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Asymptotic Stability of the Relativistic Boltzmann Equation for the Soft Potentials

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Abstract: In this paper it is shown that unique solutions to the relativistic Boltzmann equation exist for all time and decay with any polynomial rate towards their steady state relativistic Maxwellian provided that the initial data starts out sufficiently close in L^∞_ℓ. If the initial data are continuous then so is the corresponding solution. We work in the case of a spatially periodic box. Conditions on the collision kernel are generic in the sense of Dudyński and Ekiel-Jeżewska (Commun Math Phys 115(4):607–629, 1985); this resolves the open question of global existence for the soft potentials.

Contents

1. Introduction

The relativistic Boltzmann equation is a fundamental model for fast moving particles; it can be written with appropriate initial conditions as

$$p^\mu \partial_\mu F = \mathcal{C}(F, F).$$

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The collision operator [4,9] is given by

$$\mathcal{C}(f,h) = \frac{c}{2}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q')[f(p')h(q') - f(p)h(q)].$$

The transition rate, $W(p,q|p',q')$, can be expressed as

$$W(p,q|p',q') = s\sigma(g,\theta)\delta^{(4)}(p^\mu + q^\mu - p^{\mu\prime} - q^{\mu\prime}),$$

where $\sigma(g,\theta)$ is the differential cross-section or scattering kernel; it measures the interactions between particles. The speed of light is the physical constant denoted c. Standard references in relativistic Kinetic theory include [8,9,21,48,54]. The rest of the notation is defined in the sequel.

1.1. A brief history of relativistic kinetic theory. Early results include those on derivations [40], local existence [3], and linearized solutions [14,18].

DiPerna-Lions renormalized weak solutions [13] were shown to exist in 1992 by Dudyński and Ekiel-Jeżewska [19] globally in time for large data, using the causality of the relativistic Boltzmann equation [16,17]. See also [1,57] and [37,38]. In particular [1] proves the strong L^1 convergence to a relativistic Maxwellian, after taking a subsequence, for weak solutions with large initial data that is not necessarily close to an equilibrium solution. There are also results in the context of local [7] and global [49] Newtonian limits, and near vacuum results [22,35,49] and blow-up [2] for the gain term only. We also mention a study of the collision map and the pre-post collisional change of variables in [23]. For more discussion of historical results, we refer to [49].

We review in more detail the results most closely related to those in this paper. In 1993, Glassey and Strauss [24] proved for the first time global existence and uniqueness of smooth solutions which are initially close to a relativistic Maxwellian and in a periodic box. They also established exponential convergence to the Maxwellian. Their assumptions on the differential cross-section, σ, fell into the regime of “hard potentials” as discussed below. In 1995, they extended that result to the whole space case [20] where the convergence rate is polynomial. More recent results with reduced restrictions on the cross-section were proven in [36], using the energy method from [29–32]; these results also apply to the hard potentials.

For relativistic interactions–when particles are fast moving–an important physical regime is the “soft potentials”; see [15] for a physical discussion. Despite their importance, prior to the results in this paper there were no existence results for the soft potentials in the context of strong nearby relativistic Maxwellian global solutions. In 1988 a general physical assumption was given in [18]; see (2.7) and (2.8). In this paper we will prove global existence of unique L^∞ near equilibrium solutions to the relativistic Boltzmann equation and rapid time decay under the full physical assumption on the cross-section from [18]. Our main focus is of course the soft potentials; and we do not require any angular cut-off (although the angular singularity will not be worse than just barely integrable).

1.2. Notation. Prior to discussing our main results, we will now define the notation of the problem carefully. In special relativity the momentum of a particle is denoted by p^μ, $\mu = 0, 1, 2, 3$. Let the signature of the metric be $(-{+}{+}{+})$. Without loss of generality, we set the rest mass for each particle $m = 1$. The momentum for each particle is restricted to the mass shell $p^\mu p_\mu = -c^2$ with $p_0 > 0$.

Further with $p \in \mathbb{R}^3$,we may write $p^\mu = (p^0, p)$and similarly $q^\mu = (q^0, q)$. Then the energy of a relativistic particle with momentum p is $p_0 = \sqrt{c^2 + |p|^2}$. We are raising and lowering indices with the Minkowski metric $p_\mu = g_{\mu\nu} p^\nu$, where $(g_{\mu\nu}) \overset{\text{def}}{=} \text{diag}(-1\ 1\ 1\ 1)$. We use the Einstein convention of implicit summation over repeated indices. The Lorentz inner product is then given by

$$p^\mu q_\mu \overset{\text{def}}{=} -p_0 q_0 + \sum_{i=1}^{3} p_i q_i.$$

Then also $q_0 = \sqrt{c^2 + |q|^2} > 0$. Note our convention for raising and lowering indices; we only use it in this paragraph and in the Appendix. Now the streaming term of the relativistic Boltzmann equation is given by

$$p^\mu \partial_\mu = p_0 \partial_t + p \cdot \nabla_x.$$

We thus write the relativistic Boltzmann equation as

$$\partial_t F + \hat{p} \cdot \nabla_x F = \mathcal{Q}(F, F). \tag{1.1}$$

Here $\mathcal{Q}(F, F) = \mathcal{C}(F, F)/p_0$, with $\mathcal{C}$ defined at the top of this paper.

Above we consider $F = F(t, x, p)$ to be a function of time $t \in [0, \infty)$, space $x \in \mathbb{T}^3$ and momentum $p \in \mathbb{R}^3$. The normalized velocity of a particle is denoted

$$\hat{p} = c \frac{p}{p_0} = \frac{p}{\sqrt{1 + |p|^2/c^2}}. \tag{1.2}$$

Steady states of this model are the well known Jüttner solutions, also known as the relativistic Maxwellian. They are given by

$$J(p) \overset{\text{def}}{=} \frac{\exp\left(-cp_0/(k_B T)\right)}{4\pi c k_B T K_2(c^2/(k_B T))},$$

where $K_2(\cdot)$ is the Bessel function $K_2(z) \overset{\text{def}}{=} \frac{z^2}{2} \int_1^\infty e^{-zt}(t^2 - 1)^{3/2} dt$, T is the temperature and k_B is Boltzmann's constant.

In the rest of this paper, without loss of generality but for the sake of simplicity, we will now normalize all physical constants to one, including the speed of light to $c = 1$. So that in particular we denote the relativistic Maxwellian by

$$J(p) = \frac{e^{-p_0}}{4\pi}. \tag{1.3}$$

Henceforth we let C, and sometimes c denote generic positive inessential constants whose value may change from line to line.

We will now define the quantity s, which is the square of the energy in the "center of momentum" system, $p + q = 0$, as

$$s = s(p^\mu, q^\mu) \overset{\text{def}}{=} -(p^\mu + q^\mu)(p_\mu + q_\mu) = 2\left(-p^\mu q_\mu + 1\right) \geq 0. \tag{1.4}$$

The relative momentum is denoted

$$g = g(p^\mu, q^\mu) \overset{\text{def}}{=} \sqrt{(p^\mu - q^\mu)(p_\mu - q_\mu)} = \sqrt{2(-p^\mu q_\mu - 1)}. \tag{1.5}$$

Notice that $s = g^2 + 4$. We warn the reader that this notation, which is used in [9], may differ from other author's notation by a constant factor.

Conservation of momentum and energy for elastic collisions is expressed as

$$p^\mu + q^\mu = p^{\mu\prime} + q^{\mu\prime}. \tag{1.6}$$

The angle θ is defined by

$$\cos\theta \stackrel{\text{def}}{=} (p^\mu - q^\mu)(p'_\mu - q'_\mu)/g^2. \tag{1.7}$$

This angle is well defined under (1.6), see [21, Lemma 3.15.3].

We now consider the center of momentum expression for the collision operator below. An alternate expression for the collision operator was derived in [24]; see [49] for an explanation of the connection between the expression from [24] and the one we give just now. One may use Lorentz transformations as described in [9] and [50] to reduce the delta functions and obtain

$$\mathcal{Q}(f,h) = \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \;\; v_ø \, \sigma(g,\theta) \, [f(p')h(q') - f(p)h(q)], \tag{1.8}$$

where $v_ø = v_ø(p,q)$ is the Møller velocity given by

$$v_ø = v_ø(p,q) \stackrel{\text{def}}{=} \sqrt{\left|\frac{p}{p_0} - \frac{q}{q_0}\right|^2 - \left|\frac{p}{p_0} \times \frac{q}{q_0}\right|^2} = \frac{g\sqrt{s}}{p_0 q_0}. \tag{1.9}$$

The post-collisional momentum in the expression (1.8) can be written:

$$\begin{aligned} p' &= \frac{p+q}{2} + \frac{g}{2}\left(\omega + (\gamma - 1)(p+q)\frac{(p+q)\cdot\omega}{|p+q|^2}\right), \\ q' &= \frac{p+q}{2} - \frac{g}{2}\left(\omega + (\gamma - 1)(p+q)\frac{(p+q)\cdot\omega}{|p+q|^2}\right), \end{aligned} \tag{1.10}$$

where $\gamma = (p_0 + q_0)/\sqrt{s}$. The energies are then

$$\begin{aligned} p'_0 &= \frac{p_0 + q_0}{2} + \frac{g}{2\sqrt{s}}\omega \cdot (p+q), \\ q'_0 &= \frac{p_0 + q_0}{2} - \frac{g}{2\sqrt{s}}\omega \cdot (p+q). \end{aligned} \tag{1.11}$$

These clearly satisfy (1.6). The angle further satisfies $\cos\theta = k \cdot \omega$ with $k = k(p,q)$ and $|k| = 1$. This is the expression for the collision operator that we will use.

