

Boundary Value Problems for the Stationary Axisymmetric Einstein Equations: A Disk Rotating Around a Black Hole

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Abstract: We solve a class of boundary value problems for the stationary axisymmetric Einstein equations involving a disk rotating around a central black hole. The solutions are given explicitly in terms of theta functions on a family of hyperelliptic Riemann surfaces of genus 4. In the absence of a disk, they reduce to the Kerr black hole. In the absence of a black hole, they reduce to the Neugebauer-Meinell disk.

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1. Introduction

Two of the most famous solutions of the stationary axisymmetric Einstein equations are the Kerr black hole and the Neugebauer-Meinell disk. The former was discovered by Kerr in 1963 [13] and the latter by Neugebauer and Meinel in the 1990s [21–23]. In this paper, we construct analytic solutions of a class of boundary value problems (BVPs) for the stationary axisymmetric Einstein equations which combine the Kerr and Neugebauer-Meinell spacetimes. Thus, the BVPs considered involve a finite disk of dust rotating uniformly around a central black hole. In the limit of a vanishing disk, the solutions tend to the Kerr black hole. In the absence of a black hole, they reduce to the Neugebauer-Meinell disk. The constructed disk/black-hole systems are given explicitly in terms of theta functions on a family of hyperelliptic Riemann surfaces of genus 4. Given the importance of the Kerr and Neugebauer-Meinell solutions, we believe that the class of solutions presented here could also be of interest. We emphasize, however, that the new solutions involve a disk whose inner rim starts right at the event horizon of the black hole, whereas a physically correct BVP would allow for a small gap between the disk and the horizon. Thus, we expect the presented solutions to involve some dust particles traveling at superluminal speeds near the horizon.

The general analysis of rotating relativistic bodies is exceedingly complicated because it involves the study of free BVPs for the Einstein equations, which are nonlinear partial differential equations in four dimensions. However, in cases where the surface of the body is known and the motion is stationary and axisymmetric (a reasonable assumption in many astrophysical situations), the physical problem gives rise to a BVP for a single *integrable* equation in two dimensions—the celebrated Ernst equation. The integrability of the Ernst equation implies that powerful solution-generating techniques are at hand. Thus, through the application of suitable nonlinear transformations, new stationary axisymmetric spacetimes can be generated from already known ones. Furthermore, a large class of solutions of the Ernst equation can be given explicitly in terms of theta functions on Riemann surfaces [18]. In this way, it is possible to write down a large number of exact analytic solutions to the stationary axisymmetric Einstein equations and to study them using the methods of algebraic geometry cf. [17].

Nevertheless, for the solution of a concrete BVP, the power of this approach is often limited. Indeed, although a large class of exact solutions can be produced, the problem of determining the particular solution within this class that satisfies the given BVP is in general a highly nonlinear problem. It is therefore remarkable that Neugebauer and Meinel in the 1990s were able to solve explicitly, using constructive methods, the BVP corresponding to the physically relevant situation of a rotating dust disk. The structure of

an infinitesimally thin, relativistic disk of dust particles which rotate uniformly around a common center was first explored numerically by Bardeen and Wagoner 40 years ago [3], who also pointed out that “there may be some hope of finding an analytic solution” [2]. Bičák notes in the comprehensive review [4] that the subsequent construction of such an analytic solution by Neugebauer and Meinel represents “the first example of solving the BVP for a rotating object in Einstein’s theory by analytic methods.” Let us point out that since the Neugebauer-Meinel solution can be written in terms of theta functions on Riemann surfaces of genus 2, it belongs to the general class of solutions introduced in [18] and can therefore be analyzed by means of algebro-geometric methods. On the other hand, the Kerr black hole is the most famous example of a stationary axisymmetric spacetime and has had an immense impact on the development of general relativity and astrophysics (see e.g. [5]).

The approach in this paper is primarily inspired by the work of Neugebauer, Meinel, and collaborators [24], but also by a novel method for the analysis of BVPs for integrable PDEs which has been developed by Fokas and his collaborators within the past 15 years. A central development in the theory of nonlinear PDEs in the second half of the 20th century, and continuing to the present, has been the introduction of the Inverse Scattering Transform (IST). This technique was put forward in the famous 1967 paper [11] in connection with the Korteweg-de Vries (KdV) equation and the range of its applicability began to unfold with the investigation of the nonlinear Schrödinger (NLS) equation [ZS]. One of the most important later developments in this area has been the generalization of the IST formalism from initial-value to initial-boundary value problems introduced by Fokas [8,9] and subsequently developed further by several authors cf. [10]. The Fokas method consists of two steps: (a) Construct an integral representation of the solution characterized via a matrix Riemann-Hilbert (RH) problem formulated in the complex k -plane, where k denotes the spectral parameter of the associated Lax pair. Since this representation involves, in general, some unknown boundary values, the solution formula is *not* yet effective. (b) Characterize the unknown boundary values by analyzing a certain equation called the *global relation*. In general, this characterization involves the solution of a nonlinear problem; however, for certain so-called *linearizable* boundary conditions, step (b) can be solved in closed form.

In a recent work [20] steps (a) and (b) were implemented for the class of BVPs of the Ernst equation corresponding to a thin rotating disk of finite radius. In particular, it was found that the boundary conditions of the particular BVP corresponding to the uniformly rotating Neugebauer-Meinel disk are linearizable. The present paper is to some extent a continuation of [20]: The main novel observation is that the BVP which combines the Kerr black hole boundary conditions with those of a uniformly rotating disk is also linearizable.

Physically, disk/black-hole systems are important as models for black hole accretion disks in the context of active galactic nuclei and X-ray binaries [1]. Accretion disks are flattened astronomical objects made of rapidly rotating gas which slowly spirals onto a central gravitating body. Accretion onto a black hole is generally assumed to be thin and axisymmetric (see e.g. [25]) and many of the most energetic phenomena in the universe have been attributed to the accretion of matter onto black holes. In particular, active galactic nuclei and quasars are believed to be the accretion disks of supermassive black holes. We refer to [1] and references therein for more information on the physical aspects of black hole accretion disks. Our approach here is mathematical and we do not investigate any possible physical relevance of the presented solutions. In particular, we will refer to the disk as a ‘disk of dust’ although the assumption that the disk stretches all the way to the horizon presumably forces some of the dust particles to travel at

superluminal velocities. We mention in this regard that the BVP corresponding to a dust ring with finite inner radius (i.e. the BVP obtained by inserting a small gap between the horizon and the innermost dust particles) is believed to not be exactly solvable. In fact, the calculations in [14] suggest that it is impossible, even in the Newtonian limit, to obtain explicit solutions describing a self-gravitating dust ring around a central point mass.

The manuscript is organized as follows. In Sect. 2, we formulate our disk/black-hole BVP and write down its full solution in terms of theta functions. In Sect. 3, we consider a particular example. In Sects. 4–7, the derivation of the solution is presented in several steps. In Sect. 8, we consider the singularity structure of the solution and its dependence on various parameters. In the Appendix, we consider the relationship between the solution derived here and the general class of solutions of the Ernst equation studied in [19, 15].

2. Disk/Black-Hole Systems

In this section we introduce the Ernst equation, formulate the BVP corresponding to a dust disk rotating uniformly around a central black hole, and present its solution in terms of theta functions.

2.1. The Ernst equation. The metric of a stationary axisymmetric vacuum spacetime can be written in the Weyl-Lewis-Papapetrou form

$$ds^2 = e^{-2U} \left[e^{2\kappa} (d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2 \right] - e^{2U} (dt + ad\varphi)^2, \tag{2.1}$$

where $\rho \geq 0$ and $\zeta \in \mathbb{R}$ are Weyl’s canonical coordinates and t, φ are chosen so that ∂_t and ∂_φ are the two commuting asymptotically timelike and spacelike Killing vectors, respectively, cf. [26]. The metric functions e^{2U} , a , and $e^{2\kappa}$ are functions of ρ and ζ alone. Following standard practice, we introduce a real-valued potential b by

$$a_\rho = \rho e^{-4U} b_\zeta, \quad a_\zeta = -\rho e^{-4U} b_\rho. \tag{2.2}$$

It can be shown that the Einstein field equations for the metric (2.1) reduce to the following single nonlinear PDE in two dimensions for the complex-valued Ernst potential $f = e^{2U} + ib$:

$$\frac{f + \bar{f}}{2} \left(f_{\rho\rho} + f_{\zeta\zeta} + \frac{1}{\rho} f_\rho \right) = f_\rho^2 + f_\zeta^2, \quad \rho > 0, \quad \zeta \in \mathbb{R}. \tag{2.3}$$

The real part of f will be denoted by e^{2U} despite the fact that it may take on negative values.

We also need the concept of a corotating frame. Given $\Omega \in \mathbb{R}$, we define the coordinates $(\rho', \zeta', \varphi', t')$ corotating with the angular velocity Ω by

$$\rho' = \rho, \quad \zeta' = \zeta, \quad \varphi' = \varphi - \Omega t, \quad t' = t.$$

In these new coordinates, the metric (2.1) retains its form and the corotating metric functions $U_\Omega, a_\Omega, \kappa_\Omega$ are related to U, a, κ via

$$e^{2U_\Omega} = e^{2U} \left[(1 + \Omega a)^2 - \Omega^2 \rho^2 e^{-4U} \right], \tag{2.4a}$$

$$(1 - \Omega a_\Omega) e^{2U_\Omega} = (1 + \Omega a) e^{2U}, \quad \kappa_\Omega - U_\Omega = \kappa - U. \tag{2.4b}$$

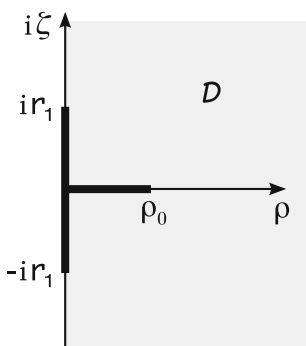


Fig. 1. The exterior domain \mathcal{D} of a disk of radius ρ_0 with the black hole horizon stretching along the imaginary axis from $-ir_1$ to ir_1

The Ernst equation retains its form in the corotating system and we denote the corotating Ernst potential by $f_\Omega := e^{2U_\Omega} + ib_\Omega$.

2.2. The boundary value problem. We now formulate the BVP for the Ernst equation (2.3) which corresponds to a finite dust disk rotating uniformly around a central black hole. The formulation involves five parameters: the radius $\rho_0 > 0$ and angular velocity $\Omega \in \mathbb{R}$ of the disk; the ‘radius’ $r_1 > 0$ and angular velocity $\Omega_h \in \mathbb{R}$ of the black hole horizon; and the (necessarily constant) value of the corotating metric function e^{2U_Ω} on the disk. We will henceforth work with a complex variable $z = \rho + i\zeta$ and write $f(z)$ for $f(\rho, \zeta)$.¹

Let \mathcal{D} denote the exterior of a finite disk of radius $\rho_0 > 0$, i.e. \mathcal{D} consists of all $z \in \mathbb{C}$ with strictly positive real part which do not belong to the interval $[0, \rho_0]$, see Fig. 1. The horizon of the black hole stretches in the z -plane along the imaginary axis from $-ir_1$ to ir_1 . We consider the problem of finding a function f such that:

- f satisfies (2.3) in \mathcal{D} . (2.6a)

- $f(z) = \overline{f(\bar{z})}$ (equatorial symmetry). (2.6b)

- $f(z) \rightarrow 1$ as $|z|^2 \rightarrow \infty$ (asymptotic flatness). (2.6c)

- $\frac{\partial f}{\partial \rho}(i\zeta) = 0$ for all $|\zeta| > r_1$ (regularity on the rotation axis).

- $f_\Omega(\rho \pm i0) = e^{2U_\Omega(+i0)}$ for $0 < \rho < \rho_0$ (boundary condition on the disk). (2.6d)

- $e^{2U_{\Omega_h}(i\zeta)} = 0$ for $0 < |\zeta| < r_1$ (boundary condition on the horizon). (2.6e)

The boundary conditions (2.6d) and (2.6e) are the boundary conditions corresponding physically to a uniformly rotating dust disk and a rotating black hole, respectively, cf. [24]. If one sets $r_1 = 0$ in (2.6) (i.e. one removes the black hole), then the solution of the obtained BVP is the Neugebauer-Meinl disk rotating with angular velocity Ω . If one sets $\rho_0 = 0$ in (2.6) (i.e. one removes the disk), then the solution of the obtained BVP is the Kerr black hole rotating with angular velocity Ω_h .

¹ In general, given a function $h(\rho, \zeta)$, we will suppress the dependence on \bar{z} and write $h(z)$ for $h(\rho, \zeta)$ even when h is not analytic.

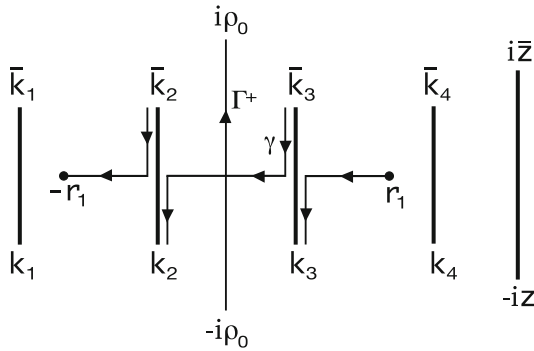


Fig. 2. The Riemann surface Σ_z presented as a two-sheeted cover of the complex k -plane with five branch cuts when $\text{Re } k_4 < \zeta$. The contours γ and Γ^+ are also shown

2.3. The solution. The formulation of the BVP (2.6) involves the five independent parameters $\rho_0, \Omega, r_1, \Omega_h,$ and the constant value of e^{2U_Ω} on the disk. However, it turns out that the condition that the solution be nonsingular at the rim of the disk imposes one relation among these parameters, so that the class of solutions is parametrized by only four parameters. It is convenient to adopt a parametrization in terms of the four parameters $\rho_0, r_1, w_2,$ and $w_4,$ where w_2 and w_4 are two real quantities related to the other parameters via Eqs. (5.9) below. For a given choice of these parameters, the corresponding solution f of the BVP (2.6) can be written in terms of theta functions on the family of Riemann surfaces $\{\Sigma_z\}_{z \in \mathcal{D}}$ defined as follows: Let $k_1, \bar{k}_1, \dots, k_4, \bar{k}_4 \in \mathbb{C}$ denote the eight zeros of $w^2(k) + 1,$ where

$$w(k) = \frac{w_4 k^4 + w_2 k^2 + \rho_0^2 (w_2 - w_4 \rho_0^2)}{(k^2 - r_1^2)}, \tag{2.7}$$

ordered so that $k_j, j = 1, \dots, 4,$ have negative imaginary parts and so that

$$\text{Re } k_1 \leq \text{Re } k_2 \leq \text{Re } k_3 \leq \text{Re } k_4.$$

Since these eight zeros are symmetrically distributed with respect to the origin, we have $-k_1 = \bar{k}_4$ and $-k_2 = \bar{k}_3.$ For each $z = \rho + i\zeta, \Sigma_z$ is defined as the hyperelliptic Riemann surface of genus 4 consisting of all points $(k, y) \in \mathbb{C}^2$ such that

$$y^2 = (k + iz)(k - i\bar{z}) \prod_{j=1}^4 (k - k_j)(k - \bar{k}_j), \tag{2.8}$$

together with two points at infinity required to make the surface compact. We introduce branch cuts in the complex k -plane from k_j to $\bar{k}_j, j = 1, \dots, 4,$ and from $-iz$ to $i\bar{z},$ see Fig. 2. For $k \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\},$ we let k^+ and k^- denote the points which project onto k and which lie in the upper and lower sheet of $\Sigma_z,$ respectively. By definition, the upper (lower) sheet is characterized by $y/k^5 \rightarrow 1$ ($y/k^5 \rightarrow -1$) as $k \rightarrow \infty.$

In view of the assumption of equatorial symmetry (2.6b), we will in the sequel always assume that $\zeta \geq 0.$ We will also, for the sake of definiteness, assume that

$$0 < \text{Re } k_3 < r_1 < \text{Re } k_4. \tag{2.9}$$

Moreover, we assume that $\zeta \neq \operatorname{Re} k_3$ and $\zeta \neq \operatorname{Re} k_4$, so that no branch cuts overlap (the solution for $\zeta = \operatorname{Re} k_j$, $j = 3, 4$, can be obtained by continuity).

For n complex numbers $\{a_j\}_1^n$, we let $[a_1, \dots, a_n]$ denote the directed contour $\cup_{j=1}^{n-1} [a_j, a_{j+1}]$. We let γ denote the contour on Σ_z which projects to the contour

$$[r_1, \operatorname{Re} k_3 + \epsilon, k_3 + \epsilon] \cup [\bar{k}_3 - \epsilon, \operatorname{Re} k_3 - \epsilon, \operatorname{Re} k_2 + \epsilon, k_2 + \epsilon] \cup [\bar{k}_2 - \epsilon, \operatorname{Re} k_2 - \epsilon, -r_1] \tag{2.10}$$

in the complex k -plane, where $\epsilon > 0$ is an infinitesimally small positive number, and which lies in the upper sheet for $\operatorname{Re} k < \zeta$ and in the lower sheet for $\operatorname{Re} k > \zeta$. We define Γ^+ as the contour in the upper sheet of Σ_z which lies above the segment $\Gamma = [-i\rho_0, i\rho_0]$. The contours Γ^+ and γ are shown in Fig. 2 in the case when $\operatorname{Re} k_4 < \zeta$.

In order to define theta functions associated with Σ_z , we need to introduce a basis of the first homology group $H_1(\Sigma_z, \mathbb{Z})$ of Σ_z . Since the Riemann surface Σ_z depends on z , there are three qualitatively different cases to consider:

- (1) The cut $[-iz, i\bar{z}]$ lies to the right of $[k_4, \bar{k}_4]$ (i.e. $\operatorname{Re} k_4 < \zeta$).
- (2) The cut $[-iz, i\bar{z}]$ lies between $[k_3, \bar{k}_3]$ and $[k_4, \bar{k}_4]$ (i.e. $\operatorname{Re} k_3 < \zeta < \operatorname{Re} k_4$).
- (3) The cut $[-iz, i\bar{z}]$ lies to the left of $[k_3, \bar{k}_3]$ (i.e. $0 \leq \zeta < \operatorname{Re} k_3$).

For these three cases, we let $\{a_j, b_j\}_{j=1}^4$ be the canonical basis of $H_1(\Sigma_z, \mathbb{Z})$ shown in (1), (2), and (3) of Figs. 3, 4 respectively. Thus, for $j = 1, \dots, 4$, a_j surrounds the cut $[k_j, \bar{k}_j]$, whereas b_j enters the upper sheet on the right side of $[-iz, i\bar{z}]$ and exits again on the right side of $[k_j, \bar{k}_j]$.

We define $\{\omega_j\}_1^4$ as the canonical basis of the space of holomorphic one-forms on Σ_z dual to $\{a_j, b_j\}$. Then

$$\int_{a_j} \omega_i = \delta_{ij}, \quad \int_{b_j} \omega_i = B_{ij}, \quad i, j = 1, \dots, 4,$$

where B is the period matrix associated with the cut system $\{a_j, b_j\}$. We let $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T$. The 4×4 matrix B is symmetric and has a positively definite imaginary part. The associated theta function $\Theta(v|B)$ is defined by

$$\Theta(v|B) = \sum_{N \in \mathbb{Z}^4} e^{2\pi i \left(\frac{1}{2} N^T B N + N^T v \right)}, \quad v \in \mathbb{C}^4. \tag{2.11}$$

We let ω_{PQ} denote the Abelian differential of the third kind on Σ_z , which has two simple poles at the points P and Q with residues $+1$ and -1 , respectively, and whose a -periods vanish, i.e. $\int_{a_j} \omega_{PQ} = 0$ for $j = 1, \dots, 4$.

We can now state our main result.

