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Existence and Uniqueness of SRB Measure on C^1 Generic Hyperbolic Attractors

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Abstract: Let M be a smooth Riemannian manifold. We show that for C^1 generic $f \in \mathrm{Diff}^1(M)$, if f has a hyperbolic attractor Λ_f, then there exists a unique SRB measure supported on Λ_f. Moreover, the SRB measure happens to be the unique equilibrium state of potential function $\psi_f \in C^0(\Lambda_f)$ defined by $\psi_f(x) = -\log|\det(Df|E^u_x)|$, $x \in \Lambda_f$, where E^u_x is the unstable space of T_xM.

1. Preliminary

Let M be a smooth Riemannian manifold. Assume m is the volume measure of M induced by Riemann metric. Denote by δ_x the probability atomic measure supported on $x \in M$. For any C^1 diffeomorphism f and ergodic measure μ, the *statistical basin* of μ is defined as

$$\begin{aligned} B(\mu) &= \{x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) = \int \varphi d\mu, \forall \varphi \in C^0(M)\} \\ &= \{x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x} = \mu\}, \end{aligned}$$

and its elements are called *generic points* of μ. If $m(B(\mu)) > 0$, we call μ an *SRB measure*.

The theory of SRB measure has been extensively studied since it was introduced by Sinai, Ruelle and Bowen in the early 1970's. The classical SRB theory says that, if dynamical systems admit sufficient differentiability and hyperbolicity, then they do have SRB measures. A particular example will be $C^{k,\alpha}$ hyperbolic attractors, where $k = 1, 2, 3, \ldots$ and $0 < \alpha \le 1$. In this situation, we have both existence and uniqueness of the SRB measure that is supported on such an attractor (see, for instance, [1,8]).

With abundance of results in the case of high differentiability, people are curious whether the theory maintains for “most” $f \in \mathrm{Diff}^1(M)$. Towards this question, Campbell and Quas obtained the following C^1 generic result for circle expanding maps (see [2]).

Theorem (Campbell, Quas). *Let $\mathcal{E}^1$ denote the set of C^1 expanding maps of the unit circle S^1 onto itself. Assume m is the normalized Lebesgue measure over S^1. Then for generic $T \in \mathcal{E}^1$, there is a unique SRB measure μ_T, with $m(B(\mu_T)) = 1$.*

In this paper, we push forward the above result to the setting of C^1 hyperbolic attractors: Let f_0 be a C^1 diffeomorphism of M. Assume there exists a compact invariant transitive hyperbolic set Λ_{f_0}, and an open neighborhood $\Omega \supset \Lambda_{f_0}$, so that $f_0(\overline{\Omega}) \subset \Omega$ and $\bigcap_{i\geq 0} f_0^i(\Omega) = \Lambda_{f_0}$. By stability theory of an isolated hyperbolic set (see [7]), there exists a C^1 neighborhood $\mathcal{U}$ of f_0, so that for any $f \in \mathcal{U}$, the f-maximal invariant set of Ω, denoted by Λ_f, is also hyperbolic. Moreover, for each $f \in \mathcal{U}$ there is a unique homeomorphism $r_f : \Lambda_{f_0} \to \Lambda_f$ that is C^0 close to $\mathrm{id}|\Lambda_{f_0}$, with

$$f|\Lambda_f \circ r_f = r_f \circ f_0|\Lambda_{f_0}.$$

The main result of the paper is

Theorem A. *There exists a generic set $\mathcal{U}'$ in $\mathcal{U}$ with the following property: for any $f \in \mathcal{U}'$, there is a unique SRB measure μ_f supported on Λ_f, with $m(B(\mu_f) \cap \Omega) = m(\Omega)$. Moreover, μ_f depends continuously in weak*-topology on $f \in \mathcal{U}'$.*

The proof of Theorem A is formulated through Sects. 3, 4. It basically follows Bowen’s convention of equilibrium state thermodynamical formalism developed in [1]. Thus we give in Sect. 2 a partial review on related concepts and results of this topic.

Notation Hypotheses:

1) For any $f \in \mathcal{U}$, denote by $E^u_{\Lambda_f} \oplus E^s_{\Lambda_f} = \bigcup_{x\in\Lambda_f} E^u_x \oplus E^s_x$ the hyperbolic splitting for $T_{\Lambda_f} M$, and $u = \dim E^u_{\Lambda_f}$.
2) For compact metric space X and continuous map T over it, denote by $\mathcal{M}(X)$ the set of Borel probability measures on X, by $\mathcal{M}(X; T)$ the set of T-invariant Borel probability measures on X, and by $\mathcal{E}(X; T)$ the set of T-ergodic Borel probability measures on X.
3) For any compact C^1 submanifold $\Delta \subset M$, denote by $T_x\Delta$ the tangent space of Δ at x, by $T\Delta$ the tangent bundle of Δ, and by m_Δ the volume measure induced by submanifold immersion.
4) For any finite set A, denote by $\sharp A$ the cardinality of A.

2. A Partial Review on Thermodynamical Formalism

Most contents of this section can be found in [1,3 and 9].

Let X be a compact metric space, and T be a continuous map over it. We call such a pair as (X, T) a *topological dynamical system*.

For any $\phi \in C^0(X)$ (ϕ is usually called a *potential function*), the *topological pressure* of ϕ (w.r.t T) is defined by

$$P(T; \phi) = \sup_{\mu\in\mathcal{M}(X;T)} \{h_\mu(T) + \int_X \phi d\mu\},$$

where $h_\mu(T)$ is the measure theoretical entropy of T with respect to μ. If the topological entropy

$$h(T) \triangleq \sup_{\mu \in \mathcal{M}(X;T)} h_\mu(T) < \infty,$$

then $|P(T;\phi)| < \infty$ for any $\phi \in C^0(X)$. In this situation, $P(T;\cdot) : C^0(X) \to \mathbb{R}$ has the following elementary properties (see Theorem 9.7 of [9]):

