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# A Frequency Localized Maximum Principle Applied to the 2D Quasi-Geostrophic Equation

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**Abstract:** In this paper, we prove a maximum principle for a frequency localized transport-diffusion equation. As an application, we prove the local well-posedness of the supercritical quasi-geostrophic equation in the critical Besov spaces $\mathring{B}^{1-\alpha}_{\infty,q}$, and global well-posedness of the critical quasi-geostrophic equation in $\mathring{B}^{0}_{\infty,q}$ for all $1 \le q < \infty$. Here $\mathring{B}^{s}_{\infty,q}$ is the closure of the Schwartz functions in the norm of $B^{s}_{\infty,q}$.

## 1. Introduction

In this paper, we consider the two dimensional quasi-geostrophic equation

$$\begin{cases} \partial_t\theta + \Lambda^\alpha\theta + v\cdot\nabla\theta = 0 \quad \text{in } (0,T)\times\mathbb{R}^2, \\ \theta(0) = \theta^0 \quad \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\alpha \in (0, 1]$, the real-valued function $\theta(t, x)$ denotes the potential temperature of the fluid. The velocity $v$ of the fluid is determined by the Riesz transforms of $\theta$:

$$v = (-\partial_2\Lambda^{-1}\theta, \partial_1\Lambda^{-1}\theta) = (-R_2\theta, R_1\theta).$$

A fractional power of the Laplacian $\Lambda^\alpha$ is defined by

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where $\widehat{f}$ denotes the Fourier transform of $f$.

The quasi-geostrophic equation is an important model in geophysical fluid dynamics, which is a special case of the general quasi-geostrophic approximation for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. We may refer to [9,19] for more details about its background in geophysics. The cases $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$ are called subcritical, critical and supercritical respectively.

Due to the deep analogy between the Eq. (1.1) with $\alpha = 1$ and the 3D Navier-Stokes equations, there are many mathematicians devoted to the study of the quasi-geostrophic equation. In the subcritical case, Constantin and Wu [10] proved the global existence of the smooth solution. In the critical case, Constantin, Cordoba, and Wu[8] proved the global existence of smooth solution for small periodic initial data. Recently, Kiselev, Nazarov and Volberg [18] proved the same result for arbitrary smooth periodic initial data. Caffarelli and Vasseur [3] established the global regularity of weak solutions associated to the initial data in $L^2(\mathbb{R}^2)$. In the supercritical case, whether smooth solutions of (1.1) develop the singularity in finite time is still an open problem. We may refer to [11–13] and references therein for some relevant results.

In this paper, we are concerned with the well-posedness problem of (1.1) in the critical spaces. It is easy to find that if $\theta$ is a solution of (1.1), then $\theta_\lambda \stackrel{\text{def}}{=} \lambda^{\alpha-1}\theta(\lambda^\alpha t, \lambda x)$ is also a solution. A functional space $X$ is said to be critical if $\|\theta_\lambda\|_X = \|\theta\|_X$ for any $\lambda > 0$. Obviously, the homogeneous Besov space $\dot{B}^{2/p+1-\alpha}_{p,q}(\mathbb{R}^2)$ is a critical space for any $p, q \in [1, \infty]$. Let us review some known well-posedness results. Chae and Lee [4] proved the global well-posedness for small initial data in the critical Besov spaces $\dot{B}^{1-\alpha}_{2,1}$. Cordoba-Cordoba [14], Ning [17] studied the well-posedness in the Sobolev spaces $H^s, s \geq 1-\alpha, \alpha \in [0, 1]$. Wu [22,23] established the well-posedness in the Besov spaces $B^s_{p,q}, s > 1-\alpha, p = 2^N$. Recently, by establishing the generalized Bernstein's inequality, Chen, Miao and Zhang [7] proved the global well-posedness for small initial data in the critical Besov space $B^{2/p+1-\alpha}_{p,q}(\mathbb{R}^2)$ for $(\alpha, p, q) \in (0, 1] \times [2, \infty) \times [1, \infty)$, and the local well-posedness for large initial data. By obtaining some regularization effects, Hmidi and Keraani [16] also proved the similar well-posedness results including the well-posedness in the limiting space $B^s_{\infty,1} \cap \dot{\mathcal{B}}^0_{\infty,1}$ for $s \geq 1-\alpha$. Very recently, Abidi and Hmidi [1] proved the global well-posedness for the critical quasi-geostrophic equation with large initial data belonging to $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2)$. Here $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2)$ is the closure of the Schwartz functions in the norm of the homogenous Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^2)$.

This paper is devoted to the well-posedness of (1.1) in the inhomogeneous critical Besov spaces $\mathring{B}^{1-\alpha}_{\infty,q}$ for $0 < \alpha \leq 1, 1 \leq q \leq \infty$, which have a certain vanishing property at infinity. That is,

$$\mathring{B}^{1-\alpha}_{\infty,q} \text{ is the closure of the Schwartz functions in the norm of } B^{1-\alpha}_{\infty,q}.$$

The key point of the proof is to establish the following maximum principle for the frequency localized transport-diffusion equation.

**Theorem 1.1** (Localized Maximum Principle). *Let $\theta, v, f$ be smooth functions with $\triangle_j\theta(t) \in C_0(\mathbb{R}^2)$ for $t > 0$ and $j \geq 0$, and let $\theta$ be a solution of the equation*

$$\partial_t\triangle_j\theta + v\cdot\nabla\triangle_j\theta + \Lambda^\alpha\triangle_j\theta = f.$$

*Then there exists a positive constant c independent of $\theta, v, f, j$ such that for a.e. $t > 0$,*

$$\partial_t\|\triangle_j\theta\|_\infty + c2^{j\alpha}\|\triangle_j\theta\|_\infty \leq \|f\|_\infty.$$

*Here $C_0(\mathbb{R}^2)$ denotes the set of the continuous function vanishing at infinity, and $\triangle_j$ is the frequency localization operator (see Sect. 2).*

*Remark 1.2.* By using the generalized Bernstein's inequality, Chen, Miao and Zhang [7] established the following inequality:

$$\partial_t \|\triangle_j \theta\|_p + c_p 2^{j\alpha} \|\triangle_j \theta\|_p \le \|f\|_p,$$

for $2 \le p < \infty$, here $c_p \to 0$ as $p \to \infty$. However, it seems difficult to adapt the method of [7] to the case when $p = \infty$.

As an application of Theorem 1.1, we obtain the well-posedness results of (1.1) by means of the Fourier localization technique and Bony's decomposition. It should be pointed out that since the Riesz transform $R_i$ is not bounded in $B^s_{\infty,q}$, we need to use a subtle commutator estimate (Lemma 6.2) and the fact that $R_i \triangle_{-1} \theta \in C_0(\mathbb{R}^2)$ if $\theta \in \mathring{B}^{1-\alpha}_{\infty,q}$ (see Remark 2.8) in order to deal with the velocity $v$.

**Theorem 1.3.** *Let $0 < \alpha \le 1$, $1 \le q \le \infty$. Assume that the initial data $\theta^0$ belongs to $\mathring{B}^{1-\alpha}_{\infty,q}$. Then there hold*

(a) *If $q < \infty$, there exist $T > 0$ and a unique solution $\theta$ to (1.1) satisfying*

$$\theta \in E_T \overset{def}{=} C([0,T]; \mathring{B}^{1-\alpha}_{\infty,q}) \cap \widetilde{L}^1_T B^1_{\infty,q}.$$

*Moreover, let $T^*$ be the maximal existence time of $\theta$, then there exists an absolute constant $\eta > 0$ such that if $T^* < \infty$, then*

$$\liminf_{t \to T^*} (T^* - t)^{\frac{\alpha}{2-\alpha}} \|\nabla \theta(t)\|_{B^0_{\infty,\infty}} \ge \eta.$$

(b) *If $q = \infty$ and $\|\theta^0\|_{B^{1-\alpha}_{\infty,\infty}} \le \epsilon_0$ for small enough $\epsilon_0$, then there exist $T > 0$ and a solution $\theta$ to (1.1) satisfying*

$$\theta \in E_T \overset{def}{=} L^\infty([0,T]; B^{1-\alpha}_{\infty,\infty}) \cap \widetilde{L}^1_T B^1_{\infty,\infty}.$$

*Moreover, the solution is unique under the following extra assumption*:

$$\sup_{t<T} \lim_{\epsilon \to 0} \|\theta\|_{\widetilde{L}^2(t-\epsilon,t; B^{\frac{1}{2}}_{\infty,\infty})} \le \epsilon_0.$$

(c) *If $\theta_0 \in \mathring{B}^s_{\infty,q}$, $q < \infty$ for $s > 1$, then the solution $\theta$ has the higher regularity, i.e.*

$$\theta \in C([0,T']; \mathring{B}^s_{\infty,q}),$$

*for any $T' < T^*$.*

*Remark 1.4.* For $1 \le q < \infty$, there holds

$$\lim_{t\to 0} \left\| 2^{j(1-\alpha)} \omega_j(t)^{\frac{1}{2}} \|\triangle_j \theta^0\|_\infty \right\|_{\ell^q} = 0, \quad \omega_j(t) = 1 - e^{-c2^{j\alpha}t},$$

which ensures that the estimates we obtained are uniform in the short time. However, it seems impossible for $q = \infty$. This is the reason why we assume that the initial data is small in Theorem 1.3 (b).

By using the argument of modulus of continuity introduced by Kiselev, Nazarov and Volberg [18], we also establish the global well-posedness for the critical quasi-geostrophic equation.

**Theorem 1.5.** *If* $\alpha = 1$ *and* $1 \le q < \infty$*, then the solution* $\theta$ *obtained in Theorem* 1.3 *is global in time.*

*Remark 1.6.* Due to the inclusion relation:

$$B^{\frac{2}{p}+1-\alpha}_{p,q} \subsetneq \mathring{B}^{1-\alpha}_{\infty,q} \quad \text{for} \quad 1 \le p < \infty, 1 \le q \le \infty,$$

Theorem 1.3 is an improvement of the well-posedness results given by [7] and [16].

Throughout this paper, $C$ stands for a positive constant which may be different from line to line. We denote by $\|\cdot\|_p$ the norm of the Lebesgue space $L^p(\mathbb{R}^2)$.

## 2. Preliminaries

First of all, we introduce the Littlewood-Paley decomposition. Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^2)$, supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^2, |\xi| \le 4/3\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^2, 3/4 \le |\xi| \le 8/3\}$ such that

$$\chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^2.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. The frequency localization operators $\triangle_j$ and $S_j$ are defined by

$$\begin{aligned}
&\triangle_j f = \varphi(2^{-j}D)f = 2^{2j}\int_{\mathbb{R}^2} h(2^j y) f(x-y)dy \quad \text{for} \quad j \ge 0,\\
&S_j f = \chi(2^{-j}D)f = \sum_{-1 \le k \le j-1} \triangle_k f = 2^{2j}\int_{\mathbb{R}^2} \tilde{h}(2^j y) f(x-y)dy, \quad \text{and}\\
&\triangle_{-1} f = S_0 f, \quad \triangle_j f = 0 \quad \text{for} \quad j \le -2.
\end{aligned}$$

With our choice of $\varphi$, it is easy to verify that

$$\begin{aligned}
\triangle_j \triangle_k f = 0 \quad &\text{if} \quad |j-k| \ge 2 \quad \text{and}\\
\triangle_j(S_{k-1} f \triangle_k f) = 0 \quad &\text{if} \quad |j-k| \ge 5.
\end{aligned} \tag{2.1}$$

In the sequel, we will constantly use Bony's decomposition from [2]:

$$uv = T_u v + T_v u + R(u,v), \tag{2.2}$$

with

$$T_u v = \sum_j S_{j-1} u \triangle_j v, \quad R(u,v) = \sum_{|j'-j| \le 1} \triangle_j u \triangle_{j'} v.$$

With the introduction of $\triangle_j$, let us recall the definition of Besov space, see [21].