Theorem 2.1. *Solution of the disk/black-hole BVP Let ρ_0, r_1, w_2, w_4 be strictly positive numbers such that (2.9) holds. [The requirement that the solution be singularity-free imposes further restrictions on these parameters, see Sect. 8.] Let the function $h(k)$ be defined by*

$$h(k) = \frac{1}{\pi i} \ln \left(\sqrt{w(k)^2 + 1} - w(k) \right), \quad k \in \Gamma = [-i\rho_0, i\rho_0]. \tag{2.12}$$

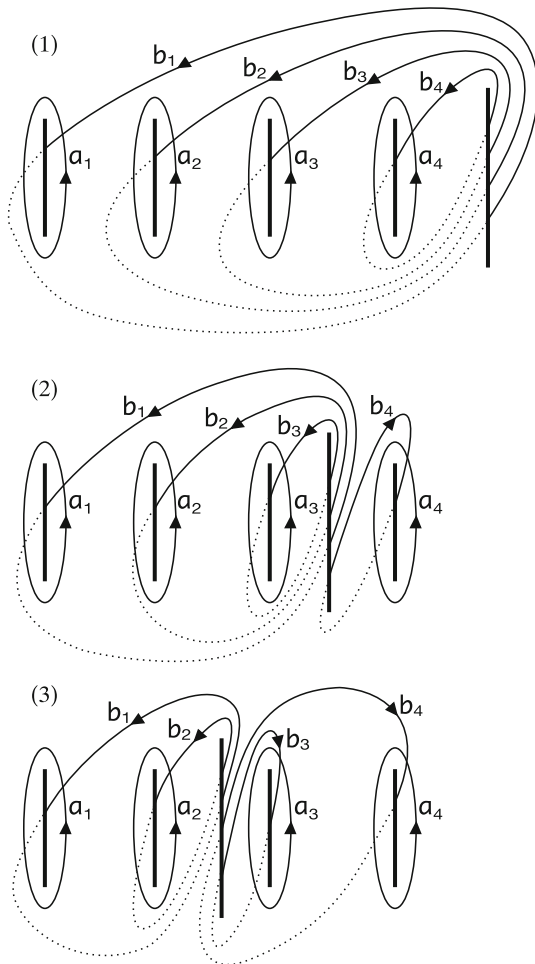


Fig. 3. The homology basis $\{a_j, b_j\}_1^4$ on the hyperelliptic Riemann surface Σ_ζ of genus $g = 4$ in the case of (1) $\text{Re } k_4 < \zeta$, (2) $\text{Re } k_3 < \zeta < \text{Re } k_4$, and (3) $0 \leq \zeta < \text{Re } k_3$

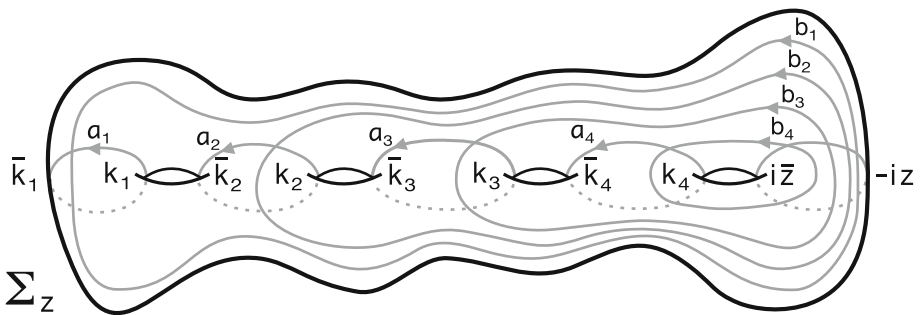


Fig. 4. Three-dimensional picture of the hyperelliptic Riemann surface Σ_ζ in the case when $\text{Re } k_4 < \zeta$

Define the z -dependent quantities $u \in \mathbb{C}^4$ and $I \in \mathbb{R}$ by

$$u = \int_{\Gamma^+} h\omega + \int_{\gamma} \omega, \quad I = \int_{\Gamma^+} h\omega_{\infty^+\infty^-} + \int_{\gamma} \omega_{\infty^+\infty^-}. \tag{2.13}$$

Then the function

$$f(z) = \frac{\Theta\left(u - \int_{-iz}^{\infty^-} \omega|B\right)}{\Theta\left(u + \int_{-iz}^{\infty^-} \omega|B\right)} e^I, \tag{2.14}$$

satisfies the BVP (2.6) with the prescribed values of ρ_0 and r_1 , and with the values of Ω_h , Ω , and $e^{2U_{\Omega}(+i0)}$ given by

$$\Omega_h = -\frac{1}{a_{hor}}, \quad \Omega = \frac{w_4\Omega_h e^{2U_0} + \sqrt{-2w_4\Omega_h^4 e^{2U_0}}}{w_4 e^{2U_0} + 2\Omega_h^2}, \quad e^{2U_{\Omega}(+i0)} = e^{2U_0} \left(1 - \frac{\Omega}{\Omega_h}\right)^2, \tag{2.15}$$

where $a_{hor} \in \mathbb{R}$ denotes the (necessarily constant) value of the metric function a on the horizon and $e^{2U_0} \in \mathbb{R}$ denotes the real part of $f(+i0)$. Explicit expressions for the constants a_{hor} and $f(+i0)$ are presented in Propositions 2.4 and 2.5 below.

Moreover, define the z -dependent quantity L by

$$L = -\frac{1}{2} \int_{\Gamma} d\kappa_1 \frac{dh}{dk}(\kappa_1) \int_{\Gamma} h(\kappa_2) \omega_{\kappa_1^+ \kappa_1^-}(\kappa_2^+) + \int_{\Gamma^+} h\omega_{-r_1^+, -r_1^-} - \text{sgn}(\zeta - r_1) \int_{\Gamma^+} h\omega_{r_1^+ r_1^-} + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\int_{\gamma_1(\epsilon)} \omega_{-r_1^+, -r_1^-} - \text{sgn}(\zeta - r_1) \int_{\gamma_2(\epsilon)} \omega_{r_1^+ r_1^-} - 2 \ln \epsilon \right), \tag{2.16}$$

where $\gamma_1(\epsilon)$ denotes the contour γ with the segment covering $[-r_1, -r_1 + \epsilon]$ removed, $\gamma_2(\epsilon)$ denotes the contour γ with the segment covering $[r_1 - \epsilon, r_1]$ removed, and the prime on the integral along Γ indicates that the integration contour should be deformed slightly before evaluation so that the pole at $\kappa_2 = \kappa_1$ is avoided.² Then the metric functions e^{2U} , a , $e^{2\kappa}$ of the line element (2.1) corresponding to the Ernst potential (2.14) are given for $z \in \mathcal{D}$ by

$$e^{2U(z)} = \frac{Q(0)}{Q(u)} e^I, \quad a(z) = a_0 - \frac{\rho}{Q(0)} \left(\frac{\Theta(u + \int_{-iz}^{\infty^-} \omega + \int_{i\bar{z}}^{\infty^-} \omega|B)}{Q(0)\Theta(u + \int_{-iz}^{\infty^-} \omega|B)} - Q(u) \right) e^{-I}, \tag{2.17a}$$

$$e^{2\kappa(z)} = K_0 \frac{\Theta(u|B)\Theta(u + \int_{-iz}^{i\bar{z}} \omega|B)}{\Theta(0|B)\Theta(\int_{-iz}^{i\bar{z}} \omega|B)} e^L, \tag{2.17b}$$

where $Q(v)$ is defined by

$$Q(v) = \frac{\Theta(v + \int_{-iz}^{\infty^-} \omega|B)\Theta(v + \int_{i\bar{z}}^{\infty^-} \omega|B)}{\Theta(v|B)\Theta(v + \int_{-iz}^{i\bar{z}} \omega|B)}, \quad v \in \mathbb{C}^4, \tag{2.18}$$

and the two constants $a_0, K_0 \in \mathbb{R}$ are given explicitly by Eqs. (2.26) and (2.27) below.

² The result is independent of whether the contour is deformed to the right or to the left of the pole.

Remark 2.2. 1. The function $h(k)$, $k \in \Gamma$, is well-defined by the right-hand side of (2.12), because $w^2 + 1 \geq 0$ and $\sqrt{w^2 + 1} - w > 0$ for $k \in \Gamma$.

2. Unless stated otherwise, all integrals in this paper along paths on Riemann surfaces for which only the endpoints are specified are assumed to lie within the fundamental polygon obtained by cutting the Riemann surface along the given cut basis. This implies that some integrals will depend on the particular choice of the a_j 's and b_j 's within their respective homology classes. It is convenient to fix the a_j 's and b_j 's so that they are invariant under the involution $k^\pm \rightarrow k^\mp$. For $\zeta > \text{Re } k_4$, this is accomplished by fixing³

$$b_4 = [i\bar{z}, k_4]^+ \cup [k_4, i\bar{z}]^-; \quad b_j = b_{j+1} \cup [\bar{k}_{j+1}, k_j]^+ \cup [k_j, \bar{k}_{j+1}]^-, \quad j = 1, 2, 3,$$

and by letting a_j , $j = 1, \dots, 4$, be the path in the homology class specified by Fig. 3 which as a point set consists of the points of Σ_z lying directly above $[k_j, \bar{k}_j]$. This implies the important symmetry $\omega(k^+) = -\omega(k^-)$. For Σ_z with $0 \leq \zeta < \text{Re } k_4$ and for other Riemann surfaces below, we will assume that an analogous fixing of the cut basis which assures $\omega(k^+) = -\omega(k^-)$ has been made.

3. The limit as $\epsilon \rightarrow 0$ of the expression within brackets on the right-hand side of (2.16) always exists and is finite because of the pole structure of $\omega_{-r_1^+, -r_1^-}$ and $\omega_{r_1^+, r_1^-}$.
4. The assumption that r_1 satisfies (2.9) is made for simplicity and is not essential. The relevant formulas in the case when (2.9) does not hold can be obtained from those presented here by analytic continuation.

2.4. Axis and horizon values. Of particular interest are the values of the Ernst potential and of the metric functions on the regular axis $\{i\zeta \mid \zeta > r_1\}$ and on the black hole horizon $\{i\zeta \mid 0 < \zeta < r_1\}$. In the limit $\rho \rightarrow 0$, the Riemann surface Σ_z degenerates since the branch cut $[-iz, i\bar{z}]$ shrinks to a point. Thus, the values of f on the ζ -axis are given in terms of quantities defined on the Riemann surface Σ' defined by the equation

$$y^2 = \prod_{j=1}^4 (k - k_j)(k - \bar{k}_j), \tag{2.19}$$

i.e., Σ' is the Riemann surface Σ_z with the cut $[-iz, i\bar{z}]$ removed.

We introduce a canonical cut basis $\{a'_j, b'_j\}_1^3$ on Σ' according to Fig. 5 and let $\omega' = (\omega'_1, \omega'_2, \omega'_3)^T$ denote the dual basis of holomorphic one-forms. We let B' denote the associated period matrix and introduce the short-hand notation $\Theta'(v) := \Theta(v|B')$, $v \in \mathbb{C}^3$, for the associated theta function.

Let γ' denote the ζ -dependent contour on Σ' which projects to the contour (2.10) in the complex k -plane and which lies in the upper sheet for $\text{Re } k < \zeta$ and in the lower sheet for $\text{Re } k > \zeta$. Define the ζ -dependent quantities $u' \in \mathbb{C}^3$, $I' \in \mathbb{R}$, and $K' \in \mathbb{C}$ by

$$u' = \int_{\Gamma^+} h\omega' + \int_{\gamma'} \omega', \quad I' = \int_{\Gamma^+} h\omega'_{\infty+\infty^-} + \int_{\gamma'} \omega'_{\infty+\infty^-}, \quad K' = \int_{k_4}^{\infty^-} \omega'_{\zeta+\zeta^-}. \tag{2.20}$$

³ For two complex numbers z_1 and z_2 , $[z_1, z_2]^+$ and $[z_1, z_2]^-$ denote the covers of $[z_1, z_2]$ in the upper and lower sheets of Σ_z , respectively.

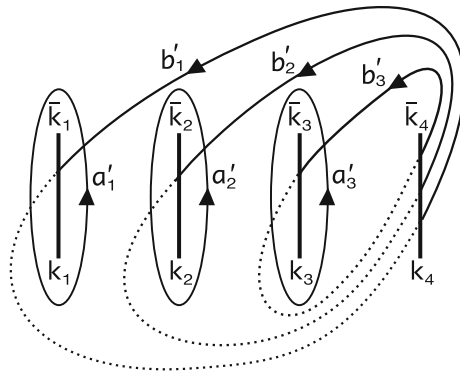


Fig. 5. The cut system $\{a'_j, b'_j\}_{j=1}^3$ on the degenerated Riemann surface Σ' of genus $g = 3$

Let γ^+ denote the contour on Σ' with the same projection onto the complex k -plane as γ' , but which lies entirely in the upper sheet, and define $J' \in \mathbb{R}$ by

$$J' = \begin{cases} \int_{\Gamma^+} h\omega'_{\zeta+\zeta^-} + \int_{\gamma'} \omega'_{\zeta+\zeta^-}, & \zeta > r_1, \\ \int_{\Gamma^+} h\omega'_{\zeta+\zeta^-} + \left(\int_{r_1^+}^{\zeta^+} + \int_{\gamma^+} \right) \omega'_{\zeta+\zeta^-}, & 0 < \zeta < r_1. \end{cases} \tag{2.21}$$

The prime on the integral along γ^+ indicates that the path should be deformed slightly before evaluation, so that it avoids the pole of the integrand at $k = \zeta^+$. It is irrelevant for the formulas below if this deformation is performed so that the pole lies to the right or to the left of γ^+ , since these two choices yield values of J' which differ by a multiple of $2\pi i$ and J' only appears exponentiated. In the same way, the value of J' changes by irrelevant multiples of $2\pi i$ if loops surrounding ζ^\pm are added to the contour from r_1^- to r_1^+ .

Proposition 2.3 (Solution on the regular axis). *The behavior of the solution (2.14) near the regular axis $\{i\zeta \mid \zeta > r_1\}$ is given by*

$$f(\rho + i\zeta) = f(i\zeta) + O(\rho^2), \quad \rho \rightarrow 0, \quad \zeta > r_1, \tag{2.22}$$

where

$$f(i\zeta) = \frac{\Theta'(u' - \int_{\zeta^-}^{\infty^-} \omega') - \Theta'(u' - \int_{\zeta^+}^{\infty^-} \omega')e^{J'-K'}}{\Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega') - \Theta'(u' + \int_{\zeta^+}^{\infty^-} \omega')e^{-J'-K'}} e^{I'-J'}, \quad \zeta > r_1. \tag{2.23}$$

The behavior of the metric functions e^{2U} , a , and $e^{2\kappa}$ in (2.17) near the regular axis is given by

$$\begin{aligned} e^{2U(\rho+i\zeta)} &= e^{2U(i\zeta)} + O(\rho^2), & a(\rho + i\zeta) &= O(\rho^2), \\ e^{2\kappa(\rho+i\zeta)} &= 1 + O(\rho^2), & \rho \rightarrow 0, \quad \zeta > r_1, \end{aligned}$$

where

$$e^{2U(i\zeta)} = \frac{\Theta'(u')^2 \left[\Theta'(\int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(\int_{\zeta^+}^{\infty^-} \omega')^2 e^{-2K'} \right]}{\Theta'(0)^2 \left[\Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(u' + \int_{\zeta^+}^{\infty^-} \omega')^2 e^{-2J'-2K'} \right]} e^{I'-J'}, \quad \zeta > r_1. \tag{2.24}$$

Define the constant L_0 by

$$L_0 = -\frac{1}{2} \int_{\Gamma} d\kappa_1 \frac{dh}{dk}(\kappa_1) \int_{\Gamma} h(\kappa_2) \omega'_{\kappa_1^+ \kappa_1^-}(\kappa_2^+) + \int_{\Gamma^+} h \omega'_{-r_1^+, -r_1^-} - \int_{\Gamma^+} h \omega'_{r_1^+ r_1^-} + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\int_{\gamma_1^+(\epsilon)} \omega'_{-r_1^+, -r_1^-} - \int_{\gamma_2^+(\epsilon)} \omega'_{r_1^+ r_1^-} - 2 \ln \epsilon \right), \tag{2.25}$$

where $\gamma_1^+(\epsilon)$ denotes the contour γ^+ with the segment covering $[-r_1, -r_1 + \epsilon]$ removed, and $\gamma_2^+(\epsilon)$ denotes the contour γ^+ with the segment covering $[r_1 - \epsilon, r_1]$ removed. Define the constants $a_0, K_0 \in \mathbb{R}$ by

$$a_0 = -2i \frac{\Theta'(u' + 2 \int_{k_4}^{\infty^-} \omega')}{\Theta'(u')} e^{-I'} \lim_{R \rightarrow \infty} \left(R e^{\int_{k_4}^{R^+} \omega'_{\infty^+ \infty^-}} \right), \tag{2.26}$$

$$K_0 = \frac{\Theta'(0)^2}{\Theta'(u')^2} e^{-L_0}, \tag{2.27}$$

where the right-hand sides are understood to be evaluated at some $\zeta > r_1$.⁴ Define the ζ -dependent quantities L' and M' by

$$L' = -\frac{1}{2} \int_{\Gamma^+} d\kappa_1 \frac{dh}{dk}(\kappa_1) \int_{\Gamma^+} h(\kappa_2) \omega'_{\kappa_1^+ \kappa_1^-}(\kappa_2^+) + \int_{\Gamma^+} h \omega'_{-r_1^+, -r_1^-} + \int_{\Gamma^+} h \omega'_{r_1^+ r_1^-} + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\int_{\gamma_1'(\epsilon)} \omega'_{-r_1^+, -r_1^-} + \int_{\gamma_2'(\epsilon)} \omega'_{r_1^+ r_1^-} - 2 \ln \epsilon \right) - \int_{\Gamma^+} h \omega'_{\zeta^+ \zeta^-}, \quad 0 < \zeta < r_1, \tag{2.28}$$

and

$$M' = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\int_{(\zeta-\epsilon)^-}^{(\zeta-\epsilon)^+} \omega'_{\zeta^+ \zeta^-} - 2 \ln \epsilon - \ln 4 - \pi i \right), \quad 0 < \zeta < r_1, \tag{2.29}$$

where $\gamma_1'(\epsilon)$ denotes the contour γ' with the segment covering $[-r_1, -r_1 + \epsilon]$ removed, and $\gamma_2'(\epsilon)$ denotes the contour γ' with the segment covering $[r_1 - \epsilon, r_1]$ removed.

Proposition 2.4 (Solution on the horizon). *The behavior of the solution (2.14) near the black hole horizon $\{i\zeta \mid 0 < \zeta < r_1\}$ is given by*

$$f(\rho + i\zeta) = f(i\zeta) + O(\rho^2), \quad \rho \rightarrow 0, \quad 0 < \zeta < r_1, \tag{2.30}$$

where

$$f(i\zeta) = -\frac{\Theta'(u' - \int_{\zeta^+}^{\infty^-} \omega') - \Theta'(u' - \int_{\zeta^+}^{\infty^-} \omega' + \int_{\zeta^-}^{\zeta^+} \omega') e^{J'-K'}}{\Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega') - \Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega' + \int_{\zeta^-}^{\zeta^+} \omega') e^{J'+K'}} e^{I'-K'}, \quad 0 < \zeta < r_1. \tag{2.31}$$

The behavior of the metric functions e^{2U} , a , and $e^{2\kappa}$ in (2.17) near the black hole horizon is given by

⁴ Their values are independent of the choice of $\zeta > r_1$.

$$\begin{aligned}
 e^{2U(\rho+i\zeta)} &= e^{2U(i\zeta)} + O(\rho^2), & a(\rho + i\zeta) &= a_{hor} + O(\rho^2), \\
 e^{2\kappa(\rho+i\zeta)} &= e^{2\kappa_{hor}} + O(\rho^2), & \rho \rightarrow 0, & 0 < \zeta < r_1,
 \end{aligned}
 \tag{2.32}$$

where

$$\begin{aligned}
 e^{2U(i\zeta)} &= -\frac{\Theta'(u' + \int_{\zeta^-}^{\zeta^+} \omega')^2}{\Theta'(0)^2} \\
 &\times \frac{\Theta'(\int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(\int_{\zeta^+}^{\infty^-} \omega')^2 e^{-2K'}}{\Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega' + \int_{\zeta^-}^{\zeta^+} \omega')^2 e^{2J'+2K'}} e^{I'+J'},
 \end{aligned}
 \tag{2.33}$$

$$a_{hor} = a_0 + \frac{\Theta'(u' + 2 \int_{\zeta^-}^{\infty^-} \omega')\Theta'(0)^4}{\Theta'(u' + \int_{\zeta^-}^{\zeta^+} \omega') \left(\Theta'(\int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(\int_{\zeta^+}^{\infty^-} \omega')^2 e^{-2K'} \right)^2} e^{-I'-M'},
 \tag{2.34}$$

$$e^{2\kappa_{hor}} = -K_0 \frac{\Theta'(u' + \int_{\zeta^-}^{\zeta^+} \omega')^2}{\Theta'(0)^2} e^{J'+L'}.
 \tag{2.35}$$

Moreover, a_{hor} and $e^{2\kappa_{hor}}$ are constants independent of $0 < \zeta < r_1$.

By taking limits in the above formulas we can find the values of f at the origin $z = +i0$ and at the point $z = ir_1$, where the regular axis meets the horizon.

Proposition 2.5. Solution at $z = +i0$ and $z = ir_1$ The value of the solution (2.14) at $z = ir_1$ is given by

$$f(ir_1) = -\frac{\Theta'(u' - \int_{r_1^+}^{\infty^-} \omega')}{\Theta'(u' + \int_{r_1^-}^{\infty^-} \omega')} e^{I'-K'} \Big|_{\zeta=r_1}.
 \tag{2.36}$$

Define the value of J' at $\zeta = 0$ by

$$J'|_{\zeta=0} = \int_{\Gamma^+}' h\omega'_{0^+0^-} + \left(\int_{r_1^-}^{r_1^+} + \int_{\gamma^+}' \right) \omega'_{0^+0^-},$$

where the primes on the integrals indicate that the contours should be deformed slightly before evaluation so that they pass to the left of the pole at $k = 0^+$. Then the value $f(+i0)$ of the solution (2.14) at $z = +i0$ is given by the right-hand side of (2.31) evaluated at $\zeta = 0$.

3. Example

Before presenting the derivation of the solution presented in the previous section, we wish to consider a concrete example. In this regard we note that all the quantities appearing in Sect. 2 can easily be computed explicitly using standard results for Riemann surfaces. In order to find the canonical basis $\{\omega_i\}_1^4$ and the period matrix B , we start with the holomorphic one-forms $\{\eta_i\}_1^4$ defined by

$$\eta_i = \frac{k^{i-1} dk}{y}, \quad i = 1, \dots, 4.
 \tag{3.1}$$

The η_i 's form a noncanonical basis of holomorphic differentials (see e.g. [6]). Defining the two 4×4 matrices A and Z by

$$(A^{-1})_{ij} = \int_{a_j} \eta_i, \quad Z_{ij} = \int_{b_j} \eta_i, \quad i, j = 1, \dots, 4, \tag{3.2}$$

we find

$$\omega = A\eta, \quad B = AZ. \tag{3.3}$$

The one-form $\omega_{\infty^+\infty^-}$ on Σ_z is given explicitly by

$$\omega_{\infty^+\infty^-} = -\frac{k^4 dk}{y} + \sum_{j=1}^4 \left(\int_{a_j} \frac{k^4 dk}{y} \right) \omega_j, \tag{3.4}$$

whereas the one-form $\omega'_{\zeta^+\zeta^-}$ on Σ' is given by

$$\omega'_{\zeta^+\zeta^-} = \frac{y'(\zeta^+) dk}{(k - \zeta)y'(k)} - \sum_{j=1}^3 \left(\int_{a'_j} \frac{y'(\zeta^+) dk}{(k - \zeta)y'(k)} \right) \omega'_j. \tag{3.5}$$

For k 's which lie some distance away from the branch cuts, the value of y in (2.8) can be evaluated according to

$$y(z, k^+) = (k + iz) \sqrt{\frac{k - i\bar{z}}{k + iz}} \prod_{j=1}^4 (k - k_j) \sqrt{\frac{k - \bar{k}_j}{k - k_j}}, \quad k \in \hat{\mathbb{C}},$$

where the branches with strictly positive real part are chosen for the square roots. For $k \in [-iz, i\bar{z}]$ and $\epsilon > 0$ infinitesimally small, we have

$$y(z, (k + \epsilon)^+) = (k + iz) \left(-i \sqrt{\left| \frac{k - i\bar{z}}{k + iz} \right|} \right) \prod_{j=1}^4 (k - k_j) \sqrt{\frac{k - \bar{k}_j}{k - k_j}}, \quad k \in \hat{\mathbb{C}};$$

similar expressions are valid when $k \in [k_j, \bar{k}_j]$. Using formulas of this type, it is straightforward to numerically evaluate all the expressions presented in Sect. 2. In fact, the theta functions are particularly suitable for numerical evaluation, because the strictly positive imaginary part of B implies that only a small number of terms in the sum (2.11) have to be included. In this way, we have verified for several examples all formulas of Sect. 2 to high precision.

