or terminates at a time $t_1' \in [0, t_0')$ so that the density tends to zero (vacuum state) as the time decreases to t_1'. In this case, there is an initial point $r_0 > 0$ such that $\rho_0(r_0) = 0$ and the mass between r_0 and initial boundary point is the same as the one between $r_0' > 0$ and the free boundary at the time $t_0' > 0$. It implies the finite time vanishing of initial vacuum state.

(ii) Theorem 2.2 and Theorem 2.4 imply that any connected vacuum state in later-on time separated (by non-vacuum flow regions) from the symmetry center is originated continuously from some initial one, and is separated from both sides before vanishing by particle paths along which the flow densities are strictly positive.

(iii) Similar phenomena occur for the 2D case, in particular, for the viscous Saint-Venat model.

Finally, we investigate the long time behavior of global solutions and the motion of the interface of the free surface problem (2.1)–(2.7). We have

Theorem 2.5 (Long time behaviors). *Let $T > 0$, $N \geq 2$ and $\gamma > 1$. Let (ρ, u, a) be any global (strong or weak) solution to the FBVP (2.1)–(2.7) in the sense of Definition 2.1 for $t \in [0, T]$ with $\rho^\gamma - \rho(u_r + \frac{2}{r}u) \in L^2(0, T; H^1(\Omega_\eta))$ and $\Omega_\eta = (a(t) - \eta, a(t))$ for some small constant $\eta > 0$. Then, for any $t > 0$,*

$$a_M(t) =: \max_{s \in [0,t]} a(s) \geq \begin{cases} C(1+t)^{\frac{1}{N\gamma}}, & \gamma > \frac{N+1}{N}, \\ C(1+t)^{\frac{1-\nu}{N\gamma}}, & \gamma = \frac{N+1}{N}, \\ C(1+t)^{\frac{\gamma-1}{\gamma}}, & \gamma \in (1, \frac{N+1}{N}), \end{cases} \tag{2.37}$$

$$c(1+t)^{-\frac{1}{\gamma-1}} \leq \rho(a(t), t) \leq C(1+t)^{-\frac{1}{\gamma-1}} \quad \text{holds,} \tag{2.38}$$

where $\nu > 0$ is a constant small enough, and $c > 0$, $C > 0$ are constants independent of time.

In addition, for $\gamma \geq 2$,

$$C(1+t)^{\frac{\gamma}{N(\gamma-1)}} \geq a(t) \geq C(1+t)^{\frac{1}{N\gamma}}, \tag{2.39}$$

$$C(1+t)^{-\frac{1}{(\gamma-1)}} \geq \frac{1}{a^N(t)} \int_0^{a(t)} \rho(r,t) r^2 dr \geq c(1+t)^{-\frac{\gamma}{(\gamma-1)}} \quad \text{holds}, \tag{2.40}$$

as $t \to +\infty$*, and for any fixed* $r_0 \in (0, a(t))$ *that*

$$\rho(r,t) \longrightarrow 0, \quad t \to \infty, \quad r \in [r_0, a(t)], \tag{2.41}$$

provided that (2.15)–(2.16) *hold for the solution* (ρ, u, a) *with* $N \geq 2$.

Remark 2.5. Theorem 2.5 shows that the fluid domain expands outward at the algebraic rates in time from above and below and the density decays along the free boundary, which together with the uniform entropy estimates lead to the decay of fluid density and the formation of the vacuum state almost everywhere as time goes to infinity.

3. Global Existence of Approximate FBVP Problem

The proofs of Theorems 2.1–2.4 consist of the construction of approximate solutions, the a-priori estimates and regularity analysis, and compactness arguments. These can be carried out by investigating the related properties of global approximate solutions and then passing into the limit. To this end, we first consider in this section an approximate FBVP problem on spatial exterior domain, show the global existence of solutions, and establish the Lagrangian properties, such as the existence and uniqueness of particle paths, transportation of initial regularities, and dynamics of vacuum states, etc.

3.1. Approximate FBVP problem. Consider a modified FBVP problem for Eq. (2.3) with the following initial data and boundary conditions for any fixed $\varepsilon > 0$:

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(r), \quad \varepsilon \leq r \leq a_0, \tag{3.1}$$

$$u(\varepsilon, t) = 0, \quad (\rho^\gamma - \rho(u_r + \frac{2}{r}u))(a(t), t) = 0, \quad t \geq 0, \tag{3.2}$$

where $a'(t) = u(a(t), t)(t > 0)$ and $a(0) = a_0$. Without loss of generality, it is assumed in this section that the initial data (3.1) is smooth enough and consistent with the boundary values (3.2) to high order.

The main result for the FBVP (2.3) and (3.1)–(3.2) is stated as follows.

Proposition 3.1. *Let* $T > 0, \gamma > 1$*, and* $\varepsilon > 0$ *be fixed. Assume that the initial data* (ρ_0, u_0) *satisfies*

$$\inf_{x \in [\varepsilon, a_0]} \rho_0(x) > 0, \quad (\rho_0, u_0) \in W^{1,\infty}([\varepsilon, a_0]). \tag{3.3}$$

Then, there exists a unique global strong solution (ρ, u, a) *of the FBVP problem* (2.3) *and* (3.1)–(3.2)*, which satisfies for* $t \in [0, T]$ *that*

$$\begin{cases} c x_i^{\frac{\gamma}{3(\gamma-1)}} \leq r_{x_i}(t) \leq a(t), \quad c \leq a(t) \leq C_T, \\ c(x_2 - x_1)^{\frac{\gamma}{\gamma-1}} \leq r_{x_2}^3(t) - r_{x_1}^3(t), \quad t \in [0, T], \\ C_{\varepsilon,T}^{-1} \leq \rho(r,t) \leq C_{\varepsilon,T}, \quad r \in [0, a(t)], \end{cases} \tag{3.4}$$

where $r = r_{x_i}(t)$ *is the particle path with* $r_{x_i}(0) = r_i \in [\varepsilon, a_0]$ *and* $x_i = 1 - \int_{r_i}^{a_0} r^2\rho_0(r)dr$, $i = 0, 1, 2$, *and*

$$\|(\rho, u)(t)\|_{H^1([\varepsilon,a(t)])} + \int_0^t \|(\rho_r, \rho_{tr}, u_t, u_r, u_{rr})(s)\|^2_{L^2([\varepsilon,a(s)])}ds + \int_0^t \|(\rho^\gamma - \rho(u_r + \frac{2}{r}u))(s)\|^2_{H^1([\varepsilon,a(s)])}ds + \int_0^t |(a, a')(s)|^2 ds \le C_{\varepsilon,T}\lambda_0, \quad (3.5)$$

where $\lambda_0 =: \|(\rho_0, u_0)\|_{W^{1,\infty}([\varepsilon,a_0])}$, *and* c, C_T *and* $C_{\varepsilon,T}$ *are positive constants. In addition, it holds for any* $r \in [r_0, a(t)]$ *that*

$$\rho(r, t) \to 0, \quad as \ \ t \to \infty, \quad (3.6)$$

uniformly with respect to any fixed $r_0 \in (0, a(t))$.

Furthermore, if $u_0 \in H^2([\varepsilon, a_0])$, *then*

$$\|u(t)\|_{H^2[\varepsilon,a(t)])} + \|u_t(t)\|_{L^2([\varepsilon,a(t)])} + \|\left(\rho^\gamma - \rho\left(u_r + \frac{2}{r}u\right)\right)(t)\|_{H^1([\varepsilon,a(t)])} + \int_0^t (\|u_t(s)\|^2_{H^1([\varepsilon,a(s)])} + |a(s)|^2 + |a'(s)|^2 + |a''(s)|^2)ds \le C_{\varepsilon,T}\lambda_1 \quad (3.7)$$

for $t \in [0, T]$, *with* $\lambda_1 =: \|\rho_0\|_{W^{1,\infty}([\varepsilon,a_0])} + \|u_0\|_{H^2([\varepsilon,a_0])}$.

3.2. Approximate FBVP in the Lagrangian coordinates. In this subsection, we prove Proposition 3.1. It is convenient to deal with the FBVP (2.3) and (3.1)-(3.2) in the Lagrangian coordinates. For simplicity we assume that $\int_\varepsilon^a \rho_0 r^2 dr = 1$, which implies

$$\int_\varepsilon^{a(t)} \rho r^2 dr = \int_\varepsilon^a \rho_0 r^2 dr = 1.$$

For $r \in [\varepsilon, a(t)]$ and $t \in [0, T]$, define the Lagrangian coordinates transformation

$$x(r, t) = \int_\varepsilon^r \rho y^2 dy = 1 - \int_r^{a(t)} \rho y^2 dy, \quad \tau = t, \quad (3.8)$$

which translates the domain $[0, T] \times [\varepsilon, a(t)]$ into $[0, T] \times [0, 1]$ and satisfies

$$\frac{\partial x}{\partial r} = \rho r^2, \quad \frac{\partial x}{\partial t} = -\rho u r^2, \quad \frac{\partial \tau}{\partial r} = 0, \quad \frac{\partial \tau}{\partial t} = 1, \quad (3.9)$$

and

$$r^3(x, \tau) = \varepsilon^3 + 3\int_0^x \frac{1}{\rho}(y, \tau)dy = a(t)^3 - 3\int_x^1 \frac{1}{\rho}(y, \tau)dy, \quad \frac{\partial r}{\partial \tau} = u. \quad (3.10)$$

The free boundary value problem (2.3) and (3.1)-(3.2) is changed to

$$\begin{cases} \rho_\tau + \rho^2(r^2 u)_x = 0, \\ r^{-2}u_\tau + (\rho^\gamma - \rho^2(r^2 u)_x)_x + \frac{2}{r}\rho_x u = 0, \end{cases} \quad (3.11)$$

for $(x, \tau) \in [0, 1] \times [0, T]$, with the initial data and boundary conditions given by

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in [0, 1], \tag{3.12}$$

$$u(0, \tau) = 0, \quad (\rho^\gamma - \rho^2(r^2 u)_x)(1, \tau) = 0, \quad \tau \in [0, T], \tag{3.13}$$

where $r = r(x, \tau)$ is defined by

$$\frac{d}{d\tau} r(x, \tau) = u(x, \tau), \quad x \in [0, 1], \ \tau \in [0, T], \tag{3.14}$$

and the fixed boundary $x = 1$ corresponds to the free boundary $a(\tau) = r(1, \tau)$ in Eulerian form determined by

$$\frac{d}{d\tau} a(\tau) = u(1, \tau), \ \tau \in [0, T], \quad a(0) = a_0. \tag{3.15}$$

It is clear that the initial data (3.12) is smooth enough and well consistent with the boundary data (3.13).

We now have the following global existence and uniqueness results as follows.

Proposition 3.2. *Let $T > 0$ and $\gamma > 1$. Assume that the initial data (ρ_0, u_0) satisfy*

$$\inf_{x \in [0,1]} \rho_0(x) > 0, \quad (\rho_0, u_0) \in W^{1,\infty}([0, 1]). \tag{3.16}$$

Then, there exists a unique global strong solution (ρ, u, a) to the FBVP (3.11)–(3.15) satisfying for $\tau \in [0, T]$ that

$$\begin{cases} c x_0^{\frac{\gamma}{3(\gamma-1)}} \le r(x_0, \tau) \le a(\tau) \le C(1+\tau)^{\frac{\gamma}{3(\gamma-1)}}, \quad \forall\, x_0 \in [0, 1], \\ c(x_2 - x_1)^{\frac{\gamma}{\gamma-1}} \le r^3(x_2, \tau) - r^3(x_1, \tau), \quad 0 \le x_1 < x_2 \le 1, \\ c \le a(\tau) \le C_{\varepsilon,T}, \quad C_{\varepsilon,T}^{-1} \le \rho(x, \tau) \le C_\varepsilon, \quad x \in [0, 1], \end{cases} \tag{3.17}$$

where $r = r(x_i, \tau)$, $i = 0, 1, 2$, is the particle path defined by (3.14) with $r(x_i, 0) = r_i \in [\varepsilon, a_0]$ and $x_i = 1 - \int_{r_i}^{a_0} r^2 \rho_0(r) dr$, and

$$\begin{aligned} &\|(\rho, u)(\tau)\|_{H^1} + \int_0^T \|(\rho_x, \rho_{\tau x}, u_\tau, u_x, u_{xx})(\tau)\|_{L^2}^2 d\tau \\ &+ \int_0^T (\|(\rho^\gamma - \rho^2(r^2 u)_x)(\tau)\|_{H^1}^2 + |(a, a')(\tau)|^2) d\tau \le C_{\varepsilon,T} \delta_0, \end{aligned} \tag{3.18}$$

with $C_{\varepsilon,T} > 0$ a constant and $\delta_0 =: \|(\rho_0, u_0)\|_{W^{1,\infty}([0,1])}$. Moreover, it holds for $x \in [x_0, 1]$ with any fixed $x_0 \in (0, 1)$

$$\rho(x, \tau) \to 0, \quad as \ \ \tau \to \infty, \tag{3.19}$$

uniformly with respect to $\varepsilon > 0$.

Furthermore, if $u_0 \in H^2([0, 1])$, then

$$\begin{aligned} &\|u(\tau)\|_{H^2} + \|u_\tau(\tau)\|_{L^2} + \|(\rho^\gamma - \rho^2(r^2 u)_x)(\tau)\|_{H^1} \\ &+ \int_0^T (\|u_\tau(\tau)\|_{H^1}^2 + |(a, a', a'')(\tau)|^2) d\tau \le C_{\varepsilon,T} \delta_1. \end{aligned} \tag{3.20}$$

with $\delta_1 =: \|\rho_0\|_{W^{1,\infty}([0,1])} + \|u_0\|_{H^2([0,1])}$.

The proof of Proposition 3.2 will be given in the next section.

3.3. The a-priori estimates. In this subsection, we establish the a-priori estimates for any (regular approximate) solution (ρ, u, a) with $\rho > 0$ to FBVP (3.11)–(3.15). We start with a basic energy estimate.