For a smooth function $h(p)$ the collision operator satisfies

$$\int_{\mathbb{R}^3} dp \begin{pmatrix} 1 \\ p \\ p_0 \end{pmatrix} \mathcal{Q}(h,h)(p) = 0.$$

By integrating the relativistic Boltzmann equation (1.1) and using this identity we obtain the conservation of mass, momentum and energy for solutions as

$$\frac{d}{dt} \int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \begin{pmatrix} 1 \\ p \\ p_0 \end{pmatrix} F(t) = 0.$$

Furthermore the entropy of the relativistic Boltzmann equation is defined as

$$\mathcal{H}(t) \stackrel{\text{def}}{=} -\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \; F(t,x,p) \ln F(t,x,p).$$

The celebrated Boltzmann H-Theorem is then formally

$$\frac{d}{dt}\mathcal{H}(t) \geq 0,$$

which says that the entropy of solutions is increasing as time passes. Notice that the steady state relativistic Maxwellians (1.3) maximize the entropy which formally implies convergence to (1.3) in large time. It is this formal reasoning that our main results make mathematically rigorous in the context of perturbations of the relativistic Maxwellian for a general class of cross-sections.

2. Statement of the Main Results

We are now ready to discuss in detail our main results. We define the standard perturbation $f(t,x,p)$ to the relativistic Maxwellian (1.3) as

$$F \stackrel{\text{def}}{=} J + \sqrt{J} f.$$

With (1.6) we observe that the quadratic collision operator (1.8) satisfies

$$\mathcal{Q}(J,J) = 0.$$

Then the relativistic Boltzmann equation (1.1) for the perturbation $f = f(t,x,p)$ takes the form

$$\partial_t f + \hat{p} \cdot \nabla_x f + L(f) = \Gamma(f,f), \quad f(0,x,p) = f_0(x,p). \tag{2.1}$$

The linear operator $L(f)$, as defined in (2.2), and the non-linear operator $\Gamma(f,f)$, defined in (2.5), are derived from an expansion of the relativistic Boltzmann collision operator (1.8). In particular, the linearized collision operator is given by

$$\begin{aligned} L(h) &\stackrel{\text{def}}{=} -J^{-1/2}\mathcal{Q}(J,\sqrt{J}h) - J^{-1/2}\mathcal{Q}(\sqrt{J}h, J) \\ &= \nu(p)h - K(h). \end{aligned} \tag{2.2}$$

Above the multiplication operator takes the form

$$\nu(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o} \, \sigma(g,\theta) \; J(q). \tag{2.3}$$

The remaining integral operator is

$$\begin{aligned} K(h) \stackrel{\text{def}}{=} &\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o} \, \sigma(g,\theta) \sqrt{J(q)} \left\{ \sqrt{J(q')}\, h(p') + \sqrt{J(p')}\, h(q') \right\} \\ &- \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega \; v_{\o} \, \sigma(g,\theta) \, \sqrt{J(q)J(p)}\, h(q) \\ = \; & K_2(h) - K_1(h). \end{aligned} \tag{2.4}$$

The non-linear part of the collision operator is defined as

$$\Gamma(h_1, h_2) \stackrel{\text{def}}{=} J^{-1/2}\mathcal{Q}(\sqrt{J}h_1, \sqrt{J}h_2) = \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega\, v_{ø}\, \sigma(g, \theta) \sqrt{J(q)}[h_1(p')h_2(q') - h_1(p)h_2(q)]. \quad (2.5)$$

Without loss of generality we can assume that the mass, momentum, and energy conservation laws for the perturbation, $f(t, x, p)$, hold for all $t \geq 0$ as

$$\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \begin{pmatrix} 1 \\ p \\ p_0 \end{pmatrix} \sqrt{J(p)}\, f(t, x, p) = 0. \quad (2.6)$$

We now state our conditions on the collisional cross-section.

Hypothesis on the collision kernel. *For soft potentials we assume the collision kernel in* (1.8) *satisfies the following growth/decay estimates*:

$$\begin{aligned} \sigma(g, \omega) &\lesssim g^{-b}\, \sigma_0(\omega), \\ \sigma(g, \omega) &\gtrsim \left(\frac{g}{\sqrt{s}}\right) g^{-b}\, \sigma_0(\omega). \end{aligned} \quad (2.7)$$

We also consider angular factors such that $\sigma_0(\omega) \lesssim \sin^\gamma \theta$ *with* $\gamma > -2$*. Additionally* $\sigma_0(\omega) \geq 0$ *and* $\sigma_0(\omega)$ *should be non-zero on a set of positive measure. We suppose further that* $0 < b < \min(4, 4 + \gamma)$.

For hard potentials we make the assumption

$$\begin{aligned} \sigma(g, \omega) &\lesssim \left(g^a + g^{-b}\right) \sigma_0(\omega), \\ \sigma(g, \omega) &\gtrsim \left(\frac{g}{\sqrt{s}}\right) g^a\, \sigma_0(\omega). \end{aligned} \quad (2.8)$$

In addition to the previous parameter ranges we consider $0 \leq a \leq 2 + \gamma$ *and also* $0 \leq b < \min(4, 4 + \gamma)$ (*in this case we allow the possibility of* $b = 0$).

This hypothesis includes the full range of conditions which were introduced in [18] as a general physical assumption on the kernel (of course we add the corresponding necessary lower bound in each case); see also [15] for further discussions.

Prior to stating our main theorem, we will need to introduce the following mostly standard notation. The notation $A \lesssim B$ will imply that a positive constant C exists such that $A \leq CB$ holds uniformly over the range of parameters which are present in the inequality and moreover that the precise magnitude of the constant is unimportant. The notation $B \gtrsim A$ is equivalent to $A \lesssim B$, and $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$. We work with the L^∞ norm

$$\|h\|_\infty \stackrel{\text{def}}{=} \text{ess sup}_{x \in \mathbb{T}^3,\ p \in \mathbb{R}^3} |h(x, p)|.$$

If we only wish to take the supremum in the momentum variables we write

$$|h|_\infty \stackrel{\text{def}}{=} \text{ess sup}_{p \in \mathbb{R}^3} |h(p)|.$$

We will additionally use the following standard L^2 spaces:

$$\|h\|_2 \stackrel{\text{def}}{=} \sqrt{\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp\ |h(x,p)|^2}, \quad |h|_2 \stackrel{\text{def}}{=} \sqrt{\int_{\mathbb{R}^3} dp\ |h(p)|^2}.$$

Similarly in the sequel any norm represented by one set of lines instead of two only takes into account the momentum variables. Next we define the norm which measures the (very weak) "dissipation" of the linear operator

$$\|h\|_\nu \stackrel{\text{def}}{=} \sqrt{\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp\ \nu(p)|h(x,p)|^2}.$$

The $L^2(\mathbb{R}^n)$ inner product is denoted $\langle\cdot,\cdot\rangle$. We use $(\cdot,\cdot)$ to denote the $L^2(\mathbb{T}^n\times\mathbb{R}^n)$ inner product. Now, for $\ell\in\mathbb{R}$, we define the following weight function:

$$w_\ell = w_\ell(p) \stackrel{\text{def}}{=} \begin{cases} p_0^{\ell b/2}, & \text{for the soft potentials: } (2.7) \\ p_0^{\ell}, & \text{for the hard potentials: } (2.8). \end{cases} \tag{2.9}$$

For the soft potentials $w_1(p)\approx 1/\nu(p)$ (Lemma 3.1). We consider weighted spaces

$$\|h\|_{\infty,\ell} \stackrel{\text{def}}{=} \|w_\ell h\|_\infty, \quad \|h\|_{2,\ell} \stackrel{\text{def}}{=} \|w_\ell h\|_2, \quad \|h\|_{\nu,\ell} \stackrel{\text{def}}{=} \|w_\ell h\|_\nu.$$

Here as usual $L^\infty_\ell(\mathbb{T}^3\times\mathbb{R}^3)$ is the Banach space with norm $\|\cdot\|_{\infty,\ell}$, etc. We will also use the momentum only counterparts of these spaces

$$|h|_{\infty,\ell} \stackrel{\text{def}}{=} |w_\ell h|_\infty, \quad |h|_{2,\ell} \stackrel{\text{def}}{=} |w_\ell h|_2, \quad |h|_{\nu,\ell} \stackrel{\text{def}}{=} |w_\ell h|_\nu.$$

We further need the following time decay norm:

$$|||f|||_{k,\ell} \stackrel{\text{def}}{=} \sup_{s\ge 0}\ (1+s)^k\ \|f(s)\|_{\infty,\ell}. \tag{2.10}$$

We are now ready to state our main results. We will first state our theorem for the soft potentials which is the main focus of this paper:

Theorem 2.1 (Soft Potential). *Fix $\ell > 3/b$. Given $f_0 = f_0(x,p)\in L^\infty_\ell(\mathbb{T}^3\times\mathbb{R}^3)$ which satisfies (2.6) initially, there is an $\eta>0$ such that if $\|f_0\|_{\infty,\ell}\le\eta$, then there exists a unique global mild solution, $f=f(t,x,p)$, to Eq. (2.1) with soft potential kernel (2.7). For any $k\ge 0$, there is a $k'=k'(k)\ge 0$ such that*

$$\|f\|_{\infty,\ell}(t) \le C_{\ell,k}(1+t)^{-k}\|f_0\|_{\infty,\ell+k'}.$$

These solutions are continuous if it is so initially. We further have positivity, i.e. $F = J+\sqrt{J}f\ge 0$ if $F_0 = J+\sqrt{J}f_0\ge 0$.