**Definition 2.1.** *Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B^s_{p,q}$ is defined by*

$$B^s_{p,q} = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} < \infty \Big\}.$$

*Here*

$$\|f\|_{B^s_{p,q}} \overset{def}{=} \Big\| 2^{js} \|\triangle_j f\|_p \Big\|_{\ell^q}.$$

*We denote by $\mathring{B}^s_{p,q}$ the completion of the Schwartz functions under the norm $\|\cdot\|_{B^s_{p,q}}$.*

Besides the usual space-time space $L^\rho_T B^s_{p,q}$, we also need the Chemin-Lerner-Besov space $\widetilde{L}^\rho_T B^s_{p,q}$, which is defined as the set of all distributions $f$ satisfying

$$\|f\|_{\widetilde{L}^\rho_T B^s_{p,q}} \overset{\text{def}}{=} \Big\| 2^{js} \|\triangle_j f\|_{L^\rho_T L^p} \Big\|_{\ell^q} < \infty.$$

From Minkowski's inequality, it is easy to find that

$$L^r_t(B^s_{p,q}) \subseteq \widetilde{L}^r_t(B^s_{p,q}) \quad \text{if} \quad r \leq q \quad \text{and} \quad \widetilde{L}^r_t(B^s_{p,q}) \subseteq L^r_t(B^s_{p,q}) \quad \text{if} \quad r \geq q.$$

**Lemma 2.2.** *Let $1 \leq q < \infty, \sigma \in \mathbb{R}$. If $f \in \mathring{B}^s_{\infty,q} \cap B^\sigma_{\infty,q}$ for some $s < \sigma$, then $f \in \mathring{B}^\sigma_{\infty,q}$.*

*Proof.* Take $\chi(x) \in C_0^\infty$ with $\chi(x) = 0$ for $|x| \leq 1$, and let $\chi_M(x) = \chi(x/M)$. Thanks to $f \in \mathring{B}^s_{\infty,q}$, it is easy to verify that $S_N f \in \mathring{B}^\sigma_{\infty,q}$ and for fixed $N$,

$$\|(1 - \chi_M(x)) S_N f\|_{B^\sigma_{\infty,q}} \longrightarrow 0,$$

as $M \to +\infty$. On the other hand, since $f \in B^\sigma_{\infty,q}$ and $q < \infty$, there holds

$$\|(1 - S_N) f\|_{B^\sigma_{\infty,q}} \longrightarrow 0,$$

as $N \to +\infty$. Thus,

$$\|\chi_M(x) S_N f - f\|_{B^\sigma_{\infty,q}} \leq \|(1 - S_N) f\|_{B^\sigma_{\infty,q}} + \|(1 - \chi_M(x)) S_N f\|_{B^\sigma_{\infty,q}} \longrightarrow 0,$$

by letting $M \to +\infty$, then $N \to +\infty$. The lemma is proved. □

**Lemma 2.3** [5]. *Let $1 \leq p \leq q \leq +\infty$. Assume that $f \in L^p(\mathbb{R}^2)$, then for any $\gamma \in (\mathbb{N} \cup \{0\})^2$, there exists a constant $C$ independent of $f$, $j$ such that*

$$\text{supp} \hat{f} \subseteq \{|\xi| \leq A_0 2^j\} \Rightarrow \|\partial^\gamma f\|_q \leq C 2^{j|\gamma| + 2j(\frac{1}{p} - \frac{1}{q})} \|f\|_p,$$
$$\text{supp} \hat{f} \subseteq \{A_1 2^j \leq |\xi| \leq A_2 2^j\} \Rightarrow \|f\|_p \leq C 2^{-j|\gamma|} \sup_{|\beta| = |\gamma|} \|\partial^\beta f\|_p.$$

Next we recall a result of the fractional integral operator.

**Lemma 2.4** [14]. *Let $0 \leq \alpha < 2$, and $\theta \in \mathcal{S}(\mathbb{R}^2)$. Then there holds*

$$\Lambda^\alpha \theta(x) = c_\alpha \int_{\mathbb{R}^2} \frac{\theta(x) - \theta(y)}{|x - y|^{2+\alpha}} dy,$$

*with $c_\alpha > 0$.*

Following the proof of Lemma 2.3 in [24], we can obtain

**Lemma 2.5.** *Let $m(\xi)$ be a homogeneous function of degree $\alpha$ with $\alpha > 0$. Define*

$$K(x) \overset{def}{=} \int m(\xi)\chi(\xi)e^{ix\cdot\xi}d\xi,$$

*where $\chi \in C_0^\infty(\mathbb{R}^2)$ with $supp\chi \subset \{\xi : |\xi| \leq 2\}$. Then there holds*

$$|K(x)| \leq C(1+|x|)^{-2-\alpha}.$$

*In particular, if $f \in L^p(\mathbb{R}^2)$ with supp $\widehat{f} \subset \{\xi : |\xi| \leq 2\}$, we have*

$$\|m(D)f\|_p \leq C\|f\|_p,$$

*for any $1 \leq p \leq \infty$.*

*Remark 2.6.* Thanks to Lemma 2.5, we have

$$\|\Lambda^\alpha f\|_{B^s_{\infty,q}} \leq C\|f\|_{B^{s+\alpha}_{\infty,q}},$$

for any $\alpha > 0, s \in \mathbb{R}, q \in [1,\infty]$.

Let us conclude this section by recalling the BMO and VMO spaces. A locally integrable function $f$ is said to belong to BMO if the inequality

$$\frac{1}{|B|}\int_B |f(x) - f_B|dx \leq A$$

holds for all balls $B$; here $f_B = |B|^{-1}\int_B f dx$. It is well-known that

$$R_i : L^\infty \to \text{BMO but } R_i : L^\infty \not\to L^\infty.$$

Let VMO be the closure of $C_0(\mathbb{R}^2)$ in the norm of BMO. We have the following properties about VMO space (see [20], p. 180).

**Proposition 2.7.** (1) *Let $\Phi \in \mathcal{S}(\mathbb{R}^2)$, $\int_{\mathbb{R}^2}\Phi dx = 1$. Suppose that $f \in BMO$, then $f \in VMO$ if and only if $f * \Phi_t \in C_0(\mathbb{R}^2)$ for all $t > 0$, and $\|f - f*\Phi_t\|_{BMO} \to 0$ as $t \to 0$.*
(2) *If $f \in C_0(\mathbb{R}^2)$, then $R_i f \in VMO$ for $i = 1, 2$.*

*Remark 2.8.* If $f \in \mathring{B}^s_{\infty,q}$ for all $s \in \mathbb{R}, q \in [1,\infty]$, then $R_i\triangle_{-1}f \in C_0(\mathbb{R}^2)\cap C^\infty(\mathbb{R}^2)$.

Indeed, since $f \in \mathring{B}^s_{\infty,q}$, $\triangle_{-1}f \in C_0(\mathbb{R}^2)$. Thanks to Proposition 2.7 (2), $R_i\triangle_{-1}f \in$ VMO, which together with Proposition 2.7 (1) implies $R_i\triangle_{-1}f \in C_0(\mathbb{R}^2)$. On the other hand, $R_i\triangle_{-1}f \in C^\infty(\mathbb{R}^2)$ is a direct consequence of Lemma 2.3.

## 3. A Frequency Localized Maximum Principle

We consider the transport-diffusion equation

$$\partial_t \theta + \Lambda^\alpha \theta + v \cdot \nabla \theta = f. \tag{3.1}$$

We have the following parabolic maximum principle.

**Proposition 3.1** (Maximum principle). *Let $v$ be a smooth vector field and $f$ be a smooth function. Assume that $\theta(t, x)$ is a solution of Eq. (3.1) with $\theta(t) \in C_0(\mathbb{R}^2)$. Then there holds*

$$\|\theta(t)\|_\infty - \|\theta(0)\|_\infty \le \int_0^t \|f(\tau)\|_\infty d\tau.$$

In the case when $v$ is a divergence free vector, the inequality was proved in [14] by using the positivity lemma. Here we generalize it to a general vector $v$. To prove Proposition 3.1, we need the following classical lemma.

**Lemma 3.2.** *Let $f(t, x)$ be a smooth function on $[0, +\infty) \times \mathbb{R}^2$ with $f(t) \in C_0(\mathbb{R}^2)$ for all $t \ge 0$. Then $\frac{d}{dt}\|f(t)\|_\infty$ exists for a.e. $t \ge 0$ and*

$$\frac{d}{dt}\|f(t)\|_\infty = (\partial_t f)(t, x_t) sign(f(t, x_t)),$$

*here $f(t, x_t) = \pm\|f(t)\|_\infty$ and sign $x$ is a sign function of $x$.*

*Proof.* Given $t \ge 0$, for all $h \in \mathbb{R}$ such that $t + h \ge 0$, we have

$$\begin{aligned} \big| \|f(t+h)\|_\infty - \|f(t)\|_\infty \big| &\le \|f(t+h) - f(t)\|_\infty \\ &\le \left( \sup_{\tau \in [0, t+1]} \|\partial_\tau f(\tau)\|_\infty + 2 \sup_{\tau \ge 0} \|f(\tau)\|_\infty \right) |h|. \end{aligned}$$

Thus, $g(t) \overset{\text{def}}{=} \|f(t)\|_\infty$ is a Lipschitz function in $t \ge 0$, and then $g'(t) = \frac{d}{dt}\|f(t)\|_\infty$ exists for a.e. $t \ge 0$.

For every $h > 0$, since $f(t, x)$ is a smooth function with $f(t) \in C_0(\mathbb{R}^2)$, there always exists a point $x_{t+h} \in \mathbb{R}^2$ such that $|f(t+h, x)|$ reaches its maximum at $x_{t+h}$, that is $g(t+h) = |f(t+h, x_{t+h})|$. Then we can find a sequence $h_n \to 0$ such that $x_{t+h_n} \to x_t$ and $g(t) = |f(t, x_t)|$. Assume that $f(t, x_t) > 0$ (The case when $f(t, x_t) \le 0$ can be similarly proved) and $g'(t)$ exists. Thus, for $n$ big enough,

$$\begin{aligned} \frac{\|f(t+h_n)\|_{L^\infty} - \|f(t)\|_{L^\infty}}{h_n} &= \frac{f(t+h_n, x_{t+h_n}) - f(t, x_t)}{h_n} \\ \le \frac{f(t+h_n, x_{t+h_n}) - f(t, x_{t+h_n})}{h_n} &\longrightarrow \partial_t f(t, x_t) \end{aligned}$$

as $n \to \infty$, which implies that

$$\frac{d}{dt}\|f(t)\|_\infty \le \partial_t f(t, x_t). \tag{3.2}$$

On the other hand, we have

$$\frac{\|f(t+h_n)\|_{L^\infty} - \|f(t)\|_{L^\infty}}{h_n} \geq \frac{f(t+h_n, x_t) - f(t, x_t)}{h_n} \to \partial_t f(t, x_t)$$

as $n \to \infty$, which implies that

$$\frac{d}{dt}\|f(t)\|_\infty \geq \partial_t f(t, x_t),$$

from which and (3.2), we conclude that

$$\frac{d}{dt}\|f(t)\|_\infty = \partial_t f(t, x_t).$$

This finishes the proof of Lemma 3.2. □

*Proof of Proposition 3.1.* Due to $\theta \in C_0(\mathbb{R}^2)$, without loss of generality, we may assume that there exists a point $x_t \in \mathbb{R}^2$ such that $\theta(t, x_t) = \|\theta(t)\|_\infty$. Then $\nabla\theta(t, x_t) = 0$. By Lemma 2.4, we have

$$\Lambda^\alpha \theta(x_t) = C_\alpha \int_{\mathbb{R}^2} \frac{\theta(x_t) - \theta(y)}{|x_t - y|^{2+\alpha}} dy \geq 0.$$

Thus, it follows from Lemma 3.2 that

$$\begin{aligned}\partial_t \|\theta(t)\|_\infty &= (\partial_t\theta)(t, x_t) \\ &= -v(t, x_t) \cdot \nabla\theta(t, x_t) - \Lambda^\alpha\theta(t, x_t) + f(t, x_t) \leq \|f(t)\|_\infty,\end{aligned}$$

which implies Proposition 3.1. □

Next, we consider the frequency localized transport-diffusion equation

$$\partial_t \triangle_j \theta + v \cdot \nabla \triangle_j \theta + \Lambda^\alpha \triangle_j \theta = f. \tag{3.3}$$

The following maximum principle plays a key role in the proof of Theorem 1.3.