Lemma 3.3. *Let $\gamma > 1$, $T > 0$, and (ρ, u, a) with $\rho > 0$ be any regular solution to the FBVP (3.11)–(3.15) for $\tau \in [0, T]$ under the assumptions of Proposition 3.2. Then,*

$$\int_0^1 (\frac{1}{2}u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1})dx + 2\int_0^\tau \int_0^1 \frac{u^2}{r^2}dxds + \int_0^\tau \int_0^1 \rho^2(r^2u_x)^2 dxds + 2\int_0^\tau \rho u^2 r(1,s)ds = E_0 =: \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}\rho_0^{\gamma-1}\right)dx, \quad \tau \in [0,T] \text{ holds.} \tag{3.21}$$

Proof. Taking the inner product of $(3.11)_2$ with r^2u on $[0, 1]$, and using $(3.11)_1$, one gets

$$\frac{d}{d\tau}\int_0^1 \left(\frac{u^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right)dx + \int_0^1 \rho^2(r^2u)_x^2 dx + 2\rho u^2 r(1,\tau) = 2\int_0^1 \rho(u^2r)_x dx = 4\int_0^1 \rho u u_x r dx + 2\int_0^1 \frac{u^2}{r^2}dx. \tag{3.22}$$

Due to the fact $\rho^2(r^2u)_x^2 = \frac{4u^2}{r^2} + 4\rho u u_x r + \rho^2(r^2u_x)^2$, it follows from (3.22) that

$$\frac{d}{d\tau}\int_0^1 \left(\frac{u^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1}\right)dx + \int_0^1 (\frac{2u^2}{r^2} + \rho^2(r^2u_x)^2)dx + 2\rho u^2 r(1,\tau) = 0,$$

which yields (3.21) after integration over $[0, \tau]$.

Lemma 3.4. *Under the same assumptions as Lemma 3.3,*

$$E_0^{-\frac{1}{3(\gamma-1)}} x^{\frac{\gamma}{3(\gamma-1)}} \le r(x,\tau) \le a(\tau), \quad (x,\tau) \in [0,1]\times[0,T], \tag{3.23}$$

$$E_0^{-\frac{1}{3(\gamma-1)}} (x_2 - x_1)^{\frac{\gamma}{3(\gamma-1)}} \le r^3(x_2,\tau) - r^3(x_1,\tau), \quad 0 \le x_1 < x_2 \le 1, \quad \tau \in [0,T] \text{ holds,} \tag{3.24}$$

with $E_0 = \int_0^1 (\frac{u_0^2}{2} + \frac{\rho_0^{\gamma-1}}{\gamma-1})dx$. In particular, it holds for $x = 1$ that

$$E_0^{-\frac{1}{3(\gamma-1)}} \le a(\tau) \equiv r(1,\tau), \quad \tau \in [0,T]. \tag{3.25}$$

Proof. First, for any $x \in (0, 1)$ and $\varepsilon \le r(x, \tau) \le a(\tau)$, then it is easy to deduce from (3.8) and (3.21) that

$$x = \int_\varepsilon^{r(x,\tau)} \rho y^2 dy \le (\int_\varepsilon^{r(x,\tau)} \rho^\gamma y^2 dy)^{\frac{1}{\gamma}} (\int_\varepsilon^{r(x,\tau)} y^2 dy)^{\frac{\gamma-1}{\gamma}} \le 3^{\frac{1-\gamma}{\gamma}} E_0^{\frac{1}{\gamma}} r(x,\tau)^{\frac{3(\gamma-1)}{\gamma}},$$

which implies for $(x, \tau) \in (0, 1] \times [0, T]$ that

$$r(x,\tau) \ge (3E_0^{-\frac{1}{\gamma-1}} x^{\frac{\gamma}{\gamma-1}})^{\frac{1}{3}} \ge E_0^{-\frac{1}{3(\gamma-1)}} x^{\frac{\gamma}{3(\gamma-1)}}. \tag{3.26}$$

Similarly, for any $0 < x_1 \le x_2 < 1$,

$$x_2 - x_1 = \int_{r(x_1,\tau)}^{r(x_2,\tau)} \rho y^2 dy \le 3^{\frac{1-\gamma}{\gamma}} E_0^{\frac{1}{\gamma}} (r^3(x_2,\tau) - r^3(x_1,\tau))^{\frac{\gamma-1}{\gamma}} \quad \text{holds,}$$

which implies (3.24). The proof is completed. □

Lemma 3.5. *Under the same assumptions as Lemma 3.3,*

$$\frac{1}{2}\int_0^1 (u+\rho_x r^2)^2 dx + \frac{1}{\gamma-1}\int_0^1 \rho^{\gamma-1} dx + \frac{4\gamma}{(\gamma+1)^2}\int_0^\tau\int_0^1 ((\rho^{\frac{\gamma+1}{2}})_x r^2)^2 dxds$$
$$+\frac{1}{3}\rho^\gamma(1,\tau)a^3(\tau) + \frac{\gamma}{3}\int_0^\tau \rho^{2\gamma-1}(1,s)a^3(s)ds = E_1, \quad \tau\in[0,T] \ \text{ holds,} \tag{3.27}$$

with $E_1 =: \frac{1}{2}\int_0^1 (u+\rho_x r^2)^2(x,0)dx + \frac{1}{\gamma-1}\int_0^1 \rho_0^{\gamma-1}(x)dx + \frac{1}{3}\rho_0^\gamma(1)a_0^3$, *and*

$$\rho(1,\tau) = \rho_0(a_0)(1+(\gamma-1)\rho_0^{\gamma-1}t)^{-\frac{1}{\gamma-1}}, \quad \tau\in[0,T], \tag{3.28}$$

$$E_0^{-\frac{1}{3(\gamma-1)}} \le a(\tau) \le C(1+\tau)^{\frac{\gamma}{3(\gamma-1)}}, \quad \tau\in[0,T]. \tag{3.29}$$

Proof. Differentiating Eq. $(3.11)_1$ with respect to x, rewriting it in the following form

$$\rho_{x\tau} = -[\rho^2(r^2u)_x]_x, \tag{3.30}$$

and substituting (3.30) into $(3.11)_2$, we have

$$r^2\rho_{x\tau} + 2\rho_x u r^{3-2} = -u_\tau - (\rho^\gamma)_x r^2.$$

Since $\frac{\partial r}{\partial \tau} = u$, the above equation can also be rewritten as

$$(u + r^2\rho_x)_\tau + (\rho^\gamma)_x r^2 = 0. \tag{3.31}$$

Multiplying (3.31) by $(u + r^2\rho_x)$, integrating the resulted equation over $[0,1]\times[0,\tau]$, we obtain after integration by parts that

$$\int_0^1 \left\{\frac{1}{2}(u+r^2\rho_x)^2 + \frac{\rho^{\gamma-1}}{\gamma-1}\right\}dx + \frac{4\gamma}{(\gamma+1)^2}\int_0^\tau\int_0^1 ((\rho^{\frac{\gamma+1}{2}})_x r^2)^2 dxds$$
$$= \int_0^1 \left\{\frac{1}{2}(u_0+r^2(\rho_0)_x)^2 + \frac{\rho_0^{\gamma-1}}{\gamma-1}\right\}dx - \int_0^\tau (\rho^\gamma u r^2)(1,s)ds. \tag{3.32}$$

It follows from (3.13) and Eq. $(3.11)_1$ that

$$\rho_\tau(1,\tau) + \rho^\gamma(1,\tau) = 0, \tag{3.33}$$

which yields (3.28) and $\rho(1,\tau) \le \rho_0(a_0)$. One may get from (3.32) and (3.33) that

$$-\int_0^\tau (\rho^\gamma u r^2)(1,t)dt = -\frac{1}{3}\int_0^\tau \{(\rho^\gamma r^3)_\tau - (\rho^\gamma)_\tau r^3\}(1,t)dt$$
$$= -\frac{\gamma}{3}\int_0^\tau \rho^{2\gamma-1}r^3(1,t)dt - \frac{1}{3}\rho^\gamma r^3(1,\tau) + \frac{1}{3}\rho^\gamma r^3(a_0,0). \tag{3.34}$$

Substituting (3.34) into (3.32) leads to (3.27). One can deduce from (3.27) and (3.28) that

$$a(\tau) \le C\rho^{-\gamma/3}(1,\tau) \le C(1+\tau)^{\frac{\gamma}{3(\gamma-1)}}, \tag{3.35}$$

which, together with (3.25), shows (3.29). □

Remark 3.6. All the estimates in Lemmas 3.3–3.5 hold for $N-$dimensional case. Indeed, the following entropy estimates hold

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma-1}\rho^{\gamma-1}\right)dx + (N-1)\int_0^\tau\int_0^1 \frac{u^2}{r^2}dxds + \int_0^\tau\int_0^1 \rho^2(r^{N-1}u_x)^2dxds$$
$$+(N-1)\int_0^\tau \rho u^2 r^{N-2}(a(s),s)ds = \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}\rho_0^{\gamma-1}\right)dx, \tag{3.36}$$

and

$$\int_0^1 \frac{1}{2}(u+\rho_x r^{N-1})^2dx + \frac{1}{\gamma-1}\int_0^1 \rho^{\gamma-1}dx + \frac{4\gamma}{(\gamma+1)^2}\int_0^\tau\int_0^1((\rho^{\frac{\gamma+1}{2}})_x r^2)^2dxds$$
$$+\frac{1}{N}\rho^\gamma(1,\tau)a^N(\tau) + \frac{\gamma}{N}\int_0^\tau \rho^{2\gamma-1}(1,s)a^N(s)ds$$
$$= \int_0^1 \frac{1}{2}(u+\rho_x r^{N-1})^2(x,0)dx + \frac{1}{\gamma-1}\int_0^1 \rho_0^{\gamma-1}(x,0)dx + \frac{1}{N}\rho_0^\gamma(1)a_0^N. \tag{3.37}$$

In addition, it holds

$$\rho(1,\tau) = \rho_0(a_0)(1+(\gamma-1)\rho_0^{\gamma-1}t)^{-\frac{1}{\gamma-1}}, \quad \tau\in[0,T], \tag{3.38}$$
$$E_0^{-\frac{1}{N(\gamma-1)}} \le a(\tau) \le C(1+\tau)^{\frac{\gamma}{N(\gamma-1)}}, \quad \tau\in[0,T]. \tag{3.39}$$

By Lemmas 3.4–3.5, we can establish an upper bound and long time behavior of the density for the global solution as follows.

Lemma 3.7. *Under the same assumptions as Lemma 3.3,*

$$0 \le \rho(x,\tau) \le Cx^{-\frac{2\gamma}{3(\gamma-1)}}, \quad (x,\tau)\in(0,1]\times[0,T], \tag{3.40}$$
$$0 \le \rho(x,\tau) \le C\varepsilon^{-2}, \quad (x,\tau)\in[0,1]\times[0,T] \quad \text{holds}, \tag{3.41}$$

with $C>0$ a constant, and

$$\rho(x,\tau)\to 0, \quad as \ \ \tau\to\infty, \tag{3.42}$$

for $x\in[x_0,1]$ with any fixed $x_0\in(0,1)$.

Proof. Collecting (3.23)–(3.24) and (3.27)–(3.28), we obtain for $(x,\tau)\in(0,1]\times[0,T]$ that

$$\rho(x,\tau) = \rho(1,\tau) - \int_x^1 \rho_y(y,\tau)dy \le \rho(1,\tau) + \int_x^1 r^{-2}|r^2\rho_y(y,\tau)|dy \tag{3.43}$$
$$\le C + Cx^{-\frac{2\gamma}{3(\gamma-1)}}\left(\int_0^1 \rho_x^2 r^4 dx\right)^{\frac{1}{2}} \le Cx^{-\frac{2\gamma}{3(\gamma-1)}}, \tag{3.44}$$

which yields (3.40). Equation (3.41) follows from (3.43), (3.28) and the fact

$$r(x,\tau)\geq\varepsilon>0,\quad \text{for all }\ (x,\tau)\in[0,1]\times[0,T]. \tag{3.45}$$

Next, we show the pointwise decay in time, (3.42), with the help of (3.21) and (3.27)–(3.28). Indeed, it holds for $x\in[x_0,1]$ with any fixed $x_0\in(0,1)$ that

$$\begin{aligned}|\rho^{\frac{\gamma+1}{2}}(x,\tau)-\rho^{\frac{\gamma+1}{2}}(1,\tau)| &\leq C\|\rho^{\frac{\gamma+1}{2}}(.,\tau)-\rho^{\frac{\gamma+1}{2}}(1,\tau)\|^{1/2}_{L^2([x_0,1])}\|(\rho^{\frac{\gamma+1}{2}})_x(\tau)\|^{1/2}_{L^2([x_0,1])}\\ &\leq C_{x_0}\|\rho^{\frac{\gamma+1}{2}}(.,\tau)-\rho^{\frac{\gamma+1}{2}}(1,\tau)\|^{1/2}_{L^2([x_0,1])},\end{aligned} \tag{3.46}$$

where $C_{x_0}>0$ is a constant. By (3.21), (3.23) and (3.27)-(3.28), one can verify for $g(t)=:\|\rho^{\frac{\gamma+1}{2}}(.,t)-\rho^{\frac{\gamma+1}{2}}(1,t)\|^2_{L^2([x_0,1])}$ that

$$\int_0^T g(t)dt\leq C\int_0^T\int_{x_0}^1(\rho^{\frac{\gamma+1}{2}})_x^2dxdt\leq Cx_0^{-\frac{4\gamma}{3(\gamma-1)}}\int_0^T\int_{x_0}^1(r^2(\rho^{\frac{\gamma+1}{2}})_x)^2dxdt\leq C_{x_0}, \tag{3.47}$$

and

$$\int_0^T|g'(t)|dt\leq C\int_0^T\int_{x_0}^1|(\rho^{\frac{\gamma+1}{2}}-\rho^{\frac{\gamma+1}{2}}(1,t))(\rho^{\frac{\gamma-1}{2}}\rho_\tau-\rho^{\frac{\gamma-1}{2}}(1,t)\rho_\tau(1,t))|dxdt\leq C_{x_0}.$$

Thus, by (3.46) and (3.28), we have

$$\rho(x,\tau)\leq C\rho(1,\tau)+C_{x_0}\|\rho^{\frac{\gamma+1}{2}}(.,\tau)-\rho^{\frac{\gamma+1}{2}}(1,\tau)\|^{\frac{1}{\gamma+1}}_{L^2([x_0,1])}\to0,\quad \text{as }\tau\to\infty, \tag{3.48}$$

for any $x\in[x_0,1]$ with fixed $x_0\in(0,1)$. The proof is complete. □

Lemma 3.8. *Under the same assumptions as Lemma 3.3, it holds that*

$$\int_0^1u^{2n}dx+\int_0^\tau\int_0^1\Big(\frac{u^{2n}}{r^2}+\rho^2u^{2n-2}u_x^2r^4\Big)dxds+2\int_0^\tau\rho u^{2n}r(1,s)ds\leq C_{n,\varepsilon,T}\delta_0, \tag{3.49}$$

for any integer $n\geq2$, where $C_{n,\varepsilon,T}>0$ is a constant.