We point out that $k'(0)=0$ in the above theorem, and $k'(k)\ge k$ can in general be computed explicitly from our proof. Our approach also applies to the hard potentials and in that case we state the following theorem which can be proven using the same methods.

Theorem 2.2 (Hard Potential). *Fix $\ell > 3/2$. Given $f_0 = f_0(x, p) \in L^\infty_\ell(\mathbb{T}^3 \times \mathbb{R}^3)$ which satisfies (2.6) initially, there is an $\eta > 0$ such that if $\| f_0\|_{\infty,\ell} \le \eta$, then there exists a unique global mild solution, $f = f(t, x, p)$, to Eq. (2.1) with hard potential kernel (2.8) which further satisfies for some $\lambda > 0$ that*

$$\| f \|_{\infty,\ell}(t) \le C_\ell e^{-\lambda t} \| f_0\|_{\infty,\ell}.$$

These solutions are continuous if it is so initially. We further have positivity, i.e. $F = J + \sqrt{J} f \ge 0$ if $F_0 = J + \sqrt{J} f_0 \ge 0$.

Previous results for the hard potentials are as follows. In 1993 Glassey and Strauss [24] proved asymptotic stability such as Theorem 2.2 in L^∞_ℓ with $\ell > 3/2$. They consider collisional cross-sections which satisfy (2.8) for the parameters $b \in [0, 1/2)$, $a \in [0, 2 - 2b)$ and either $\gamma \ge 0$ or

$$|\gamma| < \min \left\{ 2 - a, \frac{1}{2} - b, \frac{2 - 2b - a}{3} \right\},$$

which in particular implies $\gamma > -\frac{1}{2}$ if $b = 0$ say. They further assume a related growth bound on the derivative of the cross-section $\left|\frac{\partial \sigma}{\partial g}\right|$. In [36] this growth bound was removed while the rest of the assumptions on the cross-section from [24] remained the same. These results also sometimes work in smoother function spaces, and we note that we could also include space-time regularity to our solutions spaces.

However for the relativistic Boltzmann equation in (1.1) the issue of adding momentum derivatives is more challenging. In recent years many new tools have been developed to solve these problems. A method was developed in non-relativistic kinetic theory to study the soft potential Boltzmann equation with angular cut-off by Guo in [30]. This approach makes crucial use of the momentum derivatives, and Sobolev embeddings to control the singular kernel of the collision operator. Yet in the context of relativistic interactions, high derivatives of the post-collisional variables (1.10) create additional high singularities which are hard to control. Worse in the more common relativistic variables from [24], derivatives of the post-collisional momentum exhibit enough momentum growth to preclude hope of applying the method from [30]; these growth estimates on the derivatives were known much earlier in [23].

Notice also that the methods for proving time decay, such as [10,52,53], require working in the context of smooth solutions. We would also like to mention recent developments on Landau Damping [44] proving exponential decay with analytic regularity. Furthermore we point out very recent results proving rapid time decay of smooth perturbative solutions to the Newtonian Boltzmann equation without the Grad angular cut-off assumption as in [25–27]. In this paper however we avoid smooth function spaces in particular because of the aforementioned problem created by the relativistic post-collisional momentum. Other recent work [33] developed a framework to study near Maxwellian boundary value problems for the hard potential Newtonian Boltzmann equation in L^∞_ℓ. In particular a key component of this analysis was to consider solutions to the Boltzmann equation (2.1) after linearization as

$$\left(\partial_t + \hat{p} \cdot \nabla_x + L\right) f = 0, \quad f(0, x, p) = f_0(x, p), \tag{2.11}$$

with L defined in (2.2). The semi-group for this equation (relativistic or not) will satisfy a certain 'A-Smoothing property' which was pioneered by Vidav [56] about 40 years ago. This A-Smoothing property which appears at the level of the second iterate of the semi-group, as seen below in (4.4), has been employed effectively for instance in [21,24,33,34]. Related to this a new compactness connected to a similar iteration has been observed in the 'mixing lemma' of [41–43]. The key new step in [33] was to estimate the second iterate in L^2 rather than L^∞ and then use a linear decay theory in L^2 which does not require regularity and is exponential for the hard potentials case that paper considered. Note further that a method was developed in [52,53] to prove rapid polynomial decay for the soft potential Newtonian Boltzmann and Landau equations; this is related to the articles [5,6,10,55], all of which make use of smooth function spaces.

In the present work we adapt the method from [53] to prove rapid L^2 polynomial decay of solutions to the linear equation (2.11) without regularity, and we further adapt the L^2 estimate from [33] to control the second iterate. This approach, for the soft potentials in particular, yields global bounds and slow decay, $O(1/t)$, of solutions to (2.11) in L^∞_ℓ. The details and complexity of this program are however intricate in the relativistic setting. And fortunately, this slow decay is sufficient to just barely control the nonlinearity and prove global existence to the full non-linear problem as in Theorem 2.1.

It is not clear how to apply the above methods to establish the rapid "almost exponential" polynomial decay from Theorem 2.1 in this low regularity L^∞_ℓ framework. To prove the rapid decay in Theorem 2.1 our key contribution is to perform a new high order expansion of the remainder term, $R_1(f)$, from (4.15). This is contained in Proposition 6.1 and its proof. This term, $R_1(f)$, is the crucial problematic term which only appears to exhibit first order decay.

More generally, the main difficulty with proving rapid decay for the soft potentials is created by the high momentum values, where the time decay is diluted by the momentum decay. This results in the generation of additional weights, typically one weight for each order of time decay. At the same time the term $R_1(f)$ only allows us to absorb one weight, $w_1(p)$, and therefore only appears to allow one order of time decay. In our proof of Proposition 6.1 we are able to overcome this apparent obstruction by performing a new high order expansion for $k \geq 2$ as

$$R_k(f) = G_k(f) + D_k(f) + N_k(f) + L_k(f) + R_{k+1}(f).$$

(The expansion from $R_1(f)$ to $R_2(f)$ requires a slightly different approach.) At every level of this expansion we can peel of each of the terms $G_k(f)$, $D_k(f)$, $N_k(f)$, and $L_k(f)$ which will for distinct reasons exhibit rapid polynomial decay to any order. In particular we use an L^2_ℓ estimate for $L_k(f)$ which crucially makes use of the bounded velocities that come with special relativity. On the other hand the last term $R_{k+1}(f)$ will be able to absorb $k+1$ momentum weights, and therefore it will be able to produce time decay up to the order $k+1$. By continuing this expansion to any finite order, we are able to prove rapid polynomial decay. We hope that this expansion will be useful in other relativistic contexts.

The rest of this paper is organized as follows. In Sect. 3 we prove L^2_ℓ decay of solutions to the linearized relativistic Boltzmann equation (2.11). Then in Sect. 4 we prove global L^∞_ℓ bounds and slow decay of solutions to (2.11). Following that in Sect. 5 we prove nonlinear L^∞_ℓ bounds using the slow linear decay, and we thereby conclude global existence. In the remaining Sections 6 and 7 we prove linear and non-linear rapid decay respectively. Then in the Appendix we give an exposition of a derivation of the kernel of the compact part of the linear operator from (2.4).

3. Linear L^2 Bounds and Decay

It is our purpose in this section to prove global in time L^2_ℓ bounds for solutions to the linearized Boltzmann equation (2.11) with initial data in the same space L^2_ℓ. We will then prove high order, almost exponential decay for these solutions. We begin by stating a few important lemmas, and then we use them to prove the desired integral bounds (3.5) and the decay it implies in Theorem 3.7. We will prove these lemmas at the end of the section.

Lemma 3.1. *Consider* (2.3) *with the soft potential collision kernel* (2.7)*. Then*

$$\nu(p) \approx p_0^{-b/2}.$$

More generally, $\int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega\, v_\phi\, \sigma(g,\theta)\, J^\alpha(q) \approx p_0^{-b/2}$ *for any* $\alpha > 0$.

We will next look at the "compact" part of the linear operator K. The most difficult part is K_2 from (2.4). We will employ a splitting to cut out the singularity. The new element of this splitting is the Lorentz invariant argument: g. Given a small $\epsilon > 0$, choose a smooth cut-off function $\chi = \chi(g)$ satisfying

$$\chi(g) = \begin{cases} 1 & \text{if } \ g \geq 2\epsilon \\ 0 & \text{if } \ g \leq \epsilon. \end{cases} \tag{3.1}$$

Now with (3.1) and (2.4) we define

$$\begin{aligned} K_2^{1-\chi}(h) \overset{\text{def}}{=} & \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega\ (1-\chi(g))\ v_\phi\, \sigma(g,\theta)\sqrt{J(q)}\sqrt{J(q')}\, h(p') \\ & + \int_{\mathbb{R}^3} dq \int_{\mathbb{S}^2} d\omega\ (1-\chi(g))\ v_\phi\, \sigma(g,\theta)\sqrt{J(q)}\sqrt{J(p')}\, h(q'). \end{aligned} \tag{3.2}$$

Define $K_1^{1-\chi}(h)$ similarly. We use the splitting $K \overset{\text{def}}{=} K^{1-\chi} + K^\chi$. A splitting with the same goals has been previously used for the Newtonian Boltzmann equation in [53]. The advantage for soft potentials, on the singular region, is that one has exponential decay in all momentum variables. Then on the region away from the singularity we are able to extract a modicum of extra decay which is sufficient for the rest of the estimates in this paper, see Lemma 3.2 just below.