**Theorem 3.3** (Localized maximum principle). *Let $\theta, v, f$ be smooth functions with $\triangle_j\theta(t) \in C_0(\mathbb{R}^2)$ for $t > 0$ and $j \geq 0$, and let $\theta$ be a solution of (3.3). Then there exists a positive constant $c$ independent of $\theta, v, f, j$ such that for a.e. $t > 0$,*

$$\partial_t \|\triangle_j\theta\|_{L^\infty} + c2^{j\alpha}\|\triangle_j\theta\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

The proof of Theorem 3.3 is based on the following localized positivity lemma.

**Lemma 3.4.** *Let*

$$\mathcal{A} \overset{def}{=} \left\{ g \in C_0(\mathbb{R}^2) : \|g\|_{L^\infty} = 1, supp\ \widehat{g} \subset \{\xi : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\} \right\}.$$

*Suppose that $g \in \mathcal{A}$ and $|g(x_0)| = \|g\|_{L^\infty} = 1$ for some $x_0 \in \mathbb{R}^2$. Then there exists a constant $c$ independent of $g$ such that*

$$sign(g(x_0))\Lambda^\alpha g(x_0) \geq c.$$

*Proof.* We prove the lemma by the contradiction argument. Assume that there exist $\{g_n\} \subset \mathcal{A}$ and $x_n \in \mathbb{R}^2$ such that

$$|g_n(x_n)| = 1, \quad c_n = \Lambda^\alpha g_n(x_n) \to 0.$$

Let $\tau_h$ be a translation operator defined as $\tau_h g(x) = g(x+h)$. It is easy to verify that $\tau_h \Lambda^\alpha g(x) = \Lambda^\alpha \tau_h g(x)$. For the sequence $\{\tau_{x_n} g_n\}$, there exist a subsequence $\{\tau_{x_{n_k}} g_{n_k}\}$ and $g \in L^\infty$ such that $|\tau_{x_{n_k}} g_{n_k}(0)| = 1$ and $\tau_{x_{n_k}} g_{n_k} \rightharpoonup g$ in $L^\infty$. Since

$$\operatorname{supp} \mathcal{F}(\tau_{x_{n_k}} g_{n_k})(\xi) \subset \{\xi : \frac{3}{4} \le |\xi| \le \frac{8}{3}\},$$

we have $\operatorname{supp} \widehat{g} \subset \{\xi : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ and for any $x \in \mathbb{R}^2$,

$$\tau_{x_{n_k}} g_{n_k}(x) = \mathcal{F}^{-1}(\psi) * (\tau_{x_{n_k}} g_{n_k})(x) \to g(x).$$

Here $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfies $\psi(\xi) = 1$ in $\{\xi : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Moreover, $g \in C^\infty \cap L^\infty$, $|g(0)| = 1$, and $\|g\|_{L^\infty} = 1$. In addition, $g \not\equiv \pm 1$. Otherwise, $g = \mathcal{F}^{-1}(\psi) * g \equiv 0$ since $\int_{\mathbb{R}^2} \mathcal{F}^{-1}(\psi)(x)dx = \psi(0) = 0$. By Lemma 2.4 and the Fatou Lemma, we find that

$$\begin{aligned}
&\operatorname{sign}(g(0))\Lambda^\alpha g(0) = C_\alpha \operatorname{sign}(g(0)) \int_{\mathbb{R}^2} \frac{g(0) - g(y)}{|y|^{2+\alpha}} dy \\
&\quad \le C_\alpha \operatorname{sign}(g(0)) \varliminf_{k\to\infty} \int_{\mathbb{R}^2} \frac{\tau_{x_{n_k}} g_{n_k}(0) - \tau_{x_{n_k}} g_{n_k}(y)}{|y|^{2+\alpha}} dy = \operatorname{sign}(g(0)) \varliminf_{k\to\infty} c_{n_k} = 0.
\end{aligned}$$

Since $g \not\equiv \pm 1$, the above inequality contradicts the fact:

$$\operatorname{sign}(g(0))\Lambda^\alpha g(0) = \operatorname{sign}(g(0)) \int_{\mathbb{R}^2} \frac{g(0) - g(y)}{|y|^{2+\alpha}} dy > 0.$$

This completes the proof of Lemma 3.4. □

*Proof of Theorem 3.3.* Since $\theta_j \overset{\text{def}}{=} \triangle_j \theta \in C_0(\mathbb{R}^2)$ for $j \ge 0$, there exists a point $x_{t,j} \in \mathbb{R}^2$ so that $|\theta_j(t, x_{t,j})| = \|\theta_j(t)\|_\infty > 0$. By using a scaling argument and Lemma 3.4, there exists a positive constant $c$ independent of $j$ and $\theta$ such that

$$\operatorname{sign}(\theta_j(x_{t,j}))\Lambda^\alpha \theta_j(x_{t,j}) \ge c 2^{j\alpha} \|\theta_j\|_\infty.$$

Then we get by Lemma 3.2 and (3.3) that

$$\begin{aligned}
\partial_t \|\theta_j(t)\|_\infty &= \operatorname{sign}(\theta_j(x_{t,j}))(\partial_t \theta_j)(t, x_{t,j}) \\
&= \operatorname{sign}(\theta_j(x_{t,j})) \left[ -v(t, x_{t,j}) \cdot \nabla \theta_j(t, x_{t,j}) - \Lambda^\alpha \theta_j(t, x_{t,j}) + f(t, x_{t,j}) \right] \\
&\le -c 2^{j\alpha} \|\theta_j(t)\|_\infty + \|f(t)\|_\infty,
\end{aligned}$$

where we used $\nabla \theta_j(t, x_{t,j}) = 0$ in the last inequality. □

## 4. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. To construct the approximate solution, we need the following lemma.

**Lemma 4.1.** *Let* $v = (-R_2 f, R_1 f)$ *with* $f \in \widetilde{L}^\infty_T \mathring{B}^{s+3}_{\infty,q}$, *and* $\theta^0 \in \mathring{B}^s_{\infty,q}$ *for* $s \geq 3$. *Then the transport-diffusion equation*

$$\begin{cases} \partial_t \theta + \Lambda^\alpha \theta + v \cdot \nabla \theta = 0, \\ \theta(0) = \theta^0, \end{cases} \tag{4.1}$$

*has a unique solution* $\theta \in C([0, T]; \mathring{B}^s_{\infty,q})$.

*Proof.* Since $\theta^0 \in \mathring{B}^s_{\infty,q}$, we can choose $\theta^{0,n} \in \mathcal{S}(\mathbb{R}^2)$ such that as $n$ tends to $\infty$,

$$\theta^{0,n} \longrightarrow \theta^0 \quad \text{in} \quad B^s_{\infty,q}.$$

Then we solve (4.1) with the initial data $\theta^{0,n}$ and obtain a sequence of solutions $\theta^n \in C([0, T]; H^{s+2})$. Since $H^{s+2} \hookrightarrow \mathring{B}^s_{\infty,q}$, $\theta \in C([0, T]; \mathring{B}^s_{\infty,q})$. On the other hand, from Step 1 in the proof of Theorem 1.3 (especially (4.2) and (4.5)), we can infer that

$$\|\theta^n\|_{\widetilde{L}^\infty_t B^s_{\infty,q}} \leq \|\theta^{0,n}\|_{B^s_{\infty,q}} + \int_0^t \|2^{js}\|[\triangle_j, v] \cdot \nabla \theta^n\|_\infty\|_{\ell^q} d\tau,$$

from which and Lemma 6.2 (6.5), we infer that

$$\begin{aligned} \|\theta^n\|_{\widetilde{L}^\infty_t B^s_{\infty,q}} &\leq \|\theta^{0,n}\|_{B^s_{\infty,q}} + C \int_0^t \Big( \|\nabla \theta^n\|_{L^\infty} \|f\|_{B^s_{\infty,q}} + \|\nabla v\|_{L^\infty} \|\theta^n\|_{B^s_{\infty,q}} \Big) d\tau \\ &\leq \|\theta^{0,n}\|_{B^s_{\infty,q}} + Ct \|\nabla \theta^n\|_{L^\infty_t L^\infty} \|f\|_{\widetilde{L}^\infty_t B^s_{\infty,q}} + Ct \|\nabla v\|_{L^\infty_t L^\infty} \|\theta^n\|_{\widetilde{L}^\infty_t B^s_{\infty,q}}. \end{aligned}$$

Here we also used $\|f\|_{L^\infty_t B^s_{\infty,q}} \leq \|f\|_{\widetilde{L}^\infty_t B^s_{\infty,q}}$. Noticing that

$$\|\nabla \theta^n\|_{L^\infty_t L^\infty} \leq C \|\theta^n\|_{\widetilde{L}^\infty_t B^s_{\infty,q}}, \quad \|\nabla v\|_{L^\infty_t L^\infty} \leq C \|f\|_{\widetilde{L}^\infty_t B^s_{\infty,q}},$$

we obtain

$$\|\theta^n\|_{\widetilde{L}^\infty_t B^s_{\infty,q}} \leq \|\theta^{0,n}\|_{B^s_{\infty,q}} + Ct \|f\|_{\widetilde{L}^\infty_t B^s_{\infty,q}} \|\theta^n\|_{\widetilde{L}^\infty_t B^s_{\infty,q}},$$

which implies

$$\theta^n \longrightarrow \theta \quad \text{in} \quad C([0, T]; B^s_{\infty,q}).$$

Thus $\theta \in C([0, T]; \mathring{B}^s_{\infty,q})$. $\square$

*Proof of Theorem 1.3.* We divide the proof into several steps.