Proof. Taking inner product of $(3.11)_2$ with r^2u^{2n-1} over $[0,1]$, using $(3.11)_1$ and the fact $(r^2u)_x(r^2u^{2n-1})_x=\frac{2^2u^{2n}}{\rho^2r^2}+(2n-1)u^{2n-2}(r^2u_x)^2+\frac{4nu^{2n-1}u_xr}{\rho}$, it follows that

$$\begin{aligned}&\frac{1}{2n}\frac{d}{d\tau}\int_0^1u^{2n}dx+2\int_0^1\frac{u^{2n}}{r^2}dx+(2n-1)\int_0^1\rho^2u^{2n-2}(r^2u_x)^2dx+2\rho u^{2n}r(1,\tau)\\ &\leq C_{\varepsilon,T}+\int_0^1\frac{u^{2n}}{r^2}dx+\frac{2n-1}{2}\int_0^1\rho^2u^{2n-2}(r^2u_x)^2dx+C\|\rho\|^{2(\gamma-1)}_{L^\infty}\int_0^1u^{2n}dx,\end{aligned}$$

which, by Gronwall's lemma and (3.41), yields (3.49) after integration over $[0,\tau]$. □

Lemma 3.9. *Under the same assumptions as Lemma 3.3, it holds for any $\gamma > 1 + \frac{1}{2n}$ with $n \in \mathbb{N}$ large enough and $C_{n,\varepsilon,T} > 0$ a constant that*

$$\int_0^\tau \|\rho^{2n(\gamma-1)}u^{2n}\|_{L^\infty([0,1])}ds \leq C_{n,\varepsilon,T}, \quad 0 < \tau < T, \tag{3.50}$$

$$\int_0^\tau [r^2(\rho^\gamma)_x(x,s)]^{2n}ds \leq C_{n,\varepsilon,T}, \quad x \in [0,1], \quad 0 < \tau < T. \tag{3.51}$$

Proof. Using the Sobolev imbedding theorem and the Cauchy-Schwartz inequality, by virtue of (3.21), (3.27), (3.41) and (3.45), we obtain

$$\begin{aligned}\|\rho^{2n(\gamma-1)}u^{2n}\|_{L^\infty([0,1])} &\leq \int_0^1 \rho^{2n(\gamma-1)}u^{2n}dx + \int_0^1 |(\rho^{2n(\gamma-1)}u^{2n})_x|dx \\ &\leq C_{n,\varepsilon,T}\|(\rho_0,u_0)\|_{W^{1,\infty}([0,1])} + C_{n,\varepsilon,T}\int_0^1 \rho^2u^{2n-2}u_x^2r^4dx\end{aligned}$$

with $C_{n,\varepsilon,T} > 0$ a constant, and (3.49) yields (3.50) after integration over $[0,\tau]$.

Next, integrating (3.31) over $[0,\tau]$ to get

$$r^2\rho_x(x,\tau) = r^2(x,0)\rho_{0x}(x) - \int_0^\tau (\rho^\gamma)_x r^2(x,s)ds - u(x,\tau) + u_0(x), \tag{3.52}$$

then it follows from (3.40), (3.50) and (3.52) that

$$\begin{aligned}\int_0^\tau [(\rho^\gamma)_x r^2]^{2n}ds &= \gamma^{2n}\int_0^\tau \rho^{2n(\gamma-1)}[r_0^2(\rho_0)_x - \int_0^s (\rho^\gamma)_x r^2 d\tau - u + u_0]^{2n}ds \\ &\leq C\int_0^\tau \rho^{2n(\gamma-1)}[(\rho_0)_x^{2n} + u^{2n} + u_0^{2n}]ds + C\int_0^t \rho^{2n(\gamma-1)}\int_0^s [(\rho^\gamma)_x r^2]^{2n}d\tau ds \\ &\leq C_{n,\varepsilon,T}\|(\rho_0,u_0)\|_{W^{1,\infty}} + C_{n,\varepsilon,T}\int_0^t\int_0^s [(\rho^\gamma)_x r^2]^{2n}d\tau ds,\end{aligned}$$

from which and Gronwall's Lemma, (3.51) follows. □

With the help of Lemmas 3.3–3.9, we are now able to obtain the lower and upper bounds of the density for the global solution (ρ, u, a) to the FBVP (3.11)–(3.15) as follows.

Lemma 3.10. *Under the same assumptions as Lemma 3.3, it holds that*

$$0 < c_{\varepsilon,T} \leq \rho(x,\tau) \leq C_\varepsilon, \quad (x,\tau) \in [0,1] \times [0,T], \tag{3.53}$$

with $C_{\varepsilon,T} > 0$ and $c_{\varepsilon,T} > 0$ being constants.

Proof. Set

$$v(x,\tau) = \frac{1}{r^2(x,\tau)\rho(x,\tau)}, \quad V(\tau) = \max_{[0,1]\times[0,\tau]} v(x,s). \tag{3.54}$$

By the facts $\frac{\partial r}{\partial \tau} = u$, $\frac{\partial r}{\partial x} = \frac{1}{\rho r^2}$, and $(3.11)_1$, it is easy to verify that for any $\beta > 1$,

$$(v^\beta)_\tau = \beta v^{\beta-1}\frac{(r^2u)_x}{r^2} - \frac{2\beta v^\beta u}{r}. \tag{3.55}$$

Integrating (3.55) over $[0, 1] \times [0, \tau]$ and using (3.13), (3.52) show

$$\begin{aligned}
&\int_0^1 v^\beta dx + \beta(\beta-1)\int_0^\tau\int_0^1 v^\beta u^2 dxds + \frac{\beta}{2\beta-3} r^{1-2(\beta-1)}\rho^{1-\beta}(1,\tau)\\
&= \int_0^1 v_0^\beta dx + \frac{\beta}{2\beta-3} r_0^{1-2(\beta-1)}\rho_0^{1-\beta}(1) + \frac{\beta(\beta-1)}{2\beta-3}\int_0^\tau a^{1-2(\beta-1)}(s)\rho^{\gamma-\beta}(1,s)ds\\
&\quad + \beta(\beta-1)\int_0^\tau\int_0^1 v^\beta u r_0^2(\rho_0)_x dxds + \beta(\beta-1)\int_0^\tau\int_0^1 v^\beta u u_0 dxds\\
&\quad - \beta(\beta-1)\int_0^\tau\int_0^1 v^\beta u\Big(\int_0^s(\rho^\gamma)_x r^2 d\tau\Big)dxds + 2\beta(\beta-1)\int_0^\tau\int_0^1 \frac{uv^\beta}{r}dxds\\
&=: \int_0^1 v_0^\beta dx + \frac{\beta}{2\beta-3} r_0^{3-2\beta}\rho_0^{1-\beta}(1) + I_0 + I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{3.56}$$

where $r_0 = r(x, 0)$. The right hand side terms of (3.56) can be estimated as follows. By (3.28) and (3.29), it holds that for $\beta \geq 2$,

$$I_0 = \frac{\beta(\beta-1)}{2\beta-3}\int_0^\tau a^{1-2(\beta-1)}(s)\rho^{\gamma-\beta}(1,s)ds \leq C\int_0^\tau (1+s)^{-\frac{\gamma-\beta}{\gamma-1}}ds \leq C_T. \tag{3.57}$$

It follows from Young's inequality that

$$I_1 + I_2 \leq \frac{\beta(\beta-1)}{6}\int_0^\tau\int_0^1 v^\beta u^2 dxds + C\int_0^\tau\int_0^1 v^\beta dxds, \tag{3.58}$$

$$I_4 \leq \frac{\beta(\beta-1)}{6}\int_0^\tau\int_0^1 v^\beta u^2 dxds + C\varepsilon^{-2}\int_0^\tau\int_0^1 v^\beta dxds. \tag{3.59}$$

Note that $(\int_0^\tau(\rho^\gamma)_x r^2 ds)^2 \leq C(\int_0^\tau[(\rho^\gamma)_x r^2]^{2n}ds)^{\frac{1}{n}} \leq C_{\varepsilon,T}$ for $n \geq 1$ due to (3.51), we can get

$$I_3 \leq \frac{\beta(\beta-1)}{6}\int_0^\tau\int_0^1 v^\beta u^2 dxds + C_{\varepsilon,T}\int_0^\tau\int_0^1 v^\beta dxds. \tag{3.60}$$

Substituting all estimates above into (3.56) yields for any $\beta > 2$,

$$\int_0^1 v^\beta dx \leq C_T + C_{\varepsilon,T}\int_0^t\int_0^1 v^\beta dxds, \tag{3.61}$$

which, together with (3.61) and Gronwall's inequality, gives for any $\beta > 2$ that

$$\int_0^1 v^\beta dx \leq C_{\varepsilon,T}, \quad \tau \in [0, T]. \tag{3.62}$$

Therefore, one can deduce from (3.61), (3.62) and (3.23) that for $\beta > 2$,

$$\begin{aligned}
V(T)^\beta &= \max_{[0,1]\times[0,T]} v^\beta(x,\tau) \leq \int_0^1 v^\beta dx + \int_0^1 |(v^\beta)_x| dx\\
&\leq \int_0^1 v^\beta dx + \beta\Big(\int_0^1 v^{2(\beta+1)}dx\Big)^{\frac{1}{2}} + C\varepsilon^{-1}\int_0^1 v^{\beta+1}dx \leq C_{\varepsilon,T}(1 + V(T)^{1+\frac{\beta}{2}}),
\end{aligned}$$

which shows for $\beta = 3$ that

$$V(\tau) \leq C_{\varepsilon,T}, \quad \tau \in [0, T].$$

This, together with (3.54), (3.23), (3.29) and (3.41), leads to (3.53). $\square$

Finally, we have the following higher order regularity estimates.

Lemma 3.11. *Under the same assumptions as Lemma 3.3, it holds that*

$$\int_0^1 u_x^2 dx + \int_0^T \int_0^1 (u_\tau^2 + u_x^2 + u_{xx}^2 + \rho_{x\tau}^2) dx d\tau$$
$$+ \int_0^T \int_0^1 (\rho^\gamma - \rho^2 (r^2 u)_x)_x^2 dx d\tau + \int_0^T (|a(\tau)|^2 + |a'(\tau)|^2) d\tau \leq C_{\varepsilon,T}\delta_0, \tag{3.63}$$
$$\int_0^1 (u_{xx}^2 + u_\tau^2) dx + \int_0^1 ((\rho^\gamma - \rho^2 (r^2 u)_x)^2 + (\rho^\gamma - \rho^2 (r^2 u)_x)_x^2) dx + \int_0^T \int_0^1 u_{x\tau}^2 dx d\tau$$
$$+ \int_0^T u_\tau^2 (1, \tau) d\tau + \int_0^T (|a(\tau)|^2 + |a'(\tau)|^2 + |a''(\tau)|^2) d\tau \leq C_{\varepsilon,T}\delta_1, \tag{3.64}$$

with $C_{\varepsilon,T} > 0$ a constant.

Proof. To prove (3.63), we re-write $(3.11)_2$ as

$$r^{-2} u_\tau + (\rho^\gamma - \rho^2 r^2 u_x)_x - \rho(\frac{2u}{r})_x = 0. \tag{3.65}$$

Taking the inner product of (3.65) with $\rho^{-2} u_\tau$ over $[0, 1] \times [0, \tau]$, and making use of (3.13), Lemma 3.3 and the following facts:

$$C_{\varepsilon,T} \leq r(x, \tau) \leq a(\tau) \leq C_T, \quad C_{\varepsilon,T}^{-1} \leq \rho(x, \tau) \leq C_\varepsilon, \quad (x, \tau) \in [0, 1] \times [0, T], \tag{3.66}$$

derived from (3.23), (3.29) and (3.53), we can obtain after a tedious computation that

$$\int_0^1 r^2 u_x^2 dx + \int_0^\tau \int_0^1 \rho^{-2} r^{-2} u_\tau^2 dx ds + r^{-1} \rho^{-1} u^2 (1, \tau)$$
$$\leq C_{\varepsilon,T} \int_0^\tau (\|\rho^{\gamma-1} u(s)\|_{L^\infty} + \|[(\rho^\gamma)_x r^2(s)]^2\|_{L^\infty} + \|\rho_x r^2(s)\|_{L^2}^2) \int_0^1 r^2 u_x^2 dx ds,$$
$$+ C_{\varepsilon,T}\delta_0 + \int_0^\tau r^{-1} \rho^{-1} u^2 (1, s) d\tau,$$

which, together with Lemma 3.9 and (3.66), yields

$$\int_0^1 u_x^2 dx + \int_0^T \int_0^1 u_\tau^2 dx ds \leq C_{\varepsilon,T}\delta_0. \tag{3.67}$$

It follows from $(3.11)_2$, (3.66) and (3.67) that

$$\int_0^T \int_0^1 u_{xx}^2 dx d\tau \leq C_{\varepsilon,T} \int_0^T \int_0^1 (u_\tau^2 + u_x^2 + u^2 + \rho_x) dx d\tau \leq C_{\varepsilon,T}\delta_0. \tag{3.68}$$

The combination of (3.67), (3.68), $(3.11)_2$ and (3.15) leads to (3.63).