In the sequel we will use the Hilbert-Schmidt form for the non-singular part. The following representation is derived in the Appendix:

$$K_i^\chi(h) = \int_{\mathbb{R}^3} dq\, k_i^\chi(p,q)\, h(q), \quad i = 1, 2.$$

We will also record the kernel $k_i^\chi(p,q)$ below in (3.9) and (3.10). We have

Lemma 3.2. *Consider the soft potentials* (2.7)*. The kernel enjoys the estimate*

$$0 \leq k_2^\chi(p,q) \leq C_\chi\, (p_0 q_0)^{-\zeta}\, (p_0+q_0)^{-b/2}\, e^{-c|p-q|}, \quad C_\chi, c > 0,$$

with $\zeta \overset{def}{=} \min\{2-|\gamma|, 4-b, 2\}/4 > 0$*. This estimate also holds for* k_1^χ*.*

We remark that for certain ranges of the parameters γ and b the decay in Lemma 3.2 could be improved somewhat. In particular the term k_1^χ in (3.9) clearly yields exponential decay. However what is written above is sufficient for our purposes.

We will use the decomposition given above and the decay of the kernels in Lemma 3.2 to establish the following lemma.

Lemma 3.3. *Fix any small* $\eta > 0$, *we may decompose* K *from* (2.4) *as*

$$K = K_c + K_s,$$

where K_c *is a compact operator on* L^2_ν. *In particular for any* $\ell \geq 0$, *and for some* $R = R(\eta) > 0$ *sufficiently large we have*

$$|\langle w_\ell^2 K_c h_1, h_2\rangle| \leq C_\eta |\mathbf{1}_{\leq R} h_1|_2 |\mathbf{1}_{\leq R} h_2|_2.$$

Above $\mathbf{1}_{\leq R}$ *is the indicator function of the ball of radius* R. *Furthermore*

$$|\langle w_\ell^2 K_s h_1, h_2\rangle| \leq \eta\, |h_1|_{\nu,\ell}\, |h_2|_{\nu,\ell}\, .$$

This estimate will be important for proving the coercivity of the linearized collision operator, L, away from its null space. More generally, from the H-theorem L is non-negative and for every fixed (t, x) the null space of L is given by the five dimensional space [21]:

$$\mathcal{N} \stackrel{\text{def}}{=} \text{span}\left\{\sqrt{J},\, p_1\sqrt{J},\, p_2\sqrt{J},\, p_3\sqrt{J},\, p_0\sqrt{J}\right\}. \tag{3.3}$$

We define the orthogonal projection from $L^2(\mathbb{R}^3)$ onto the null space $\mathcal{N}$ by $\mathbf{P}$. Further expand $\mathbf{P}h$ as a linear combination of the basis in (3.3):

$$\mathbf{P}h \stackrel{\text{def}}{=} \left\{a^h(t,x) + \sum_{j=1}^{3} b_j^h(t,x) p_j + c^h(t,x) p_0\right\}\sqrt{J}. \tag{3.4}$$

We can then decompose $f(t, x, p)$ as

$$f = \mathbf{P}f + \{\mathbf{I} - \mathbf{P}\}f.$$

With this decomposition we have

Lemma 3.4. $L \geq 0$. $Lh = 0$ *if and only if* $h = \mathbf{P}h$, *and* $\exists \delta_0 > 0$ *such that*

$$\langle Lh, h\rangle \geq \delta_0 |\{\mathbf{I} - \mathbf{P}\}h|_\nu^2.$$

This last statement on coercivity holds as an operator inequality at the functional level. The following lemma is a well-known statement of the linearized H-Theorem for solutions to (2.11) which was shown in the non-relativistic case in [33].

Lemma 3.5. *Given initial data* $f_0 \in L^2_\ell\left(\mathbb{T}^3 \times \mathbb{R}^3\right)$ *for some* $\ell \geq 0$, *which satisfies* (2.6) *initially, consider the corresponding solution,* f, *to* (2.11) *in the sense of distributions. Then there is a universal constant* $\delta_\nu > 0$ *such that*

$$\int_0^1 ds\ \|\{\mathbf{I} - \mathbf{P}\}f\|_\nu^2(s) \geq \delta_\nu \int_0^1 ds\ \|\mathbf{P}f\|_\nu^2(s).$$

We will give just one more operator level inequality.

Lemma 3.6. *Given* $\delta \in (0,1)$ *and* $\ell \geq 0$*, there are constants* $C, R > 0$ *such that*

$$\langle w_\ell^2 L h, h\rangle \geq \delta |h|_{\nu,\ell}^2 - C|\mathbf{1}_{\leq R} h|_2^2.$$

Notice that Lemma 3.6 follows easily from Lemma 3.3. With these results, we can prove the following energy inequality for any $\ell \geq 0$:

$$\|f\|_{2,\ell}^2(t) + \delta_\ell \int_0^t ds\ \|f\|_{\nu,\ell}^2(s) \leq C_\ell \|f_0\|_{2,\ell}^2, \quad \exists \delta_\ell, C_\ell > 0, \tag{3.5}$$

as long as $\|f_0\|_{2,\ell}^2$ is finite. We will prove this first for $\ell = 0$, and then for arbitrary $\ell > 0$. In the first case we multiply (2.11) with f and integrate to obtain

$$\|f\|_2^2(t) + \int_0^t ds\ (Lf, f) = \|f_0\|_2^2.$$

First suppose that $t \in \{1, 2, \ldots\}$. By Lemma 3.4 we have

$$\begin{aligned}\int_0^t ds\ (Lf,f) &= \sum_{j=0}^{t-1}\int_0^1 ds\ (Lf,f)(s+j) \geq \sum_{j=0}^{t-1}\delta_0 \int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2(s+j)\\ &= \frac{\delta_0}{2}\sum_{j=0}^{t-1}\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2(s+j)\\ &\quad+\frac{\delta_0}{2}\sum_{j=0}^{t-1}\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2(s+j).\end{aligned}$$

Then by Lemma 3.5 the second term above satisfies the lower bound

$$\frac{\delta_0}{2}\sum_{j=0}^{t-1}\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2(s+j) \geq \frac{\delta_0\delta_\nu}{2}\sum_{j=0}^{t-1}\int_0^1 ds\ \|\mathbf{P}f\|_\nu^2(s+j).$$

This follows in particular because $f_j(s,x,v) \overset{\text{def}}{=} f(s+j,x,p)$ satisfies the linearized Boltzmann equation (2.11) on the interval $0 \leq s \leq 1$. Collecting the previous two estimates yields

$$\begin{aligned}\int_0^t ds\ (Lf,f) &\geq \frac{\delta_0}{2}\sum_{j=0}^{t-1}\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2(s+j)\\ &\quad+\frac{\delta_0\delta_\nu}{2}\sum_{j=0}^{t-1}\int_0^1 ds\ \|\mathbf{P}f\|_\nu^2(s+j)\\ &\geq \tilde{\delta}\sum_{j=0}^{t-1}\int_0^1 ds\ \|f\|_\nu^2(s+j) = \tilde{\delta}\int_0^t ds\ \|f\|_\nu^2(s),\end{aligned}$$

with $\tilde{\delta} = \frac{1}{2}\min\left\{\frac{\delta_0\delta_\nu}{2}, \frac{\delta_0}{2}\right\} > 0$. Plugging this estimate into the last one establishes the energy inequality (3.5) for $\ell = 0$ and $t \in \{1, 2, \dots\}$. For an arbitrary $t > 0$, we choose $m \in \{0, 1, 2, \dots\}$ such that $m \leq t \leq m+1$. We then split the time integral as $[0, t] = [0, m] \cup [m, t]$. For the time interval $[m, t]$ we have

$$\|f(t)\|_2^2 + \int_m^t ds\ (Lf, f) = \|f(m)\|_2^2. \tag{3.6}$$

Since $L \geq 0$ by Lemma 3.4, we see that $\|f(t)\|_2^2 \leq \|f(m)\|_2^2$. We then have

$$\|f\|_2^2(t) + \int_m^t ds\ (Lf, f) + \frac{\tilde{\delta}}{2}\int_0^m ds\ \|f\|_\nu^2(s) \leq C_k\|f_0\|_2^2.$$

Furthermore with Lemma 3.6 for $C_\delta > 0$ independent of t we obtain

$$\begin{aligned}\int_m^t ds\ (Lf, f) &\geq \delta\int_m^t ds\ \|f\|_\nu^2(s) - C_\delta\int_m^t ds\ \|\mathbf{1}_{\leq R} f\|_2^2(s)\\ &\geq \delta\int_m^t ds\ \|f\|_\nu^2(s) - C_\delta \sup_{m\leq s\leq t}\|f\|_2^2(s).\end{aligned} \tag{3.7}$$

However we have already shown that $\sup_{m\leq s\leq t}\|f\|_2^2(s) \leq \|f\|_2^2(m) \leq C\|f_0\|_2^2$. Collecting the last few estimates for any $t > 0$ we have (3.5) for $\ell = 0$.

For $\ell > 0$, we multiply Eq. (2.11) with $w_\ell^2 f$ and integrate to obtain

$$\|f\|_{2,\ell}^2(t) + \int_0^t ds\ (w_\ell^2 Lf, f) = \|f_0\|_{2,\ell}^2.$$

In this case using Lemma 3.6 we have

$$\int_0^t ds\ (w_\ell^2 Lf, f) \geq \delta\int_0^t ds\ \|f\|_{\nu,\ell}^2(s) - C_{\delta,\ell}\int_0^t ds\ \|f\|_\nu^2(s).$$

Adding this estimate to the line above it, we obtain

$$\|f\|_{2,\ell}^2(t) + \delta\int_0^t ds\ \|f\|_{\nu,\ell}^2(s) \leq \|f_0\|_{2,\ell}^2 + C_{\delta,\ell}\int_0^t ds\ \|f\|_\nu^2(s).$$

Now the just proven integral inequality (3.5) in the case $\ell = 0$ (as an upper bound for the integral on the right-hand side) establishes the claimed energy inequality (3.5) for any $\ell > 0$. In fact we prove a more general time decay version of this inequality in (3.51). These will imply the following rapid decay theorem.