*Step 1. A priori estimates*. We assume that $\theta(t, x) \in C_0(\mathbb{R}^2)$ is a smooth solution of (1.1). Set $\theta_j \stackrel{\text{def}}{=} \triangle_j \theta$ for $j \geq -1$, then $\theta_j$ satisfies

$$\partial_t \theta_j + \Lambda^\alpha \theta_j + v \cdot \nabla \theta_j = -[\triangle_j, v] \cdot \nabla \theta.$$

First of all, we get by Proposition 3.1 that

$$\|\theta_{-1}(t)\|_\infty \le \|\theta^0_{-1}\|_\infty + \int_0^t \|[\triangle_{-1}, v]\cdot\nabla\theta\|_{L^\infty} d\tau. \tag{4.2}$$

We infer from Lemma 6.2 (6.4) that

$$\begin{aligned}\int_0^t \|[\triangle_{-1}, v]\cdot\nabla\theta\|_{L^\infty} d\tau &\le \|2^{j(1-\alpha)}\|[\triangle_{-j}, v]\cdot\nabla\theta\|_{L^1_t L^\infty}\|_{\ell^q}\\ &\le C\|\nabla v\|_{\widetilde{L}^2_t B^{-\frac{\alpha}{2}}_{\infty,q}}\|\nabla\theta\|_{\widetilde{L}^2_t B^{-\frac{\alpha}{2}}_{\infty,q}}\\ &\le C\|\theta\|^2_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}.\end{aligned}$$

Thus we deduce that

$$\|\theta_{-1}(t)\|_\infty \le \|\theta^0_{-1}\|_\infty + C\|\theta\|^2_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}. \tag{4.3}$$

Thanks to Theorem 3.3, we get for $j \ge 0$,

$$\partial_t\|\theta_j\|_\infty + c2^{j\alpha}\|\theta_j\|_\infty \le \|[\triangle_j, v]\cdot\nabla\theta\|_\infty,$$

from which and Gronwall's inequality, we infer that

$$\|\theta_j(t)\|_\infty \le e^{-c2^{j\alpha}t}\|\theta^0_j\|_\infty + e^{-c2^{j\alpha}t} * \|[\triangle_j, v]\cdot\nabla\theta\|_\infty. \tag{4.4}$$

Here the convolution is defined as

$$f * g(t) = \int_0^t f(t-s)g(s)ds.$$

Using (4.4) and the Young's inequality, we get for $1 \le r \le \infty$ and $j \ge 0$,

$$2^{j\frac{\alpha}{r}}\|\theta_j\|_{L^r_t L^\infty} \le C\omega_j(t)^{\frac{1}{r}}\|\theta^0_j\|_{L^\infty} + C\|[\triangle_j, v]\cdot\nabla\theta\|_{L^1_t L^\infty}, \tag{4.5}$$

where $\omega_j(t) \overset{\text{def}}{=} 1 - e^{-c2^{j\alpha}t}$. Multiplying by $2^{j(1-\alpha)}$ on both sides of (4.5), and then summing over $j$, we obtain

$$\begin{aligned}&\left(\sum_{j\ge0} 2^{jq(1-\frac{\alpha}{r'})}\|\theta_j\|^q_{L^r_t L^\infty}\right)^{\frac{1}{q}}\\ &\le C\|2^{j(1-\alpha)}\omega_j(t)^{\frac{1}{r}}\|\theta^0_j\|_{L^\infty}\|_{\ell^q} + C\|2^{j(1-\alpha)}\|[\triangle_j, v]\cdot\nabla\theta\|_{L^1_t L^\infty}\|_{\ell^q}.\end{aligned} \tag{4.6}$$

While due to Lemma 6.2 and Remark 2.6, we have

$$\|2^{j(1-\alpha)}\|[\triangle_j, v]\cdot\nabla\theta\|_{L^1_t L^\infty}\|_{\ell^q} \le C\|\theta\|^2_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}},$$

which together with (4.3) and (4.6) gives for $1 \le r \le \infty$,

$$\|\theta\|_{\widetilde{L}^r_t B^{1-\alpha/r'}_{\infty,q}} \le C(1+t^{\frac{1}{r}})\|\theta^0\|_{B^{1-\alpha}_{\infty,q}} + C(1+t^{\frac{1}{r}})\|\theta\|^2_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}, \tag{4.7}$$

$$\begin{aligned}\|\theta\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}} &\le t^{\frac{1}{2}}\|\triangle_{-1}\theta^0\|_{L^\infty} + C\|2^{j(1-\alpha)}\omega_j(t)^{\frac{1}{2}}\|\theta^0_j\|_{L^\infty}\|_{\ell^q} \\ &\quad + C(1+t^{\frac{1}{2}})\|\theta\|^2_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}.\end{aligned} \tag{4.8}$$

*Step 2. Approximate solutions and uniform estimates*. We construct the approximate solutions of (1.1) by solving the following linear system:

$$\begin{cases} \partial_t\theta^{(k+1)} + \Lambda^\alpha\theta^{(k+1)} + v^{(k)}\cdot\nabla\theta^{(k+1)} = 0, \\ v^{(k)} = (-R_2\theta^{(k)}, R_1\theta^{(k)}), \\ \theta^{(k+1)}(0) = S_{k+4}\theta^{(0)}, \end{cases} \tag{4.9}$$

for all $k \ge 0$. Here we set $(\theta^{(0)}, v^{(0)}) = (\triangle_{-1}\theta^0, \triangle_{-1}v^0)$. Since $\theta^{(k)}(0) \in \mathring{B}^{s_0}_{\infty,q}$ for any $s_0$(for example, we take $s_0 \ge 3$), we infer from Lemma 4.1 that the solution $\theta^{(k)} \in C([0,+\infty); \mathring{B}^{s_0}_{\infty,q})$ for $k \ge 0$. Then a similar argument as leading to (4.7) and (4.8) ensures that

$$\begin{aligned}\|\theta^{(k+1)}\|_{\widetilde{L}^r_t B^{1-\alpha/r'}_{\infty,q}} &\le C(1+t^{\frac{1}{r}})\|\theta^0\|_{B^{1-\alpha}_{\infty,q}} \\ &\quad + C(1+t^{\frac{1}{r}})\|\theta^{(k)}\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}\|\theta^{(k+1)}\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\|\theta^{(k+1)}\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}} &\le t^{\frac{1}{2}}\|\triangle_{-1}\theta^0\|_{L^\infty} + C\|2^{j(1-\alpha)}\omega_j(t)^{\frac{1}{2}}\|\theta^0_j\|_{L^\infty}\|_{\ell^q} \\ &\quad + C(1+t^{\frac{1}{2}})\|\theta^{(k)}\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}\|\theta^{(k+1)}\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}.\end{aligned} \tag{4.11}$$

Noticing that for $1 \le q < \infty$,

$$\lim_{t\to 0}\left\|2^{j(1-\alpha)}\omega_j(t)^{\frac{1}{2}}\|\triangle_j\theta^0\|_\infty\right\|_{\ell^q} = 0,$$

and $\|\theta^0\|_{B^{1-\alpha}_{\infty,q}} \le \epsilon_0$ for $q = \infty$, then by a standard induction argument, (4.10)-(4.11) ensure that there exists $T > 0$ such that

$$\|\theta^{(k)}\|_{\widetilde{L}^r_t B^{1-\alpha/r'}_{\infty,q}} \le C\left(\|\theta^0\|_{B^{1-\alpha}_{\infty,q}} + \epsilon_0^2\right), \qquad \|\theta^{(k)}\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}} \le C\epsilon_0, \tag{4.12}$$

for $k \ge 0$ and $0 \le t \le T$.

*Step 3. Convergence and existence*. In this step, we assume that $0 \le t \le T \le 1$. Set

$$\delta\theta^{(k)} = \theta^{(k+1)} - \theta^{(k)}, \quad \delta v^{(k)} = v^{(k+1)} - v^{(k)}.$$

Then $(\delta\theta^k, \delta v^k)$ satisfies

$$\begin{cases} \partial_t\delta\theta^{(k)} + \Lambda^\alpha\delta\theta^{(k)} + v^{(k)}\cdot\nabla\delta\theta^{(k)} + \delta v^{(k)}\cdot\nabla\theta^{(k+1)} = 0, \\ \delta\theta^{(k)}(0) = \triangle_{k+3}\theta^0. \end{cases} \tag{4.13}$$

Taking $\triangle_j$ on both sides of (4.13), we obtain

$$\begin{aligned}&\partial_t \triangle_j \delta\theta^{(k)} + \Lambda^\alpha \triangle_j \delta\theta^{(k)} + v^{(k)} \cdot \nabla \triangle_j \delta\theta^{(k)} \\ &\quad = -[\triangle_j, v^{(k)}] \cdot \nabla \delta\theta^{(k)} - \triangle_j (\delta v^{(k)} \cdot \nabla \theta^{(k+1)}).\end{aligned}$$

Noticing that $\triangle_{-1}\triangle_{k+3}\theta^0 = 0$, we get by Proposition 3.1 that

$$\|\triangle_{-1}\delta\theta^{(k)}(t)\|_\infty \le \int_0^t \|[\triangle_{-1}, v^{(k)}] \cdot \nabla\delta\theta^{(k)} + \triangle_{-1}(\delta v^{(k)} \cdot \nabla\theta^{(k+1)})\|_\infty d\tau. \quad (4.14)$$

Exactly as in the proof of (4.6), we can obtain

$$\begin{aligned}&\left(\sum_{j\ge 0} 2^{jq(\frac{\alpha}{2}-s)} \|\delta\theta_j^{(k)}\|_{L_t^2 L^\infty}^q\right)^{\frac{1}{q}} \\ &\le C\|\triangle_{k+3}\theta^0\|_{B^{-s}_{\infty,q}} + C\left\|2^{-js}\|[\triangle_j, v^{(k)}] \cdot \nabla\delta\theta^{(k)}\|_{L_t^1 L^\infty}\right\|_{\ell^q} \\ &\quad + C\left\|2^{-js}\|\triangle_j(\delta v^{(k)} \cdot \nabla\theta^{(k+1)})\|_{L_t^1 L^\infty}\right\|_{\ell^q}. \end{aligned} \quad (4.15)$$

From (4.14) and (4.15), we deduce that

$$\begin{aligned}&\|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}} \\ &\le C\|\triangle_{k+3}\theta^0\|_{B^{-s}_{\infty,q}} + C\left\|2^{-js}\|[\triangle_j, v^{(k)}] \cdot \nabla\delta\theta^{(k)}\|_{L_t^1 L^\infty}\right\|_{\ell^q} \\ &\quad + C\left\|2^{-js}\|\triangle_j(\delta v^{(k)} \cdot \nabla\theta^{(k+1)})\|_{L_t^1 L^\infty}\right\|_{\ell^q}. \end{aligned} \quad (4.16)$$

Taking $s$ such that $\frac{\alpha}{2} < s < 1$, we get by Lemma 6.2 and Remark 2.6 that

$$\left\|2^{-js}\|[\triangle_j, v^{(k)}] \cdot \nabla\delta\theta^{(k)}\|_{L_t^1 L^\infty}\right\|_{\ell^q} \le C\|\theta^{(k)}\|_{\widetilde{L}_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}} \|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}}, \quad (4.17)$$

and by Lemma 6.1 we have

$$\begin{aligned}&\left\|2^{-js}\|\triangle_j(\delta v^{(k)} \cdot \nabla\theta^{(k+1)})\|_{L_t^1 L^\infty}\right\|_{\ell^q} \\ &\le C\|\delta v^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}} \|\theta^{(k+1)}\|_{\widetilde{L}_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}} \\ &\le C\Big(\|\triangle_{-1}\delta v^{(k)}\|_{L_t^\infty L^\infty} + \|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}}\Big)\|\theta^{(k+1)}\|_{\widetilde{L}_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}}. \end{aligned} \quad (4.18)$$

