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Analytic Sharp Fronts for the Surface Quasi-Geostrophic Equation

Charles Fefferman[1], **José L. Rodrigo**[2]

[1] Mathematics Department, Princeton University, Princeton, NJ 08544, USA.
E-mail: cf@math.princeton.edu
[2] Mathematics Department, Warwick University, Coventry CV4 7AL, UK.
E-mail: J.Rodrigo@warwick.ac.uk

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Abstract: We study the evolution of sharp fronts for the Surface Quasi-Geostrophic equation in the context of analytic functions. We showed that, even though the equation contains operators of order higher than 1, by carefully studying the evolution of the second derivatives it can be adapted to fit an abstract version of the Cauchy-Kowaleski Theorem.

1. Introduction

In this paper we study the existence of analytic sharp fronts for the Surface Quasi-Geostrophic (SQG) equation. SQG is given by the equations

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0, \tag{1}$$

$$u = -(-\triangle)^{-1/2} \nabla^{\perp} \theta, \tag{2}$$

and has been the subject of extensive research in recent years for its connections with 3D Euler. We refer the reader to [1] and [2] for more details.

A sharp front for SQG corresponds to the evolution of an initial data given by the characteristic function of an open set (with sufficiently regular boundary). It is easy to see that the solution remains the characteristic function of an evolving set, and so the problem reduces to the contour dynamics problem obtained by considering the evolution of the boundary of the patch. We refer the reader to [3,4,9,10 and 6] for more details.

For simplicity we will the consider periodic case in which the boundary is a graph. That is we take θ of the form

$$\theta(x, y, t) = \begin{cases} 1 & y \geq \varphi(x, t) \\ 0 & y < \varphi(x, t), \end{cases} \tag{3}$$

where $\varphi(x,t)$ is periodic in x of period 1. In [9] the following equation is derived for the evolution of the sharp front in this periodic setting

$$\frac{\partial\varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\bar{x}}(\bar{x},t)}{[(x-\bar{x})^2 + (\varphi(x,t)-\varphi(\bar{x},t))^2]^{1/2}} \chi(x-\bar{x}, \varphi(x,t)-\varphi(\bar{x},t))d\bar{x}$$
$$+ \int_{\mathbb{R}/\mathbb{Z}} \left[\frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\bar{x}}(\bar{x},t)\right]\eta(x-\bar{x}, \varphi(x,t)-\varphi(\bar{x},t))d\bar{x}. \tag{4}$$

In addition local existence results are obtained for smooth initial data [9]. In [6] a similar equation is obtained for the case in which the curve is closed (and hence no longer a graph) and local existence results were obtained in Sobolev space.

The purpose of this article is to study the existence of analytic solutions of (4). The motivation is two-fold. Analytic solutions are important in the study of almost sharp fronts, as described in [3], in the upcoming work of the authors [5]. An additional motivation is of theoretical nature. In particular studying whether the system can fit in the Cauchy-Kowaleski scheme even though the right hand side of (4) is of order higher than one. We recall that in [9] the author showed that the most singular term in the right hand side of the equation is a nonlinear, nonlocal version of the operator given by the multiplier

$$\mathrm{i}\, k \log |k|,$$

which in principle does not fit in the standard Cauchy-Kowaleski machinery. We will show, by studying the evolution of $\varphi''(x,t)$ (where the primes represent derivatives with respect to x) that the new system fits the scheme of the abstract version of the Cauchy-Kowaleski Theorem of Sammartino and Caflisch [11,12].

2. Adapting the Equation

In order to simplify the presentation we consider the equation

$$\frac{\partial\varphi}{\partial t}(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial\varphi}{\partial x}(x,t) - \frac{\partial\varphi}{\partial\bar{x}}(\bar{x},t)}{[(x-\bar{x})^2 + (\varphi(x,t)-\varphi(\bar{x},t))^2]^{1/2}} \chi(x-\bar{x})d\bar{x}, \tag{5}$$

with $\varphi(x,0) = \varphi_0(x)$ an analytic initial data.

Remark 1. Equation (5) arises from (4) by making the following simplifications:

- We ignore the correction term η as, in the context of analytic functions, it can be handled by any version of the Cauchy-Kowaleski Theorem.
- We take the cut-off function χ to be an analytic function of $x-\bar{x}$ alone (notice that analyticity is not an additional constraint in this case).
- We also assume that $\chi(0) = 1$ and $\chi(\pm\frac{1}{2}) = 0$ to some fixed order.

In addition, and to simplify the notation we will write $\varphi(x)$, suppressing the explicit t dependence. Also, we will use a prime to denote partial derivatives with respect to the space variables. Finally we will use $\Xi(x,\bar{x})$ or simply Ξ to denote $(x-\bar{x})^2 + (\varphi(x,t) - \varphi(\bar{x},t))^2$.

The main result we will obtain is the following

Theorem 1. *Given an analytic, periodic (of period 1) function $\varphi_0(x)$ there exists $T > 0$ and a unique solution of Eq. (5) in $C([0,T); H^k(\mathbb{R}))$ (for any $k \geq 1$) that is analytic with respect to the space variable. Additional time regularity can, of course, be read directly from the equation.*

The main tool in the proof will be a version of the Cauchy-Kowaleski Theorem due to Sammartino and Caflisch. In general these arguments are restricted to equations with a right-hand side of first order. Notice that operator on the right hand side of (5) is of order greater than 1, as it can be seen by observing that the linearization will correspond to the Fourier multiplier given by $\mathrm{i}\,k \log |k|$. To recast (5) in the setting of Cauchy-Kowaleski we differentiate the equation twice and study the evolution equations for φ, φ_x and φ_{xx}, as independent unknowns. In terms on these new unknowns the system will be recast as an operator of order 1.

The following results will be useful in calculating a suitable expression for the evolution equations of φ, φ_x and φ_{xx}.

Lemma 1. *Given a smooth function $f(x,\bar{x})$ and under the assumptions above on χ the derivative of*

$$\int_{\mathbb{R}/\mathbb{Z}} \operatorname{sgn}(x-\bar{x}) f(x,\bar{x}) \chi(x-\bar{x}) d\bar{x}$$

is given by

$$\begin{aligned} &\frac{d}{dx}\int_{\mathbb{R}/\mathbb{Z}} \operatorname{sgn}(x-\bar{x}) f(x,\bar{x}) \chi(x-\bar{x}) d\bar{x} \\ &= 2f(x,x) + \int_{\mathbb{R}/\mathbb{Z}} \operatorname{sgn}(x-\bar{x}) \frac{d}{dx}\Big[f(x,\bar{x})\chi(x-\bar{x})\Big] d\bar{x}. \end{aligned} \tag{6}$$

Lemma 2. *We will use the following Taylor expansions with remainder*:

$$\varphi(\bar{x}) = \varphi(x) + \varphi'(x)(\bar{x}-x) + \int_0^1 \varphi''((1-\tau)\bar{x} + \tau\, x)\tau \, d\tau (\bar{x}-x)^2,$$

$$\begin{aligned} \varphi(\bar{x}) = \varphi(x) + \varphi'(x)(\bar{x}-x) + \frac{1}{2}\varphi''(x)(\bar{x}-x)^2 \\ + \frac{1}{2}\int_0^1 \varphi'''((1-\tau)\bar{x} + \tau\, x)\tau^2 \, d\tau (\bar{x}-x)^3 \end{aligned}$$

and

$$\varphi'(\bar{x}) = \varphi'(x) + \varphi''(x)(\bar{x}-x) + \int_0^1 \varphi'''((1-\tau)\bar{x} + \tau\, x)\tau \, d\tau (\bar{x}-x)^2.$$

To simplify the notation when we use the above Taylor expansions we define, for $a, b \in \mathbb{N}$,

$$I_{a,b} = I_{a,b}(x,\bar{x},t) = \int_0^1 \varphi^{(a)}((1-\tau)\bar{x} + \tau x, t)\tau^b d\tau. \tag{7}$$

As indicated above we are going to study the evolution of φ' and φ''. Differentiating Eq. (5) with respect to x and using the lemma above we obtain

$$\varphi_t'(x) = 2\frac{\varphi''(x)}{[1+(\varphi'(x))^2]^{1/2}} + \int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi''(x)}{\Xi^{1/2}}\chi(x-\bar{x}) \\ -\frac{[\varphi'(x)-\varphi'(\bar{x})][(x-\bar{x})+(\varphi(x)-\varphi(\bar{x}))\varphi'(x)]}{\Xi^{3/2}}\chi(x-\bar{x})d\bar{x} \\ +\int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'(x)-\varphi'(\bar{x})}{\Xi^{1/2}}\chi'(x-\bar{x})d\bar{x}, \tag{8}$$

and so an additional differentiation yields

$$\varphi_t''(x) = 2\frac{\varphi'''(x)}{[1+(\varphi'(x))^2]^{1/2}} - 2\frac{\varphi'(x)(\varphi''(x))^2}{[1+(\varphi'(x))^2]^{3/2}} \\ +\lim_{\bar{x}\to x^+} 2\left[\frac{\varphi''(x)}{(x-\bar{x})\left[1+\left(\frac{\varphi(x)-\varphi(\bar{x})}{x-\bar{x}}\right)^2\right]^{1/2}} - \frac{\frac{\varphi'(x)-\varphi'(\bar{x})}{x-\bar{x}}\left[1+\frac{\varphi(x)-\varphi(\bar{x})}{x-\bar{x}}\varphi'(x)\right]}{(x-\bar{x})\left[1+\left(\frac{\varphi(x)-\varphi(\bar{x})}{x-\bar{x}}\right)^2\right]^{3/2}}\right] \\ +\int_{\mathbb{R}/\mathbb{Z}} \operatorname{sgn}(x-\bar{x})\partial_x\Big\{\operatorname{sgn}(x-\bar{x})\Big(\frac{\varphi''(x)}{\Xi^{1/2}}\chi(x-\bar{x}) \\ -\frac{[\varphi'(x)-\varphi'(\bar{x})][(x-\bar{x})+(\varphi(x)-\varphi(\bar{x}))\varphi'(x)]}{\Xi^{3/2}}\chi(x-\bar{x})\Big)\Big\}d\bar{x} \\ +\int_{\mathbb{R}/\mathbb{Z}} \partial_x\left\{\frac{\varphi'(x)-\varphi'(\bar{x})}{\Xi^{1/2}}\chi'(x-\bar{x})\right\}d\bar{x}. \tag{9}$$

Notice that this last term is zero.

Using Taylor's Theorem we will rewrite Eqs. (5), (8) and (9). In the case of (5) we obtain

$$\varphi_t(x,t) = \int_{\mathbb{R}/\mathbb{Z}} \frac{-\varphi''(x)-I_{3,1}(\bar{x}-x)}{[1+(\varphi'(x)+I_{2,1}(\bar{x}-x))^2]^{1/2}}\operatorname{sgn}(\bar{x}-x)\chi(\bar{x}-x)d\bar{x}. \tag{10}$$

In order to rewrite Eq. (8) for φ_t' we start by considering the integrand of the most singular expression. We have

$$\frac{\varphi''(x)}{\Xi^{1/2}} - \frac{[\varphi'(x)-\varphi'(\bar{x})][(x-\bar{x})+(\varphi(x)-\varphi(\bar{x}))\varphi'(x)]}{\Xi^{3/2}} \\ = (x-\bar{x})^2\frac{\varphi''(x)\left(1+\left(\frac{\varphi(x)-\varphi(\bar{x})}{x-\bar{x}}\right)^2\right) - \frac{\varphi'(x)-\varphi'(\bar{x})}{x-\bar{x}}\left[1+\frac{\varphi(x)-\varphi(\bar{x})}{x-\bar{x}}\varphi'(x)\right]}{\Xi^{3/2}} \\ = (x-\bar{x})^2\frac{\varphi''(x)[1+(\varphi'(x)+I_{2,1}(\bar{x}-x))^2] - [\varphi''(x)+I_{3,1}(\bar{x}-x)][1+(\varphi'(x)+I_{2,1}(\bar{x}-x))\varphi'(x)]}{\Xi^{3/2}} \\ = (x-\bar{x})^2\frac{\varphi'(x)\varphi''(x)I_{2,1}(\bar{x}-x)+\varphi''(x)I_{2,1}^2(\bar{x}-x)^2 - [1+(\varphi'(x))^2]I_{3,1}(\bar{x}-x) - \varphi'(x)I_{2,1}I_{3,1}(\bar{x}-x)^2}{\Xi^{3/2}} \\ = (\bar{x}-x)^3\frac{\varphi'(x)\varphi''(x)I_{2,1} - [1+(\varphi'(x))^2]I_{3,1} + \varphi''(x)I_{2,1}^2(\bar{x}-x) - \varphi'(x)I_{2,1}I_{3,1}(\bar{x}-x)}{|x-\bar{x}|^3[1+(\varphi'(x)+I_{2,1}(\bar{x}-x))^2]^{3/2}},$$

and so Eq. (8) becomes

$$\begin{aligned}\varphi_t'(x) &= 2\frac{\varphi''(x)}{[1+(\varphi'(x))^2]^{1/2}}\\ &+\int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'(x)\varphi''(x)I_{2,1}-[1+(\varphi'(x))^2]I_{3,1}+\varphi''(x)I_{2,1}^2(\bar{x}-x)-\varphi'(x)I_{2,1}I_{3,1}(\bar{x}-x)}{[1+(\varphi'(x)+I_{2,1}(\bar{x}-x))^2]^{3/2}}\\ &\times \operatorname{sgn}(\bar{x}-x)\chi(x-\bar{x})d\bar{x}+\int_{\mathbb{R}/\mathbb{Z}}\frac{\varphi'(x)-\varphi'(\bar{x})}{\Xi^{1/2}}\chi'(x-\bar{x})d\bar{x}. \qquad (11)\end{aligned}$$