3.1. Numerical data. Here we consider the particular example for which

$$\rho_0 = 1, \quad r_1 = \frac{1}{2}, \quad w_2 = 3, \quad w_4 = \frac{1}{10}. \tag{3.6}$$

Numerically, we find

$$k_1 \approx -0.95 - 5.48i, \quad k_2 \approx -0.21 - 0.95i.$$

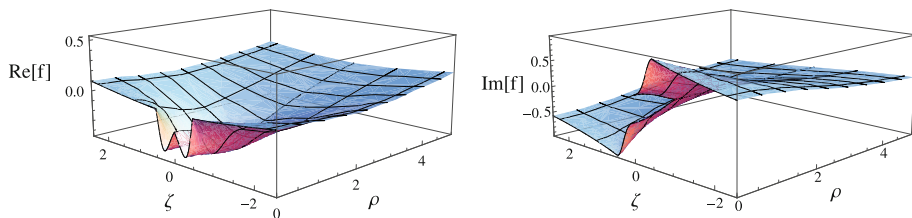


Fig. 6. The real and imaginary parts of the solution (2.14) for the choice of parameters specified in (3.6)

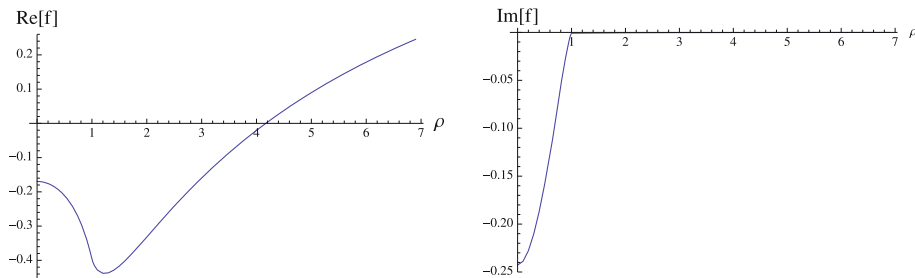


Fig. 7. The values of f in the equatorial plane $\zeta = 0^+$ for the choice of parameters specified in (3.6)

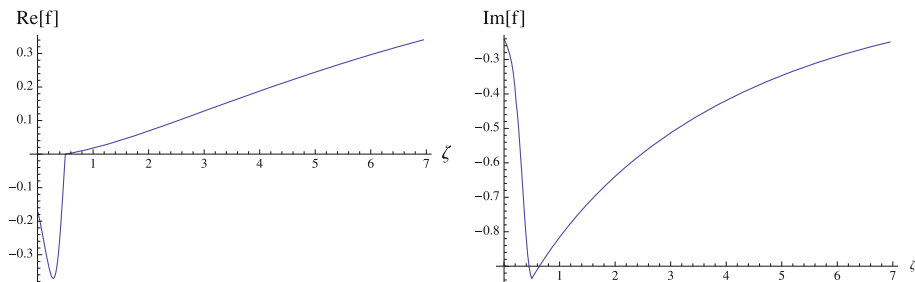


Fig. 8. The axis and horizon values of f for the choice of parameters specified in (3.6)

Thus, (2.9) is satisfied and we compute

$$\Omega \approx 0.055, \quad \Omega_h \approx 0.14, \quad f(+i0) \approx -0.17, \quad f(ir_1) \approx -0.94i,$$

and

$$a_0 \approx -18.17, \quad K_0 \approx 9.43, \quad e^{2\kappa_{hor}} \approx -93.46.$$

The real and imaginary parts of f are shown in Fig. 6. Note that the real part of f is continuous but not smooth at the endpoints $\pm ir_1$ of the horizon. Moreover, as expected, the imaginary part of f has a jump across the disk. The values of f in the equatorial plane $\zeta = 0^+$ are shown in Figs. 7, 8.

The metric functions ae^{2U} and $e^{2\kappa}$ are shown in Fig. 9. Note that $ae^{2U} = 0$ on the regular part of the axis. Moreover, $e^{2\kappa} = 1$ on the regular axis and $e^{2\kappa} = e^{2\kappa_{hor}}$ on the horizon. Figure 10 shows the real parts of the corotating potentials f_Ω and f_{Ω_h} in the equatorial plane and along the ζ -axis, respectively. In accordance with the boundary conditions (2.6d) and (2.6e), e^{2U_Ω} is constant along the disk and $e^{2U_{\Omega_h}}$ vanishes along

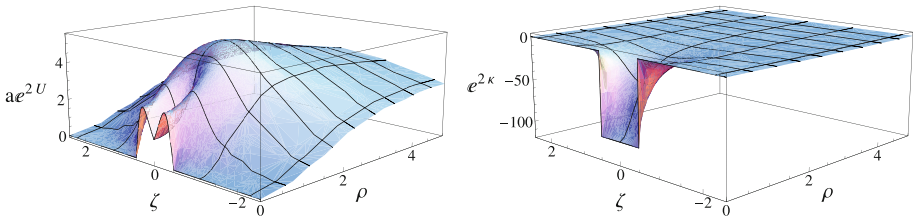


Fig. 9. The metric functions ae^{2U} and $e^{2\kappa}$ for the choice of parameters specified in (3.6)

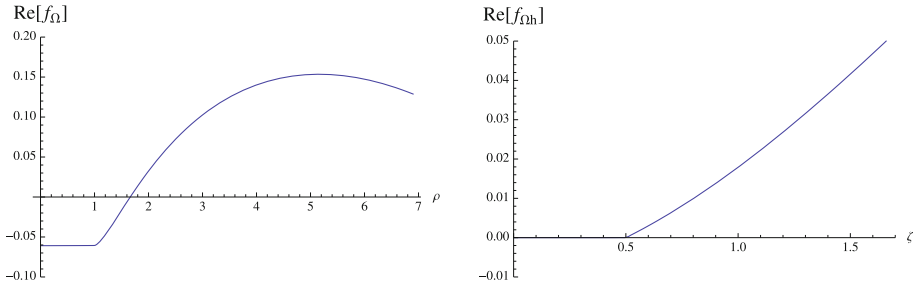


Fig. 10. The real parts of the corotating potentials f_Ω and f_{Ω_h} in the equatorial plane and along the ζ -axis, respectively

the horizon. It can also be verified to high accuracy that the metric functions a and κ defined by (2.17) satisfy the appropriate equations, i.e. (cf. [17])

$$a_z = \frac{i\rho}{e^{4U}} b_z \quad \text{and} \quad \kappa_z = \frac{\rho}{2e^{4U}} f_z \bar{f}_z, \tag{3.7}$$

and that

$$\frac{da_\Omega}{d\zeta} \Big|_{z=\rho+i0} = 0, \quad 0 < \rho < \rho_0. \tag{3.8}$$

Equation (3.8) implies that the imaginary part of f_Ω is constant along the disk in accordance with (2.6d).

4. Spectral Theory

We now turn to the proof of the results of Sect. 2. The proof will proceed through four main steps, presented in Sects. 4, 5, 6, and 7, respectively. The first step consists of analyzing the Lax pair for Eq. (2.3) and formulating two *matrix* Riemann-Hilbert (RH) problems: one main RH problem (which can be formulated for any choice of boundary conditions) and one auxiliary RH problem (which can be formulated because the boundary conditions of the BVP (2.6) are linearizable). In the second step, we show that these two matrix RH problems can be combined into a single *scalar* RH problem. The third step consists of solving this scalar RH problem explicitly in terms of theta functions. In the final fourth step, we prove Propositions 2.3–2.5 concerning the behavior of the solution near the ζ -axis; this step will follow from a study of the theta function formulas of Theorem 2.1 in the limit $\rho \rightarrow 0$.

4.1. *A bounded and analytic eigenfunction.* The elliptic Ernst equation (2.3) admits the Lax pair

$$\begin{cases} \Phi_z(z, k) = U(z, k)\Phi(z, k), \\ \Phi_{\bar{z}}(z, k) = V(z, k)\Phi(z, k), \end{cases} \tag{4.1}$$

where the 2×2 -matrix valued function $\Phi(z, k)$ is an eigenfunction, k is a spectral parameter, and the 2×2 -matrix valued functions U and V are defined by

$$U = \frac{1}{f + \bar{f}} \begin{pmatrix} \bar{f}_z & \lambda \bar{f}_z \\ \lambda f_z & f_z \end{pmatrix}, \quad V = \frac{1}{f + \bar{f}} \begin{pmatrix} \bar{f}_{\bar{z}} & \frac{1}{\lambda} \bar{f}_{\bar{z}} \\ \frac{1}{\lambda} f_{\bar{z}} & f_{\bar{z}} \end{pmatrix}, \quad \lambda(z, k) = \sqrt{\frac{k - i\bar{z}}{k + iz}}.$$

We write the Lax pair (4.1) in the form

$$d\Phi = W\Phi \quad \text{where} \quad W := Udz + Vd\bar{z}. \tag{4.2}$$

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose f is a solution of the BVP (2.6). Following the same procedure as in [20], we define a solution $\Phi(z, k)$ of (4.2) with the following properties:

- For each z , $\Phi(z, \cdot)$ is a map from the Riemann surface \mathcal{S}_z to the space of 2×2 matrices, where \mathcal{S}_z is defined by the equation

$$\lambda^2 = \frac{k - i\bar{z}}{k + iz}.$$

We view \mathcal{S}_z as a two-sheeted covering of the Riemann k -sphere endowed with a branch cut from $-iz$ to $i\bar{z}$; the upper (lower) sheet is characterized by $\lambda \rightarrow 1$ ($\lambda \rightarrow -1$) as $k \rightarrow \infty$.

- Φ satisfies the initial conditions

$$\lim_{z \rightarrow i\infty} [\Phi(z, k^-)]_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lim_{z \rightarrow i\infty} [\Phi(z, k^+)]_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad k \in \hat{\mathbb{C}}, \tag{4.3}$$

where, for a 2×2 -matrix A , $[A]_1$ and $[A]_2$ denote the first and second columns of A , respectively.

- Φ obeys the symmetries

$$\Phi(z, k^+) = \sigma_3 \Phi(z, k^-) \sigma_1, \quad \Phi(z, k^+) = \sigma_1 \overline{\Phi(z, \bar{k}^+)} \sigma_3, \quad k \in \hat{\mathbb{C}}. \tag{4.4}$$

- Φ is analytic for $k \in \mathcal{S}_z$ away from the set $\Gamma^+ \cup \Gamma^- \cup \{-r_1^\pm, r_1^\pm\}$, where Γ^+ and Γ^- denote the coverings of $\Gamma = [-i\rho_0, i\rho_0]$ in the upper and lower sheets of \mathcal{S}_z , respectively.

We emphasize that Φ , in general, has singularities (simple poles) at the points k^\pm for $k = -r_1$ and $k = r_1$. These poles arise since the Lax pair (4.1) is singular at points where $e^{2U} = 0$. Physically, the points at which e^{2U} vanishes make up the boundary of the *ergospheres* of the spacetime (within these surfaces there can be no static observer with respect to infinity). To see that $e^{2U} = 0$ at $z = ir_1$ and $z = -ir_1$, we note that the metric function a vanishes identically on the regular axis (cf. [26]). Thus, evaluating

(2.4a) at $\rho = 0$, we find $U_\Omega(i\zeta) = U(i\zeta)$ for $\zeta > r_1$. The boundary condition (2.6e) together with the continuity of f_Ω imply that $e^{2U(i\zeta)} \rightarrow 0$ as $\zeta \downarrow r_1$.

In addition to the eigenfunction $\Phi(z, k)$, we will also need its corotating analog $\Phi_\Omega(z, k)$. This eigenfunction satisfies the Lax pair equations (4.1) with f replaced by f_Ω and the initial conditions (4.3) with Φ replaced by Φ_Ω . The eigenfunctions Φ and Φ_Ω are related by [24]

$$\Phi_\Omega(z, k) = \Lambda_\Omega(z, k)\Phi(z, k), \quad k \in \mathcal{S}_z, \tag{4.5}$$

where

$$\Lambda_\Omega(z, k) = (1 + \Omega a)\mathbb{I} - \Omega \rho e^{-2U} \sigma_3 + i(k + iz)\Omega e^{-2U} (\lambda(z, k)\sigma_1 - \mathbb{I})\sigma_3 \tag{4.6}$$

and \mathbb{I} denotes the 2×2 identity matrix. The corotating eigenfunction $\Phi_{\Omega_h}(z, k)$ is defined analogously.

4.2. The main Riemann-Hilbert problem. Evaluation at $\rho = 0$ of Eq. (2.4a) with Ω replaced by Ω_h yields

$$e^{2U_{\Omega_h}(i\zeta)} = e^{2U(i\zeta)}(1 + \Omega_h a(i\zeta))^2, \quad 0 < \zeta < r_1. \tag{4.7}$$

Note that $e^{2U} < 0$ along the horizon, which lies inside the ergosphere. Thus, in view of the boundary condition (2.6e), we find that $a \equiv a_{hor}$ on the black hole horizon, where a_{hor} is a constant given by

$$a_{hor} = -1/\Omega_h. \tag{4.8}$$

The next proposition expresses the values of Φ on the ζ -axis in terms of two spectral functions $F(k)$ and $G(k)$. We let f_1 denote the value of f at $z = ir_1$.

Proposition 4.1. *The values of Φ on the ζ -axis can be expressed in terms of two spectral functions $F(k)$ and $G(k)$ as*

$$\Phi(i\zeta, k^+) = \begin{pmatrix} \overline{f(i\zeta)}1 \\ f(i\zeta) - 1 \end{pmatrix} A(k), \quad \zeta > r_1, \quad k \in \hat{\mathbb{C}}, \tag{4.9a}$$

$$\Phi(i\zeta, k^+) = \begin{pmatrix} \overline{f(i\zeta)}1 \\ f(i\zeta) - 1 \end{pmatrix} T_1(k)A(k), \quad 0 < \zeta < r_1, \quad k \in \hat{\mathbb{C}}, \tag{4.9b}$$

$$\Phi(i\zeta, k^+) = \begin{pmatrix} \overline{f(i\zeta)}1 \\ f(i\zeta) - 1 \end{pmatrix} T_2(k)\sigma_1 A(k)\sigma_1, \quad -r_1 < \zeta < 0, \quad k \in \hat{\mathbb{C}}, \tag{4.9c}$$

$$\Phi(i\zeta, k^+) = \begin{pmatrix} \overline{f(i\zeta)}1 \\ f(i\zeta) - 1 \end{pmatrix} \sigma_1 A(k)\sigma_1, \quad \zeta < -r_1, \quad k \in \hat{\mathbb{C}}, \tag{4.9d}$$

where the 2×2 -matrix valued functions $A(k)$, $T_1(k)$, and $T_2(k)$ are defined by

$$A(k) = \begin{pmatrix} F(k) & 0 \\ G(k) & 1 \end{pmatrix}, \quad k \in \hat{\mathbb{C}}, \tag{4.10}$$

$$T_1(k) = \frac{1}{2(k - r_1)\Omega_h} \begin{pmatrix} 2(k - r_1)\Omega_h - if_1 & i \\ -if_1^2 & 2(k - r_1)\Omega_h + if_1 \end{pmatrix}, \quad k \in \hat{\mathbb{C}}, \tag{4.11}$$

$$T_2(k) = \overline{T_1(-\bar{k})}, \quad k \in \hat{\mathbb{C}}.$$

The functions $F(k)$ and $G(k)$ have the following properties:

- F and G are unique functions of $k \in \hat{\mathbb{C}}$, i.e. viewed as functions on \mathcal{S}_z they satisfy

$$F(k^+) = F(k^-), \quad G(k^+) = G(k^-), \quad k \in \hat{\mathbb{C}}. \tag{4.12}$$

- $F(k)$ and $G(k)$ are analytic for $k \in \hat{\mathbb{C}} \setminus (\Gamma \cup \{r_1, -r_1\})$.
- Under the conjugation $k \mapsto \bar{k}$, F and G obey the symmetries

$$F(k) = \overline{F(\bar{k})}, \quad G(k) = -\overline{G(\bar{k})}, \quad k \in \hat{\mathbb{C}}. \tag{4.13}$$

- In the limit $k \rightarrow \infty$,

$$F(k) = 1 + O(1/k), \quad G(k) = O(1/k), \quad k \rightarrow \infty. \tag{4.14}$$

Proof. For $z = i\zeta$, $\lambda = 1$ for all k on the upper sheet of \mathcal{S}_z . The axis values of Φ are thus determined by integration of the equation

$$d\Phi = W(i\zeta, k^+) \Phi, \tag{4.15}$$

where

$$W(i\zeta, k^+) = \frac{1}{f + \bar{f}} \begin{pmatrix} d\bar{f} & d\bar{f} \\ df & df \end{pmatrix}. \tag{4.16}$$

Since the real part of f vanishes at $z = \pm ir_1$, Eq. (4.15) breaks down at $\zeta = \pm r_1$. Integration of (4.15) for ζ in each of the four intervals $(-\infty, -r_1)$, $(-r_1, 0)$, $(0, r_1)$, and (r_1, ∞) yields

$$\begin{aligned} \Phi(i\zeta, k^+) &= \begin{pmatrix} \overline{f(i\zeta)} & 1 \\ f(i\zeta) & -1 \end{pmatrix} U_1(k), & \zeta > r_1, \quad k \in \hat{\mathbb{C}}, \\ \Phi(i\zeta, k^+) &= \begin{pmatrix} \overline{f(i\zeta)} & 1 \\ f(i\zeta) & -1 \end{pmatrix} U_2(k), & 0 < \zeta < r_1, \quad k \in \hat{\mathbb{C}}, \\ \Phi(i\zeta, k^+) &= \begin{pmatrix} \overline{f(i\zeta)} & 1 \\ f(i\zeta) & -1 \end{pmatrix} U_3(k), & -r_1 < \zeta < 0, \quad k \in \hat{\mathbb{C}}, \\ \Phi(i\zeta, k^+) &= \begin{pmatrix} \overline{f(i\zeta)} & 1 \\ f(i\zeta) & -1 \end{pmatrix} U_4(k), & \zeta < -r_1, \quad k \in \hat{\mathbb{C}}, \end{aligned}$$

where the matrices $U_j(k)$, $j = 1, \dots, 4$, are independent of ζ . The initial conditions (4.3) imply that $U_1 = A$ for some functions $F(k)$ and $G(k)$. This establishes (4.9a). The value of Φ at $z = -i\infty$ is obtained from the value at $z = i\infty$ by integrating $W\Phi$ along a large semicircle at infinity. During this integration k changes sheets. Thus, using (4.9a) and the fact that $W\Phi$ vanishes for large z , we compute

$$\lim_{z \rightarrow -i\infty} \Phi(z, k^+) = \lim_{z \rightarrow i\infty} \Phi(z, k^-) = \lim_{z \rightarrow i\infty} \sigma_3 \Phi(z, k^+) \sigma_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & G(k) \\ 0 & F(k) \end{pmatrix}. \tag{4.17}$$

This shows that $U_4 = \sigma_1 A(k) \sigma_1$ and proves (4.9d).

We now use continuity of the matrices Φ and Φ_{Ω_h} at the points $z = \pm ir_1$ to find U_2 and U_3 . Let f_2 denote the value of f at $z = -ir_1$. The conditions that $\Phi(i\zeta, k^+)$ be continuous at $\zeta = r_1$ and $\zeta = -r_1$ are

$$\begin{pmatrix} \bar{f}_1 & 1 \\ f_1 & -1 \end{pmatrix} (A(k) - U_2(k)) = 0 \quad \text{and} \quad \begin{pmatrix} \bar{f}_2 & 1 \\ f_2 & -1 \end{pmatrix} (\sigma_1 A(k) \sigma_1 - U_3(k)) = 0, \quad (4.18a)$$

respectively. In view of Eqs. (4.5), (4.6), and (4.8), the conditions that $\Phi_{\Omega_h}(i\zeta, k^+)$ be continuous at $\zeta = r_1$ and $\zeta = -r_1$ are

$$\left[\begin{pmatrix} \bar{f}_1 & 1 \\ f_1 & -1 \end{pmatrix} + 2i\Omega_h(k - r_1) \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \right] A(k) = 2i\Omega_h(k - r_1) \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} U_2(k), \quad (4.18b)$$

and

$$\left[\begin{pmatrix} \bar{f}_2 & 1 \\ f_2 & -1 \end{pmatrix} + 2i\Omega_h(k + r_1) \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \right] \sigma_1 A(k) \sigma_1 = 2i\Omega_h(k + r_1) \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} U_3(k), \quad (4.18c)$$

respectively. The top and bottom rows of each of the four matrix equations in (4.18) are linearly dependent since f_1 and f_2 are purely imaginary. Combining the four bottom rows into two matrix equations, we find

$$\begin{pmatrix} f_1 & -1 \\ f_1 + 2i\Omega_h(k - r_1) & -1 \end{pmatrix} A(k) = \begin{pmatrix} f_1 & -1 \\ 2i\Omega_h(k - r_1) & 0 \end{pmatrix} U_2(k),$$

$$\begin{pmatrix} f_2 & -1 \\ f_2 + 2i\Omega_h(k + r_1) & -1 \end{pmatrix} \sigma_1 A(k) \sigma_1 = \begin{pmatrix} f_2 & -1 \\ 2i\Omega_h(k + r_1) & 0 \end{pmatrix} U_3(k).$$

Using that $f_1 = -f_2$ in view of the equatorial symmetry, we deduce from these equations that $U_2(k) = T_1(k)A(k)$ and $U_3(k) = T_2(k)\sigma_1 A(k)\sigma_1$, where T_1 and T_2 are given by (4.11). This proves (4.9b) and (4.9c).