From (4.16)–(4.18), we infer that

$$\begin{aligned}\|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}} &\le C2^{-k(1+s-\alpha)}\|\theta^0\|_{B^{1-\alpha}_{\infty,q}} \\ &\quad + C\sum_{m=k}^{k+1} \|\theta^{(m)}\|_{\widetilde{L}_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}} \Big(\|\triangle_{-1}\delta v^{(k)}\|_{L_t^\infty L^\infty} + \|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}}\Big). \end{aligned} \quad (4.19)$$

It remains to estimate $\|\triangle_{-1}\delta v^{(k)}\|_{L_t^\infty L^\infty}$. In order to do this, we take $R_i\triangle_{-1}(i=1,2)$ on both sides of (4.13) to obtain

$$\partial_t R_i\triangle_{-1}\delta\theta^{(k)} + v^{(k)}\cdot\nabla R_i\triangle_{-1}\delta\theta^{(k)} + \Lambda^\alpha R_i\triangle_{-1}\delta\theta^{(k)} \\ = -[R_i\triangle_{-1}, v^{(k)}]\cdot\nabla\delta\theta^{(k)} - R_i\triangle_{-1}(\delta v^{(k)}\cdot\nabla\theta^{(k+1)}),$$

from which and the fact that $R_i\triangle_{-1}\theta^{(k)}\in C_0(\mathbb{R}^2)$, we get by Proposition 3.1 that

$$\|\triangle_{-1}\delta v^{(k)}(t)\|_\infty \le \sum_{i=1}^2\int_0^t \|[R_i\triangle_{-1}, v^{(k)}]\cdot\nabla\delta\theta^{(k)}\|_\infty d\tau \\ +\sum_{i=1}^2\int_0^t \|R_i\triangle_{-1}(\delta v^{(k)}\cdot\nabla\theta^{(k+1)})\|_\infty d\tau. \tag{4.20}$$

Firstly, from Lemma 6.3 and $s<1$, we infer that

$$\int_0^t \|[R_i\triangle_{-1}, v^{(k)}]\cdot\nabla\delta\theta^{(k)}\|_\infty d\tau \\ \le C\int_0^t\Big(\|\triangle_{-1}\delta\theta^{(k)}\|_\infty\sum_{j>4}\|\triangle_j\theta^{(k)}\|_\infty + \sum_{|j-j'|\le 6}\|\triangle_j\theta^{(k)}\|_\infty\|\triangle_{j'}\delta\theta^{(k)}\|_\infty\Big)d\tau \\ \le C\|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}}\|\theta^{(k)}\|_{\widetilde{L}_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}}. \tag{4.21}$$

Secondly, from Lemma 2.5 and Bony's decomposition (2.2), we can deduce that

$$\|R_i\triangle_{-1}(\delta v^{(k)}\cdot\nabla\theta^{(k+1)})\|_\infty = \|R_i\nabla\triangle_{-1}\cdot(\delta v^{(k)}\theta^{(k+1)})\|_\infty \le C\|\triangle_{-1}(\delta v^{(k)}\theta^{(k+1)})\|_\infty \\ \le C\Big(\|\triangle_{-1}(T_{\delta v^{(k)}}\theta^{(k+1)})\|_\infty + \|\triangle_{-1}(T_{\theta^{(k+1)}}\delta v^{(k)})\|_\infty + \|\triangle_{-1}R(\theta^{(k+1)},\delta v^{(k)})\|_\infty\Big) \\ \le C\sum_{j\le 4}\Big(\|S_{j-1}\delta v^{(k)}\|_\infty\|\triangle_j\theta^{(k+1)}\|_\infty + \|S_{j-1}\theta^{(k+1)}\|_\infty\|\triangle_j\delta v^{(k)}\|_\infty\Big) \\ +C\sum_{|j-j'|\le 1}\|\triangle_j\theta^{(k+1)}\|_\infty\|\triangle_{j'}\delta v^{(k)}\|_\infty,$$

which implies that

$$\int_0^t \|R_i\triangle_{-1}(\delta v^{(k)}\cdot\nabla\theta^{(k+1)})\|_\infty d\tau \\ \le C\|\delta v^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}}\|\theta^{(k+1)}\|_{L_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}} \\ \le C\Big(\|\triangle_{-1}\delta v^{(k)}\|_{L_t^\infty L^\infty} + \|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}}\Big)\|\theta^{(k+1)}\|_{\widetilde{L}_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}}. \tag{4.22}$$

Summing up (4.20)-(4.22), we obtain

$$\|\triangle_{-1}\delta v^{(k)}(t)\|_\infty \le C\sum_{m=k}^{k+1}\|\theta^{(m)}\|_{\widetilde{L}_t^2 B^{1-\frac{\alpha}{2}}_{\infty,q}}\Big(\|\triangle_{-1}\delta v^{(k)}\|_{L_t^\infty L^\infty} + \|\delta\theta^{(k)}\|_{\widetilde{L}_t^2 B^{\frac{\alpha}{2}-s}_{\infty,q}}\Big),$$

which together with (4.19) gives

$$\begin{aligned}\|\delta\theta^{(k)}\|_{\widetilde{L}^2_t B^{\frac{\alpha}{2}-s}_{\infty,q}} + \|\triangle_{-1}\delta v^{(k)}(t)\|_\infty &\le C2^{-k(1+s-\alpha)}\|\theta^0\|_{B^{1-\alpha}_{\infty,q}} \\ &+ C\sum_{m=k}^{k+1}\|\theta^{(m)}\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}\left(\|\triangle_{-1}\delta v^{(k)}\|_{L^\infty_t L^\infty} + \|\delta\theta^{(k)}\|_{\widetilde{L}^2_t B^{\frac{\alpha}{2}-s}_{\infty,q}}\right). \end{aligned} \tag{4.23}$$

The above inequality and (4.12) ensure that $\theta^{(k)}$ and $\triangle_{-1}v^{(k)}$ are two Cauchy sequences in $\widetilde{L}^2(0,T;B^{\frac{\alpha}{2}-s}_{\infty,q}) \times L^\infty_T L^\infty$. Thus, there exists a limit $\theta(t,x)$ such that as $n \to \infty$,

$$\begin{aligned} \theta^{(k)} &\longrightarrow \theta \quad \text{in} \quad \widetilde{L}^2(0,T;B^{\frac{\alpha}{2}-s}_{\infty,q}), \\ \triangle_{-1}v^{(k)} &\longrightarrow \triangle_{-1}v \quad \text{in} \quad L^\infty_T L^\infty, \end{aligned} \tag{4.24}$$

with $v = (-R_2\theta, R_1\theta)$. Then by (4.12) and interpolation, we get for any $\sigma < 1 - \frac{\alpha}{2}$,

$$\theta^{(k)} \longrightarrow \theta \quad \text{in} \quad \widetilde{L}^2(0,T;B^{\sigma}_{\infty,q}), \tag{4.25}$$

and $\theta \in \widetilde{L}^r(0,T;B^{1-\frac{\alpha}{r'}}_{\infty,q}) \cap \widetilde{L}^2(0,T;B^{\sigma}_{\infty,q})$ for $1 \le r \le \infty$ with

$$\|\theta\|_{\widetilde{L}^r_t B^{1-\frac{\alpha}{r'}}_{\infty,q}} \le C\left(\|\theta^0\|_{B^{1-\alpha}_{\infty,q}} + \epsilon_0^2\right), \quad \|\theta\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}} \le C\epsilon_0. \tag{4.26}$$

With (4.25) and (4.26), a standard limit argument will ensure that $(\theta(t,x), v(t,x))$ satisfies (1.1) in the sense of distribution.

Next, we prove $\theta \in C([0,T], \mathring{B}^{1-\alpha}_{\infty,q})$ for $1 \le q < \infty$. Thanks to the definition of Besov spaces, we have

$$\|\theta(t) - \theta(t')\|_{B^{1-\alpha}_{\infty,q}} \le \left\{\sum_{j<N} + \sum_{j\ge N}\left(2^{j(1-\alpha)}\|\theta_j(t) - \theta_j(t')\|_\infty\right)^q\right\}^{1/q}.$$

Let $\epsilon > 0$ be arbitrarily small. Since $\theta \in \widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}$, there exists $N > 0$ such that

$$\left\{\sum_{j\ge N}\left(2^{j(1-\alpha)}\|\theta_j(t) - \theta_j(t')\|_\infty\right)^q\right\}^{1/q} < \epsilon. \tag{4.27}$$

On the other hand, from Eq. (1.1), we obtain

$$\begin{aligned} &\left\{\sum_{j<N}\left(2^{j(1-\alpha)}\|\theta_j(t) - \theta_j(t')\|_\infty\right)^q\right\}^{1/q} \\ &\le \left\{\sum_{j<N}\left(2^{j(1-\alpha)}\int_{t'}^t \|-\Lambda^\alpha\theta_j - \triangle_j(v\cdot\nabla\theta)\|_\infty ds\right)^q\right\}^{1/q} \\ &\le C2^{2N}|t-t'|^{\frac{1}{2}}\left(\|\theta\|_{\widetilde{L}^2_t(t',t;B^{1-\frac{\alpha}{2}}_{\infty,q})} + \|\theta\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}\|v\|_{\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}}\right) \\ &< \epsilon, \end{aligned} \tag{4.28}$$

for $|t-t'|$ small enough, which together with (4.27) implies the continuity of $\theta(t)$. For $\sigma<1-\alpha$, we have

$$\|\theta^{(k)}(t)-\theta(t)\|_{B^{\sigma}_{\infty,q}}\le\Big\{\sum_{j<N}+\sum_{j\ge N}\Big(2^{j\sigma}\|\theta^{(k)}_j(t)-\theta_j(t)\|_\infty\Big)^q\Big\}^{1/q}.$$

Since $\theta\in\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}$ and $\theta^{(k)}$ is bounded in $\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}$, there exists $N>0$ such that

$$\Big\{\sum_{j\ge N}\Big(2^{j\sigma}\|\theta^{(k)}_j(t)-\theta_j(t)\|_\infty\Big)^q\Big\}^{1/q}<C2^{-N(1-\alpha-\sigma)}<\epsilon.$$

Similar to the proof of (4.28), we get by (4.24) and (4.25) that

$$\Big\{\sum_{j<N}\Big(2^{j\sigma}\|\theta^{(k)}_j(t)-\theta_j(t)\|_\infty\Big)^q\Big\}^{1/q}\le\|\theta^{(k)}(0)-\theta^0\|_{B^{1-\alpha}_{\infty,q}}$$
$$+C_N\Big(\|\theta^{(k)}-\theta\|_{\widetilde{L}^2_t B^{\sigma}_{\infty,q}}+\|v^{(k-1)}-v\|_{\widetilde{L}^2_t B^{\sigma}_{\infty,q}}\Big)<\epsilon,$$

for $k$ large enough. Thus,

$$\theta^{(k)}\longrightarrow\theta\quad\text{in}\quad C([0,T];B^{\sigma}_{\infty,q}),$$

which together with Lemma 2.2 implies $\theta\in C([0,T];\mathring{B}^{1-\alpha}_{\infty,q})$.