Next, we prove (3.64). Differentiating (3.65) with respect to τ gives

$$r^{-2}u_{\tau\tau} - 2r^{-3}uu_{\tau} + (\rho^{\gamma} - \rho^2 r^2 u_x)_{x\tau} + \left(\rho\left(\frac{2u}{r}\right)_x\right)_{\tau} = 0. \tag{3.69}$$

Taking the inner product of (3.69) with u_τ over $[0, 1]$ and making use of (3.13), Lemma 3.3, Lemma 3.5, (3.63) and (3.66), one may get after a complicated computation that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 r^{-2}u_\tau^2 dx + \frac{1}{2}\int_0^1 \rho^2 r^2 u_{x\tau}^2 dx + \rho\frac{u_\tau^2}{r}(1,\tau) \le C_{\varepsilon,T} + C_{\varepsilon,T}\int_0^1 (r^{-2}u_\tau^2 + u_{xx}^2)dx. \tag{3.70}$$

Applying Gronwall's inequality to (3.70) and using (3.21), (3.23), (3.29) and (3.53) show that

$$\int_0^1 u_\tau^2 dx + \int_0^T\int_0^1 u_{x\tau}^2 dx d\tau + \int_0^T u_\tau^2(1,\tau)d\tau \le C_{\varepsilon,T}\delta_1. \tag{3.71}$$

Furthermore, it follows from $(3.11)_2$, (3.71), (3.66), Lemma 3.3 and Lemma 3.5 that

$$\int_0^1 u_{xx}^2 dx + \int_0^1 (\rho^{\gamma} - \rho^2(r^2u)_x)_x^2 dx \le C_{\varepsilon,T}\int_0^1 (u_\tau^2 + \rho_x^2 + u^2)dx \le C_{\varepsilon,T}\delta_1.$$

The combination of (3.71)–(3.72), (3.63) and (3.15) leads to (3.64). The proof is completed. □

The proofs of Propositions 3.1–3.2. With the help of Lemmas 3.3–3.10, Proposition 3.2 can be proved quite easily in terms of short time existence, a-priori estimates, and a continuity argument. Indeed, the short time existence of the unique classical solution (ρ, u, a) to the FBVP (3.11)–(3.15) under the assumptions of Proposition 3.2 can be shown by the standard argument as in [26]. By the a-priori estimates established in Lemmas 3.3–3.11 for (ρ, u, a) and a continuity argument, we show that it is indeed a global classical solution to the FBVP (3.11)–(3.13) satisfying (3.17)–(3.20).

The proof of Proposition 3.1 follows from Proposition 3.2 and the coordinates transform (3.9)–(3.10). The proofs are completed.

4. Uniform Estimates Away from Symmetry Center

This section is devoted to the proof of the uniform Lagrangian properties of the approximate global solutions to the FBVP (2.3) and (3.1)–(3.2) constructed in Sect. 3. As shown in Sect. 5, these properties can be maintained for the global approximate solutions to the original FBVP (2.1)–(2.7) and thus hold also for the global weak solution to FBVP (2.1)–(2.7) after passing into the limit in the approximate solution sequences.

4.1. Uniformly localized spatial estimates. In this sub-section, we derive some desired uniform estimates for (ρ, u, a) to the modified FBVP (2.3) and (3.1)–(3.2) as follows.

Proposition 4.1. *Let $T > 0$ and $\gamma \geq 2$. In addition to the assumptions of Proposition 3.1, assume further that for $0 < \varepsilon < r_1^- < r_2^+ \leq a_0$ and a constant $\rho_* > 0$,*

$$(\rho_0, u_0) \in W^{1,\infty}([r_1^-, r_2^+]), \qquad \inf_{r\in[r_1^-, r_2^+]} \rho_0(r) \geq \rho_* > 0 \quad \text{holds.} \tag{4.1}$$

Then the solution (ρ, u, a) to the FBVP (2.3) and (3.1)-(3.2) satisfies the following additional properties:

(i) (**Non-formation of vacuum state**) *For any $r_0 \in [r_1, r_2] \subset (r_1^-, r_2^+)$, there exists a unique particle path $r = r_{x_0}(t)$ with $r_{x_0}(0) = r_0$ and $x_0 = 1 - \int_{r_0}^{a_0} r^2\rho_0(r)dr \in (0, 1]$, such that*

$$\begin{cases} 0 < cx_0^{\frac{\gamma}{3(\gamma-1)}} \leq r_{x_0}(t) \leq a(t) \leq C_T, \quad t \in [0, T], \\ 0 < c_{x_0,T} \leq \rho(r_{x_0}(t), t) \leq Cx^{-\frac{2\gamma}{3(\gamma-1)}}, \quad t \in [0, T], \end{cases} \tag{4.2}$$

where $c_{x_0,T} > 0$ satisfying $c_{x_0,T} \to 0$ as $x_0 \to 0_+$. In particular, for any $r_1 \leq r_3 < r_0 \leq r_2$,

$$c(x_i - x_j)^{\frac{\gamma}{\gamma-1}} \leq r_{x_i}^3(t) - r_{x_j}^3(t), \quad i, j \in \{0, 1, 2, 3\}, \quad x_i < x_j, \quad t \in [0, T], \tag{4.3}$$

$$0 < c_{x_i,T} \leq \rho(r_{x_i}(t), t) \leq Cx_i^{-\frac{2\gamma}{3(\gamma-1)}}, \quad t \in [0, T], \tag{4.4}$$

$$0 < c_{x_1,T} \leq \rho(r, t) \leq Cx_1^{-\frac{2\gamma}{3(\gamma-1)}}, \quad r \in [r_{x_1}(t),\ r_{x_2}(t)], \quad t \in [0, T], \tag{4.5}$$

$$\|(\rho, u)\|_{L^\infty(0,T;H^1([r_{x_1}(t),r_{x_2}(t)]))} + \|u\|_{L^2(0,T;H^2([r_{x_1}(t), r_{x_2}(t)]))} \leq C_{x_1,T}\lambda_3 \quad \text{holds,} \tag{4.6}$$

where $r = r_{x_i}(t)$ is the particle path with $r_{x_i}(0) = r_i$ and $x_i = 1 - \int_{x_i}^{a_0} r^2\rho_0(r)dr$, $i = 1, 2, 3$, and $\lambda_3 =: \|(\rho_0, u_0)\|_{W^{1,\infty}([r_1^-, r_2^+])}$. The constants $c_{x_i,T} > 0$ and $C_{x_1,T} > 0$ satisfy $c_{x_i,T} \to 0$ as $x_i \to 0_+$ and $C_{x_1,T} \to \infty$ as $x_1 \to 0_+$.

(ii) (**Interior regularity**) *Assume that $r_2^+ < a_0$ in (4.1) and $u_0 \in H^2([r_1^-, r_2^+])$. Then (ρ, u) possess the following interior regularities*

$$\begin{cases} (\rho, u) \in C^0([r_{x_1}(t), r_{x_2}(t)] \times [0, T]), \\ \rho \in L^\infty(0, T; H^1([r_{x_1}(t), r_{x_2}(t)])), \quad \rho_t \in L^\infty(0, T; L^2([r_{x_1}(t), r_{x_2}(t)])), \\ u \in L^\infty(0, T; H^2([r_{x_1}(t), r_{x_2}(t)])) \cap L^2(0, T; H^3([r_{x_1}(t), r_{x_2}(t)])), \\ u_t \in L^\infty(0, T; L^2([r_{x_1}(t), r_{x_2}(t)])) \cap L^2(0, T; H^1([r_{x_1}(t), r_{x_2}(t)])), \end{cases} \tag{4.7}$$

and

$$\|(\rho_r,\ u_r, u_{rr}, u_t)(t)\|^2_{L^2([r_{x_1}(t),\, r_{x_2}(t)])} + \int_0^T \|(\rho_r, \rho_{tr})(s)\|^2_{L^2([r_{x_1}(s),\, r_{x_2}(s)])} ds$$
$$+ \int_0^T (\|(u_r, u_{xx}, u_t)(s)\|^2_{L^2([r_{x_1}(s),\, r_{x_2}(s)])} + |(a, a')(s)|^2) ds \le C_{x_1,T}\lambda_4, \quad (4.8)$$

with $\lambda_4 =: \|\rho_0\|_{W^{1,\infty}([r_1^-, r_2^+])} + \|u_0\|_{H^2([r_1^-, r_2^+])}$.

(iii) (**Boundary regularity**) *Assume that* $r_2^+ = a_0$ *in* (4.1) *and* $u_0 \in H^2([r_1^-, a_0])$ *hold. Then,* (ρ, u, a) *satisfies the following boundary regularities*

$$\begin{cases} 0 < c_{x_1,T} \le \rho(r,t) \le C x_1^{-\frac{2\gamma}{3(\gamma-1)}}, \quad (r,t) \in [r_{x_1}(t),\ a(t)] \times [0,T], \\ (\rho, u) \in C^0([r_{x_1}(t), a(t)] \times [0,T]), \\ \rho \in L^\infty(0,T; H^1([r_{x_1}(t), a(t)])), \quad \rho_t \in L^\infty(0,T; L^2([r_{x_1}(t), a(t)])), \\ u \in L^\infty(0,T; H^2([r_{x_1}(t), a(t)])) \cap L^2(0,T; H^3([r_{x_1}(t), a(t)])), \\ u_t \in L^\infty(0,T; L^2([r_{x_1}(t), a(t)])) \cap L^2(0,T; H^1([r_{x_1}(t), a(t)])), \\ \rho^\gamma - \rho u_r \in C^0([0,T] \times [r_{x_1}(t), a(t)]) \cap L^\infty(0,T; H^1([r_{x_1}(t), a(t)])), \\ a(t) \in H^2([0,T]) \cap C^1([0,T]), \end{cases} \quad (4.9)$$

and (4.5) holds with $x_2 = a_0$*, and*

$$\|(\rho_r, u_r, u_{rr},\ u_t)(t)\|^2_{L^2([r_{x_1}(t),\, a(t)])} + \|(\rho^\gamma - \rho u_r)_r(t)\|_{L^2([r_{x_1}(t),\, a(t)])}$$
$$+ \int_0^T (\|u_t(\tau)\|^2_{H^1([r_{x_1}(s),\, a(s)])} + |(a, a', a'')(s)|^2) ds \le C_{x_1,T}\lambda_5, \quad (4.10)$$

with $\lambda_5 =: \|\rho_0\|_{W^{1,\infty}([r_1^-,\, a_0])} + \|u_0\|_{H^2([r_1^-,\, a_0])}$.

4.2. The uniform a-priori estimates. In order to prove Proposition 4.1, it suffices to establish the corresponding uniform estimates for the reformulated FBVP (3.11)–(3.15) in Lagrange coordinates, which together with the coordinates transformation (3.9)–(3.10) gives rise to the expected estimates in Proposition 4.1.

We start with

Lemma 4.2. *Let* $T > 0$ *and* $\gamma \ge 2$*. Let* (ρ, u, a) *with* $\rho(x,\tau) > 0$ *be the solution to FBVP* (3.11)–(3.15) *for* $(x,\tau) \in [0,1] \times [0,T]$ *constructed in Proposition* 3.2*. Assume further that*

$$(\rho_0, u_0) \in W^{1,\infty}([x_1^-, x_2^+]), \quad \rho_0(x) \ge \rho_* > 0,\ x \in [x_1^-, x_2^+], \quad (4.11)$$

for $0 < x_1^- < x_2^+ \le 1$ *and* ρ_* *constant.*

Then (ρ, u, a) *satisfies*

$$\int_0^\tau [\rho^{\gamma-1}u]^2(x,s)ds \le C_{x_1,T,1}, \quad (x,\tau) \in [x_1, x_2] \times [0,T], \quad (4.12)$$

$$\int_0^\tau [(\rho^\gamma)_x r^2]^2(x,s)ds \le C_{x_1,T,2}(1+\delta_2), \quad (x,\tau) \in [x_1, x_2] \times [0,T], \quad (4.13)$$

for any $[x_1, x_2]$ *satisfying that either* $x_1^- < x_1 < x_2 < x_2^+$ *in the case* $x_2^+ < 1$*, or* $x_1^- < x_1 < x_2 \le 1$ *in the case* $x_2^+ = 1$*, where* $\delta_2 =: \|(\rho_0, u_0)\|_{W^{1,\infty}([x_1^-, x_2^+])}$*, and* $C_{x_1,T,i} > 0$ *satisfying* $C_{x,T,i} \to \infty$ *as* $x_i \to 0_+$.

Proof. We only show (4.12)–(4.13) for $x_2^+ = 1$, the case $x_2^+ < 1$ can be proved similarly. It follows from Lemma 3.4 and Lemma 3.7 that for any $(x, \tau) \in [x_1, 1] \times [0, T]$,

$$\rho(x, \tau) \le C x_1^{-\frac{2\gamma}{3(\gamma-1)}}, \quad c x_1^{\frac{\gamma}{3(\gamma-1)}} \le r(x, \tau) \le C_T. \tag{4.14}$$

Then, by virtue of (3.21), (3.27), for any $\gamma \ge 2$, one can show that for $x \in [x_1, 1]$,

$$\begin{aligned} |\rho^{\gamma-1} u(x, \tau)| &\le \frac{1}{1 - x_1} \int_{x_1}^1 \rho^{\gamma-1} |u|(x, \tau) dx + \int_{x_1}^1 |(\rho^{\gamma-1} u)_x| dx \\ &\le C_T x_1^{-\frac{2\gamma}{3}} + C_T x_1^{-\frac{2\gamma}{3}} (\|\rho_x r^2\|_{L^2([0,1])} \|u\|_{L^2([0,1])} + \|\rho u_x r^2\|_{L^2([0,1])}) \le C_T x_1^{-\frac{4\gamma}{3}}, \end{aligned} \tag{4.15}$$

which leads to (4.12) with $C_T > 0$.

Next, one deduces from (4.12), (4.14) and (3.52) that for any $(x, \tau) \in [x_1, x_2] \times [0, T]$,

$$\begin{aligned} \int_0^\tau [(\rho^\gamma)_x r^2]^2(x, s) ds &= \gamma^2 \int_0^\tau \rho^{2(\gamma-1)} [r^2(x, 0) \rho_{0x}(x) - \int_0^s (\rho^\gamma)_x r^2 d\tau - u + u_0]^2 ds \\ &\le C_T x_1^{-\frac{4\gamma}{3}} + C_T x_1^{-\frac{4\gamma}{3}} \delta_2 + C_T x_1^{-\frac{4\gamma}{3}} \int_0^\tau \int_0^s [(\rho^\gamma)_x r^2]^2(x, z) dz ds, \end{aligned} \tag{4.16}$$

which implies (4.13) by Gronwall's inequality.