A simple use of Taylor's formula shows that the limit in Eq. (9) equals

$$-\frac{\varphi'(x)(\varphi''(x))^2}{[1+(\varphi'(x))^2]^{3/2}}+\frac{\varphi'''(x)}{[1+(\varphi'(x))^2]^{1/2}},$$

and using this expression we can rewrite Eq. (9) for the evolution of $\varphi''(x)$ as follows:

$$\begin{aligned}\varphi_t''(x) &= 3\frac{\varphi'''(x)}{[1+(\varphi'(x))^2]^{1/2}}-3\frac{\varphi'(x)(\varphi''(x))^2}{[1+(\varphi'(x))^2]^{3/2}}\\ &+\int \frac{\chi(x-\bar{x})}{\Xi^{5/2}}\Big\{\varphi'''(x)\Xi^2-2\varphi''(x)\big[(x-\bar{x})+(\varphi(x)-\varphi(\bar{x}))\varphi'(x)\big]\Xi\\ &-[\varphi'(x)-\varphi'(\bar{x})][1+(\varphi'(x))^2+(\varphi(x)-\varphi(\bar{x}))\varphi''(x)]\Xi\\ &+3[\varphi'(x)-\varphi'(\bar{x})][(x-\bar{x})+(\varphi(x)-\varphi(\bar{x}))\varphi'(x)]^2\Big\}d\bar{x}\\ &+\int_{\mathbb{R}/\mathbb{Z}}\Big\{\frac{\varphi''(x)}{\Xi^{1/2}}-\frac{[\varphi'(x)-\varphi'(\bar{x})][(x-\bar{x})+(\varphi(x)-\varphi(\bar{x}))\varphi'(x)]}{\Xi^{3/2}}\Big\}\chi'(x-\bar{x})d\bar{x}.\end{aligned} \qquad (12)$$

We want to systematically apply the Taylor's expansions of Lemma 2 to simplify (2). We will show the details for the first term, leaving the details of the rest to the interested reader. We have

$$\begin{aligned}\varphi'''(x)\Xi^2 &= (\bar{x}-x)^4\varphi'''(x)\Big[1+\Big(\frac{\varphi(x)-\varphi(\bar{x})}{x-\bar{x}}\Big)^2\Big]^2\\ &= (\bar{x}-x)^4\varphi'''(x)\Big[1+\Big(\varphi'(x)+I_{2,1}(\bar{x}-x)\Big)^2\Big]^2\\ &= (\bar{x}-x)^4\varphi'''(x)\Big[1+(\varphi'(x))^2+2\varphi'(x)I_{2,1}(\bar{x}-x)+\Big(I_{2,1}\Big)^2(\bar{x}-x)^2\Big]^2\\ &= (\bar{x}-x)^4\varphi'''(x)\Big[[1+(\varphi'(x))^2]^2+4(1+(\varphi'(x))^2)\varphi'(x)I_{2,1}(\bar{x}-x)\\ &\quad +2[1+(\varphi'(x))^2]\Big(I_{2,1}\Big)^2(\bar{x}-x)^2+4(\varphi'(x))^2\Big(I_{2,1}\Big)^2(\bar{x}-x)^2\\ &\quad +4\varphi'(x)\Big(I_{2,1}\Big)^3(\bar{x}-x)^3+\Big(I_{2,1}\Big)^4(\bar{x}-x)^4\Big].\end{aligned}$$

Since we will need the following expression later we also include the details for the expression multiplying the curly brackets in (2), ignoring χ:

$$\frac{1}{\Xi^{5/2}} = \frac{1}{|\bar{x}-x|^5}\frac{1}{[1+(\varphi'(x))^2]^{5/2}} + \frac{1}{|\bar{x}-x|^5}\frac{1}{[1+(\varphi'(x))^2]^{5/2}} \times\left[\frac{1}{\left[1+\frac{2\varphi'(x)(\bar{x}-x)}{[1+(\varphi'(x))^2]}I_{2,1}+\frac{(\bar{x}-x)^2}{[1+(\varphi'(x))^2]}\left(I_{2,1}\right)^2\right]^{5/2}} - 1\right]. \tag{13}$$

We can rewrite the integrand of the most singular term in (2) as

$$\frac{\chi(\bar{x}-x)}{\Xi^{5/2}}\Big\{\varphi'''(x)[1+(\varphi'(x))^2]^2(\bar{x}-x)^4 - 2[1+(\varphi'(x))^2]^2 I_{3,1}(\bar{x}-x)^4 + (powers\ \ of\ \ (\bar{x}-x)\ \ of\, degree\ \ 5\ \ or\ \ higher)\Big\}$$
$$= \frac{\chi(\bar{x}-x)}{\Xi^{5/2}}\Big\{[1+(\varphi'(x))^2]^2(\bar{x}-x)^4[\varphi'''(x)-2I_{3,1}] + other\ \ terms...\Big\}.$$

We can rewrite equation as

$$\varphi_t''(x) = +\int_{\mathbb{R}/\mathbb{Z}} \frac{\chi(\bar{x}-x)}{\Xi^{5/2}}[1+(\varphi'(x))^2]^2(\bar{x}-x)^4\Big[\varphi'''(x)-2I_{3,1}\Big]d\bar{x} + U_1(\varphi,x,t), \tag{14}$$

where U_1 can be written as

$$3\frac{\varphi'''(x)}{[1+(\varphi'(x))^2]^{1/2}} - 3\frac{\varphi'(x)(\varphi''(x))^2}{[1+(\varphi'(x))^2]^{3/2}} + \int_{\mathbb{R}/\mathbb{Z}}\Big\{\frac{\varphi''(x)}{(1+(\varphi'(x)+I_{2,1}(\bar{x}-x))^2)^{1/2}} - \frac{[\varphi''(x)+I_{3,1}(\bar{x}-x)][1+(\varphi'(x)+I_{2,1}(\bar{x}-x))\varphi'(x)]}{(1+(\varphi'(x)+I_{2,1}(\bar{x}-x))^2)^{3/2}}\Big\}\frac{\chi'(x-\bar{x})}{|\bar{x}-x|}d\bar{x}$$
$$+\int_{\mathbb{R}/\mathbb{Z}}\frac{(\bar{x}-x)^5}{\Xi^{5/2}}\chi(\bar{x}-x) \times\Big\{\text{ a polynomial in } \varphi(x), \varphi'(x), \varphi''(x), \varphi'''(x), (\bar{x}-x), \text{ and } I_{2,1}, I_{3,1}, I_{3,2}\Big\}d\bar{x}. \tag{15}$$

Now, using (13) we have

$$\varphi_t''(x) = +\int_{\mathbb{R}/\mathbb{Z}}\frac{\chi(\bar{x}-x)}{|\bar{x}-x|[1+(\varphi'(x))^2]^{1/2}}\Big[\varphi'''(x)-2I_{3,1}\Big]d\bar{x} + \int_{\mathbb{R}/\mathbb{Z}}\chi(\bar{x}-x)(\bar{x}-x)^4[1+(\varphi'(x))^2]^2\Big[\varphi'''(x)-2I_{3,1}\Big] \times\left[\frac{1}{\Xi^{5/2}} - \frac{1}{|\bar{x}-x|^5[1+(\varphi'(x))^2]^{5/2}}\right]d\bar{x} + U_1(x,t). \tag{16}$$

We use the following lemma to deal with the most singular term:

Lemma 3.

$$\int_{\mathbb{R}/\mathbb{Z}} \chi(\bar{x}-x)\frac{1}{|\bar{x}-x|}\Big[\varphi'''(x)-2I_{3,1}\Big]d\bar{x} = -\int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'''(z)-\varphi'''(x)}{|z-x|}\chi(z-x)dz$$
$$+\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\Big[(z-x)\int_z^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_z^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\Big]dz$$
$$+\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\Big[(z-x)\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\Big]dz.$$

Proof. Notice that since $2\int_0^1 \tau d\tau = 1$ we can rewrite the left-hand side as

$$-2\int_{\mathbb{R}/\mathbb{Z}}\int_0^1 \chi(\bar{x}-x)\frac{\varphi'''((1-\tau)\bar{x}+\tau x)-\varphi'''(x)}{|\bar{x}-x|}\tau\, d\tau d\bar{x}$$
$$=-2\int_x^{x+\frac{1}{2}}\int_0^1\cdots-2\int_{x-\frac{1}{2}}^{x}\int_0^1\cdots.$$

For the first of the two double integrals we consider the change of variables $z=(1-\tau)\bar{x}+\tau x$ and $\bar{x}=\bar{x}$. Then we have $d\tau d\bar{x}=(\bar{x}-x)^{-1}dzd\bar{x}$ and so

$$-2\int_x^{x+\frac{1}{2}}\int_0^1\cdots = -2\int_x^{x+\frac{1}{2}}\int_x^{\bar{x}}\frac{\varphi'''(z)-\varphi'''(x)}{|\bar{x}-x|}\chi(\bar{x}-x)\frac{z-\bar{x}}{x-\bar{x}}\frac{1}{\bar{x}-x}dzd\bar{x}$$
$$=2\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\int_z^{x+\frac{1}{2}}\chi(\bar{x}-x)\frac{z-\bar{x}}{(\bar{x}-x)^3}d\bar{x}dz$$
$$=2\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\int_z^{x+\frac{1}{2}}\chi(\bar{x}-x)\frac{z-x}{(\bar{x}-x)^3}+\chi(\bar{x}-x)\frac{x-\bar{x}}{(\bar{x}-x)^3}d\bar{x}dz$$
$$=\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\int_z^{x+\frac{1}{2}}(z-x)\frac{d}{d\bar{x}}\Big(-\frac{\chi(\bar{x}-x)}{(\bar{x}-x)^2}\Big)+(z-x)\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}dz$$
$$+2\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\int_z^{x+\frac{1}{2}}\frac{d}{d\bar{x}}\Big(\frac{\chi(\bar{x}-x)}{(\bar{x}-x)}\Big)-\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}dz$$
$$=\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\Big[(z-x)\frac{\chi(z-x)}{(z-x)^2}+(z-x)\int_z^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}\Big]dz$$
$$+\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\Big[-2\frac{\chi(z-x)}{(z-x)}-2\int_z^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\Big]dz$$
$$=-\int_x^{x+\frac{1}{2}}\frac{\varphi'''(z)-\varphi'''(x)}{(z-x)}\chi(z-x)dz$$
$$+\int_x^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\Big[(z-x)\int_z^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_z^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\Big]dz.$$

As for the second integral (notice that $d\tau d\bar{x} = (x-\bar{x})^{-1} dz d\bar{x}$),

$$
\begin{aligned}
-2\int_{x-\frac{1}{2}}^{x}\int_{0}^{1}\cdots &= -2\int_{x-\frac{1}{2}}^{x}\int_{\bar{x}}^{x}\frac{\varphi'''(z)-\varphi'''(x)}{|\bar{x}-x|}\chi(\bar{x}-x)\frac{z-\bar{x}}{x-\bar{x}}\frac{1}{x-\bar{x}}dzd\bar{x}\\
&= -2\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\int_{x-\frac{1}{2}}^{z}\chi(\bar{x}-x)\frac{z-\bar{x}}{(x-\bar{x})^3}d\bar{x}dz\\
&= -2\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\int_{x-\frac{1}{2}}^{z}\chi(\bar{x}-x)\frac{z-x}{(x-\bar{x})^3}+\chi(\bar{x}-x)\frac{x-\bar{x}}{(x-\bar{x})^3}d\bar{x}dz\\
&= \int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\int_{x-\frac{1}{2}}^{z}(z-x)\frac{d}{d\bar{x}}\left(-\frac{\chi(\bar{x}-x)}{(\bar{x}-x)^2}\right)+(z-x)\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}dz\\
&\quad +2\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\int_{x-\frac{1}{2}}^{z}\frac{d}{d\bar{x}}\left(\frac{\chi(\bar{x}-x)}{(\bar{x}-x)}\right)-\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}dz\\
&= \int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\left[-(z-x)\frac{\chi(z-x)}{(z-x)^2}+(z-x)\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}\right]dz\\
&\quad +\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\left[2\frac{\chi(z-x)}{(z-x)}-2\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\right]dz\\
&= \int_{x-\frac{1}{2}}^{x}\frac{\varphi'''(z)-\varphi'''(x)}{(z-x)}\chi(z-x)dz\\
&\quad +\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\left[(z-x)\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\right]dz,
\end{aligned}
$$

and so the integral becomes

$$
\begin{aligned}
&= -\int_{\mathbb{R}/\mathbb{Z}}\frac{\varphi'''(z)-\varphi'''(x)}{|z-x|}\chi(z-x)dz\\
&\quad +\int_{x}^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\left[(z-x)\int_{z}^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_{z}^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\right]dz\\
&\quad +\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\left[(z-x)\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\right]dz.
\end{aligned}
$$

□

We remark that

$$
\begin{aligned}
&+\int_{x}^{x+\frac{1}{2}}(\varphi'''(z)-\varphi'''(x))\left[(z-x)\int_{z}^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_{z}^{x+\frac{1}{2}}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\right]dz\\
&+\int_{x-\frac{1}{2}}^{x}(\varphi'''(z)-\varphi'''(x))\left[(z-x)\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)^2}d\bar{x}-2\int_{x-\frac{1}{2}}^{z}\frac{\chi'(\bar{x}-x)}{(\bar{x}-x)}d\bar{x}\right]dz
\end{aligned}
\tag{17}
$$

can be written as

$$
\int_{\mathbb{R}/\mathbb{Z}}(\varphi'''(z)-\varphi'''(x))K_1(z,x)dz,
$$

where K_1 is continuous, and moreover smooth outside $z = x$. Actually it can be written as the sum of a smooth function in x and z and $|z - x|$ times a smooth funcation in x and z. Notice that since we have taken χ to be even we have

$$\int_x^{x+\frac{1}{2}} \frac{\chi'(\bar{x}-x)}{(\bar{x}-x)} d\bar{x} = \int_{x-\frac{1}{2}}^{x} \frac{\chi'(\bar{x}-x)}{(\bar{x}-x)} d\bar{x},$$

and this shows that the expressions in the square brackets in (17) agree when $z = x$.

