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On the Global Existence of Mild Solutions to the Boltzmann Equation for Small Data in L^D

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Abstract: We develop a new theory of existence of global solutions to the Boltzmann equation for small initial data. These new mild solutions are analogous to the mild solutions for the Navier-Stokes equations. The existence comes as a result of the study of the competing phenomena of dispersion, due to the transport operator, and of singularity formation, due to the nonlinear Boltzmann collision operator. It is the joint use of the so-called dispersive estimates with new convolution inequalities on the gain term of the collision operator that allows to obtain uniform bounds on the solutions and thus demonstrate the existence of solutions.

1. Mild Solutions

The Boltzmann equation is a model for gases in a low density regime. That is, a many particles system where we assume $Nr^2 \approx 1$ as $N \to \infty$ and $r \to 0$, N being the number of particles and r their diameter. Denoting by $F(t, x, v) \geq 0$ the density function describing the microscopic state of the gas at time $t \in [0, \infty)$, position $x \in \Omega \subset \mathbb{R}^D$ and velocity $v \in \mathbb{R}^D$, where $D \geq 2$ is the dimension, then the initial value problem for Boltzmann's equation reads as follows:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{Q}(F, F), \qquad F(0, x, v) = F^{\text{in}}(x, v), \tag{1.1}$$

for some initial density $F^{\text{in}} \geq 0$. The Boltzmann operator $\mathcal{Q}(F, F)$ is a quadratic integral operator and accounts for the variations of the density F due to the interparticle elastic collisions in the gas. It is defined by

$$\mathcal{Q}(F, G)(v) = \int_{\mathbb{R}^D} \int_{\mathbb{S}^{D-1}} \left(F' G'_* - F G_*\right) b(v - v_*, \sigma)\, d\sigma dv_*, \tag{1.2}$$

where the unit sphere $\mathbb{S}^{D-1} \subset \mathbb{R}^D$ is endowed with its standard surface measure and we have denoted $F = F(v)$, $F' = F(v')$, $G_* = G(v_*)$ and $G'_* = G(v'_*)$, and

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma. \tag{1.3}$$

It is natural to split $\mathcal{Q}(F,G)$ into its gain part $\mathcal{Q}^+(F,G)(v) = \int F'G'_* b\, d\sigma dv_*$ and its loss part $\mathcal{Q}^-(F,G)(v) = \int FG_* b\, d\sigma dv_*$.

One can show that the quadruple of pre-collisional and post-collisional velocities (v, v_*, v', v'_*) parametrized by $\sigma \in \mathbb{S}^{D-1}$ provides the family of all solutions to the system of $D+1$ equations

$$\begin{aligned} v + v_* &= v' + v'_*, \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2, \end{aligned} \tag{1.4}$$

which, at the kinetic level, expresses the fact that interparticle collisions are assumed elastic and thus conserve momentum and energy. The so-called collision kernel or collisional cross section $b(z,\sigma) \in L^1_{\mathrm{loc}}\left(\mathbb{R}^D \times \mathbb{S}^{D-1}\right)$ is a nonnegative measurable function that is determined by the molecular forces that are being considered in the gas. In fact, by the Galilean invariance of collisions, the kernel only depends on the magnitude of the relative velocity $|v - v_*|$ and the deflection angle θ defined by $\cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma$, so that we will often write $b(v - v_*, \sigma) = b(|v - v_*|, \cos\theta)$ interchangeably, by abuse of language. Moreover, noticing that the change of variable $\sigma \mapsto -\sigma$ only exchanges v' with v'_*, it is always possible to replace the collision kernel $b(z,\sigma)$ in the Boltzmann collision operator $\mathcal{Q}(F,F)$ by its symmetrized version $[b(z,\sigma) + b(z,-\sigma)]\,\mathbb{1}_{\{\cos\theta \geq 0\}}$ without changing its nature. Therefore, without any loss of generality, we will always assume that the cross section is supported on $\{\cos\theta \geq 0\}$.

The Boltzmann equation is a cornerstone of collisional kinetic theory and its Cauchy problem has thus always attracted much interest. However, due to its nonlinear nature, constructing solutions to (1.1) remains a challenging problem, even though it has already been tackled with numerous strategies. Most notably, the DiPerna-Lions theory of renormalized solutions [11] provides the existence, globally in time, of solutions to (1.1) based solely on the physical a priori estimates and for large initial data. Unfortunately, the uniqueness of renormalized solutions is unknown. On the other hand, several local well-posedness results are available, of which we will only mention the weak solutions of Illner, Kaniel and Shinbrot [17,18] and the strong solutions of Guo [13,14].

In this work, we establish the existence (without uniqueness) for small initial data of a new class of weak solutions to (1.1), which we call mild solutions and are analogous to the mild solutions for the incompressible Navier-Stokes equations developed by Kato [19].

1.1. Strategy. Throughout this work, we will assume that the spatial domain Ω is in fact the entire space $\mathbb{R}^D$, which will be crucial in order to allow for the mechanisms of transfer of integrability to act on the solutions as explained later on in Sect. 3.

Considering any $f(t,x,v), g(t,x,v) \in L^1_{\mathrm{loc}}\left(\mathbb{R} \times \mathbb{R}^D \times \mathbb{R}^D\right)$, it is well-known from the linear transport theory that

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = g(t,x,v), \qquad \text{in the sense of distributions} \tag{1.5}$$

if and only if

$$\partial_t(f(t,x+tv,v)) = g(t,x+tv,v), \qquad \text{in the sense of distributions.} \tag{1.6}$$

In particular, it holds that $f(t,x+tv,v) \in W^{1,1}_{\text{loc}}\left(\mathbb{R}; L^1_{\text{loc}}\left(\mathbb{R}^D \times \mathbb{R}^D\right)\right)$, which in turn implies (after possibly redefining $f(t,x+tv,v)$ on a set of measure zero) the time continuity $f(t,x+tv,v) \in C_{\text{loc}}\left(\mathbb{R}; L^1_{\text{loc}}\left(\mathbb{R}^D \times \mathbb{R}^D\right)\right)$ and thus allows to give an unambiguous sense to weak solutions of the Cauchy problem for the transport equation (1.5).

Considering then any specific initial value $f(0,x,v) = f^{\text{in}}(x,v)$ in the space $L^1_{\text{loc}}\left(\mathbb{R}^D \times \mathbb{R}^D\right)$, it holds that, for almost every $(t,x,v) \in \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}^D$,

$$f(t,x,v) = f^{\text{in}}(x-tv,v) + \int_0^t g(s, x-(t-s)v, v)\,ds. \tag{1.7}$$

This representation is merely Duhamel's formula for the transport equation and is obtained by simply integrating (1.5) along its characteristic curves. For convenience, we will utilize the operators $\mathcal{T}$ and $\mathcal{N}$ defined by

$$\begin{aligned} \mathcal{T} f^{\text{in}}(t,x,v) &= f^{\text{in}}(x-tv,v) \\ \text{and} \quad \mathcal{N} g(t,x,v) &= \int_0^t g(s,x-(t-s)v,v)\,ds, \end{aligned} \tag{1.8}$$

so that Duhamel's formula may be written as $f = \mathcal{T} f^{\text{in}} + \mathcal{N} g$.

Then, equipped with this integral formulation, it becomes simple to demonstrate a basic local well-posedness result for the Boltzmann equation (1.1). Indeed, by possibly assuming that the collision kernel is globally integrable, it is readily seen that the Boltzmann operator satisfies, for any given $T > 0$,

$$\begin{aligned} &\|\mathcal{N}\mathcal{Q}(F,F) - \mathcal{N}\mathcal{Q}(G,G)\|_{L^\infty_t([0,T];L^\infty_{x,v})} \\ &\quad = \|\mathcal{N}\mathcal{Q}(F,F-G) + \mathcal{N}\mathcal{Q}(F-G,G)\|_{L^\infty_t([0,T];L^\infty_{x,v})} \\ &\quad \leq T\,\|b\|_{L^1(\mathbb{R}^D\times\mathbb{S}^{D-1})} \left(\|F\|_{L^\infty_t([0,T];L^\infty_{x,v})} + \|G\|_{L^\infty_t([0,T];L^\infty_{x,v})}\right) \\ &\qquad \times \|F-G\|_{L^\infty_t([0,T];L^\infty_{x,v})}. \end{aligned} \tag{1.9}$$

Consequently, the nonlinear operator $\mathcal{K}F = \mathcal{T}F^{\text{in}} + \mathcal{N}\mathcal{Q}(F,F)$ is a well-defined contraction on the complete metric space $\{\|F\|_{L^\infty} \leq R\}$ for some $R > 0$, provided $T > 0$ is small enough. It follows then from the Banach fixed point theorem that the above operator $\mathcal{K}$ has a unique fixed point, thus showing the local well-posedness of Boltzmann's equation, i.e. the existence and uniqueness of weak solutions in $L^\infty_t\left([0,T]; L^\infty_{x,v}\right)$ for some small time $T > 0$. It had already been noticed by Lions in [20] that such local well-posedness results could be proven in L^∞ as a simple and direct way to construct solutions that could be applied to the weak-strong uniqueness principle for the Boltzmann equation demonstrated therein. As emphasized by Lions, we do not make any claim regarding the originality of this simple result.

It is then legitimate to attempt to extend this method of construction of mild solutions to other function spaces. Thus, we wish to find fixed points of the operator $\mathcal{K}$ in other functional settings such as the mixed Lebesgue spaces $L^p_t L^q_x L^r_v$ and the success of this approach therefore relies on obtaining norm preserving estimates similar to (1.9) for some mixed Lebesgue norm. It turns out that such stable controls are only available for the gain term $\mathcal{Q}^+(F,G)$ of the Boltzmann operator as demonstrated in Sect. 3.3 and it will therefore not be possible to prove contractivity properties for the operator $\mathcal{K}$, which is the reason why our methods employed here will fail to yield the uniqueness of solutions. The uniqueness may however be recovered in some situations with some mildly restrictive assumptions, which is currently under study.