The properties of F and G are proved as in [20]. \square

The functions $F(k)$ and $G(k)$ jump across $\Gamma = [-i\rho_0, i\rho_0]$. Let F^+ , G^+ and F^- , G^- denote the values of F and G for k to the right and left of Γ , respectively. It follows as in [24] (see also [20]) that

$$\Phi^-(z, k) = \Phi^+(z, k)D(k), \quad k \in \Gamma^+; \quad \Phi^-(z, k) = \Phi^+(z, k)\sigma_1 D(k)\sigma_1, \quad k \in \Gamma^-, \quad (4.19)$$

where the jump matrix D is given in terms of F^\pm and G^\pm by

$$D(k) = \begin{pmatrix} F^+(k) & 0 \\ G^+(k) & 1 \end{pmatrix}^{-1} \begin{pmatrix} F^-(k) & 0 \\ G^-(k) & 1 \end{pmatrix}. \quad (4.20)$$

For a given z , Eq. (4.19) provides the jump condition for a matrix RH problem on the Riemann surface \mathcal{S}_z satisfied by $\Phi(z, k)$. We will refer to this as the *main* RH problem.⁵ In general, given *both* the Dirichlet and Neumann boundary values for a BVP for

⁵ A complete formulation of this problem also involves specifying residue conditions at the four points $\pm ir_1^\pm$ as well as a normalization condition.

the Ernst equation, it is possible to determine the spectral functions F and G , compute the jump matrix D , and then obtain the Ernst potential f from the asymptotics of the solution of the main RH problem. However, for a well-posed problem only one of these boundary values is specified. In our analysis of (2.6) we will therefore instead use the global relation and the symmetry of the boundary conditions to formulate an *auxiliary* RH problem from which F and G can be determined.

4.3. *The global relation.* The equatorial symmetry of the solution f of (2.6) implies that the spectral functions $F(k)$ and $G(k)$ satisfy an important relation. Recalling the axis values (4.9) of Φ , the following proposition is proved in the same way as Proposition 4.3 in [20].

Proposition 4.2. *The spectral functions $F(k)$ and $G(k)$ defined in Proposition 4.1 satisfy*

$$\overline{T_1 A_+(k) \sigma_1 A_+^{-1}(k) T_1^{-1}} = T_2 \sigma_1 A_-(k) \sigma_1 A_-^{-1}(k) \sigma_1 T_2^{-1}, \quad k \in \Gamma, \tag{4.21}$$

where $A(k)$ is defined in terms of $F(k)$ and $G(k)$ by Eq. (4.10) and A_{\pm} denote the values of A to the right and left of Γ , respectively.

Equation (4.21) is referred to as the *global relation*.

4.4. *Linearizable boundary conditions.* In general, the global relation alone is not sufficient for determining F and G . However, for boundary conditions satisfying sufficient symmetry, so-called linearizable boundary conditions, there exist another algebraic relation satisfied by F and G . The boundary conditions specified in (2.6) turn out to be linearizable. Indeed, recalling the axis values (4.9) of Φ , the following proposition is proved in the same way as Proposition 5.3 in [20].

Proposition 4.3. *The spectral functions $F(k)$ and $G(k)$ satisfy the relation*

$$\begin{aligned} & (B^{-1} \Lambda^{-1} \sigma_1 \sigma_3 \overline{\Lambda B}) \overline{(T_1 A_+ \sigma_1 A_+^{-1} T_1^{-1})} \\ & = -(T_1 A_+ \sigma_1 A_+^{-1} T_1^{-1}) (B^{-1} \Lambda^{-1} \sigma_1 \sigma_3 \overline{\Lambda B}), \quad k \in \Gamma, \end{aligned} \tag{4.22}$$

where we use the short-hand notation B and Λ for

$$B := \begin{pmatrix} \overline{f(+i0)} & 1 \\ \overline{f(+i0)} & -1 \end{pmatrix}, \quad \Lambda := \Lambda_{\Omega}(+i0, k^+) = \left(1 - \frac{\Omega}{\Omega_h}\right) \mathbb{I} + ik\Omega e^{-2U_0} (\sigma_1 - \mathbb{I}) \sigma_3. \tag{4.23}$$

4.5. *The auxiliary Riemann-Hilbert problem.* Combining the relations (4.21) and (4.22), we can formulate a RH problem for the 2×2 -matrix valued function $\mathcal{M}(k)$ defined by

$$\mathcal{M}(k) = A(k) \sigma_1 A^{-1}(k) = \begin{pmatrix} -G(k) & F(k) \\ \frac{1-G(k)^2}{F(k)} & G(k) \end{pmatrix}, \quad k \in \hat{\mathbb{C}}. \tag{4.24}$$

Proposition 4.4. *Suppose f is a solution of the BVP (2.6). Let $f_1 := f(ir_1) \in i\mathbb{R}$ denote the value of f at $z = ir_1$. Then the spectral functions $F(k)$ and $G(k)$ are given by*

$$F(k) = \mathcal{M}_{12}(k), \quad G(k) = \mathcal{M}_{22}(k), \quad k \in \hat{\mathbb{C}},$$

where \mathcal{M} is the unique solution of the following RH problem:

- $\mathcal{M}(k)$ is analytic for $k \in \hat{\mathbb{C}} \setminus (\Gamma \cup \{-r_1, r_1\})$.
- Across Γ , $\mathcal{M}(k)$ satisfies the jump condition

$$\mathcal{S}(k)\mathcal{M}^-(k) = -\mathcal{M}^+(k)\mathcal{S}(k), \quad k \in \Gamma, \tag{4.25}$$

where \mathcal{M}^+ and \mathcal{M}^- denote the values of \mathcal{M} to the right and left of Γ , respectively, and $\mathcal{S}(k)$ is defined by

$$\mathcal{S}(k) = T_1^{-1} B^{-1} \Lambda^{-1} \sigma_1 \sigma_3 \overline{\Lambda B} T_2 \sigma_1, \quad k \in \Gamma. \tag{4.26}$$

- \mathcal{M} has the asymptotic behavior

$$\mathcal{M}(k) = \sigma_1 + O(1/k), \quad k \rightarrow \infty. \tag{4.27}$$

- The entries of \mathcal{M} have simple poles at $k = r_1$ and $k = -r_1$. The associated residues are given by

$$\operatorname{Res}_{r_1} \mathcal{M}(k) = \frac{1}{\alpha} \begin{pmatrix} -f_1 & 1 \\ -f_1^2 & f_1 \end{pmatrix}, \quad \operatorname{Res}_{-r_1} \mathcal{M}(k) = \frac{1}{\alpha} \begin{pmatrix} f_1 & -|f_1|^2 \\ f_1/\bar{f}_1 & -f_1 \end{pmatrix}, \tag{4.28}$$

where

$$\alpha = \left. \frac{d^+}{d\zeta} \right|_{\zeta=r_1} e^{2U(i\zeta)} \tag{4.29}$$

and $d^+/d\zeta$ denotes the right-sided derivative.

Proof. We deduce from (4.21) and (4.22) that the function \mathcal{M} defined in (4.24) satisfies the jump condition (4.25). The asymptotic behavior (4.27) follows from the properties of F and G .

By evaluating the first symmetry in (4.4) at the branch point $i\bar{z}$ and taking the limit as z approaches the regular axis, we find (cf. Eqs. (2.63)–(2.64) in [24])

$$F(\zeta) = \frac{1}{\operatorname{Re} f(i\zeta)}, \quad G(\zeta) = \frac{i\operatorname{Im} f(i\zeta)}{\operatorname{Re} f(i\zeta)}, \quad \zeta > r_1, \tag{4.30a}$$

$$F(\zeta) = \frac{|f(i\zeta)|^2}{\operatorname{Re} f(i\zeta)}, \quad G(\zeta) = \frac{-i\operatorname{Im} f(i\zeta)}{\operatorname{Re} f(i\zeta)}, \quad \zeta < -r_1. \tag{4.30b}$$

The poles of F and G arise since $\operatorname{Re} f(i\zeta) = 0$ at $\zeta = \pm r_1$. Equations (4.30) together with the equatorial symmetry of f yield

$$\operatorname{Res}_{r_1} F(k) = \frac{1}{\alpha}, \quad \operatorname{Res}_{r_1} G(k) = \frac{f_1}{\alpha}, \quad \operatorname{Res}_{-r_1} F(k) = -\frac{|f_1|^2}{\alpha}, \quad \operatorname{Res}_{-r_1} G(k) = -\frac{f_1}{\alpha},$$

where α is given by (4.29). The residue conditions (4.28) follow immediately from these relations.

In the last step of the proof, we show that $\mathcal{M}_{21}(k)$ does *not* have poles at the possible zeros of F , despite the form of (4.24). We first extend the definition (4.26) of $S(k)$ to all $k \in \hat{\mathbb{C}}$ by

$$S(k) = T_1^{-1} B^{-1} \Lambda_\Omega(+i0, k^+)^{-1} \sigma_1 \sigma_3 \Lambda_\Omega(-i0, k^+) \bar{B} T_2 \sigma_1. \tag{4.31}$$

This definition is consistent with (4.26). Indeed, since f is equatorially symmetric,

$$\overline{\Lambda_\Omega(\bar{z}, k^+)} = \Lambda_\Omega(z, -\bar{k}^+), \tag{4.32}$$

so that $\bar{\Lambda} = \overline{\Lambda_\Omega(+i0, k^+)} = \Lambda_\Omega(-i0, k^+)$ for $k \in \Gamma$. We claim that the matrices S and \mathcal{M} satisfy

$$\text{tr}(S\mathcal{M}) = 0, \quad k \in \hat{\mathbb{C}}. \tag{4.33}$$

Indeed, the same type of argument used to prove Proposition 4.3 shows that the function

$$\mathcal{R} = \Phi_\Omega^{-1}(\rho + i0, k) \sigma_1 \sigma_3 \Phi_\Omega(\rho - i0, k)$$

is independent of ρ . Evaluation at $\rho = 0$ using the axis values yields

$$\mathcal{R}(k^+) = A^{-1} T_1^{-1} B^{-1} \Lambda_\Omega^{-1}(+i0, k^+) \sigma_1 \sigma_3 \Lambda_\Omega(-i0, k^+) \bar{B} T_2 \sigma_1 A \sigma_1.$$

Evaluation at $\rho = \rho_0$ yields

$$\text{Tr } \mathcal{R} = 0.$$

The preceding two equations imply (4.33). It follows from (4.33) that $G^2 - 1$ must vanish whenever F has a zero. \square

The auxiliary RH problem presented in Proposition 4.4 can be used to determine the spectral functions F and G . These spectral functions can then be used to compute the jump matrix and to set up the main RH problem. However, in analogy with linearizable BVPs for other integrable PDEs, we expect that the jump condition of the auxiliary RH problem can also be substituted directly into the main RH problem with the result that the unknown quantities in the main RH problem disappear. In fact, an example of this mechanism was observed by Neugebauer and Meinel in the case of a rigidly rotating disk. They discovered that the analogs of the main and auxiliary RH problems can be combined into a single scalar RH problem from which the Ernst potential f can be directly recovered, see [24]. It turns out that a similar approach can be adopted in the present case—in the next section, we will combine the main and auxiliary RH problems with respective jump conditions (4.19) and (4.25) into a scalar RH problem on the Riemann surface Σ_z introduced in Sect. 2.

5. A Scalar Riemann-Hilbert Problem

We let the scalar-valued function $w(k)$ be defined by

$$w(k) = -\frac{1}{2}\text{tr}(\mathcal{S}(k)), \quad k \in \hat{\mathbb{C}}, \tag{5.1}$$

and define two 2×2 -matrix valued functions \mathcal{L} and \mathcal{Q} by

$$\mathcal{L}(z, k) = \Phi(z, k)\sigma_1\Phi^{-1}(z, k), \quad k \in \mathcal{S}_z, \tag{5.2}$$

and

$$\mathcal{Q}(z, k) = -\Phi(z, k)A(k)^{-1}\mathcal{S}(k)A(k)\Phi(z, k)^{-1} - w(k)\mathbb{I}, \quad k \in \mathcal{S}_z. \tag{5.3}$$

Lemma 5.1. *The functions \mathcal{L} , \mathcal{Q} , and w have the following properties:*

- *The traces and determinants of \mathcal{Q} and \mathcal{L} satisfy*

$$\text{tr } \mathcal{Q} = 0, \quad \text{tr } \mathcal{L} = 0, \quad \det \mathcal{L} = -1, \quad \det \mathcal{Q} = -1 - w^2. \tag{5.4}$$

- *\mathcal{Q} can be alternatively written as*

$$\mathcal{Q}(z, k) = \Phi(z, k)\sigma_1A(k)^{-1}\mathcal{S}(k)A(k)\sigma_1\Phi(z, k)^{-1} + w(k)\mathbb{I}. \tag{5.5}$$

- *\mathcal{Q} and \mathcal{L} admit the symmetries*

$$\mathcal{Q}(z, k^-) = -\sigma_3\mathcal{Q}(z, k^+)\sigma_3, \quad \mathcal{L}(z, k^-) = \sigma_3\mathcal{L}(z, k^+)\sigma_3. \tag{5.6}$$

- *\mathcal{Q} has no jump across Γ^\pm , whereas \mathcal{L} satisfies the following jump conditions across Γ^\pm :*

$$\begin{aligned} (\mathcal{Q} + w\mathbb{I})\mathcal{L}_- &= -\mathcal{L}_+(\mathcal{Q} + w\mathbb{I}), & k \in \Gamma^+, \\ (\mathcal{Q} - w\mathbb{I})\mathcal{L}_- &= -\mathcal{L}_+(\mathcal{Q} - w\mathbb{I}), & k \in \Gamma^-. \end{aligned} \tag{5.7}$$

- *$\mathcal{Q}\mathcal{L} = -\mathcal{L}\mathcal{Q}$; in particular, $\text{tr}(\mathcal{Q}\mathcal{L}) = 0$.*

- *Let $\hat{\mathcal{L}}_{22} = \mathcal{L}_{21}\mathcal{Q}_{11} + \mathcal{L}_{22}\mathcal{Q}_{21}$. Then*

$$\hat{\mathcal{L}}_{22}^2 - \mathcal{L}_{21}^2(1 + w^2) = \mathcal{Q}_{21}^2. \tag{5.8}$$

- *w has the form*

$$w(k) = \frac{w_4k^4 + w_2k^2 + w_0}{k^2 - r_1^2},$$

where w_4, w_2, w_0 are real coefficients explicitly given by

$$w_4 = -\frac{2\Omega^2\Omega_h^2}{e^{2U_0}(\Omega - \Omega_h)^2}, \tag{5.9a}$$

$$w_2 = \frac{|f_0|^2(\Omega - \Omega_h)^2 + f_1^2\Omega(\Omega - 2\Omega_h) + 4if_1r_1\Omega^2\Omega_h + \Omega_h^2(4r_1^2\Omega^2 - 1)}{2e^{2U_0}(\Omega - \Omega_h)^2}, \tag{5.9b}$$

$$\begin{aligned} w_0 = &-\frac{1}{8e^{2U_0}\Omega_h^2} \left(-2ib_0 \left(f_1^3 - 2if_1^2r_1\Omega_h + f_1 + 2ir_1\Omega_h \right) \right. \\ &\left. - |f_0|^2 \left(f_1^2 - 4if_1r_1\Omega_h - 4r_1^2\Omega_h^2 + 1 \right) + f_1^4 + f_1^2 + 4if_1r_1\Omega_h - 4r_1^2\Omega_h^2 \right), \end{aligned} \tag{5.9c}$$

where f_0 and f_1 denote the values of f at $z = +i0$ and $z = ir_1$, respectively.

Proof. The first three properties in (5.4) are immediate from Eqs. (5.1)–(5.3). The fourth property follows since

$$\det(S) = -1$$

and

$$\det Q = \det(-\Phi A^{-1}(S + w\mathbb{I})A\Phi^{-1}) = \det(S + w\mathbb{I}) = \det S - w^2.$$

Using (4.33) and the definitions of A and \mathcal{M} , a computation shows that

$$A^{-1}(S + w\mathbb{I})A = -\sigma_1 A^{-1}(S + w\mathbb{I})A\sigma_1.$$

Using this identity in the definition (5.3) of \mathcal{Q} , we find (5.5).

The symmetries (5.6) follow from (5.2), (5.3), and (5.5) together with the first symmetry in (4.4).

By (4.19) and (4.20), ΦA^{-1} and $\Phi\sigma_1 A^{-1}$ do not jump across Γ^+ and Γ^- , respectively. It follows from the expressions (5.3) and (5.5) that \mathcal{Q} does not jump across Γ^\pm . For $k \in \Gamma^+$, the definitions (5.2) and (5.3) of \mathcal{L} and \mathcal{Q} show that

$$(\mathcal{Q}_- + w\mathbb{I})\mathcal{L}_- = -\Phi_- A_-^{-1} S A_- \sigma_1 \Phi_-^{-1} = -\Phi_- A_-^{-1} S \mathcal{M}_- A_- \Phi_-^{-1} \tag{5.10}$$

and

$$-\mathcal{L}_+(Q_+ + w\mathbb{I}) = \Phi_+ \sigma_1 A_+^{-1} S A_+ \Phi_+^{-1} = \Phi_+ A_+^{-1} \mathcal{M}_+ S A_+ \Phi_+^{-1}. \tag{5.11}$$

Using that ΦA^{-1} does not jump across Γ^+ together with the jump condition (4.25), we see that the right-hand sides of (5.10) and (5.11) are equal. Similarly, using (5.5), the fact that $\Phi\sigma_1 A^{-1}$ has no jump across Γ^- , and (4.25), we find the jump across Γ^- . This proves (5.7).

Since \mathcal{M} is tracefree and $\text{tr}(S\mathcal{M}) = 0$, we deduce that $MS + SM + 2wM = 0$. In view of the definitions of \mathcal{Q} and \mathcal{L} , this implies $\mathcal{Q}\mathcal{L} = -\mathcal{L}\mathcal{Q}$.

Equation (5.8) follows by direct computation using the identity $\text{tr}(\mathcal{Q}\mathcal{L}) = 0$ and the four properties in (5.4).

The last statement concerning the form of w follows from (4.31) and (5.1) by direct computation. \square

The condition that \mathcal{M} does not jump at the endpoints of Γ implies that $\text{tr } S(\pm i\rho_0) = 0$, i.e.

$$w_0 = \rho_0^2(w_2 - w_4\rho_0^2). \tag{5.12}$$

In particular, the function $h(k)$ defined in (2.12) vanishes at the endpoints of Γ .

Lemma 5.2. *There exist points $\{m_j\}_1^4 \subset \mathbb{C}$ such that*

$$\mathcal{Q}_{21}(k) = \frac{8f\Omega^2\Omega_h^2}{(f + \bar{f})(f_0 + \bar{f}_0)(\Omega - \Omega_h)^2} \frac{\prod_{j=1}^4(k - m_j)}{(k^2 - r_1^2)}. \tag{5.13}$$

Proof. By (5.6), \mathcal{Q}_{21} is a unique function of k , i.e. $\mathcal{Q}_{21}(k^+) = \mathcal{Q}_{21}(k^-)$. Thus $(k^2 - r_1^2)\mathcal{Q}_{21}$ is an entire function of $k \in \mathbb{C}$. The existence of $\{m_j\}_1^4$ satisfying (5.13) therefore follows if we can show that

$$(k^2 - r_1^2)\mathcal{Q}_{21}(k) = \frac{8f\Omega^2\Omega_h^2}{(f + \bar{f})(f_0 + \bar{f}_0)(\Omega - \Omega_h)^2}k^4 + O(k^3), \quad k \rightarrow \infty. \quad (5.14)$$

As $k \rightarrow \infty$, we have

$$\Phi(z, k^+) = \begin{pmatrix} \overline{f(z)} & 1 \\ f(z) & -1 \end{pmatrix} + O(1/k), \quad A(k) = \mathbb{I} + O(1/k), \quad k \rightarrow \infty. \quad (5.15)$$

Thus, by (5.3) and (4.31),

$$\begin{aligned} (k^2 - r_1^2)\mathcal{Q}(z, k^+) &= - \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{8k^4\Omega^2\Omega_h^2}{(f_0 + \bar{f}_0)(\Omega - \Omega_h)^2} \end{pmatrix} \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix}^{-1} - w_4k^4\mathbb{I} + O(k^3) \\ &= \frac{4\Omega^2\Omega_h^2}{(f + \bar{f})(f_0 + \bar{f}_0)(\Omega - \Omega_h)^2} \begin{pmatrix} \bar{f} - f & 2\bar{f} \\ 2f & f - \bar{f} \end{pmatrix} k^4 + O(k^3), \quad k \rightarrow \infty. \end{aligned} \quad (5.16)$$

The (21)-entry of this equation yields (5.14). \square

Define four points $\{k_j\}_{j=1}^4 \subset \hat{\mathbb{C}}$ by

$$w^2 + 1 = \frac{w_4^2 \prod_{j=1}^4 (k - k_j)(k - \bar{k}_j)}{(k^2 - r_1^2)^2}.$$

We assume that the k_j 's are ordered as in Subsect. 2.3. Let $\hat{\mathcal{S}}_z$ denote the double cover of the Riemann surface \mathcal{S}_z defined by adding cuts $[k_j, \bar{k}_j]$, $j = 1, \dots, 4$, both on the upper and lower sheets of \mathcal{S}_z . Thus a point $(k, \pm\lambda, \pm\mu)$ of $\hat{\mathcal{S}}_z$ is specified by giving a point $k \in \hat{\mathbb{C}}$ together with a choice of sign of λ and of

$$\mu = \sqrt{\prod_{j=1}^4 (k - k_j)(k - \bar{k}_j)}.$$

We specify the sheets so that $\lambda \rightarrow 1$ ($\lambda \rightarrow -1$) as $k \rightarrow \infty$ on sheets 1 and 2 (sheets 3 and 4), and $\mu \sim k^4$ ($\mu \sim -k^4$) as $k \rightarrow \infty$ on sheets 1 and 3 (sheets 2 and 4). As k crosses the cut $[-iz, iz]$, λ changes sign whereas the sign of μ remains unchanged. As k crosses any of the other cuts, μ changes sign whereas the sign of λ remains unchanged.