Using this lemma we can rewrite (16) as follows:

$$\begin{aligned}
\varphi_t''(x) = &-\frac{1}{[1+(\varphi'(x))^2]^{1/2}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'''(z)-\varphi'''(x)}{|z-x|} \chi(z-x)dz \\
&+\frac{1}{[1+(\varphi'(x))^2]^{1/2}} \int_{\mathbb{R}/\mathbb{Z}} (\varphi'''(z)-\varphi'''(x)) K_1(x,z)dz \\
&+\int_{\mathbb{R}/\mathbb{Z}} \chi(\bar{x}-x)(\bar{x}-x)^4[1+(\varphi'(x))^2]^2 \Big[\varphi'''(x) - 2I_{3,1}\Big] \\
&\times \left[\frac{1}{\Xi^{5/2}} - \frac{1}{|\bar{x}-x|^5[1+(\varphi'(x))^2]^{5/2}}\right] d\bar{x} + U_1(x,t).
\end{aligned}$$

We obtain the equation

$$\begin{aligned}
\varphi_t''(x) = &-\frac{1}{[1+(\varphi'(x))^2]^{1/2}} \int_{\mathbb{R}/\mathbb{Z}} \frac{\varphi'''(z)-\varphi'''(x)}{|z-x|} \chi(z-x)dz \\
&+U_1(x,t) + U_2(x,t),
\end{aligned}$$

where U_2 is implicitly defined by the equality above. Notice that U_2 can be rewritten as

$$\begin{aligned}
U_2 = &\frac{1}{[1+(\varphi'(x))^2]^{1/2}} \int_{\mathbb{R}/\mathbb{Z}} (\varphi'''(z)-\varphi'''(x)) K_1(x,z)dz \\
&+[1+(\varphi'(x))^2]^2 \int_{\mathbb{R}/\mathbb{Z}} \chi(\bar{x}-x) \Big[\varphi'''(x) - 2I_{3,1}\Big] \\
&\times \frac{1}{|\bar{x}-x|} \left[\frac{1}{\left(1+\left(\frac{\varphi(x)-\varphi(\bar{x})}{x-\bar{x}}\right)^2\right)^{5/2}} - \frac{1}{[1+(\varphi'(x))^2]^{5/2}}\right] d\bar{x}.
\end{aligned} \tag{18}$$

In order to make the main term simpler we will introduce new coordinates, based on arch length, as this will make the term $[1+(\varphi'(z))^2]^{-1/2}$ disappear. Before we change coordinates into arc-length we make one additional algebraic manipulation and introduce new unknowns. We rewrite the equation as

$$\begin{aligned}
\varphi_t''(x) = &-\int_{\mathbb{R}/\mathbb{Z}} \left(\frac{\varphi'''(z)}{[1+(\varphi'(z))^2]^{1/2}} - \frac{\varphi'''(x)}{[1+(\varphi'(x))^2]^{1/2}}\right) \frac{1}{|z-x|} \chi(z-x)dz \\
&+\int_{\mathbb{R}/\mathbb{Z}} \varphi'''(z) \left(\frac{1}{[1+(\varphi'(z))^2]^{1/2}} - \frac{1}{[1+(\varphi'(x))^2]^{1/2}}\right) \frac{1}{|z-x|} \chi(z-x)dz \\
&+U_1(x,t) + U_2(x,t),
\end{aligned}$$

which we can rewrite as

$$\varphi_t''(x) = -\int_{\mathbb{R}/\mathbb{Z}} \left(\frac{\varphi'''(z)}{[1+(\varphi'(z))^2]^{1/2}} - \frac{\varphi'''(x)}{[1+(\varphi'(x))^2]^{1/2}} \right) \frac{1}{|z-x|} \chi(z-x)dz$$
$$+ U_1(x,t) + U_2(x,t) + U_3(x,t), \tag{19}$$

where

$$U_3 = \int_{\mathbb{R}/\mathbb{Z}} \varphi'''(z) \left(\frac{1}{[1+(\varphi'(z))^2]^{1/2}} - \frac{1}{[1+(\varphi'(x))^2]^{1/2}} \right) \frac{1}{|z-x|} \chi(z-x)dz. \tag{20}$$

We will prove the existence of analytic sharp fronts by studying a system involving Eqs. (10), (11) and (19) (to which we will eventually add an equation for the length of the curve) where we want to consider the functions φ, φ' and φ'' as independent unknowns. We introduce the following functions

$$\bar{\bar{f}}(x,t) := \varphi(x,t), \quad \bar{\bar{g}}(x,t) := \varphi'(x,t), \quad \bar{\bar{h}}(x,t) := \varphi''(x,t). \tag{21}$$

We want to rewrite the system formed by Eqs. (10), (11) and (19) in terms of the new unknowns in (21). We obtain (we keep the notation $I_{a,b}$ as it makes no explicit mention to φ or $\bar{\bar{f}}$)

$$\bar{\bar{f}}_t(x,t) = \int \frac{\bar{\bar{h}}(x) + I_{3,1}(\bar{x}-x)}{[1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2]^{1/2}} \operatorname{sgn}(x-\bar{x})\chi(\bar{x}-x)d\bar{x} \tag{22}$$

$$\bar{\bar{g}}_t(x) = 2\frac{\bar{\bar{h}}(x)}{[1+(\bar{\bar{g}}(x))^2]^{1/2}}$$
$$+ \int_{\mathbb{R}/\mathbb{Z}} \frac{\bar{\bar{g}}(x)\bar{\bar{h}}(x)I_{2,1} - [1+(\bar{\bar{g}}(x))^2]I_{3,1} + \bar{\bar{h}}(x)I_{2,1}^2(\bar{x}-x) - \bar{\bar{g}}(x)I_{2,1}I_{3,1}(\bar{x}-x)}{[1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2]^{3/2}}$$
$$\times \operatorname{sgn}(\bar{x}-x)\chi(x-\bar{x})d\bar{x} + \int_{\mathbb{R}/\mathbb{Z}} \frac{\bar{\bar{g}}(x)-\bar{\bar{g}}(\bar{x})}{[1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2]^{1/2}} \frac{\chi'(x-\bar{x})}{|\bar{x}-x|}d\bar{x} \tag{23}$$

and finally

$$\bar{\bar{h}}_t(x) = -\int_{\mathbb{R}/\mathbb{Z}} \left(\frac{\bar{\bar{h}}'(z)}{[1+(\bar{\bar{g}}(z))^2]^{1/2}} - \frac{\bar{\bar{h}}'(x)}{[1+(\bar{\bar{g}}(x))^2]^{1/2}} \right) \frac{1}{|z-x|}\chi(z-x)dz$$
$$+ U_1(\bar{\bar{f}}, \bar{\bar{g}}, \bar{\bar{h}}, x, t) + U_2(\bar{\bar{f}}, \bar{\bar{g}}, \bar{\bar{h}}, x, t) + U_3(\bar{\bar{f}}, \bar{\bar{g}}, \bar{\bar{h}}, x, t). \tag{24}$$

We need to keep careful track of the expressions for U_i. From (15) we have

$$U_1 = 3\frac{\bar{\bar{h}}'(x)}{[1+(\bar{\bar{g}}(x))^2]^{1/2}} - 3\frac{\bar{\bar{g}}(x)(\bar{\bar{h}}(x))^2}{[1+(\bar{\bar{g}}(x))^2]^{3/2}} + \int_{\mathbb{R}/\mathbb{Z}} \left\{ \frac{\bar{\bar{h}}(x)}{(1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2)^{1/2}} \right.$$
$$\left. - \frac{[\bar{\bar{h}}(x)+I_{3,1}(\bar{x}-x)][1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))\bar{\bar{g}}(x)]}{(1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2)^{3/2}} \right\} \frac{\chi'(x-\bar{x})}{|\bar{x}-x|}d\bar{x}$$

$$+\int_{\mathbb{R}/\mathbb{Z}} \frac{\operatorname{sgn}(\bar{x}-x)}{(1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2)^{5/2}}\chi(\bar{x}-x)$$
$$\times\left\{\text{ pol in } \bar{\bar{f}}(x), \bar{\bar{g}}(x), \bar{\bar{h}}(x), \bar{\bar{h}}'(x), (\bar{x}-x), \text{ and } I_{2,1}, I_{3,1}, I_{3,2}\right\}d\bar{x}. \tag{25}$$

As for U_2, (18) becomes

$$U_2 = \frac{1}{[1+(\bar{\bar{g}}(x))^2]^{1/2}}\int_{\mathbb{R}/\mathbb{Z}}(\bar{\bar{h}}'(z)-\bar{\bar{h}}'(x))K_1(x,z)dz$$
$$+[1+(\bar{\bar{g}}(x))^2]^2\int_{\mathbb{R}/\mathbb{Z}}\chi(\bar{x}-x)\left[\bar{\bar{h}}'(x)-2I_{3,1}\right]$$
$$\times\frac{1}{|\bar{x}-x|}\left[\frac{1}{\left(1+[\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x)]^2\right)^{5/2}}-\frac{1}{[1+(\bar{\bar{g}}(x))^2]^{5/2}}\right]d\bar{x}. \tag{26}$$

As for U_3 (20) becomes

$$U_3 = \int_{\mathbb{R}/\mathbb{Z}}\bar{\bar{h}}'(z)\left(\frac{1}{[1+(\bar{\bar{g}}(z))^2]^{1/2}}-\frac{1}{[1+(\bar{\bar{g}}(x))^2]^{1/2}}\right)\frac{1}{|z-x|}\chi(z-x)dz. \tag{27}$$

We will incorporate the evolution equation for the length $L(t)=\int_0^1[1+(\varphi'(y,t))^2]^{1/2}dy$ of the curve to our system (as it will appear in the change of coordinates involving arc length). We have

$$L'(t) = \int_0^1 \frac{\varphi'(x)\varphi_t'(x)}{[1+(\varphi'(x))^2]^{1/2}}dx$$
$$= \int_0^1 \frac{\varphi'(x)}{[1+(\varphi'(x))^2]^{1/2}}2\frac{\varphi''(x)}{[1+(\varphi'(x))^2]^{1/2}}dx + \int_0^1 \frac{\varphi'(x)}{[1+(\varphi'(x))^2]^{1/2}}$$
$$\times\int_{\mathbb{R}/\mathbb{Z}}\frac{\varphi'(x)\varphi''(x)I_{2,1}-[1+(\varphi'(x))^2]I_{3,1}+\varphi''(x)I_{2,1}^2(\bar{x}-x)-\varphi'(x)I_{2,1}I_{3,1}(\bar{x}-x)}{[1+(\varphi'(x)+I_{2,1}(\bar{x}-x))^2]^{3/2}}$$
$$\times\operatorname{sgn}(\bar{x}-x)\chi(x-\bar{x})d\bar{x}dx$$
$$+\int_0^1\frac{\varphi'(x)}{[1+(\varphi'(x))^2]^{1/2}}\int_{\mathbb{R}/\mathbb{Z}}\frac{\varphi'(x)-\varphi'(\bar{x})}{\Xi^{1/2}}\chi'(x-\bar{x})d\bar{x}dx, \tag{28}$$

which in terms of the new unknowns becomes

$$L'(t) = 2\int_0^1 \frac{\bar{\bar{g}}(x)}{[1+(\bar{\bar{g}}(x))^2]^{1/2}}\frac{\bar{\bar{h}}(x)}{[1+(\bar{\bar{g}}(x))^2]^{1/2}}dx + \int_0^1 \frac{\bar{\bar{g}}(x)}{[1+(\bar{\bar{g}}(x))^2]^{1/2}}$$
$$\times\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{\bar{g}}(x)\bar{\bar{h}}(x)I_{2,1}-[1+(\bar{\bar{g}}(x))^2]I_{3,1}+\bar{\bar{h}}(x)I_{2,1}^2(\bar{x}-x)-\bar{\bar{g}}(x)I_{2,1}I_{3,1}(\bar{x}-x)}{[1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2]^{3/2}}$$
$$\times\operatorname{sgn}(\bar{x}-x)\chi(x-\bar{x})d\bar{x}dx$$
$$+\int_0^1\frac{\bar{\bar{g}}(x)}{[1+(\bar{\bar{g}}(x))^2]^{1/2}}\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{\bar{g}}(x)-\bar{\bar{g}}(\bar{x})}{[1+(\bar{\bar{g}}(x)+I_{2,1}(\bar{x}-x))^2]^{1/2}}\frac{\chi'(x-\bar{x})}{|\bar{x}-x|}d\bar{x}dx. \tag{29}$$

We will rewrite the 4 equations of our system (22), (23), (24) and (29) in new coordinates to simplify the most singular term in (24). We introduce a renormalized arc length, that is we divide by the total length, to keep the period of the new functions constant in time. We define (we use $\bar{\bar{g}}$ and φ' interchangeably) new coordinates s and τ given by

$$s := R(x,t) = \frac{1}{L(t)}\int_0^x \left[1+(\varphi'(y,t))^2\right]^{1/2} dy \quad \tau := t.$$

Notice that

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{1}{\frac{\partial x}{\partial s}}\frac{\partial}{\partial s} = \frac{\partial R}{\partial x}\frac{\partial}{\partial s},\\ \frac{\partial}{\partial t} &= -\frac{\frac{\partial x}{\partial \tau}}{\frac{\partial x}{\partial s}}\frac{\partial}{\partial s} + \frac{\partial}{\partial \tau} = \frac{\partial R}{\partial t}\frac{\partial}{\partial s} + \frac{\partial}{\partial \tau},\end{aligned}$$

and so

$$\frac{\partial}{\partial s} = \frac{L(t)}{[1+(\varphi'(x))^2]^{1/2}}\frac{\partial}{\partial x}.$$

We now consider $\frac{dz}{|z-x|}$. We need to write it in terms of the new variables. Notice that $dz = \frac{L(t)\,d\bar{s}}{[1+(\varphi'(z))^2]^{1/2}}$, and so

$$\begin{aligned}\frac{dz}{|z-x|} &= \frac{L(t)\,d\bar{s}}{[1+(\varphi'(z(\bar{s})))^2]^{1/2}}\frac{1}{|z(\bar{s},t)-z(s,t)|}\\ &= \frac{L(t)\,d\bar{s}}{[1+(\varphi'(z(\bar{s})))^2]^{1/2}}\left(\frac{1}{\left|\frac{\partial z}{\partial \bar{s}}\right||s-\bar{s}|} + \frac{1}{|z(\bar{s},\tau)-z(s,\tau)|} - \frac{1}{\left|\frac{\partial z}{\partial \bar{s}}\right||s-\bar{s}|}\right)\\ &= \frac{d\bar{s}}{|s-\bar{s}|} + \frac{L(t)\,d\bar{s}}{[1+(\varphi'(z(\bar{s})))^2]^{1/2}}\left(\frac{1}{|z(\bar{s},\tau)-z(s,\tau)|} - \frac{1}{\left|\frac{\partial z}{\partial \bar{s}}\right||s-\bar{s}|}\right).\end{aligned}$$

Also, we write $\chi(z-x) = \tilde{\chi}(s-\bar{s}) + \chi(z-x) - \tilde{\chi}(s-\bar{s})$.

