Commun. Math. Phys. 301, 563–581 (2011)
Digital Object Identifier (DOI) 10.1007/s00220-010-1147-z

Communications in
Mathematical
Physics

Hölder Continuity of Absolutely Continuous Spectral Measures for One-Frequency Schrödinger Operators*

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Received: 18 December 2009 / Accepted: 7 April 2010
Published online: 2 November 2010 –

Abstract: We establish sharp results on the modulus of continuity of the distribution of the spectral measure for one-frequency Schrödinger operators with Diophantine frequencies in the region of absolutely continuous spectrum. More precisely, we establish 1/2-Hölder continuity near almost reducible energies (an essential support of absolutely continuous spectrum). For non-perturbatively small potentials (and for the almost Mathieu operator with subcritical coupling), our results apply for all energies.

1. Introduction

In this work we study absolutely continuous spectral measures of (one-frequency) quasiperiodic Schrödinger operators $H = H_{\lambda v,\alpha,\theta}$ defined on $\ell^2(\mathbb{Z})$,

$$(Hu)_n = u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha)u_n, \tag{1.1}$$

where v is the potential, $\lambda \in \mathbb{R}$ is the coupling constant, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is the frequency and $\theta \in \mathbb{R}$ is the phase. A central example is given by the almost Mathieu operator, when $v(x) = 2\cos(2\pi x)$.

Except where otherwise noted, below we assume the frequency α to be Diophantine in the usual sense (see definition in Sect. 3), and v analytic.

Absolutely continuous spectrum occurs only rarely in one-dimensional Schrödinger operators [R]. Until recently it was expected that in the class of ergodic Schrödinger operators it only occurs for almost periodic potentials, a conjecture recently disproved [A2]. Quasiperiodic operators with analytic potential stand out in this respect as the family $\{H_{\lambda v,\alpha,\theta}\}_{\lambda\in\mathbb{R}}$, for small couplings λ, is always in the metallic phase (has good transport properties) with zero Lyapunov exponents and absolutely continuous spectrum.

* This work was supported in part by NSF, grant DMS-0601081, and BSF, grant 2006483. This research was partially conducted during the period A.A. served as a Clay Research Fellow.

We will be concerned with the regularity of spectral measures. More precisely, given a function $f \in \ell^2(\mathbb{Z})$ with $\|f\| = 1$, and letting $\mu^f = \mu^f_{\lambda v,\alpha,\theta}$ be the associated spectral measure[1], what can be said of the modulus of continuity of the distribution of μ^f? We will assume that f is a reasonably localized function in the sense that $f \in \ell^1(\mathbb{Z})$ (notice that without regularity assumptions, there are no non-trivial restrictions on μ^f: any probability measure absolutely continuous with respect to some spectral measure is still a spectral measure). Our first result concerns small potentials:

Theorem 1.1. *For every $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, there exists $\lambda_0 = \lambda_0(v) > 0$ such that if $|\lambda| < \lambda_0$ and α is Diophantine, then $\mu^f_{\lambda v,\alpha,\theta}(J) \le C(\alpha, \lambda v)|J|^{1/2}\|f\|^2_{\ell^1}$, for all intervals J and all θ. For the almost Mathieu operator, one can take $\lambda_0 = 1$.*

Remark 1.1. The smallness constant λ_0 only depends on bounds on the analytic extension of v to some band $|\Im x| < \epsilon$. This is important for applications to arbitrary potentials (see below). The constant C depends on bounds on the analytic extension of λv and on the Diophantine properties of α.

Recall that averaging the distributions of spectral measures with respect to the phase θ yields the integrated density of states (i.d.s.), whose regularity is therefore significantly simpler to analyze. Indeed in [AJ], it is shown that the i.d.s. is 1/2-Hölder (and no more, see below) in the setting of Theorem 1.1. The averaged 1/2-Hölder behavior is compatible with point spectrum (consider the almost Mathieu operator with $\lambda > 1$, [J, AJ]), and hence discontinuous distributions. The key point of Theorem 1.1 is that here we are able to control the behavior of each individual spectral measure, uniformly on θ.

The study of small potentials is not merely interesting on its own: it gives information about the absolutely continuous spectrum of an arbitrary potential. To make this precise, one introduces the notion of *almost reducibility*: roughly speaking an energy is almost reducible if the associated cocycle $(\alpha, A^{(E-\lambda v)})$ (a dynamical system

$$(x, w) \mapsto (x + \alpha, A^{(E-\lambda v)} \cdot w), \tag{1.2}$$

$$A^{(E-\lambda v)} = \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.3}$$

that describes the behavior of solutions of the eigenvalue equation $H_{\lambda v,\alpha,\theta} u = Eu$), is analytically conjugate (in a uniform band) to the associated cocycle of some $(\alpha, A^{(E'-v')})$ with v' arbitrarily small. It follows from renormalization [AK1, AK2], that almost reducible energies (indeed *reducible* energies, for which v' can be taken as 0) form an essential support of absolutely continuous spectrum. In [AJ], almost reducibility was proved for all energies in the case of small potentials (indeed the same setting of Theorem 1.1), which implies that almost reducibility is *stable* (in particular, the set of almost reducible energies is open).

Theorem 1.2. *Let $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and let α be Diophantine. Then for any almost reducible energy $E \in \Sigma_{v,\alpha}$, (thus for a.e. energy in $\Sigma^{ac}_{v,\alpha}$) there exists $C, \epsilon > 0$ such that if $J \subset (E - \epsilon, E + \epsilon)$ is an interval then, for all θ, $\mu^f_{v,\alpha,\theta}(J) \le C|J|^{1/2}\|f\|_{\ell^1}$.*

As far as we know Theorems 1.2, 1.1 are the first results on fine properties of individual absolutely continuous spectral measures of ergodic operators.

[1] That is, $\mu^f(X) = \|\Pi_X(f)\|^2$ where $\Pi_X : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the spectral projection associated to the Borel set $X \subset \mathbb{R}$.

Let us call attention to the following conjecture that clarifies the fundamental importance of understanding almost reducibility:

Spectral Dichotomy Conjecture. For typical v, α, θ, $H_{v,\alpha,\theta}$ is the direct sum of operators H_+ and H_- with disjoint spectra such that H_+ is "localized" and H_- is "almost reducible".

Remark 1.2. (1) Typical should be understood in the measure-theoretical sense of *prevalence*. In particular frequencies may be assumed to be Diophantine.

(2) Localization for H_+ means both what is usually understood as Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) or dynamical localization. Almost reducibility for H_- just means that the spectrum of H_- is the closure of almost reducible energies for H, but as described above, it indeed provides a very fine spectral description: in particular the results of [AJ] and this paper apply to H_-, e.g., it has absolutely continuous spectral measures with $1/2$-Hölder distributions.

(3) By Kotani theory, see e.g. [LS], if the conjecture holds, then H_+/H_- must be defined by spectral projection on the parts of the spectrum where the associated cocycle has positive/zero Lyapunov exponent,

$$L(E) = \lim \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n^{(E-v)}(x)\| dx, \tag{1.4}$$

$$A_n^{(E-v)}(x) = A^{(E-v)}(x + (n-1)\alpha) \cdots A^{(E-v)}(x). \tag{1.5}$$

The result of disjointness of the spectra for this decomposition was recently established (in the typical setting) [A3,A4].