Theorem 3.7. *Consider a solution $f(t, x, p)$ to the linear Boltzmann equation* (2.11) *with data $\|f_0\|_{2,\ell+k} < \infty$ for some $\ell, k \geq 0$. Then*

$$\|f\|_{2,\ell}(t) \leq C_{\ell,k}(1+t)^{-k}\|f_0\|_{2,\ell+k}.$$

This allows "almost exponential" polynomial decay of any order.

Theorem 3.7 is the main result of this section. We now proceed to prove each of Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.5, and then Theorem 3.7 in order. These proofs will complete this section on linear decay.

Proof of Lemma 3.1. We will use the soft potential hypothesis for the collision kernel (2.7) to estimate (2.3). For $\alpha > 0$ we more generally consider

$$\nu_\alpha(p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq\ v_ø\ J^\alpha(q) \int_0^\pi d\theta\ \sigma(g,\theta)\ \sin\theta.$$

Initially we record the following pointwise estimates:

$$\frac{|p-q|}{\sqrt{p_0 q_0}} \le g \le |p-q|, \quad \text{and} \quad g \le 2\sqrt{p_0 q_0}, \tag{3.8}$$

see [24, Lemma 3.1]. However notice that in [24] their "g" is actually defined to be $1/2$ times our "g". With the Møller velocity (1.9), we thus also have

$$s = 4 + g^2 \lesssim p_0 q_0, \quad v_ø \lesssim 1.$$

For $b \in (1,4)$, these estimates including (3.8) yield

$$v_ø\ g^{-b} = \frac{\sqrt{s}}{p_0 q_0} g^{1-b} \lesssim \frac{\sqrt{s}}{p_0 q_0} \frac{(p_0 q_0)^{(b-1)/2}}{|p-q|^{b-1}} \lesssim \frac{(p_0 q_0)^{(b-2)/2}}{|p-q|^{b-1}}.$$

We thus obtain

$$\begin{aligned}
\nu_\alpha(p) &\lesssim \int_{\mathbb{R}^3} dq\ \frac{(p_0 q_0)^{(b-2)/2}}{|p-q|^{b-1}}\ J^\alpha(q) \int_0^\pi d\theta\ \sin^{1+\gamma}\theta \\
&\lesssim p_0^{(b-2)/2} \int_{\mathbb{R}^3} dq\ \frac{J^{\alpha/2}(q)}{|p-q|^{b-1}} \int_0^\pi d\theta\ \sin^{1+\gamma}\theta \\
&\lesssim p_0^{(b-2)/2} p_0^{1-b} \approx p_0^{-b/2}.
\end{aligned}$$

We note that the angular integral is finite since $\gamma > -2$:

$$\int_0^\pi d\theta\ \sin^{1+\gamma}\theta = C_\gamma < \infty.$$

In this case above and the cases below a key point is that

$$\int_{\mathbb{R}^3} dq\ \frac{J^{\alpha/2}(q)}{|p-q|^\beta} \approx p_0^{-\beta}, \quad \forall \beta < 3.$$

For $b \in (0,1)$, with (3.8) we alternatively have

$$v_ø\ g^{-b} = \frac{\sqrt{s}}{p_0 q_0} g^{1-b} \lesssim \frac{(p_0 q_0)^{(1-b)/2}}{\sqrt{p_0 q_0}} \lesssim (p_0 q_0)^{-b/2}.$$

Now in a slightly easier way than for the previous case we have $\nu_\alpha(p) \lesssim p_0^{-b/2}$.

For the lower bound with $b \in (2,4)$, we use (2.7), (3.8) and the estimate

$$v_ø \left(\frac{g}{\sqrt{s}}\right) g^{-b} = \frac{g^{2-b}}{p_0 q_0} \gtrsim \frac{(p_0 q_0)^{(2-b)/2}}{p_0 q_0} \approx (p_0 q_0)^{-b/2}.$$

Alternatively for $b \in (0,2)$, we use (3.8) to get the estimate

$$v_ø \left(\frac{g}{\sqrt{s}}\right) g^{-b} = \frac{g^{2-b}}{p_0 q_0} \gtrsim \frac{1}{p_0 q_0} \left(\frac{|p-q|}{\sqrt{p_0 q_0}}\right)^{2-b} \approx (p_0 q_0)^{(b-4)/2} |p-q|^{2-b}.$$

We use these estimates to obtain $\nu_\alpha(p) \gtrsim p_0^{-b/2}$ as for the upper bound. □

Now that the proof of Lemma 3.1 is complete, we develop the necessary formulation for the proof of Lemma 3.2. It is trivial to write the Hilbert-Schmidt form for the cut-off (3.1) part of K_1 from (2.4) as

$$k_1^\chi(p,q) = \sqrt{J(q)J(p)}\,\chi(g)\int_{\mathbb{S}^2} d\omega\, v_\o\,\sigma(g,\theta). \tag{3.9}$$

Furthermore, we can write the Hilbert-Schmidt form for the cut-off (3.1) part of K_2 from (2.4) in the following somewhat complicated integral form

$$k_2^\chi(p,q) \stackrel{\text{def}}{=} c_2\frac{s^{3/2}}{gp_0q_0}\chi(g)\int_0^\infty dy\, e^{-l\sqrt{1+y^2}}\,\sigma\left(\frac{g}{\sin(\psi/2)},\psi\right) \times\frac{y\left(1+\sqrt{1+y^2}\right)}{\sqrt{1+y^2}}I_0(jy). \tag{3.10}$$

Here $c_2>0$, and the modified Bessel function of index zero is defined by

$$I_0(y)=\frac{1}{2\pi}\int_0^{2\pi} e^{y\cos\varphi}\,d\varphi. \tag{3.11}$$

We are also using the simplifying notation

$$\sin(\psi/2) \stackrel{\text{def}}{=} \frac{\sqrt{2}g}{[g^2-4+(g^2+4)\sqrt{1+y^2}]^{1/2}}. \tag{3.12}$$

Additionally

$$l=(p_0+q_0)/2,\quad j=\frac{|p\times q|}{g}.$$

This derivation for a pure k_2 operator appears to go back to [9,11], where it was done in the case of the alternate linearization $F=J(1+f)$. The author gave a similar derivation with many details explained in full in [50], including the explicit form of the necessary Lorentz transformation; in particular Eq. (5.51) in this thesis. For the benefit of the reader, we have provided the derivation of k_2^χ in the case of the linearization $F=J+\sqrt{J}f$ in the Appendix to this paper.

Note that it is elementary to verify that (3.9) under (2.7) satisfies the estimate in Lemma 3.2. In the proof below we focus on the more involved estimate for (3.10).

Proof of Lemma 3.2. We consider K_2 from (2.4) with Hilbert-Schmidt form represented by the kernel (3.10). From (2.7), we have the bound

$$\sigma\left(\frac{g}{\sin(\psi/2)},\psi\right)\lesssim\left(\frac{\sin(\psi/2)}{g}\right)^b\sin^\gamma\psi \lesssim g^\gamma\left(\frac{\sin(\psi/2)}{g}\right)^{b+\gamma}\cos^\gamma(\psi/2). \tag{3.13}$$

We have just used the trigonometric identity

$$\sin\psi = 2\sin(\psi/2)\cos(\psi/2).$$

We estimate these angles in three cases. In each of the cases below we will repeatedly use the following known [24, p.317] estimates

$$\frac{y}{2\sqrt{1+y^2}} \le \cos(\psi/2) \le 1. \tag{3.14}$$

The estimates above and below are proved for instance in [24, Lemma 3.1]:

$$\frac{1}{\sqrt{s}(1+y^2)^{1/4}} \lesssim \frac{\sin(\psi/2)}{g} \lesssim \frac{1}{g(1+y^2)^{1/4}}. \tag{3.15}$$

Notice that in general from (3.10) and (3.13) we have the bound

$$\begin{aligned} k_2^{\chi}(p,q) &\lesssim \frac{s^{3/2}}{g p_0 q_0} g^{\gamma} \chi(g) \int_0^{\infty} dy\, e^{-l\sqrt{1+y^2}} y I_0(jy) \\ &\quad \times \left(\frac{\sin(\psi/2)}{g}\right)^{b+\gamma} \cos^{\gamma}(\psi/2). \end{aligned} \tag{3.16}$$

We will estimate this upper bound in three cases.