1. **(Continuity)** For any $\phi', \phi'' \in C^0(X)$,

$$\|P(T;\phi') - P(T;\phi'')| \le \|\phi' - \phi''\|_{C^0(X)}. \tag{2.1}$$

2. **(Convexity)** For any $\phi', \phi'' \in C^0(X)$ and $0 \le t \le 1$

$$P(T; t\phi' + (1-t)\phi'') \le tP(T;\phi') + (1-t)P(T;\phi'').$$

As a consequence of convexity, for any $\phi, \varphi \in C^0(X)$, and $t_1 < t_2 < t_3$, we have

$$\frac{P(T;\phi + t_2\varphi) - P(T;\phi + t_1\varphi)}{t_2 - t_1} \le \frac{P(T;\phi + t_3\varphi) - P(T;\phi + t_1\varphi)}{t_3 - t_1}, \tag{2.2}$$

and

$$\frac{P(T;\phi + t_2\varphi) - P(T;\phi + t_1\varphi)}{t_2 - t_1} \le \frac{P(T;\phi + t_3\varphi) - P(T;\phi + t_2\varphi)}{t_3 - t_2}. \tag{2.3}$$

In particular, taking $t_1 = 0$, (2.2) implies that $(P(T;\phi + t\varphi) - P(T;\phi))/t$ monotonically decreases as $t \to 0^+$. Moreover, taking $t_2 = 0$, (2.3) implies that $(P(T;\phi + t\varphi) - P(T;\phi))/t, t > 0$ is bounded from below. Thus $\lim_{t\to 0^+}(P(T;\phi + t\varphi) - P(T;\phi))/t$ exists, and equals $\inf_{t>0}(P(T;\phi + t\varphi) - P(T;\phi)/t$. We denote the limit by $\tau(T;\phi,\varphi)$, i.e.,

$$\tau(T;\phi,\varphi) = \inf_{t>0} \frac{P(T;\phi + t\varphi) - P(T;\phi)}{t} = \lim_{t\to 0^+} \frac{P(T;\phi + t\varphi) - P(T;\phi)}{t}. \tag{2.4}$$

Lemma 2.1. *Assume $h(T) < \infty$ and $\phi \in C^0(X)$. Then*

1. *For any $\varphi \in C^0(X)$,*

$$\tau(T;\phi,\varphi) \ge -\tau(T;\phi,-\varphi). \tag{2.5}$$

2. *$\tau(T;\phi,\cdot) : C^0(X) \to \mathbb{R}$ is continuous. More precisely, for any $\varphi', \varphi'' \in C^0(X)$,*

$$|\tau(T;\phi,\varphi') - \tau(T;\phi,\varphi'')| \le \|\varphi' - \varphi''\|_{C^0(X)}.$$

Proof. Let $t_2 = 0$ and take the limit as $t_1 \to 0^-$, respectively $t_3 \to 0^+$ in (2.3). Then the first statement is clear by definition of $\tau(T;\phi,\varphi)$.

The second statement is straightforward by (2.1) and direct computation. □

Lemma 2.2. *Assume $h(T) < \infty$ and $\varphi \in C^0(X)$. Then*

1. *$\tau(T;\cdot,\varphi) : C^0(X) \to \mathbb{R}$ is upper semicontinuous.*
2. *For any $\phi \in C^0(X)$, if $\tau(T;\phi,\varphi) = -\tau(T;\phi,-\varphi)$, then $\tau(T;\cdot,\varphi)$ is continuous at ϕ.*

Proof. Upper semicontinuity of $\tau(T;\cdot,\varphi)$ is clear from the first "=" of (2.4). For the second statement, let $\phi_k \overset{C^0}{\to} \phi$, then upper semicontinuity of $\tau(T;\cdot,\varphi)$ gives

$$\limsup_{k\to\infty} \tau(T;\phi_k,\varphi) \le \tau(T;\phi,\varphi),$$

and

$$\limsup_{k\to\infty} \tau(T;\phi_k,-\varphi) \le \tau(T;\phi,-\varphi).$$

Therefore, if $\tau(T;\phi,\varphi) = -\tau(T;\phi,-\varphi)$, we have

$$\begin{aligned}\tau(T;\phi,\varphi) &= -\tau(T;\phi,-\varphi) \le -\limsup_{k\to\infty} \tau(T;\phi_k,-\varphi) = \liminf_{k\to\infty} -\tau(T;\phi_k,-\varphi)\\ &\overset{(2.5)}{\le} \liminf_{k\to\infty} \tau(T;\phi_k,\varphi) \le \limsup_{k\to\infty} \tau(T;\phi_k,\varphi) \le \tau(T;\phi,\varphi).\end{aligned}$$

Then the above "$\le$" must all be "=". In particular,

$$\liminf_{k\to\infty} \tau(T;\phi_k,\varphi) = \limsup_{k\to\infty} \tau(T;\phi_k,\varphi) = \tau(T;\phi,\varphi),$$

thus $\lim_{k\to\infty} \tau(T;\phi_k,\varphi) = \tau(T;\phi,\varphi)$. □

An *equilibrium state* of ϕ (w.r.t. T) is a T-invariant probability measure ν satisfying

$$P(T;\phi) = h_\nu(T) + \int_X \phi d\nu.$$

A *tangent functional* to $P(T;\cdot)$ at ϕ is a finite signed measure μ on X such that

$$P(T;\phi+\varphi) - P(T;\phi) \ge \int_X \varphi d\mu, \quad \forall \varphi \in C^0(X)$$

Let $Eq(T;\phi)$ be the collection of equilibrium states of ϕ w.r.t. T, $t(T;\phi)$ be the collection of tangent functionals to $P(T;\cdot)$ at ϕ.

Lemma 2.3. *Assume $h_\cdot(T) : \mathcal{M}(X;T) \to \mathbb{R}$ is upper semicontinuous. Then for any $\phi \in C^0(X)$, $Eq(T;\phi) = t(T;\phi)$.*

Proof. See Theorem 9.15 of [9]. □

Lemma 2.4. *Assume $h_\cdot(T) : \mathcal{M}(X;T) \to \mathbb{R}$ is upper semicontinuous and $\phi \in C^0(X)$. Then the following statements are equivalent*:

1) $\sharp Eq(T;\phi) = 1$.
2) $\tau(T;\phi,\varphi) = -\tau(T;\phi,-\varphi), \forall \varphi \in C^0(X)$.
3) *For any $\nu \in Eq(T,\phi)$, we have $\int \varphi d\nu = \tau(T;\phi,\varphi), \forall \varphi \in C^0(X)$.*