Consider the function H defined by

$$H(z, k) = \frac{\hat{\mathcal{L}}_{22} - \mathcal{L}_{21}\sqrt{w^2 + 1}}{\hat{\mathcal{L}}_{22} + \mathcal{L}_{21}\sqrt{w^2 + 1}}, \quad k \in \hat{\mathcal{S}}_z, \quad (5.17)$$

where $\hat{\mathcal{L}}_{22} = \mathcal{L}_{21}\mathcal{Q}_{11} + \mathcal{L}_{22}\mathcal{Q}_{21}$. We fix the sign of the root $\sqrt{w^2 + 1}$ in (5.17) by requiring that $\sqrt{w^2 + 1} = -w_4k^2 + O(k)$ as $k \rightarrow \infty^+$. Since $w_4 > 0$, this implies that $\sqrt{w^2 + 1} \geq 0$ for $k \in \Gamma^+$. The eigenfunction Φ of the Lax pair (4.1) satisfies $\det \Phi = -2e^{2U}F(k)$ (see Eq. (2.65) in [24]), so that the entries of \mathcal{L} may have poles at

the points which project to the zeros of $F(k)$ in the Riemann k -sphere. However, H has no singularities at these points. Therefore, in view of (5.8), the possible zeros and poles of H belong to the set in \hat{S}_z which projects to $\{\pm r_1\} \cup \{m_j\}_1^4$, and if H has a double pole at (m_j, λ, μ) , then it has a double zero at $(m_j, \lambda, -\mu)$, $j = 1, \dots, 4$.

By the symmetries (5.6), we have

$$\hat{\mathcal{L}}_{22}(k, \lambda, \mu) = \hat{\mathcal{L}}_{22}(k, -\lambda, \mu), \quad \mathcal{L}_{21}(k, \lambda, \mu) = -\mathcal{L}_{21}(k, -\lambda, \mu).$$

Therefore

$$H(k, \lambda, \mu) = \frac{1}{H(k, -\lambda, \mu)}.$$

Similarly, we have $H(k, \lambda, \mu) = 1/H(k, \lambda, -\mu)$. Consequently, $H \rightarrow 1/H$ whenever k crosses one of the cuts of the two-sheeted Riemann surface Σ_z defined by (2.8). We can therefore view H as a single-valued function on Σ_z with the values on the upper sheet given by the values of H on sheet 1 of \hat{S}_z , and the values on the lower sheet given by the inverses of these values.

5.1. Formulation of the scalar RH problem. We want to formulate a scalar RH problem in terms of the complex-valued function $\psi(z, k)$ defined by

$$\psi(z, k) = \frac{\log H(z, k)}{y}, \quad k \in \Sigma_z. \tag{5.18}$$

However, since $\log H$ is a multi-valued function on Σ_z , this definition of ψ needs to be supplemented by a choice of branches for the logarithm. We will fix a single-valued representative of ψ on Σ_z by introducing cuts which connect the zeros and poles of H . Across these cuts ψ will jump by multiples of $2\pi i/y$. The problem is that even though (5.8) implies that all zeros and poles of H lie in the cover of the set $\{\pm r_1\} \cup \{m_j\}_1^4$, the exact distribution of these zeros and poles is not known. It is therefore not clear at this stage how to make a consistent choice of branches.

We address this problem by considering the limit in which the solution f approaches the Kerr solution. For a solution near the Kerr solution, we can utilize the Kerr expressions for F and G to compute H explicitly to first order. This will give us the correct choice of branches in the Kerr limit and by continuity this choice extends also to more general solutions.

The Ernst potential for the Kerr black hole rotating with angular velocity Ω_h and with a horizon stretching from $-ir_1$ to ir_1 is given by

$$f^{kerr} = \frac{R_+ e^{-i\delta} + R_- e^{i\delta} - 2r_1}{R_+ e^{-i\delta} + R_- e^{i\delta} + 2r_1},$$

where R_\pm are defined by

$$R_\pm = \sqrt{(\pm r_1 - \zeta)^2 + \rho^2},$$

and the parameter $\delta \in (-\pi/2, 0)$ is related to Ω_h by

$$\Omega_h = \frac{if_1^{kerr}(1 + (f_1^{kerr})^2)}{2r_1(1 - (f_1^{kerr})^2)}, \quad f_1^{kerr} = i \tan(\delta/2).$$

Table 1. The change $\Delta \arg H$ in $\arg H$ as k traverses each of the cycles in the cut basis $\{a_j, b_j\}_{j=1}^4$

Cycle	$\Delta \arg H$	Cycle	$\Delta \arg H$
a_1	-4π	b_1	-8π
a_2	-4π	b_2	-8π
a_3	4π	b_3	-12π
a_4	4π	b_4	-4π

The value at the origin is given by

$$f_0^{kerr} = \frac{\cos(\delta) - 1}{\cos(\delta) + 1}.$$

We consider adding a slowly rotating disk to the Kerr solution. Using the Kerr values for f_0 and f_1 , we compute w_4, w_2, w_0 according to (5.9) with $\Omega \ll 1$. The branch points $\{k_j\}_1^4$ are found by solving the equation $w^2 + 1 = 0$. As $\Omega \rightarrow 0$, k_1 and k_4 tend to infinity, whereas k_2 and k_3 approach finite values. The spectral functions $F^{kerr}(k)$ and $G^{kerr}(k)$ are given explicitly by (cf. Sect. 2.4 in [24])

$$F^{kerr}(k) = \frac{2\Omega_h^2(k^2 - r_1^2) + 2i\Omega_h f_1^{kerr} k - (f_1^{kerr})^2}{2\Omega_h^2(k^2 - r_1^2)},$$

$$G^{kerr}(k) = \frac{(2i\Omega_h r_1 - f_1^{kerr})(f_1^{kerr})^2}{2\Omega_h^2(k^2 - r_1^2)}.$$

For definiteness, we consider the example of $r_1 = 1/2$ and $\delta = -1/2$. Assuming that $z = \rho + i\zeta$ with $\rho \ll 1$ and $\zeta \gg 1$, we compute \mathcal{Q} and \mathcal{L} to first order by substituting the axis values (4.9a) for Φ together with the values of the Kerr solution into the right-hand sides of (5.2) and (5.3). We find that H has double poles and double zeros at the points in the sets

$$\{m_1^-, m_2^-, -r_1^-, m_3^+, m_4^+, r_1^+\} \quad \text{and} \quad \{m_1^+, m_2^+, -r_1^+, m_3^-, m_4^-, r_1^-\},$$

respectively. As $\Omega \rightarrow 0$, $m_1 \rightarrow k_1$ and $m_4 \rightarrow k_4$, whereas m_2 and m_3 converge to values close (but not equal) to k_2 and k_3 , respectively. Let $\{a_j, b_j\}_1^4$ be the particular cycles on Σ_z specified in Remark 2.2. As k traverses each of these cycles, the argument of H changes by the amount $\Delta \arg H$ according to Table 1.

A choice of branches for $\log H$ consistent with the above properties is obtained by introducing cuts on Σ_z according to Fig. 11. The introduced cuts run from the poles of H to the zeros of H . Letting $(\log H)^+$ and $(\log H)^-$ denote the values of $\log H$ for z just to the right and to the left of a cut, respectively, we have $(\log H)^+ + 4\pi i = (\log H)^-$. An overall choice of branch is made by requiring that $\log H \rightarrow 2 \log f$ as $k \rightarrow \infty^+$, where an appropriate branch is chosen for $\log f$, see Eq. (5.20) below.

For this choice of branches, we have $\log H(k^+) = -\log H(k^-)$, $k \in \hat{\mathbb{C}}$. Consequently, $\psi(z, k)$ is a unique function of k , i.e.

$$\psi(z, k^+) = \psi(z, k^-), \quad k \in \hat{\mathbb{C}}.$$

We therefore view $\psi(z, \cdot)$ as a function $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$.

Proposition 5.3. *Let ψ be defined by (5.18) with the choice of branches for $\log H$ specified above. Then the function $\psi(z, \cdot) : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ has the following properties:*

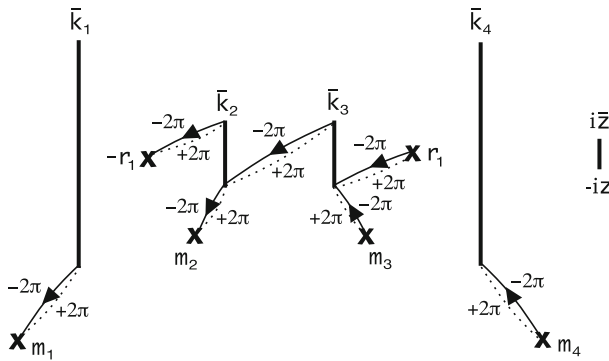


Fig. 11. The additional cuts introduced on Σ_z in order to make $\log H$ a single-valued function

- $\psi(z, k)$ is analytic for $k \in \hat{\mathbb{C}} \setminus \left[\Gamma \cup \left(\bigcup_{j=1}^4 [m_j, k_j] \right) \cup \{\pm r_1\} \right]$.
- Across Γ , $\psi(z, k)$ satisfies the jump condition

$$\psi^-(z, k) = \psi^+(z, k) + \frac{2}{y(z, k^+)} \ln \left(\frac{\sqrt{1+w^2-w}}{\sqrt{1+w^2+w}} \right), \quad k \in \Gamma, \quad (5.19)$$

where ψ^+ and ψ^- denote the values of ψ to the right and left of the cut, respectively.

- Across the directed intervals $[k_j, m_j]$, $j = 1, 2$; $[m_j, k_j]$, $j = 3, 4$; $[r_1, k_3]$, $[\bar{k}_3, k_2]$, and $[\bar{k}_2, -r_1]$, $\psi(z, k)$ satisfies the jump condition

$$\psi^-(z, k) = \psi^+(z, k) + \frac{4\pi i}{y(z, k^+)}.$$

- As $k \rightarrow m_j$, $j = 1, 2$, $\psi(z, k)$ satisfies

$$\psi(z, k) = \frac{2}{y(z, m_j^+)} \log(k - m_j), \quad k \rightarrow m_j, \quad j = 1, 2.$$

- As $k \rightarrow m_j$, $j = 3, 4$, $\psi(z, k)$ satisfies

$$\psi(z, k) = \frac{-2}{y(z, m_j^+)} \log(k - m_j), \quad k \rightarrow m_j, \quad j = 3, 4.$$

- As $k \rightarrow -r_1$, $\psi(z, k)$ satisfies

$$\psi(z, k) = \frac{2}{y(z, -r_1^+)} \log(k + r_1), \quad k \rightarrow -r_1.$$

- As $k \rightarrow r_1$, $\psi(z, k)$ satisfies

$$\psi(z, k) = \frac{-2}{y(z, r_1^+)} \log(k - r_1), \quad k \rightarrow r_1.$$

- As $k \rightarrow k_j$, $j = 1, \dots, 4$, $\psi(z, k)$ satisfies

$$\psi(z, k) = \frac{2\pi i}{y}, \quad k \rightarrow k_j, \quad j = 1, \dots, 4,$$

where $y = y(z, k^+)$ for k just to the left of the cut $[k_j, m_j]$ for $j = 1, 2$, and just to the left of the cut $[m_j, k_j]$ for $j = 3, 4$ and is analytically continued around the endpoint k_j so that $y = y(z, k^-)$ to the right of the cut.

- As $k \rightarrow \infty$,

$$\psi \rightarrow \frac{2 \log(f)}{k^5} + O(1/k^6), \quad k \rightarrow \infty. \tag{5.20}$$

Proof. We first show that ψ satisfies the jump condition (5.19). Algebraic manipulation of (5.7) using the identity $\text{tr}(\mathcal{Q}\mathcal{L}) = 0$ and the properties in (5.4) shows that the functions \mathcal{L}_{21} and $\hat{\mathcal{L}}_{22} = \mathcal{L}_{21}\mathcal{Q}_{11} + \mathcal{L}_{22}\mathcal{Q}_{21}$ satisfy

$$\begin{cases} \mathcal{L}_{21+} = -2w\hat{\mathcal{L}}_{22-} + (1 + 2w^2)\mathcal{L}_{21-}, \\ \hat{\mathcal{L}}_{22+} = (1 + 2w^2)\hat{\mathcal{L}}_{22-} - 2w(1 + w^2)\mathcal{L}_{21-}, \end{cases} \quad k \in \Gamma^+, \tag{5.21}$$

$$\begin{cases} \mathcal{L}_{21+} = 2w\hat{\mathcal{L}}_{22-} + (1 + 2w^2)\mathcal{L}_{21-}, \\ \hat{\mathcal{L}}_{22+} = (1 + 2w^2)\hat{\mathcal{L}}_{22-} + 2w(1 + w^2)\mathcal{L}_{21-}, \end{cases} \quad k \in \Gamma^-. \tag{5.22}$$

Thus

$$H_-(k) = \left(\frac{\sqrt{1+w^2} - w}{\sqrt{1+w^2} + w} \right)^{\pm 2} H_+(k), \quad k \in \Gamma^\pm.$$

Equation (5.19) follows from here since $1 + w^2 \geq 0$ and $\sqrt{1+w^2} \pm w > 0$ for $k \in \Gamma$.

The behavior as $k \rightarrow \pm r_1$ and $k \rightarrow m_j$ follows since H has double zeros at $-r_1, m_j^+$, $j = 1, 2$, and double poles at r_1, m_j^+ , $j = 3, 4$.

In order to find the behavior of ψ as $k \rightarrow \infty$, we note that, by (5.15),

$$\mathcal{L}(z, k^+) = \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix} \sigma_1 \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix}^{-1} + O(1/k), \quad k \rightarrow \infty.$$

Thus, by (5.16),

$$\hat{\mathcal{L}}_{22}(z, k^+) = -\frac{4(1+f^2)\Omega^2\Omega_h^2}{(f+\bar{f})(f_0+\bar{f}_0)(\Omega-\Omega_h)^2}k^2 + O(k), \quad k \rightarrow \infty.$$

Since $\sqrt{w^2+1} = -w_4k^2 + O(k)$ as $k \rightarrow \infty^+$, we find

$$H(z, k^+) = f(z)^2 + O(1/k), \quad k \rightarrow \infty.$$

Therefore

$$\frac{\log H}{y} \rightarrow \frac{2 \log f}{k^5} + O(1/k^6), \quad k \rightarrow \infty.$$

□

5.2. *Solution of the scalar RH problem.* The solution of the scalar RH problem presented in Proposition 5.3 is

$$\begin{aligned} \psi(z, k) &= \frac{2}{\pi i} \int_{\Gamma} \frac{dk'}{y(z, k'^+)(k' - k)} \ln \left(\sqrt{1 + w(k')^2} - w(k') \right) \\ &+ 2 \sum_{j=1}^2 \int_{[k_j, m_j]} \frac{dk'}{y(z, k'^+)(k' - k)} - 2 \sum_{j=3}^4 \int_{[k_j, m_j]} \frac{dk'}{y(z, k'^+)(k' - k)} \\ &+ 2 \left(\int_{[r_1, k_3]} + \int_{[\bar{k}_3, k_2]} + \int_{[\bar{k}_2, -r_1]} \right) \frac{dk'}{y(z, k'^+)(k' - k)}, \quad k \in \Sigma_z, \end{aligned} \tag{5.23}$$

where we used that

$$\ln \left(\sqrt{1 + w^2} + w \right) = - \ln \left(\sqrt{1 + w^2} - w \right), \quad k \in \Gamma.$$

By deforming contours, we can replace the last three integrals on the right-hand side of (5.23) with an integral along γ , where γ is the contour on Σ_z defined in Sect. 2. We define two divisors \mathfrak{K} and \mathfrak{M} on Σ_z . \mathfrak{K} is defined by

$$\mathfrak{K} = \sum_{j=1}^4 k_j, \tag{5.24}$$

whereas \mathfrak{M} is defined as the sum of the points in Σ_z which lie above the set $\{m_j\}_1^4$ and which are double poles of H , i.e.

$$\mathfrak{M} = m_1^- + m_2^- + m_3^+ + m_3^+. \tag{5.25}$$

We can then write (5.23) as

$$\begin{aligned} \psi(z, k) &= \frac{2}{\pi i} \int_{\Gamma^+} \frac{dk'}{y(z, k')(k' - k)} \ln \left(\sqrt{1 + w(k')^2} - w(k') \right) \\ &- 2 \int_{\mathfrak{K}}^{\mathfrak{M}} \frac{dk'}{y(z, k')(k' - k)} + 2 \int_{\gamma} \frac{dk'}{y(z, k')(k' - k)}, \end{aligned}$$

where the integrals are contour integrals on Σ_z and the prime on the integral from \mathfrak{K} to \mathfrak{M} indicates that the paths of integration do not necessarily lie in the complement of the cut basis $\{a_j, b_j\}$. In view of (5.20), this leads to

$$\log f = \int_{\mathfrak{K}}^{\mathfrak{M}} \frac{k^4 dk}{y} - \int_{\Gamma^+} h(k) \frac{k^4 dk}{y} - \int_{\gamma} \frac{k^4 dk}{y}, \tag{5.26a}$$

$$\int_{\mathfrak{K}}^{\mathfrak{M}} \frac{k^{n-1} dk}{y} = \int_{\Gamma^+} h(k) \frac{k^{n-1} dk}{y} + \int_{\gamma} \frac{k^{n-1} dk}{y}, \quad n = 1, \dots, 4, \tag{5.26b}$$

where $h(k)$ is defined by (2.12).

Remark 5.4. Although Eqs. (5.26) were derived under the assumption that the solution is a small perturbation of the Kerr solution, they are valid more generally. Indeed, the crucial facts used in the derivation are that H has a double pole at r_1^+ and a double zero at $-r_1^+$, and these properties are preserved under a continuous deformation. It is conceivable that the double poles of H that make up \mathfrak{M} will change sheets under such a deformation so that (5.25) has to be modified, but the resulting Eqs. (5.26) remain unchanged.

6. Theta Functions

In this section we derive explicit expressions for the Ernst potential f and the metric functions e^{2U} and a in terms of theta functions.

6.1. Explicit expression for the Ernst potential. We will show that the right-hand side of (5.26a) can be expressed in terms of the theta function on Σ_z . We will first assume that the integration paths from \mathfrak{K} to \mathfrak{M} in (5.26) lie in the fundamental polygon determined by the cut basis; later we will see that the result is the same also when this is not the case.

Let $\{\eta_j\}_{j=1}^4$ denote the noncanonical basis of holomorphic one-forms on Σ_z defined in (3.1) and let A be the matrix defined in (3.2). Then the canonical basis is $\omega = A\eta$. Let $u, I \in \mathbb{C}^4$ be defined by (2.13). Applying A to (5.26b), we find

$$u = \int_{\mathfrak{K}}^{\mathfrak{M}} \omega. \tag{6.1}$$

Using that

$$\omega_{\infty^+\infty^-} = -\frac{k^4 dk}{y} + \gamma^T \eta,$$

for some vector $\gamma \in \mathbb{C}^4$, Eq. (5.26a) yields

$$f = e^{-\int_{\mathfrak{K}}^{\mathfrak{M}} \omega_{\infty^+\infty^-} + I}, \tag{6.2}$$

where the terms involving γ cancelled because of (5.26b). Formula (2.14) for f will follow if we can prove that

$$e^{-\int_{\mathfrak{K}}^{\mathfrak{M}} \omega_{\infty^+\infty^-}} = \frac{\Theta\left(u - \int_{-iz}^{\infty^-} \omega\right)}{\Theta\left(u + \int_{-iz}^{\infty^-} \omega\right)}, \tag{6.3}$$

where $\Theta(v) := \Theta(v|B)$. Let $e^{(j)}$ and $\pi^{(j)}$ denote the j^{th} columns of the 4×4 identity matrix I and the period matrix B , respectively. Then

$$\Theta(v + e^{(j)}|B) = \Theta(v|B), \quad \Theta(v + \pi^{(j)}|B) = e^{-2\pi i(v_j + \frac{1}{2}B_{jj})} \Theta(v|B), \quad v \in \mathbb{C}^4. \tag{6.4}$$

The Jacobian $Jac(\Sigma_z)$ of Σ_z is defined as the complex torus \mathbb{C}^4/\mathbb{L} , where \mathbb{L} is the discrete lattice generated by the $e^{(j)}$'s and the $\pi^{(j)}$'s. We define the map $\varphi : \Sigma_z \rightarrow \mathbb{C}^4$ by

$$\varphi(k) = \int_{-iz}^k \omega, \quad k \in \Sigma_z,$$

with the contour fixed to lie within the fundamental polygon. Then φ composed with the projection $\mathbb{C}^4 \rightarrow Jac(\Sigma_z)$ is the Abel map with base point $-iz$. We write $\mathfrak{M} = \sum_{j=1}^4 M_j$, where, for $j = 1, \dots, 4$, $M_j = m_j^+$ or $M_j = m_j^-$. Let $\mathcal{K} = \varphi(\mathfrak{K}) \in \mathbb{C}^4$. An

argument following pp. 322–325 in [6] shows that \mathcal{K} projects to the vector of Riemann constants in $Jac(\Sigma_z)$. Thus the functions

$$\frac{\Theta(\varphi(P) - \varphi(\mathfrak{R}) + \mathcal{K})}{\Theta(\varphi(P) - \varphi(\mathfrak{M}) + \mathcal{K})}, \quad P \in \Sigma_z, \tag{6.5}$$

and

$$e^{-\sum_{j=1}^4 \int_{\infty^-}^P \omega_{M_j k_j}}, \quad P \in \Sigma_z, \tag{6.6}$$

both have simple poles at the points of \mathfrak{M} and simple zeros at the points of \mathfrak{R} . Moreover, the general identity ([6], p. 67)

$$\int_{b_j} \omega_{RS} = 2\pi i \int_S^R \omega_j, \quad j = 1, \dots, 4, \quad R, S \in \Sigma_z, \tag{6.7}$$

implies that as a_j is traversed the functions (6.5) and (6.6) both get multiplied by 1 and as b_j is traversed they both get multiplied by $e^{-2\pi i \int_{\mathfrak{R}}^{\mathfrak{M}} \omega_j}$. Hence their quotient is a constant and we deduce that

$$e^{-\sum_{j=1}^4 \int_{\infty^-}^P \omega_{M_j k_j}} = \frac{\Theta(\varphi(P) - \varphi(\mathfrak{R}) + \mathcal{K})\Theta(\varphi(\infty^-) - \varphi(\mathfrak{M}) + \mathcal{K})}{\Theta(\varphi(P) - \varphi(\mathfrak{M}) + \mathcal{K})\Theta(\varphi(\infty^-) - \varphi(\mathfrak{R}) + \mathcal{K})}. \tag{6.8}$$

Using the identity

$$\int_S^R \omega_{\infty^+ \infty^-} = \int_{\infty^-}^{\infty^+} \omega_{RS}, \quad R, S \in \Sigma_z, \tag{6.9}$$

and the fact that $\mathcal{K} = \varphi(\mathfrak{R})$, evaluation of (6.8) at $P = \infty^+$ yields

$$e^{-\int_{\mathfrak{R}}^{\mathfrak{M}} \omega_{\infty^+ \infty^-}} = \frac{\Theta(\varphi(\infty^+))\Theta\left(\varphi(\infty^-) - \int_{\mathfrak{R}}^{\mathfrak{M}} \omega\right)}{\Theta\left(\varphi(\infty^+) - \int_{\mathfrak{R}}^{\mathfrak{M}} \omega\right)\Theta(\varphi(\infty^-))}. \tag{6.10}$$

Our choice of the cut basis $\{a_j, b_j\}$ implies that

$$\varphi(\infty^+) = -\varphi(\infty^-) \text{ modulo } a\text{-periods}. \tag{6.11}$$

Thus, since $\Theta(v)$ is an even function, we arrive at (6.3).