Case 1. Take $\gamma = -|\gamma| < 0$ and $b+\gamma = b-|\gamma| < 0$. Then we have

$$\begin{aligned} \left(\frac{\sin(\psi/2)}{g}\right)^{b+\gamma} \cos^{\gamma}(\psi/2) &\lesssim s^{-(b-|\gamma|)/2} \left(\frac{1}{(1+y^2)^{1/4}}\right)^{b-|\gamma|} \left(\frac{y}{\sqrt{1+y^2}}\right)^{-|\gamma|} \\ &\lesssim s^{-(b-|\gamma|)/2} y^{-|\gamma|} (1+y^2)^{-(b-3|\gamma|)/4} \\ &\lesssim s^{-(b-|\gamma|)/2} y^{-|\gamma|} (1+y^2)^{-(b+3\gamma)/4}. \end{aligned}$$

Above we have used (3.14) and (3.15). In this case from (3.16) we have

$$\begin{aligned} k_2^{\chi}(p,q) &\lesssim \frac{s^{3/2}}{g p_0 q_0} \frac{s^{-(b-|\gamma|)/2}}{g^{|\gamma|}} \chi(g) \int_0^{\infty} dy\, e^{-l\sqrt{1+y^2}} y^{1+\gamma} I_0(jy) (1+y^2)^{-(b+3\gamma)/4} \\ &\le C_{\epsilon} \frac{s^{(3+|\gamma|-b)/2}}{p_0 q_0} \chi(g) \int_0^{\infty} dy\, e^{-l\sqrt{1+y^2}} y^{1-|\gamma|} I_0(jy) (1+y^2)^{-(b-3|\gamma|)/4}. \end{aligned}$$

Note that we have just used the $\epsilon > 0$ from (3.1). From (2.7), $b \in (0, 4-|\gamma|)$ and $\gamma \in (-2, 0)$. We have in this case $b \in (0, |\gamma|)$. Hence $b - 3|\gamma| \in (-6, 0)$.

We evaluate the relevant integral above as

$$\begin{aligned} \int_0^{\infty} dy\, e^{-l\sqrt{1+y^2}} y^{1-|\gamma|} I_0(jy) (1+y^2)^{(3|\gamma|-b)/4} &= \int_0^1 dy + \int_1^{\infty} dy \\ \lesssim \int_0^1 dy\, e^{-l\sqrt{1+y^2}} y^{1-|\gamma|} I_0(jy) &+ \int_1^{\infty} dy\, e^{-l\sqrt{1+y^2}} y I_0(jy) (1+y^2)^{(|\gamma|-b)/4}. \end{aligned} \tag{3.17}$$

For the unbounded integral we have used the estimate $y^{-|\gamma|} \lesssim (1+y^2)^{-2|\gamma|/4}$. In this case $|\gamma| - b \in (0, 2)$ since $0 < b < |\gamma|$.

To estimate the remaining integrals above we use the precise theory of special functions, see e.g. [39,46]. We define

$$\tilde{K}_\alpha(l,j) \stackrel{\text{def}}{=} \int_0^\infty dy\, e^{-l\sqrt{1+y^2}} y I_0(jy)(1+y^2)^{\alpha/4}.$$

Then for $\alpha \in [-2,2]$ from [24, Cor. 1 and Cor. 2] it is known that

$$\tilde{K}_\alpha(l,j) \le C l^{1+\alpha/2} e^{-c|p-q|}. \tag{3.18}$$

We also define

$$\tilde{I}_\eta(l,j) \stackrel{\text{def}}{=} \int_0^1 dy\, e^{-l\sqrt{1+y^2}} y^{1-\eta} I_0(jy).$$

Then for $\eta \in [0,2)$ from [24, Lemma 3.6] we have the asymptotic estimate:

$$\tilde{I}_\eta(l,j) \le C e^{-c\sqrt{l^2-j^2}} \le C e^{-c|p-q|/2}. \tag{3.19}$$

We will use these estimates in each of the cases below.

Thus in this Case 1 by (3.19) and (3.18) the integral in (3.17) is

$$\lesssim e^{-c|p-q|/2} + l^{1+(|\gamma|-b)/2} e^{-|p-q|/2} \lesssim l^{1+(|\gamma|-b)/2} e^{-c|p-q|/2}.$$

We may collect the last few estimates together to obtain

$$k_2^\chi(p,q) \le C_\epsilon \frac{s^{(3+|\gamma|-b)/2}}{p_0 q_0} \chi(g)\,(p_0+q_0)^{1+(|\gamma|-b)/2}\, e^{-c|p-q|/2}.$$

Note that for any $\ell \in \mathbb{R}$ we have

$$s^\ell e^{-c|p-q|/2} \le C_\ell e^{-c|p-q|/4}.$$

This follows trivially from (3.8) and $s = 4 + g^2 \le 4 + |p-q|^2$. Furthermore, for any $\ell \in [0,2]$, we *claim* the following estimate

$$\frac{(p_0+q_0)^\ell}{p_0 q_0} e^{-c|p-q|/4} \lesssim (p_0 q_0)^{\ell/2-1}\, e^{-c|p-q|/8}. \tag{3.20}$$

Using (3.20) with $\ell = 1 + |\gamma|/2$, in this Case 1, we have the general estimate

$$k_2^\chi(p,q) \le C_\epsilon \left(\sqrt{p_0 q_0}\right)^{|\gamma|/2-1} (p_0+q_0)^{-b/2}\, e^{-c|p-q|}.$$

This is the desired estimate in the current range of exponents for Case 1.

Before moving on to the next case, we establish the *claim*. Suppose that

$$\frac{1}{2}|q| \le |p| \le 2|q|.$$

In this case (3.20) is obvious. If $\frac{1}{2}|q| \ge |p|$, then we have

$$|p-q| \ge |q| - |p| \ge \frac{1}{2}|q|. \tag{3.21}$$

Whence

$$\frac{(p_0+q_0)^\ell}{p_0q_0}e^{-c|p-q|/4} \le C_\ell\,(p_0q_0)^{-1}\,e^{-c|p-q|/8}e^{-c|q|/64}.$$

In the last splitting $|p| \ge 2|q|$, then we alternatively have

$$|p-q| \ge |p| - |q| \ge \frac{1}{2}|p|. \tag{3.22}$$

Similarly in this situation

$$\frac{(p_0+q_0)^\ell}{p_0q_0}e^{-c|p-q|/4} \le C_\ell\,(p_0q_0)^{-1}\,e^{-c|p-q|/8}e^{-c|p|/64}.$$

These last two stronger estimates establish (3.20). We move on to the next case.

Case 2. We still consider $\gamma = -|\gamma| < 0$ but now $b+\gamma = b-|\gamma| \ge 0$. We have

$$\begin{aligned}\left(\frac{\sin(\psi/2)}{g}\right)^{b+\gamma}\cos^\gamma(\psi/2) &\lesssim g^{-b+|\gamma|}\left(\frac{1}{(1+y^2)^{1/4}}\right)^{b-|\gamma|}\left(\frac{y}{2\sqrt{1+y^2}}\right)^{-|\gamma|}\\ &\lesssim g^{-(b-|\gamma|)}y^{-|\gamma|}(1+y^2)^{-(b-3|\gamma|)/4}\\ &\le C_\epsilon\; y^{-|\gamma|}(1+y^2)^{-(b-3|\gamma|)/4}.\end{aligned}$$

We have used $g \ge \epsilon$ on the support of $\chi(g)$ in (3.1). We also used (3.14) and (3.15). Then again from (3.16) we have

$$k_2^\chi(p,q) \le C_\epsilon\frac{s^{3/2}}{p_0q_0}\int_0^\infty dy\; e^{-l\sqrt{1+y^2}}y^{1-|\gamma|}I_0(jy)(1+y^2)^{-(b-3|\gamma|)/4}.$$

From (2.7), we have in this case that $\gamma \in (-2,0)$ and $b \in [|\gamma|, 4-|\gamma|)$. Hence for the exponent above $b-3|\gamma| \in (-4,4)$.

Then the relevant integral above is bounded by

$$\begin{aligned}\int_0^\infty dy\; e^{-l\sqrt{1+y^2}}y^{1-|\gamma|}I_0(jy)(1+y^2)^{-(b-3|\gamma|)/4} &= \int_0^1 dy + \int_1^\infty dy\\ \lesssim \int_0^1 dy\; e^{-l\sqrt{1+y^2}}y^{1-|\gamma|}I_0(jy) + \int_1^\infty dy\; e^{-l\sqrt{1+y^2}}&yI_0(jy)(1+y^2)^{(|\gamma|-b)/4}.\end{aligned}$$

Here we used the same estimates as in Case 1. In this case $|\gamma|-b \in (-4,0)$. Let $\zeta_2 = \max(-2, |\gamma|-b)$. Then in this case by (3.19) and (3.18) the above is

$$\le Ce^{-c|p-q|/2} + Cl^{1+\zeta_2/2}e^{-|p-q|/2} \le Cl^{1+\zeta_2/2}e^{-c|p-q|/2}.$$

We may collect the last few estimates together to obtain

$$k_2^\chi(p,q) \le C_\epsilon\frac{s^{3/2}}{p_0q_0}(p_0+q_0)^{1+\zeta_2/2}\,e^{-c|p-q|/2} \le C_\epsilon\frac{(p_0+q_0)^{1+\zeta_2/2}}{p_0q_0}e^{-c|p-q|/4}.$$

We will further estimate the quotient.

If $\zeta_2 = |\gamma| - b$, then $1 + |\gamma|/2 \in [1, 2)$ and (3.20) implies

$$\frac{(p_0+q_0)^{1+\zeta_2/2}}{p_0 q_0} e^{-c|p-q|/4} = \frac{(p_0+q_0)^{1+|\gamma|/2}}{p_0 q_0} (p_0+q_0)^{-b/2} e^{-c|p-q|/4} \leq (p_0 q_0)^{(|\gamma|-2)/4} (p_0+q_0)^{-b/2} e^{-c|p-q|/8}.$$

Alternatively, if $\zeta_2 = -2$ then $1 + \zeta_2/2 = 0$ and (3.20) implies

$$\frac{(p_0+q_0)^{1+\zeta_2/2}}{p_0 q_0} e^{-c|p-q|/4} = \frac{(p_0+q_0)^{b/2}}{p_0 q_0} (p_0+q_0)^{-b/2} e^{-c|p-q|/4} \lesssim (p_0 q_0)^{(b-4)/4} (p_0+q_0)^{-b/2} e^{-c|p-q|/8}.$$

In either situation

$$k_2^\chi(p,q) \leq C_\epsilon (p_0 q_0)^{-\zeta} (p_0+q_0)^{-b/2} e^{-c|p-q|/8},$$

with $\zeta \overset{\text{def}}{=} \min\{2 - |\gamma|, 4 - b\}/4 > 0$.