The key idea in obtaining norm preserving estimates on the operator $F \mapsto \mathcal{N}\mathcal{Q}^+(F,F)$ is based on the analysis of two distinct mechanisms of transfer of integrability. On the one hand, we have the dispersion estimates, which we present in Sect. 3.1 and were initially developed by Castella and Perthame in [9]. These estimates express the fact that densities are transported along the microscopic trajectories under the action of the transport operator. Loosely speaking, this dispersive character due to the action of the operator $\mathcal{N}$ will allow the transfer of some integrability from the velocity variable into the time and space variables. On the other hand, even though the nonlinear nature of the Boltzmann operator may lead to the formation of singularities and blow-ups, which is physically due to the collisions in the gas, we have a convoluting effect of the gain term of the Boltzmann operator whose action allows us to tame and control the singularities with appropriate norms. These effects are expressed through new convolution inequalities on the Boltzmann operator exposed in Sect. 3.2, which will let us transfer the integrability back into the velocity variable and thus obtain norm preserving estimates. In fact, the gain term is already known to possess some convoluting properties as in [1,12,15,16,24] and even to provide smoothing effects as seen in [7,20,21,27].

Thus, what will be possible to achieve is in fact the stability of a complete metric space $\left\{\|F\|_{L^p_t L^q_x L^r_v} \leq R\right\}$ for some $R > 0$ under the action of the operator $\mathcal{K}$. This will provide the weak relative compactness of the solutions $F_n, n \in \mathbb{N}$, to an approximate Boltzmann equation $F_n = \mathcal{K}_n F_n$, where $\mathcal{K}_n$ is a truncated operator where the nonlinearity is tamed. By a compactness argument, we will then let n tend to infinity and thus recover a fixed point $F = \mathcal{K}F$ in the limit, thus yielding the existence of mild solutions for Boltzmann's equation in the small, i.e. for small time or small initial values. In fact, we will focus on obtaining existence results valid globally in time for small initial data, but it will be clear that the same method may be applied to yield the existence of solutions locally in time and for large initial data.

2. Global Existence for Small Initial Data

The main result in this work is expressed in the following theorem.

Theorem 2.1. *Let the dimension be $D = 2$ or $D = 3$ only and $2 < \lambda_0 \leq 3$ be a fixed parameter. Let the cross section $b(z,\sigma) \geq 0$ satisfy*

$$\begin{aligned} &b(z,\sigma) \in L_z^{\frac{D}{D-\left(1+\frac{1}{\alpha}\right)}}\left(\mathbb{R}^D; L^1_\sigma\left(\mathbb{S}^{D-1}\right)\right) \quad \text{for some } \alpha \geq \lambda_0 \text{ such that } \alpha > D, \\ &\text{and} \quad b(z,\sigma) = b\left(|z|, \cos\theta\right) \leq a_0(|z|) b_0\left(\cos\theta\right), \\ &\text{for some} \quad a_0(|z|) \in L_z^{\frac{D}{D-2}}\left(\mathbb{R}^D\right) \quad \text{and some} \quad \frac{b_0\left(\cos\theta\right)}{\sin^{\frac{\lambda_0+1}{2\lambda_0}} \frac{\theta}{2}} \in L^1_\sigma\left(\mathbb{S}^{D-1}\right). \end{aligned} \tag{2.1}$$

Then, there exists a constant $C_0 > 0$ *such that for every initial data satisfying*

$$F^{\text{in}}(x,v) \in L^D_{x,v}\left(\mathbb{R}^D \times \mathbb{R}^D\right), \quad F^{\text{in}} \geq 0$$
$$\text{and} \quad \left\| \frac{b_0(\cos\theta)}{\sin^{\frac{\lambda_0+1}{2\lambda_0}} \frac{\theta}{2}} \right\|_{L^1_\sigma} \|a_0\|_{L_z^{\frac{D}{D-2}}} \left\| F^{\text{in}} \right\|_{L^D_{x,v}} < C_0, \tag{2.2}$$

there exists a weak solution $F(t,x,v)$ *to the Boltzmann equation* (1.1) *with initial data* F^{in} *and satisfying*

$$F \in L_t^\lambda\left([0,\infty); L_x^{D\frac{\lambda}{\lambda-1}}\left(\mathbb{R}^D; L^{D\frac{\lambda}{\lambda+1}}\left(\mathbb{R}^D\right)\right)\right)$$
$$\text{for all } \lambda_0 \leq \lambda \leq \infty \text{ such that } \lambda > D. \tag{2.3}$$

Furthermore, the Boltzmann collision operator satisfies

$$\mathcal{Q}(F,F) \in L_t^{\frac{\alpha}{2}}\left([0,\infty); L_x^{\frac{D\alpha}{2(\alpha-1)}} L_v^{D\frac{\alpha}{\alpha+1}}\right). \tag{2.4}$$

We provide the demonstration of the above theorem in Sect. 4. A few comments are now in order:

1. First of all, as explained in Sect. 1.1, notice that the above weak solutions enjoy, at the very least, the temporal continuity along the characteristics
$$F(t,x+tv,v) \in C_{\text{loc}}\left([0,\infty); L^1_{\text{loc}}\left(\mathbb{R}^D \times \mathbb{R}^D\right)\right), \tag{2.5}$$
so that the initial value problem makes sense.
2. Note that the integrability assumption on the angular kernel b_0 in (2.2) becomes more stringent as the parameter λ_0 tends closer to two. On the other hand, a smaller value for λ_0 yields more spaces characterizing the weak solution in (2.3). In three dimensions, it is pointless to choose any value other than $\lambda_0 = 3$ because of the restriction $\lambda > D$. However, in two dimensions, if one is willing to impose stricter conditions on the angular kernel b_0, it is thus possible to enlarge the existence spaces in (2.3) by letting λ_0 approach the value $\lambda_0 \to 2$.
3. The integrability assumption (2.1) on the collision kernel makes the above existence result only suitable for cross sections that have a decaying kinetic kernel at infinity. However, it allows for some local integrable singularities. Thus, examples of kernels suitable for the application of the above theorem are given by
$$b(|z|,\cos\theta) = \frac{1}{|z|^A(1+|z|)^{B-A}} \sin^C\theta \mathbb{1}_{\{\cos\theta\geq 0\}} \tag{2.6}$$
and
$$b(|z|,\cos\theta) = \left(\chi(|z|)\frac{1}{|z|^A} + (1-\chi)(|z|)\frac{1}{|z|^B}\right)\sin^C\theta \mathbb{1}_{\{\cos\theta\geq 0\}}, \tag{2.7}$$
for any $A < D-2$, $B > D - \left(1 + \frac{1}{\alpha}\right)$ and $C > \frac{\lambda_0+1}{2\lambda_0} - (D-1)$, where $\chi(|z|)$ is a cutoff function satisfying $\mathbb{1}_{\{|z|\leq 1\}} \leq \chi(|z|) \leq \mathbb{1}_{\{|z|\leq 2\}}$, say. These examples remain rather artificial as they are not physical (see [26] for more details on cross sections).

4. Even though the above results provide the basic framework for understanding the mechanisms of transfer of integrability between the transport and collision operators, it is clear that they can be largely improved. For instance, by using spaces with weights and regularity, as was done in the homogeneous case in [24] by Mouhot and Villani for instance, it should be possible to extend the well-posedness results to more function spaces and, by the same token, considerably relax the assumptions on the cross section.
5. The smallness condition (2.2) on the norm $L^D_{x,v}$ of the initial data illustrates the existing competition between the dispersive effects of the transport operator and the singularity formations due to collisions. Indeed, suppose that the initial data has finite mass, i.e. finite $L^1_{x,v}$ norm, no matter how large. Then, loosely speaking, it is still possible to ensure that the $L^D_{x,v}$ norm is arbitrarily small by sufficiently spreading this mass about the whole space so that the inequality (2.2) is satisfied. Thus, the condition (2.2) guarantees that the ensuing dispersion may overcome the large amount of collisions measured by the norm of the collision kernel.
6. The significance of these existence results lies in the fact that they yield global solutions for a very large class of initial data. Indeed, only an integrability condition without any regularity or pointwise bound is imposed on the initial value. Furthermore, there is no need to renormalize the equation.
7. Some major drawbacks have to be acknowledged. Indeed, the existence is only true for small initial data and necessitates stringent hypotheses on the collision kernel. However, due to the sometimes crude methods of proof we employed, it seems that the results are not yet optimal. Our methods are based on the splitting of the Boltzmann operator into its gain and its loss part and thus do not take advantage of the cancellation properties between these two components. Still, this work truly exhibits the mechanisms of transfer of integrability between the dispersive effects of the transport operator and the convoluting effects of the Boltzmann operator, thus leaving much room for interesting research perspectives. Finally, it is crucial to consider the whole space in order to exploit the dispersive effects and so it seems difficult to adapt the present methods to other spatial domains.
8. We have stated Theorem 2.1 as an existence result near vacuum. However, it is possible to prove a similar existence assertion near a Maxwellian equilibrium $M(x - tv, v)$, i.e. $M(x, v)$ is a Gaussian distribution in both variables x and v, by using the fact that

$$\begin{aligned} \mathcal{Q}(F,F) &= \mathcal{Q}(F,F) - \mathcal{Q}(M,M) \\ &= \mathcal{Q}(F-M,F-M) + \mathcal{Q}(F-M,M) + \mathcal{Q}(M,F-M), \end{aligned} \tag{2.8}$$

   and adapting the proof where necessary. This is precisely the kind of mild solution that may be used to perform hydrodynamic limits of the Boltzmann equation. This is currently under study.
9. Notice the similarity of the present theory with the mild solutions for the incompressible Navier-Stokes equations built by Kato [19], which are also set in an L^D context, where D is the space dimension. It is strongly expected that a link through hydrodynamic limits exists between the mild solutions of the Boltzmann and Navier-Stokes equations.
10. It is possible to obtain similar existence results for large initial data but for a small time of existence. Moreover, one can show, at least in some cases, the uniqueness of mild solutions by exploiting the global conservation of mass, which is a natural a

priori estimate on the solutions of the Boltzmann equation. This is currently under study.

11. With an elementary Picard iteration scheme and using the same controls on the gain term of the Boltzmann collision operator that are utilized in the proof of Theorem 2.1, it is possible to prove a global existence and uniqueness result for the gain-term-only Boltzmann equation for small initial data. This result is not in contradiction with the blowup results for the same equation from [2]. Furthermore, the ensuing weak solutions provide very convenient candidates for the so-called beginning condition for the Kaniel-Shinbrot iteration scheme from [17,18].

3. Transfer of Integrability

In this section, we first expose the mechanisms of transfer of integrability in Sects. 3.1 and 3.2 and then draw some consequences in the form of norm preserving estimates in Sect. 3.3.