*Step 4. Uniqueness*. Assume that $\theta_1,\theta_2\in E_T$ are two solutions of (1.1) with the same initial data. Set

$$\delta\theta=\theta_2-\theta_1,\quad \delta v=v_2-v_1.$$

Then $(\delta\theta,\delta v)$ satisfies

$$\begin{cases}\partial_t\delta\theta+\Lambda^\alpha\delta\theta+v_2\cdot\nabla\delta\theta+\delta v\cdot\nabla\theta_1=0,\\ \delta\theta(0)=0.\end{cases}$$

A similar proof of (4.23) gives for $\frac{\alpha}{2}<s<1$,

$$\|\delta\theta\|_{\widetilde{L}^2_t B^{\frac{\alpha}{2}-s}_{\infty,q}}+\|\Delta_{-1}\delta v\|_{L^\infty_t L^\infty}$$
$$\le C\sum_{k=1}^{2}\|\theta^k\|_{\widetilde{L}^2_t B^{1-\frac{\alpha}{2}}_{\infty,q}}\Big(\|\Delta_{-1}\delta v\|_{L^\infty_t L^\infty}+\|\delta\theta\|_{\widetilde{L}^2_t B^{\frac{\alpha}{2}-s}_{\infty,q}}\Big),$$

which implies $\delta\theta=0$, i.e. $\theta_1=\theta_2$. In the case of $q=\infty$, we need the following extra assumption:

$$\sup_{t<T}\lim_{\epsilon\to0}\|\theta\|_{\widetilde{L}^2(t-\epsilon,t;B^{1-\alpha/2}_{\infty,\infty})}\le\epsilon_0.$$

*Step 5. Blow-up criteria*. We follow the proof of Proposition 5.2 in [1]. From the local existence theory and especially (4.8), we see that if $T^* < \infty$, then there exists a positive constant $\eta$ such that

$$\liminf_{t\to T^*}\left((T^*-t)^{\frac{1}{2}}\|\Delta_{-1}\theta(t)\|_\infty + \sum_j \omega_j^{\frac{q}{2}}(T^*-t)2^{jq(1-\alpha)}\|\theta_j(t)\|_\infty^q\right)$$
$$= \liminf_{t\to T^*}\sum_j \omega_j^{\frac{q}{2}}(T^*-t)2^{jq(1-\alpha)}\|\theta_j(t)\|_\infty^q \ge \eta,$$

where we used the fact that $\|\theta(t)\|_\infty \le \|\theta(\delta)\|_\infty$ for $\delta > 0$ such that

$$\lim_{t\to T^*}(T^*-t)^{\frac{1}{2}}\|\Delta_{-1}\theta(t)\|_\infty = 0.$$

While, thanks to Lemma 2.3, we get

$$\eta \le \liminf_{t\to T^*}\Big(\sum_{j\le N}\omega_j^{\frac{q}{2}}(T^*-t)2^{jq(1-\alpha)}\|\theta_j(t)\|_\infty^q + \sum_{j>N}\omega_j^{\frac{q}{2}}(T^*-t)2^{jq(1-\alpha)}\|\theta_j(t)\|_\infty^q\Big)$$
$$\le \liminf_{t\to T^*}\Big((T^*-t)^{\frac{q}{2}}2^{qN(1-\frac{\alpha}{2})}\|\theta(\delta)\|_\infty^q + 2^{-qN\alpha}\|\nabla\theta(t)\|_{B^0_{\infty,\infty}}^q\Big).$$

Choosing judiciously $N$, we obtain the desired result.

*Step 6. Persistency of regularity*. In this step, we will prove (c). A similar argument as leading to (4.6) and Lemma 6.2 ensures that

$$\|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}} \le \|\theta^0\|_{B^s_{\infty,q}} + \int_0^t \|2^{js}\|[\Delta_j, v]\cdot\nabla\theta\|_\infty\|_{\ell^q}d\tau$$
$$\le \|\theta^0\|_{B^s_{\infty,q}} + C\int_0^t(\|\nabla v\|_\infty + \|\nabla\theta\|_\infty)\|\theta\|_{\widetilde{L}_\tau^\infty B^s_{\infty,q}}d\tau. \qquad (4.29)$$

Then Gronwall's inequality gives

$$\|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}} \le \|\theta^0\|_{B^s_{\infty,q}}e^{C\int_0^t(\|\nabla v\|_\infty + \|\nabla\theta\|_\infty)d\tau},$$

which is equivalent to

$$\ln(e + \|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}}) \le \ln(e + \|\theta^0\|_{B^s_{\infty,q}}) + C\int_0^t(\|\nabla v\|_\infty + \|\nabla\theta\|_\infty)d\tau.$$

For any $N \in \mathbb{N}$, we have by Lemma 2.3 and Remark 2.6 that

$$\|\nabla v\|_\infty + \|\nabla\theta\|_\infty \le \sum_{j\le N}(\|\Delta_j\nabla v\|_\infty + \|\Delta_j\nabla\theta\|_\infty) + \sum_{j>N}(\|\Delta_j\nabla v\|_\infty + \|\Delta_j\nabla\theta\|_\infty)$$
$$\le C\sum_{j\le N}2^j\|\Delta_j\theta\|_\infty + \sum_{j>N}2^{-j(s-1)}\|\theta\|_{B^s_{\infty,q}}.$$

Then, taking $N$ such that $2^{-N(s-1)}\|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}} \le 1$ (i.e., $N = \frac{\log(e+\|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}})}{s\log 2}$), we obtain

$$\begin{aligned}\int_0^t (\|\nabla v\|_\infty + \|\nabla\theta\|_\infty)d\tau &\le \sum_{j\le N}\int_0^t 2^j\|\triangle_j\theta\|_\infty d\tau + t\\ &\le C\|\theta\|_{\widetilde{L}_t^1 B^1_{\infty,q}} \ln(e+\|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}}) + t.\end{aligned}$$

Thus we have

$$\ln(e+\|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}}) \le \ln(e+\|\theta^0\|_{B^s_{\infty,q}}) + C\|\theta\|_{\widetilde{L}_t^1 B^1_{\infty,q}} \ln(e+\|\theta\|_{\widetilde{L}_t^\infty B^s_{\infty,q}}) + Ct.$$

Since $q<\infty$ and $\theta\in\widetilde{L}_t^1 B^1_{\infty,q}$, we can choose $t$ small enough such that

$$\|\theta\|_{\widetilde{L}_t^1 B^1_{\infty,q}} \le \frac{1}{2C}.$$

Thus, $\theta\in\widetilde{L}_t^\infty B^s_{\infty,q}$. A standard continuity argument concludes that $\theta\in\widetilde{L}^\infty_{T^*} B^s_{\infty,q}$. Then $\theta\in C([0,T^*);\mathring{B}^s_{\infty,q})$ from the proof at the end of Step 3. This completes the proof of Theorem 1.3. □

*Remark 4.2.* If the initial data $\theta^0\in\mathring{B}^s_{\infty,q}$ for $s>1-\alpha$, there exist $T_0>0$ and a unique solution $\theta$ of (1.1) such that

$$\theta\in C([0,T_0];\mathring{B}^s_{\infty,q})\cap\widetilde{L}^1_{T_0}B^{s+\alpha}_{\infty,q}.$$

This remark can be deduced by following the proof of the existence and uniqueness part of Theorem 1.3.

Let us conclude this section by establishing the higher regularity of the local solution of (1.1).

**Theorem 4.3.** *Assume that $\theta$ is a solution of* (1.1) *on* $[0,T^*)$ *, as stated in Theorem* 1.3 (a). *Then $\theta(t)\in C([\delta,T^*);\mathring{B}^s_{\infty,q})$ for any $\delta\in(0,T^*)$ and $s>0$.*

*Proof.* Thanks to Theorem 1.3 (a), $\theta\in\widetilde{L}^1_{T^*}B^1_{\infty,q}$, thus $\theta\in L^1_{T^*}B^{1-\epsilon}_{\infty,q}$ for any $\epsilon>0$, which together with Lemma 2.2 implies that for any $\varepsilon<T^*$, there exists $t_0\in(0,\varepsilon)$ such that $\theta(t_0)\in\mathring{B}^{1-\epsilon}_{\infty,q}$. Then from Remark 4.2 and the uniqueness of the solution, we infer that $\theta\in\widetilde{L}^1(t_0,T_0;B^{1-\epsilon+\alpha}_{\infty,q})$ for some $T_0>t_0$. Repeating the above argument again, we can conclude that there exists $t_0\in(0,\delta)$ such that the local solution $\theta(t_0)$ obtained in Theorem 1.3 belongs to $\mathring{B}^{1+\frac{\alpha}{2}}_{\infty,q}$. Theorem 1.3 (c) will ensure that the solution $\theta$ satisfies $\theta\in C([t_0,T^*);\mathring{B}^{1+\frac{\alpha}{2}}_{\infty,q})$.

For any $\beta\in\mathbb{R}^+$, let us consider the equation

$$\begin{cases}\partial_t[(t-t_0)^\beta\theta] + \Lambda^\alpha(t-t_0)^\beta\theta + v\cdot\nabla(t-t_0)^\beta\theta = \beta(t-t_0)^{\beta-1}\theta,\\ (t-t_0)^\beta\theta|_{t=t_0} = 0.\end{cases}$$

In a similar way as (4.29), we get for $t_0 \leq t < T^*$,

$$\begin{aligned}
&\|(t-t_0)\theta\|_{\widetilde{L}^\infty(t_0,t;B^1_{\infty,q})} \\
&\leq \int_{t_0}^t \|2^j\|[\triangle_j, v]\cdot(\tau-t_0)\nabla\theta\|_\infty\|_{\ell^q}d\tau + C\|\theta\|_{\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}} \\
&\leq C\int_{t_0}^t (\|\nabla v\|_\infty + \|\nabla\theta\|_\infty)\,\|(\tau-t_0)\theta\|_{\widetilde{L}^\infty(t_0,\tau;B^1_{\infty,q})}d\tau + C\|\theta\|_{\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}},
\end{aligned}$$

from which and Gronwall's inequality, we infer that

$$\begin{aligned}
\|(t-t_0)\theta\|_{\widetilde{L}^\infty(t_0,t;B^1_{\infty,q})} &\leq C\|\theta\|_{\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}} e^{C\int_{t_0}^t(\|\nabla v\|_\infty+\|\nabla\theta\|_\infty)d\tau} \\
&\leq C\|\theta\|_{\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}} e^{C\int_{t_0}^t\|\theta\|_{B^1_{\infty,1}}d\tau},
\end{aligned}$$

where we used in the last inequality

$$\|\nabla v\|_\infty \leq C\|\nabla v\|_{B^0_{\infty,1}} \leq C\|\theta\|_{B^1_{\infty,1}}.$$

Then we can get by an induction argument that

$$\begin{aligned}
\|(t-t_0)^{n+1}\theta\|_{\widetilde{L}^\infty_t B^{n+1}_{\infty,q}} &\leq C(n+1)e^{C\int_{t_0}^t\|\theta\|_{B^1_{\infty,1}}d\tau}\|(t-t_0)^n\theta\|_{\widetilde{L}^\infty_t B^n_{\infty,q}} \\
&\leq C_n e^{C(n+2)\int_{t_0}^t\|\theta\|_{B^1_{\infty,1}}d\tau}\|\theta\|_{\widetilde{L}^\infty_t B^{1-\alpha}_{\infty,q}},
\end{aligned}$$

from which and interpolation, we can deduce that $(t-t_0)^\beta\theta \in \widetilde{L}^\infty(t_0,t;\mathring{B}^\beta_{\infty,q})$ for any $\beta \in \mathbb{R}^+$ and $t \in [t_0, T^*)$. Thus $\theta \in C([\delta, T^*); \mathring{B}^\beta_{\infty,q})$. $\square$

## 5. Proof of Theorem 1.5

This section is devoted to the global well-posedness of (1.1) with the critical dissipation. We shall use the argument of modulus of continuity introduced by Kiselev, Nazarov and Volberg[18]. Here we will only sketch the proof, see [1] for more details.