We introduce new unknowns

$$\bar{f}(s,\tau) = \bar{\bar{f}}(x(s,\tau),\tau) \qquad \bar{g}(s,\tau) = \bar{\bar{g}}(x(s,\tau),\tau) \qquad \bar{h}(s,\tau) = \bar{\bar{h}}(x(s,\tau),\tau)$$

and

$$J_{a,b}(s,\bar{s},\tau) = I_{a,b}(x(s,\tau),\bar{x}(\bar{s},\tau),\tau).$$

We can rewrite the system formed by Eqs. (22), (23), (24) and (29) as

$$\begin{aligned}\bar{f}_\tau(s,\tau) + \frac{\partial R}{\partial t}\bar{f}_s(s,\tau) = \int &\frac{\bar{h}(s) + J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+(\bar{g}(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}}\\ &\times \operatorname{sgn}(s-\bar{s})\chi(\bar{x}(\bar{s})-x(s))\frac{L(t)d\bar{s}}{[1+(\bar{g}(\bar{s}))^2]^{1/2}},\end{aligned} \tag{30}$$

where we have used $\operatorname{sgn}(x(s)-\bar{x}(\bar{s}))=\operatorname{sgn}(s-\bar{s})$. Equation (23) becomes

$$\begin{aligned}
&\bar{g}_\tau(s,\tau)+\frac{\partial R}{\partial t}\bar{g}_s(s,\tau)=2\frac{\bar{h}(s)}{[1+(\bar{g}(s))^2]^{1/2}}\\
&\quad+L(t)\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{g}(s)\bar{h}(s)J_{2,1}-[1+(\bar{g}(s))^2]J_{3,1}+\bar{h}(s)J_{2,1}^2(\bar{x}(\bar{s})-x(s))-\bar{g}(s)J_{2,1}J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+(\bar{g}(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}}\\
&\quad\times\operatorname{sgn}(\bar{s}-s)\frac{\chi(x(s)-\bar{x}(\bar{s}))d\bar{s}}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}\\
&\quad+L(t)\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{g}(s)-\bar{g}(\bar{s})}{[1+(\bar{g}(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}}\frac{\chi'(x(s)-\bar{x}(\bar{s}))}{|\bar{x}(\bar{s})-x(s)|}\frac{d\bar{s}}{[1+(\bar{g}(\bar{s}))^2]^{1/2}},
\end{aligned}\tag{31}$$

(24) becomes

$$\begin{aligned}
&\bar{h}_\tau(s,\tau)+\frac{\partial R}{\partial t}\bar{h}_s(s,\tau)\\
&\quad=-\frac{1}{L(\tau)}\int_{\mathbb{R}/\mathbb{Z}}(\bar{h}'(\bar{s})-\bar{h}'(s))[\tilde{\chi}(s-\bar{s})+\chi(z(\bar{s})-z(s))-\tilde{\chi}(s-\bar{s})]\\
&\qquad\times\left\{\frac{1}{|s-\bar{s}|}+\frac{L(t)}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}\left(\frac{1}{|z(\bar{s},\tau)-z(s,\tau)|}-\frac{1}{\left|\frac{\partial z}{\partial\bar{s}}\right||s-\bar{s}|}\right)\right\}d\bar{s}\\
&\qquad+\bar{U}_1+\bar{U}_2+\bar{U}_3,
\end{aligned}\tag{32}$$

which we can rewrite as

$$\bar{h}_\tau(s,\tau)=-\frac{1}{L(\tau)}\int_{\mathbb{R}/\mathbb{Z}}(\bar{h}'(\bar{s})-\bar{h}'(s))\tilde{\chi}(s-\bar{s})\frac{1}{|\bar{s}-s|}d\bar{s}+\bar{U}_1+\bar{U}_2+\bar{U}_3+\bar{U}_4,\tag{33}$$

with

$$\begin{aligned}
\bar{U}_4=&-\frac{\partial R}{\partial t}\bar{h}_s(s,\tau)\\
&-\int_{\mathbb{R}/\mathbb{Z}}(\bar{h}'(\bar{s})-\bar{h}'(s))\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}\left(\frac{1}{|z(\bar{s},\tau)-z(s,\tau)|}-\frac{1}{\left|\frac{\partial z}{\partial\bar{s}}\right||s-\bar{s}|}\right)\tilde{\chi}(s-\bar{s})d\bar{s}\\
&-\int_{\mathbb{R}/\mathbb{Z}}(\bar{h}'(\bar{s})-\bar{h}'(s))[\chi(z(\bar{s})-z(s))-\tilde{\chi}(s-\bar{s})]\frac{1}{|z(\bar{s},t)-z(s,t)|}\frac{d\bar{s}}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}.
\end{aligned}\tag{34}$$

Finally for L, (29) becomes,

$$\begin{aligned}
L'(\tau)=&\,2L(t)\int_0^1\frac{\bar{g}(s)\bar{h}(s)}{[1+(\bar{g}(s))^2]^{3/2}}ds+(L(\tau))^2\int_0^1\frac{\bar{g}(s)}{[1+(\bar{g}(s))^2]}\\
&\times\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{g}(s)\bar{h}(s)J_{2,1}-[1+(\bar{g}(s))^2]J_{3,1}+\bar{h}(s)J_{2,1}^2(\bar{x}(\bar{s})-x(s))-\bar{g}(s)J_{2,1}J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+(\bar{g}(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}}\\
&\times\operatorname{sgn}(\bar{s}-s)\chi(x(s)-\bar{x}(\bar{s}))\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}d\bar{s}ds+(L(\tau))^2\int_0^1\frac{\bar{g}(s)}{[1+(\bar{g}(s))^2]}\\
&\times\int_{\mathbb{R}/\mathbb{Z}}\frac{\bar{g}(s)-\bar{g}(\bar{s})}{[1+(\bar{g}(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}}\frac{\chi'(x(s)-\bar{x}(\bar{s}))}{|\bar{x}(\bar{s})-x(s)|}\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}d\bar{s}ds.
\end{aligned}\tag{35}$$

The final transformation that we make in the equation is defining new unknowns f, g, h and l as

$$\begin{aligned}\bar{f}(s,\tau) &= f(s,\tau) + \bar{f}_0(s), \quad \text{where} \quad \bar{f}_0(s) = \bar{f}(s,0),\\ \bar{g}(s,\tau) &= g(s,\tau) + \bar{g}_0(s), \quad \text{where} \quad \bar{g}_0(s) = \bar{g}(s,0),\\ \bar{h}(s,\tau) &= h(s,\tau) + \bar{h}_0(s), \quad \text{where} \quad \bar{h}_0(s) = \bar{h}(s,0),\\ L(t) &= l(t) + L(0).\end{aligned}$$

The system of equations becomes

$$\begin{aligned}&f_\tau(s,\tau) + \frac{\partial R}{\partial t} f_s(s,\tau) + \frac{\partial R}{\partial t}\bar{f}_0'(s)\\ &= [l(\tau)+L_0]\int_{\mathbb{R}/\mathbb{Z}} \frac{h(s)+\bar{h}_0(s)+J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}}\operatorname{sgn}(s-\bar{s})\chi(\bar{x}(\bar{s})\\ &\quad -x(s))\frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}}, \qquad (36)\end{aligned}$$

$$\begin{aligned}&g_\tau(s,\tau) + \frac{\partial R}{\partial t} g_s(s,\tau) + \frac{\partial R}{\partial t} g_0'(s) = 2\frac{h(s)+\bar{h}_0(s)}{[1+(g(s)+\bar{g}_0(s))^2]^{1/2}}\\ &\quad +[l(\tau)+L_0]\int_{\mathbb{R}/\mathbb{Z}} \left[\frac{(g(s)+\bar{g}_0(s))(h(s)+\bar{h}_0(s))J_{2,1} - [1+(g(s)+\bar{g}_0(s))^2]J_{3,1}}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}}\right.\\ &\quad \left. + \frac{(h(s)+\bar{h}_0(s))J_{2,1}^2(\bar{x}(\bar{s})-x(s)) - (g(s)+\bar{g}_0(s))J_{2,1}J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}}\right]\\ &\quad \times \operatorname{sgn}(\bar{s}-s)\frac{\chi(x(s)-\bar{x}(\bar{s}))d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}}\\ &\quad + [l(\tau)+L_0]\int_{\mathbb{R}/\mathbb{Z}} \frac{(g(s)+\bar{g}_0(s))-(g(\bar{s})+\bar{g}_0(\bar{s}))}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}}\frac{\chi'(x(s)-\bar{x}(\bar{s}))}{|\bar{x}(\bar{s})-x(s)|}\\ &\quad \times\frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}}, \qquad (37)\end{aligned}$$

$$\begin{aligned}h_\tau(s,\tau) &= -\frac{1}{l(\tau)+L_0}\int_{\mathbb{R}/\mathbb{Z}} (h'(\bar{s},\tau)-h'(s,\tau))\tilde{\chi}(s-\bar{s})\frac{1}{|\bar{s}-s|}d\bar{s}\\ &\quad + \bar{U}_1 + \bar{U}_2 + \bar{U}_3 + \bar{U}_4 + \bar{U}_5, \qquad (38)\end{aligned}$$

where

$$\bar{U}_5 = -\frac{1}{l(\tau)+L_0}\int_{\mathbb{R}/\mathbb{Z}} (\bar{h}_0'(\bar{s})-\bar{h}_0'(s))\tilde{\chi}(s-\bar{s})\frac{1}{|\bar{s}-s|}d\bar{s}, \qquad (39)$$

and finally

$$\begin{aligned}l'(\tau) &= 2[l(t)+L_0]\int_0^1 \frac{(g(s)+\bar{g}_0(s))(h(s)+\bar{h}_0(s))}{[1+(g(s)+\bar{g}_0(s))^2]^{3/2}}ds\\ &\quad + [l(t)+L_0]^2\int_0^1 \frac{(g(s)+\bar{g}_0(s))}{[1+(g(s)+\bar{g}_0(s))^2]}\end{aligned}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}/\mathbb{Z}} \Bigg[\frac{(g(s)+\bar{g}_0(s))(h(s)+\bar{h}_0(s))J_{2,1} - [1+(g(s)+\bar{g}_0(s))^2]J_{3,1}}{[1+((g(s)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}} \\
&+ \frac{(h(s)+\bar{h}_0(s))J_{2,1}^2(\bar{x}(\bar{s})-x(s)) - (g(s)+\bar{g}_0(s))J_{2,1}J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+((g(s)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}} \Bigg] \\
&\times \operatorname{sgn}(\bar{s}-s)\chi(x(s)-\bar{x}(\bar{s})) \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} ds \\
&+ [l(t)+L_0]^2 \int_0^1 \frac{(g(s)+\bar{g}_0(s))}{[1+(g(s)+\bar{g}_0(s))^2]} \\
&\times \int_{\mathbb{R}/\mathbb{Z}} \frac{(g(s)+\bar{g}_0(s)) - (g(\bar{s})+\bar{g}_0(\bar{s}))}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}} \\
&\times \frac{\chi'(x(s)-\bar{x}(\bar{s}))}{|\bar{x}(\bar{s})-x(s)|} \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} ds.
\end{aligned}
\tag{40}
$$