Proof. Consider "1) $\Rightarrow$ 2)" first. Assume $\sharp Eq(T;\phi) = 1$. Suppose $\exists \varphi' \in C^0(X)$ so that $\tau(T;\phi,\varphi') \neq -\tau(T;\phi,-\varphi')$. Then by (2.5) $\tau(T;\phi,\varphi') > -\tau(T;\phi,-\varphi')$. We claim that for any $a \in [-\tau(T;\phi,-\varphi'), \tau(T;\phi,\varphi')]$, there exist $\nu \in Eq(T;\phi)$ so that $\int \varphi' d\nu = a$. In fact, consider $\langle\varphi'\rangle$ the one-dimensional linear space generated by φ'. We define the linear functional $\tilde{A} : \langle\varphi'\rangle \to \mathbb{R}$ by $\tilde{A}(t\varphi') = at$. Then the first "=" of (2.4) yields

$$\begin{cases} P(T;\phi + t\varphi') - P(T;\phi) \geq t\tau(T;\phi,\varphi') \geq at = \tilde{A}(t\varphi'), \\ P(T;\phi - t\varphi') - P(T;\phi) \geq t\tau(T;\phi,-\varphi') \geq -at = \tilde{A}(-t\varphi'), \end{cases} \tag{2.6}$$

for $t \geq 0$. This implies that the graph of $\tilde{A}$ is under the graph of $P(T;\phi+\cdot)-P(T;\phi)|_{\langle\varphi'\rangle}$. Applying the Hahn-Banach theorem and due to convexity of $P(T;\phi+\cdot) - P(T;\phi)$, we can extend $\tilde{A}$ to $A \in C^0(X)^*$, so that $A(t\varphi') = at$, and the graph of A is under the graph of $P(T;\phi+\cdot) - P(T;\phi)$, i.e.

$$P(T;\phi + \varphi) - P(T;\phi) \geq A(\varphi), \quad \forall \varphi \in C^0(X).$$

Let ν be the signed measure associated to A by the Riesz representation theorem, then $\nu \in t(T;\phi)$, and by Lemma 2.3, $\nu \in Eq(T;\phi)$. Clearly, $\int \varphi' d\nu = a$.

Therefore, for arbitrary $-\tau(T;\phi,-\varphi') \leq a_1 < a_2 \leq \tau(T;\phi,\varphi')$, there must be $\nu_1, \nu_2 \in Eq(T;\phi)$, so that $\int \varphi' d\nu_1 = a_1$ and $\int \varphi' d\nu_2 = a_2$. This contradicts $\sharp Eq(T;\phi) = 1$.

For "2) $\Rightarrow$ 3)", let ν be arbitrary equilibrium state of ϕ. Viewing it as tangential functional, we have

$$P(T;\phi + t\varphi) - P(T;\phi) \geq \int t\varphi d\nu = t\int \varphi d\nu, \quad \forall \varphi \in C^0(X). \tag{2.7}$$

Dividing (2.7) by $t > 0$, respectively $t < 0$, and taking limit as $t \to 0^+$, respectively 0^-, we obtain

$$\tau(T;\phi,\varphi) \geq \int \varphi d\nu \geq -\tau(T;\phi,-\varphi), \quad \forall \varphi \in C^0(X). \tag{2.8}$$

Then if $\tau(T;\phi,\varphi) = -\tau(T;\phi,-\varphi)$ for any $\varphi \in C^0(X)$, (2.8) yields

$$\int \varphi d\nu = \tau(T;\phi,\varphi), \quad \forall \varphi \in C^0(X).$$

"3) $\Rightarrow$ 1)" is trivial. □

Corollary 2.5. *Assume $h.(T) : \mathcal{M}(X;T) \to \mathbb{R}$ is upper semicontinuous. Denote by $\mathcal{R} \subset C^0(X)$ the set of potential functions that have unique equilibrium state. Then $\mathcal{R}$ is a G_δ set in $C^0(X)$.*

Proof. Let $\{\varphi_i\}_i$ be a countable and dense subset of $C^0(X)$. By Lemma 2.4 and 2) of Lemma 2.1, $\mathcal{R}$ can be represented as

$$\mathcal{R} = \bigcap_i \{\phi \in C^0(X) \mid \tau(T;\phi,\varphi_i) = -\tau(T;\phi,-\varphi_i)\}.$$

Since $\tau(T;\phi,\varphi_i) \geq -\tau(T;\phi,-\varphi_i)$,

$$\begin{aligned}
&\{\phi \in C^0(X) \mid \tau(T;\phi,\varphi_i) = -\tau(T;\phi,-\varphi_i)\} \\
&\quad = \bigcap_{\varepsilon>0} \{\phi \in C^0(X) \mid \tau(T;\phi,\varphi_i) + \tau(T;\phi,-\varphi_i) < \varepsilon\} \\
&\quad = \bigcap_{\varepsilon>0} \{\phi \in C^0(X) : \inf_{t>0} \frac{P(T;\phi+t\varphi_i) + P(T;\phi-t\varphi_i) - 2P(T;\phi)}{t} < \varepsilon\} \\
&\quad = \bigcap_{\varepsilon>0} \bigcup_{t>0} \{\phi \in C^0(X) : P(T;\phi+t\varphi_i) + P(T;\phi-t\varphi_i) - 2P(T;\phi) < t\varepsilon\}.
\end{aligned}$$

This implies that $\mathcal{R}$ is G_δ. □

Remark 2.6. In fact, one may go further to prove that $\mathcal{R}$ is a dense G_δ set in $C^0(X)$, see Corollary 9.15.1 of [9].

Under the condition of Corollary 2.5, for any $\phi \in \mathcal{R}$ we denote by μ_ϕ the unique equilibrium state of ϕ.

Corollary 2.7. *μ_ϕ depends continuously in weak*-topology on $\phi \in \mathcal{R}$.*

Proof. By 3) of Lemma 2.4, we have $\int \varphi d\mu_\phi = \tau(T;\phi,\varphi)$ for any $\varphi \in C^0(X)$. Thus it is sufficient to prove for any $\varphi \in C^0(X)$, $\tau(T;\cdot,\varphi)$ is continuous at ϕ, and this is derived from 2) of Lemma 2.2.

Denote by $d : X \times X \to \mathbb{R}$ the distance function of X. Call $E \subset X$, (n, ε) *separated*, if whenever x, y are two distinct points in E, one can find $0 \leq i \leq n-1$ with $d(T^i x, T^i y) > \varepsilon$.