Now suppose the integration paths from \mathfrak{R} to \mathfrak{M} in (5.26) do not lie within the fundamental polygon. Then there exist integer vectors $p, q \in \mathbb{Z}^4$ such that equations (6.1) and (6.2) get replaced by

$$u = \int_{\mathfrak{R}}^{\mathfrak{M}} \omega + Bp + q \quad \text{and} \quad f = e^{-\int_{\mathfrak{R}}^{\mathfrak{M}} \omega_{\infty^+ \infty^-} - \sum_{j=1}^4 p_j \int_{b_j} \omega_{\infty^+ \infty^-} + I},$$

respectively. However, a computation using (6.4), (6.7), and (6.10) shows that the terms involving p and q cancel, so that f is still given by (2.14).

We can now complete most of the proof of Theorem 2.1; the derivation of the formula for $e^{2\kappa}$ will be postponed to the Appendix. We first establish the formulas in (2.15): The expression for Ω_h follows from (4.8); the expression for Ω follows by solving (5.9a) for Ω recalling that $e^{2U_0} < 0$ and $w_4 > 0$; and the expression for $e^{2U_{\Omega(+i0)}}$ follows by evaluating (2.4a) at $z = +i0$.

6.2. *The metric functions e^{2U} and a .* Using formula (2.14), which was established in the previous subsection, the expression for the metric function e^{2U} in (2.17) can be derived as follows cf. [15]. Since the entries of u are purely imaginary, $I \in \mathbb{R}$, and $\Theta(\bar{v}) = \overline{\Theta(v)}$ for $v \in \mathbb{C}^4$ ([17], p. 203), Eq. (2.14) yields

$$f + \bar{f} = \left(\frac{\Theta(u - \int_{-iz}^{\infty^-} \omega)}{\Theta(u + \int_{-iz}^{\infty^-} \omega)} + \frac{\Theta(u - \int_{i\bar{z}}^{\infty^-} \omega)}{\Theta(u + \int_{i\bar{z}}^{\infty^-} \omega)} \right) e^I. \tag{6.12}$$

Let $E(P, Q)$ denote the prime form on Σ_z . Applying Fay’s identity ([17], p. 205)

$$\begin{aligned} & \frac{E(P_3, P_1)E(P_2, P_4)}{E(P_3, P_4)E(P_2, P_1)} \Theta\left(v + \int_{P_2}^{P_3} \omega\right) \Theta\left(v + \int_{P_1}^{P_4} \omega\right) \\ & + \frac{E(P_3, P_2)E(P_1, P_4)}{E(P_3, P_4)E(P_1, P_2)} \Theta\left(v + \int_{P_1}^{P_3} \omega\right) \Theta\left(v + \int_{P_2}^{P_4} \omega\right) \\ & = \Theta(v) \Theta\left(v + \int_{P_2}^{P_3} \omega + \int_{P_1}^{P_4} \omega\right), \quad v \in \mathbb{C}^4, \end{aligned}$$

with $(P_1, P_2, P_3, P_4) = (i\bar{z}, \infty^+, -iz, \infty^-)$ to (6.12), we find

$$\begin{aligned} & 2Q(0) \Theta\left(v + \int_{\infty^+}^{-iz} \omega\right) \Theta\left(v + \int_{i\bar{z}}^{\infty^-} \omega\right) - \Theta\left(v + \int_{i\bar{z}}^{-iz} \omega\right) \Theta\left(v + \int_{\infty^+}^{\infty^-} \omega\right) \\ & = \Theta(v) \Theta\left(v + \int_{\infty^+}^{-iz} \omega + \int_{i\bar{z}}^{\infty^-} \omega\right), \end{aligned} \tag{6.13}$$

where

$$Q(0) = \frac{1}{2} \frac{E(-iz, i\bar{z})E(\infty^+, \infty^-)}{E(-iz, \infty^-)E(\infty^+, i\bar{z})},$$

and we used that (Lemma 3.12 in [17])

$$\frac{E(-iz, \infty^+)E(i\bar{z}, \infty^-)}{E(-iz, \infty^-)E(i\bar{z}, \infty^+)} = -1.$$

By Proposition 3.11 in [17], $Q(0)$ can be written as in (2.18). Letting $v = u + \int_{-iz}^{\infty^+} \omega = u - \int_{-iz}^{\infty^-} \omega$ and dividing by $\Theta(u + \int_{-iz}^{\infty^-} \omega)\Theta(u + \int_{i\bar{z}}^{\infty^-} \omega)$, Eq. (6.13) yields

$$2Q(0) \frac{\Theta(u) \Theta(u + \int_{i\bar{z}}^{-iz} \omega)}{\Theta(u + \int_{-iz}^{\infty^-} \omega) \Theta(u + \int_{i\bar{z}}^{\infty^-} \omega)} - \frac{\Theta(u + \int_{i\bar{z}}^{\infty^+} \omega)}{\Theta(u + \int_{i\bar{z}}^{\infty^-} \omega)} = \frac{\Theta(u + \int_{-iz}^{\infty^+} \omega)}{\Theta(u + \int_{-iz}^{\infty^-} \omega)}. \tag{6.14}$$

Equations (6.12) and (6.14) lead to the expression for e^{2U} in (2.17).

By (5.7) in [16], we have

$$(a - a_0)e^{2U} = -\rho \left(\frac{\Theta(u + \int_{-iz}^{\infty^-} \omega + \int_{i\bar{z}}^{\infty^-} \omega)}{Q(0)Q(u)\Theta(u + \int_{-iz}^{i\bar{z}} \omega)} - 1 \right), \tag{6.15}$$

where $a_0 \in \mathbb{R}$ is a constant determined by the condition that $a = 0$ on the regular axis. In view of the expression for e^{2U} , this yields the expression for the metric function a given in (2.17a). An alternative derivation of Eq. (6.15) is presented in the Appendix.

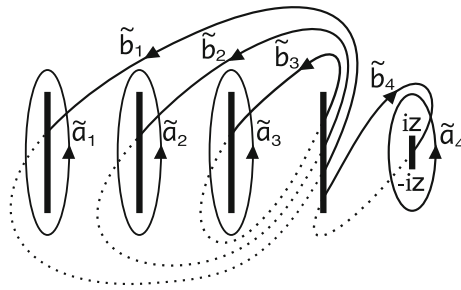


Fig. 12. The axis-adapted homology basis $\{\tilde{a}_j, \tilde{b}_j\}_{j=1}^4$ on Σ_z

7. Axis and Horizon Values

In this section we consider the limits of the formulas in Theorem 2.1 as z approaches a point on the ζ -axis. As $\rho \downarrow 0$, the Riemann surface Σ_z degenerates since the branch cut $[-iz, iz]$ shrinks to a point. This type of degeneration of a Riemann surface is analyzed in Chap. III of [7]. In order to utilize the results of [7], we introduce an axis-adapted cut basis $\{\tilde{a}_j, \tilde{b}_j\}_{j=1}^4$ on Σ_z by

$$\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} \mathcal{A}^T & 0 \\ 0 & \mathcal{A} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{where} \quad \mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The axis-adapted cut basis is displayed in Fig. 12 in the case when $\zeta > \text{Re } k_4$. Note that \tilde{a}_4 surrounds the collapsing cut $[-iz, iz]$.

According to the transformation formula for theta functions (Eq. (12) in [7]), there exists a constant c_0 independent of v and B such that

$$\Theta(\tilde{v}|\tilde{B}) = c_0 \Theta(v|B) \tag{7.1}$$

whenever $\tilde{v} = \mathcal{A}^{-1}v = \mathcal{A}v$. We define \tilde{u} and \tilde{I} as the analogs of u and I in the axis-adapted basis, i.e.

$$\tilde{u} = \int_{\Gamma^+} h\tilde{\omega} + \int_{\gamma} \tilde{\omega}, \quad \tilde{I} = \int_{\Gamma^+} h\tilde{\omega}_{\infty^+\infty^-} + \int_{\gamma} \tilde{\omega}_{\infty^+\infty^-}.$$

Since $\tilde{\omega} = \mathcal{A}\omega$, we have $\tilde{u} = \mathcal{A}u$. Since

$$\omega_{\infty^+\infty^-} = \tilde{\omega}_{\infty^+\infty^-} - 2\pi i \tilde{\omega}_4,$$

we have $I = \tilde{I} - 2\pi i \tilde{u}_4$. Thus, introducing the shorthand notation $\tilde{\Theta}(v) := \Theta(v|\tilde{B})$, the Ernst potential (2.14) can be expressed in terms of the axis-adapted basis as

$$f(z) = \frac{\tilde{\Theta}(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega})}{\tilde{\Theta}(\tilde{u} + \int_{-iz}^{\infty^-} \tilde{\omega})} e^{-2\pi i \tilde{u}_4 + \tilde{I}}. \tag{7.2}$$

Let Σ' denote the degenerated Riemann surface defined in (2.19). According to [7], we have the following expansions as $\rho \downarrow 0$:

$$\tilde{\omega}_j = \omega'_j + O(\rho^2), \quad j = 1, 2, 3; \quad \tilde{\omega}_4 \rightarrow \frac{1}{2\pi i} \omega'_{\zeta^+\zeta^-} + O(\rho^2), \quad (7.3a)$$

$$\tilde{B}_{ij} \rightarrow B'_{ij} + O(\rho^2), \quad i, j = 1, 2, 3; \quad \tilde{B}_{i4} \rightarrow \int_{\zeta^-}^{\zeta^+} \omega'_i + O(\rho^2), \quad i = 1, 2, 3, \quad (7.3b)$$

$$\tilde{B}_{44} = \frac{1}{\pi i} \ln \rho + \frac{M'}{\pi i} + O(\rho^2), \quad (7.3c)$$

where $M' \in \mathbb{C}$ is a constant. The path of integration from ζ^- to ζ^+ in (7.3b) must be chosen as the limit of the cycle \tilde{b}_4 . For example, for $\text{Re } k_3 < \zeta < \text{Re } k_4$, this path is $[\zeta, \bar{k}_4]^- \cup [\bar{k}_4, \zeta]^+$. Whether this path lies within the fundamental polygon on Σ' depends on the particular representatives of the homology cycles a'_j and b'_j .

7.1. Values near the regular axis. We first consider the case when z approaches a point on the regular axis. We define $c \in \mathbb{C}^4$ and $c' \in \mathbb{C}^3$ by

$$c = \int_{k_4}^{\infty^-} \tilde{\omega}, \quad c' = \int_{k_4}^{\infty^-} \omega'. \quad (7.4)$$

Lemma 7.1. *The following limits hold as z approaches a point on the regular axis (i.e. as $\rho \rightarrow 0$ with $r_1 < \zeta$):*

$$\tilde{u} = \left(\frac{u'}{2\pi i} \right) + O(\rho^2), \quad c = \left(\frac{c'}{2\pi i} \right) + O(\rho^2), \quad (7.5a)$$

$$\tilde{\Theta} \left(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega} \right) = \Theta' \left(u' - \int_{\zeta^-}^{\infty^-} \omega' \right) - \Theta' \left(u' - \int_{\zeta^+}^{\infty^-} \omega' \right) e^{J'-K'} + O(\rho^2), \quad (7.5b)$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{-iz}^{\infty^-} \tilde{\omega} \right) = \Theta' \left(u' + \int_{\zeta^-}^{\infty^-} \omega' \right) - \Theta' \left(u' + \int_{\zeta^+}^{\infty^-} \omega' \right) e^{-J'-K'} + O(\rho^2), \quad (7.5c)$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{i\bar{z}}^{\infty^-} \tilde{\omega} \right) = \Theta' \left(u' + \int_{\zeta^-}^{\infty^-} \omega' \right) + \Theta' \left(u' + \int_{\zeta^+}^{\infty^-} \omega' \right) e^{-J'-K'} + O(\rho^2), \quad (7.5d)$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{-iz}^{i\bar{z}} \tilde{\omega} \right) = \Theta'(u') + \beta\rho + O(\rho^2), \quad \tilde{\Theta}(\tilde{u}) = \Theta'(u') - \beta\rho + O(\rho^2), \quad (7.5e)$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{-iz}^{\infty^-} \tilde{\omega} + \int_{i\bar{z}}^{\infty^-} \tilde{\omega} \right) = -\frac{1}{\rho} \Theta'(u' + 2c') e^{-J'-2K'-M'} + O(1), \quad (7.5f)$$

where $\beta \in \mathbb{C}$ is a constant and u', K', J' , and M' are defined in (2.20), (2.21), and (2.29). The equations obtained from (7.5) by replacing \tilde{u}, u' , and J' by 0 everywhere are also valid.

Proof. The expansions in (7.5a) follow immediately from (7.3a). We will also show (7.5b); the proofs of the other expansions are similar. We first assume that $\zeta > \text{Re } k_4$. In order for the argument of the theta function to have a finite limit, we shift the integration limit from $-iz$ to k_4 . Our choice of the cut system $\{\tilde{a}_j, \tilde{b}_j\}$ implies that

$$\int_{-iz}^{\infty^-} \tilde{\omega} = c + \tilde{B}r + s, \tag{7.6}$$

where $r, s \in \mathbb{R}^4$ are defined by

$$r = (0, 0, 0, 1/2)^T, \quad s = (0, 0, 0, -1/2)^T. \tag{7.7}$$

Therefore,

$$\tilde{\Theta} \left(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega} \right) = \tilde{\Theta}(\tilde{u} - c - \tilde{B}r - s) = \sum_{N \in \mathbb{Z}^4} e^{2\pi i (\frac{1}{2} N^T \tilde{B} N + N^T (\tilde{u} - c - \tilde{B}r - s))}. \tag{7.8}$$

Using (7.7), we can write the right-hand side of (7.8) as

$$\sum_{N \in \mathbb{Z}^4} e^{2\pi i \left(\frac{1}{2} \sum_{i,j=1}^3 N_i \tilde{B}_{ij} N_j + \frac{1}{2} \tilde{B}_{44} N_4 (N_4 - 1) + \sum_{i=1}^3 \tilde{B}_{i4} N_i (N_4 - \frac{1}{2}) + \sum_{i=1}^3 N_i (\tilde{u}_i - c_i) + N_4 (\tilde{u}_4 - c_4) + \frac{1}{2} N_4 \right)}.$$

In view of (7.3c), only the terms with $N_4 = 0$ and $N_4 = 1$ give nonzero contributions in the limit $\rho \rightarrow 0$. Equations (7.3) and (7.5a) imply that the subleading terms, which also receive contributions from the terms with $N_4 = -1$ and $N_4 = 2$, are of $O(\rho^2)$. We find

$$\begin{aligned} \tilde{\Theta} \left(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega} \right) &= \sum_{N' \in \mathbb{Z}^3} e^{2\pi i \left(\frac{1}{2} N'^T B' N' - \frac{1}{2} N'^T \int_{\zeta^-}^{\zeta^+} \omega' + N'^T (u' - c') \right)} \\ &\quad + \sum_{N' \in \mathbb{Z}^3} e^{2\pi i \left(\frac{1}{2} N'^T B' N' + \frac{1}{2} N'^T \int_{\zeta^-}^{\zeta^+} \omega' + N'^T (u' - c') \right) + J' - K' + \pi i} + O(\rho^2) \\ &= \Theta' \left(u' - c' - \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' \right) - \Theta' \left(u' - c' + \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' \right) e^{J' - K'} + O(\rho^2). \end{aligned} \tag{7.9}$$

Since

$$c' + \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' = \int_{\zeta^-}^{\infty^-} \omega', \quad c' - \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' = \int_{\zeta^+}^{\infty^-} \omega', \tag{7.10}$$

this proves (7.5b) in the case when $\zeta > \text{Re } k_4$.

Similar arguments apply when $\zeta < \text{Re } k_3$ or $\text{Re } k_3 < \zeta < \text{Re } k_4$. For example, if $\text{Re } k_3 < \zeta < \text{Re } k_4$, then Eq. (7.6) gets replaced by

$$\int_{-iz}^{\infty^-} \tilde{\omega} = c + \tilde{B}r + t,$$

where

$$r = (0, 0, 0, 1/2)^T, \quad t = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)^T. \tag{7.11}$$

Letting $t' = (-1/2, -1/2, -1/2)$, this leads to the following analog of Eq. (7.9):

$$\begin{aligned} \tilde{\Theta}\left(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega}\right) &= \Theta'\left(u' - c' - \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' - t'\right) \\ &\quad - \Theta'\left(u' - c' + \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' - t'\right) e^{J'-K'} + O(\rho^2). \end{aligned} \tag{7.12}$$

Taking into account that, for $\text{Re } k_3 < \zeta < \text{Re } k_4$,

$$c' + \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' + t' = \int_{\zeta^-}^{\infty^-} \omega', \quad c' - \frac{1}{2} \int_{\zeta^-}^{\zeta^+} \omega' + t' = \int_{\zeta^+}^{\infty^-} \omega', \tag{7.13}$$

we again arrive at (7.5b).

Lemma 7.2. *Let Q be given by (2.18). As $\rho \rightarrow 0$ with $r_1 < \zeta$,*

$$Q(u) = \frac{1}{\Theta'(u')^2} \left[\Theta'\left(u' + \int_{\zeta^-}^{\infty^-} \omega'\right)^2 - \Theta'\left(u' + \int_{\zeta^+}^{\infty^-} \omega'\right)^2 e^{-2J'-2K'} \right] + O(\rho^2). \tag{7.14}$$

The behavior of $Q(0)$ as $\rho \rightarrow 0$ is given by the expression obtained by replacing u' and J' with zero in the right-hand side of (7.14).

Proof. In view of (7.1), the expression for $Q(u)$ is invariant under the change of cut basis from $\{a_j, b_j\}$ to $\{\tilde{a}_j, \tilde{b}_j\}$, i.e.

$$Q(u) = \frac{\tilde{\Theta}(\tilde{u} + \int_{-iz}^{\infty^-} \tilde{\omega}) \tilde{\Theta}(\tilde{u} + \int_{i\bar{z}}^{\infty^-} \tilde{\omega})}{\tilde{\Theta}(\tilde{u}) \tilde{\Theta}(\tilde{u} + \int_{-iz}^{i\bar{z}} \tilde{\omega})}. \tag{7.15}$$

Utilizing the limits of Lemma 7.1, we find (7.14). \square

By applying the results of Lemma 7.1 to formula (7.2) and using that $e^I = e^{\tilde{I}-2\pi i \tilde{u}_4} = e^{I'-J'} + O(\rho^2)$ as $\rho \rightarrow 0$, we find that f is given by (2.22) near the regular axis. The expression (2.24) for e^{2U} on the regular axis follows by applying the results of Lemma 7.2 to the equation $e^{2U} = \frac{Q(0)}{Q(u)} e^I$. The limiting behavior $a = O(\rho^2)$ follows from (6.15); the fact that the terms of $O(\rho)$ vanish in the expansion of a is most easily seen from (2.2). The behavior $e^{2\kappa} = 1 + O(\rho^2)$ near the regular axis follows from (2.17b) and the condition that $\kappa = 0$ on the regular axis; the fact that the terms of $O(\rho)$ vanish in the expansion of $e^{2\kappa}$ is most easily seen from (3.7). This completes the proof of Proposition 2.3.

7.2. *Values of a_0 and K_0 .* The constant a_0 is determined by (6.15) and the condition that $a = 0$ on the regular axis. We find

$$a_0 = \lim_{\rho \rightarrow 0} \frac{\rho e^{-2U}}{Q(0)} \frac{\Theta(u + \int_{-iz}^{\infty^-} \omega + \int_{i\bar{z}}^{\infty^-} \omega)}{Q(u) \Theta(u + \int_{-iz}^{i\bar{z}} \omega)}, \quad \zeta > r_1.$$

Substituting into this equation the expression (2.17a) for e^{2U} and passing to the axis-adapted basis, we find

$$a_0 = \lim_{\rho \rightarrow 0} \frac{\rho}{Q(0)^2} \frac{\tilde{\Theta}(\tilde{u} + \int_{-iz}^{\infty^-} \tilde{\omega} + \int_{i\bar{z}}^{\infty^-} \tilde{\omega})}{\tilde{\Theta}(\tilde{u} + \int_{-iz}^{i\bar{z}} \tilde{\omega})} e^{2\pi i \tilde{u}_4 - \tilde{I}}.$$

By Lemmas 7.1 and 7.2, this yields

$$a_0 = - \frac{\Theta'(0)^4}{\left(\Theta'(\int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(\int_{\zeta^+}^{\infty^-} \omega')^2 e^{-2K'}\right)^2} \frac{\Theta'(u' + 2c')}{\Theta'(u')} e^{-M' - I' - 2K'}. \tag{7.16}$$

Expression (2.26) for a_0 is obtained by letting $\zeta \rightarrow \infty$ in (7.16). Indeed, since

$$\lim_{\zeta \rightarrow \infty} \frac{\Theta'(0)^4}{\left(\Theta'(\int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(\int_{\zeta^+}^{\infty^-} \omega')^2 e^{-2K'}\right)^2} = 1$$

and u', c', I' are independent of $\zeta > r_1$, we find

$$a_0 = - \frac{\Theta'(u' + 2c')}{\Theta'(u')} e^{-I'} \left(\lim_{\zeta \rightarrow \infty} e^{-M' - 2K'} \right). \tag{7.17}$$

The constant M' is given by⁶

$$M' = \frac{1}{2} \lim_{x \rightarrow 0} \left(\int_{(\zeta-x)^-}^{(\zeta-x)^+} \omega'_{\zeta^+ \zeta^-} - 2 \ln x - \ln 4 - \pi i \right).$$

The combination $M' + 2K'$ remains finite in the limit $\zeta \rightarrow \infty$ and we find

$$\lim_{\zeta \rightarrow \infty} (M' + 2K') = \frac{1}{2} \lim_{R \rightarrow \infty} \left[- \int_{R^-}^{R^+} \omega'_{\infty^+ \infty^-} - 2 \ln R - \ln 4 - \pi i \right]. \tag{7.18}$$

Equations (7.17) and (7.18) imply (2.26).