(4) With H_+ defined as above, the precise spectral and dynamical description, particularly dynamical localization, follows (in the typical setting) from a minoration of the Lyapunov exponent through the spectrum of H_+, using [BG] and [BJ3]. Such minoration is a consequence of disjointness of spectra [A3,A4] and continuity of the Lyapunov exponent [BJ1]. More is known in this regime [JL3,GS2,GS3].

(5) What is still incomplete in the above picture is the description of H_-. Zero Lyapunov exponent does not necessarily imply almost reducibility (consider the critical almost Mathieu operator). In [A3,A4], it is shown that (in the typical setting) energies in the spectrum of of H_- satisfy not only $L(E) = 0$ but the stronger condition (called *subcriticality*)

$$\ln \|A_n(x)\| = o(n) \tag{1.6}$$

uniformly in some band $|\Im x| < \epsilon$. The Spectral Dichotomy Conjecture is thus reduced to the Almost Reducibility Conjecture (the main outstanding problem in the theory): subcriticality implies almost reducibility.

1.1. Further perspective. One should distinguish between two possible regimes of small $|\lambda|$ (similar considerations can be applied to the analysis of large coupling). One is *perturbative*, meaning that the smallness condition on $|\lambda|$ depends not only on the potential v, but also on the frequency α: the key resulting limitation is that the analysis at a given coupling, however small, has to exclude a positive Lebesgue measure set of α. Such exclusions are inherent to the KAM-type methods that have been traditionally used in this context. The other, stronger regime, is called *non-perturbative*, meaning that the smallness condition on $|\lambda|$ only depends on the potential, leading to results that hold for almost every α.

A thorough study of absolutely continuous spectrum of operators (1.1) in the case of small analytic potentials in the perturbative regime was done by Eliasson [E]. He proved the reducibility of the associated cocycle for almost all energies in the spectrum and fine estimates on solutions for the other energies, by developing a sophisticated KAM scheme, which avoided the limitations of earlier KAM methods (that go back to the work of Dinaburg-Sinai [DiS] and that excluded parts of the spectrum from consideration). This allowed him in particular to conclude purely absolutely continuous spectrum.

A thorough study of absolutely continuous spectrum of operators (1.1) in the non-perturbative regime of [BJ2] was done in [AJ] where we used some techniques of [BJ2] to obtain localization estimates for all energies for the dual model, and developed quantitative Aubry duality theory, which allowed us, in particular, to conclude almost reducibility for all energies (including those for which neither dual localization nor reducibility hold). The smallness condition on the coupling constant in [AJ] coincides with that of [BJ2]. In particular, for the almost Mathieu operator, all the estimates and conclusions hold throughout the subcritical regime $\lambda < 1$.

The analyses of [E] and [AJ] allowed to obtain sharp bounds (Hölder-1/2 continuity) for the integrated density of states, for Diophantine frequencies. This was done, in perturbative and non-perturbative regimes in correspondingly [Am] and [AJ].

Earlier, Goldstein-Schlag [GS2] had shown Hölder continuity of the integrated density of states for a full Lebesgue measure subset of Diophantine frequencies in the regime of positive Lyapunov exponents, with the result becoming almost sharp for the super-critical almost Mathieu operator: $(1/2 - \epsilon)$-Hölder for any ϵ, and $|\lambda| > 1$. For this model their result also gives the same bound in the sub-critical regime $|\lambda| < 1$, by duality. Before that Bourgain [B1] had obtained almost 1/2-Hölder continuity for almost Mathieu type potentials in the perturbative regime, for Diophantine α and $\ln|\lambda|$ large (depending on α).

There were no results however, neither recently nor previously, on the modulus of continuity of the individual spectral measures, even in the perturbative regime. In this paper we achieve this by applying methods developed in [AJ] combined with a dynamical reformulation of the power-law subordinacy techniques of [JL2,JL3]. As mentioned above, our all energy results hold throughout the regime of [BJ2], and in particular, for all sub-critical almost-Mathieu operators. The general absolutely continuous case is obtained through a reduction to the small potential case and almost reducibility result of [AK1].

Our estimate is optimal in several ways. First, there are square-root singularities at the boundaries of gaps (e.g., [P2]), so the modulus of continuity cannot be improved. Also, since the integrated density of states satisfies

$$N(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu^{\sigma^k(f)}(0, E] \tag{1.7}$$

($\sigma : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denotes the shift), the spectral measures of ℓ^1 functions cannot have higher modulus of continuity than $N(E)$. There are examples with lower regularity of $N(E)$ that demonstrate that Diophantine condition on α as well as a condition on λ are essential here. In particular, it is known that for the almost Mathieu operator for a certain non-empty set of α which satisfy good Diophantine properties (but has zero Lebesgue measure) and $\lambda = 1$, the integrated density of states is not Hölder ([B3], Remark after

Corollary 8.6). Additionally, for any $\lambda \neq 0$ and generic α, the integrated density of states is not Hölder (this is because the Lyapunov exponent is discontinuous at rational α, which easily implies that it is not Hölder for generic α. Such discontinuity holds for the almost Mathieu operator and presumably generically).

Remark 1.3. As our approach is non-perturbative and non-KAM, it is not expected to break down at the Brjuno condition and can potentially be extended much further. While, as mentioned above, the exact modulus of continuity should depend on the Diophantine properties for very well approximated α we expect the same methods to work for small rate of exponential approximation as well. We do not pursue it here though.

2. Preliminaries

For a bounded analytic (perhaps matrix valued) function f defined on a strip $\{|\Im z| < \epsilon\}$ and extending continuously to the boundary, we let $\|f\|_\epsilon = \sup_{|\Im z|<\epsilon} |f(z)|$. If f is a bounded continuous function on $\mathbb{R}$, we let $\|f\|_0 = \sup_{x\in\mathbb{R}} |f(x)|$.

2.1. Cocycles. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$. We call (α, A) a *(complex) cocycle*. The *Lyapunov exponent* is given by the formula

$$L(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int \ln \|A_n(x)\| dx, \tag{2.1}$$

where A_n, $n \in \mathbb{Z}$, is defined by $(\alpha, A)^n = (n\alpha, A_n)$, so that for $n \geq 0$,

$$A_n(x) = A(x + (n-1)\alpha) \cdots A(x). \tag{2.2}$$

We say that (α, A) is *uniformly hyperbolic* if there exists a continuous splitting $\mathbb{C}^2 = E^s(x) \oplus E^u(x)$, $x \in \mathbb{R}/\mathbb{Z}$ such that for some $C > 0$, $c > 0$, and for every $n \geq 0$, $\|A_n(x) \cdot w\| \leq Ce^{-cn}\|w\|$, $w \in E^s(x)$ and $\|A_{-n}(x) \cdot w\| \leq Ce^{-cn}\|w\|$, $w \in E^u(x)$. In this case, of course $L(\alpha, A) > 0$.