Lemma 2.8. *Given $\varepsilon > 0$ and $\psi \in C^0(X)$, for each $n \in \mathbb{N}$, let $E_n \subset X$ be an (n, ε) separated set, and $\mu_n \in \mathcal{M}(X)$ be defined by*:

$$\mu_n = \sum_{x \in E_n} \frac{e^{S_n\psi(x)}}{\sum_{x\in E_n} e^{S_n\psi(x)}} \cdot \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x},$$

where $S_n\psi = \sum_{i=0}^{n-1} \psi \circ T^i$. Assume $\mu_{n_i} \to \mu$ in weak-topology, then $\mu \in \mathcal{M}(X;T)$ and*

$$h_\mu(T) + \int_X \psi d\mu \geq \limsup_{i\to\infty} \frac{1}{n_i} \log \sum_{x\in E_{n_i}} e^{S_{n_i}\psi(x)}.$$

Proof. See part (2) of proof of Theorem 9.10 in [9]. □

3. Generic Properties of $P(f|\Lambda_f;\psi_f)$ and $Eq(f|\Lambda_f;\psi_f)$ for $f \in \mathcal{U}$

For any $f \in \mathcal{U}$, we define $\psi_f \in C^0(\Lambda_f)$ by

$$\psi_f(x) = -\log|\det(Df|E^u_x)|, \quad x \in \Lambda_f.$$

With preparations in the previous section, we are going to study $P(f|\Lambda_f;\psi_f)$ and $Eq(f|\Lambda_f;\psi_f)$ for generic $f \in \mathcal{U}$. Indeed, since $f|\Lambda_f$ is expansive, the entropy map

$$h_\cdot(f|\Lambda_f) : \mathcal{M}(\Lambda_f; f|\Lambda_f) \to \mathbb{R}$$

is upper semicontinuous, thus $h(f|\Lambda_f) < \infty$ (see Theorem 8.2 of [9]). Then all the results presented in the previous section hold in this situation.

Recall that by classical SRB theory, if $f \in \mathcal{U} \cap \mathrm{Diff}^2(M)$, we have

$$P(f|\Lambda_f;\psi_f) = 0, \quad \sharp Eq(f|\Lambda_f;\psi_f) = 1. \tag{3.9}$$

Indeed, this is another presentation of the Ruelle-Pesin formula (see [4]). The next proposition says that this property holds for "most" $f \in \mathcal{U}$.

Proposition 3.1. 1. *For any $f \in \mathcal{U}$, $P(f|\Lambda_f;\psi_f) = 0$.*
2. *There exists a generic subset $\mathcal{U}' \subset \mathcal{U}$, so that for any $f \in \mathcal{U}'$, $\sharp Eq(f|\Lambda_f;\psi_f) = 1$.*

Proof. We introduce a continuous map $\Phi : \mathcal{U} \to C^0(\Lambda_{f_0})$ defined by $\Phi(f) = \psi_f \circ r_f$. By invariance of topological pressure under conjugation, we have

$$P(f|\Lambda_f;\psi_f) = P(f_0|\Lambda_{f_0};\Phi(f)), \quad Eq(f|\Lambda_f;\psi_f) = r_{f*}Eq(f_0|\Lambda_{f_0};\Phi(f)). \tag{3.10}$$

For the first statement, let f be an arbitrary diffeomorphism in $\mathcal{U}$, and $\{f_k\}_k$ be C^2 diffeomorphisms so that $f_k \overset{C^1}{\to} f$. Therefore

$$\begin{aligned} P(f|\Lambda_f;\psi_f) &= P(f_0|\Lambda_{f_0};\Phi(f)) = \lim_{k\to\infty} P(f_0|\Lambda_{f_0};\Phi(f_k)) \\ &= \lim_{k\to\infty} P(f_k|\Lambda_{f_k};\psi_{f_k}) \overset{(3.9)}{=} 0. \end{aligned}$$

For the second statement, abusing the notations in Corollary 2.5, we denote by $\mathcal{R}$ the set of potentials in $C^0(\Lambda_{f_0})$ that have unique equilibrium state w.r.t. $f_0|\Lambda_{f_0}$. Let $\mathcal{U}' = \Phi^{-1}(\mathcal{R})$. Clearly, for any $f' \in \mathcal{U}'$, $\sharp Eq(f'|\Lambda_{f'}, \psi_{f'}) = 1$.

By Corollary 2.5, $\mathcal{R}$ is a G_δ set, thus $\mathcal{U}'$ is a G_δ set in $\mathcal{U}$. Moreover, by (3.9), $\mathcal{U} \cap \mathrm{Diff}^2(M) \subset \mathcal{U}'$. This implies that $\mathcal{U}'$ is dense in $\mathcal{U}$. □

In the sequel, for any $f \in \mathcal{U}'$, we denote by μ_f the unique equilibrium state for ψ_f w.r.t. $f|\Lambda_f$. Derived directly from (3.10) and Corollary 2.7, we have:

Corollary 3.2. *μ_f depends continuously in weak*-topology on $f \in \mathcal{U}'$.* □

Corollary 3.3. *μ_f is ergodic (w.r.t f).*

Proof. Let $\mu_f = \int_{\mathcal{E}(\Lambda_f; f|\Lambda_f)} \mu d\eta(\mu)$ be the ergodic decomposition of μ_f, where $\eta \in \mathcal{M}(\mathcal{M}(\Lambda_f; f|\Lambda_f))$ with $\eta(\mathcal{E}(\Lambda_f; f|\Lambda_f)) = 1$. Therefore by Theorem 8.4 of [9],

$$0 = h_{\mu_f}(f|\Lambda_f) + \int_{\Lambda_f} \psi_f d\mu_f = \int_{\mathcal{E}(\Lambda_f;f|\Lambda_f)} \{h_\mu(f|\Lambda_f) + \int_{\Lambda_f} \psi_f d\mu\} d\eta(\mu). \tag{3.11}$$

By (3.9), $h_\mu(f|\Lambda_f) + \int_{\Lambda_f} \psi_f d\mu \le 0$, and "=" holds if and only if $\mu = \mu_f$. Then (3.11) implies that $\mu = \mu_f$ for η a.e. μ. Thus μ_f is ergodic. □