In order to adapt the system to the version of Cauchy-Kowaleski that we will use we rewrite the system as

$$
\begin{aligned}
f(s,\tau) = \int_0^\tau \Bigg\{ &- \frac{\partial R}{\partial \bar{t}} f_s(s,\bar{t}) - \frac{\partial R}{\partial \bar{t}} \bar{f}_0'(s) \\
&+ [l(\bar{t})+L_0] \int_{\mathbb{R}/\mathbb{Z}} \frac{h(s)+\bar{h}_0(s)+J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}} \\
&\times \operatorname{sgn}(s-\bar{s})\chi(\bar{x}(\bar{s})-x(s)) \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} \Bigg\} d\bar{t},
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
g(s,\tau) = \int_0^\tau \Bigg\{ &- \frac{\partial R}{\partial \bar{t}} g_s(s,\bar{t}) - \frac{\partial R}{\partial \bar{t}} g_0'(s) + 2\frac{h(s)+\bar{h}_0(s)}{[1+(g(s)+\bar{g}_0(s))^2]^{1/2}} \\
&+ [l(\bar{t})+L_0] \int_{\mathbb{R}/\mathbb{Z}} \Bigg[\frac{(g(s)+\bar{g}_0(s))(h(s)+\bar{h}_0(s))J_{2,1} - [1+(g(s)+\bar{g}_0(s))^2]J_{3,1}}{[1+(g(s)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}} \\
&+ \frac{(h(s)+\bar{h}_0(s))J_{2,1}^2(\bar{x}(\bar{s})-x(s)) - (g(s)+\bar{g}_0(s))J_{2,1}J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}} \Bigg] \\
&\times \operatorname{sgn}(\bar{s}-s) \frac{\chi(x(s)-\bar{x}(\bar{s}))d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} \\
&+ [l(\bar{t})+L_0] \int_{\mathbb{R}/\mathbb{Z}} \frac{(g(s)+\bar{g}_0(s)) - (g(\bar{s})+\bar{g}_0(\bar{s}))}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}} \\
&\times \frac{\chi'(x(s)-\bar{x}(\bar{s}))}{|\bar{x}(\bar{s})-x(s)|} \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} \Bigg\} d\bar{t},
\end{aligned}
\tag{42}
$$

$$
h(s,\tau) = \int_0^\tau e^{im(k)a(\tau-\bar{t})}(\bar{U}_1(\bar{t}) + \bar{U}_2(\bar{t}) + \bar{U}_3(\bar{t}) + \bar{U}_4(\bar{t}) + \bar{U}_5(\bar{t}))d\bar{t}, \tag{43}
$$

where

$$
a(t) := \int_0^t \frac{1}{l(r)+L_0} dr,
$$

and $m(k) \sim k \ln|k|$.

$$
\begin{aligned}
l(\tau) = \int_0^\tau \Bigg\{ & 2[l(\bar{t}) + L_0] \int_0^1 \frac{(g(s)+\bar{g}_0(s))(h(s)+\bar{h}_0(s))}{[1+(g(s)+\bar{g}_0(s))^2]^{3/2}} ds \\
& + [l(\bar{t}) + L_0]^2 \int_0^1 \frac{(g(s)+\bar{g}_0(s))}{[1+(g(s)+\bar{g}_0(s))^2]} \\
& \times \int_{\mathbb{R}/\mathbb{Z}} \Bigg[\frac{(g(s)+\bar{g}_0(s))(h(s)+\bar{h}_0(s))J_{2,1} - [1+(g(s)+\bar{g}_0(s))^2]J_{3,1}}{[1+((g(s)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}} \\
& + \frac{(h(s)+\bar{h}_0(s))J_{2,1}^2(\bar{x}(\bar{s})-x(s)) - (g(s)+\bar{g}_0(s))J_{2,1}J_{3,1}(\bar{x}(\bar{s})-x(s))}{[1+((g(s)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{3/2}} \Bigg] \\
& \times \operatorname{sgn}(\bar{s}-s)\chi(x(s)-\bar{x}(\bar{s})) \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} ds \\
& + [l(\bar{t}) + L_0]^2 \int_0^1 \frac{(g(s)+\bar{g}_0(s))}{[1+(g(s)+\bar{g}_0(s))^2]} \\
& \times \int_{\mathbb{R}/\mathbb{Z}} \frac{(g(s)+\bar{g}_0(s)) - (g(\bar{s})+\bar{g}_0(\bar{s}))}{[1+((g(s)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2]^{1/2}} \\
& \times \frac{\chi'(x(s)-\bar{x}(\bar{s}))}{|\bar{x}(\bar{s})-x(s)|} \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} ds \Bigg\} d\bar{t}. \qquad (44)
\end{aligned}
$$

We want to show that the right hand sides of Eqs. (41)–(44) can be written as an analytic function whose arguments only involve $f, g, h, l, f', g', h', l'$ with no higher derivatives of the unknowns being involved. Clearly right hand sides of that form would satisfy the Cauchy estimates required by the Cauchy-Kowaleski Theorem.

We start by considering the terms $\bar{U}_1, \ldots, \bar{U}_5$. We have

$$
\bar{U}_5(s) = -\int_{\mathbb{R}/\mathbb{Z}} (\bar{h}_0'(\bar{s}) - \bar{h}_0'(s))\tilde{\chi}(s-\bar{s}) \frac{1}{|\bar{s}-s|} d\bar{s}. \qquad (45)
$$

The expression for U_4 becomes

$$
\begin{aligned}
\bar{U}_4 = & -\frac{\partial R}{\partial \tau}(h'(s,\tau) + \bar{h}_0'(s)) - \int_{\mathbb{R}/\mathbb{Z}} (h'(\bar{s},\tau) + \bar{h}_0'(\bar{s}) - (h'(s,\tau) + \bar{h}_0'(s))) \\
& \times \frac{1}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} \left(\frac{1}{|z(\bar{s},\tau) - z(s,\tau)|} - \frac{1}{\left|\frac{\partial z}{\partial \bar{s}}\right| |s-\bar{s}|} \right) \tilde{\chi}(s-\bar{s}) d\bar{s} \\
& - \int_{\mathbb{R}/\mathbb{Z}} (h'(\bar{s},\tau) + \bar{h}_0'(\bar{s}) - (h'(s,\tau) + \bar{h}_0'(s)))[\chi(z(\bar{s}) - z(s)) \\
& - \tilde{\chi}(s-\bar{s})] \frac{1}{|z(\bar{s},t) - z(s,t)|} \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}}. \qquad (46)
\end{aligned}
$$

We need to rewrite

$$
\frac{1}{|z(\bar{s},\tau) - z(s,\tau)|} - \frac{1}{\left|\frac{\partial z}{\partial \bar{s}}\right| |s-\bar{s}|}
$$

as an analytic function of the arguments described before.

Recall that $z(s) = L(t)\int_0^s \frac{1}{([1+(\bar{g}(\bar{\bar{s}}))^2])^{1/2}} d\bar{\bar{s}}$, and so we have

$$\begin{aligned}
&\frac{1}{L(t)}\operatorname{sgn}(\bar{s}-s)\frac{1}{\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}}} - \frac{1}{\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}(\bar{s}-s)} \\
&= \frac{1}{L(t)}\frac{\operatorname{sgn}(\bar{s}-s)}{\bar{s}-s}\frac{-\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} - \frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}} d\bar{\bar{s}}}{\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}} \\
&= \frac{1}{L(t)}\frac{\operatorname{sgn}(\bar{s}-s)}{\bar{s}-s}\frac{\int_s^{\bar{s}}\int_{\bar{\bar{s}}}^{\bar{s}} \frac{\bar{g}(\tilde{s})\bar{g}'(\tilde{s})}{[1+(\bar{g}(\tilde{s}))^2]^{3/2}} d\bar{\bar{s}}}{\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}} \\
&= \frac{1}{L(t)}\frac{\operatorname{sgn}(\bar{s}-s)}{\bar{s}-s}\frac{\int_s^{\bar{s}} (\tilde{s}-s)\frac{\bar{g}(\tilde{s})\bar{g}'(\tilde{s})}{[1+(\bar{g}(\tilde{s}))^2]^{3/2}} d\bar{\bar{s}}}{\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}},
\end{aligned}$$

and taking $\tilde{s} = (1-\rho)s + \rho\bar{s} = s + \rho(\bar{s}-s)$ (which leads to $\tilde{s} - s = \rho(\bar{s}-s)$ and $d\tilde{s} = d\rho(\bar{s}-s)$) we obtain

$$\begin{aligned}
&= \frac{1}{L(t)}\frac{\operatorname{sgn}(\bar{s}-s)}{\bar{s}-s}\frac{\int_0^1 \rho(\bar{s}-s)\frac{\bar{g}((1-\rho)s+\rho\bar{s})\bar{g}'((1-\rho)s+\rho\bar{s})}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{3/2}} d\rho(\bar{s}-s)}{\int_0^1 \frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho(\bar{s}-s)\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}} \\
&= \frac{1}{L(t)}\operatorname{sgn}(\bar{s}-s)\frac{\int_0^1 \rho\frac{\bar{g}((1-\rho)s+\rho\bar{s})\bar{g}'((1-\rho)s+\rho\bar{s})}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{3/2}} d\rho}{\int_0^1 \frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}}}.
\end{aligned}$$

Using this expression U_4 becomes

$$\begin{aligned}
\bar{U}_4 &= -\frac{\partial R}{\partial \tau}(h'(s,\tau)+\bar{h}_0'(s)) - \frac{1}{L(t)}\int_{\mathbb{R}/\mathbb{Z}}(h'(\bar{s},\tau)+\bar{h}_0'(\bar{s}) - (h'(s,\tau)+\bar{h}_0'(s))) \\
&\times\left(\operatorname{sgn}(\bar{s}-s)\frac{\int_0^1 \rho\frac{[g((1-\rho)s+\rho\bar{s})+\bar{g}_0((1-\rho)s+\rho\bar{s})][g'((1-\rho)s+\rho\bar{s})-\bar{g}'((1-\rho)s+\rho\bar{s})]}{[1+(g((1-\rho)s+\rho\bar{s})+\bar{g}_0((1-\rho)s+\rho\bar{s}))^2]^{3/2}} d\rho}{\int_0^1 \frac{1}{[1+(g((1-\rho)s+\rho\bar{s})+\bar{g}_0((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho}\right)\tilde{\chi}(s-\bar{s})d\bar{s} \\
&-\int_{\mathbb{R}/\mathbb{Z}}(h'(\bar{s},\tau)+\bar{h}_0'(\bar{s}) - (h'(s,\tau)+\bar{h}_0'(s)))[\chi(z(\bar{s})-z(s)) \\
&-\tilde{\chi}(s-\bar{s})]\frac{1}{|z(\bar{s},t)-z(s,t)|}\frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}}. \qquad (47)
\end{aligned}$$

We still need to consider $\frac{\partial R}{\partial \tau}$ in the above expression. We have

$$\frac{\partial R}{\partial \tau} = \frac{-L'(\tau)}{L^2(\tau)}\int_0^x [1+(\varphi'(y,t))^2]^{1/2} dy + \frac{1}{L(\tau)}\int_0^x \frac{\varphi'(y)\varphi_t'(y)}{[1+(\varphi'(y,t))^2]^{1/2}} dy,$$

and so in terms of $\bar{f}$ and $\bar{g}$ it becomes (the last integral in the RHS above is L' when $x = 1$, and so we can use expression (35))

$$\frac{\partial R}{\partial \tau} = \frac{-L'(\tau)}{L^2(\tau)} \int_0^s [1+(\bar{g}'(\bar{s},t))^2]^{1/2} \frac{L(\tau)}{[1+(\bar{g}(\bar{s},t))^2]^{1/2}} d\bar{s}$$
$$+\frac{1}{L(\tau)} 2L(t) \int_0^s \frac{\bar{g}(r)\bar{h}(r)}{[1+(\bar{g}(r))^2]^{3/2}} dr + \frac{1}{L(\tau)} (L(\tau))^2 \int_0^s \frac{\bar{g}(r)}{[1+(\bar{g}(r))^2]}$$
$$\times \int_{\mathbb{R}/\mathbb{Z}} \frac{\bar{g}(r)\bar{h}(r)J_{2,1} - [1+(\bar{g}(r))^2]J_{3,1} + \bar{h}(r)J_{2,1}^2(\bar{x}(\bar{s}) - x(r)) - \bar{g}(r)J_{2,1}J_{3,1}(\bar{x}(\bar{s}) - x(r))}{[1+(\bar{g}(r)+J_{2,1}(\bar{x}(\bar{s}) - x(r)))^2]^{3/2}}$$
$$\times \operatorname{sgn}(\bar{s} - r)\chi(x(r) - \bar{x}(\bar{s})) \frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}} d\bar{s}dr + \frac{1}{L(\tau)} (L(\tau))^2 \int_0^s \frac{\bar{g}(r)}{[1+(\bar{g}(r))^2]}$$
$$\times \int_{\mathbb{R}/\mathbb{Z}} \frac{\bar{g}(r) - \bar{g}(\bar{s})}{[1+(\bar{g}(r)+J_{2,1}(\bar{x}(\bar{s}) - x(r)))^2]^{1/2}} \frac{\chi'(x(r) - \bar{x}(\bar{s}))}{|\bar{x}(r) - x(\bar{s})|} \frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}} d\bar{s}dr, \tag{48}$$

which in terms of g and $\bar{g}_0$ becomes

$$\frac{\partial R}{\partial \tau} = \frac{-L'(\tau)}{L(\tau)} s + 2\int_0^s \frac{(g(r)+\bar{g}_0(r))(h(r)+\bar{h}_0(r))}{[1+(g(r)+\bar{g}_0(r))^2]^{3/2}} dr$$
$$+[l(\tau)+L_0] \int_0^s \frac{(g(r)+\bar{g}_0(r))}{[1+(g(r)+\bar{g}_0(r))^2]}$$
$$\times \int_{\mathbb{R}/\mathbb{Z}} \left[\frac{(g(r)+\bar{g}_0(r))(h(r)+\bar{h}_0(r))J_{2,1} - [1+(g(r)+\bar{g}_0(r))^2]J_{3,1}}{[1+((g(r)+\bar{g}_0(r))+J_{2,1}(\bar{x}(\bar{s}) - x(r)))^2]^{3/2}} \right.$$
$$\left. + \frac{(h(r)+\bar{h}_0(r))J_{2,1}^2(\bar{x}(\bar{s}) - x(r)) - (g(r)+\bar{g}_0(r))J_{2,1}J_{3,1}(\bar{x}(\bar{s}) - x(r))}{[1+((g(r)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s}) - x(r)))^2]^{3/2}} \right]$$
$$\times \operatorname{sgn}(\bar{s} - r)\chi(x(r) - \bar{x}(\bar{s})) \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} dr$$
$$+[l(\tau)+L_0] \int_0^s \frac{(g(r)+\bar{g}_0(r))}{[1+(g(r)+\bar{g}_0(r))^2]} \int_{\mathbb{R}/\mathbb{Z}} \frac{(g(r)+\bar{g}_0(r)) - (g(\bar{s})+\bar{g}_0(\bar{s}))}{[1+((g(r)+\bar{g}_0(s))+J_{2,1}(\bar{x}(\bar{s}) - x(r)))^2]^{1/2}}$$
$$\times \frac{\chi'(x(r) - \bar{x}(\bar{s}))}{|x(r) - \bar{x}(\bar{s})|} \frac{d\bar{s}}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} dr.$$