Given two cocycles $(\alpha, A^{(1)})$ and $(\alpha, A^{(2)})$, a (*complex*) *conjugacy* between them is a continuous $B : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ such that

$$A^{(2)}(x) = B(x+\alpha)A^{(1)}(x)B(x)^{-1}. \tag{2.3}$$

We assume now that (α, A) is a *real* cocycle, that is, $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. The notion of real conjugacy (between real cocycles) is the same as before, except that we ask for $B \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$. Real conjugacies still preserve the Lyapunov exponent.

We say that a (real) cocycle (α, A) is analytically *reducible* if it is (real) conjugate to a constant cocycle, and the conjugacy is analytic. We say that it is *almost reducible* if there exists a sequence $A^{(n)} \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ converging (uniformly in some band $\{|\Im z| < \epsilon\}$) to a constant, such that $(\alpha, A^{(n)})$ is conjugated to (α, A), and the conjugacies extend holomorphically to some fixed band: $B^{(n)} \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2,\mathbb{R}))$.[2]

2.2. Schrödinger operators. We consider now Schrödinger operators $\{H_{v,\alpha,\theta}\}_{\theta\in\mathbb{R}}$ (we incorporate the coupling constant into v). The spectrum $\Sigma = \Sigma_{v,\alpha}$ does not depend on

[2] In fact this last property is automatic, but this is non-trivial (it follows from the openness of almost reducibility).

θ, and it is the set of E such that $(\alpha, A^{(E-v)})$ is not uniformly hyperbolic, with $A^{(E-v)}$ as in the Introduction.

For $f \in l^2(\mathbb{Z})$ the spectral measure $\mu = \mu_x^f$ is defined so that

$$\langle (H_x - E)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{E' - E} d\mu(E') \tag{2.4}$$

holds for E in the resolvent set $\mathbb{C} \setminus \Sigma$. Alternatively, for a Borel set X,

$$\mu_{v,\alpha,\theta}^f(X) = \|\Pi_X f\|^2, \tag{2.5}$$

where Π_X is the corresponding spectral projection.

The integrated density of states is the function $N : \mathbb{R} \to [0, 1]$ that can be defined by (1.7). It is a continuous non-decreasing surjective (for bounded potentials) function. The Thouless formula relates the Lyapunov exponent to the integrated density of states,

$$L(E) = \int_{\mathbb{R}} \ln |E' - E| dN(E'). \tag{2.6}$$

2.3. Almost reducibility and the support of absolutely continuous spectrum. We justify the claim made in the Introduction that almost reducible energies support the absolutely continuous part of the spectral measures.

By [LS], the set $\Sigma_0 = \{L(E) = 0\}$ (which is closed by [BJ1]) is the essential support of ac spectrum, so that the ac spectral measures are precisely those probability measures on Σ_0 which are equivalent to Lebesgue.

Theorem 2.1 ([AFK]). *If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then for almost every $E \in \Sigma_0$, $(\alpha, A^{(E-v)})$ is real analytically conjugated to a cocycle of rotations, i.e., taking values in* $\mathrm{SO}(2, \mathbb{R})$.

This result was proved in [AK1] under a full measure condition on α (which is stronger than Diophantine). For Diophantine α, it can also be obtained as a consequence of [AJ] and [AK2].

It is easy to see that for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, analytic cocycles of rotations are almost reducible. Moreover, if α is Diophantine (more generally, if the best rational approximations to α are subexponential), then analytic cocycles of rotations are reducible.

2.4. Almost reducibility in Schrödinger form. While almost reducibility allows one to conjugate the dynamics of the cocycle close to a constant, it is rather convenient to have the conjugated cocycle in Schrödinger form, since many results (particularly the ones depending on Aubry duality, as the ones obtained in [AJ]) are obtained only in this setting. The following result takes care of this.

Lemma 2.2. *Let $(\alpha, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ be almost reducible. Then there exists $\epsilon_0 > 0$ such that for every $\gamma > 0$, there exists $v \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ with $\|v\|_{\epsilon_0} < \gamma$, $E \in \mathbb{R}$ and $B \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2, \mathbb{R}))$ such that $B(x + \alpha)A(x)B(x)^{-1} = A^{(E-v)}(x)$. Moreover, for every $0 < \epsilon \leq \epsilon_0$, there exists $\delta > 0$ such that if $\tilde{A} \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ is such that $\|\tilde{A} - A\|_\epsilon < \delta$ then there exists $\tilde{v} \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, such that $\|\tilde{v}\|_\epsilon < \gamma$ and $\tilde{B} \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2, \mathbb{R}))$ such that $\|\tilde{B} - B\| < \gamma$ and $\tilde{B}(x + \alpha)\tilde{A}(x)\tilde{B}(x)^{-1} = A^{(E-\tilde{v})}(x)$.*

For the proof, one basically just needs to be able to convert non-Schrödinger perturbations of Schrödinger cocycles to Schrödinger form. This problem is studied in [A4]. For completeness, we will give a much simpler (unpublished) argument of Avila-Krikorian which is enough for our purposes.

Lemma 2.3. *Let $v \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, be such that $1/v \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. If $A \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ and $\|A - A^{(v)}\|_\epsilon$ is sufficiently small (depending on $\|v\|_\epsilon$ and $\|1/v\|_\epsilon$), then there exists $v' \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $B \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$ such that $\|v' - v\|_\epsilon$ and $\|B - \mathrm{id}\,\|_\epsilon$ are small and $B(x+\alpha)A(x)B(x)^{-1} = A^{(v')}(x)$.*

Proof. Let $w = \begin{pmatrix} w_1 & w_2 \\ w_3 & -w_1 \end{pmatrix} \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{sl}(2, \mathbb{R}))$ be such that $\|w\|_\epsilon$ is small and $A = A^{(v)}e^w$. Let $s = \begin{pmatrix} s_1 & s_2 \\ s_3 & -s_1 \end{pmatrix} \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathrm{sl}(2, \mathbb{R}))$ be defined by $s_1 = 0$, $s_2(x) = w_2(x) + \frac{w_1(x)}{v(x)}$, $s_3(x) = -\frac{w_1(x-\alpha)}{v(x-\alpha)}$, and let $\tilde{v} \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ be given by

$$\tilde{v}(x) = v(x) - w_3(x) + w_2(x+\alpha) + \frac{w_1(x+\alpha)}{v(x+\alpha)} + v(x)w_1(x) - \frac{w_1(x-\alpha)}{v(x-\alpha)}.$$

Then $\|\tilde{v} - v\|_\epsilon \le C\|w\|_\epsilon$ and $e^{s(x+\alpha)}A(x)e^{-s(x)}$ is of the form $A^{(\tilde{v})}e^{\tilde{w}}$, where $\|\tilde{w}\|_\epsilon \le C\|w\|_\epsilon^2$, for some constant C depending on $\|v\|_\epsilon$ and $\|1/v\|_\epsilon$. The result follows by iteration. □

Remark 2.1. (1) This result with only assuming $v \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ to be non-identically zero is proved in [A4].
(2) For Diophantine α the result holds with no conditions on $v \in C^\omega_\epsilon(\mathbb{R}/\mathbb{Z}, \mathbb{R})$.