4. Volume Estimate of $B(\mu_f) \cap \Omega$ for $f \in \mathcal{U}'$

Now we carry on to compute, for any fixed $f \in \mathcal{U}'$, the volume of $B(\mu_f) \cap \Omega$. Our aim is to derive estimate

$$m(B(\mu_f) \cap \Omega) = m(\Omega) \tag{4.12}$$

through the thermodynamical properties

$$P(f|\Lambda_f; \psi_f) = 0, \quad Eq(f|\Lambda_f; \psi_f) = \{\mu_f\}. \tag{4.13}$$

Recall that if we consider a local unstable manifold Δ, by Bowen's standard technique developed in [1], one can obtain the following estimate:

$$m_\Delta(B(\mu_f) \cap \Delta) = m_\Delta(\Delta) \tag{4.14}$$

from (4.13). Then, when f is of C^2 class, by an absolutely continuous holonomy map derived by stable foliation of Λ_f, one can transfer (4.14) to every u-dimensional C^1 compact submanifold that is transverse to stable foliation (in the sequel, we call them *u-transversal C^1 compact submanifold* or *u-TCSM* in abbreviation). Observe that Ω can be foliated by a smooth family of u-TCSM's. Thus applying Fubini's Theorem, one can integrate (4.14) over this family to obtain estimate (4.12).

However, for $f \in \mathrm{Diff}^1(M)$, the above holonomy map is, in general, not absolutely continuous (see [6]). Our strategy in this situation is to generalize Bowen's technique for every u-TCSM in Ω to obtain (4.14). More specifically, we will prove:

Lemma 4.1. *Let $\Delta \subset \Omega$ be a u-TCSM. Then $m_\Delta(B(\mu_f) \cap \Delta) = m_\Delta(\Delta)$.*

As an immediate consequence of Lemma 4.1 and Fubini's Theorem, we have:

Proposition 4.2. $m(B(\mu_f) \cap \Omega) = m(\Omega)$. □

Then Proposition 4.2, Proposition 3.1, Corollary 3.2 and Corollary 3.3 jointly accomplish the proof of Theorem A.

Now we only need to prove Lemma 4.1. To illustrate the argument in a simple case, we first prove the lemma for those Δ's so that:

$$\text{case *) for any } i \in \mathbb{N},\ f^i\Delta \cap \Delta = \emptyset \text{ and } f^i\Delta \cap \Lambda_f = \emptyset.$$

The proof of the general case is very similar.

Before the formal argument, we need the following preparative lemma:

Lemma 4.3. *Let $\Delta \subset \Omega$ be a u-TCSM. Then,*

1. *Given $C_1 > 0$, there exist $\delta_1 > 0$, so that for any $i, j \in \mathbb{N}$ and any compact disk $D \subset f^i\Delta$,*

$$diam(D) \le \delta_1 \Longrightarrow m_{f^i\Delta}(D) \le C_1.$$

2. *Given $C_2 > 1$, there exist $\delta_2 > 0$, so that for any $i \in \mathbb{N}$ and any $x \in f^i\Delta$, $y \in f^j\Delta$,*

$$d(x, y) \le \delta_2 \Longrightarrow C_2^{-1} \le |\det(Df|T_x f^i\Delta)| \cdot |\det(Df|T_y f^j\Delta)|^{-1} \le C_2,$$

where $d(\cdot, \cdot)$ is the distance function of M.

Proof. The detail of the proof is omitted. The key observation is that, due to λ-lemma (see p. 82 of [5]), $f^i\Delta$ "C^1-converges" to Λ_f as $i \to \infty$. Thus we can apply the argument of compactness over $\Lambda_f \cup \bigcup_{i\ge0} f^i\Delta$. □

*4.1. Proof of Lemma 4.1 for Case *).* For Δ of case *), we consider the positive invariant set $\Xi = \Lambda_f \cup \bigcup_{i\geq 0} f^i\Delta$ and potential $\psi \in C^0(\Xi)$ defined by

$$\psi(x) = \begin{cases} \psi_f(x), & \text{if } x \in \Lambda_f; \\ -\log|\det(Df|T_x f^i\Delta)|, & \text{if } x \in f^i\Delta, i = 0, 1, 2, \ldots. \end{cases}$$

By definition of case *), one sees that ψ is well defined.

By λ-lemma, Ξ is a compact set, thus $(\Xi, f|\Xi)$ is a topological dynamical system. Furthermore, $\bigcap_{i\geq 0} f^i\Xi = \Lambda_f$, which implies that any invariant measure on Ξ must be supported on Λ_f. Then the thermodynamical properties for $f|\Lambda_f$ with potential ψ_f can be handed to $f|\Xi$ with potential ψ. Therefore, by (4.13) and upper-semicontinuity of $h_\cdot(f|\Lambda_f) : \mathcal{M}(\Lambda_f, f|\Lambda_f) \to \mathbb{R}$, we have

$$\begin{cases} 1)\ P(f|\Xi;\psi) = 0, \quad Eq(f|\Xi;\psi) = \{\mu_f\}, \\ 2)\ h_\cdot(f|\Xi) : \mathcal{M}(\Xi, f|\Xi) \to \mathbb{R} \text{ is upper semicontinuous.} \end{cases} \tag{4.15}$$

For any $r > 0$, let $\mathcal{K}_r \subset \mathcal{M}(\Xi; f|\Xi)$ be defined by

$$\{\nu \in \mathcal{M}(\Xi; f|\Xi) : h_\nu(f|\Xi) + \int_\Xi \psi d\nu \geq -r\}.$$

Then by 1) of (4.15), $\bigcap_{r>0} \mathcal{K}_r = \{\mu_f\}$. Furthermore, by 2) of (4.15), $\mathcal{K}_r$ is closed in $\mathcal{M}(\Xi; f|\Xi)$, thus closed in $\mathcal{M}(\Xi)$. This implies $\mathcal{M}(\Xi)\backslash\mathcal{K}_r$ is open in $\mathcal{M}(\Xi)$. Therefore by local compactness and local convexity of $\mathcal{M}(\Xi)$, $\mathcal{M}(\Xi)\backslash\mathcal{K}_r$ can be covered by a countable family of open sets $\{\mathcal{V}_i\}_i$ in $\mathcal{M}(\Xi)$, so that each $\mathcal{V}_i$ is convex, and the closure of $\mathcal{V}_i$ is contained in $\mathcal{M}(\Xi)\backslash\mathcal{K}_r$.