As for $\bar{U}_3$ we have

$$\bar{U}_3 = \int \bar{h}'(\bar{s}) \left(\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}} - \frac{1}{[1+(\bar{g}(s))^2]^{1/2}} \right) \frac{1}{|z(\bar{s}) - z(s)|} \chi(z(\bar{s}) - z(s)) d\bar{s}.$$

We need to rewrite

$$\left(\frac{1}{[1+(\bar{g}(\bar{s}))^2]^{1/2}} - \frac{1}{[1+(\bar{g}(s))^2]^{1/2}} \right) \frac{1}{|z(\bar{s}) - z(s)|}$$
$$= \frac{1}{L(t)} \frac{[1+(\bar{g}(s))^2]^{1/2} - [1+(\bar{g}(\bar{s}))^2]^{1/2}}{[1+(\bar{g}(s))^2]^{1/2}[1+(\bar{g}(\bar{s}))^2]^{1/2}} \frac{1}{\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}}}$$
$$= \frac{1}{L(t)} \frac{-\int_s^{\bar{s}} \frac{\bar{g}(\bar{\bar{s}})\bar{g}'(\bar{\bar{s}})}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}}}{[1+(\bar{g}(s))^2]^{1/2}[1+(\bar{g}(\bar{s}))^2]^{1/2}} \frac{1}{\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}}},$$

and taking $\bar{\bar{s}} = (1-\rho)s + \rho\bar{s} = s + \rho(\bar{s}-s)$ (which leads to $\bar{\bar{s}} - s = \rho(\bar{s}-s)$ and $d\bar{\bar{s}} = d\rho(\bar{s}-s)$) we obtain

$$= \frac{1}{L(t)} \frac{-\int_0^1 \frac{\bar{g}((1-\rho)s+\rho\bar{s})\bar{g}'((1-\rho)s+\rho\bar{s})}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}}\, d\rho(\bar{s}-s)}{[1+(\bar{g}(s))^2]^{1/2}[1+(\bar{g}(\bar{s}))^2]^{1/2} \int_0^1 \frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho(\bar{s}-s)}$$

$$= \frac{1}{L(t)} \frac{-\int_0^1 \frac{\bar{g}((1-\rho)s+\rho\bar{s})\bar{g}'((1-\rho)s+\rho\bar{s})}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}}\, d\rho}{[1+(\bar{g}(s))^2]^{1/2}[1+(\bar{g}(\bar{s}))^2]^{1/2}} \frac{1}{\int_0^1 \frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho}.$$

Using this expression U_3 becomes

$$\bar{U}_3 = \int [h'(\bar{s}) + \bar{h}_0'(\bar{s})] \frac{1}{L(t)} \frac{-\int_0^1 \frac{[g((1-\rho)s+\rho\bar{s})+\bar{g}_0((1-\rho)s+\rho\bar{s})][g'((1-\rho)s+\rho\bar{s})+\bar{g}_0'((1-\rho)s+\rho\bar{s})]}{[1+(g((1-\rho)s+\rho\bar{s})+\bar{g}_0((1-\rho)s+\rho\bar{s}))^2]^{1/2}}\, d\rho}{[1+(g(s)+\bar{g}_0(s))^2]^{1/2}[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}}$$
$$\times \frac{1}{\int_0^1 \frac{1}{[1+(g((1-\rho)s+\rho\bar{s})+\bar{g}_0((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho} \chi(z(\bar{s}) - z(s)) d\bar{s}.$$

As for $\bar{U}_2$ we have

$$\bar{U}_2 = \frac{1}{[1+(\bar{g}(s))^2]^{1/2}} \int_{\mathbb{R}/\mathbb{Z}} \Big(\frac{[1+(\bar{g}(\bar{s}))^2]^{1/2}}{L(\tau)} \bar{h}'(\bar{s})$$
$$- \frac{[1+(\bar{g}(s))^2]^{1/2}}{L(\tau)} \bar{h}'(s) \Big) K_1(x(s), z(\bar{s})) \frac{L(\tau)}{[1+(\bar{g}(\bar{s}))^2]^{1/2}} d\bar{s}$$
$$+ [1+(\bar{g}(s))^2]^2 \int_{\mathbb{R}/\mathbb{Z}} \chi(\bar{x}(\bar{s}) - x(s)) \Big[\frac{[1+(\bar{g}(s))^2]^{1/2}}{L(\tau)} \bar{h}'(s) - 2J_{3,1} \Big]$$
$$\times \frac{1}{|\bar{x}(\bar{s}) - x(s)|} \Bigg[\frac{1}{\Big(1 + [\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s))]^2\Big)^{5/2}}$$
$$- \frac{1}{[1+(\bar{g}(s))^2]^{5/2}} \Bigg] \frac{L(\tau)}{[1+(\bar{g}(\bar{s}))^2]^{1/2}} d\bar{s}.$$

And so we need to rewrite

$$\frac{1}{|\bar{x}(\bar{s}) - x(s)|} \Bigg[\frac{1}{\Big(1 + [\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s))]^2\Big)^{5/2}} - \frac{1}{[1+(\bar{g}(s))^2]^{5/2}} \Bigg]$$

$$= \frac{1}{|\bar{x}(\bar{s}) - x(s)|} \frac{[1+(\bar{g}(s))^2]^{5/2} - \Big(1 + [\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s))]^2\Big)^{5/2}}{[1+(\bar{g}(s))^2]^{5/2} \Big(1 + [\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s))]^2\Big)^{5/2}}.$$

Define $f(\tau) = \left(1 + [\bar{g}(s) + J_{2,1}\tau]^2\right)^{5/2}$, and use $f(0) - f(\tau) = -\int_0^1 f'(\rho\tau)d\rho\tau$ to obtain

$$= \frac{1}{|\bar{x}(\bar{s}) - x(s)|} \times \frac{-3\int_0^1 (1 + (\bar{g}(s) + J_{2,1}\rho(\bar{x}(\bar{s}) - x(s)))^2)^{3/2}[\bar{g}(s) + J_{2,1}\rho(\bar{x}(\bar{s}) - x(s))]J_{2,1}d\rho(\bar{x}(\bar{s}) - x(s))}{[1 + (\bar{g}(s))^2]^{5/2}\left(1 + [\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s))]^2\right)^{5/2}}$$

$$= \frac{-3\int_0^1 (1 + (\bar{g}(s) + J_{2,1}\rho(\bar{x}(\bar{s}) - x(s)))^2)^{3/2}[\bar{g}(s) + J_{2,1}\rho(\bar{x}(\bar{s}) - x(s))]J_{2,1}d\rho}{[1 + (\bar{g}(s))^2]^{5/2}\left(1 + [\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s))]^2\right)^{5/2}} \operatorname{sgn}(\bar{s} - s).$$

Using this expression U_2 becomes

$$\begin{aligned}
\bar{U}_2 = {} & \frac{1}{[1 + (g(s) + \bar{g}_0(s))^2]^{1/2}} \int_{\mathbb{R}/\mathbb{Z}} \Big([1 + (g(\bar{s}) + \bar{g}_0(\bar{s}))^2]^{1/2}[h'(\bar{s}) + \bar{h}_0'(\bar{s})] \\
& -[1 + (g(s) + \bar{g}_0(s))^2]^{1/2}[h'(s) + \bar{h}_0(s)]\Big) \times K_1(x(s), z(\bar{s})) \frac{1}{[1 + (g(\bar{s}) + \bar{g}_0(\bar{s}))^2]^{1/2}} d\bar{s} \\
& +[1 + (g(s) + \bar{g}_0(s))^2]^2 \int_{\mathbb{R}/\mathbb{Z}} \chi(\bar{x}(\bar{s}) - x(s)) \Big[[1 + (g(s) + \bar{g}_0(s))^2]^{1/2}[h'(s) + \bar{h}_0'(s)] \\
& -2(l(\tau) + L_0)J_{3,1}\Big] \\
& \times \frac{-3\int_0^1 (1 + (g(s) + \bar{g}_0(s) + J_{2,1}\rho(\bar{x}(\bar{s}) - x(s)))^2)^{3/2}[g(s) + \bar{g}_0(s) + J_{2,1}\rho(\bar{x}(\bar{s}) - x(s))]J_{2,1}d\rho}{[1 + (g(s) + \bar{g}_0(s))^2]^{5/2}\left(1 + [\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s))]^2\right)^{5/2}} \\
& \times \frac{\operatorname{sgn}(\bar{s} - s)}{[1 + (g(\bar{s}) + \bar{g}_0(\bar{s}))^2]^{1/2}} d\bar{s}.
\end{aligned} \tag{49}$$

As for $\bar{U}_1$ (25) becomes

$$\begin{aligned}
\bar{U}_1 = {} & \frac{3}{L(\tau)}\bar{h}'(s) - 3\frac{\bar{g}(s)(\bar{h}(s))^2}{[1 + (\bar{g}(\bar{s}))^2]^{3/2}} \\
& + \int_{\mathbb{R}/\mathbb{Z}} \Big\{\frac{\bar{h}(s)}{(1 + (\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s)))^2)^{1/2}} \\
& - \frac{[\bar{h}(s) + I_{3,1}(\bar{x}(\bar{s}) - x(s))][1 + (\bar{g}(s) + I_{2,1}(\bar{x}(\bar{s}) - x(s)))\bar{g}(s)]}{(1 + (\bar{g}(s) + I_{2,1}(\bar{x}(\bar{s}) - x(s)))^2)^{3/2}}\Big\} \\
& \times \frac{\chi'(x(s) - \bar{x}(\bar{s}))}{|\bar{x}(\bar{s}) - x(s)|} \frac{L(\tau)}{[1 + (\bar{g}(\bar{s}))^2]^{1/2}} d\bar{s} \\
& + \int_{\mathbb{R}/\mathbb{Z}} \frac{\operatorname{sgn}(\bar{s} - s)}{[1 + (\bar{g}(s) + J_{2,1}(\bar{x}(\bar{s}) - x(s)))^2]^{5/2}} \chi(\bar{x}(\bar{s}) - x(s)) \\
& \times \Big\{\text{pol in } \bar{f}(s), \bar{g}(s), \bar{h}(s), \frac{[1 + (\bar{g}(s))^2]^{1/2}}{L(\tau)} \bar{\bar{h}}'(s), (\bar{x}(\bar{s}) \\
& -x(s)), \text{ and } J_{2,1}, J_{3,1}, J_{3,2}\Big\} \frac{L(\tau)}{[1 + (\bar{g}(\bar{s}))^2]^{1/2}} d\bar{s},
\end{aligned}$$

which leads to

$$\begin{aligned}
\bar{U}_1 &= \frac{3}{l(t)+L(0)}[h'(s)+\bar{h}_0'(s)] - 3\frac{[g(s)+\bar{g}_0(s)][h(s)+\bar{h}_0(s)]^2}{[1+(g(s)+\bar{g}_0(s))^2]^{3/2}} \\
&\quad + \int_{\mathbb{R}/\mathbb{Z}} \Bigg\{ \frac{h(s)+\bar{h}_0(s)}{(1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2)^{1/2}} \\
&\quad - \frac{[h(s)+\bar{h}_0(s)+I_{3,1}(\bar{x}(\bar{s})-x(s))][1+(g(s)+\bar{g}_0(s)+I_{2,1}(\bar{x}(\bar{s})-x(s)))(g(s)+\bar{g}_0(s))]}{(1+(g(s)+\bar{g}_0(s)+I_{2,1}(\bar{x}(\bar{s})-x(s)))^2)^{3/2}} \Bigg\} \\
&\quad \times \frac{\chi'(x(s)-\bar{x}(\bar{s}))}{|\bar{x}(\bar{s})-x(s)|} \frac{l(t)+L(0)}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} d\bar{s} \\
&\quad + \int_{\mathbb{R}/\mathbb{Z}} \frac{\operatorname{sgn}(\bar{s}-s)}{[1+(g(s)+\bar{g}_0(s)+J_{2,1}(\bar{x}(\bar{s})-x(s)))^2])^2]^{5/2}} \chi(\bar{x}(\bar{s})-x(s)) \\
&\quad \times \Bigg\{ \text{pol in } [f(\bar{s})+\bar{f}_0(\bar{s})], [g(s)+\bar{g}_0(s)], [h(s)+\bar{h}_0(s)], \frac{[1+(g(s)+\bar{g}_0(s))^2]^{1/2}}{l(t)+L(0)}[h'(s) \\
&\quad + \bar{h}_0'(s)], (\bar{x}(\bar{s})-x(s)), \text{ and } J_{2,1}, J_{3,1}, J_{3,2} \Bigg\} \frac{l(t)+L(0)}{[1+(g(\bar{s})+\bar{g}_0(\bar{s}))^2]^{1/2}} d\bar{s}.
\end{aligned}$$

In order to complete the task of checking that the right hand sides of (41)–(44) have the correct analytic structure, the final point to consider is $J_{a,b}$. Recall that we have

$$\begin{gathered}
I_{a,b} = I_{a,b}(x,\bar{x},t) = \int_0^1 \varphi^{(a)}((1-\rho)\bar{x}+\rho x, t)\rho^b d\rho, \\
J_{a,b}(s,\bar{s},\tau) = I_{a,b}(x(s,\tau),\bar{x}(\bar{s},\tau),\tau).
\end{gathered}$$

In the rewriting of these expressions we will need to consider expressing $\bar{f}'(s)$, $\bar{f}''(s)$, $\bar{f}'''(s)$, $\bar{g}'(s)$ and $\bar{g}''(s)$ in terms of expressions that involve at most one derivative.