Proof of Lemma 2.2. Since (α, A) is almost reducible, if $\epsilon_0 > 0$ is small then there exists a sequence $B^{(n)} \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathrm{PSL}(2, \mathbb{R}))$ and $A_* \in \mathrm{SL}(2, \mathbb{R})$ such that $\|B^{(n)}(x+\alpha)A(x)B^{(n)}(x)^{-1} - A_*\|_{\epsilon_0} \to 0$. Let us show that, up to changing $B^{(n)}$ to $C^{(n)}B^{(n)}$ for an appropriate choice of $C^{(n)} \in C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2, \mathbb{R}))$, we may assume that A_* is of the form $\begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$ with $E \neq 0$. Indeed:

(1) If $|\mathrm{tr}A_*| > 2$, by converting first to the diagonal form, we can find $\tilde{C}_{A_*} \in \mathrm{SL}(2, \mathbb{R})$ such that $\tilde{C}_{A_*}A_*\tilde{C}_{A_*}^{-1} = \begin{pmatrix} \mathrm{tr}A_* & -1 \\ 1 & 0 \end{pmatrix}$, so we can just take $C^{(n)} = C_{A_*}$.
(2) If $|\mathrm{tr}A_*| < 2$, there exists $\tilde{C}_{A_*} \in \mathrm{SL}(2, \mathbb{R})$ such that $\tilde{C}_{A_*}A_*\tilde{C}_{A_*}^{-1} = R_\theta$ for some $\theta \neq k/2$, $k \in \mathbb{Z}$. If $0 < \sin 2\pi\theta < 1$, then let

$$C_\theta^{-1} = \frac{1}{(\sin 2\pi\theta)^{1/2}} \begin{pmatrix} 0 & -\sin 2\pi\theta \\ 1 & -\cos 2\pi\theta \end{pmatrix},$$

so that $C_\theta R_\theta C_\theta^{-1} = \begin{pmatrix} 2\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}$, and we can take $C^{(n)} = C_\theta\tilde{C}_{A_*}$. Otherwise, let $k \in \mathbb{Z}$ be such that $0 < \sin 2\pi(\theta + k\alpha) < 1$, and take $C^{(n)}(x) = C_{\theta+k\alpha}R_{kx}\tilde{C}_{A_*}$.

(3) If $|\mathrm{tr} A_*| = 2$, there exists $\tilde{C}^{(n)} \in \mathrm{SL}(2,\mathbb{R})$ such that $\tilde{C}^{(n)} A_* (\tilde{C}^{(n)})^{-1} \to R_\theta$, where $\theta = 0$ or $\theta = 1/2$. Indeed, either A_* is equal to such R_θ or we can assume it is in the Jordan form, in which case one can take

$$\tilde{C}^{(n)} = \begin{pmatrix} \epsilon_n & 0 \\ \epsilon_n & \frac{1}{\epsilon n} \end{pmatrix}.$$

By choosing ϵ_n appropriately, we may also assume that $\|\tilde{C}^{(n)}\|^2 \|B^{(n)}(x+\alpha)A(x)B^{(n)}(x)^{-1} - A_*\|_{\epsilon_0} \to 0$. Choosing again $k \in \mathbb{Z}$ such that $0 < \sin 2\pi(\theta + k\alpha) < 1$ we can take $C^{(n)} = C_{\theta+k\alpha} R_{kx} \tilde{C}^{(n)}$.

Now the first statement follows from Lemma 2.3. For the second statement, apply again Lemma 2.3. □

3. Estimates on the Dynamics

Here we describe the [AJ] estimates on the dynamics of almost reducible cocycles.

3.1. Rational approximations. Let q_n be the denominators of the approximants of α. We recall the basic properties:

$$\|q_n \alpha\|_{\mathbb{R}/\mathbb{Z}} = \inf_{1 \le k \le q_{n+1}-1} \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}, \tag{3.1}$$

$$1 \ge q_{n+1} \|q_n \alpha\|_{\mathbb{R}/\mathbb{Z}} \ge 1/2. \tag{3.2}$$

We say that α is Diophantine if $\frac{\ln q_{n+1}}{\ln q_n} = O(1)$. Let $\mathrm{DC} \subset \mathbb{R}$ be the set of Diophantine numbers.

3.2. Resonances. Let $\alpha \in \mathbb{R}$, $\theta \in \mathbb{R}$, $\epsilon_0 > 0$. We say that k is an ϵ_0-resonance if

$$\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \le e^{-|k|\epsilon_0}$$

and

$$\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|j| \le |k|} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

Remark 3.1. In particular, there always exists at least one resonance, 0. If $\alpha \in \mathrm{DC}(\kappa, \tau)$, $\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} \le e^{-|k|\epsilon_0}$ implies

$$\|2\theta - k\alpha\|_{\mathbb{R}/\mathbb{Z}} = \min_{|j| \le |k|} \|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}}$$

for $k > C(\kappa, \tau)$.

For fixed α and θ, we order the ϵ_0-resonances $0 = n_0 < |n_1| \le |n_2| \le \cdots$. We say that θ is ϵ_0-resonant if the set of resonances is infinite. If θ is non-resonant, with the set of resonances $\{n_0, \ldots, n_j\}$ we formally set $n_{j+1} = \infty$. The Diophantine condition immediately implies exponential repulsion of resonances:

Lemma 3.1. *If $\alpha \in \mathrm{DC}$, then*

$$|n_{j+1}| \ge c\|2\theta - n_j \alpha\|_{\mathbb{R}/\mathbb{Z}}^{-c} \ge c e^{c\epsilon_0 |n_j|},$$

where $c = c(\alpha, \epsilon_0) > 0$.

3.3. Dynamical estimates. Let us say that a cocycle $(\alpha, A) \in \mathbb{R}\backslash\mathbb{Q}\times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is (C, c, ϵ_0)-good if there exists $\theta \in \mathbb{R}$ with the following property: for any finite ϵ_0-resonance n_j associated to α and θ, denoting $n = |n_j| + 1$ and $N = |n_{j+1}|$, there exists $\Phi : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2,\mathbb{C})$ analytic with $\|\Phi\|_{cn^{-C}} \le Cn^C$ such that

$$\Phi(x+\alpha)A(x)\Phi(x)^{-1} = \begin{pmatrix} e^{2\pi i\theta} & 0 \\ 0 & e^{-2\pi i\theta} \end{pmatrix} + \begin{pmatrix} q_1(x) & q(x) \\ q_3(x) & q_4(x) \end{pmatrix}, \tag{3.3}$$

with

$$\|q_1\|_{cn^{-C}}, \|q_3\|_{cn^{-C}}, \|q_4\|_{cn^{-C}} \le Ce^{-cN} \tag{3.4}$$

and

$$\|q\|_{cn^{-C}} \le Ce^{-cn(\ln(1+n))^{-C}}. \tag{3.5}$$

The following is one of the main estimates of [AJ] (combining Theorems 3.3, 3.4 and 5.1 of [AJ]):

Theorem 3.2 [see Theorems 3.4 and 5.1 of [AJ]]. *There exists a constant $c_0 > 0$ with the following property. Let $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $E \in \Sigma_{v,\alpha}$, (α, A). If for some $0 < \epsilon < 1$, $\|v\|_\epsilon < c_0\epsilon^3$ then (α, A) is (C, c, ϵ_0)-good for some constants $c = c(\epsilon, \alpha) > 0$, $C = C(\epsilon, \alpha) > 0$ and $\epsilon_0 = \epsilon_0(\epsilon)$.*