3.1. Dispersion estimates. We recall here the so-called dispersion estimates developed by Castella and Perthame in [9], which are reminiscent of the well-known Strichartz estimates for the wave, Klein-Gordon and Schrödinger equations from [25]. We also provide proofs of these results for the convenience of the reader. One may also consult [8] for a clear exposition of the subject.

Lemma 3.1. *Let* $1 \le r \le p \le \infty$ *and* $f \in L^r_v\left(\mathbb{R}^D; L^p_x\left(\mathbb{R}^D\right)\right)$. *Then,*

$$\|f(x,v)\|_{L^p_x L^r_v} \le \|f(x,v)\|_{L^r_v L^p_x}. \tag{3.1}$$

Proof. The above assertion is readily verified in the case $p = r$ and in the case $p = \infty$. We conclude by applying the Riesz-Thorin interpolation theorem for mixed Lebesgue spaces (see [4, p. 316]). □

Proposition 3.2. *Let* $1 \le r \le p \le \infty$ *and* $f \in L^r_x\left(\mathbb{R}^D; L^p_v\left(\mathbb{R}^D\right)\right)$. *Then, for any* $t \neq 0$, *we have that*

$$\|f(x-tv,v)\|_{L^p_x L^r_v} \le \frac{1}{|t|^{D\left(\frac{1}{r}-\frac{1}{p}\right)}} \|f(x,v)\|_{L^r_x L^p_v}. \tag{3.2}$$

Proof. Using first the change of variable $v = \frac{x-y}{t}$ and then using Lemma 3.1, we infer

$$\begin{aligned} \|f(x-tv,v)\|_{L^p_x L^r_v} &= \left\| |t|^{-D\frac{1}{r}} f\left(y, \frac{x-y}{t}\right) \right\|_{L^p_x L^r_y} \\ &\le \left\| |t|^{-D\frac{1}{r}} f\left(y, \frac{x-y}{t}\right) \right\|_{L^r_y L^p_x} = \left\| |t|^{-D\left(\frac{1}{r}-\frac{1}{p}\right)} f(y,x) \right\|_{L^r_y L^p_x}, \end{aligned} \tag{3.3}$$

which concludes the proof of the proposition. □

Proposition 3.3. *Let* $1 \le a, p, q, r \le \infty$ *be such that*

$$\frac{2}{q} = D\left(\frac{1}{r} - \frac{1}{p}\right), \qquad \frac{2}{a} = \frac{1}{p} + \frac{1}{r} \qquad \text{and} \qquad a < q. \tag{3.4}$$

Then, for any $f \in L^a_{x,v}\left(\mathbb{R}^D \times \mathbb{R}^D\right)$*, we have that*

$$\|f(x-tv,v)\|_{L^q_t(\mathbb{R};L^p_x L^r_v)} \leq C\|f(x,v)\|_{L^a_{x,v}}, \tag{3.5}$$

for some fixed constant $C > 0$*.*

Proof. First, let us assume that $a = 2$ and $k = q$. It follows that $r = p'$. Then, for any $g \in L^{q'}_t L^{p'}_x L^p_v$, we have that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}} g(t, x+tv, v)\, dt\right\|^2_{L^2_{x,v}} \\
&= \iint \int_{\mathbb{R}}\int_{\mathbb{R}} g(t,x+tv,v)g(s,x+sv,v)\, dtdsdxdv \\
&= \iint \int_{\mathbb{R}} g(t,x,v)\int_{\mathbb{R}} g(s,x-(t-s)v,v)\, dsdtdxdv \\
&\leq \|g(t,x,v)\|_{L^{q'}_t L^{p'}_x L^p_v}\left\|\int_{\mathbb{R}} g(s,x-(t-s)v,v)ds\right\|_{L^q_t L^p_x L^{p'}_v}.
\end{aligned} \tag{3.6}$$

Next, using Proposition 3.2 and the classical Hardy-Littlewood-Sobolev inequality, we deduce

$$\begin{aligned}
\left\|\int_{\mathbb{R}} g(s,x-(t-s)v,v)\, ds\right\|_{L^q_t L^p_x L^{p'}_v} &\leq \left\|\int_{\mathbb{R}} \|g(s,x-(t-s)v,v)\|_{L^p_x L^{p'}_v}\, ds\right\|_{L^q_t} \\
\leq \left\|\int_{\mathbb{R}} |t-s|^{-\frac{2}{q}}\|g(s,x,v)\|_{L^{p'}_x L^p_v}\, ds\right\|_{L^q_t} &\leq C\|g(t,x,v)\|_{L^{q'}_t L^{p'}_x L^p_v},
\end{aligned} \tag{3.7}$$

where $C > 0$ is a fixed constant. Note that the use above of the classical Hardy-Littlewood-Sobolev inequality is valid because it is assumed that $\frac{2}{q} < 1$. Combining now the estimates (3.6) and (3.7), we obtain

$$\left\|\int_{\mathbb{R}} g(t,x+tv,v)\, dt\right\|_{L^2_{x,v}} \leq C^{\frac{1}{2}}\|g(t,x,v)\|_{L^{q'}_t L^{p'}_x L^p_v}. \tag{3.8}$$

We can now turn to estimating the norm of $f(x-tv,v)$. Using (3.8), we infer

$$\begin{aligned}
&\left|\iint\int_{\mathbb{R}} f(x-tv,v)g(t,x,v)\, dtdxdv\right| \\
&= \left|\iint f(x,v)\int_{\mathbb{R}} g(t,x+tv,v)\, dtdxdv\right| \\
&\leq \|f(x,v)\|_{L^2_{x,v}}\left\|\int_{\mathbb{R}} g(t,x+tv,v)\, dt\right\|_{L^2_{x,v}} \\
&\leq C^{\frac{1}{2}}\|f(x,v)\|_{L^2_{x,v}}\|g(t,x,v)\|_{L^{q'}_t L^{p'}_x L^p_v}.
\end{aligned} \tag{3.9}$$

Taking the supremum over all $g \in L^{q'}_t L^{p'}_x L^p_v$ in (3.9) permits us to conclude

$$\|f(x-tv,v)\|_{L^q_t L^p_x L^{p'}_v} \leq K^{\frac{1}{2}}\|f(x,v)\|_{L^2_{x,v}}, \tag{3.10}$$

thus proving that (3.5) holds in the case $a = 2$.

If $a \neq 2$, it suffices to apply inequality (3.10) to $f^{\frac{a}{2}}$ to deduce

$$\|f(x-tv,v)\|_{L_t^q L_x^p L_v^r} = \left\| f(x-tv,v)^{\frac{a}{2}} \right\|_{L_t^{\frac{2q}{a}} L_x^{\frac{2p}{a}} L_v^{\frac{2r}{a}}}^{\frac{2}{a}}$$
$$\leq K^{\frac{1}{a}} \|f(x,v)^{\frac{a}{2}}\|_{L^2_{x,v}}^{\frac{2}{a}} = K^{\frac{1}{a}} \|f(x,v)\|_{L^a_{x,v}}, \tag{3.11}$$

which concludes the proof of the proposition. □

Proposition 3.4. *Let* $1 \leq k, l, p, q, r \leq \infty$ *be such that*

$$1 < l < k < \infty, \qquad \frac{1}{q} = D\left(\frac{1}{r} - \frac{1}{p}\right) \quad \textit{and} \quad 1 + \frac{1}{k} = \frac{1}{q} + \frac{1}{l}. \tag{3.12}$$

Then, for any $g \in L_t^l\left(\mathbb{R}; L_x^r\left(\mathbb{R}^D; L_v^p\left(\mathbb{R}^D\right)\right)\right)$, *we have the following estimate*:

$$\left\| \int_0^t g(s, x-(t-s)v, v)\, ds \right\|_{L_t^k L_x^p L_v^r} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}, \tag{3.13}$$

for some fixed constant $C > 0$.

Proof. First, using Proposition 3.2, we estimate

$$\left\| \int_0^t g(s, x-(t-s)v, v)\, ds \right\|_{L_x^p L_v^r} \leq \int_0^t \|g(s, x-(t-s)v, v)\|_{L_x^p L_v^r}\, ds$$
$$\leq \int_0^t (t-s)^{-\frac{1}{q}} \|g(s,x,v)\|_{L_x^r L_v^p}\, ds. \tag{3.14}$$

Next, applying the Hardy-Littlewood-Sobolev inequality to the last integral in (3.14), we infer

$$\left\| \int_0^t (t-s)^{-\frac{1}{q}} \|g(s,x,v)\|_{L_x^r L_v^p}\, ds \right\|_{L_t^k} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}, \tag{3.15}$$

which concludes the proof. □

Proposition 3.5. *Let* $1 \leq l, p, q, r \leq \infty$ *be such that*

$$1 < q, \qquad \frac{1}{q} = D\left(\frac{1}{r} - \frac{1}{p}\right), \qquad 1 = \frac{1}{l} + \frac{1}{2q} \quad \textit{and} \quad \frac{1}{p} + \frac{1}{r} \leq 1. \tag{3.16}$$

Then, for any $g \in L_t^l\left(\mathbb{R}; L_x^r\left(\mathbb{R}^D; L_v^p\left(\mathbb{R}^D\right)\right)\right)$, *we have the following estimate*:

$$\left\| \int_0^t g(s, x-(t-s)v, v)\, ds \right\|_{L_t^\infty L_{x,v}^{\frac{2pr}{p+r}}} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}, \tag{3.17}$$

for some fixed constant $C > 0$.