Let $T^*$ be the maximal existence time of the solution in the space $C([0,T^*);\mathring{B}^0_{\infty,q}) \cap \widetilde{L}^1(0,T^*;B^1_{\infty,q})$. Let $\lambda$ be a real positive number determined later and $T_1 \in (0,T^*)$. We define the set

$$I \stackrel{\text{def}}{=} \Big\{T \in [T_1,T^*); \forall t \in [T_1,T], x \neq y \in \mathbb{R}^2, |\theta(t,x)-\theta(t,y)| < \omega_\lambda(|x-y|)\Big\},$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly non-decreasing, concave, $\omega(0) = 0, \omega'(0) < +\infty$, $\lim_{\xi\to 0^+}\omega''(\xi) = -\infty$ and $\omega_\lambda(|x-y|) = \omega(\lambda|x-y|)$. First of all, if we take

$$\lambda = \frac{\omega^{-1}(3\|\theta^0\|_\infty)}{2\|\theta^0\|_\infty}\|\nabla\theta(T_1)\|_\infty,$$

then we can deduce from the maximum principal and the properties of $\omega$ that $T_1 \in I$, thus the set $I$ is non-empty. From the construction, the set $I$ is an interval of the form $[T_1, T_*)$. Now there are three possibilities:

The first case is $T_* = T^*$ and in this case we have the necessary $T^* = +\infty$ because the Lipschitz norm of $\theta$ does not blow up.

The second case is $T_* \in I$ which is impossible. In fact, using the higher regularity of $\theta$(see Theorem 4.3), we can infer that there exists a $\delta > 0$ such that for all $t \in [T_*, T_*+\delta]$,

$$|\theta(t,x) - \theta(t,y)| < \omega_\lambda(|x-y|).$$

Thus, we obtain that $T_* + \delta \in I$ which contradicts the fact that $T_*$ is maximal.

The last case is $T_* \notin I$. By the time continuity of $\theta$, there exists $x \neq y$ such that

$$\theta(T_*, x) - \theta(T_*, y) = \omega_\lambda(\xi), \quad \text{with} \quad \xi = |x-y|.$$

As in [18], we choose the continuous function $\omega$ as follows:

$$\begin{cases} \omega(\xi) = \xi - \xi^{3/2}, & \text{when } 0 \leq \xi \leq \delta, \\ \omega'(\xi) = \frac{\gamma}{\xi(4+\log(\xi/\delta))}, & \text{when } \xi > \delta, \end{cases}$$

here $\delta, \gamma$ are small enough and satisfy $\delta > \gamma > 0$. With this choice, it is shown in [18] that

$$f'(T_*) < 0, \quad \text{where} \quad f(t) = \theta(t,x) - \theta(t,y).$$

Thus, this scenario can not occur since $f(t) \leq f(T_*)$, $\forall t \in [0, T_*]$. This completes the proof of Theorem 1.5. □

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## 6. Appendix

In this Appendix, we shall present the product estimate and commutator estimate in the inhomogeneous Besov spaces. We can refer to Lecture notes [6,15] for a more detailed introduction.

**Lemma 6.1.** *Let* $1 \leq q \leq \infty$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$, $\rho_1, \rho_2 < 0$, $\rho_1 + \rho_2 + 1 > 0$, *and* $v$ *be a solenoidal vector field. Then there holds*

$$\|v \cdot \nabla\theta\|_{L^r_t B^{\rho_1+\rho_2}_{\infty,q}} \leq C \|v\|_{L^{r_1}_t B^{\rho_1}_{\infty,q}} \|\theta\|_{L^{r_2}_t B^{1+\rho_2}_{\infty,q}}.$$

*Proof.* Using Bony's decomposition (2.2), we write

$$v \cdot \nabla\theta = T_{v^i}\partial_i\theta + T_{\partial_i\theta}v^i + R(v^i, \partial_i\theta).$$

Thanks to (2.1), we have

$$\Delta_j(T_{v^i}\partial_i\theta) = \sum_{|j'-j|\leq 4} \Delta_j(S_{j'-1}v^i\partial_i\Delta_{j'}\theta).$$

From Lemma 2.3 and $\rho_1 < 0$, it follows that

$$\|\triangle_j(T_{v^i}\partial_i\theta)\|_\infty \le C\sum_{|j'-j|\le 4} 2^{j'}\sum_{k\le j'-2}\|\triangle_k v\|_\infty\|\triangle_{j'}\theta\|_\infty$$
$$\le C2^{j(1-\rho_1)}\|v\|_{B^{\rho_1}_{\infty,q}}\sum_{|j'-j|\le 4}\|\triangle_{j'}\theta\|_\infty. \tag{6.1}$$

Similarly, we have

$$\triangle_j(T_{\partial_i\theta}v^i) = \sum_{|j'-j|\le 4}\triangle_j(S_{j'-1}(\partial_i\theta)\triangle_{j'}v^i).$$

From $\rho_2 < 0$, Lemma 2.3 applied gives

$$\|\triangle_j(T_{\partial_i\theta}v^i)\|_\infty \le C\sum_{|j'-j|\le 4}\sum_{k\le j'-2}2^k\|\triangle_k\theta\|_\infty\|\triangle_{j'}v\|_\infty$$
$$\le C2^{-j\rho_2}\|\theta\|_{B^{1+\rho_2}_{\infty,q}}\sum_{|j'-j|\le 4}\|\triangle_{j'}v\|_\infty. \tag{6.2}$$

Since $\mathrm{div} v = 0$, we have

$$\triangle_j R(v^i, \partial_i\theta) = \sum_{j',j''\ge j-3;|j'-j''|\le 1}\partial_i\triangle_j(\triangle_{j'}v^i\triangle_{j''}\theta),$$

from which and Lemma 2.3, it follows that

$$\|\triangle_j R(v^i, \partial_i\theta)\|_\infty \le C\sum_{j',j''\ge j-3;|j'-j''|\le 1}2^j\|\triangle_{j'}v\|_\infty\|\triangle_{j''}\theta\|_\infty$$
$$\le C2^j\|v\|_{B^{\rho_1}_{\infty,q}}\sum_{j'\ge j-3}2^{-j'\rho_1}\|\triangle_{j'}\theta\|_\infty. \tag{6.3}$$

Summing up (6.1)–(6.3), we obtain

$$\|v\cdot\nabla\theta\|_{L^r_t B^{\rho_1+\rho_2}_{\infty,q}} \le C\|v\|_{L^{r_1}_t B^{\rho_1}_{\infty,q}}\|\theta\|_{L^{r_2}_t B^{1+\rho_2}_{\infty,q}}.$$

Indeed, we have by the Young's inequality for the series and $\rho_1+\rho_2+1>0$ that

$$\|2^{j(\rho_1+\rho_2)}\|\triangle_j R(v^i, \partial_i\theta)\|_\infty\|_{\ell^q} \le C\|v\|_{B^{\rho_1}_{\infty,q}}\|\sum_{j'\ge j-3}2^{(\rho_1+\rho_2+1)(j-j')}2^{j'(\rho_2+1)}\|\triangle_{j'}\theta\|_\infty\|_{\ell^q}$$
$$\le C\|v\|_{B^{\rho_1}_{\infty,q}}\|\theta\|_{B^{1+\rho_2}_{\infty,q}}.$$

This finishes the proof of Lemma 6.1. □

**Lemma 6.2.** *Let* $1\le q\le\infty$, $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}\le 1$, $\rho_1,\rho_2<1$, $\rho_1+\rho_2>0$, $\rho_1>0$, *and* $v$ *be a solenoidal vector field. Then there holds*

$$\|\{2^{j(\rho_1+\rho_2-1)}\|[\triangle_j, v]\cdot\nabla\theta\|_{L^r_t L^\infty}\}\|_{\ell^q} \le C\|\nabla v\|_{\widetilde{L}^{r_1}_t B^{\rho_1-1}_{\infty,q}}\|\nabla\theta\|_{\widetilde{L}^{r_2}_t B^{\rho_2-1}_{\infty,q}}. \tag{6.4}$$

*Moreover, if* $v=(-R_2 f, R_1 f)$, *we have for all* $s>0$,

$$\|2^{js}\|[\triangle_j, v\cdot\nabla]\theta\|_\infty\|_{\ell^q} \le C\Big(\|\nabla\theta\|_\infty\|f\|_{B^s_{\infty,q}}+\|\nabla v\|_\infty\|\theta\|_{B^s_{\infty,q}}\Big). \tag{6.5}$$

*Proof.* We firstly prove (6.4). Using Bony's decomposition (2.2), we write

$$\begin{aligned}[\triangle_j, v\cdot\nabla]\theta &= [\triangle_j, T_{v^i}]\partial_i\theta + \Big(\triangle_j(T_{\partial_i\theta}v^i) - T_{\partial_i\triangle_j\theta}v^i\Big)\\ &\quad + \Big(\triangle_j R(v^i,\partial_i\theta) - R(v^i,\partial_i\triangle_j\theta)\Big)\\ &\overset{\text{def}}{=} I + II + III.\end{aligned}$$

Firstly, in view of the definition of $\triangle_j$, we write

$$\begin{aligned} I &= \sum_k \Big\{\triangle_j(S_{k-1}v\cdot\nabla\triangle_k\theta) - S_{k-1}v\cdot\triangle_j\nabla\triangle_k\theta\Big\}\\ &= \sum_{|k-j|\le 4} 2^{2j}\int h(2^j(x-y))(S_{k-1}v(y) - S_{k-1}v(x))\cdot\nabla\triangle_k\theta(y)dy\\ &= \sum_{|k-j|\le 4} 2^{2j}\int h(2^j y)\int_0^1 y\cdot\nabla S_{k-1}v(x+\tau y)d\tau\cdot\nabla\triangle_k\theta(y+x)dy,\end{aligned}$$

which implies

$$\|I\|_\infty \le C2^{-j}\sum_{|k-j|\le 4}\|\nabla S_{k-1}v\|_\infty\|\nabla\triangle_k\theta\|_{L^\infty}, \tag{6.6}$$

from which and $\rho_1 < 1$, it follows that

$$\begin{aligned}&\Big\|\{2^{j(\rho_1+\rho_2-1)}\|I\|_{L^r_tL^\infty}\}\Big\|_{\ell^q}\\ &\quad\le C\Bigg\|\{2^{j(\rho_1+\rho_2-2)}\sum_{|k-j|\le 4}\|\nabla S_{k-1}v\|_{L^{r_1}_tL^\infty}\|\nabla\triangle_k\theta\|_{L^{r_2}_tL^\infty}\}\Bigg\|_{\ell^q}\\ &\quad\le C\|\nabla v\|_{\widetilde{L}^{r_1}_t B^{\rho_1-1}_{\infty,q}}\|\nabla\theta\|_{\widetilde{L}^{r_2}_t B^{\rho_2-1}_{\infty,q}}.\end{aligned} \tag{6.7}$$