A more precise result is available for the almost Mathieu operator (still a combination of Theorems 3.4 and 5.1 of [AJ]):

Theorem 3.3 [see Theorems 3.4 and 5.1 of [AJ]]. *For every $0 < \lambda_0 < 1$ and α Diophantine, there exists $C = C(\lambda_0, \alpha)$, $c = c(\lambda_0, \alpha)$, $\epsilon_0 = \epsilon_0(\lambda_0) > 0$ such that for $v = 2\lambda \cos 2\pi(x+\theta)$ with $|\lambda| < \lambda_0$, and $E \in \Sigma_{v,\alpha}$, $(\alpha, A^{(E-v)})$ is (C, c, ϵ_0)-good.*

Coupling Theorem 3.2 and Lemma 2.2 we immediately get:

Theorem 3.4. *Let α be Diophantine and let $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$. If (α, A) is almost reducible, then there exists $\bar\epsilon > 0$ such that for every $0 < \epsilon < \bar\epsilon$ there exist $\delta, C, c, \epsilon_0 > 0$ such that if $\tilde A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(2,\mathbb{R}))$ is such that $\|\tilde A - A\|_\epsilon < \delta$ and $(\alpha, \tilde A)$ is not uniformly hyperbolic then $(\alpha, \tilde A)$ is (C, c, ϵ_0)-good.*

Remark 3.2. Using [A1], Theorem 3.8, one can consider a stronger definition of goodness, so that Theorem 3.2, and hence Theorem 3.4, and Theorem 3.3, still hold: $\|\Phi\|_c \le Cn^C$, $\|q_j\|_c \le Ce^{-cN}$, $j = 1, 3, 4$, and $\|q\|_c \le Ce^{-cn}$.

An immediate consequence of (C, c, ϵ_0)-goodness is (see [AJ] for the easy argument):

Lemma 3.5. *If (α, A) is (C, c, ϵ_0)-good, then for every $s \ge 0$ we have $\|A_s\|_0 \le C'(C, c, \epsilon_0, \alpha)(1+s)$.*

4. Regularity of the Spectral Measures at Good Energies

Let $v \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $E \in \Sigma_{v,\alpha}$. Let $\mu_x = \mu^{e_{-1}}_{v,\alpha,x} + \mu^{e_0}_{v,\alpha,x}$ and e_i is the Dirac mass at $i \in \mathbb{Z}$.

Our main estimate is:

Theorem 4.1. *If* $(\alpha, A^{(E-v)})$ *is* (C_0, c_0, ϵ_0)*-good, then for every* $0 < \epsilon < 1, \mu_x(E-\epsilon, E+\epsilon) \leq C'(C_0, c_0, \epsilon_0, \alpha)\epsilon^{1/2}$.

Proof of Theorems 1.1 and 1.2. We first prove Theorem 1.2. By Theorem 3.4, $(\alpha, A^{(E'-v)})$ is (C_0, c_0, ϵ_0)-good for any E' near E which is in the spectrum. By Theorem 4.1, we get

$$\mu_x(J) \leq C'|J|^{1/2} \tag{4.1}$$

for any interval containing such an E', and hence (since μ_x is supported on the spectrum), for any interval contained in a neighborhood of E. Let $\sigma : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ be the shift $f(i+1) = \sigma f(i)$. Then $\sigma H_{v,\alpha,x}\sigma^{-1} = H_{v,\alpha,x+\alpha}$. Thus $\mu^{\sigma f}_{x+\alpha} = \mu^f_x$ and $\mu^{e_k}_x = \mu^{e_0}_{x+k\alpha} \leq \mu_{x+k\alpha}$. By (2.5), $\mu^f_x(E-\epsilon, E+\epsilon)^{1/2}$ defines a semi-norm on $l^2(\mathbb{Z})$. Therefore, by the triangle inequality, $\mu^f_x(J)^{1/2} \leq \sum_{k\in\mathbb{Z}} |f(k)|(\mu_{x+k\alpha}(J))^{1/2}$, and the result follows immediately from (4.1).

Theorem 1.1 is proved analogously, using Theorems 3.2 and 3.3 to establish appropriate (C, c)-goodness. □

It therefore remains to prove Theorem 4.1 which we do in Sect. 4.2.

Through the end of this section, $A = A^{(E-v)}$. We will use C and c for large and small constants that only depend on C_0, c_0, ϵ_0, and α.

4.1. Spectral measures and m-functions. In the study of $\mu = \mu_x$, we will use a result of [JL2] (or its improvement in [KKL]), interpreted in terms of cocycles. In the definition of the m-functions below, we follow the notation of [JL3].

We will consider energies $E + i\epsilon$, $E \in \mathbb{R}$, $\epsilon > 0$. Then there are non-zero solutions $u^\pm$ of $Hu^\pm = (E+i\epsilon)u^\pm$ which are l^2 at $\pm\infty$, well defined up to normalization. We define

$$m^\pm = \mp \frac{u^\pm_1}{u^\pm_0}. \tag{4.2}$$

It coincides with the Weyl-Titchmarsh m-function which is the Borel transform of the spectral measure $\mu^\pm = \mu^\pm_{e_0}$ of the corresponding half-line problem with Dirichlet boundary conditions:

$$m^\pm(z) = \int \frac{d\mu^\pm(x)}{x - z},$$

(e.g. [CL]).

Thus $m^\pm$ has positive imaginary part for every $\epsilon > 0$.

Let

$$M(E+i\epsilon) = \int \frac{1}{E' - (E+i\epsilon)} d\mu(E'). \tag{4.3}$$

Notice that $M(E+i\epsilon) \in \mathbb{H} = \{z,\ \Im z > 0\}$. We have

$$\Im M(E+i\epsilon) \geq \frac{1}{2\epsilon}\mu(E-\epsilon, E+\epsilon). \tag{4.4}$$

Then, as discussed in [JL3],

$$M = \frac{m^+m^- - 1}{m^+ + m^-}. \tag{4.5}$$

As in [JL3], we define $m_\beta^+ = R_{-\beta/2\pi} \cdot m^+$, or, more generally,

$$z_\beta = R_{-\beta/2\pi} z.$$

Those are Borel transforms of the half-line spectral measures $\mu^\beta = \mu_{e_0}^\beta$ of operator H on $l^2([0,\infty))$ with boundary conditions $u_0 \cos\beta + u_1 \sin\beta = 0$. Here we make use of the action of $\mathrm{SL}(2,\mathbb{C})$ on $\overline{\mathbb{C}}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Let $\psi(z) = \sup_\beta |z_\beta|$. We have

$$\psi(z)^{-1} \leq \Im z \leq |z| \leq \psi(z), \tag{4.6}$$

where the first inequality easily follows from the invariance of ϕ, see below. It was shown in [DKL] that, as a corollary of the maximal modulus principle, one obtains

$$|M| \leq \psi(m^+). \tag{4.7}$$

This also can be shown directly by the following computation, that gives some more quantitative estimates. Let

$$\phi(z) = \frac{1+|z|^2}{2\Im z}.$$

If $z \in \mathbb{H}$ then $\phi(z) \geq 1$. $\phi(z)$ is invariant with respect to the action of R_β. Thus the maximum of $|z_\beta|$ is attained when z_β is purely imaginary with $\Im z_\beta > 1$ and is easily checked to be equal to $\phi(z) + (\phi(z)^2 - 1)^{1/2}$. Thus

$$\psi(z) = \phi(z) + (\phi(z)^2 - 1)^{1/2}.$$

We can compute

$$\phi(M) = \frac{\phi(m^+)\phi(m^-) + 1}{\phi(m^+) + \phi(m^-)}, \tag{4.8}$$

which implies $\phi(M) \leq \phi(m^+)$ and hence

$$\psi(M) \leq \psi(m^+) \tag{4.9}$$

(whatever the value of $m^- \in \mathbb{H}$). By (4.6), this gives (4.7).