Proof. First, using Hölder's inequality and Proposition 3.2, we find that

$$\begin{aligned}
&\left\| \int_0^t g(s, x-(t-s)v, v)\, ds \right\|_{L^{\frac{2pr}{p+r}}_{x,v}} \\
&= \left\| \int_0^t \int_0^t g(s, x-(t-s)v, v) g(u, x-(t-u)v, v)\, ds du \right\|^{\frac{1}{2}}_{L^{\frac{pr}{p+r}}_{x,v}} \\
&\leq \left(\int_0^t \int_0^t \| g(s, x-(t-s)v, v) g(u, x-(t-u)v, v) \|_{L^{\frac{pr}{p+r}}_{x,v}}\, ds du \right)^{\frac{1}{2}} \\
&= \left(\int_0^t \int_0^t \| g(s, x, v) g(u, x-(s-u)v, v) \|_{L^{\frac{pr}{p+r}}_{x,v}}\, ds du \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^t \int_0^t \| g(s, x, v) \|_{L^r_x L^p_v} \| g(u, x-(s-u)v, v) \|_{L^p_x L^r_v}\, du ds \right)^{\frac{1}{2}} \\
&\leq \left(\int_0^t \int_0^t \| g(s, x, v) \|_{L^r_x L^p_v} |s-u|^{-\frac{1}{q}} \| g(u, x, v) \|_{L^r_x L^p_v}\, du ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.18}$$

If $q < \infty$, applying the classical Hardy-Littlewood-Sobolev inequality to the last integral of (3.18) yields

$$\left\| \int_0^t g(s, x-(t-s)v, v)\, ds \right\|_{L^{\frac{2pr}{p+r}}_{x,v}} \leq C \| g(s, x, v) \|_{L_s^{(2q)'} L^r_x L^p_v}, \tag{3.19}$$

where $C > 0$ is a constant. On the other hand, if $q = \infty$, the same holds true by direct computation, which concludes the proof. □

3.2. Convolution inequalities. Recall that the gain and loss operators, $\mathcal{Q}^+$ and $\mathcal{Q}^-$ respectively, are defined by

$$\begin{aligned}
\mathcal{Q}^+(F, G) &= \int_{\mathbb{R}^D} \int_{\mathbb{S}^{D-1}} F' G'_* b(v - v_*, \sigma)\, d\sigma dv_*, \\
\mathcal{Q}^-(F, G) &= \int_{\mathbb{R}^D} \int_{\mathbb{S}^{D-1}} F G_* b(v - v_*, \sigma)\, d\sigma dv_*.
\end{aligned} \tag{3.20}$$

Employing the well-known pre-post-collisional change of variables, which merely permutes (v, v_*) and (v', v'_*) and has unit jacobian, these operators become, in their Maxwellian (weak) formulation,

$$\begin{aligned}
\int_{\mathbb{R}^D} \mathcal{Q}^+(F, G) \varphi\, dv &= \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \int_{\mathbb{S}^{D-1}} F(v) G(v_*) \varphi(v') b(v - v_*, \sigma)\, d\sigma dv_* dv, \\
\int_{\mathbb{R}^D} \mathcal{Q}^-(F, G) \varphi\, dv &= \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \int_{\mathbb{S}^{D-1}} F(v) G(v_*) \varphi(v) b(v - v_*, \sigma)\, d\sigma dv_* dv.
\end{aligned} \tag{3.21}$$

Clearly, the loss operator acts as a convolution in the variables v and v_*. Thus, defining $a(z) = \int_{\mathbb{S}^{D-1}} b(z,\sigma)\,d\sigma$ and using Young's inequality, one easily finds that

$$\begin{aligned} \|\mathcal{Q}^-(F,G)\|_{L^s_v} &\le \left\| \int_{\mathbb{R}^n} F(v)G(v_*)a(v-v_*)\,dv_* \right\|_{L^s_v} \\ &\le \|F\|_{L^p}\|G\|_{L^q}\|a\|_{L^r}, \end{aligned} \tag{3.22}$$

where $1+\frac{1}{s} = \frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ and $s \le p$. Since $\mathcal{Q}^-(F,G)$ is merely a product between $a*G$ and F, it is not possible to improve the integrability of F for parameters $s > p$. Otherwise, we would be able to choose some fixed a and G so that there exists $\varphi \in C_0^\infty(\mathbb{R}^D)$ satisfying $0 \le \varphi \le a*G$, thus yielding $\|F\varphi\|_{L^s} \le C\|F\|_{L^p}$, where $C = \|G\|_{L^q}\|a\|_{L^r}$, which is obviously a contradiction when $s > p$ as soon as φ is not trivial.

At first, due to its intricate nature, it is unclear whether $\mathcal{Q}^+$ will satisfy an identical estimate or not. In fact, $\mathcal{Q}^+$ bears a much nicer structure since it is known to have some convoluting effects (see [1,12,15,16,24]) and even to provide a gain of regularity (see [7,20,21,27]) and so, it is reasonable to hope for a similar inequality to hold. It turns out that a slight modification of the above argument shows that a convolution inequality holds for the gain operator as well. Indeed, writing the Maxwellian formulation and using Hölder's and Young's inequalities, we deduce

$$\begin{aligned} &\left| \int_{\mathbb{R}^D} \mathcal{Q}^+(F,G)(v)\varphi(v)\,dv \right| \le \int_{\mathbb{R}^D}\int_{\mathbb{R}^D}\int_{\mathbb{S}^{D-1}} |FG_*\varphi'| b(v-v_*,\sigma)\,d\sigma dv_* dv \\ &\le \iiint \left| F(v)G(v-V)\varphi\left(v-\frac{V}{2}+\frac{|V|}{2}\sigma\right)\right| b(V,\sigma)dVdvd\sigma \\ &\le \|\varphi\|_{L^{s'}} \int \|F(v)G(v-V)\|_{L^s_v}\, a(V)\,dV \\ &\le \|\varphi\|_{L^{s'}} \left\| \, \|F(v)G(v-V)\|_{L^s_v} \right\|_{L^{r'}_V} \|a\|_{L^r} \\ &\le \|\varphi\|_{L^{s'}}\|F\|_{L^p}\|G\|_{L^q}\|a\|_{L^r}, \end{aligned} \tag{3.23}$$

where $1 \le s \le p, q \le r' \le \infty$ and $1+\frac{1}{s} = \frac{1}{p}+\frac{1}{q}+\frac{1}{r}$. Taking the supremum in (3.23) over all $\varphi \in L^{s'}$, we conclude

$$\begin{aligned} \left\|\mathcal{Q}^+(F,G)\right\|_{L^s} &\le \|F\|_{L^p}\|G\|_{L^q}\|a\|_{L^r}, \\ \left\|\mathcal{Q}^-(F,G)\right\|_{L^s} &\le \|F\|_{L^p}\|G\|_{L^q}\|a\|_{L^r}, \end{aligned} \tag{3.24}$$

for any $1 \le p,q,r,s \le \infty$ such that $1+\frac{1}{s} = \frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ and $s \le p, q \le r'$. However, this simple argument retains the restriction on the parameters $s \le p, q$ for $\mathcal{Q}^+$, which is absolutely not sufficient in order to carry out our arguments on the mechanisms of transfer of integrability leading to the existence of mild solutions. It is fortunate that a better convolution inequality for the gain term $\mathcal{Q}^+$ including the full range of parameters $1 \le p,q,r,s \le \infty$ is available, as shown in Proposition 3.6 below. Its validity comes however under some further integrability condition on the angular collision kernel. It is to be emphasized that this extension of the parameters range constitutes the originality of this new inequality. Even though it may seem of rather technical nature, it is crucial in our work and its demonstration remains elementary since it merely involves the utilization of Hölder's inequality and some changes of variables (just as Young's convolution inequality is a consequence of Hölder's inequality).

Finally, we remark that the convoluting nature of the gain term had been noticed in several previous studies, especially in the works of Gustafsson [15,16], Mouhot and Villani [24], and Duduchava, Kirsch and Rjasanow [12]. Simultaneously to our work, another similar result has been obtained independently by Alonso and Carneiro [1]. However, none of these results included the whole parameter range for p, q, r, s in (3.24) for $\mathcal{Q}^+$, since they regarded the cross section as a weight rather than as an element of the convolution.

Proposition 3.6. *Let* $b(z,\sigma) \geq 0$ *be a collision kernel satisfying*

$$b(z,\sigma) = b\left(|z|, \cos\theta\right) = a_0\left(|z|\right) b_0(\cos\theta), \tag{3.25}$$

where $\cos\theta = \frac{z}{|z|} \cdot \sigma$, *and let the parameters* $1 \leq p, q, r, s \leq \infty$ *be such that*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{s}. \tag{3.26}$$

Then,

$$\left\| \mathcal{Q}^+(F,G) \right\|_{L^s} \leq 2^{\frac{D}{2}} C_0 \left\|F\right\|_{L^p} \left\|G\right\|_{L^q} \left\|a_0\right\|_{L^r}, \tag{3.27}$$

where

$$\begin{aligned} C_0 &= \int_{\mathbb{S}^{D-1}} b_0\left(\cos\theta\right)\, d\sigma \quad \text{if } s \leq p, \\ \text{and} \quad C_0 &= \int_{\mathbb{S}^{D-1}} \frac{b_0\left(\cos\theta\right)}{\sin^{D\left(\frac{1}{p}-\frac{1}{s}\right)}\frac{\theta}{2}}\, d\sigma \quad \text{if } s \geq p. \end{aligned} \tag{3.28}$$

Proof. Let $\varphi \in L^s_v\left(\mathbb{R}^D\right)$. By duality and using the collisional changes of variables, we will need to control

$$\begin{aligned} \int \mathcal{Q}^+(F,G)(v)\varphi(v)\, dv &= \iiint F' G'_* \varphi(v) a_0(v-v_*) b_0(\cos\theta)\, d\sigma dv dv_* \\ &= \iint F(v)G(v_*)a_0(v-v_*) \int \varphi\left(v'\right) b_0(\cos\theta)\, d\sigma dv dv_* \\ &= \iint F(v)G(v_*)a_0(v-v_*)\Phi_0(v,v_*)\, dv dv_*, \end{aligned} \tag{3.29}$$

where we have written $\Phi_0\left(v,v_*\right) = \int \varphi\left(v'\right) b_0(\cos\theta)\, d\sigma$.

To this end, we employ the set of parameters $1 \leq p_1, p_2, p_3, p_4, p_5, p_6 \leq \infty$ given by Lemma 3.7 with s replaced by s' and we define

$$\begin{aligned} \alpha_1 &= \frac{p}{p_1} & \alpha_2 &= \frac{p}{p_2} & \alpha_3 &= \frac{p}{p_3} \\ \beta_1 &= \frac{q}{p_1} & \beta_2 &= \frac{q}{p_4} & \beta_3 &= \frac{q}{p_5} \\ \gamma_1 &= \frac{r}{p_2} & \gamma_2 &= \frac{r}{p_4} & \gamma_3 &= \frac{r}{p_6} \\ \rho_1 &= \frac{s'}{p_3} & \rho_2 &= \frac{s'}{p_5} & \rho_3 &= \frac{s'}{p_6} \end{aligned} \tag{3.30}$$

so that $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\beta_1 + \beta_2 + \beta_3 = 1$, $\gamma_1 + \gamma_2 + \gamma_3 = 1$ and $\rho_1 + \rho_2 + \rho_3 = 1$. Furthermore, in accordance with that lemma, we may choose $\frac{1}{p_3} = \max\left\{0, \frac{1}{p} - \frac{1}{s}\right\}$.