The constant K_0 in (2.17b) is determined by the condition that $e^{2\kappa} = 1$ on the regular part of the axis. In order to compute K_0 we first rewrite (2.17b) in terms of the axis-adapted cut system. Note that

$$\omega_{-r_1^+, -r_1^-} = \tilde{\omega}_{-r_1^+, -r_1^-} - 2\pi i \tilde{\omega}_4, \quad \omega_{r_1^+ r_1^-} = \tilde{\omega}_{r_1^+ r_1^-} - 2\pi i \tilde{\omega}_4, \quad \omega_{\kappa_1^+ \kappa_1^-} = \tilde{\omega}_{\kappa_1^+ \kappa_1^-} - 2\pi i \tilde{\omega}_4. \tag{7.19}$$

Since $\int_{\Gamma^+} d\kappa_1 \frac{dh}{dk}(\kappa_1) = 0$, we find

$$e^{2\kappa} = K_0 \frac{\tilde{\Theta}(\tilde{u}) \tilde{\Theta}(\tilde{u} + \int_{-iz}^{i\bar{z}} \tilde{\omega})}{\tilde{\Theta}(0) \tilde{\Theta}(\int_{\xi}^{\xi} \tilde{\omega})} e^{-\frac{1}{2} \int_{\Gamma^+} d\kappa_1 \frac{dh}{dk}(\kappa_1) \int_{\Gamma^+} h(\kappa_2) \tilde{\omega}_{\kappa_1^+ \kappa_1^-} (\kappa_2^+)^+ + \int_{\Gamma^+} h \tilde{\omega}_{-r_1^+, -r_1^-} - \int_{\Gamma^+} h \tilde{\omega}_{r_1^+ r_1^-}} \\ \times e^{\frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\int_{\gamma_1(\epsilon)} \tilde{\omega}_{-r_1^+, -r_1^-} - \int_{\gamma_2(\epsilon)} \tilde{\omega}_{r_1^+ r_1^-} - 2 \ln \epsilon \right)}, \quad \zeta > r_1.$$

Taking the limit as $\rho \rightarrow 0$ of this expression, we find (2.27).

⁶ Expressions of this type are considered in [28].

7.3. *Values near the black hole horizon.* The limits as z approaches the black hole horizon have a slightly different flavor than those considered in the previous subsection, because \tilde{u}_4 diverges as $\rho \downarrow 0$ with $0 < \zeta < r_1$. In fact,

$$\tilde{u}_4 = -\frac{1}{\pi i} \ln \rho + \frac{P'}{\pi i} + O(\rho^2), \quad \rho \downarrow 0, \quad 0 < \zeta < r_1, \tag{7.20}$$

where

$$P' = \frac{1}{2} \int_{\Gamma^+} h \omega'_{\zeta^+ \zeta^-} + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\int_{\gamma'(\epsilon)} \omega'_{\zeta^+ \zeta^-} + 2 \ln \epsilon + \ln 4 \right),$$

and $\gamma'(\epsilon)$ denotes the contour γ' with the segments which lie above the interval $[\zeta - \epsilon, \zeta + \epsilon]$ removed.⁷ Note that

$$e^{2P'+2M'} = e^{J'}, \quad 0 < \zeta < r_1. \tag{7.21}$$

Lemma 7.3. *The following limits hold as z approaches a point on the black hole horizon (i.e. as $\rho \rightarrow 0$ with $0 < \zeta < r_1$):*

$$\begin{aligned} \tilde{\Theta} \left(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega} \right) &= \frac{1}{\rho^2} \left[-\Theta' \left(u' - \int_{\zeta^+}^{\infty^-} \omega' \right) e^{J'-2M'-K'} \right. \\ &\quad \left. + \Theta' \left(u' - \int_{\zeta^+}^{\infty^-} \omega' + \int_{\zeta^-}^{\zeta^+} \omega' \right) e^{2J'-2M'-2K'} \right] + O(1), \end{aligned} \tag{7.22}$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{-iz}^{\infty^-} \tilde{\omega} \right) = \Theta' \left(u' + \int_{\zeta^-}^{\infty^-} \omega' \right) - \Theta' \left(u' + \int_{\zeta^-}^{\infty^-} \omega' + \int_{\zeta^-}^{\zeta^+} \omega' \right) e^{J'+K'} + O(\rho^2),$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{i\bar{z}}^{\infty^-} \tilde{\omega} \right) = \Theta' \left(u' + \int_{\zeta^-}^{\infty^-} \omega' \right) + \Theta' \left(u' + \int_{\zeta^-}^{\infty^-} \omega' + \int_{\zeta^-}^{\zeta^+} \omega' \right) e^{J'+K'} + O(\rho^2),$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{-iz}^{i\bar{z}} \tilde{\omega} \right) = -\frac{1}{\rho} \Theta' \left(u' + \int_{\zeta^-}^{\zeta^+} \omega' \right) e^{J'-M'} + \delta + O(\rho),$$

$$\tilde{\Theta}(\tilde{u}) = \frac{1}{\rho} \Theta' \left(u' + \int_{\zeta^-}^{\zeta^+} \omega' \right) e^{J'-M'} + \delta + O(\rho), \tag{7.23}$$

$$\tilde{\Theta} \left(\tilde{u} + \int_{-iz}^{\infty^-} \tilde{\omega} + \int_{i\bar{z}}^{\infty^-} \tilde{\omega} \right) = \Theta' \left(u' + 2 \int_{\zeta^-}^{\infty^-} \omega' \right) + O(\rho),$$

where δ is a constant.

Proof. We prove (7.22) in the case when $\text{Re } k_3 < \zeta < r_1$; the proofs of the other identities are similar. For $\text{Re } k_3 < \zeta < r_1$,

$$\tilde{\Theta} \left(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega} \right) = \tilde{\Theta}(\tilde{u} - c - \tilde{B}r - t),$$

⁷ For $0 < \zeta < r_1$, the contour γ' contains the covering in the upper sheet of $[\zeta - \epsilon, \zeta]$ and the covering in the lower sheet of $[\zeta, \zeta + \epsilon]$.

where r and t are defined in (7.11). We can write the right-hand side as

$$\sum_{N \in \mathbb{Z}^4} e^{2\pi i \left(\sum_{i,j=1}^3 N_i \tilde{B}_{ij} N_j + \frac{1}{2} \tilde{B}_{44} N_4 (N_4 - 1) \right)} \times e^{2\pi i \left(\sum_{i=1}^3 \tilde{B}_{i4} N_i (N_4 - \frac{1}{2}) + \sum_{i=1}^3 N_i (\tilde{u}_i - c_i) + N_4 (\tilde{u}_4 - c_4) - \sum_{i=1}^3 N_i t_i + \frac{1}{2} N_4 \right)}.$$

The factor involving the divergent quantities \tilde{B}_{44} and \tilde{u}_4 is

$$e^{2\pi i \left(\frac{1}{2} \tilde{B}_{44} N_4 (N_4 - 1) + N_4 \tilde{u}_4 \right)} = \rho^{N_4(N_4 - 3)} e^{N_4(N_4 - 1)M' + 2N_4P'} (1 + O(\rho^2)).$$

Thus the diverging terms are of $O(\rho^{-2})$ and arise when $N_4 = 1, 2$. We find

$$\tilde{\Theta} \left(\tilde{u} - \int_{-iz}^{\infty^-} \tilde{\omega} \right) = -\frac{1}{\rho^2} e^{2P' - K'} \sum_{N' \in \mathbb{Z}^3} e^{2\pi i \left(\frac{1}{2} N'^T B' N' + \frac{1}{2} N'^T \int_{\zeta^-}^{\zeta^+} \omega' + N'^T (u' - c' - t') \right)} + \frac{1}{\rho^2} e^{2M' + 4P' - 2K'} \sum_{N' \in \mathbb{Z}^3} e^{2\pi i \left(\frac{1}{2} N'^T B' N' + \frac{3}{2} N'^T \int_{\zeta^-}^{\zeta^+} \omega' + N'^T (u' - c' - t') \right)} + O(1).$$

Using (7.13) and (7.21), we find (7.22). \square

Lemma 7.4. *Let Q be given by (2.18). As $\rho \rightarrow 0$ with $0 < \zeta < r_1$,*

$$Q(u) = -\rho^2 \frac{\Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(u' + \int_{\zeta^-}^{\infty^-} \omega' + \int_{\zeta^-}^{\zeta^+} \omega')^2 e^{2J' + 2K'}}{\Theta'(u' + \int_{\zeta^-}^{\zeta^+} \omega')^2 e^{2J' - 2M'}} + O(\rho^4), \tag{7.24}$$

$$Q(0) = \frac{\Theta'(\int_{\zeta^-}^{\infty^-} \omega')^2 - \Theta'(\int_{\zeta^+}^{\infty^-} \omega')^2 e^{-2K'}}{\Theta'(0)^2} + O(\rho^2). \tag{7.25}$$

Proof. By applying the limits in Lemma 7.3 to (7.15), we find the statement for $Q(u)$. The limit of $Q(0)$ is obtained as in the case of the regular axis, since the diverging factors involving \tilde{u}_4 are not present.

By applying the limits of Lemma 7.3 to Eq. (7.2), we find that f is given by (2.30) near the horizon. Similarly, the behavior of e^{2U} near the horizon follows by applying Lemma 7.4 and the expansion (7.20) to the equation $e^{2U} = \frac{Q(0)}{Q(u)} e^{-2\pi i \tilde{u}_4 + \tilde{I}}$. The expression for a_{hor} is established by applying the results of Lemmas 7.3 and 7.4 to the axis-adapted version of (6.15).

We next show formula (2.32) for the behavior of $e^{2\kappa}$ near the horizon. Equation (3.7) implies that the terms of $O(\rho)$ in the expansion of $e^{2\kappa}$ vanish, so we only have to determine the leading term. Suppose $0 < \zeta < r_1$. Then, recalling (7.19) and using that $\int_{\Gamma^+} d\kappa_1 \frac{dh}{dk}(\kappa_1) = 0$, we can write L as defined in (2.16) in terms of axis-adapted quantities as

$$L = -\frac{1}{2} \int_{\Gamma} d\kappa_1 \frac{dh}{dk}(\kappa_1) \int_{\Gamma} h(\kappa_2) \tilde{\omega}_{\kappa_1^+ \kappa_1^-}(\kappa_2^+) + \int_{\Gamma^+} h \tilde{\omega}_{-r_1^+, -r_1^-} + \int_{\Gamma^+} h \tilde{\omega}_{r_1^+ r_1^-} + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left(\int_{\gamma_1(\epsilon)} \tilde{\omega}_{-r_1^+, -r_1^-} + \int_{\gamma_2(\epsilon)} \tilde{\omega}_{r_1^+ r_1^-} - 2 \ln \epsilon \right) - 2\pi i \int_{\Gamma^+} h \tilde{\omega}_4 - 2\pi i \tilde{u}_4.$$

In view of (7.20) and (7.21), we find

$$e^L = \rho^2 e^{L'+2M'-J'} + O(\rho), \quad \rho \rightarrow 0, \quad 0 < \zeta < r_1,$$

where L' is defined by (2.28). Applying this expansion together with Lemma 7.3 to the expression for $e^{2\kappa}$ in Theorem 2.1, we arrive at (2.32). This completes the proof of Proposition 2.4.

We conclude this section by proving Proposition 2.5. As $\zeta \downarrow r_1$, we have $e^{-J'} \rightarrow 0$ whereas $e^{K'}$ tends to a bounded constant. The expression (2.36) for $f_1 := f(ir_1)$ therefore follows immediately from (2.23). Similarly, the statement in Proposition 2.5 regarding $f_0 := f(+i0)$ follows by taking the limit $\rho \downarrow 0$ in (2.31).

8. Parameter Ranges

In this section we consider the singularity structure of the solution (2.14) and its dependence on the four parameters ρ_0, r_1, w_2 , and w_4 .

8.1. Singularity structure. The solution f presented in (2.14) is continuous but not smooth at the point $z = \pm ir_1$, where the regular axis meets the horizon. Moreover, $\text{Im } f$ has a jump across the disk. Away from these points f is smooth except possibly at points in the set where the denominator of (2.14) vanishes. Physically, we are interested in solutions which are singularity-free away from the disk and the horizon. A complete characterization of the singularity-free solutions involves determining for which choices of ρ_0, r_1, w_2, w_4 the set $\left\{ z \in \mathcal{D} \mid \Theta(u + \int_{-iz}^\infty \omega) = 0 \right\}$ is empty. We will not complete this analysis here, but we will indicate how a large class of singularity-free solutions can be constructed starting with parameters corresponding to a Kerr background.

In Subsec. 5.1, the Kerr solutions were parametrized in terms of the parameters $r_1 > 0$ and $-\pi/2 < \delta < 0$. However, since the map

$$\delta \mapsto w_2 = \frac{2}{\tan \delta \sin \delta} : \left(-\frac{\pi}{2}, 0\right) \rightarrow \mathbb{R}_{>0} \tag{8.1}$$

is one-to-one, we may also adopt a parametrization in terms of $r_1 > 0$ and $w_2 = 2/(\tan \delta \sin \delta) > 0$. Let $f_{r_1 w_2}^{kerr}$ denote the unique Kerr solution corresponding to the parameters $r_1 > 0$ and $w_2 > 0$. Moreover, let f denote the solution in (2.14) corresponding to some strictly positive parameters ρ_0, r_1, w_2, w_4 . Then $f \rightarrow f_{r_1 w_2}^{kerr}$ as $\rho_0 \downarrow 0$ and $w_4 \downarrow 0$ with r_1, w_2 held fixed. Indeed, consider perturbing a Kerr background solution f^{kerr} by adding a small disk of radius ρ_0 rotating with angular velocity Ω . In the limit $\rho_0 \downarrow 0$ and $\Omega \downarrow 0$, the jump contour Γ^+ in the RH problem disappears and Λ_Ω reduces to the identity matrix. Thus, the BVP (2.6) reduces to the Kerr black hole BVP and the perturbed solution f approaches f^{kerr} in this limit. Substituting the Kerr values of f_0 and f_1 and letting $\Omega \downarrow 0$ in (5.9), we find that

$$w_0 \rightarrow 0, \quad w_2 \rightarrow \frac{(f_0^{kerr})^2 - 1}{2 f_0^{kerr}} = \frac{2}{\tan \delta \sin \delta}, \quad \text{and} \quad w_4 \rightarrow 0, \tag{8.2}$$

as the Kerr background is approached. This leads to the relation (8.1) between δ and w_2 . In view of (5.12), the vanishing limiting value of w_0 is achieved by letting $\rho_0 \downarrow 0$.

Thus, for small values of $w_4 > 0$ and $\rho_0 > 0$, the solution f corresponding to $\{\rho_0, r_1, w_2, w_4\}$ is a small perturbation of the Kerr background solution $f_{r_1 w_2}^{kerr}$. In particular, f is singularity-free for sufficiently small perturbations. By increasing w_4 and

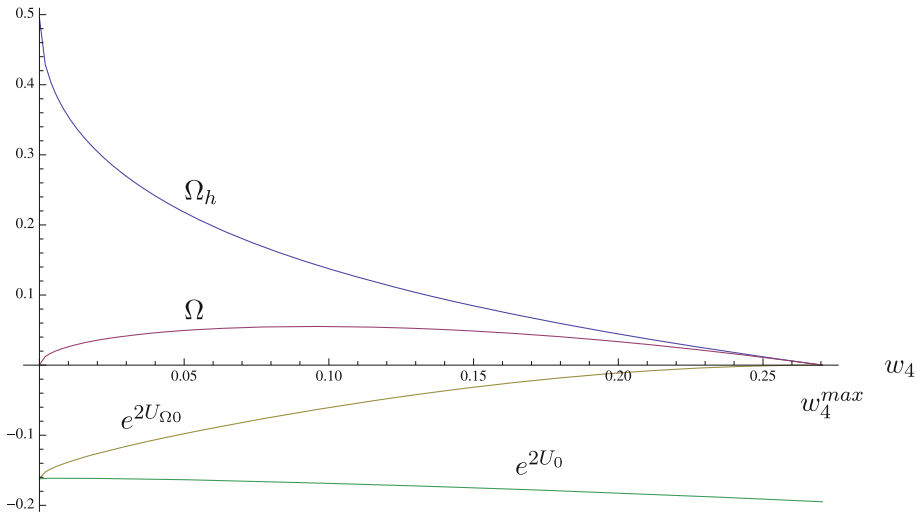


Fig. 13. The dependence on w_4 of the parameters Ω_h , Ω , $e^{2U_0} = \text{Re } f(+i0)$, and $e^{2U_{\Omega 0}} = \text{Re } f_{\Omega}(+i0)$ for the example specified by (8.3)

ρ_0 , larger perturbations of the background are obtained until the construction eventually breaks down and the solutions become singular. In this way, a large class of singularity-free solutions can be constructed. Numerical data suggest that given strictly positive values of the parameters ρ_0 , r_1 , and w_2 , there exists an interval $[0, w_4^{max}]$, $w_4^{max} > 0$, such that all solutions f corresponding to $\{\rho_0, r_1, w_2, w_4\}$ with $w_4 \in (0, w_4^{max})$ are free of singularities. In the following subsection, we illustrate the general situation by considering a typical example.

8.2. *Dependence on w_4 .* We let

$$\rho_0 = 1, \quad r_1 = \frac{1}{2}, \quad w_2 = 3, \tag{8.3}$$

and consider the solution f given in (2.14) corresponding to $\{\rho_0, r_1, w_2, w_4\}$ as $w_4 > 0$ varies. We find that the solution is free of singularities for $0 < w_4 < w_4^{max}$, where $w_4^{max} \approx 0.27051$. The example presented in Sect. 3 corresponds to taking $w_4 = 1/10$. The dependence on w_4 of several parameters is displayed in Figs. 13 and 14. The parameter w_4 is analogous to the variable μ used in [24] to parametrize the Neugebauer-Meinell solutions and Figs. 13 and 14 are the analogs of Fig. 2.9 in [24]. To see how the solution f becomes singular as w_4 increases beyond w_4^{max} , we note that as $w_4 < w_4^{max}$ increases, the ergosphere of the solution f grows larger and larger until it eventually, in the limit $w_4 \uparrow w_4^{max}$, envelops all of spacetime. As w_4 increases beyond w_4^{max} a singularity of f enters the domain \mathcal{D} at $z = +\infty$ and moves inward along the positive real axis. The graph of the singular function $\text{Re } f$ for $w_4 = 1/2 > w_4^{max}$ is shown in Fig. 15.

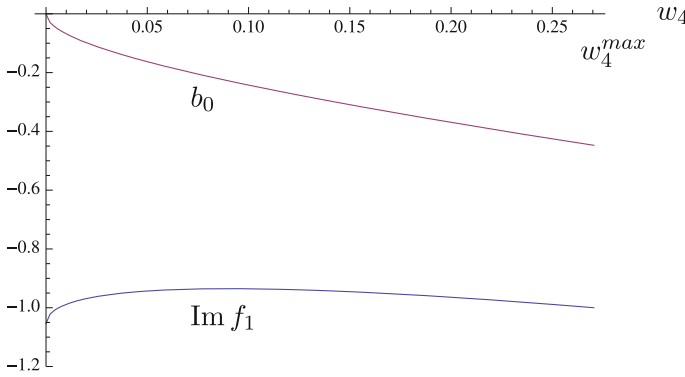


Fig. 14. The dependence on w_4 of the parameters $b_0 = \text{Im } f(+i0)$ and $f_1 = f(ir_1)$ for the example specified by (8.3)

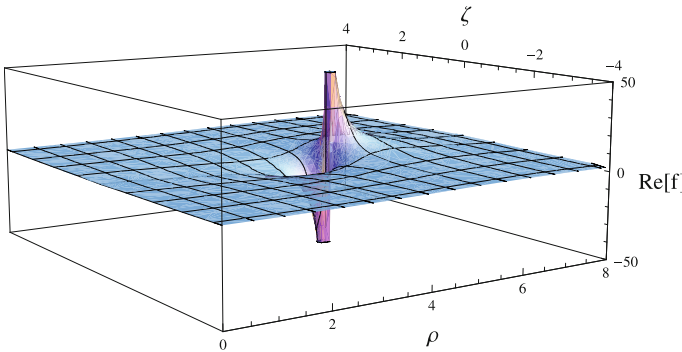


Fig. 15. The graph of $\text{Re } f$ for $w_4 = 1/2 > w_4^{max}$. The disk and the black hole are too small to be visible

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Appendix A. Condensation of Branch Points

In this Appendix we show that the Ernst potential (2.14) is related via a certain limiting procedure to the class of solutions of the Ernst equation studied in [19, 15]. By applying this limiting procedure to the formula for the metric function $e^{2\kappa}$ given in [15], we will also establish the expression (2.17b) for $e^{2\kappa}$ and so complete the proof of Theorem 2.1. The limiting operation involves partially degenerating a Riemann surface by letting branch points coalesce along the curve Γ^+ and at the points $\pm r_1$. The construction of new solutions of the Ernst equation through this type of ‘condensation’ of branch points along curves was first described in [19].