For $k \geq 1$ integer, let

$$P_{(k)} = \sum_{j=1}^{k} A_{2j-1}^{*}(x+\alpha)A_{2j-1}(x+\alpha). \tag{4.10}$$

Then $P_{(k)}$ is an increasing family of positive self-adjoint linear maps. In particular, $\|P_{(k)}\|$, $\frac{\det P_{(k)}}{\|P_{(k)}\|}$ and $\det P_{(k)}$ are increasing positive functions. It is not difficult to see that $\|P_{(k)}\|$ (and hence $\det P_{(k)}$) is also unbounded (since $A_j \in \mathrm{SL}(2,\mathbb{R})$ implies $\mathrm{tr} P_{(k)} \geq 2k$).

Lemma 4.2. *Let* ϵ *be such that* $\det P_{(k)} = \frac{1}{4\epsilon^2}$. *Then*

$$C^{-1} < \frac{\psi(m^{+}(E+i\epsilon))}{2\epsilon\|P_{(k)}\|} < C. \tag{4.11}$$

Proof. Let $(u_j^{\beta})_{j\geq 0}$ satisfy

$$A(x+j\alpha)\cdot\begin{pmatrix} u_j^{\beta} \\ u_{j-1}^{\beta}\end{pmatrix} = \begin{pmatrix} u_{j+1}^{\beta} \\ u_j^{\beta}\end{pmatrix}, \tag{4.12}$$
$$u_0^{\beta}\cos\beta + u_1^{\beta}\sin\beta = 0, \quad |u_0^{\beta}|^2 + |u_1^{\beta}|^2 = 1.$$

For integer L define

$$\|u\|_L = \left(\sum_{j=1}^{L}|u_j|^2\right)^{1/2}. \tag{4.13}$$

Unlike [JL2] it will be sufficient for us here to deal with the “discrete” definition (4.13) of $\|u\|_L$, because we only deal with bounded potentials, see proof of Corollary 4.7.

Theorem 1.1 of [JL2] can be stated as follows (see (2.13) in [JL3]). If

$$\|u^{\beta}\|_L\|u^{\beta+\pi/2}\|_L = \frac{1}{2\epsilon},$$

then

$$5-\sqrt{24} < |m_{\beta}^{+}(E+i\epsilon)|\frac{\|u^{\beta}\|_L}{\|u^{\beta+\pi/2}\|_L} < 5+\sqrt{24}. \tag{4.14}$$

In other words,

$$5-\sqrt{24} < \frac{|m_{\beta}^{+}(E+i\epsilon)|}{2\epsilon\|u^{\beta+\pi/2}\|_L^2} < 5+\sqrt{24}. \tag{4.15}$$

It is immediate to see that if $L = 2k$ then

$$\|u^{\beta}\|_L^2 = \langle P_{(k)}\begin{pmatrix} u_1^{\beta} \\ u_0^{\beta}\end{pmatrix}, \begin{pmatrix} u_1^{\beta} \\ u_0^{\beta}\end{pmatrix}\rangle \leq \|P_{(k)}\|, \tag{4.16}$$

with equality for β maximizing $\|u^\beta\|_L^2$. Thus

$$\det P_{(k)} = \inf_\beta \|u^\beta\|_L^2 \|u^{\beta+\pi/2}\|_L^2, \tag{4.17}$$

the infimum being attained at the critical points of $\beta \mapsto \|u^\beta\|_L^2$. We conclude that if $\|\det P_{(k)}\| = \frac{1}{4\epsilon^2}$, then for every β,

$$\frac{|m_\beta^+(E+i\epsilon)|}{2\epsilon\|P_{(k)}\|} < 5 + \sqrt{24},$$

and if β is such that $\|u^{\beta+\pi/2}\|_L^2$ is maximal then

$$\frac{|m_\beta^+(E+i\epsilon)|}{2\epsilon\|P_{(k)}\|} > 5 - \sqrt{24}.$$

This together gives (4.11) with $C = 5 + \sqrt{24}$. □

Remark 4.1. Replacing (4.14) with a result of [KKL] we can obtain by the same argument that if ϵ is such that $\det P_{(k)} = \frac{1}{\epsilon^2}$, then

$$2 - \sqrt{3} < \frac{\psi(m^+(E+i\epsilon))}{\epsilon\|P_{(k)}\|} < 2 + \sqrt{3}. \tag{4.18}$$

We note that $\det P_{(k)}$ is precisely the Hilbert-Schmidt norm of operator K in [KKL] at scale L. The exact value of C in (4.11) is not important for the present argument.

4.2. Proof of Theorem 4.1. We need to estimate $\|P_{(k)}\|$ and $\det P_{(k)} = \|P_{(k)}\|\|P_{(k)}^{-1}\|^{-1}$. More precisely, we will show that $\|P_{(k)}\| \le C\|P_{(k)}^{-1}\|^{-3}$, which is just enough for our purposes.

Remark 4.2. By Lemma 3.5, we have $\|P_{(k)}\| \le Ck^3$, so to estimate $\|P_{(k)}\| \le C\|P_{(k)}^{-1}\|^{-3}$ it would be enough to show that $\|P_k^{-1}\| \le Ck^{-1}$. We do not know whether the latter estimate holds.

A key point will be to compare the dynamics of (α, A) with the dynamics of (α, T), where T is in some particularly simple triangular form. Below, the notation $a \approx b$ $(a, b > 0)$ denotes $C^{-1}a \le b \le Ca$.