Then, defining an auxiliary kernel

$$b_1(\cos\theta) = \frac{b_0(\cos\theta)}{\sin^{\frac{D}{p_3}}\frac{\theta}{2}} \tag{3.31}$$

and writing

$$\begin{aligned} c_0 &= \left(\int_{\mathbb{S}^{D-1}} b_1(\cos\theta)\, d\sigma\right)^{\frac{1}{s}}, \\ \Phi_1(v, v_*) &= c_0 \left(\int_{\mathbb{S}^{D-1}} \left|\varphi\left(v'\right)\right|^{s'} \sin^D \frac{\theta}{2} b_1(\cos\theta)\, d\sigma\right)^{\frac{1}{s'}}, \\ \Phi_2(v, v_*) = \Phi_3(v, v_*) &= c_0 \left(\int_{\mathbb{S}^{D-1}} \left|\varphi\left(v'\right)\right|^{s'} b_1(\cos\theta)\, d\sigma\right)^{\frac{1}{s'}}, \end{aligned} \tag{3.32}$$

we obtain, simply using Hölder's inequality, that

$$|\Phi_0| \leq \Phi_1^{\rho_1} \cdot \Phi_2^{\rho_2} \cdot \Phi_3^{\rho_3}. \tag{3.33}$$

Therefore, we may decompose the last integrand in (3.29) as

$$|F(v)G(v_*)a_0(v - v_*)\Phi_0(v, v_*)| \leq |P_1 \cdot P_2 \cdot P_3 \cdot P_4 \cdot P_5 \cdot P_6|, \tag{3.34}$$

where

$$\begin{aligned} P_1 &= F(v)^{\alpha_1} G(v_*)^{\beta_1} & P_2 &= F(v)^{\alpha_2} a_0(v - v_*)^{\gamma_1} \\ P_3 &= F(v)^{\alpha_3} \Phi_1(v, v_*)^{\rho_1} & P_4 &= G(v_*)^{\beta_2} a_0(v - v_*)^{\gamma_2} \\ P_5 &= G(v_*)^{\beta_3} \Phi_2(v, v_*)^{\rho_2} & P_6 &= a_0(v - v_*)^{\gamma_3} \Phi_3(v, v_*)^{\rho_3} \end{aligned} \tag{3.35}$$

so that, by Hölder's inequality again,

$$\begin{aligned} &\left|\int \mathcal{Q}^+(F, G)(v)\varphi(v)\, dv\right| \\ &\leq \|P_1\|_{L^{p_1}_{v,v_*}} \cdot \|P_2\|_{L^{p_2}_{v,v_*}} \cdot \|P_3\|_{L^{p_3}_{v,v_*}} \cdot \|P_4\|_{L^{p_4}_{v,v_*}} \cdot \|P_5\|_{L^{p_5}_{v,v_*}} \cdot \|P_6\|_{L^{p_6}_{v,v_*}}. \end{aligned} \tag{3.36}$$

Next, we estimate each of the six resulting terms separately, which is trivial for P_1, P_2 and P_4 since their variables are separated because they do not contain the functions Φ_k. Indeed, one easily verifies that

$$\begin{aligned} \|P_1\|_{L^{p_1}_{v,v_*}} &= \|F\|^{\alpha_1}_{L^p_v} \|G\|^{\beta_1}_{L^q_v}, \\ \|P_2\|_{L^{p_2}_{v,v_*}} &= \|F\|^{\alpha_2}_{L^p_v} \|a_0\|^{\gamma_1}_{L^r_v}, \\ \|P_4\|_{L^{p_4}_{v,v_*}} &= \|G\|^{\beta_2}_{L^q_v} \|a_0\|^{\gamma_2}_{L^r_v}. \end{aligned} \tag{3.37}$$

On the other hand, the terms P_3, P_5 and P_6 only satisfy

$$\begin{aligned}
\|P_3\|_{L^{p_3}_{v,v_*}} &= \left\| F^{\alpha_3}(v)\, \|\Phi_1(v,v_*)\|^{\rho_1}_{L^{s'}_{v_*}} \right\|_{L^{p_3}_v}, \\
\|P_5\|_{L^{p_5}_{v,v_*}} &= \left\| G^{\beta_3}(v_*)\, \|\Phi_2(v,v_*)\|^{\rho_2}_{L^{s'}_{v}} \right\|_{L^{p_5}_{v_*}}, \\
\|P_6\|_{L^{p_6}_{v,v_*}} &= \left\| a_0^{\gamma_3}(v)\, \|\Phi_3(v+v_*,v_*)\|^{\rho_3}_{L^{s'}_{v_*}} \right\|_{L^{p_6}_v}.
\end{aligned} \tag{3.38}$$

Thus, in order to carry on these estimates, we will need to exploit the explicit definition of each Φ_k.

To this end, for any given $\sigma \in \mathbb{S}^{D-1}$, we consider the function

$$R_\sigma(v) = \frac{v}{2} + \frac{|v|}{2}\sigma \tag{3.39}$$

defined for any $v \in \mathbb{R}^D$. It is then easy to see that R_σ is a well-defined bijection from $\mathbb{R}^D \setminus \{1 : \sigma \cdot v = -|v|\}$ onto $\left\{u \in \mathbb{R}^D : \sigma \cdot u > 0\right\}$ with an inverse given by

$$R_\sigma^{-1}(u) = 2u - \frac{|u|^2}{\sigma \cdot u}\sigma. \tag{3.40}$$

Furthermore, it is readily seen, with the use of spherical coordinates, that the Jacobian of R_σ^{-1} is given by $\frac{2^{D-1}|u|^2}{(\sigma\cdot u)^2}$. In other words, for any measurable function $P : \left\{u \in \mathbb{R}^D : \sigma \cdot u > 0\right\} \to \mathbb{R}$, it holds that

$$\int_{\mathbb{R}^D \setminus \{\sigma\cdot v = -|v|\}} P\left(R_\sigma(v)\right) dv = \int_{\{\sigma\cdot u>0\}} P(u) \frac{2^{D-1}|u|^2}{(\sigma\cdot u)^2}\, du. \tag{3.41}$$

Finally, it is straightforward to check that if θ is the angle between v and σ, then $\frac{\theta}{2}$ is the angle between $R_\sigma(v)$ and σ. Therefore, it holds that $\frac{v}{|v|}\cdot\sigma = \cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 2\left(\frac{R_\sigma(v)\cdot\sigma}{|R_\sigma(v)|}\right)^2 - 1$.

Thus, employing the function R_σ with the explicit expression for Φ_1 and then the change of variable formula (3.41), we arrive at

$$\begin{aligned}
&\|\Phi_1(v,v_*)\|^{s'}_{L^{s'}_{v_*}} \\
&= c_0^{s'} \int |\varphi(v + R_\sigma(V))|^{s'} \left(\frac{R_\sigma(V)\cdot\sigma}{|R_\sigma(V)|}\right)^D b_1\left(1 - 2\left(\frac{R_\sigma(V)\cdot\sigma}{|R_\sigma(V)|}\right)^2\right) d\sigma dV \\
&= c_0^{s'} \int_{\{\sigma\cdot u>0\}} |\varphi(v+u)|^{s'} \left(\frac{u\cdot\sigma}{|u|}\right)^D b_1\left(1 - 2\left(\frac{u\cdot\sigma}{|u|}\right)^2\right) \frac{2^{D-1}|u|^2}{(\sigma\cdot u)^2}\, d\sigma du \\
&= c_0^{s'} \int |\varphi(v+u)|^{s'} \left|\mathbb{S}^{D-2}\right| \int_0^{\frac{\pi}{2}} b_1(-\cos 2\theta)\, 2^{D-1} \cos^{D-2}\theta \sin^{D-2}\theta\, d\theta du \\
&= c_0^{s'+s} \int |\varphi(u)|^{s'}\, du.
\end{aligned} \tag{3.42}$$

As to the term Φ_2, we treat it similarly, recalling that $b_1(\cos\theta)$ is supported on $\{\cos\theta \geq 0\}$,

$$\begin{aligned}
&\|\Phi_2(v, v_*)\|_{L_v^{s'}}^{s'} \\
&= c_0^{s'} \int |\varphi(v_* + R_\sigma(V))|^{s'} b_1\left(2\left(\frac{R_\sigma(V)\cdot\sigma}{|R_\sigma(V)|}\right)^2 - 1\right) d\sigma dV \\
&= c_0^{s'} \int_{\{\sigma\cdot u>0\}} |\varphi(v_* + u)|^{s'} b_1\left(2\left(\frac{u\cdot\sigma}{|u|}\right)^2 - 1\right) \frac{2^{D-1}|u|^2}{(\sigma\cdot u)^2} d\sigma du \\
&= c_0^{s'} \int |\varphi(v_* + u)|^{s'} \left|\mathbb{S}^{D-2}\right| \int_0^{\frac{\pi}{2}} b_1(\cos 2\theta) \frac{2^{D-1}\cos^{D-2}\theta \sin^{D-2}\theta}{\cos^D\theta} d\theta du \\
&\leq 2^{\frac{D}{2}} c_0^{s'+s} \int |\varphi(u)|^{s'} du. \qquad (3.43)
\end{aligned}$$

Finally, the term Φ_3 receives the simpler treatment

$$\begin{aligned}
\|\Phi_3(v + v_*, v_*)\|_{L_{v_*}^{s'}}^{s'} &= c_0^{s'} \iint |\varphi(V + R_\sigma(v))|^{s'} b_1\left(\frac{v}{|v|}\cdot\sigma\right) d\sigma dV \\
&= c_0^{s'+s} \int |\varphi(u)|^{s'} du. \qquad (3.44)
\end{aligned}$$

Therefore, incorporating (3.42), (3.43) and (3.44) into (3.38), we arrive at

$$\begin{aligned}
\|P_3\|_{L_{v,v_*}^{p_3}} &= c_0^{s\rho_1} \|F\|_{L_v^p}^{\alpha_3} \|\varphi\|_{L_v^{s'}}^{\rho_1}, \\
\|P_5\|_{L_{v,v_*}^{p_5}} &\leq 2^{\frac{D}{2p_5}} c_0^{s\rho_2} \|G\|_{L_v^q}^{\beta_3} \|\varphi\|_{L_v^{s'}}^{\rho_2}, \qquad (3.45) \\
\|P_6\|_{L_{v,v_*}^{p_6}} &= c_0^{s\rho_3} \|a_0\|_{L_v^r}^{\gamma_3} \|\varphi\|_{L_v^{s'}}^{\rho_3}.
\end{aligned}$$