For any $\mathcal{W} \subset \mathcal{M}(\Xi)$, let $\Delta(\mathcal{W}, n)$ and $\Delta(\mathcal{W})$ be defined by

$$\Delta(\mathcal{W}, n) = \{x \in \Delta : \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x} \in \mathcal{W}\},$$

$$\Delta(\mathcal{W}) = \{x \in \Delta : \lim_{i\to\infty} \frac{1}{n_i}\sum_{i=0}^{n_i-1} \delta_{f^{n_i} x} \in \mathcal{W}, \text{for some}\{n_i\}_i\}.$$

It is easy to see that $\Delta(\mathcal{W}) \subset \bigcap_{n\geq 0}\bigcup_{i\geq n} \Delta(\mathcal{W}, i)$ whenever $\mathcal{W}$ is open.

Claim. *For any $\mathcal{V} \in \{\mathcal{V}_i\}_i$, $m_\Delta(\Delta(\mathcal{V})) = 0$.*

Proof. We choose arbitrary $C_1 > 0, C_2 > 1$, and determine $\delta_1 = \varepsilon_1(C_1, \Delta), \delta_2 = \varepsilon_2(C_2, \Delta)$ by Lemma 4.3. Let $\delta = \min\{\delta_1, \delta_2\}$. Moreover, choose $0 < \varepsilon < \delta$ so that for any $x, y \in M$, $d(fx, fy) < \delta$ whenever $d(x, y) < \varepsilon$. For each $n \in \mathbb{N}$, select E_n an (n, ε) separated set that is maximal in $\Delta(\mathcal{V}, n)$. For each $x \in E_n$, let $B_{n,\varepsilon}(x) = \{y \in \Delta : d(f^i x, f^i y) \leq \varepsilon, 0 \leq i \leq n-1\}$. Due to maximality, $\Delta(\mathcal{V}, n) \subset \bigcup_{x\in E_n} B_{n,\varepsilon}(x)$. Then

$$\begin{aligned} m_\Delta(\Delta(\mathcal{V}, n)) &\leq \sum_{x\in E_n} m_\Delta(B_{n,\varepsilon}(x)) = \sum_{x\in E_n}\int_{B_{n,\varepsilon}(x)} dm_\Delta(y) \\ &= \sum_{x\in E_n}\int_{f^n(B_{n,\varepsilon}(x))} \prod_{i=0}^{n-1} |\det(Df|T_{f^{-n+i}y'} f^i\Delta)|^{-1} dm_{f^n\Delta}(y') \end{aligned}$$

$$\leq C_2^n \sum_{x\in E_n} e^{S_n\psi(x)} m_{f^n\Delta}(f^n(B_{n,\varepsilon}(x)))$$
$$\leq C_1 C_2^n \sum_{x\in E_n} e^{S_n\psi(x)}, \tag{4.16}$$

where $S_n\psi = \sum_{i=0}^{n-1} \psi \circ f^i$.

Now we apply Lemma 2.8 to $(\Xi, f|\Xi)$, ψ and $\{E_n\}_n$. For each $n \in \mathbb{N}$, let

$$\nu_n = \sum_{x\in E_n} \frac{e^{S_n\psi(x)}}{\sum_{x\in E_n} e^{S_n\psi(x)}} \cdot \frac{1}{n}\sum_{i=0}^{n} \delta_{f^i x},$$

and $\{\nu_{n_i}\}_i$ be a subsequence converging to some ν in weak*-sense. Then Lemma 2.8 gives

$$\limsup_{i\to\infty} \frac{1}{n_i} \log \sum_{x\in E_{n_i}} e^{S_{n_i}\psi(x)} \leq h_\nu(f|\Xi) + \int_\Xi \psi d\nu. \tag{4.17}$$

Observe that ν_n is a convex combination of $\{\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x}, x \in E_n\}$, and $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x} \in \mathcal{V}$ for any $x \in E_n$. By convexity of $\mathcal{V}$ we have $\nu_n \in \mathcal{V}$, thus $\nu = \lim_{i\to\infty}\nu_{n_i} \in \overline{\mathcal{V}} \subset \mathcal{M}(\Xi)\backslash\mathcal{K}_r$. Then by definition of $\mathcal{K}_r$, $h_\nu(f|\Xi) + \int_\Xi \psi d\nu < -r$. Therefore

$$\limsup_{i\to\infty} \frac{1}{n_i} \log \sum_{x\in E_{n_i}} e^{S_{n_i}\psi(x)} < -r. \tag{4.18}$$

Clearly, (4.18) holds for any $\{n_i\}_i$ such that ν_{n_i} converges. Substituting n_i in (4.18) by n,

$$\limsup_{n\to\infty} \frac{1}{n} \log m_\Delta(\Delta(\mathcal{V}, n)) < -r. \tag{4.19}$$

Combine (4.19) with (4.16),

$$\limsup_{n\to\infty} \frac{1}{n} \log m_\Delta(\Delta(\mathcal{V}, n)) \leq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E_n} e^{S_n\psi(x)} + \log C_2$$
$$< -r + \log C_2. \tag{4.20}$$

Let $C_2 \to 1$, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log m_\Delta(\Delta(\mathcal{V}, n)) \leq -r. \tag{4.21}$$

Then, given $0 < \sigma < r$, there exist $C > 1$, so that

$$m_\Delta(\Delta(\mathcal{V}, n)) \leq Ce^{-(r-\sigma)n}. \tag{4.22}$$

Note that $\Delta(\mathcal{V}) \subset \bigcap_{n\geq 0}\bigcup_{i\geq n} \Delta(\mathcal{V}, i)$ because $\mathcal{V}$ is open,

$$m_\Delta(\Delta(\mathcal{V})) \leq \limsup_{n\to\infty} \sum_{i\geq n} m_\Delta(\Delta(\mathcal{V}, i)) \leq \limsup_{n\to\infty} \sum_{i\geq n} Ce^{-(r-\sigma)i} = 0. \tag{4.23}$$