Case 3. In this last case $\gamma = |\gamma| \geq 0$ and $b + \gamma \geq 0$. From (3.14) and (3.15):

$$\left(\frac{\sin(\psi/2)}{g}\right)^{b+\gamma} \cos^\gamma(\psi/2) \lesssim g^{-b-|\gamma|} \left(\frac{1}{(1+y^2)^{1/4}}\right)^{b+|\gamma|} \lesssim g^{-(b+|\gamma|)} (1+y^2)^{-(b+|\gamma|)/4} \leq C_\epsilon (1+y^2)^{-(b+|\gamma|)/4}.$$

We have again used $g \geq \epsilon$ on the support of $\chi(g)$ in (3.1). In this case from (3.16) we have

$$k_2^\chi(p,q) \leq C_\epsilon \frac{s^{3/2} g^{|\gamma|}}{p_0 q_0} \int_0^\infty dy\, e^{-l\sqrt{1+y^2}} y I_0(jy) (1+y^2)^{-(b+|\gamma|)/4}.$$

From (2.7), $b \in [0, 4)$ and $b + |\gamma| \in [0, 4 + |\gamma|)$. Let $\zeta_3 = \min(2, b + |\gamma|) \geq 0$.

Again using (3.18) we have

$$\int_0^\infty dy\, e^{-l\sqrt{1+y^2}} y I_0(jy) (1+y^2)^{-(b+|\gamma|)/4} \leq C l^{1-\zeta_3/2} e^{-c|p-q|/2}.$$

Hence

$$k_2^\chi(p,q) \leq C_\epsilon \frac{s^{3/2} g^{|\gamma|}}{p_0 q_0} (p_0+q_0)^{1-\zeta_3/2} e^{-c|p-q|/2} \leq C_\epsilon \frac{(p_0+q_0)^{1-\zeta_3/2}}{p_0 q_0} e^{-c|p-q|/4}.$$

If $\zeta_3 = 2$ this estimate can be handled exactly as in Case 2. If $\zeta_3 = b + |\gamma|$:

$$\frac{(p_0+q_0)^{1-\zeta_3/2}}{p_0 q_0} e^{-c|p-q|/4} = \frac{(p_0+q_0)^{1-|\gamma|/2}}{p_0 q_0} (p_0+q_0)^{-b/2} e^{-c|p-q|/4} \leq \frac{(p_0+q_0)}{p_0 q_0} (p_0+q_0)^{-b/2} e^{-c|p-q|/4} \lesssim (p_0 q_0)^{-1/2} (p_0+q_0)^{-b/2} e^{-c|p-q|/16}.$$

We have again used the estimate (3.20). In all of the cases we see that $k_2^\chi(p,q)$ satisfies the claimed bound from Lemma 3.2 with $\zeta = \min\{2 - |\gamma|, 4 - b, 2\}/4 > 0$. We could obtain a larger ζ in Case 3 if it was needed. □

With the estimate for the Hilbert-Schmidt form just proven in Lemma 3.2, we will now prove the decomposition from Lemma 3.3.

Proof of Lemma 3.3. We recall the splitting $K = K^{1-\chi} + K^{\chi}$ from (3.2). For K^{χ} we have the kernel $k^{\chi} = k_2^{\chi} - k_1^{\chi}$ from (3.9) and (3.10). For a given $R \geq 1$, choose another smooth cut-off function $\phi_R = \phi_R(p, q)$ satisfying

$$\begin{aligned} &\phi_R \equiv 1, \quad \text{if} \quad |p| + |q| \leq R/2, \quad |\phi_R| \leq 1, \\ &\text{supp}(\phi_R) \subset \{(p, q) \mid |p| + |q| \leq R\}. \end{aligned} \tag{3.23}$$

We will use this cut-off with several different R's in the cases below. Now we split the kernels $k^{\chi}(p, q)$ of the operator K^{χ} into

$$\begin{aligned} k^{\chi}(p, q) &= k^{\chi}(p, q)\phi_R(p, q) + k^{\chi}\,(1 - \phi_R) \\ &= k_c^{\chi}(p, q) + k_s^{\chi}(p, q). \end{aligned}$$

We further define

$$K_s \overset{\text{def}}{=} K^{1-\chi} + K_s^{\chi},$$

where $K_s^{\chi}(h) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dq\; k_s^{\chi}(p, q)\; h(q)$. Then the compact part is given by

$$K_c \overset{\text{def}}{=} K_c^{\chi},$$

where $K_c^{\chi}(h) \overset{\text{def}}{=} \int_{\mathbb{R}^3} dq\; k_c^{\chi}(p, q)\; h(q)$. Note that the compactness of $K_c(h)$ is evident from the integrability of the kernel. In the following we will show that the operators K_c and K_s satisfy the estimates claimed in Lemma 3.3.

First off, for K_c, from the Cauchy-Schwartz inequality we have

$$\begin{aligned} \left|\langle w_\ell^2 K_c(h_1), h_2\rangle\right| &\leq \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp\; w_\ell^2(p)\; \left|k_c^{\chi}(p, q)\right| |h_1(q)h_2(p)| \\ &\leq \left(\int dqdp\; w_\ell^2(p) \left|k_c^{\chi}(p, q)\right| \; |h_1(q)|^2\right)^{1/2} \\ &\quad \times \left(\int dqdp\; w_\ell^2(p) \left|k_c^{\chi}(p, q)\right| |h_2(p)|^2\right)^{1/2}. \end{aligned}$$

From the definition of $k_c^{\chi}(p, q)$ and Lemma 3.2, we see that

$$w_\ell^2(p) \left|k_c^{\chi}(p, q)\right| \leq C_R\; e^{-c|p-q|}\; \mathbf{1}_{\leq R}(p)\; \mathbf{1}_{\leq R}(q),$$

where $\mathbf{1}_{\leq R}$ is the indicator function of the ball of radius R centered at the origin as defined in Lemma 3.3. By combining the last few estimates we clearly have the claimed estimate for K_c from Lemma 3.3.

In the remainder of this proof we estimate $K_s = K^{1-\chi} + K_s^{\chi}$. For K_s^{χ} we have

$$\left|\langle w_\ell^2 K_s^{\chi}(h_1), h_2\rangle\right| \leq \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp\; w_\ell^2(p)\; \left|k_s^{\chi}(p, q)\right| \; |h_1(q)h_2(p)|.$$

With the definition of $k_s^{\chi}(p, q)$ and Lemma 3.2, we obtain

$$w_\ell^2(p) \left|k_s^{\chi}(p, q)\right| = w_\ell^2(p) \left|k^{\chi}(p, q)\right| (1 - \phi_R) \lesssim \frac{w_\ell^2(p)}{R^{\zeta}}\; (p_0 + q_0)^{-b/2}\; e^{-c|p-q|}.$$

Furthermore, we *claim* that

$$w_\ell^2(p)e^{-c|p-q|} \lesssim w_\ell(p)w_\ell(q)e^{-c|p-q|/2}. \tag{3.24}$$

By combining the last few estimates including (3.24), with Lemma 3.1, we have

$$\begin{aligned}\left|\langle w_\ell^2 K_s^\chi(h_1), h_2\rangle\right| &\lesssim \frac{1}{R^\zeta}\int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp\ w_\ell(p)w_\ell(q)\ \frac{e^{-c|p-q|}}{(p_0q_0)^{b/4}}\ |h_1(q)h_2(p)| \\ &\lesssim \frac{|h_1|_{\nu,\ell}|h_2|_{\nu,\ell}}{R^\zeta}.\end{aligned}$$

Since $\zeta > 0$ we conclude our estimate here by choosing $R > 0$ sufficiently large. Notice that the size of this R above clearly depends upon $\epsilon > 0$ from (3.1).

To prove the *claim* in (3.24), we use the same general strategy which was used to prove (3.20). Indeed, if $\frac{1}{2}|q| \geq |p|$ then because of (3.21) we have

$$w_\ell^2(p)e^{-c|p-q|/2}e^{-c|p-q|/2} \leq w_\ell^2(q)e^{-c|q|/4}e^{-c|p-q|/2} \leq C_\ell e^{-c|p-q|/2},$$

which is better than (3.24). Alternatively if $|p| \geq 2|q|$, then with (3.22) we have

$$w_\ell^2(p)e^{-c|p-q|/2}e^{-c|p-q|/2} \leq w_\ell^2(p)e^{-c|p|/4}e^{-c|p-q|/2} \leq C_\ell e^{-c|p-q|/2}.$$

The only remaining case is $\frac{1}{2}|q| \leq |p| \leq 2|q|$ for which the estimate (3.24) is obvious.