Lemma 4.3. *Under the assumptions of Lemma 4.2,*

$$c_{x_1,T} \le \rho(x, \tau) \le C x^{-\frac{2\gamma}{3(\gamma-1)}}, \quad (x, \tau) \in [x_1, 1] \times [0, T] \text{ holds}, \tag{4.17}$$

in the case $x_2^+ = 1$ *in* (4.11)*, while for* $x_2^+ < 1$ *in* (4.11)*, then*

$$c_{x_1,T} \le \rho(x, \tau) \le C x^{-\frac{2\gamma}{3(\gamma-1)}}, \quad (x, \tau) \in [x_1, x_2] \times [0, T], \tag{4.18}$$

with $[x_1, x_2] \subset (x_1^-, x_2^+)$*, where* $C > 0$ *and* $c_{x_1,T}$ *are constants such that* $c_{x_1,T} \to 0$ *as* $x_1 \to 0_+$.

Proof. We only prove (4.17). First, we consider the mass transportation in Eulerian coordinates and Lagrangian coordinates respectively. Without loss of generality, we assume that $\rho_0(r) > 0$ for $r \in [\varepsilon, a_0]$ with $\int \rho_0(r) r^2 dr = 1$ in Eulerian coordinates, namely, $\rho_0(x) > 0$ for $x \in [0, 1]$ in Lagrangian coordinates. Then, for any constant $\eta \in (1/2, 1)$ with $1 - \eta$ small enough so that $x_5 =: \eta x_1 > x_1^-$, we define particle paths $r_{x_i}(t) = r(x_i, t)$ as

$$\frac{d}{dt} r_{x_i}(t) = u(r_{x_i}(t), t), \quad r_{x_i}(0) = r_i \in (\varepsilon, a_0), \quad i = 1, 5, \tag{4.19}$$

where $r_i = (a_0^3 - 3\int_{x_i}^1 \rho_0^{-1} dy)^{\frac{1}{3}}$ satisfies $\varepsilon < r_5 < r_1 < a_0$ due to (3.10). The conservation of mass between the particle paths $r_{x_5}(t)$ and $r_{x_1}(t)$ implies the existence of $m_{5,1} = x_1 - x_5 = (1-\eta)x_1$ so that

$$\int_{r_{x_5}(t)}^{r_{x_1}(t)} r^2 \rho(r,t) dr = \int_{r_5}^{r_1} r^2 \rho_0(r) dr = m_{5,1} = x_1 - x_5 > \frac{1}{2} x_1, \quad t \geq 0. \tag{4.20}$$

Thus, there is a particle path $r = r_{x_3}(t) \in [r_{x_5}(t), r_{x_1}(t)]$ for $t \in [0, T]$ defined by

$$\frac{d}{dt} r_{x_3}(t) = u(r_{x_3}(t), t), \quad t > 0, \quad r_{x_3}(0) = r_3 \in [r_5, r_1], \tag{4.21}$$

with $u_{0r}(r_3) = \frac{u_0(r_1) - u_0(r_5)}{r_1 - r_5}$ and $x_3 = 1 - \int_{r_3}^{a_0} r^2 \rho_0(r) dr \in [x_5, x_1]$, so that it holds in Eulerian coordinates

$$r_{x_3}^2(t) \rho(r_{x_3}(t), t) = \frac{1}{r_{x_1}(t) - r_{x_5}(t)} \int_{r_{x_5}(t)}^{r_{x_1}(t)} r^2 \rho(r,t) dr = \frac{x_1 - x_5}{r_{x_1}(t) - r_{x_5}(t)}, \quad t \geq 0, \tag{4.22}$$

and in Lagrangian coordinates

$$(r^2\rho)^{-1}(x_3, \tau) = \frac{r(x_1, \tau) - r(x_5, \tau)}{x_1 - x_5}, \quad \tau \geq 0. \tag{4.23}$$

Similar to (3.54), one can define

$$v(x, \tau) = \frac{1}{r^2(x,\tau)\rho(x,\tau)}, \quad V(\tau) = \max_{[x_3, 1] \times [0, \tau]} v(x, s) \tag{4.24}$$

for any $(x, \tau) \in [x_3, 1] \times [0, T]$.

It is easy to verify that v satisfies (3.55). Integrating (3.55) over $[0, \tau] \times [x_3, 1]$, and using (3.13), (4.21) and (3.52), we get after a complicated but straightforward computation that

$$\begin{aligned}
&\int_{x_3}^1 v^\beta dx + \beta(\beta-1) \int_0^\tau \int_{x_3}^1 v^\beta u^2 dx ds \\
&= \int_{x_3}^1 v_0^\beta dx + \beta(\beta-1) \int_0^\tau \int_{x_3}^1 v^\beta u r^2 \rho_x(x, 0) dx ds \\
&\quad + \beta(\beta-1) \int_0^\tau \int_{x_3}^1 v^\beta u u_0 dx ds - \beta(\beta-1) \int_0^\tau \int_{x_3}^1 v^\beta u (\int_0^s (\rho^\gamma)_x r^2 dt) dx ds \\
&\quad + 2\beta(\beta-1) \int_0^\tau \int_{x_3}^1 v^\beta \frac{u}{r} dx ds + \beta \int_0^\tau v^{\beta-1} u(y,s)|_{y=x_3}^1 ds \\
&= \int_{x_c(0)}^1 v_0^\beta dx + I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{4.25}$$

The right-hand side terms of (4.25) can be estimated as follows. Similar to (3.58)–(3.59), we have

$$I_1 + I_2 \le \frac{\beta(\beta-1)}{3}\int_0^\tau\int_{x_3}^1 v^\beta u^2 dxds + C\int_0^\tau\int_{x_3}^1 v^\beta dxds, \tag{4.26}$$

$$I_3 + I_4 \le \frac{\beta(\beta-1)}{6}\int_0^\tau\int_{x_3}^1 v^\beta u^2 dxds + C_{x_1}\int_0^\tau\int_{x_3}^1 v^\beta dxds, \tag{4.27}$$

by virtue of (4.13) and the fact that $r(x,\tau) \ge cx^{\frac{\gamma}{3(\gamma-1)}} \ge cx_1^{\frac{\gamma}{3(\gamma-1)}}$ for $x \in [x_3, 1] \subset [x_5, x_1]$.

One can show by (3.29) and (3.28) that

$$v(1,\tau) = r^{-2}(1,\tau)\rho^{-1}(1,\tau) \le C_T, \quad \tau \in [0,T], \tag{4.28}$$

and by (4.23) and the fact, $r(x_5,\tau) \le r(x_1,\tau) \le a(\tau) \le C_T$, that

$$v(x_3,\tau) = (r^2\rho(x_3,\tau))^{-1} = \frac{r(x_1,t) - r(x_5,t)}{x_1 - x_5} \le C_T x_1^{-1}, \quad \tau \in [0,T]. \tag{4.29}$$

Due to (3.21) and the Sobolev embedding theorem, it holds that

$$\int_0^T \|u\|^2_{L^\infty([x_3,1])} ds \le \frac{C}{1-x_1}\int_0^T\int_{x_3}^1 u^2 dxds + C\int_0^T\int_{x_3}^1 |uu_x| dxds \tag{4.30}$$

$$\le C_T + C_T\Big(\int_0^T\int_0^1 \rho^2 u_x^2 r^4 dxds\Big)^{\frac{1}{2}}\Big(\int_0^T\int_{x_3}^1 v^2u^2 dxds\Big)^{\frac{1}{2}}$$

$$\le C_T(1 + V(T)). \tag{4.31}$$

One concludes from (4.28), (4.29) and (4.31) that for any $\beta \ge 2$,

$$I_5 \le \beta\int_0^T |v^{\beta-1}u(x_3,\tau)| d\tau + \beta\int_0^T |v^{\beta-1}u(1,\tau)| d\tau \le C_T x_1^{-(\beta-1)}(1 + V(T)^{\frac{1}{2}}). \tag{4.32}$$

Substituting (4.26), (4.27), and (4.32) into (4.25) and using Young's inequality yield that for any $\beta \ge 2$,

$$\int_{x_3}^1 v^\beta dx \le C_{x_1,T,4}(1 + V(T)^{\frac{1}{2}}), \quad \tau \in [0,T], \tag{4.33}$$

with $C_{x_1,T,4} > 0$ satisfying $C_{x_1,T,4} \to \infty$ as $x_1 \to 0_+$.

Finally, by Sobolev imbedding, (4.33) and (4.28), we deduce that for $\beta > 3$,

$$V(T)^\beta = \max_{[x_3,1]\times[0,T]} v^\beta(x,\tau) \le v^\beta(1,\tau) + \int_{x_3}^1 |(v^\beta)_x| dx$$

$$\le C_T + \beta\Big(\int_{x_3}^1 v^{2(\beta+1)} dx\Big)^{\frac{1}{2}} + Cx_1^{-1}\int_{x_3}^1 v^{\beta+1} dx \le C_{x_1,T,5}(1 + V(T)^{\frac{\beta+3}{2}}),$$

with $C_{x_1,T,5} > 0$ satisfying $C_{x_1,T,4} \to \infty$ as $x_1 \to 0_+$, which implies in particular that for $\beta = 4$,

$$V(T) \le C_{x_1,T,0}, \tag{4.34}$$

where $C_{x_1,T,0} > 0$ satisfying $C_{x_1,T,0} \to \infty$ as $x_1 \to 0_+$. The combination of (4.24), (4.34), (3.23) and (3.29) yields (4.17).

Furthermore, repeating the above arguments with few modifications on the domain $[(1-\eta)x_1, (1+\eta)x_2] \times [0, T]$ with $\eta \in (0, 1)$ a constant small enough, we can prove (4.18). The details are omitted here. The proof of Lemma 4.3 is complete. □

With the help of Lemma 4.3, we can further establish the higher order regularities of the global solution (ρ, u, r) as follows.

Lemma 4.4. *Under the assumptions of Lemma 4.2, it holds for $x_2^+ = 1$ in (4.11),*

$$\sup_{\tau\in[0,T]} \|(\rho_x, u_x)(\tau)\|^2_{L^2([x_1,x_2])} + \int_0^T \|(\rho_x, u_x, u_{xx}, u_\tau, \sigma_x)(\tau)\|^2_{L^2([x_1,x_2])} d\tau \le C_{x_1,T}\delta_4, \tag{4.35}$$

$$\int_0^T (|a(\tau)|^2 + |a'(\tau)|^2) d\tau \le C_{x_1,T}\delta_4, \tag{4.36}$$

for any $[x_1, x_2] \subset (x_1^-, 1]$, where $\sigma = \rho^\gamma - \rho^2(r^2u)_x$. Assume further that $u_0 \in H^2([x_1^-, 1])$. Then

$$\sup_{\tau\in[0,T]} \|(\rho_x, u_x, u_{xx}, u_\tau, \sigma_x)(\tau)\|^2_{L^2([x_1,x_2])} + \int_0^T \|u_{x\tau}(\tau)\|^2_{L^2([x_1,x_2])} d\tau \le C_{x_1,T}\delta_5, \tag{4.37}$$

$$\int_0^T (|a(\tau)|^2 + |a'(\tau)|^2 + |a''(\tau)|^2) d\tau \le C_{x_1,T}\delta_5, \tag{4.38}$$

with $C_{x_1,T} > 0$ independent of $\varepsilon > 0$, $\delta_4 =: \|\rho_0\|_{W^{1,\infty}([x_1^-,x_2^+])} + \|u_0\|_{H^1([x_1^-,x_2^+])}$, and $\delta_5 =: \|\rho_0\|_{W^{1,\infty}([x_1^-,x_2^+])} + \|u_0\|_{H^2([x_1^-,x_2^+])}$.

In addition, (4.35) and (4.37) hold on any interior domain $[x_1, x_2] \subset (x_1^-, x_2^+)$ in the case that $x_2^+ < 1$ in (4.11).

Proof. We only show (4.35)–(4.38) for the case $x_2^+ = 1$, the other case can be proven similarly. Rewrite $(3.11)_2$ as

$$r^{-2}u_\tau + (\rho^\gamma - \rho^2 r^2 u_x)_x - \rho\left(\frac{2u}{r}\right)_x = 0. \tag{3.65}$$

Take the inner product of (3.65) with $\phi^2\rho^{-2}u_\tau$, where $\phi = \psi^2(x)$ and $\psi \in C^\infty([0, 1])$ satisfies $0 \le \psi(x) \le 1$, $\psi(x) = 1$ for $x \in [(1-\eta)x_1, 1]$, and $\psi(x) = 0$ for $x \in [0, 1-2\eta]$ with $\eta \in (0, 1)$ small enough so that $(1-2\eta)x_1 > x_1^-$. It follows from (3.13) that

$$\frac{d}{dt}\int_0^1 (\frac{1}{2}\phi r^2 u_x^2 - \phi\rho^{\gamma-2}u_x)dx + \int_0^1 \phi\rho^{-2}r^{-2}u_\tau^2 dx + \frac{d}{d\tau}(r^{-1}\rho^{-1}u^2(1,\tau))$$
$$= \int_0^1 \phi(ruu_x^2 - (\rho^{\gamma-2})_\tau u_x)dx + \int_0^1 (\rho^\gamma - \rho^2 r^2 u_x)(\phi\rho^{-2})_x u_\tau dx + (r^{-1}\rho^{-1})_\tau u^2(1,\tau). \tag{4.39}$$

Integrating (4.39) over $[0, \tau]$, making use of Lemmas 3.3–3.5, Lemma 4.2 and the following facts:

$$c_{x_1} \le r(x,\tau) \le C_T, \quad c_{x_1,T} \le \rho(x,\tau) \le C_{x_1}, \quad (x,\tau) \in [(1-2\eta)x_1, 1] \times [0,T], \tag{4.40}$$

derived from (3.23), (3.29) and (4.17), we obtain after a tedious computation that

$$\begin{aligned}&\int_0^1 \phi r^2 u_x^2 dx + \int_0^\tau \int_0^1 \phi \rho^{-2} r^{-2} u_\tau^2 dx ds \\ &\quad \le C_{x_1,T}\delta_4 + C_{x_1,T} \int_0^\tau \|(\rho^{\gamma-1}u, [(\rho^\gamma)_x r^2]^2)(s)\|_{L^\infty_{([(1-2\eta)x_1,1])}} \int_0^1 \phi r^2 u_x^2 dx ds.\end{aligned} \tag{4.41}$$

This, together with Lemma 4.2, (4.40) and the fact that $\phi(x) = 1$ for $x \in [(1-\eta)x_1, 1]$, yields

$$\int_{(1-\eta)x_1}^1 u_x^2 dx + \int_0^T \int_{(1-\eta)x_1}^1 u_\tau^2 dx ds \le C_{x_1,T}\delta_4. \tag{4.42}$$

It follows from (4.42), $(3.11)_2$, (4.40), and Lemma 4.2 that

$$\int_0^T \int_{(1-\eta)x_1}^1 [u_{xx}^2 + (\rho^\gamma - \rho^2(r^2u)_x)_x^2 + \rho_{x\tau}^2] dx d\tau \le C_{x_1,T}\delta_4. \tag{4.43}$$

Now (4.35) and (4.36) follow from (4.42)–(4.43) and the definition (3.15) of the free boundary.