Thus, on the whole, combining (3.36), (3.37) and (3.45), we have shown that

$$\left|\int \mathcal{Q}^+(F, G)(v)\varphi(v)\, dv\right| \leq 2^{\frac{D}{2p_5}} c_0^s \|F\|_{L_v^p} \|G\|_{L_v^q} \|a_0\|_{L_v^r} \|\varphi\|_{L_v^{s'}}. \qquad (3.46)$$

We conclude by taking the supremum over all $\varphi \in L_v^{s'}$ and setting $C_0 = c_0^s$. □

Lemma 3.7. *Let* $1 \leq p, q, r, s \leq \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2$. *Then, there are parameters* $1 \leq p_1, p_2, p_3, p_4, p_5, p_6 \leq \infty$ *satisfying*

$$\begin{aligned}
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p}, \qquad \frac{1}{p_1} + \frac{1}{p_4} + \frac{1}{p_5} = \frac{1}{q}, \\
\frac{1}{p_2} + \frac{1}{p_4} + \frac{1}{p_6} = \frac{1}{r}, \qquad \frac{1}{p_3} + \frac{1}{p_5} + \frac{1}{p_6} = \frac{1}{s}.
\end{aligned} \qquad (3.47)$$

In particular, they satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{1}{p_5} + \frac{1}{p_6} = 1. \qquad (3.48)$$

Moreover, it is always possible to set $\frac{1}{p_3} = \max\left\{0, \frac{1}{s} + \frac{1}{p} - 1\right\}$*, which is the best possible choice for* $\frac{1}{p_3}$ *in the sense that it minimizes its value. This way, it holds that* $p_3 = \infty$ *whenever* $\frac{1}{p} + \frac{1}{s} \leq 1$.

Proof. It is possible to show that the general solution to the system (3.47) is given by

$$\begin{pmatrix} \frac{1}{p_1} \\ \frac{1}{p_2} \\ \frac{1}{p_3} \\ \frac{1}{p_4} \\ \frac{1}{p_5} \\ \frac{1}{p_6} \end{pmatrix} = \begin{pmatrix} \frac{1}{p} + \frac{1}{q} - 1 \\ \frac{1}{p} + \frac{1}{r} - 1 \\ \frac{1}{s} \\ 1 - \frac{1}{p} \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \tag{3.49}$$

for all $\alpha, \beta \in \mathbb{R}$. In order to conclude, it suffices to suitably choose α and β so that $0 \leq \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \frac{1}{p_4}, \frac{1}{p_5}, \frac{1}{p_6} \leq 1$. Therefore, α and β need to satisfy

$$\begin{aligned} \max\left\{1 - \left(\frac{1}{p} + \frac{1}{r}\right), 0\right\} &\leq \alpha \leq \min\left\{2 - \left(\frac{1}{p} + \frac{1}{r}\right), 1\right\}, \\ \max\left\{1 - \left(\frac{1}{p} + \frac{1}{q}\right), 0\right\} &\leq \beta \leq \min\left\{2 - \left(\frac{1}{p} + \frac{1}{q}\right), 1\right\}, \\ \max\left\{-\frac{1}{p}, \frac{1}{s} - 1\right\} &\leq \alpha + \beta \leq \min\left\{\frac{1}{s}, 1 - \frac{1}{p}\right\} \end{aligned} \tag{3.50}$$

and one straightforwardly checks that this is always possible.

Finally, it is readily seen that we may always choose α and β so that $\alpha + \beta = \min\left\{\frac{1}{s}, 1 - \frac{1}{p}\right\}$. Since $\frac{1}{p_3} = \frac{1}{s} - (\alpha + \beta)$, this choice clearly maximizes the value of p_3 and $\alpha + \beta = \frac{1}{s}$ as soon as $\frac{1}{p} + \frac{1}{s} \leq 1$. This concludes the proof. □

3.3. Norm preserving estimates. We will now make use of the mechanisms of transfer of integrability exposed in the previous sections to demonstrate some norm preserving estimates on the operator $F \mapsto \mathcal{N}\mathcal{Q}^+(F, F)$ that are crucial to our work.

Lemma 3.8. *Let* $b(z, \sigma) \geq 0$ *be a collision kernel satisfying a decomposition* $b(z, \sigma) = a_0(|z|)\, b_0(\cos\theta)$*. Then, for any time* $T > 0$ *and any fixed parameter* $2 \leq a \leq \infty$*, the gain operator satisfies the quadratic estimate*

$$\begin{aligned} &\left\| \mathcal{Q}^+(F, G) \right\|_{L_t^{\frac{a}{2}}\left([0,T]; L_x^{\frac{D}{2}a'} L_v^{\frac{D}{2}a}\right)} \\ &\leq K \left\| F \right\|_{L_t^a\left([0,T]; L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)} \left\| G \right\|_{L_t^a\left([0,T]; L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)}, \end{aligned} \tag{3.51}$$

where the constant $K > 0$ *is defined by*

$$K = 2^{\frac{D}{2}} \left\| \frac{b_0(\cos\theta)}{\sin^{\frac{a-1}{a}} \frac{\theta}{2}} \right\|_{L_\sigma^1} \left\| a_0 \right\|_{L_z^{\frac{D}{D-2}}}. \tag{3.52}$$

Proof. A straightforward application of Proposition 3.6 yields

$$\left\| \mathcal{Q}^{+}(F,G) \right\|_{L_v^{\frac{D}{2}a}} \leq K \left\| F \right\|_{L_v^{D\frac{a}{a+1}}} \left\| G \right\|_{L_v^{D\frac{a}{a+1}}} . \tag{3.53}$$

The result then follows from the Cauchy-Schwarz inequality on the time and space variables only. □

Lemma 3.9. *Let $b(z,\sigma) \geq 0$ be a collision kernel satisfying a decomposition $b(z,\sigma) = a_0(|z|)\, b_0(\cos\theta)$. Then, for any time $T > 0$ and any fixed parameter $2 < a < 4$, the gain operator satisfies the quadratic estimate, valid for all $\frac{a}{a-2} \leq \lambda \leq \infty$,*

$$\begin{aligned} &\left\| \mathcal{N}\mathcal{Q}^{+}(F,G) \right\|_{L_t^{\lambda}\left([0,T]; L_x^{D\frac{\lambda}{\lambda-1}} L_v^{D\frac{\lambda}{\lambda+1}}\right)} \\ &\quad \leq K \left\| F \right\|_{L_t^{a}\left([0,T]; L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)} \left\| G \right\|_{L_t^{a}\left([0,T]; L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)} , \end{aligned} \tag{3.54}$$

where the constant $K > 0$ is defined by

$$K = C 2^{\frac{D}{2}} \left\| \frac{b_0(\cos\theta)}{\sin^{\frac{a-1}{a}} \frac{\theta}{2}} \right\|_{L_\sigma^1} \left\| a_0 \right\|_{L_z^{\frac{D}{D-2}}} , \tag{3.55}$$

for some constant $C > 0$ independent of $T > 0$.

Proof. First, by an application of Proposition 3.4 and denoting conjugate exponents by $\frac{1}{a} + \frac{1}{a'} = 1$, we obtain that

$$\begin{aligned} &\left\| \mathcal{N}\mathcal{Q}^{+}(F,G) \right\|_{L_t^{\frac{a}{a-2}}\left([0,T]; L_x^{\frac{D}{2}a} L_v^{\frac{D}{2}a'}\right)} \\ &\quad \leq C \left\| \mathcal{Q}^{+}(F,G) \right\|_{L_t^{\frac{a}{2}}\left([0,T]; L_x^{\frac{D}{2}a'} L_v^{\frac{D}{2}a}\right)} , \end{aligned} \tag{3.56}$$

where C is independent of $T > 0$. Notice that the use above of Proposition 3.4 is valid because the parameter a lies in the range $2 < a < 4$. Then, defining for convenience an auxiliary parameter $b = \frac{a}{a-2}$ so that $2 < b < \infty$, the estimate (3.56) becomes

$$\begin{aligned} &\left\| \mathcal{N}\mathcal{Q}^{+}(F,G) \right\|_{L_t^{b}\left([0,T]; L_x^{D\frac{b}{b-1}} L_v^{D\frac{b}{b+1}}\right)} \\ &\quad \leq C \left\| \mathcal{Q}^{+}(F,G) \right\|_{L_t^{\frac{a}{2}}\left([0,T]; L_x^{\frac{D}{2}a'} L_v^{\frac{D}{2}a}\right)} . \end{aligned} \tag{3.57}$$

Similarly, by an application of Proposition 3.5, we obtain that

$$\begin{aligned} &\left\| \mathcal{N}\mathcal{Q}^{+}(F,G) \right\|_{L_t^{\infty}\left([0,T]; L_{x,v}^{D}\right)} \\ &\quad \leq C \left\| \mathcal{Q}^{+}(F,G) \right\|_{L_t^{\frac{a}{2}}\left([0,T]; L_x^{\frac{D}{2}a'} L_v^{\frac{D}{2}a}\right)} , \end{aligned} \tag{3.58}$$

where $C > 0$ is independent of $T > 0$. Notice that the use above of Proposition 3.5 is valid because the parameter a lies in the range $2 \leq a < 4$. Furthermore, for any $b < \lambda < \infty$, we may apply Hölder's inequality to deduce

$$\begin{aligned}&\left\|\mathcal{N}\mathcal{Q}^+(F,G)\right\|_{L_t^\lambda\left([0,T];L_x^{D\frac{\lambda}{\lambda-1}}L_v^{D\frac{\lambda}{\lambda+1}}\right)}\\&\leq \left\|\mathcal{N}\mathcal{Q}^+(F,G)\right\|_{L_t^\infty([0,T];L_{x,v}^D)}^{1-\frac{b}{\lambda}}\left\|\mathcal{N}\mathcal{Q}^+(F,G)\right\|_{L_t^b\left([0,T];L_x^{D\frac{b}{b-1}}L_v^{D\frac{b}{b+1}}\right)}^{\frac{b}{\lambda}},\end{aligned}\tag{3.59}$$

so that, incorporating (3.57) and (3.58) into (3.59), we infer, for any $\frac{a}{a-2} \leq \lambda \leq \infty$,

$$\left\|\mathcal{N}\mathcal{Q}^+(F,G)\right\|_{L_t^\lambda\left([0,T];L_x^{D\frac{\lambda}{\lambda-1}}L_v^{D\frac{\lambda}{\lambda+1}}\right)} \leq C\left\|\mathcal{Q}^+(F,G)\right\|_{L_t^{\frac{a}{2}}\left([0,T];L_x^{\frac{D}{2}a'}L_v^{\frac{D}{2}a}\right)}.\tag{3.60}$$

Finally, the conclusion of the lemma follows from a direct application of Lemma 3.8 to the last term above. □

Lemma 3.10. *For any $F^{\text{in}} \in L_{x,v}^D\left(\mathbb{R}^D \times \mathbb{R}^D\right)$ and any $D < \lambda \leq \infty$, it holds that*

$$\left\|\mathcal{T}F^{\text{in}}\right\|_{L_t^\lambda\left([0,T];L_x^{D\frac{\lambda}{\lambda-1}}L_v^{D\frac{\lambda}{\lambda+1}}\right)} \leq C\|F^{\text{in}}\|_{L_{x,v}^D},\tag{3.61}$$

where $C > 0$ is independent of $T > 0$.