Lemma 4.3. *Let*

$$T(x) = \begin{pmatrix} e^{2\pi i\theta} & t(x) \\ 0 & e^{-2\pi i\theta} \end{pmatrix},$$

where t has a single non-zero Fourier coefficient, $t(x) = \hat{t}_r e^{2\pi i r x}$. Let $X = \sum_{j=1}^k T_{2j-1}^* T_{2j-1}$. Then

$$\|X\| \approx k(1 + |\hat{t}_r|^2 \min\{k^2, \|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}}^{-2}\}), \tag{4.19}$$

$$\|X^{-1}\|^{-1} \approx k. \tag{4.20}$$

Proof. We can compute explicitly

$$T_j = \begin{pmatrix} e^{2\pi i j\theta} & t_j \\ 0 & e^{-2\pi i j\theta} \end{pmatrix}$$

with

$$t_j(x) = \hat{t}_r e^{2\pi i(rx+(j-1)\theta)} \frac{e^{2\pi i j\delta} - 1}{e^{2\pi i\delta} - 1},$$

where $\delta = r\alpha - 2\theta$. Write $X = \begin{pmatrix} k & x_1 \\ \overline{x}_1 & x_2 \end{pmatrix}$. By a straightforward computation,

$$x_1 = \hat{t}_r e^{2\pi i(rx-\theta)} \sum_{j=1}^{k} \frac{e^{2\pi i(2j-1)\delta} - 1}{e^{2\pi i\delta} - 1} = \frac{\hat{t}_r}{e^{2\pi i\delta} - 1} e^{2\pi i(rx-\theta)} \left(e^{2\pi i\delta} \frac{e^{4\pi i k\delta} - 1}{e^{4\pi i\delta} - 1} - k \right), \tag{4.21}$$

$$x_2 = k + |\hat{t}_r|^2 \sum_{j=1}^{k} \left(\frac{\sin \pi(2j-1)\delta}{\sin \pi\delta} \right)^2 = k \left(1 + \frac{2|\hat{t}_r|^2}{|e^{2\pi i\delta} - 1|^2} \left(1 - \frac{\sin 4\pi k\delta}{2k \sin 2\pi\delta} \right) \right). \tag{4.22}$$

From those formulas we conclude that

$$\det X = k^2 \left(1 + \frac{|\hat{t}_r|^2}{|e^{2\pi i\delta - 1}|^2} \left(1 - \left(\frac{\sin 2\pi k\delta}{k \sin 2\pi\delta} \right)^2 \right) \right). \tag{4.23}$$

We first estimate x_2. If $k\|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{12}$, then

$$1 - \frac{\sin 4\pi k\delta}{2k \sin 2\pi\delta} \approx 1$$

and

$$x_2 \approx k(1 + \frac{|\hat{t}_r|^2}{\|2\theta - r\alpha\|^2_{\mathbb{R}/\mathbb{Z}}}).$$

If $k\|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{3}$ then

$$1 - \frac{\sin 4\pi k\delta}{2k \sin 2\pi\delta} \approx (k\|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}})^2,$$

and $x_2 \approx k(1 + k^2|\hat{t}_r|^2)$. Since X is positive, we have $kx_2 \geq |x_1|^2$, and since $x_2 \geq k$ we have $x_2 \geq \max\{k, x_1\}$. Thus $\|X\| \approx x_2$, and the first estimate follows.

We now estimate $\det X$. For $k = 1$ we have $\det X = 1$. Assume that $k > 1$. If $k\|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{12}$ then

$$1 - \left(\frac{\sin 2\pi k\delta}{k \sin 2\pi\delta} \right)^2 \approx 1$$

and

$$\det X \approx k^2(1 + \frac{|\hat{t}_r|^2}{\|2\theta - r\alpha\|^2_{\mathbb{R}/\mathbb{Z}}}).$$

If $k\|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{12}$ then

$$1 - \left(\frac{\sin 2\pi k\delta}{k \sin 2\pi\delta}\right)^2 \approx (k\|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}})^2,$$

and

$$\det X \approx k^2(1 + k^2|\hat{t}_r|^2).$$

This implies the second estimate, using that $\|X^{-1}\|^{-1} = \frac{\det X}{\|X\|}$. □

Lemma 4.4. *Let T, t and X be as in the previous lemma. Let $\tilde{T} : \mathbb{R}/\mathbb{Z} \to \mathrm{SL}(2, \mathbb{C})$. Let $\tilde{X} = \sum_{j=1}^{k} \tilde{T}^*_{2j-1}\tilde{T}_{2j-1}$. Then*

$$\|\tilde{X} - X\| \le 1, \quad \textit{provided that} \quad \|\tilde{T} - T\|_0 \le ck^{-2}(1 + 2k\|t\|_0)^{-2}. \tag{4.24}$$

Proof. Notice that $\|T_j\|_0 \le 1 + j\|t\|_0$, and

$$\|\tilde{T}_j - T_j\|_0 \le \sum_{s=1}^{j} \binom{j}{s} \|\tilde{T} - T\|_0^s \max_{1\le i\le j} \|T_i\|_0^{1+s}. \tag{4.25}$$

Thus, if $\|\tilde{T} - T\|_0 k^2(1 + 2k\|t\|_0)^2$ is small we have

$$\|\tilde{T}_j - T_j\|_0 \le ck^{-1}(1 + 2k\|t\|_0)^{-1},\ 1 \le j \le 2k - 1.$$

This implies

$$\|\tilde{T}_j^*\tilde{T}_j - T_j^*T_j\|_0 \le ck^{-1},\ 1 \le j \le 2k - 1,$$

which gives the estimate. □

Theorem 4.5. *Let $n = |n_j| + 1 < \infty$, $N = |n_{j+1}|$. Then*

$$\frac{\|P_{(k)}\|}{\|P_{(k)}^{-1}\|^{-3}} \le C, \quad Cn^C < k < ce^{cN}. \tag{4.26}$$

Proof. Let Φ, q_1, q, q_3, q_4 be as in the definition of (C_0, c_0, ϵ_0)-goodness.

Let $\Delta > n$. Let $|r| \le \Delta$ minimize $\|2\theta - r\alpha\|_{\mathbb{R}/\mathbb{Z}}$. Then $|r| \ge n - 1$. By the Diophantine condition,

$$\|2\theta - j\alpha\|_{\mathbb{R}/\mathbb{Z}} \ge c\max\{1 + |r|, |j|\}^{-C}, \quad \text{for} \quad j \ne r \quad \text{such that} \quad |j| \le \Delta. \tag{4.27}$$

Decompose $q = t + g + h$ so that t has only the Fourier mode r, g has only the Fourier modes $j \ne r$ with $|j| \le \Delta$ and h is the rest. Then

$$\Phi(x + \alpha)A(x)\Phi(x)^{-1} = T + G + H, \tag{4.28}$$

where

$$T = \begin{pmatrix} e^{2\pi i\theta} & t \\ 0 & e^{-2\pi i\theta} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} q_1 & h \\ q_3 & q_4 \end{pmatrix}.$$

By (3.4) and (3.5),

$$\|H\|_0 \leq Ce^{-cn^{-C}\Delta} + Ce^{-cN}. \tag{4.29}$$

Let $Y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ be such that

$$Y(x+\alpha)(T+G)(x)Y(x)^{-1} = T(x).$$

Then we have

$$\hat{y}_j = -\hat{q}_j \frac{e^{-2\pi i\theta}}{1 - e^{-2\pi i(2\theta - j\alpha)}}, \quad \text{for } j \neq r \quad \text{such that } |j| \leq \Delta, \tag{4.30}$$

and $\hat{y}_j = 0$ if $|j| > \Delta$ or $j = r$. Thus, by (3.5) and (4.27),

$$\|Y - \mathrm{id}\|_0 = \|y\|_0 \leq \sum_{j \leq C(1+|r|)^C} |\hat{y}_j| + \sum_{C(1+|r|)^C < j \leq \Delta} |\hat{y}_j| \leq Ce^{-cn(\ln(1+n))^{-C}}(1+|r|)^C. \tag{4.31}$$