The higher order regularities of the solution near the free boundary can be shown by similar arguments as proving (3.64) with some modifications. Indeed, taking the inner product of (3.69) with ϕu_τ over $[0, 1]$, where $\phi = \psi^2(x)$ and $\psi \in C_0^\infty([0, 1])$ satisfies $0 \le \psi(x) \le 1$, $\psi(x) = 1$ for $x \in [x_1, 1]$, and $\psi(x) = 0$ for $x \in [0, (1-\eta)x_1]$ with $\eta > 0$ small enough so that $(1-2\eta)x_1 > x_1^-$, and using (3.13), we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\int_0^1 \phi r^{-2} u_\tau^2 dx - \int_0^1 \phi(\rho^\gamma - \rho^2 r^2 u_x)_\tau u_{x\tau} dx + \left(\rho\frac{2u}{r}\right)_\tau u_\tau(1,\tau) \\ &\quad = \int_0^1 \phi_x(\rho^\gamma - \rho^2 r^2 u_x)_\tau u_\tau dx + \int_0^1 \phi r^{-3} u u_\tau^2 dx + \int_0^1 \phi \left(\rho\left(\frac{2u}{r}\right)_x\right)_\tau u_\tau dx.\end{aligned} \tag{4.44}$$

By a similar argument for (3.63) and using (3.13), (4.35), (4.40), (4.42)–(4.43), we can obtain from (4.44) after a complicated computation that

$$\int_0^1 \phi(\rho^{\gamma+1}u_x^2 + r^{-2}u_\tau^2) dx + \int_0^T \int_0^1 \phi\rho^2 r^2 u_{x\tau}^2 dx d\tau + \int_0^T u_\tau^2(1,\tau) d\tau \le C_{x_1,T}\delta_5, \tag{4.45}$$

which with (4.40) imply that

$$\int_{x_1}^1 (u_x^2 + u_\tau^2) dx + \int_0^T \int_{x_1}^1 u_{x\tau}^2 dx d\tau + \int_0^T u_\tau^2(1,\tau) d\tau \le C_{x_1,T}\delta_5. \tag{4.46}$$

It follows from (4.46), $(3.11)_2$, (4.40) and Lemma 4.2 that

$$\int_{x_1}^{1} u_{xx}^2 dx + \int_{x_1}^{1} (\rho^\gamma - \rho^2 (r^2 u)_x)_x^2 dx \le C_{x_1,T}\delta_5. \tag{4.47}$$

The combination of (4.45)–(4.47), (4.35)–(4.36), (3.11), and (3.15) gives rise to (4.37)–(4.38). The proof of Lemma 4.4 is completed. □

The proof of Proposition 4.1. Proposition 4.1 follows from Proposition 3.1, the uniform estimates established in Lemmas 4.3–4.4 for the solution (ρ, u, a), and the coordinates transformation (3.9)–(3.10), the details are omitted.

5. Proof of the Main Results

This section is devoted to the proofs of Theorems 2.1–2.4. Indeed, we can construct global approximate solutions to the FBVP (2.1)–(2.7), establish uniform a-priori estimates based on Proposition 3.1 and Proposition 4.1, show their convergence to a solution of the original FBVP problem, and then justify the expected properties in Theorems 2.1–2.4 for the limiting solution.

5.1. Construction of global approximate solutions. For any $\varepsilon > 0$ fixed, we can modify the initial data (ρ_0, u_0) in (2.5) and construct a sequence of global approximate solutions $(\rho^\varepsilon, u^\varepsilon, a^\varepsilon)$ to the FBVP (2.3)–(2.7) as

$$(\rho^\varepsilon, u^\varepsilon) = \begin{cases} (\tilde{\rho}^{\varepsilon,\delta}, \tilde{u}^{\varepsilon,\delta})(r,t), & (r,t) \in [\varepsilon, a^\varepsilon(t)] \times [0,T], \\ (\tilde{\rho}^{\varepsilon,\delta}(\varepsilon,t), 0), & (r,t) \in [0,\varepsilon] \times [0,T], \end{cases} \tag{5.1}$$

respectively, where $a^\varepsilon(t) = \tilde{a}^{\varepsilon,\delta}(t)$ defined by

$$\frac{d}{dt}\tilde{a}^{\varepsilon,\delta}(t) = \tilde{u}^{\varepsilon,\delta}(\tilde{a}^{\varepsilon,\delta}(t), t), \quad \tilde{a}^{\varepsilon,\delta}(0) = a_0, \tag{5.2}$$

and $(\tilde{\rho}^{\varepsilon,\delta}, \tilde{u}^{\varepsilon,\delta}, \tilde{a}^{\varepsilon,\delta})$ is the unique global strong solution on $[\varepsilon, a^{\varepsilon,\delta}(t)] \times [0,T]$ to the modified FBVP (2.3) with following initial data and boundary conditions:

$$(\rho, u)(r, 0) = (\rho_0^\delta, u_0^\delta), \quad r \in [\varepsilon, a_0], \tag{5.3}$$

$$u^\delta(\varepsilon, t) = 0, \quad ((\rho^\delta)^\gamma - \rho^\delta u_r^\delta)(a(t), t) = 0, \quad t \ge 0, \tag{5.4}$$

so that initial data $(\rho_0^\delta, u_0^\delta)$ satisfies all assumptions in Proposition 3.1 and Proposition 4.1 on $[\varepsilon, a_0]$ (uniformly with respect to $\delta > 0$ and $\varepsilon > 0$) and the following properties:

$$\inf_{r\in[\varepsilon,a_0]} \rho_0^\delta(r) > 0, \int_\varepsilon^{a_0} r^2 \rho_0^\delta(r) dr = \int_\varepsilon^{a_0} r^2 \rho_0(r) dr; \quad u_0^\delta(r) = 0, r \in [0,\varepsilon],$$

and is well consistent with the boundary values (5.4). In particular, $(\rho_0^\delta, \rho_0^\delta u_0^\delta) \to (\rho_0, m_0)$ strongly in $W^{1,\infty}([\varepsilon, a_0])$, $\sqrt{\rho_0^\delta} \to \sqrt{\rho_0}$ strongly in $H^1([\varepsilon, a_0])$, and $\frac{(m_0^\delta)^{2+\eta}}{(\rho_0^\delta)^{1+\eta}} \to \frac{m_0^{2+\eta}}{\rho_0^{1+\eta}}$ strongly in $L^1(\Omega_0)$ as $\delta \to 0_+$. These can be carried out by the standard arguments as used in [11,26,29], we omit the details. Thus, by Proposition 3.1

and Proposition 4.1, the FBVP (2.3) and (5.3)–(5.4) admits a unique global strong solution $(\tilde{\rho}^{\varepsilon,\delta}, \tilde{u}^{\varepsilon,\delta}, \tilde{a}^{\varepsilon,\delta})$ on the domain $[\varepsilon, a^{\varepsilon,\delta}(t)] \times [0, T]$, which satisfies (3.4)–(3.7) and (4.2)–(4.10) (uniformly with respect to $\varepsilon > 0$ and $\delta > 0$), in particular, the free boundary $r = \tilde{a}^{\varepsilon,\delta}(t)$ is uniformly bounded from below and above with respect to $\varepsilon > 0$ and $\delta > 0$.

By (5.1)–(5.4), we obtain a sequence of spherically symmetric approximate solutions to the original FBVP (2.1)–(2.7) for any $\varepsilon > 0$ as

$$(\rho^\varepsilon(\mathbf{x}, t), \rho^\varepsilon \mathbf{U}^\varepsilon(\mathbf{x}, t), a^\varepsilon(t)) = (\rho^\varepsilon(|\mathbf{x}|, t), \rho^\varepsilon u^\varepsilon(\mathbf{x}, t)\frac{\mathbf{x}}{|\mathbf{x}|}, a^\varepsilon(t)), \quad (\mathbf{x}, t) \in \Omega_T^\varepsilon, \tag{5.5}$$

where $(\rho^\varepsilon, u^\varepsilon, a^\varepsilon)$ satisfies (5.1)–(5.4) with $\delta = o(\varepsilon^\eta)$ with $\eta > 0$ a constant (refer to [11,29] for details), and Ω_t^ε is given by

$$\Omega_t^\varepsilon =: \{(\mathbf{x}, s)|\ 0 \le |\mathbf{x}| \le a^\varepsilon(s),\ s \in [0, t]\}, \quad t \in (0, T].$$

In addition, it is easy to verify that $(\rho^\varepsilon, \mathbf{U}^\varepsilon)$ is differentiable with respect to $\mathbf{x}$ in terms of $(\rho^\varepsilon, u^\varepsilon)$ in the following sense:

$$(\partial_r\sqrt{\rho^\varepsilon}, \partial_r u^\varepsilon) = \begin{cases} (\partial_r\sqrt{\rho^\varepsilon}, \partial_r u^\varepsilon), & r \in (\varepsilon, a^\varepsilon(t)], \\ (0, 0), & r \in [0, \varepsilon]. \end{cases} \tag{5.6}$$

Similar to [11,29], we can show that the solution $(\rho^\varepsilon(\mathbf{x}, t), \mathbf{U}^\varepsilon(\mathbf{x}, t), a^\varepsilon(t))$ also satisfies the following estimates with respect to $\varepsilon > 0$.

Lemma 5.1. *Under the assumptions of Theorem 2.1,* $(\rho^\varepsilon, \mathbf{U}^\varepsilon, a^\varepsilon)$ *satisfies that*

$$\int_{\Omega_t^\varepsilon} \rho^\varepsilon(\mathbf{x}, t)d\mathbf{x} = \int_{\Omega_0^\varepsilon} \rho_0^\varepsilon(\mathbf{x})d\mathbf{x}, \tag{5.7}$$

$$\sup_{t\in[0,T]} \int_{\Omega_t^\varepsilon} (|\sqrt{\rho^\varepsilon}\,\mathbf{U}^\varepsilon|^2 + (\rho^\varepsilon)^\gamma)(\mathbf{x}, t)d\mathbf{x} + \int_0^T \int_{\Omega_t^\varepsilon} |\sqrt{\rho^\varepsilon}\,\nabla\mathbf{U}^\varepsilon|^2 d\mathbf{x}dt \le C, \tag{5.8}$$

$$\sup_{t\in[0,T]} \int_{\Omega_t^\varepsilon} |\nabla\sqrt{\rho^\varepsilon}|^2(\mathbf{x}, t)d\mathbf{x} + \int_0^T \int_{\Omega_t^\varepsilon} (|\nabla(\rho^\varepsilon)^{\frac{\gamma}{2}}|^2 + (\rho^\varepsilon)^{\frac{5\gamma}{3}})d\mathbf{x}dt \le C, \tag{5.9}$$

$$\sup_{t\in[0,T]} \int_{\Omega_t^\varepsilon} \rho^\varepsilon|\mathbf{U}^\varepsilon|^{2+\nu}(\mathbf{x}, t)dx \le C_T, \tag{5.10}$$

where $C > 0$ and $C_T > 0$ are two constants independent of $\varepsilon > 0$, and $\nu > 0$ is a small constant.

5.2. Compactness and dynamical behavior of solutions.

The proof of Theorem 2.1. It remains to show the convergence, as $\varepsilon \to 0_+$, of the approximate solutions $(\rho^\varepsilon, \mathbf{U}^\varepsilon, a^\varepsilon)$ constructed in (5.5) to the FBVP (2.1)–(2.7) under the assumptions of Theorem 2.1. This consists of the strong convergence near the free boundary in Lagrangian coordinates and the convergence in the whole domain in Euerlian coordinates.

We start with the strong convergence of $(\rho^\varepsilon, \rho^\varepsilon \mathbf{U}^\varepsilon, a^\varepsilon) = (\rho^\varepsilon(|\mathbf{x}|, t), \rho^\varepsilon u^\varepsilon(\mathbf{x}, t) \frac{\mathbf{x}}{|\mathbf{x}|}, a^\varepsilon)$ near the free boundary. It suffices to prove the strong convergence on the domain $[r^\varepsilon_{x_b}(t), a^\varepsilon(t)] \times [0, T]$, where $r = r^\varepsilon_{x_b}(t)$ is a particle path with $r^\varepsilon_{x_b}(0) = r_b \in (a_1, a_0]$ and $x_b = \int_{r_b}^{a_0} r^2 \rho_0(r) dr$, and the initial data satisfies $(\rho_0, u_0) \in W^{1,\infty}([a_1, a_0])$ and $\inf_{r \in [a_1, a_0]} \rho_0(r) > 0$. It is convenient to show the strong convergence of $(\rho^\varepsilon, u^\varepsilon, a^\varepsilon)$ in Lagrangian coordinates on $[x_b, 1] \times [0, T]$ in terms of the coordinate transformations (3.8)–(3.9). Indeed, one can show under the assumptions of Theorem 2.1 that $(\rho^\varepsilon, u^\varepsilon, a^\varepsilon)$ satisfies the uniform estimates established in Lemmas 3.3–3.5 and Lemmas 4.3–4.4 on $[x_b, 1] \times [0, T]$. Thus, by Lions-Aubin's Lemma, there is a limiting function $(\rho_b(x, \tau), u_b(x, \tau), a(\tau))$ so that up to a subsequence $(\rho^{\varepsilon_j}, u^{\varepsilon_j}, a^{\varepsilon_j})$, it holds as $\varepsilon_j \to 0$,

$$\begin{cases} (\rho^{\varepsilon_j}, u^{\varepsilon_j}) \to (\rho_b, u_b) & \text{strongly in } C([0, T] \times [x_b, 1]) \times C([0, T] \times [x_b, 1]), \\ (r^{\varepsilon_j}, \sigma^\varepsilon) \to (r, \sigma) & \text{strongly in } H^1([0, T] \times [x_b, 1]) \times L^2([0, T] \times [x_b, 1]), \\ a^{\varepsilon_j} \to a, & \text{strongly in } C^\alpha([0, T]), \quad \alpha \in (0, 1/2), \end{cases} \tag{5.11}$$

where $r_\tau = u_b$ and $(r^3)_x = \frac{3}{\rho_b}$, $\sigma = \rho_b^\gamma - \rho_b^2 (r^2 u_b)_x$.