Proof. First, notice that it trivially holds that

$$\left\|\mathcal{T}F^{\text{in}}\right\|_{L_t^\infty([0,T];L_{x,v}^D)} = \|F^{\text{in}}\|_{L_{x,v}^D}.\tag{3.62}$$

Furthermore, since $\lambda > D$, a direct application of Proposition 3.3 yields that (3.61) holds, which concludes the proof of the lemma. □

4. Proof of the Main Theorem

In this section, we provide a proof of the Main Theorem 2.1. The key idea of the demonstration consists in utilizing the estimates developed in Sect. 3 on the Boltzmann operator to obtain the weak compactness of an approximating sequence of solutions to a truncated Boltzmann equation.

4.1. Weak compactness of truncated approximations. Many different choices for the truncated equation are available at this point and most of them would suit our demonstration. However, we will choose for convenience the truncated equation that was employed in the DiPerna-Lions theory of renormalized solutions [11]. We will detail now this truncation procedure.

We consider an approximating sequence of regularized and compactly supported collision kernels $\{b_n(z,\sigma)\}_{n=1}^{\infty} \subset C_0^{\infty}\left(\mathbb{R}^D \times \mathbb{S}^{D-1}\right)$ such that

$$b_n(z,\sigma) = b_n\left(|z|, \frac{z}{|z|}\cdot\sigma\right), \quad 0 \le b_n \le b_{n+1} \le b,$$
$$\text{and} \quad b_n \to b \quad \text{almost everywhere as } n \to \infty, \tag{4.1}$$

and a suitable approximating sequence of initial data

$$\left\{F_n^{\text{in}}(x,v)\right\}_{n=1}^{\infty} \subset \mathcal{S}\left(\mathbb{R}^D \times \mathbb{R}^D\right) \tag{4.2}$$

(here $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions) such that

$$0 \le F_n^{\text{in}} \le F^{\text{in}} \quad \text{and} \quad F_n^{\text{in}} \to F^{\text{in}} \quad \text{almost everywhere as } n \to \infty. \tag{4.3}$$

Furthermore, let $\delta_n > 0$ be such that $\lim_{n\to\infty} \delta_n \left\| F_n^{\text{in}} \right\|_{L^1_{x,v}} = 0$.

Notice that the general properties and estimates on the Boltzmann collision operator remain unchanged if one allows the collision kernel to depend on the time and space variables. Thus, it is possible to show, as was performed in [11] (see also [10]), that there exists a unique nonnegative sequence $\{F_n(t,x,v)\}_{n=1}^{\infty} \subset C\left([0,\infty); \mathcal{S}\left(\mathbb{R}^D \times \mathbb{R}^D\right)\right)$ of solutions to the truncated equation

$$\partial_t F_n + v\cdot\nabla_x F_n = \frac{\mathcal{Q}_n(F_n,F_n)}{1+\delta_n \int_{\mathbb{R}^D} |F_n| \, dv}, \qquad F_n(0,x,v) = F_n^{\text{in}}(x,v), \tag{4.4}$$

where the regularized Boltzmann operator $\mathcal{Q}_n$ is simply defined by replacing the collision kernel $b(z,\sigma)$ by its regularized version $b_n(z,\sigma)$ in Definition (1.2). In particular, by the collisional symmetries, the approximating solutions F_n satisfy the global conservation of mass $\int F_n(t,x,v)dxdv = \int F_n^{\text{in}}(x,v)dxdv$, for each $t \ge 0$, so that $\lim_{n\to\infty} \delta_n \|F_n(t)\|_{L^1_{x,v}} = 0$. Thus, up to extraction of a subsequence, we may assume that

$$\frac{1}{1+\delta_n \int_{\mathbb{R}^D} |F_n| \, dv} \to 1 \quad \text{almost everywhere in } (t,x) \in [0,\infty)\times\mathbb{R}^D. \tag{4.5}$$

We will now obtain important uniform estimates on the solutions F_n. Thus, according to Duhamel's formula (1.7), we have the following representation:

$$F_n = \mathcal{T} F_n^{\text{in}} + \mathcal{N} \frac{\mathcal{Q}_n(F_n,F_n)}{1+\delta_n \int_{\mathbb{R}^D} |F_n| \, dv}. \tag{4.6}$$

Consequently, for any fixed parameter $3 \le a < 4$, by virtue of Lemmas 3.9 and 3.10, it holds that, for any $\frac{a}{a-2} \le \lambda \le \infty$ such that $\lambda > D$,

$$\begin{aligned} &\left\| \mathcal{N} \frac{\mathcal{Q}_n^{+}(F_n,F_n)}{1+\delta_n \int_{\mathbb{R}^D} |F_n| \, dv} \right\|_{L_t^{\lambda}\left([0,T]; L_x^{D\frac{\lambda}{\lambda-1}} L_v^{D\frac{\lambda}{\lambda+1}}\right)} \\ &\le C \cdot K \, \|F_n\|^2_{L_t^{a}\left([0,T]; L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)} \end{aligned} \tag{4.7}$$

and

$$\left\| \mathcal{T} F_n^{\mathrm{in}} \right\|_{L_t^\lambda\left([0,T]; L_x^{D\frac{\lambda}{\lambda-1}} L_v^{D\frac{\lambda}{\lambda+1}}\right)} \leq C \| F^{\mathrm{in}} \|_{L^D_{x,v}}, \tag{4.8}$$

where $K > 0$ is determined by (3.55) and $C > 0$ is independent of $T > 0$. Thus, on the whole, we conclude that

$$\| F_n \|_{L_t^\lambda\left([0,T]; L_x^{D\frac{\lambda}{\lambda-1}} L_v^{D\frac{\lambda}{\lambda+1}}\right)} \leq C \| F^{\mathrm{in}} \|_{L^D_{x,v}} + C \cdot K \, \| F_n \|^2_{L_t^a\left([0,T]; L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)}. \tag{4.9}$$

In particular, since $a \geq 3$, the above estimate holds true for $\lambda = a$, so that defining the function $\rho_n(T) = \| F_n \|_{L_t^a\left([0,T]; L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)}$ which is continuous on $T \in [0, \infty)$ and satisfies $\rho(0) = 0$, we see that

$$0 \leq C \cdot K \rho_n(T)^2 - \rho_n(T) + C \| F^{\mathrm{in}} \|_{L^D_{x,v}}. \tag{4.10}$$

Provided $4C^2 K \| F^{\mathrm{in}} \|_{L^D_{x,v}} < 1$, which is guaranteed by the smallness condition (2.2) with an appropriate choice of constant C_0, it follows that $\rho_n(T) \in [0, \eta_1] \cup [\eta_2, \infty)$, for every $T > 0$, where $0 \leq \eta_1 < \eta_2$ are the two real roots of the quadratic equation $CK\eta^2 - \eta + C \| F^{\mathrm{in}} \|_{L^D_{x,v}}$ and may thus be expressed as

$$\eta_1 = \frac{1 - \sqrt{1 - 4C^2 K \| F^{\mathrm{in}} \|_{L^D_{x,v}}}}{2CK} \quad \text{and} \quad \eta_2 = \frac{1 + \sqrt{1 - 4C^2 K \| F^{\mathrm{in}} \|_{L^D_{x,v}}}}{2CK}. \tag{4.11}$$

Hence, by virtue of the continuity of $\rho_n(T)$, we infer that

$$\| F_n \|_{L_t^a\left([0,\infty); L_x^{D\frac{a}{a-1}} L_v^{D\frac{a}{a+1}}\right)} = \sup_{T>0} \rho_n(T) \leq \eta_1 = \frac{1 - \sqrt{1 - 4C^2 K \| F^{\mathrm{in}} \|_{L^D_{x,v}}}}{2CK}, \tag{4.12}$$

which yields, when incorporated into (4.9),

$$\| F_n \|_{L_t^\lambda\left([0,\infty); L_x^{D\frac{\lambda}{\lambda-1}} L_v^{D\frac{\lambda}{\lambda+1}}\right)} \leq \frac{1 - \sqrt{1 - 4C^2 K \| F^{\mathrm{in}} \|_{L^D_{x,v}}}}{2CK}, \tag{4.13}$$

for every $\frac{a}{a-2} \leq \lambda \leq \infty$ such that $\lambda > D$.