Similarly, we have

$$\begin{aligned}\|II\|_\infty &= \Bigg\|\sum_k\Big(\triangle_j(S_{k-1}\partial_i\theta\triangle_k v^i) - S_{k-1}\triangle_j\partial_i\theta\triangle_k v^i\Big)\Bigg\|_\infty\\ &\le \Bigg\|\sum_{|k-j|\le 4}\triangle_j(S_{k-1}\partial_i\theta\triangle_k v^i) - S_{k-1}\triangle_j\partial_i\theta\triangle_k v^i\Bigg\|_\infty\\ &\quad + \Bigg\|\sum_{k>j+4}S_{k-1}\triangle_j\partial_i\theta\triangle_k v^i\Bigg\|_\infty\\ &\le C2^{-j}\sum_{|k-j|\le 4}\|\nabla\triangle_k v\|_\infty\|\nabla S_{k-1}\theta\|_\infty + C\|\nabla\triangle_j\theta\|_\infty\sum_{k>j+4}\|\triangle_k v\|_\infty,\end{aligned}$$

from which and the assumption that $\rho_2 < 1$ and $\rho_1 > 0$, we infer that

$$\begin{aligned}
&\left\|\{2^{j(\rho_1+\rho_2-1)}\|II\|_{L_t^r L^\infty}\}\right\|_{\ell^q} \\
&\leq C\left\|\{2^{j(\rho_1+\rho_2-2)}\sum_{|k-j|\leq 4}\|\nabla\triangle_k v\|_{L_t^{r_1}L^\infty}\|\nabla S_{k-1}\theta\|_{L_t^{r_2}L^\infty}\right\|_{\ell^q} \\
&\quad + C\|\nabla\theta\|_{\widetilde{L}_t^{r_2}B_{\infty,q}^{\rho_2-1}}\left\|\{2^{j\rho_1}\sum_{k>j+4}\|\triangle_k v\|_{L_t^{r_1}L^\infty}\}\right\|_{\ell^q} \\
&\leq C\|\nabla v\|_{\widetilde{L}_t^{r_1}B_{\infty,q}^{\rho_1-1}}\|\nabla\theta\|_{\widetilde{L}_t^{r_2}B_{\infty,q}^{\rho_2-1}},
\end{aligned}\tag{6.8}$$

where we used Lemma 2.3 in the last inequality.

Since $\operatorname{div} v = 0$, we can rewrite $III$ as

$$\begin{aligned}
III = &\sum_{|k-j|\leq 4;|k-k'|\leq 1}\left\{\triangle_j(\triangle_k v\cdot\triangle_{k'}\nabla\theta) - \triangle_k v\cdot\triangle_j\triangle_{k'}\nabla\theta\right\} \\
&+\sum_{k>j+4;|k-k'|\leq 1}\operatorname{div}\triangle_j(\triangle_k v\triangle_{k'}\theta).
\end{aligned}$$

Then a similar proof of (6.6) gives

$$\begin{aligned}
\|III\|_\infty \leq C2^{-j}&\sum_{|k-j|\leq 4;|k-k'|\leq 1}\|\nabla\triangle_k v\|_\infty\|\nabla\triangle_{k'}\theta\|_\infty \\
&+C2^j\sum_{k>j+4;|k-k'|\leq 1}\|\triangle_k v\|_\infty\|\triangle_{k'}\theta\|_\infty,
\end{aligned}$$

which together with Lemma 2.3 and $\rho_1+\rho_2 > 0$ implies that

$$\|\{2^{j(\rho_1+\rho_2-1)}\|III\|_{L_t^r L^\infty}\}\|_{\ell^q} \leq C\|\nabla v\|_{\widetilde{L}_t^{r_1}B_{\infty,q}^{\rho_1-1}}\|\nabla\theta\|_{\widetilde{L}_t^{r_2}B_{\infty,q}^{\rho_2-1}}. \tag{6.9}$$

Summing up (6.7)–(6.9), we conclude the proof of (6.4). Since the proof of (6.5) is completely similar, here we omit it. □

Let us close this section by the following commutator estimate.

**Lemma 6.3.** *Let $v = (-R_2 f, R_1 f)$. Then there holds*

$$\|[R_k\triangle_{-1}, v\cdot\nabla]\theta\|_\infty \leq C\|\triangle_{-1}\theta\|_\infty\sum_{j>4}\|\triangle_j f\|_\infty + C\sum_{|j-j'|\leq 6}\|\triangle_j\theta\|_\infty\|\triangle_{j'}f\|_\infty.$$

*Proof.* Using Bony's decomposition (2.2), we write

$$\begin{aligned}
[R_k\triangle_{-1}, v\cdot\nabla]\theta &= [R_k\triangle_{-1}, T_{v^i}]\partial_i\theta + \left(R_k\triangle_{-1}(T_{\partial_i\theta}v^i) - T_{\partial_i R_k\triangle_{-1}\theta}v^i\right) \\
&\quad + \left(R_k\triangle_{-1}R(v^i,\partial_i\theta)\right) - R\left(v^i,\partial_i R_k\triangle_{-1}\theta\right) \\
&\overset{\text{def}}{=} I + II + III.
\end{aligned}$$

We denote by $K(x)$ the kernel of the operator $\nabla\triangle_{-1}R_k$. From Lemma 2.5, we have

$$|K(x)| \le C(1+|x|)^{-3}.$$

Given $0<\delta<1$, thanks to $\operatorname{div} v = 0$, we have

$$\begin{aligned}
\|I\|_\infty &= \sum_{j\le 4}\left\|\int K(x-y)\cdot(S_{j-1}v(y)-S_{j-1}v(x))\triangle_j\theta(y)dy\right\|_\infty \\
&= \sum_{j\le 4}\left\|\int |K(x-y)||x-y|^\delta\frac{|S_{j-1}v(y)-S_{j-1}v(x)|}{|x-y|^\delta}|\triangle_j\theta(y)|dy\right\|_\infty \\
&\le \sum_{j\le 4}\int |K(x)||x|^\delta dx \sup_{x\ne y}\frac{|S_{j-1}v(y)-S_{j-1}v(x)|}{|x-y|^\delta}\|\triangle_j\theta\|_\infty \\
&\le C\sum_{j\le 4}\|S_{j-1}f\|_\infty\|\triangle_j\theta\|_\infty. \qquad (6.10)
\end{aligned}$$

Here in the last inequality we used the fact that for $j\le 3$,

$$\begin{aligned}
\sup_{x\ne y}\frac{|S_jv(y)-S_jv(x)|}{|x-y|^\delta} &\le C\sup_{x\ne y}\frac{|S_jf(y)-S_jf(x)|}{|x-y|^\delta} \\
&\le C\|S_jf\|_{C^\delta}\le C2^{j\delta}\|S_jf\|_\infty\le C\|S_jf\|_\infty, \qquad (6.11)
\end{aligned}$$

which follows from the boundedness of $R_k$ on the homogeneous Hölder spaces and Lemma 2.3.

Similarly, from $\operatorname{div} v = 0$ and the fact that $\|\triangle_j v\|_\infty \le C\|\triangle_j f\|_\infty$ for $j\ge 0$, we deduce that

$$\begin{aligned}
\|II\|_\infty &\le \sum_{j\le 4}\left\|\int K(x-y)\cdot(\triangle_jv(y)-\triangle_jv(x))S_{j-1}\theta(y)dy\right\|_\infty \\
&\quad+\sum_{j>4}\|S_{j-1}(\nabla R_k\triangle_{-1})\theta\|_\infty\|\triangle_jv\|_\infty \\
&\le C\sum_{j\le 4}\|S_{j-1}f\|_\infty\|\triangle_j\theta\|_\infty + C\|\triangle_{-1}\theta\|_\infty\sum_{j>4}\|\triangle_jf\|_\infty, \qquad (6.12)
\end{aligned}$$

and

$$\begin{aligned}
\|III\|_\infty &= \left\|\sum_{j\le 4;|j-j'|\le 1}\int K(x-y)\cdot(\triangle_jv(y)-\triangle_jv(x))\triangle_{j'}\theta(y)dy\right\|_\infty \\
&\quad+\sum_{j>4;|j-j'|\le 1}\|\operatorname{div} R_k\triangle_{-1}(\triangle_jv\triangle_{j'}\theta)\|_\infty \\
&\le C\sum_{|j-j'|\le 1}\|\triangle_jf\|_\infty\|\triangle_{j'}\theta\|_\infty. \qquad (6.13)
\end{aligned}$$

Summing up (6.10)–(6.13), we conclude the proof of this lemma. □

## References


1. Abidi, H., Hmidi, T.: On the global well-posedness of the critical quasi-geostrophic equation. SIAM J. Math. Anal. **40**, 167–185 (2008)
2. Bony, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. École Norm. Sup. **14**, 209–246 (1981)
3. Caffarelli, L., Vasseur, A.: Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. Ann. Math. **171**(3), 1903–1930 (2010)
4. Chae, D., Lee, J.: Global well-posedness in the super-critical dissipative quasi-geostrophic equations. Commun. Math. Phys. **233**, 297–311 (2003)
5. Chemin, J.Y.: *Perfect Incompressibe Fluids*, New York: Oxford University Press, 1998
6. Chemin, J.Y.: Localization in Fourier space and Navier-Stokes system. In: *Phase Space Analysis of Partial Differential Equations*, Proceedings 2004, CRM Series, Pisa, pp. 53–136
7. Chen, Q., Miao, C., Zhang, Z.: A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation. Commun. Math. Phys. **271**, 821–838 (2007)
8. Constantin, P., Córdoba, D., Wu, J.: On the critical dissipative quasi-geostrophic thermal active scalar. Indiana Univ. Math. J. **50**, 97–107 (2001)
9. Constantin, P., Majda, A., Tabak, E.: Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar. Nonlinearity **7**, 1495–1533 (1994)
10. Constantin, P., Wu, J.: Behavior of solutions of 2D quasi-geostrophic equations. SIAM J. Math. Anal. **30**, 937–948 (1999)
11. Constantin, P., Wu, J.: Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation. Annal. L'Inst. H. Poincaré Analyse Non Linéaire **25**, 1103–1110 (2008)
12. Córdoba, D.: Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation. Ann. of Math. **148**, 1135–1152 (1998)
13. Córdoba, D., Fefferman, C.: Growth of solutions for QG and 2D Euler equations. J. Amer. Math. Soc. **15**, 665–670 (2002)
14. Córdoba, A., Córdoba, D.: A maximum principle applied to quasi-geostrophic equations. Commun. Math. Phys. **249**, 511–528 (2004)
15. Danchin, R.: Fourier analysis methods for PDEs, http://perso-math.univ-mlv.fr/users/danchin.raphael/courschine.pdf, 2005
16. Hmidi, T., Keraani, S.: Global solutions of the super-critical 2D Q-G equation in Besov spaces. Adv. in Math. **214**, 618–638 (2007)
17. Ju, N.: Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space. Commun. Math. Phys. **251**, 365–376 (2004)
18. Kiselev, A., Nazarov, F., Volberg, A.: Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. Invent. Math. **167**, 445–453 (2007)
19. Pedlosky, J.: Geophysical Fluid Dynamics, New York: Springer, 1987
20. Stein, E.M.: *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton, NJ: Princeton Univ Press, 1993
21. Triebel, H.: *Theory of Function Spaces. Monograph in Mathematics*, Vol. **78**, Basel: Birkhauser Verlag, 1983
22. Wu, J.: Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces. SIAM J. Math. Anal. **36**, 1014–1030 (2004)
23. Wu, J.: The two-dimensional quasi-geostrophic equation with critical or supercritical dissipation. Nonlinearity **18**, 139–154 (2005)
24. Zhang, P., Zhang, Z.: On the local well-posedness for the dumbbell model of polymeric fluids. J. Part. Diff. Eqs. **21**, 234–252 (2008)


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