It is easy to verify that (ρ_b, u_b, a) satisfies (3.11) on $[x_b, 1] \times [0, T]$ and (3.13). By (3.8)–(3.10), Lemmas 3.3–3.5 and Lemmas 4.3–4.4, we easily deduce that (ρ_b, u_b, a) satisfies

$$\begin{cases} \rho_b \in C([0, T] \times [r_{x_b}(t), a(t)]), \quad u_b \in C([0, T] \times [r_{x_b}(t), a(t)]), \\ \rho_b \in L^\infty(0, T; H^1([r_{x_b}(t), a(t)])), \quad \rho_\tau \in L^2(0, T; L^2([r_{x_b}(t), a(t)])), \\ u_b \in L^\infty(0, T; H^1([r_{x_b}(t), a(t)])) \cap L^2(0, T; H^2([r_{x_b}(t), a(t)])), \\ a(t) \in H^1([0, T]), \quad c \le a(t) \le C_T, \quad t \in [0, T], \end{cases} \tag{5.12}$$

and

$$\begin{cases} c_{x_b, T} \le \rho_b(r, t) \le C x_b^{-\frac{2\gamma}{3(\gamma - 1)}}, \quad r \in [r_{x_b}(t), a(t)], \; t \in [0, T], \\ \|\rho_b\|_{L^\infty(0, T; H^1([r_{x_b}(t), a(t)]))} + \|u_b\|_{L^2(0, T; H^1([r_{x_b}(t), a(t)]))} + \|a\|_{H^1([0, T])} \le C_{x_b, T}, \end{cases} \tag{5.13}$$

where $r = r_{x_b}(t)$ is the particle path with $r_{x_b}(0) = r_b \in (a_1, a_0]$.

Denote $(\rho_b, \mathbf{U}_b, a)$ by

$$(\rho_b, \mathbf{U}_b, a)(\mathbf{x}, t) = (\rho_b(|\mathbf{x}|, t), u_b(|\mathbf{x}|, t) \frac{\mathbf{x}}{|\mathbf{x}|}, a(t)), \quad r_{x_1}(t) \le |\mathbf{x}| \le a(t), \; t \in [0, T]. \tag{5.14}$$

We conclude from the above analysis that $(\rho^{\varepsilon_j}, \mathbf{U}^{\varepsilon_j}, a^{\varepsilon_j})$ converges to $(\rho_b, \mathbf{U}_b, a)$ strongly on the domain $[r_{x_b}(t), a(t)] \times [0, T]$. In addition, $(\rho_b, \mathbf{U}_b, a)$ satisfies (2.3), the free boundary condition (2.7), and (3.4)–(3.7) and (4.2)–(4.10) on the domain $[r_{x_b}(t), a(t)] \times [0, T]$.

Next, we show the convergence of $(\rho^{\varepsilon_j}, \mathbf{U}^{\varepsilon_j}, a^{\varepsilon_j})$ on an interior domain $\Omega^{\varepsilon_j}_{in}$ defined by

$$\Omega^{\varepsilon_j}_{in} = \Omega^{\varepsilon_j}_t \cap \{(\mathbf{x}, t) \mid 0 \le |\mathbf{x}| < a(t), \; t \in [0, T]\}. \tag{5.15}$$

Due to the strong convergence (5.11) of the velocity and particle path as $\varepsilon_j \to 0_+$, it holds that for $\varepsilon_j > 0$ small enough,

$$\Omega_{in} =: \{(\mathbf{x}, t) \,|\, 0 \le |\mathbf{x}| \le r_{x_{in}}(t),\ t \in [0, T]\} \subset\subset \Omega^{\varepsilon}_{in}, \tag{5.16}$$

where $r = r_{x_{in}}(t)$ is a particle path defined by

$$\frac{d}{dt} r_{x_{in}}(t) = u_b(r_{x_{in}}(t), t), \quad r_{x_{in}}(0) = r_{in} \in (r_b, a_0), \tag{5.17}$$

which satisfies that for $x_b < x_{in} = 1 - \int_{r_{in}}^{a_0} r^2 \rho_0(r) dy$,

$$0 < c(x_{in} - x_b)^{\frac{\gamma}{\gamma-1}} \le r^3_{x_{in}}(t) - r^3_{x_b}(t), \quad t \in [0, T]. \tag{5.18}$$

It is easy to show that $(\rho^{\varepsilon_j}, \rho^{\varepsilon_j}\mathbf{U}^{\varepsilon_j})$ satisfies (5.11) on the domain $[r_{x_b}(t), a(t)] \times [0, T]$ and admits a converging sub-subsequence, still denoted by $(\rho^{\varepsilon_j}, \rho^{\varepsilon_j}\mathbf{U}^{\varepsilon_j})$, on Ω_{in}. Indeed, since (5.7)–(5.10) hold uniformly for $(\rho^{\varepsilon_j}, \rho^{\varepsilon_j}\mathbf{U}^{\varepsilon_j})$ on Ω_{in}, using a similar compactness argument as in [11,29,36], one can show that there is a limiting function

$$(\rho_{in}, \rho_{in}\mathbf{U}_{in})(\mathbf{x}, t) = \left(\rho_{in}(|\mathbf{x}|, t), \rho_{in}u_{in}(|\mathbf{x}|, t)\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad (\mathbf{x}, t) \in \Omega_{in}, \tag{5.19}$$

so that up to a sub-subsequence $(\rho^{\varepsilon_j}, \rho^{\varepsilon_j}\mathbf{U}^{\varepsilon_j})$ converges to $(\rho_{in}, \rho_{in}\mathbf{U}_{in})$ in the sense

$$\begin{cases} \rho^{\varepsilon_j} \to \rho_{in} & \text{strongly in } C([0, T]; L^{3/2}(\Omega_{in})), \\ (\rho^{\varepsilon_j})^\gamma \to \rho^\gamma_{in} & \text{strongly in } L^1(0, T; L^1(\Omega_{in})), \\ \nabla\sqrt{\rho^{\varepsilon_j}} \to \nabla\sqrt{\rho_{in}} & \text{weakly in } L^2(0, T; L^2(\Omega_{in})), \\ \sqrt{\rho^{\varepsilon_j}}\,\mathbf{U}^{\varepsilon_j} \to \sqrt{\rho}_{in}\,\mathbf{U}_{in} & \text{strongly in } L^1((0, T) \times \Omega_{in}), \\ \sqrt{\rho^{\varepsilon_j}}\,\nabla\mathbf{U}^{\varepsilon_j} \to \sqrt{\rho_{in}}\,\nabla\mathbf{U}_{in} & \text{weakly in } L^2((0, T) \times \Omega_{in}), \\ \rho^{\varepsilon_j}\mathbf{U}^{\varepsilon_j} \to \rho_{in}\mathbf{U}_{in} & \text{strongly in } L^2(0, T; L^p(\Omega_{in})), \quad p \in (1, 2), \end{cases} \tag{5.20}$$

as $\varepsilon_j \to 0_+$, details are omitted. In addition, the momentum $\mathbf{m}^{\varepsilon_j} = \rho^{\varepsilon_j}\mathbf{U}^{\varepsilon_j}$ converges almost everywhere to $\mathbf{m}_{in}(\mathbf{x}, t) = \rho_{in}\mathbf{U}_{in}$, and $\mathbf{m}_{in}(x, t) = 0$ a.e. on $\{\rho_{in}(\mathbf{x}, t) = 0\}$. It is easy to verify by a similar argument as in [11] that $(\rho_{in}, \rho_{in}\mathbf{U}_{in})$ solves (2.3) on Ω_{in} in the sense of distributions.

Finally, set

$$(\rho, \rho\mathbf{U}) = \begin{cases} (\rho_b, \rho_b\mathbf{U}_b)(\mathbf{x}, t), & r_{x_1}(t) \le |\mathbf{x}| \le a(t),\ t \in [0, T], \\ (\rho_{in}, \rho_{in}\mathbf{U}_{in})(\mathbf{x}, t), & 0 \le |\mathbf{x}| \le r_{x_2}(t),\ t \in [0, T]. \end{cases} \tag{5.21}$$

This is well-defined due to (5.18) and

$$(\rho_b, \rho_b\mathbf{U}_b) = (\rho_{in}, \rho_{in}\mathbf{U}_{in}), \quad a.e. \quad x \in [r_{x_1}(t), r_{x_2}(t)],\ t \in [0, T].$$

We can easily deduce that $(\rho, \rho\mathbf{U}, a)$ is a solution to the FBVP (2.3)–(2.7) in the sense of Definition 2.1, which also satisfies (2.13)–(2.16) and the free boundary condition. The proof of Theorem 2.1 is completed. □

The proof of Theorem 2.2. To obtain the desired properties (2.17)–(2.24) for the global weak solution $(\rho, \rho\mathbf{U}, a)$ to the FBVP (2.1)–(2.7) constructed in Theorem 2.2, it suffices to justify them for approximate solutions $(\rho^\varepsilon, \mathbf{U}^\varepsilon, a^\varepsilon)$ defined by (5.5) uniformly with respect to $\varepsilon > 0$.

Indeed, for any $0 < r_1^- < r_1 \le r_0 < r_3 \le r_2 < r_2^+ \le a_0$ with $\inf_{r\in[r_1^-, r_2^+]} \rho_0(r) \ge \rho_* > 0$, there is a $\varepsilon_0 > 0$ so that $r_1^- > \varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$. In addition, the modified and regularized initial data $(\rho_0^\delta, u_0^\delta)$ in (5.3) satisfies all the assumptions of Proposition 4.1, in particular, $\inf_{r\in[r_1^-, r_2^+]} \rho_0^\delta(r) > 0$ uniformly with respect to $\varepsilon > 0$. Therefore, for $x_1^- < x_1^\varepsilon \le x_0^\varepsilon < x_3^\varepsilon \le x_2^\varepsilon < x_2^+ \le 1$ determined by $x_i^\varepsilon = \int_\varepsilon^{a_0} r^2 \rho_0^\delta(r)dr - \int_{r_i}^{a_0} r^2 \rho_0^\delta(r)dr$, $i = 0, 1, 2, 3$, and $x_i^\pm = \int_\varepsilon^{a_0} r^2 \rho_0^\delta(r)dr - \int_{r_i^\pm}^{a_0} r^2 \rho_0(r)dr$, $i = 1, 2$, we can define the particle path

$$\frac{d}{dt} r_{x_i^\varepsilon}(t) = \tilde{u}^\varepsilon(r_{x_i^\varepsilon}(t), t), \quad r_{x_i^\varepsilon}(0) = r_i \in [r_1^-, r_2^+], \tag{5.22}$$

so that the global approximate solution $(\rho^\varepsilon(r, t), u^\varepsilon(r, t), a^\varepsilon(t))$ satisfies all properties (4.2)–(4.4) along particle paths $r = r_{x_i^\varepsilon}(t)$, (4.5)–(4.8) on $[r_{x_1^\varepsilon}(t), r_{x_2^\varepsilon}(t)] \times [0, T]$, and (4.9)–(4.10) on $[r_{x_1^\varepsilon}(t), a^\varepsilon(t)] \times [0, T]$ uniformly with respect to $\varepsilon > 0$. These lead to the expected properties (2.18)–(2.31) for the global weak solution $(\rho, \rho\mathbf{U}, a)$, after passing into the limit $\varepsilon \to 0_+$. Equation (2.17) follows from the continuity of $r^2\rho$ on the domain $[0, a(t)] \times [0, T]$ derived based on $(2.3)_1$, (2.15) and (2.16). The point-wise decays (2.22)–(2.24) of the solution follow from (3.6) and Theorem 2.5 for $N = 3$, for which we omit the details. The proof of Theorem 2.2 is completed. □

The proof of Theorems 2.4. Let $(\rho, \rho\mathbf{U}, a) = (\rho(r, t), \rho u(r, t)\frac{\mathbf{x}}{r}, a(t))$ with $r = |\mathbf{x}|$ be the global weak solution to (2.1)–(2.7) constructed in Theorem 2.1. First, we show (2.32)–(2.35). Indeed, due to the assumption that $\rho(r_0', t_0') > 0$ for some point $(r_0', t_0') \in (0, a(t_0')) \times (0, T]$ and the continuity (2.14) of the density away from symmetry center, we deduce that there exist a small constant $\eta_0 > 0$ and a constant $\rho_1 > 0$ so that $[r_0' - \eta_0, r_0' + \eta_0] \subset (0, a_0)$, and

$$\inf_{r\in[r_0'-\eta_0, r_0'+\eta_0]} \rho(r, t_0') \ge \rho_1 > 0, \quad x_0 = 1 - \int_{r_0'}^{a(t_0')} r^2 \rho(r, t_0')dr \in (0, 1), \tag{5.23}$$

where we recall that the conservation of total mass holds

$$\int_0^{a_0} r^2 \rho_0(r)dr = \int_0^{a(t_0)} r^2 \rho(r, t_0)dr = 1.$$

In particular, we have

(a) either for some initial point $0 < r_0' < a_0$,

$$\rho_0(r_0') > 0, \quad x_0 = \int_{r_0'}^{a_0} r^2 \rho_0(r)dr, \tag{5.24}$$

(b) or for some points $0 < r_1 < r_3 \le r_4 < r_2 < a_0$,

$$x_0 = 1 - \int_{r_j}^{a_0} r^2 \rho_0(r)dr, \quad j = 3, 4, \tag{5.25}$$

and

$$\begin{cases} \rho_0(r) = 0, & r \in \{r : r_3 \le r \le r_4\}, \\ \rho_0(r) > 0, & r \in [r_1, r_3) \cup (r_4, r_2]. \end{cases} \tag{5.26}$$

We deal with the case (5.24) first. By (2.18) there is a particle path $r = r_{x_0}(t)$ defined by

$$\frac{d}{dt} r_{x_0}(t) = u(r_{x_0}(t), t), \quad r_{x_0}(0) = r_0' \in (0, a_0),$$

so that

$$\begin{cases} 0 < c x^{\frac{\gamma}{3(\gamma-1)}} \le r_{x_0}(t) < a(t) \le C_T, & t \in [0, t_0], \\ 0 < c_{x_0, t_0} \le \rho(r_{x_0}(t), t) \le C_{t_0} x_0^{-\frac{2\gamma}{3(\gamma-1)}}, & t \in [0, t_0], \end{cases} \tag{5.27}$$

and the conservation of mass between $r = r_{x_i}(t)$ and $r = a(t)$, and (5.24) imply

$$x_0 = 1 - \int_{r_0'}^{a_0} r^2 \rho_0(r) dr = 1 - \int_{r_{x_0}(t)}^{a(t)} r^2 \rho(r, t) dr, \quad t \in [0, t_0]. \tag{5.28}$$

As a consequence of (5.23), (5.27), (5.28) and the uniqueness of particle paths, $r_0 = r_{x_i}(t_0)$ and (2.32) holds.