Consequently, by possibly extracting a subsequence and setting $\lambda_0 = \frac{a}{a-2}$ so that $2 < \lambda_0 \leq 3$, we find that

$$F_n \to F \quad \text{weakly in } L_t^\lambda\left([0,\infty); L_x^{D\frac{\lambda}{\lambda-1}} L_v^{D\frac{\lambda}{\lambda+1}}\right) \quad \text{for every } \lambda_0 \leq \lambda \leq \infty \text{ such that } \lambda > D. \tag{4.14}$$

We will prove that F is a weak solution to Boltzmann's equation. Finally, notice that an application of Lemma 3.8 shows that

$$\begin{aligned}&\mathcal{Q}_n^{+}(F_n, F_n) \text{ and } \mathcal{Q}^{+}(F, F) \text{ are uniformly bounded}\\ &\text{in } L_t^{\frac{\lambda}{2}}\left([0,\infty); L_x^{\frac{D}{2}\lambda'} L_v^{\frac{D}{2}\lambda}\right) \text{ for any } \lambda_0 \le \lambda \le \infty \text{ such that } \lambda > D.\end{aligned} \tag{4.15}$$

4.2. Strong compactness by velocity averaging. We wish now to pass to the limit in the truncated equation (4.4) and thus recover a weak solution of the Boltzmann equation (1.1). To this end, we need to show the convergence of the nonlinear terms in the right-hand side of (4.4),

$$\frac{\mathcal{Q}_n^{\pm}(F_n, F_n)}{1 + \delta_n \int_{\mathbb{R}^D} |F_n| \, dv} \to \mathcal{Q}^{\pm}(F, F) \quad \text{weakly in } L^1_{\mathrm{loc}}\left([0,\infty) \times \mathbb{R}^D \times \mathbb{R}^D\right). \tag{4.16}$$

Recall now that a bounded sequence u_n in $L^\infty\left(\mathbb{R}^D\right)$ converging almost everywhere to some u and a sequence v_n converging weakly to some v in $L^1\left(\mathbb{R}^D\right)$ satisfy the nonlinear convergence of the product $u_n v_n \to uv$ weakly in $L^1\left(\mathbb{R}^D\right)$. This result is a basic combination of Egorov's theorem with the Dunford-Pettis criterion for weak relative compactness in $L^1\left(\mathbb{R}^D\right)$. Essentially, we use the equi-integrability and the tightness of v_n to reduce the domain to a region where u_n converges uniformly towards u.

Thus, in view of the almost everywhere convergence of the denominators (4.5), the limit (4.16) will be verified as soon as we show that

$$\mathcal{Q}_n^{\pm}(F_n, F_n) \to \mathcal{Q}^{\pm}(F, F) \quad \text{weakly in } L^1_{\mathrm{loc}}\left([0,\infty) \times \mathbb{R}^D \times \mathbb{R}^D\right). \tag{4.17}$$

Furthermore, by virtue of the basic convolution inequalities (3.24), it holds that, for any $k \le n$,

$$\begin{aligned}&\left\| \mathcal{Q}_n^{\pm}(F_n, F_n) - \mathcal{Q}_k^{\pm}(F_n, F_n) \right\|_{L_t^{\frac{\alpha}{2}}\left([0,\infty); L_x^{\frac{D\alpha}{2(\alpha-1)}} L_v^{D\frac{\alpha}{\alpha+1}}\right)}\\
&\le \|F_n\|^2_{L_t^{\alpha}\left([0,\infty); L_x^{D\frac{\alpha}{\alpha-1}} L_v^{D\frac{\alpha}{\alpha+1}}\right)} \|b_n - b_k\|_{L_z^{\frac{D\alpha}{\alpha(D-1)-1}} L^1_\sigma}\\
&\le \|F_n\|^2_{L_t^{\alpha}\left([0,\infty); L_x^{D\frac{\alpha}{\alpha-1}} L_v^{D\frac{\alpha}{\alpha+1}}\right)} \|b - b_k\|_{L_z^{\frac{D\alpha}{\alpha(D-1)-1}} L^1_\sigma},\end{aligned} \tag{4.18}$$

where α is the parameter in the assumptions of the Main Theorem 2.1. Utilizing now that $b(z,\sigma) \in L_z^{\frac{D\alpha}{\alpha(D-1)-1}} L^1_\sigma$ together with the convergence properties of the approximating kernels (4.1), we see that the norm of the difference $b - b_k$ above can be made arbitrarily small if k is chosen large enough. Therefore, thanks to the control (4.14) with $\lambda = \alpha$, we deduce that it will be enough to show for a fixed k that, as n tends to infinity,

$$\mathcal{Q}_k^{\pm}(F_n, F_n) \to \mathcal{Q}_k^{\pm}(F, F) \quad \text{weakly in } L^1_{\mathrm{loc}}\left([0,\infty) \times \mathbb{R}^D \times \mathbb{R}^D\right), \tag{4.19}$$

which will be achieved by means of velocity averaging.

Notice first, that the estimate (4.18) also shows, if one mentally replaces b_k by zero, that the right-hand side of the truncated Eq. (4.4) is bounded locally in $L^1_{loc}\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right)$. This will allow us to apply the following basic velocity averaging lemma (see [8, 11]), which is not optimal but sufficient for our purpose.

Lemma 4.1. *Suppose that*

$$\begin{aligned}&\{F_n(t,x,v)\}_{n=1}^{\infty} \quad \textit{is weakly relatively compact in } L^1_{loc}\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right)\\ &\textit{and} \quad \{\partial_t F_n + v\cdot\nabla_x F_n\}_{n=1}^{\infty} \quad \textit{is bounded in } L^1_{loc}\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right). \quad (4.20)\end{aligned}$$

Then, for any $\psi(t,x,v_*,v)\in L^\infty\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D;L^1\left(\mathbb{R}^D\right)\right)$ such that $\psi(t,x,v_*,\cdot)$ is compactly supported,

$$\begin{aligned}&\left\{\int_{\mathbb{R}^D} F_n(t,x,v_*)\psi(t,x,v_*,v)\,dv_*\right\}_{n=1}^{\infty}\\ &\textit{is strongly relatively compact in } L^1_{loc}\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right). \quad (4.21)\end{aligned}$$

We apply now the above averaging lemma twice. First, with

$$\psi(t,x,v_*,v)=\varphi(t,x,v)\int_{\mathbb{S}^{D-1}} b_k(v-v_*,\sigma)\,d\sigma \quad (4.22)$$

and second with

$$\psi(t,x,v_*,v)=\int_{\mathbb{S}^{D-1}}\varphi(t,x,v')b_k\left(v-v_*,\sigma\right)d\sigma, \quad (4.23)$$

where $\varphi\in C_0^\infty\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right)$. Thus, in the first case, we conclude that

$$\begin{aligned}&\varphi\int_{\mathbb{R}^D} F_{n*}\int_{\mathbb{S}^{D-1}} b_k(v-v_*,\sigma)\,d\sigma dv_* \to \varphi\int_{\mathbb{R}^D} F_*\int_{\mathbb{S}^{D-1}} b_k(v-v_*,\sigma)\,d\sigma dv_*\\ &\quad\text{strongly in } L^1\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right), \quad (4.24)\end{aligned}$$

while, in the second case, we find that

$$\begin{aligned}&\int_{\mathbb{R}^D} F_{n*}\int_{\mathbb{S}^{D-1}}\varphi' b_k(v-v_*,\sigma)\,d\sigma dv_* \to \int_{\mathbb{R}^D} F_*\int_{\mathbb{S}^{D-1}}\varphi' b_k(v-v_*,\sigma)\,d\sigma dv_*\\ &\quad\text{strongly in } L^1\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right). \quad (4.25)\end{aligned}$$

Moreover, notice that the above sequences are compactly supported in all variables because the kernel b_k itself is compactly supported. This implies, when combined with the uniform bounds on the F_n's obtained from (4.14), that the convergences (4.24) and (4.25) hold in the strong topology of $L^2\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right)$ as well. Furthermore, the $L^\infty_t L^D_x L^D_v$ control obtained by setting $\lambda=\infty$ in (4.14) implies that the F_n's are weakly compact locally in $L^2_{loc}\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right)$. Consequently, as n tends to infinity, we see that, thanks to the collision symmetries,

$$\begin{aligned}&\int_{\mathbb{R}^D}\mathcal{Q}_k^-\left(F_n,F_n\right)\varphi\,dv=\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D}\varphi F_n F_{n*} b_k(v-v_*,\sigma)\,d\sigma dv_* dv\\ &\quad\longrightarrow\int_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D}\varphi F F_* b_k(v-v_*,\sigma)\,d\sigma dv_* dv=\int_{\mathbb{R}^D}\mathcal{Q}_k^-\left(F,F\right)\varphi\,dv, \quad (4.26)\end{aligned}$$

and that

$$\int_{\mathbb{R}^D} \mathcal{Q}_k^+(F_n, F_n)\,\varphi\, dv = \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \varphi' F_n F_{n*} b_k(v-v_*,\sigma)\, d\sigma dv_* dv$$
$$\longrightarrow \int_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \varphi' F F_* b_k(v-v_*,\sigma)\, d\sigma dv_* dv = \int_{\mathbb{R}^D} \mathcal{Q}_k^+(F,F)\,\varphi\, dv. \tag{4.27}$$

Since we already know that the loss and gain operators $\mathcal{Q}_k^{\pm}(F_n, F_n)$ form families that are weakly precompact in $L^1_{\mathrm{loc}}\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right)$ by the estimate (4.18), where we mentally replace b_n by zero, we conclude that the weak convergence (4.19) holds and so, that the weak convergence of the truncated operators (4.16) holds as well.

We are now ready to easily pass to the limit in the truncated Eq. (4.4). To this end, we consider any $\varphi(t,x,v) \in C_0^\infty\left([0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D\right)$ and, integrating Eq. (4.4) against φ, we infer

$$-\int_{\mathbb{R}^D\times\mathbb{R}^D} F_n^{\mathrm{in}}\varphi(0)\, dxdv - \int_{[0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D} F_n\partial_t\varphi\, dtdxdv$$
$$= \int_{[0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D} F_n v\cdot\nabla_x\varphi + \frac{\mathcal{Q}_n(F_n,F_n)}{1+\delta_n\int_{\mathbb{R}^D}|F_n|\, dv}\varphi\, dtdxdv. \tag{4.28}$$

Therefore, since F_n^{in} converges to F^{in} in $L^1_{\mathrm{loc}}\left(\mathrm{R}^D\times\mathbb{R}^D\right)$, letting n tend to infinity, we arrive at

$$-\int_{\mathbb{R}^D\times\mathbb{R}^D} F^{\mathrm{in}}\varphi(0)\, dxdv - \int_{[0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D} F\partial_t\varphi\, dtdxdv$$
$$= \int_{[0,\infty)\times\mathbb{R}^D\times\mathbb{R}^D} F v\cdot\nabla_x\varphi + \mathcal{Q}(F,F)\varphi\, dtdxdv, \tag{4.29}$$

which shows that F is a weak solution of the Boltzmann equation (1.1) and thus concludes the proof of Theorem 2.1.

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