Recall that

$$\partial_s = \frac{L(t)}{(1+(\varphi'(x))^2)^{1/2}}\partial_x = \frac{L(t)}{(1+(\bar{g}(s))^2)^{1/2}}\partial_x.$$

We obtain

$$\begin{aligned}
\bar{f}'(s) &= \frac{L(t)}{(1+(\varphi'(x))^2)^{1/2}}\partial_x \bar{\bar{f}} = \frac{L(t)}{(1+(\bar{g}(s))^2)^{1/2}}\bar{g}(s), \\
\bar{g}'(s) &= \frac{L(t)}{(1+(\bar{g}(s))^2)^{1/2}}\bar{h}(s), \\
\bar{f}''(s) &= \partial_s \bar{f}'(s) = -\frac{L(t)\bar{g}(s)\bar{g}'(s)}{[1+(\bar{g}(s))^2]^{3/2}}\bar{g}(s) + \frac{L(t)}{[1+(\bar{g}(s))^2]^{1/2}}\bar{g}'(s) \\
&= -\frac{(L(t))^2(\bar{g}(s))^2\bar{h}(s)}{[1+(\bar{g}(s))^2]^2} + \frac{(L(t))^2}{[1+(\bar{g}(s))^2]}\bar{h}(s), \\
\bar{g}''(s) &= \partial_s \bar{g}'(s) = -\frac{L(t)\bar{g}(s)\bar{g}'(s)}{[1+(\bar{g}(s))^2]^{3/2}}\bar{h}(s) + \frac{L(t)}{[1+(\bar{g}(s))^2]^{1/2}}\bar{h}'(s) \\
&= -\frac{(L(t))^2(\bar{h}(s))^2\bar{g}(s)}{[1+(\bar{g}(s))^2]^2} + \frac{L(t)}{[1+(\bar{g}(s))^2]^{1/2}}\bar{h}'(s),
\end{aligned}$$

and finally (a simple calculation shows)

$$\bar{f}'''(s) = -\frac{4(L(t))^2\bar{g}(s)\bar{g}'(s)h(s)}{[1+(\bar{g}(s))^2]^3} + \frac{(L(t))^2}{[1+(\bar{g}(s))^2]^2}\bar{h}'(s).$$

Other recurrent expressions that will be needed are considered now:

$$\begin{aligned}\bar{x}(\bar{s}) - x(s) &= L(t)\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}} \\ &= L(t)\int_0^1 \frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho(\bar{s}-s),\end{aligned}$$

where we have taken $\bar{\bar{s}} = s + \rho(\bar{s}-s) = \rho\bar{s} + (1-\rho)s$, yielding $d\bar{\bar{s}} = d\rho(\bar{s}-s)$.

Also

$$\frac{\partial x}{\partial s} = L(t)\frac{1}{[1+(\bar{g}(s))^2]^{1/2}},$$

and hence

$$\begin{aligned}&\bar{x}(\bar{s}) - x(s) - \frac{\partial x}{\partial s}(\bar{s}-s) = L(t)\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}} - L(t)\frac{1}{[1+(\bar{g}(s))^2]^{1/2}}(\bar{s}-s) \\ &= L(t)\Big[\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} - \frac{1}{[1+(\bar{g}(s))^2]^{1/2}} d\bar{\bar{s}}\Big] \\ &= L(t)\Big[\int_s^{\bar{s}}\int_s^{\bar{\bar{s}}} -\frac{\bar{g}(\tilde{s})\bar{g}'(\tilde{s})}{[1+(\bar{g}(\tilde{s}))^2]^{3/2}} d\tilde{s}\, d\bar{\bar{s}}\Big] = -L(t)\int_s^{\bar{s}} (s-\tilde{s})\frac{\bar{g}(\tilde{s})\bar{g}'(\tilde{s})}{[1+(\bar{g}(\tilde{s}))^2]^{3/2}} d\tilde{s} \\ &= -L(t)\int_0^1 (1-\rho)\frac{\bar{g}(\rho\bar{s}+(1-\rho)s)\bar{g}'(\rho\bar{s}+(1-\rho)s)}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{3/2}} d\rho(\bar{s}-s)^2.\end{aligned}$$

Also

$$\frac{\partial^2 x}{\partial s^2} = -\frac{L(t)\bar{g}(s)\bar{g}'(s)}{[1+(\bar{g}(s))^2]^{3/2}},$$

and so

$$\begin{aligned}&\bar{x}(\bar{s}) - x(s) - \frac{\partial x}{\partial s}(\bar{s}-s) - \frac{1}{2}\frac{\partial^2 x}{\partial s^2}(\bar{s}-s)^2 \\ &= L(t)\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} d\bar{\bar{s}} - L(t)\frac{1}{[1+(\bar{g}(s))^2]^{1/2}}(\bar{s}-s) \\ &\quad -\frac{1}{2}\Big(-\frac{L(t)\bar{g}(s)\bar{g}'(s)}{[1+(\bar{g}(s))^2]^{3/2}}\Big)(\bar{s}-s)^2 \\ &= L(t)\int_s^{\bar{s}} \frac{1}{[1+(\bar{g}(\bar{\bar{s}}))^2]^{1/2}} - \frac{1}{[1+(\bar{g}(s))^2]^{1/2}} + \frac{\bar{g}(s)\bar{g}'(s)}{[1+(\bar{g}(s))^2]^{1/2}}(\bar{\bar{s}}-s)d\bar{\bar{s}},\end{aligned}$$

which by Taylor's Theorem becomes

$$
\begin{aligned}
&= L(t)\int_s^{\bar{s}}\int_s^{\bar{\bar{s}}}\frac{\partial^2}{\partial\tilde{s}^2}\Big(\frac{1}{[1+(\bar{g}(\tilde{s}))^2]^{1/2}}\Big)(\bar{\bar{s}}-\tilde{s})d\tilde{s}d\bar{\bar{s}}\\
&= L(t)\int_s^{\bar{s}}\frac{\partial^2}{\partial\tilde{s}^2}\Big(\frac{1}{[1+(\bar{g}(\tilde{s}))^2]^{1/2}}\Big)\frac{1}{2}(\bar{s}-\tilde{s})^2d\tilde{s}\\
&= L(t)\int_s^{\bar{s}}\Big[\frac{(\bar{g}'(\tilde{s}))^2}{[1+(\bar{g}(\tilde{s}))^2]^{3/2}}+\frac{\bar{g}(\tilde{s})L(t)\bar{h}'(\tilde{s})}{[1+(\bar{g}(\tilde{s}))^2]^2}-3\frac{(\bar{g}(\tilde{s}))^2(\bar{g}'(\tilde{s}))^2}{[1+(\bar{g}(\tilde{s}))^2]^{5/2}}\Big]\frac{1}{2}(\bar{s}-s)^2d\bar{s}\\
&= L(t)\int_0^1\Big[\frac{(\bar{g}'(\tilde{s}))^2}{[1+(\bar{g}(\tilde{s}))^2]^{3/2}}+\frac{\bar{g}(\tilde{s})L(t)\bar{h}'(\tilde{s})}{[1+(\bar{g}(\tilde{s}))^2]^2}-3\frac{(\bar{g}(\tilde{s}))^2(\bar{g}'(\tilde{s}))^2}{[1+(\bar{g}(\tilde{s}))^2]^{5/2}}\Big]\frac{1}{2}(1-\rho)^2d\rho(\bar{s}-s)^3.
\end{aligned}
$$

We start by considering $I_{2,1}$. It is originally defined by Taylor's formula,

$$\varphi(\bar{x})=\varphi(x)+\varphi'(x)(\bar{x}-x)+I_{2,1}(\bar{x}-x)^2,$$

which in terms of $\bar{f},\bar{g},\ldots$ becomes

$$\bar{f}(\bar{x})=\bar{f}(x)+\bar{g}(s)(\bar{x}(\bar{s})-x(s))+J_{2,1}(\bar{x}(\bar{s})-x(s))^2.$$

Also, Taylor-expanding $f(s)$ directly we obtain

$$\bar{f}(\bar{s})=\bar{f}(s)+\bar{f}'(s)(\bar{s}-s)+\int_0^1\bar{f}''((1-\rho)\bar{s}+\rho s)\rho d\rho(\bar{s}-s)^2.$$

And so

$$
\begin{aligned}
&\bar{g}(s)\frac{\partial x}{\partial s}(\bar{s}-s)+g(s)[\bar{x}(\bar{s})-x(s)-\frac{\partial x}{\partial s}(\bar{s}-s)]+J_{2,1}(\bar{x}(\bar{s})-x(s))^2\\
&\quad=\bar{f}'(s)(\bar{s}-s)+\int_0^1\bar{f}''((1-\rho)\bar{s}+\rho s)\rho d\rho(\bar{s}-s)^2,
\end{aligned}
$$

which yields

$$
\begin{aligned}
J_{2,1}=&\frac{1}{(\bar{x}(\bar{s})-x(s))^2}\Big[\bar{g}(s)[\bar{x}(\bar{s})-x(s)-\frac{\partial x}{\partial s}(\bar{s}-s)]\\
&+\int_0^1\bar{f}''((1-\rho)\bar{s}+\rho s)\rho d\rho(\bar{s}-s)^2\Big]\\
=&\frac{1}{\Big(L(t)\int_0^1\frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}}d\rho(\bar{s}-s)\Big)^2}\Big[-\bar{g}(s)L(t)\\
&\times\int_0^1(1-\rho)\frac{\bar{g}(\rho\bar{s}+(1-\rho)s)\bar{g}'(\rho\bar{s}+(1-\rho)s)}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{3/2}}d\rho\\
&+\int_0^1-\frac{(L(t))^2(\bar{g}((1-\rho)\bar{s}+\rho s))^2\bar{h}((1-\rho)\bar{s}+\rho s)}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^2}\\
&+\frac{(L(t))^2}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]}\bar{h}((1-\rho)\bar{s}+\rho s)d\rho\Big].
\end{aligned}
$$

The next term to consider is $J_{3,1}$. We have

$$\varphi'(\bar{x}) = \varphi'(x) + \varphi''(x)(\bar{x} - x) + I_{3,1}(\bar{x} - x)^2,$$

which in terms of $\bar{f}, \ldots$ becomes

$$\bar{g}(\bar{s}) = \bar{g}(s) + \bar{h}(s)(\bar{x}(\bar{s}) - x(s)) + J_{3,1}(\bar{x}(\bar{s}) - x(s))^2,$$
$$\bar{g}(\bar{s}) = \bar{g}(s) + \bar{h}(s)\frac{\partial x}{\partial s} + \bar{h}(s)(\bar{x}(\bar{s}) - x(s) - \frac{\partial x}{\partial s}) + J_{3,1}(\bar{x}(\bar{s}) - x(s))^2.$$

Taylor for $\bar{g}$ yields

$$\bar{g}(\bar{s}) = \bar{g}(s) + \bar{g}'(s)(\bar{s} - s) + \int_0^1 \bar{g}''((1-\rho)\bar{s} + \rho s)\rho d\rho(\bar{s} - s)^2,$$

and so

$$\begin{aligned}
J_{3,1} = {} & \frac{1}{(\bar{x}(\bar{s}) - x(s))^2}\Big[- \bar{h}(s)\left(\bar{x}(\bar{s}) - x(s) - \frac{\partial x}{\partial s}\right) \\
& + \int_0^1 \bar{g}''((1-\rho)\bar{s} + \rho s)\rho d\rho(\bar{s} - s)^2\Big] \\
& \times \frac{1}{\left(L(t)\int_0^1 \frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}} d\rho(\bar{s} - s)\right)^2}\Big[- \bar{g}(s)L(t) \\
& \times \int_0^1 (1-\rho)\frac{\bar{g}(\rho\bar{s} + (1-\rho)s)\bar{g}'(\rho\bar{s} + (1-\rho)s)}{[1 + (\bar{g}((1-\rho)s + \rho\bar{s}))^2]^{3/2}} d\rho \\
& + \int_0^1 -\frac{(L(t))^2(\bar{h}(\rho\bar{s} + (1-\rho)s))^2\bar{g}(\rho\bar{s} + (1-\rho)s)}{[1 + (\bar{g}((1-\rho)s + \rho\bar{s}))^2]^2} \\
& + \frac{L(t)}{[1 + (\bar{g}((1-\rho)s + \rho\bar{s}))^2]^{1/2}}\bar{h}'(\rho\bar{s} + (1-\rho)s)d\rho\Big].
\end{aligned}$$