Now assume that (5.25) and (5.26) hold, and define $\Omega_{x_0}^{t_0}$ and $\mathcal{V}_{x_0}^t$ as

$$\Omega_{x_0}^{t_0'} =: \{ (r, t) | \rho(r, t) \ge 0, \ r \in (0, a(t)), \ t \in [0, t_0']; \int_r^{a(t)} s^2 \rho(s, t) ds = 1 - x_0 \}, \tag{5.29}$$

$$\mathcal{V}_{x_0}^t =: \{ (r, t) | \rho(r, t) = 0, \ r \in (0, a(t)); \int_r^{a(t)} s^2 \rho(s, t) ds = 1 - x_0 \}. \tag{5.30}$$

By the continuity (2.14) of the density away from symmetry center, (5.23), (5.25)–(5.26) and (2.18), one can show that there exists a time $t_1' \in [0, t_0')$ so that $\mathcal{V}_{x_0}^t$ is a non-empty closed subset on $[0, t_1']$ with $\mathcal{V}_{x_0}^0 = [r_3, r_4]$ and $\mathcal{V}_{x_0}^t \times [0, t_1'] \subset \Omega_{x_0}^{t_0'}$. In addition, there exists a particle path $r = r_{x_0}(t)$ uniquely defined backward in time by

$$\frac{d}{dt} r_{x_0}(t) = u(r_{x_0}(t), t), \ t \le t_0, \quad r_{x_0}(t_0') = r_0,$$

along which it holds

$$\begin{cases} 0 < c x_0^{\frac{\gamma}{3(\gamma-1)}} \le r_{x_0}(t) < a(t) \le C_T, & t \in (t_1', t_0'], \\ 0 < \rho(r_{x_0}(t), t) \le C_{t_0'} x_0^{-\frac{2\gamma}{3(\gamma-1)}}, & t \in (t_1', t_0'], \\ 1 - x_0 = \int_{r_{x_0}(t)}^{a(t)} r^2 \rho(r, t) dr, & t \in (t_1', t_0']. \end{cases} \tag{5.31}$$

It is easy to show that $\{(r_{x_0}(t), t) | t \in (t_1', t_0']\} \subset \Omega_{x_0}^{t_0'}$, and

$$\Omega_{x_0}^{t_0'} = \{\mathcal{V}_{x_0}^t \times [0, t_1']\} \cup \{(r_{x_0}(t), t) | t \in (t_1', t_0']\}. \tag{5.32}$$

Indeed, for any $(r, t) \in \Omega_{x_0}^{t_0}$, it holds either $\rho(r, t) = 0$ with $(r, t) \in \mathcal{V}_{x_0}^{t}$, or $\rho(r, t) > 0$ with $r = r_{x_0}(t)$ due to the uniqueness of particle path, which implies (5.32). The above facts lead to (2.33)–(2.35).

The blow-up phenomena (2.36) can be shown by the contradiction argument as used in [29], the details are omitted. The proof of Theorem 2.4 is completed. □

6. Long Time Expanding and Decay Rate

In this section, we investigate the large time behavior of any global spherical symmetric weak solutions to FBVP (2.1)–(2.7). Indeed, we can obtain an expanding rate of the domain occupied by the fluid and the pointwise decay of density away from the symmetry center as follows.

Lemma 6.1. *Let the assumptions in Theorem 2.5 hold. Then the estimates (2.37) and (2.39)-(2.41) holds. Furthermore,*

$$a_M(t) \le C(1+t)^{\frac{\gamma}{N(\gamma-1)}}, \tag{6.1}$$

and

$$\rho(a(t), t) \le C(1+t)^{-\frac{1}{\gamma-1}}, \quad \gamma > 1, \quad t > 0. \tag{6.2}$$

Proof. Define an energy functional for a spherically symmetric solution as

$$H_\delta(t) = \int_0^{a(t)} \big(r - (\delta+t)u\big)^2 \rho r^{N-1} dr + \frac{2}{\gamma-1}(\delta+t)^2 \int_0^{a(t)} \rho^\gamma r^{N-1} dr \tag{6.3}$$

$$= \int_0^{a(t)} \rho r^{N+1} dr - 2(\delta+t)\int_0^{a(t)} \rho u r^N dr + (\delta+t)^2 \int_0^{a(t)} \rho u^2 r^{N-1} dr$$
$$+\frac{2}{\gamma-1}(\delta+t)^2 \int_0^{a(t)} \rho^\gamma r^{N-1} dr, \tag{6.4}$$

where $a'(t) = u(a(t), t)$ is the free boundary. For simplicity, we only consider the case $\delta = 1$ and set $H(t) = H_1(t)$ below. A direct computation gives

$$H'(t) = \int_0^{a(t)} (\rho_t r^{N+1} - 2\rho u r^N) dr + (1+t)^2 \int_0^{a(t)} \Big((\rho u^2)_t + \frac{2}{\gamma-1}(\rho^\gamma)_t\Big) r^{N-1} dr$$
$$+2(1+t)\int_0^{a(t)} \Big(\rho u^2 r^{N-1} - (\rho u)_t r^N + \frac{2}{\gamma-1}\rho^\gamma r^{N-1}\Big) dr$$
$$+(\rho u r^{N+1} - 2(1+t)\rho u^2 r^N + (1+t)^2 \rho u^3 r^{N-1})|_{r=a(t)}$$
$$+\frac{2}{\gamma-1}(1+t)^2 \rho^\gamma u r^{N-1}|_{r=a(t)} =: I_1 + I_2 + I_3 + I_{BD}. \tag{6.5}$$

By $(2.3)_1$ and (2.6)–(2.7), one has

$$I_1 = -\int_0^{a(t)} \big(r^2(r^{N-1}\rho u)_r + 2\rho u r^N\big)dr = -\int_0^{a(t)} (\rho u r^{N+1})_r dr = -\rho u r^{N+1}(a(t),t),$$

$$I_2 = -2(1+t)^2 \int_0^{a(t)} (\rho u_r^2 r^{N-1} + (N-1)\rho u^2 r^{N-3})dr$$
$$-(1+t)^2\big[\rho u^3 r^{N-1} + \tfrac{2}{\gamma-1}\rho^\gamma u r^{N-1} + 2(N-1)\rho u^2 r^{N-2}\big](a(t),t),$$

$$I_3 = 2(1+t)\int_0^{a(t)} \big(\rho u_r r^{N-1} + (N-1)\rho u r^{N-2} + \tfrac{2-N(\gamma-1)}{\gamma-1}\rho^\gamma r^{N-1}\big)dr$$
$$+2(1+t)\rho u^2 r^N(a(t),t) + 2(N-1)(1+t)\rho u r^{N-1}(a(t),t).$$

Substituting above estimates into (6.5) yields

$$H'(t) \le \frac{2(N+2-N\gamma)}{\gamma-1}(1+t)\int_0^{a(t)} \rho^\gamma r^{N-1}dr + \frac{N}{2}\int_0^{a(t)} \rho r^{N-1}dr$$
$$-2(N-1)(1+t)^2\rho u^2(1,t)a^{N-2}(t) + 2(N-1)(1+t)\rho u(1,t)a^{N-1}(t) \tag{6.6}$$

$$\le \frac{2(N+2-N\gamma)}{\gamma-1}(1+t)\int_0^{a(t)} \rho^\gamma r^{N-1}dr + \frac{N}{2} + \frac{1}{2}(N-1)\rho a^N(t). \tag{6.7}$$

Therefore, we deduce from (6.7) that for $\gamma \ge \frac{N+2}{N}$,

$$H'(t) \le \frac{N}{2} + \frac{1}{2}(N-1)\rho(1,t)a^N(t) \quad \Leftrightarrow \quad H(t) \le C(1+t) + C(1+t)^{1-\frac{1}{\gamma-1}}a_M^N(t),$$

where

$$a_M(t) = \max_{s\in[0,t]} a(s) \ge c_0 > 0.$$

This leads to

$$\int_0^{a(t)} \rho^\gamma r^{N-1}dr \le C(1+t)^{-1}a_M^N(t), \quad \gamma \ge \frac{N+2}{N}. \tag{6.8}$$

For $1 < \gamma < \frac{N+2}{N}$, re-write (6.7) as

$$H'(t) \le \frac{N+2-N\gamma}{1+t}H(t) + \frac{N}{2} + \frac{1}{2}(N-1)\rho(1,t)a^N(t),$$

from which it follows that for $1 < \gamma < \frac{N+2}{N}$ with $\gamma \ne \frac{N+1}{N}$,

$$H(t) \le C(1+t)^{N+2-N\gamma} + C(1+t)a_M^N(t), \tag{6.9}$$

and for $\gamma = \frac{N+1}{N}$

$$H(t) \le C(1+t)a_M^N(t) + C(1+t)\log(1+t). \tag{6.10}$$

These, together with the fact $a(t) \geq c > 0$, give rise to

$$\int_0^{a(t)} \rho^\gamma r^{N-1} dr \leq \begin{cases} C(1+t)^{-1} a_M^N(t), & \gamma > \frac{N+1}{N}, \\ C(1+t)^{-1} \log(1+t) a_M^N(t), & \gamma = \frac{N+1}{N}, \\ C(1+t)^{-N(\gamma-1)} a_M^N(t), & \gamma \in (1, \frac{N+1}{N}). \end{cases} \tag{6.11}$$

Note that

$$1 = \int_0^a \rho_0(r) r^{N-1} dr = \int_0^{a(t)} \rho r^{N-1} dr \leq C a(t)^{\frac{N(\gamma-1)}{\gamma}} \left(\int_0^{a(t)} \rho^\gamma r^{N-1} dr \right)^{\frac{1}{\gamma}}. \tag{6.12}$$

Combining (6.12) with (6.13) implies that

$$a_M(t) = \max_{s \in [0,t]} a(s) \geq \begin{cases} C(1+t)^{\frac{1}{N\gamma}}, & \gamma > \frac{N+1}{N}, \\ C(1+t)^{\frac{1-\nu}{N\gamma}}, & \gamma = \frac{N+1}{N}, \\ C(1+t)^{\frac{\gamma-1}{\gamma}}, & \gamma \in (1, \frac{N+1}{N}), \end{cases} \tag{6.13}$$

where we have used $(1+t)^\nu \sim \log(1+t)$ for any $\nu > 0$ small enough, and

$$a_M(t) \to +\infty, \quad \text{as } t \to +\infty. \tag{6.14}$$

Next, we show the exact expanding rate of the interface $r = a(t)$ for $\gamma \geq 2 \geq \frac{N+2}{N}$. Indeed, applying a similar argument as for (6.6), we can obtain

$$\begin{aligned} H_\delta'(t) &\leq \frac{N}{2} + \frac{2(N-1)}{N}((1+t)\rho(1,t)a^N(t))' - \frac{2(N-1)}{N} a^N(t)[(\delta+t)\rho(1,t)]' \\ &\leq \frac{N}{2} + \frac{2(N-1)}{N}((\delta+t)\rho(1,t)a^N(t))' \end{aligned}$$

which, together with the facts that $[(\delta+t)\rho(1,t)]' \geq 0$ for $\gamma \geq 2$ and $\delta = (\gamma-1)^{-\frac{1}{\gamma-1}} \rho_0^{1-\gamma}(a_0)$, leads to

$$H_\delta(t) \leq H_\delta(0) + \frac{N}{2} t + \frac{2(N-1)}{N} (\delta+t)\rho(1,t)a^N(t). \tag{6.15}$$

We deduce by (6.15), (3.28) and (6.3) that

$$\int_0^{a(t)} \rho^\gamma r^{N-1} dr \leq C(1+t)^{-1}(1+(N-1)a^N(t)), \tag{6.16}$$

which with (6.12) imply that

$$a(t) \geq C(1+t)^{\frac{1}{N\gamma}}, \quad \gamma \geq 2, \ N \geq 2. \tag{6.17}$$

The upper bound on the expanding rate of the free boundary follows similarly to (3.27), and the point-wise decay of density can be shown as for (3.42); we omit the details. □

Acknowledgements. The authors are grateful to the referee for his/her helpful comments and suggestions on the improvements of the paper.

This research was done when Z.H.Guo and H.L.Li were visiting The Institute of Mathematical Sciences (IMS) of The Chinese University of Hong Kong. The financial support from the IMS and the hospitality of the staff at the IMS are appreciated greatly. The research of Z.H.Guo is partially supported by the NNSFC No.10771170, 11071195, and Zheng Ge Ru Fundation. The research of H.L.Li is partially supported by the NNSFC No. 10871134 and 11011130029, the AHRDIHL Project of Beijing Municipality No.PHR201006107, the Huo Ying Dong Fund No.111033, and Zheng Ge Ru Fundation. The research of Z.P. Xin is partially supported by Hong Kong RGC Earmarked Research Grants CUHK4040/06P, CUHK4042/08P, the RGC Central Allocation Grant CA05/06.SC01, Zheng Ge Ru Fundation, NNSFC No.10771170 and a research grant at the Northwest University.

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Communicated by P. Constantin