Let $\Psi = Y\Phi$. By (C_0, c_0, ϵ_0)-goodness, $\|\Phi\|_0 \leq Cn^C$, which together with (4.31) gives

$$\|\Psi\|_0 \leq C(n^C + e^{-cn(\ln(1+n))^{-C}}(1+|r|)^C). \tag{4.32}$$

Let $k_\Delta \geq 1$ be maximal such that for $1 \leq k < k_\Delta$, if we let

$$\tilde{T}(x) = \Psi(x+\alpha)A(x)\Psi(x)^{-1} \quad \text{and} \quad \tilde{X} = \sum_{j=1}^{k} \tilde{T}^*_{2j-1}\tilde{T}_{2j-1}, \tag{4.33}$$

then

$$\|\tilde{X} - X\|_0 \leq 1, \tag{4.34}$$

where X is as in Lemma 4.3. Notice that

$$\tilde{T} - T = Y(x+\alpha)H(x)Y(x)^{-1},$$

so

$$\|\tilde{T} - T\|_0 \leq \|Y\|_0^2\|H\|_0.$$

By Lemma 4.4, (4.34) implies

$$\|Y\|_0^2\|H\|_0 \geq ck_\Delta^{-2}(1+2k_\Delta|\hat{q}_r|)^{-2} \geq ck_\Delta^{-4} \tag{4.35}$$

(since (3.5) implies $|\hat{q}_r| \leq C$), so that (4.29) and (4.31) imply

$$k_\Delta \geq c\min\{e^{cn^{-C}\Delta}, \Delta^{-C}e^{cN}\}. \tag{4.36}$$

Notice that

$$\|P_{(k)}(x)\| \le \|\Psi\|_0^4 \|\tilde{X}(x+\alpha)\|$$

and

$$\|P_{(k)}^{-1}\|^{-1} \ge \|\Psi\|_0^{-4} \|\tilde{X}(x+\alpha)^{-1}\|^{-1}.$$

Since $\|\tilde{X}\| \le \|X\| + 1$ and $\|\tilde{X}^{-1}\|^{-1} \ge \|X^{-1}\|^{-1} - 1$ for $1 \le k < k_\Delta$, Lemma 4.3 and (4.32) imply

$$\|P_{(k)}\| \le C(n^C + e^{-cn(\ln(1+n))^{-C}}(1+|r|)^C)k(1+|\hat{q}_r|^2k^2), \quad C \le k < k_\Delta, \tag{4.37}$$

$$\|P_{(k)}^{-1}\|^{-1} \ge c(n^C + e^{-cn(\ln(1+n))^{-C}}(1+|r|)^C)^{-1}k, \quad C \le k < k_\Delta. \tag{4.38}$$

Thus

$$\frac{\|P_{(k)}\|}{\|P_{(k)}^{-1}\|^{-3}} \le C^{(2)} \frac{n^{C^{(2)}} + e^{-c^{(3)}n(\ln(1+n))^{-C^{(2)}}}(1+|r|)^{C^{(2)}}}{k^2} + C^{(2)}(n^{C^{(2)}} + e^{-c^{(3)}n(\ln(1+n))^{-C^{(2)}}}(1+|r|)^{C^{(2)}})|\hat{q}_r|^2, \quad \text{for } C^{(2)} < k < k_\Delta. \tag{4.39}$$

By (3.5),

$$C^{(2)}(n^{C^{(2)}} + e^{-c^{(3)}n(\ln(1+n))^{-C^{(2)}}}(1+|r|)^{C^{(2)}})|\hat{q}_r|^2 \le C. \tag{4.40}$$

Let

$$k_\Delta^- = (n^{C^{(2)}} + e^{-c^{(3)}n(\ln(1+n))^{-C^{(2)}}}(1+\Delta)^{C^{(2)}})^{1/2}. \tag{4.41}$$

Then (4.39) and (4.40) imply

$$\frac{\|P_{(k)}\|}{\|P_{(k)}^{-1}\|^{-3}} \le C, \quad k_\Delta^- < k < k_\Delta. \tag{4.42}$$

In order to conclude, we have to show that for $Cn^C < k < ce^{cN}$ there exists $\Delta > n$ such that $k_\Delta^- < k < k_\Delta$. But this is clear from (4.36) and (4.41) that one can find such Δ with $\Delta \approx n^C \ln k$. □

Corollary 4.6. *For $k \ge 1$, we have $\|P_{(k)}\| \le C\|(P_{(k)})^{-1}\|^{-3}$.*

Proof. It follows from Theorem 4.5 and Lemma 3.1. □

Set $\epsilon_k = \sqrt{\frac{1}{4 \det P_{(k)}}}$.

Corollary 4.7. *We have $\psi(m^+(E + i\epsilon_k)) \le C\epsilon_k^{-1/2}$.*

Proof. We have $\|P_{(k)}\| = \det P_{(k)}\|(P_{(k)})^{-1}\| < \frac{C}{\epsilon_k^2}\|P_{(k)}\|^{-1/3}$. Thus $\|P_{(k)}\| \le C\epsilon_k^{-3/2}$ and the statement follows from (4.11). □

Proof of Theorem 4.1. By (4.4) it is enough to show that

$$c\epsilon^{1/2} \leq \Im M(E+i\epsilon) \leq C\epsilon^{-1/2}. \tag{4.43}$$

For any bounded potential and any solution u we have $\|u\|_{L+1} \leq C\|u\|_L$, thus by (4.17), $\epsilon_{k+1} > c\epsilon_k$. Since $\frac{1}{\epsilon}\Im M(E+i\epsilon)$ is monotonic in ϵ it therefore suffices to prove (4.43) for $\epsilon = \epsilon_k$. But it follows immediately from Corollary 4.7 and (4.6), (4.9). □

Remark 4.3. Corollary 4.7 can be refined: for any $0 < \epsilon < 1$ we have $\psi(m^+(E+i\epsilon)) \leq C\epsilon^{-1/2}$ (thus it is not necessary to restrict to a subsequence of the ϵ_k). Indeed, by the definition of $\psi(z)$ it is enough to show that $|m^+_\beta| \leq C\epsilon^{1/2}$ for arbitrary β. Our proof can be easily adapted to show $1/2$-Hölder continuity of spectral measures μ^+_β associated to half-line problems (with appropriate boundary conditions) whose Borel transform is m^+_β, so that

$$\begin{aligned}|m^+_\beta(E+i\epsilon)| &\leq \int \frac{d\mu^+_\beta}{\sqrt{(x-E)^2+\epsilon^2}}\\ &= \int_0^{\frac{1}{\epsilon}} dt\ \mu\left(\left\{x \in \mathbb{R}\ \middle|\ |x-E| < \sqrt{\frac{1}{t^2}-\epsilon^2}\right\}\right)\\ &< C\epsilon^{-1/2}\int_0^\infty dx\ \frac{x^{1/4}}{(x+1)^{3/2}}.\end{aligned} \tag{4.44}$$

Remark 4.4. It would be interesting to obtain estimates on the modulus of absolute continuity of the spectral measures. It does not seem unreasonable that for all X, $\mu(X) \leq C|X|^{1/2}$. Heuristically, the densities of the spectral measures are unbounded just because of the presence of those countably many (but quickly decaying) square-root singularities located at the gap boundaries. We point out that this is extremely similar to what is expected from the densities of physical measures of typical chaotic unimodal maps.

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Communicated by B. Simon