This ends the proof of the claim. □

As a consequence of the claim,

$$m_\Delta(\Delta(\mathcal{M}(\Xi)\backslash\mathcal{K}_r)) \le \sum_i m_\Delta(\Delta(\mathcal{V}_i)) = 0,$$

then by $\bigcap_{r>0}\mathcal{K}_r = \{\mu_f\}$,

$$m_\Delta(\Delta(\mathcal{M}(\Xi)\backslash\{\mu_f\})) = \lim_{r\to 0} m_\Delta(\Delta(\mathcal{M}(\Xi)\backslash\mathcal{K}_r)) = 0.$$

Clearly, we have $\Delta = \Delta(\mathcal{M}(\Xi)\backslash\{\mu_f\})\cup(B(\mu_f)\cap\Delta)$ and $\Delta(\mathcal{M}(\Xi)\backslash\{\mu_f\})\cap(B(\mu_f)\cap\Delta) = \emptyset$. Thus

$$m_\Delta(B(\mu_f)\cap\Delta) = m_\Delta(\Delta) - m_\Delta(\Delta(\mathcal{M}(\Xi)\backslash\{\mu_f\})) = m_\Delta(\Delta). \tag{4.24}$$

This completes the proof of Lemma 4.1 for case *). □

4.2. Proof of Lemma 4.1. Now we are going to apply the above argument in the general case. Note that the crucial point in the previous proof is that we "naturally" extend ψ_f to ψ, in the sense that $\psi|\Delta$ is "compatible" with volume measure on Δ. However, without the assumption in case *), such an extension may be unrealizable. For example, assume there exists $x \in \Delta\cap\Lambda_f$ so that $T_x\Delta \neq E^u_x$, $\psi(x)$ should equal $-\log|\det(Df|E^u_x)|$ if x is referred to a point in Λ_f, while $\psi(x)$ must be $-\log|\det(Df|T_x\Delta)|$ if x is considered contained in Δ, and $|\det(Df|E^u_x)| \neq |\det(Df|T_x\Delta)|$ in general. Similar problem happens when there exists $y \in \Delta\cap f^i\Delta$ with $T_y\Delta \neq T_y f^i\Delta$.

To overcome this problem, we introduce the framework of the Grassmann bundle, in which the previously mentioned $T_x\Delta$ and E^u_x (respectively, $T_y\Delta$ and $T_y f^i\Delta$) are forced apart. In precise words, let $\pi : \mathcal{G}(M,u) \to M$ be the u-dimensional Grassmann bundle over M. For any $V \subset TM$ a u-dimensional linear subspace, we write $[V]$ to denote the corresponding element in $\mathcal{G}(M,u)$. The topology of $\mathcal{G}(M,u)$ is determined by the distance function

$$\hat{d}([V],[V']) = \min\{l(\gamma) + \angle_{\pi([V])}(V, P_\gamma V')|\gamma : [0,1] \to M \text{ is piecewise smooth with } \gamma(0) = \pi([V']), \gamma(1) = \pi([V])\},$$

where $l(\gamma)$ is the length of γ, P_γ the parallel translation along γ, and

$$\angle_{\pi([V])}(V, P_\gamma V') \triangleq \sup\{\|v - v'\| \mid v \in V, v' \in P_\gamma V', \|v\| = \|v'\| = 1\}.$$

Under this topology $\pi : \mathcal{G}(M,u) \to M$ is a continuous map.

Let $\hat{f} : \mathcal{G}(M,u) \to \mathcal{G}(M,u)$ be a homeomorphism defined by

$$\hat{f}[V] = [Df(V)].$$

Then $f\circ\pi = \pi\circ\hat{f}$. Let potential $\hat{\psi} \in C^0(\mathcal{G}(M,u))$ be defined by

$$\hat{\psi}([V]) = -\log|\det(Df|V)|.$$

Proof of Lemma 4.1. Still as in case *), we consider $\Xi = \Lambda_f \cup \bigcup_{i\ge 0} f^i\Delta$. Moreover, define the following sets of $\mathcal{G}(M,u)$ that are related to Ξ:

$$\hat{\Lambda}_f = \bigcup_{x\in\Lambda_f}[E^u_x], \quad \hat{\Delta} = \bigcup_{x\in\Delta}[T_x\Delta], \quad \hat{\Xi} = \hat{\Lambda}_f \cup \bigcup_{i\ge 0}\hat{f}^i\hat{\Delta}.$$

Clearly, π maps $\hat{\Lambda}_f$, $\hat{\Delta}$ and $\hat{\Xi}$ respectively onto Λ_f, Δ and Ξ. In particular, $\pi|\hat{\Lambda}_f : \hat{\Lambda}_f \to \Lambda_f$ is a homeomorphism. Then by upper semicontinuity of $h.(f|\Lambda_f)$, $h.(\hat{f}|\hat{\Lambda}_f)$ is upper semicontinuous. Moreover, since $(\hat{\psi} \circ \pi)|\Lambda_f = \psi_f$, by (4.13) and invariance of topological pressure,

$$P(\hat{f}|\hat{\Lambda}_f; \hat{\psi}) = 0, \quad Eq(\hat{f}|\hat{\Lambda}_f; \hat{\psi}) = \{\hat{\mu}_f\}, \tag{4.25}$$

where $\hat{\mu}_f \triangleq (\pi|\hat{\Lambda}_f)_*^{-1}\mu_f$.

By λ-lemma, $\hat{\Xi}$ is a compact set, thus $(\hat{\Xi}, \hat{f}|\hat{\Xi})$ is a topological dynamical system. Furthermore, $\bigcap_{i\geq 0} \hat{f}^i \hat{\Xi} = \hat{\Lambda}_f$. Then for a similar reason mentioned before (4.15), we have

$$\begin{cases} 1)\ P(\hat{f}|\hat{\Xi}; \hat{\psi}) = 0, \quad Eq(\hat{f}|\hat{\Xi}; \hat{\psi}) = \{\hat{\mu}_f\}, \\ 2)\ h.(\hat{f}|\hat{\Xi}) : \mathcal{M}(\hat{\Xi}; \hat{f}|\hat{\Xi}) \to \mathbb{R} \text{ is upper semicontinuous.} \end{cases} \tag{4.26}$$

For any $r > 0$, let $\hat{\mathcal{K}}_r \subset \mathcal{M}(\hat{\Xi}; \hat{f}|\hat{\Xi})$ be defined by

$$\{\hat{\nu} \in \mathcal{M}(\hat{\Xi}; \hat{f}|\hat{\Xi}) : h_{\hat{\nu}}(\hat{f}|\hat{\Xi}) + \int_{\hat{\Xi}} \hat{\psi} d\hat{\nu} \geq -r\}.$$