The last term that we need to consider is $I_{3,2}$

$$\varphi(\bar{x}) = \varphi(x) + \varphi'(x)(\bar{x} - x) + \frac{1}{2}\varphi''(x)(\bar{x} - x)^2 + \frac{1}{2}I_{3,2}(\bar{x} - x)^3,$$

which becomes

$$\begin{aligned}
\bar{f}(\bar{s}) = {} & \bar{f}(s) + \bar{g}(s)(\bar{x}(\bar{s}) - x(s)) + \frac{1}{2}\bar{h}(s)(\bar{x}(\bar{s}) - x(s))^2 + \frac{1}{2}J_{3,2}(\bar{x}(\bar{s}) - x(s))^3, \\
\bar{f}(\bar{s}) = {} & \bar{f}(s) + \bar{g}(s)(\frac{\partial x}{\partial s}(\bar{s} - s) + \frac{1}{2}\frac{\partial^2 x}{\partial s^2}(\bar{s} - s)^2)) + \frac{1}{2}\bar{h}(s)\Big(\frac{\partial x}{\partial s}(\bar{s} - s)\Big)^2 \\
& + \bar{g}(s)(\bar{x}(\bar{s}) - x(s) - \frac{\partial x}{\partial s}(\bar{s} - s) - \frac{1}{2}\frac{\partial^2 x}{\partial s^2}(\bar{s} - s)^2) \\
& + \frac{1}{2}\bar{h}(s)\Big[(\bar{x}(\bar{s}) - x(s))^2 - \Big(\frac{\partial x}{\partial s}(\bar{s} - s)\Big)^2\Big] \\
& + \frac{1}{2}J_{3,2}(\bar{x}(\bar{s}) - x(s))^3.
\end{aligned}$$

On the other hand Taylor for $\bar{f}$ yields

$$\bar{f}(\bar{s}) = \bar{f}(s) + \bar{f}'(s)(\bar{s}-s) + \frac{1}{2}\bar{f}''(s)(\bar{s}-s)^2 + \frac{1}{2}\int_0^1 \bar{f}'''(\rho\bar{s}+(1-\rho)s)\rho^2 d\rho.$$

This yields

$$\begin{aligned}
J_{3,2} &= \frac{2}{(\bar{x}(\bar{s})-x(s))^3}\Big[-\bar{g}(s)(\bar{x}(\bar{s})-x(s)-\frac{\partial x}{\partial s}(\bar{s}-s)-\frac{1}{2}\frac{\partial^2 x}{\partial s^2}(\bar{s}-s)^2)\\
&\quad -\frac{1}{2}\bar{h}(s)\Big[(\bar{x}(\bar{s})-x(s))^2-\Big(\frac{\partial x}{\partial s}(\bar{s}-s)\Big)^2\Big] + \frac{1}{2}\int_0^1 \bar{f}'''(\rho\bar{s}+(1-\rho)s)\rho^2 d\rho\Big]\\
&= \frac{2}{(\bar{x}(\bar{s})-x(s))^3}\Big[-\bar{g}(s)(\bar{x}(\bar{s})-x(s)-\frac{\partial x}{\partial s}(\bar{s}-s)-\frac{1}{2}\frac{\partial^2 x}{\partial s^2}(\bar{s}-s)^2)\\
&\quad -\frac{1}{2}\bar{h}(s)\Big[(\bar{x}(\bar{s})-x(s)-\frac{\partial x}{\partial s}(\bar{s}-s))^2+2\frac{\partial x}{\partial s}(\bar{s}-s)\Big(\bar{x}(\bar{s})-x(s)-\frac{\partial x}{\partial s}(\bar{s}-s)\Big)\Big]\\
&\quad +\frac{1}{2}\int_0^1 \bar{f}'''(\rho\bar{s}+(1-\rho)s)\rho^2 d\rho\Big],
\end{aligned}$$

and so we obtain

$$\begin{aligned}
&\frac{2}{\Big(L(t)\int_0^1 \frac{1}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{1/2}}d\rho\Big)^3(\bar{s}-s)^3}\\
&\times\Big[-\bar{g}(s)\Big(L(t)\int_0^1\Big[\frac{(\bar{g}'(\tilde{s}))^2}{[1+(\bar{g}(\tilde{s}))^2]^{3/2}}+\frac{\bar{g}(\tilde{s})L(t)\bar{h}'(\tilde{s})}{[1+(\bar{g}(\tilde{s}))^2]^2}\\
&-3\frac{(\bar{g}(\tilde{s}))^2(\bar{g}'(\tilde{s}))^2}{[1+(\bar{g}(\tilde{s}))^2]^{5/2}}\Big]\frac{1}{2}(1-\rho)^2 d\rho\Big)(\bar{x}-x)^3\\
&-\frac{1}{2}\bar{h}(s)\Big(-L(t)\int_0^1(1-\rho)\frac{\bar{g}(\rho\bar{s}+(1-\rho)s)\bar{g}'(\rho\bar{s}+(1-\rho)s)}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{3/2}}d\rho(\bar{s}-s)^2\Big)^2\\
&-\bar{h}(s)\frac{L(t)}{[1+(\bar{g}(s))^2]^{1/2}}(\bar{s}-s)\Big(-L(t)\int_0^1(1-\rho)\\
&\times\frac{\bar{g}(\rho\bar{s}+(1-\rho)s)\bar{g}'(\rho\bar{s}+(1-\rho)s)}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^{3/2}}d\rho(\bar{s}-s)^2\Big)\\
&+\frac{1}{2}\int_0^1 -\frac{4(L(t))^2\bar{g}(\rho\bar{s}+(1-\rho)s)\bar{g}'(\rho\bar{s}+(1-\rho)s)\bar{h}(\rho\bar{s}+(1-\rho)s)}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^3}\\
&+\frac{(L(t))^2}{[1+(\bar{g}((1-\rho)s+\rho\bar{s}))^2]^2}\bar{h}'(\rho\bar{s}+(1-\rho)s)\rho^2 d\rho(\bar{s}-s)^3\Big].
\end{aligned}$$

This completes the analysis of the system of 4 equations to which we will apply the Cauchy-Kowaleski Theorem. In the next section we will briefly describe the abstract scheme and check that the system formed by (41)–(44) fits that scheme.

3. Cauchy-Kowaleski Theorem

We begin with some definitions.

Definition 1. *A Banach scale* $\{X_\rho, 0 < \rho < \rho_0\}$ *with norms* $\| \, \|_\rho$ *is a collections of Banach spaces such that* $X_{\rho'} \subset X_{\rho''}$ *with* $\| \, \|_{\rho''} \leq \| \, \|_{\rho'}$ *whenever* $\rho'' \leq \rho' \leq \rho_0$.

Definition 2. *Given* $\tau > 0, 0 < \rho \leq \rho_0$ *and* $R > 0$ *we define*:

1. $X_{\rho,\tau}$ *to be the set of all functions* $u(t)$ *from* $[0, \tau]$ *to* X_ρ *endowed with the norm*

$$\|u\|_{\rho,\tau} = \sup_{0\leq t\leq\tau} \|u(t)\|_\rho.$$

2. $Y_{\rho,\beta,\tau}$ *is the set of functions* $u(t)$ *from* $[0, T]$ *to* X_ρ *with the norm*

$$\|u(t)\|_{\rho,\beta,\tau} = \sup_{0\leq t\leq\tau} \|u(t)\|_{\rho-\beta\tau}.$$

3. *We will denote by* $X_{\rho,\tau}(R)$ *and* $Y_{\rho,\beta,\tau}(R)$ *the balls of radius* R *in* $X_{\rho,\tau}$ *and* $Y_{\rho,\beta,\tau}$ *respectively.*

Theorem 2 (Sammartino–Caflisch). *Suppose that there exist* $R > 0$*,* $T > 0$ *and* $\beta_0 > 0$ *such that for* $0 < t \leq T$ *the following holds:*

1. *For every* $0 < \rho' < \rho < \rho_0 - \beta_0 T$ *and every* $u \in X_{\rho,T}(R)$ *the function* $F(t, u) : [0, T] \to X_{\rho'}$ *is continuous.*
2. *For every* $0 < \rho \leq \rho_0 - \beta_0 T$ *the function* $F(t, 0) : [0, T] \to X_{\rho,T}(R)$ *is continuous in* $[0, T]$ *and*

$$\|F(t, 0)\|_{\rho_0-\beta_0 T} \leq R_0 < R.$$

3. *For every* $0 < \rho' < \rho(s) \leq \rho_0 - \beta_0 s$ *and every* $u_1, u_2 \in Y_{\rho_0,\beta_0,T}(T)$ *we have*

$$\|F(t, u_1) - F(t, u_2)\|_{\rho'} \leq C \int_0^t \frac{\|u_1 - u_2\|_{\rho(s)}}{\rho(s) - \rho} ds.$$

Then there exist $\beta > \beta_0$ *and* $T^* > 0$ *such that*

$$u + F(t, u) = 0$$

has a unique solution in Y_{ρ_0,β,T^*}.

We begin by defining the spaces of functions. We will be complexifying only the space variable. We look at functions in $\mathbb{C}$, periodic in $\Re x$.

Definition 3. *Given* $l \in \mathbb{N}$ *and* $\rho > 0$ *we say that a function* $f(x)$ *is in* $K^{l,\rho}$ *if and only if*

- f *is periodic in* $\Re x$ *and analytic in* $|\Im x| < \rho$.
- *For every* $|\Im x| < \rho$, $\partial_x^\alpha f \in L^2$ *for every fixed* $\Im x$*, that is, as a function of the real part only.*
- *The norm in* $\|f\|_{l,\rho}$ *is finite, where*

$$\|f\|_{l,\rho} := \sum_{\alpha\leq l} \sup_{|\Im x|<\rho} \|\partial_x^\alpha f(\cdot + \Im x)\|_{L^2(\mathbb{R}/\mathbb{Z})}.$$

The following two lemmas about analytic functions will be used below to prove the main theorem.

Lemma 4. *Suppose that* $\Phi(z_1, \ldots, z_n)$ *is analytic with* $\{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| \le CA_1, \ldots, |z_n| \le CA_n\} \subset\subset U$. *Then the map*

$$(f_1, \ldots, f_n) \to \Phi(f_1, \ldots, f_l)$$

(*where* f_i *belongs to the Banach space under consideration* (*be precise*)) *is Lip(1) on*

$$\{(f_1, \ldots, f_n) \in \mathbb{C}^l : \|f_1\| \le CA_1, \ldots, \|f_n\| \le CA_n\},$$

with Lip(1)-norm bounded a priori by Φ *and* A_i.

Lemma 5. *Define*

$$F(x) := \int_\Omega \Phi(f_1(x + \tau_1(\rho)), \ldots, f_n(x + \tau_n(\rho)))d\mu(\rho),$$

where f_i *are in the Banach space under consideration and satisfy* $\|f_i\| \le CA_i$, *and we assume that all* $\tau_i(\rho)$ *are smooth real functions,* Φ *is as in the previous lemma and* $(\Omega, d\mu)$ *is a probability measure. Then the map*

$$(f_1, \ldots, f_n) \to F(x)$$

is Lip(1), with Lip(1)-norm a priori bounded by Φ *and* A_i.

In order to prove Theorem 1 we start by defining the spaces appearing in the Theorem of Sammartino and Caflisch. We define

$$X^{k,\rho} = H^{k,\rho} \times H^{k,\rho} \times H^{k,\rho} \times \mathbb{R}.$$

We rewrite the system of equations in the form

$$\begin{pmatrix} f \\ g \\ h \\ l \end{pmatrix} = \begin{pmatrix} F_1(t, f, g, h, l) \\ F_2(t, f, g, h, l) \\ F_3(t, f, g, h, l) \\ F_4(t, f, g, h, l) \end{pmatrix},$$

where F_1, F_2, F_3 and F_4 are given by the right hand sides of Eqs. (41), (42), (43) and (44).

Conditions 1 and 2 of the theorem are easily checked. We leave the details to the interested reader. We concentrate on the more complicated Cauchy estimate 3.

By simple inspection (using the careful rewriting of the right hand sides at the end of the previous section) it is straightforward to check that F_1, F_2 and F_4 can be written as analytic functions of the arguments

$$f, g, h, l, f'g', h', l' \text{ and integrals of the form}$$
$$\int_0^1 \Phi(f_1(x + \tau(\rho)), \ldots, f_n(x + \tau_n(\rho)))d\rho, \tag{50}$$

where Φ is analytic and the functions f_i can only be taken from the list f, g, hf', g', h'.

Lemmas 4 and 5 show that F_1, F_2 and F_4 satisfy the Lipschitz estimate required by the Cauchy-Kowaleski Theorem.

As for F_3 we notice that the equation for h is given by (38) which can be rewritten as

$$h_\tau(s,\tau) + \frac{1}{l(\tau)+L_0}\int_{\mathbb{R}/\mathbb{Z}} (h'(\bar{s},\tau) - h'(s,\tau))\tilde{\chi}(s-\bar{s})\frac{1}{|\bar{s}-s|}d\bar{s}$$
$$= \bar{U}_1 + \bar{U}_2 + \bar{U}_3 + \bar{U}_4 + \bar{U}_5,$$

which using notation from pseudo-differential operators can be rewritten as

$$(\partial_t + \frac{1}{l(\tau)+L_0} \mathrm{i}m(k))h = \bar{U}_1 + \bar{U}_2 + \bar{U}_3 + \bar{U}_4 + \bar{U}_5,$$

and so

$$h = (\partial_t + \frac{1}{l(\tau)+L_0} \mathrm{i}m(k))^{-1}(\bar{U}_1 + \bar{U}_2 + \bar{U}_3 + \bar{U}_4 + \bar{U}_5),$$

which has the integral representation form given in (43). Notice that the operator

$$(\partial_t + \frac{1}{l(\tau)+L_0} \mathrm{i}m(k))^{-1}$$

preserves all L^2-based Sobolev norms in the spatial variable. This is a simple consequence of the fact that the multiplier is purely imaginary (since $m(k)$ is real), and can also be seen given the integral representation in (43) where the exponent of e is purely imaginary (notice that t has not been complexified).

Since $U_1, \ldots, U_5$ can be written as analytic functions of the arguments described in (50) we obtain the required Lipschitz estimate concluding the proof.

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