Let $\hat{\Sigma}_z$ be a Riemann surface of genus $g > 4$ obtained by adding $g - 4$ branch cuts $\{[E_j, F_j]\}_{j=1}^{g-4}$ to Σ_z . Let $\xi = -iz$. Then $\hat{\Sigma}_z$ is defined by the equation

$$\hat{y}^2 = (k - \xi)(k - \bar{\xi}) \prod_{i=1}^4 (k - k_i)(k - \bar{k}_i) \prod_{i=1}^{g-4} (k - E_i)(k - F_i).$$

Let $\{\hat{a}_j, \hat{b}_j\}_{j=1}^g$ be the cut basis on $\hat{\Sigma}_z$ which is the natural generalization of the basis $\{a_j, b_j\}_1^4$ on Σ_z , i.e. for $j = 1, \dots, 4$, \hat{a}_j surrounds the cut $[k_j, \bar{k}_j]$; for $j = 5, \dots, g$, \hat{a}_j surrounds the cut $[E_{j-4}, F_{j-4}]$; the cycle \hat{b}_j enters the upper sheet on the right side of $[-iz, i\bar{z}]$ and exits again on the right side of $[k_j, \bar{k}_j]$ for $j = 1, \dots, 4$ and on the right side of $[E_{j-4}, F_{j-4}]$ for $j = 5, \dots, g$. For simplicity, we will assume that $\zeta > \text{Re } k_4$. Let $\hat{\omega} = (\hat{\omega}_1, \dots, \hat{\omega}_g)^T$ denote the canonical dual basis and let $\hat{\Theta}(\hat{w}) := \Theta(\hat{w}|\hat{B})$ be the associated theta function. Let $p, q \in \mathbb{C}^g$ be vectors which are independent of z and which satisfy the reality condition $\hat{B}p + q \in \mathbb{R}^g$. The theta function with characteristics $p, q \in \mathbb{R}^g$ is defined by

$$\Theta \begin{bmatrix} p \\ q \end{bmatrix} (\hat{v}|\hat{B}) = \Theta(\hat{v} + \hat{B}p + q|\hat{B})e^{2\pi i \left(\frac{1}{2}p^T \hat{B}p + p^T(\hat{v}+q)\right)}, \quad \hat{v} \in \mathbb{C}^g.$$

Then

$$\hat{f} = \frac{\hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} (\int_{\xi}^{\infty^+} \hat{\omega})}{\hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} (\int_{\xi}^{\infty^-} \hat{\omega})}, \tag{A.1}$$

is a solution of the Ernst equation (2.3) and the corresponding metric function $e^{2\hat{k}}$ is given by

$$e^{2\hat{k}} = \hat{K}_0 \frac{\hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} (0)\hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} (\int_{\xi}^{\bar{\xi}} \hat{\omega})}{\hat{\Theta}(0)\hat{\Theta}(\int_{\xi}^{\bar{\xi}} \hat{\omega})}, \tag{A.2}$$

where $\hat{K}_0 \in \mathbb{C}$ is a constant determined by the condition that $e^{2\hat{k}} = 1$ on the regular axis [19, 15].

We choose

$$E_1 = -r_1 - i\epsilon, \quad F_1 = \bar{E}_1, \quad E_2 = r_1 - i\epsilon, \quad F_2 = \bar{E}_2,$$

where $\epsilon > 0$ is a small number, and define the Riemann surface $\check{\Sigma}_z$ of genus 6 by

$$\check{y}^2 = (k - \xi)(k - \bar{\xi}) \prod_{i=1}^4 (k - k_i)(k - \bar{k}_i) \prod_{i=1}^2 (k - E_i)(k - F_i).$$

In other words, $\check{\Sigma}_z$ is obtained from Σ_z by adding two short vertical cuts centered at $-r_1$ and r_1 , respectively. The cut basis $\{\check{a}_j, \check{b}_j\}_{j=1}^6$ is shown in Fig. 16.

The condensation of branch points will now proceed in two steps: In the first step, we let the branch points $E_{j+2}, F_{j+2}, j = 1, \dots, g - 6$ condense along the curve Γ . In doing this, the Riemann surface $\hat{\Sigma}_z$ degenerates to $\check{\Sigma}_z$ and the Ernst potential \hat{f} approaches a solution \check{f} defined in terms of theta functions on $\check{\Sigma}_z$. Intuitively \check{f} has a disk, but no black hole. In the second step, we let $\epsilon \rightarrow 0$. Then $\check{\Sigma}_z$ degenerates to Σ_z and we will find that \check{f} approaches the solution f in (2.14).

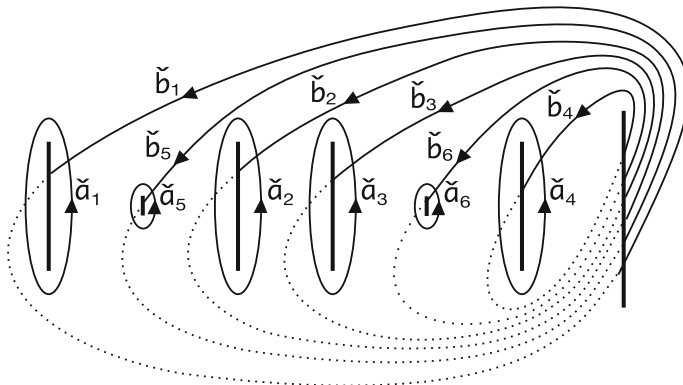


Fig. 16. The homology basis $\{\check{a}_j, \check{b}_j\}_{j=1}^6$ on the Riemann surface $\check{\Sigma}_z$

A. 1. *The first degeneration.* Let

$$p = (\check{p}, m) \in \mathbb{C}^6 \times \mathbb{R}^{g-6}, \quad q = 0,$$

where the components of the vector $m \in \mathbb{R}^{g-6}$ satisfy $0 < m_j < 1/2, j = 1, \dots, g-6$. We consider the limit $E_{j+2}, F_{j+2} \rightarrow \kappa_j, j = 1, \dots, g-6$, in which the branch cut $[E_{j+2}, F_{j+2}]$ shrinks to a point $\kappa_j \in \Gamma$. In this limit, (cf. Eq. (7.3))

$$(\hat{\omega}_1, \dots, \hat{\omega}_g) \rightarrow \left(\check{\omega}_1, \dots, \check{\omega}_6, \frac{1}{2\pi i} \check{\omega}_{\kappa_1^+ \kappa_1^-}, \dots, \frac{1}{2\pi i} \check{\omega}_{\kappa_{g-6}^+ \kappa_{g-6}^-} \right);$$

$$\hat{B}_{ij} \rightarrow \check{B}_{ij}, \quad i, j = 1, \dots, 6; \quad \hat{B}_{i, j+6} \rightarrow \int_{\kappa_j^-}^{\kappa_j^+} \check{\omega}_i, \quad i = 1, \dots, 6, \quad j = 1, \dots, g-6;$$

$$\hat{B}_{i+6, j+6} \rightarrow \frac{1}{2\pi i} \int_{\kappa_j^-}^{\kappa_j^+} \check{\omega}_{\kappa_i^+ \kappa_i^-}, \quad i, j = 1, \dots, g-6, \quad i \neq j;$$

$$\hat{B}_{j+6, j+6} = \frac{1}{\pi i} \ln |E_{j+2} - F_{j+2}| + O(1), \quad j = 1, \dots, g-6.$$

For two points $P, Q \in \hat{\Sigma}_z$, we have

$$\hat{\Theta} \begin{bmatrix} p \\ 0 \end{bmatrix} \left(\int_P^Q \hat{\omega} \right) = \left(\sum_{N \in \mathbb{Z}^g} e^{2\pi i \left(\frac{1}{2} N^T \hat{B} N + N^T \int_P^Q \hat{\omega} + B p \right)} \right) e^{2\pi i \left(\frac{1}{2} p^T \hat{B} p + p^T \int_P^Q \hat{\omega} \right)}. \quad (\text{A.4})$$

Letting $N = (\check{N}, n) \in \mathbb{Z}^6 \times \mathbb{Z}^{g-6}$ and using that $p = (\check{p}, m)$, we find that the factor in the sum on the right-hand side involving the diverging quantities $\hat{B}_{j+6, j+6}, j = 1, \dots, g-6$, is

$$e^{\pi i \sum_{j=1}^{g-6} n_j (n_j + 2m_j) \hat{B}_{j+6, j+6}}.$$

Consequently, since $0 < m_j < 1/2$ by assumption, all terms in the sum in (A.4) approach zero except the ones with $n \equiv 0$. We infer that the sum on the right-hand side of (A.4) converges to

$$\check{\Theta} \left(\int_P^Q \check{\omega} + \sum_{j=1}^{g-6} m_j \int_{\kappa_j^-}^{\kappa_j^+} \check{\omega} + \check{B} \check{p} \right). \quad (\text{A.5})$$

We let the κ_j 's condense onto the curve Γ with a density determined by the measure $dm(\kappa)$ defined by

$$dm(\kappa) = -\frac{1}{2} \frac{dh}{d\kappa}(\kappa) d\kappa, \quad \kappa \in \Gamma,$$

where h is the function defined in (2.12). Then, integrating by parts and using that h vanishes at the endpoints of Γ , we find

$$\sum_{j=1}^{g-6} m_j \int_{\kappa_j^-}^{\kappa_j^+} \check{\omega} \rightarrow \int_{\Gamma} dm(\kappa) \int_{\kappa^-}^{\kappa^+} \check{\omega} = \check{u}, \tag{A.6}$$

where $\check{u} \in \mathbb{C}^6$ is defined by

$$\check{u} = \int_{\Gamma^+} h \check{\omega}.$$

Combining (A.4)-(A.6), we find

$$\hat{\Theta} \begin{bmatrix} p \\ 0 \end{bmatrix} \left(\int_P^Q \hat{\omega} \right) \rightarrow \check{\Theta} \begin{bmatrix} \check{p} \\ 0 \end{bmatrix} \left(\check{u} + \int_P^Q \check{\omega} \right) e^{\check{L}/2 + \int_{\Gamma} dm(\kappa) \int_P^Q \check{\omega}_{\kappa^+ \kappa^-}}, \tag{A.7}$$

where \check{L} is defined by

$$\check{L} = -\frac{1}{2} \int_{\Gamma} d\kappa_1 \frac{dh}{d\kappa}(\kappa_1) \int_{\Gamma} h(\kappa_2) \omega_{\kappa_1^+ \kappa_1^-}(\kappa_2^+),$$

and the prime on the integral along Γ indicates that the integration contour should be deformed slightly before evaluation so that the pole at $\kappa_2 = \kappa_1$ is avoided.⁸

Applying this formula to (A.1), we arrive at the following limit of \hat{f} :

$$\hat{f} \rightarrow \check{f} = \frac{\check{\Theta} \begin{bmatrix} \check{p} \\ 0 \end{bmatrix} (\check{u} + \int_{\xi}^{\infty^+} \check{\omega})}{\check{\Theta} \begin{bmatrix} \check{p} \\ 0 \end{bmatrix} (\check{u} + \int_{\xi}^{\infty^-} \check{\omega})} e^{\check{I}}, \tag{A.8}$$

where $\check{I} \in \mathbb{R}$ is defined by

$$\check{I} = \int_{\Gamma} dm(\kappa) \int_{\infty^-}^{\infty^+} \check{\omega}_{\kappa^+ \kappa^-} = \int_{\Gamma^+} h \check{\omega}_{\infty^+ \infty^-}.$$

Moreover, applying Eq. (A.7) to the expression for $e^{2\hat{k}}$ in (A.2), we find

$$e^{2\hat{k}} \rightarrow e^{2\check{k}} = \check{K}_0 \frac{\check{\Theta}(\check{u}) \check{\Theta}(\check{u} + \int_{\xi}^{\xi} \check{\omega})}{\check{\Theta}(0) \check{\Theta}(\int_{\xi}^{\xi} \check{\omega})} e^{\check{L}} e^{\int_{\Gamma} dm(\kappa) \int_{\xi}^{\xi} \check{\omega}_{\kappa^+ \kappa^-}}, \tag{A.9}$$

⁸ The result is independent of whether the contour is deformed to the right or to the left of the pole.

where \check{K}_0 is a constant independent of z . For some constant C , we have

$$e^{-\int_{P_0}^P \omega_{\xi\bar{\xi}}} = C \sqrt{\frac{P - \bar{\xi}}{P - \xi}}, \quad P \in \check{\Sigma}_z,$$

because both sides have simple poles at ξ , simple zeros at $\bar{\xi}$, and are analytic elsewhere on $\check{\Sigma}_z$. Hence,

$$\int_{\xi}^{\bar{\xi}} \check{\omega}_{\kappa^+\kappa^-} = - \int_{\kappa^-}^{\kappa^+} \omega_{\xi\bar{\xi}} = \log \sqrt{\frac{P - \bar{\xi}}{P - \xi}} \Big|_{P=\kappa^-}^{\kappa^+} \in \pi i + 2\pi i \mathbb{Z}.$$

It follows that the last exponential factor in (A.9) is independent of z and can be absorbed into \check{K}_0 . Thus,

$$e^{2\check{\kappa}} = \check{K}_0 \frac{\check{\Theta}(\check{u})\check{\Theta}(\check{u} + \int_{\xi}^{\bar{\xi}} \check{\omega})}{\check{\Theta}(0)\check{\Theta}(\int_{\xi}^{\bar{\xi}} \check{\omega})} e^{\check{L}}. \tag{A.10}$$

A. 2. *The second degeneration.* We now consider the degeneration of $\check{\Sigma}_z$ as the cuts centered at $\pm r_1$ collapse. In the limit $\epsilon \rightarrow 0$,

$$\begin{aligned} (\check{\omega}_1, \dots, \check{\omega}_6) &\rightarrow \left(\omega_1, \dots, \omega_4, \frac{1}{2\pi i} \omega_{-r_1^+, -r_1^-}, \frac{1}{2\pi i} \omega_{r_1^+ r_1^-} \right); \\ \check{B} &= \begin{pmatrix} B & B_1 \\ B_1^T & B_2 \end{pmatrix} + O(\epsilon^2), \end{aligned}$$

where B is the period matrix on Σ_z , the 4×2 matrix B_1 is defined by

$$B_1 = \begin{pmatrix} \int_{-r_1^-}^{-r_1^+} \omega & \int_{r_1^-}^{r_1^+} \omega \end{pmatrix},$$

and the 2×2 matrix B_2 is given by

$$B_2 = \begin{pmatrix} \frac{1}{\pi i} (\ln \epsilon + c_-) & \frac{1}{2\pi i} \int_{r_1^-}^{r_1^+} \omega_{-r_1^+, -r_1^-} \\ \frac{1}{2\pi i} \int_{-r_1^-}^{-r_1^+} \omega_{r_1^+ r_1^-} & \frac{1}{\pi i} (\ln \epsilon + c_+) \end{pmatrix},$$

where $c_+, c_- \in \mathbb{C}$ are constants. Moreover,

$$\check{u} \rightarrow \begin{pmatrix} \int_{\Gamma^+} h\omega \\ \frac{1}{2\pi i} \int_{\Gamma^+} h\omega_{-r_1^+, -r_1^-} \\ \frac{1}{2\pi i} \int_{\Gamma^+} h\omega_{r_1^+ r_1^-} \end{pmatrix}.$$

We have

$$\check{\Theta} \begin{bmatrix} P \\ 0 \end{bmatrix} \left(\check{u} + \int_P^Q \hat{\omega} \right) = \left(\sum_{\check{N} \in \mathbb{Z}^6} e^{2\pi i \left(\frac{1}{2} \check{N}^T \check{B} \check{N} + \check{N}^T (\check{u} + \int_P^Q \hat{\omega} + \check{B} \check{p}) \right)} \right) e^{2\pi i \left(\frac{1}{2} \check{p}^T \check{B} \check{p} + \check{p}^T (\check{u} + \int_P^Q \hat{\omega}) \right)}. \tag{A.11}$$

Letting $\check{N} = (N, n) \in \mathbb{Z}^4 \times \mathbb{Z}^2$ and choosing, for some $0 < \alpha < 1/2$,

$$\check{p} = (0, 0, 0, 0, \alpha, -\alpha),$$

the same type of argument that led to (A.5) shows that all terms in the sum approach 0 as $\epsilon \rightarrow 0$ except those with $n \equiv 0$.

It follows that the sum in (A.11) converges to

$$\Theta \left(u + \int_P^Q \omega \right) \quad \text{where} \quad u = \int_{\Gamma^+} h\omega + \alpha \left(\int_{-r_1^-}^{-r_1^+} - \int_{r_1^-}^{r_1^+} \right) \omega.$$

On the other hand, the rightmost exponential factor in (A.11) can be written as

$$e^{2\pi i \left(\frac{1}{2} \check{p}^T \check{B} \check{p} + \check{p}^T (\check{u} + \int_P^Q \check{\omega}) \right)} = e^{\pi i \alpha^2 (\check{B}_{55} - 2\check{B}_{56} + \check{B}_{66})} \times e^{\alpha \int_{\Gamma^+} h\omega_{-r_1^+, -r_1^-} - \alpha \int_{\Gamma^+} h\omega_{r_1^+, r_1^-} + \alpha \int_P^Q \omega_{-r_1^+, -r_1^-} - \alpha \int_P^Q \omega_{r_1^+, r_1^-}}.$$

Applying these formulas to (A.8), we find

$$\check{f} \rightarrow f = \frac{\Theta(u + \int_{\xi}^{\infty^+} \omega)}{\Theta(u + \int_{\xi}^{\infty^-} \omega)} e^I, \tag{A.12}$$

where

$$I = \int_{\Gamma^+} h\omega_{\infty^+ \infty^-} + \alpha \left(\int_{-r_1^-}^{-r_1^+} - \int_{r_1^-}^{r_1^+} \right) \omega_{\infty^+ \infty^-}.$$

The contours in the integrals $\int_{-r_1^-}^{-r_1^+}$ and $\int_{r_1^-}^{r_1^+}$ are the limits of the cycles \check{b}_5 and \check{b}_6 , respectively. By deforming these contours, we find that

$$\left(\int_{-r_1^-}^{-r_1^+} - \int_{r_1^-}^{r_1^+} \right) \omega_{\infty^+ \infty^-} = \int_{\gamma} \omega_{\infty^+ \infty^-}, \quad \left(\int_{-r_1^-}^{-r_1^+} \omega - \int_{r_1^-}^{r_1^+} \omega \right) = \int_{\gamma} \omega,$$

where γ is the contour on Σ_z defined in Sect. 2. Therefore, in the limit $\alpha \rightarrow 1/2$, u and I become exactly the u and I of Theorem 2.1 and the solution f in (A.12) becomes the Ernst potential in (2.14). This provides the promised link between the solutions in [19] and the solution presented in this paper.

By applying the same limiting procedure to Eq. (A.10) we will determine the corresponding metric function $e^{2\kappa}$. We have

$$e^{\pi i \alpha^2 (\check{B}_{55} - 2\check{B}_{56} + \check{B}_{66})} = \epsilon^{2\alpha^2} e^{\alpha^2 \left(c_+ + c_- - \int_{-r_1^-}^{-r_1^+} \omega_{r_1^+, r_1^-} \right)} (1 + O(\epsilon)), \quad \epsilon \rightarrow 0.$$

Hence, for the quotient in (A.10) we find

$$\check{K}_0 \frac{\check{\Theta}(\check{u})\check{\Theta}(\check{u} + \int_{\xi}^{\check{\xi}} \check{\omega})}{\check{\Theta}(0)\check{\Theta}(\int_{\xi}^{\check{\xi}} \check{\omega})} = \frac{\Theta(u)\Theta(u + \int_{\xi}^{\check{\xi}} \omega)}{\Theta(0)\Theta(\int_{\xi}^{\check{\xi}} \omega)} \epsilon^{4\alpha^2} e^{2\alpha^2(c_+ + c_-) - 2\alpha^2 \int_{-r_1^-}^{-r_1^+} \omega_{r_1^+, r_1^-}} \times e^{\int_{\Gamma^+} h\omega_{-r_1^+, -r_1^-} - \int_{\Gamma^+} h\omega_{r_1^+, r_1^-}} e^{\frac{1}{2} \int_{\xi}^{\check{\xi}} \omega_{-r_1^+, -r_1^-} - \frac{1}{2} \int_{\xi}^{\check{\xi}} \omega_{r_1^+, r_1^-}} (1 + O(\epsilon)). \tag{A.13}$$

This expression vanishes in the limit $\epsilon \rightarrow 0$. However, this behavior is compensated by the fact that the constant \check{K}_0 diverges as $\epsilon \rightarrow 0$, so that the limit of $e^{2\check{k}}$ is finite and non-zero. The last exponential factor on the right-hand side of (A.13) can be absorbed into K_0 . Indeed, the same type of argument that we used to find (A.10) shows that this factor is independent of z . The constants c_+ and c_- are given by the expressions obtained by replacing ζ in the right-hand side of (2.29) with r_1 and $-r_1$, respectively. Letting $\alpha \rightarrow 1/2$ and using that

$$\begin{aligned} \lim_{\delta \rightarrow 0} & \left(\int_{(-r_1-\delta)^-}^{(-r_1-\delta)^+} \omega_{-r_1^+, -r_1^-} + \int_{(r_1-\delta)^-}^{(r_1-\delta)^+} \omega_{r_1^+ r_1^-} - 4 \ln \delta \right) - 2 \int_{r_1^-}^{r_1^+} \omega_{-r_1^+, -r_1^-} \\ & = 2 \lim_{\delta \rightarrow 0} \left(\int_{\gamma_1(\delta)} \omega_{-r_1^+, -r_1^-} - \int_{\gamma_2(\delta)} \omega_{r_1^+ r_1^-} - 2 \ln \delta \right), \end{aligned}$$

we infer that the limit of $e^{2\check{k}}$ is given by (2.17b).

We finally point out that formula (6.15) for the metric function a can be derived in a similar way. Indeed, the metric function \hat{a} corresponding to the solution \hat{f} in (A.1) is given by [15]

$$(\hat{a} - \hat{a}_0)e^{2\hat{U}} = -\rho \left(\frac{\hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} (0) \hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} \left(\int_{\xi}^{\infty^-} \hat{\omega} + \int_{\xi}^{\infty^-} \hat{\omega} \right)}{\hat{Q}(0) \hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} \left(\int_{\xi}^{\infty^-} \hat{\omega} \right) \hat{\Theta} \begin{bmatrix} p \\ q \end{bmatrix} \left(\int_{\xi}^{\infty^-} \hat{\omega} \right)} - 1 \right),$$

where

$$\hat{Q}(0) = \frac{\hat{\Theta} \left(\int_{\xi}^{\infty^-} \hat{\omega} \right) \hat{\Theta} \left(\int_{\xi}^{\infty^-} \hat{\omega} \right)}{\hat{\Theta}(0) \hat{\Theta} \left(\int_{\xi}^{\infty^-} \hat{\omega} \right)}.$$

An application of the above limiting procedure to this expression yields (6.15).

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