Then by 1) of (4.26), $\bigcap_{r>0} \hat{\mathcal{K}}_r = \{\hat{\mu}_f\}$. Furthermore, by 2) of (4.26) $\hat{\mathcal{K}}_r$ is closed in $\mathcal{M}(\hat{\Xi}; \hat{f}|\hat{\Xi})$, thus closed in $\mathcal{M}(\hat{\Xi})$. This implies $\mathcal{M}(\hat{\Xi})\backslash\hat{\mathcal{K}}_r$ is open in $\mathcal{M}(\hat{\Xi})$. Therefore by local compactness and local convexity of $\mathcal{M}(\hat{\Xi})$, $\mathcal{M}(\hat{\Xi})\backslash\hat{\mathcal{K}}_r$ can be covered by a countable family of open sets $\{\hat{\mathcal{V}}_i\}_i$, so that each $\hat{\mathcal{V}}_i$ is convex, and the closure of $\hat{\mathcal{V}}_i$ is contained in $\mathcal{M}(\hat{\Xi})\backslash\hat{\mathcal{K}}_r$.

In the sequel, for any $x \in \Delta$, we write $\hat{x}$ to represent $[T_x\Delta]$ for simplicity. For any $\hat{\mathcal{W}} \subset \mathcal{M}(\hat{\Xi})$, let $\hat{\Delta}(\hat{\mathcal{W}}, n)$ and $\hat{\Delta}(\hat{\mathcal{W}})$ be defined by

$$\hat{\Delta}(\hat{\mathcal{W}}, n) = \{\hat{x} \in \hat{\Delta} : \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\hat{f}^i\hat{x}} \in \hat{\mathcal{W}}\},$$

$$\hat{\Delta}(\hat{\mathcal{W}}) = \{\hat{x} \in \hat{\Delta} : \lim_{i\to\infty} \frac{1}{n_i}\sum_{i=0}^{n_i-1} \delta_{\hat{f}^{n_i}\hat{x}} \in \hat{\mathcal{W}}, \text{ for some } \{n_i\}_i\}.$$

Claim. *For any* $\hat{\mathcal{V}} \in \{\hat{\mathcal{V}}_i\}_i$, $m_\Delta(\pi(\hat{\Delta}(\hat{\mathcal{V}}))) = 0$.

Proof. Again, we choose arbitrary $C_1 > 0$, $C_2 > 1$, determine $\delta_1 = \varepsilon_1(C_1, \Delta)$, $\delta_2 = \delta_2(C_2, \Delta)$ and $\delta = \min\{\delta_1, \delta_2\}$ by Lemma 4.3, and choose $0 < \varepsilon < \delta$ as in case *). For each $n \in \mathbb{N}$, select E_n an (n, ε) separated set (w.r.t. f) that is maximal in $\pi(\hat{\Delta}(\hat{\mathcal{V}}, n))$. We write $\hat{E}_n = \{\hat{x} : x \in E_n\}$, then $\hat{E}_n$ is (n, ε) separated (w.r.t. $\hat{f}$). For each $x \in E_n$, let $B_{n,\varepsilon}(x) = \{y \in \Delta : d(f^i x, f^i y) \leq \varepsilon, 0 \leq i \leq n-1\}$. Due to maximality, $\pi(\hat{\Delta}(\hat{\mathcal{V}}, n)) \subset \bigcup_{x\in E_n} B_{n,\varepsilon}(x)$. Then similar to (4.16),

$$m_\Delta(\pi(\hat{\Delta}(\hat{\mathcal{V}}, n))) \leq \sum_{x\in E_n} m_\Delta(B_{n,\varepsilon}(x)) \leq C_1 C_2^n \sum_{\hat{x}\in\hat{E}_n} e^{S_n\hat{\psi}(\hat{x})}, \tag{4.27}$$

where $S_n\hat{\psi} = \sum_{i=0}^{n-1} \hat{\psi} \circ \hat{f}^i$.

Now we apply Lemma 2.8 to $(\hat{\Xi}, \hat{f}|\hat{\Xi})$, $\hat{\psi}$ and $\hat{E}_n$. With same argument as in between (4.17) and (4.21), we have

$$\limsup_{n\to\infty} \frac{1}{n} \log m_\Delta(\pi(\hat{\Delta}(\hat{\mathcal{V}}, n))) \leq -r, \tag{4.28}$$

which implies, as in (4.23), that

$$m_\Delta(\pi(\hat{\Delta}(\hat{\mathcal{V}}))) = 0. \tag{4.29}$$

This ends the proof of claim. □

As a consequence of the claim,

$$m_\Delta(\pi(\hat{\Delta}(\mathcal{M}(\hat{\Xi})\backslash\hat{\mathcal{K}}_r))) \leq \sum_i m_\Delta(\pi(\hat{\Delta}(\hat{\mathcal{V}}_i))) = 0,$$

then by $\bigcap_{r>0} \hat{\mathcal{K}}_r = \{\hat{\mu}_f\}$,

$$m_\Delta(\pi(\hat{\Delta}(\mathcal{M}(\hat{\Xi})\backslash\{\hat{\mu}_f\}))) = \lim_{r\to 0} m_\Delta(\pi(\hat{\Delta}(\mathcal{M}(\hat{\Xi})\backslash\hat{\mathcal{K}}_r))) = 0.$$

Moreover, it is easy to check that

$$\pi(\hat{\Delta}(\mathcal{M}(\hat{\Xi})\backslash\{\hat{\mu}_f\})) = \Delta(\mathcal{M}(\Xi)\backslash\{\mu_f\}).$$

Then similar to (4.24),

$$\begin{aligned} m_\Delta(B(\mu_f)\cap\Delta) &= m_\Delta(\Delta) - m_\Delta(\Delta(\mathcal{M}(\Xi)\backslash\{\mu_f\}) \\ &= m_\Delta(\Delta) - m_\Delta(\pi(\hat{\Delta}(\mathcal{M}(\hat{\Xi})\backslash\{\hat{\mu}_f\}))) = m_\Delta(\Delta). \end{aligned} \tag{4.30}$$

This completes the proof of Lemma 4.1. □

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