The last term to estimate is $K^{1-\chi} = K_2^{1-\chi} - K_1^{1-\chi}$. Notice that

$$K_1^{1-\chi}(h) = \int_{\mathbb{R}^3} dq\ k_1^{1-\chi}(p,q)\ h(q),$$

where from (2.4) and (3.1),

$$k_1^{1-\chi}(p,q) = (1-\chi(g))\sqrt{J(q)J(p)}\int_{\mathbb{S}^2} d\omega\ v_\o\ \sigma(g,\theta).$$

To estimate $K_1^{1-\chi}$ we apply Cauchy-Schwartz to obtain

$$\begin{aligned}\left|\langle w_\ell^2 K_1^{1-\chi}(h_1), h_2\rangle\right| &\leq \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dp\ w_\ell^2(p)\ k_1^{1-\chi}(p,q)\ \ |h_1(q)h_2(p)| \\ &\leq \left(\int dpdq\ w_\ell^2(p)k_1^{1-\chi}(p,q)|h_1(q)|^2\right)^{1/2} \\ &\quad\times\left(\int dpdq\ w_\ell^2(p)k_1^{1-\chi}(p,q)|h_2(p)|^2\right)^{1/2}.\end{aligned}$$

We will estimate the kernel of each term above. We further split

$$k_1^{1-\chi}(p,q) = k_1^{1-\chi}(p,q)\phi_R(p,q) + k_1^{1-\chi}\ (1-\phi_R) = k_{1S}^{1-\chi} + k_{1L}^{1-\chi}.$$

The value of $R \geq 1$ used here is independent of the case considered previously. The R here will be independent of ϵ. From (2.7) and (1.9), in general we have

$$\int_{\mathbb{S}^2} d\omega\ v_\o\ \sigma(g,\theta) \lesssim \frac{\sqrt{s}}{p_0q_0}g^{1-b}\int_0^\pi d\theta\ \sin^{1+\gamma}\theta \lesssim \frac{\sqrt{s}}{p_0q_0}g^{1-b}. \tag{3.25}$$

For $g \le 2\epsilon$ as in (3.1), with (3.8) we conclude that

$$|p-q| \le 2\epsilon\sqrt{p_0 q_0}. \tag{3.26}$$

Furthermore, on the support of ϕ_R we notice that additionally $|p-q| \le 4\epsilon R$. Then if $b \in (1,4)$, with the formula for $k_1^{1-\chi}(p,q)$, (3.25) and then (3.8), we obtain

$$\begin{aligned}
\int dp\, w_\ell^2(p)\, k_{1S}^{1-\chi}(p,q) &\le C_R \int_{|p-q|\le 4\epsilon R} dp\, \frac{\sqrt{s}}{p_0 q_0} g^{1-b}\sqrt{J(q)J(p)} \\
&\le C_R \int_{|p-q|\le 4\epsilon R} dp\, g^{1-b} J^{1/4}(q) J^{1/4}(p) \\
&\le C_R \int_{|p-q|\le 4\epsilon R} dp\, \left(\frac{\sqrt{p_0 q_0}}{|p-q|}\right)^{b-1} J^{1/4}(q) J^{1/4}(p) \\
&\le C R^{4-b} \epsilon^{4-b} J^{1/8}(q).
\end{aligned}$$

The last inequality above follows easily from $\left(\sqrt{p_0 q_0}\right)^{b-1} J^{1/8}(q) J^{1/8}(p) \le C$ and also

$$\int_{|p-q|\le 4\epsilon R} dp\, \frac{J^{1/8}(p)}{|p-q|^{b-1}} \le C R^{4-b}\epsilon^{4-b}.$$

Thus when $b \in (1,4)$ we have

$$\left|\langle w_\ell^2 K_{1S}^{1-\chi}(h_1), h_2\rangle\right| \le C_R \epsilon^{4-b} |J^{1/16} h_1|_2 |J^{1/16} h_2|_2. \tag{3.27}$$

This is much stronger than the desired estimate for $\epsilon = \epsilon(R) > 0$ chosen sufficiently small. Alternatively if $b \in [0,1]$ then with (3.25) we have

$$\int_{\mathbb{R}^3} dp\, w_\ell^2(p)\, k_{1S}^{1-\chi}(p,q) \le C \int_{|p-q|\le 4\epsilon R} dp\, J^{1/4}(q) J^{1/4}(p) \le C R^3 \epsilon^3 J^{1/4}(q).$$

Thus when $b \in [0,1]$ we have $\left|\langle w_\ell^2 K_1^{1-\chi}(h_1), h_2\rangle\right| \le C_R \epsilon^3 |J^{1/8} h_1|_2 |J^{1/8} h_2|_2$. This concludes our estimate for the part containing $k_{1S}^{1-\chi}(p,q)$ for any fixed R after choosing $\epsilon = \epsilon(R) > 0$ small enough (depending on the size of R).

For the term involving $k_{1L}^{1-\chi}(p,q)$ the estimate is much easier. In this case

$$\begin{aligned}
\int_{\mathbb{R}^3} dp\, w_\ell^2(p)\, k_{1L}^{1-\chi}(p,q) &\lesssim \int_{\mathbb{R}^3} dp\, g^{1-b}\,(1-\phi_R(p,q))\, J^{1/4}(q) J^{1/4}(p) \\
&\lesssim e^{-R/16}.
\end{aligned} \tag{3.28}$$

The same estimates hold for the other term in the inner product above. These estimates are independent of ϵ. We thus obtain the desired estimate for this term in the same way as for the last term; here we first choose $R > 0$ sufficiently large.

The last term to estimate is $K_2^{1-\chi}$ from (3.2). With (3.26) we see that

$$p_0 \le |p-q| + q_0 \le 2\epsilon\sqrt{p_0 q_0} + q_0 \le \epsilon p_0 + (1+\epsilon)\, q_0.$$

The first inequality in this chain can be found in [18, Ineq. A.1]. We conclude $p_0 \lesssim q_0$ and similarly $q_0 \lesssim p_0$. For $0 < \epsilon < 1/4$ say the constant in these inequalities can be chosen to not depend upon ϵ. Furthermore from (1.11), if $g \le 2\epsilon$ and ϵ is small (say less than 1/8), then it is easy to show that

$$p_0' \ge \frac{p_0 + q_0}{4}, \quad q_0' \ge \frac{p_0 + q_0}{4}. \tag{3.29}$$

These post-collisional energies are also clearly bounded from above by p_0 and q_0, so that all of these variables are comparable on (3.2). We thus have

$$\left|\langle K_2^{1-\chi}(h_1), h_2\rangle\right| \lesssim \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} d\omega dq dp \ (1-\chi(g)) \ v_{\o}\, \sigma(g,\theta) e^{-cq_0 - cp_0} \times \left(\left|h_1(p')\right| + \left|h_1(q')\right|\right) |h_2(p)|.$$

With Cauchy-Schwartz we obtain

$$\begin{aligned}
&\lesssim \left(\int_{g\le 2\epsilon} d\omega dq dp \ v_{\o}\, \sigma(g,\theta) e^{-cq_0 - cp_0} \left|h_1(p')\right|^2\right)^{1/2} \\
&\quad\times \left(\int_{g\le 2\epsilon} d\omega dq dp \ v_{\o}\, \sigma(g,\theta) e^{-cq_0 - cp_0} \left|h_2(p)\right|^2\right)^{1/2} \\
&\quad+ \left(\int_{g\le 2\epsilon} d\omega dq dp \ v_{\o}\, \sigma(g,\theta) e^{-cq_0 - cp_0} \left|h_1(q')\right|^2\right)^{1/2} \\
&\quad\times \left(\int_{g\le 2\epsilon} d\omega dq dp \ v_{\o}\, \sigma(g,\theta) e^{-cq_0 - cp_0} \left|h_2(p)\right|^2\right)^{1/2}.
\end{aligned}$$

From (3.25) and the arguments just below it, for any small $\eta > 0$ we can estimate

$$\int_{g\le 2\epsilon} d\omega dq \ v_{\o}\, \sigma(g,\theta) e^{-cq_0 - cp_0} \le \eta e^{-cp_0/2}.$$

Above of course we have $\eta = \eta(\epsilon) \to 0$ as $\epsilon \to 0$, and by symmetry the same estimate holds if the roles of p and q are reversed. Since the kernels of the integrals above are invariant with respect to the relativistic pre-post collisional change of variables [23], which is justified for (1.10), we may apply it as

$$dp dq = \frac{p_0' q_0'}{p_0 q_0} dp' dq'.$$

Putting all of this together with (1.6) we have

$$\left|\langle K_2^{1-\chi}(h_1), h_2\rangle\right| \le \eta(\epsilon) \left(\int dp \ e^{-cp_0} \left|h_1(p)\right|^2\right)^{1/2} \left(\int dp \ e^{-cp_0} \left|h_2(p)\right|^2\right)^{1/2}.$$

For $\epsilon > 0$ small enough, this is stronger than the estimate which we wanted to prove. We have now completed the proof of this lemma. □

We will at this time use the prior lemma to prove the next lemma.

Proof of Lemma 3.4. Most of this lemma is standard, see e.g. [21]. We only prove the coercive lower bound for the linear operator. Assuming the converse grants a sequence of functions $h^n(p)$ satisfying $\mathbf{P}h^n = 0$, $|h^n|_\nu^2 = \langle \nu h^n, h^n \rangle = 1$ and

$$\langle Lh^n, h^n \rangle = |h^n|_\nu^2 - \langle Kh^n, h^n \rangle \leq \frac{1}{n}.$$

Thus $\{h^n\}$ is weakly compact in $|\cdot|_\nu$ with limit point h^0. By weak lower-semi continuity $|h^0|_\nu \leq 1$. Furthermore,

$$\langle Lh^n, h^n \rangle = 1 - \langle Kh^n, h^n \rangle.$$

We *claim* that

$$\lim_{n \to \infty} \langle Kh^n, h^n \rangle = \langle Kh^0, h^0 \rangle.$$

The *claim* will follow from the prior Lemma 3.3. This *claim* implies

$$0 = 1 - \langle Kh^0, h^0 \rangle.$$

Or equivalently

$$\langle Lh^0, h^0 \rangle = |h^0|_\nu^2 - 1.$$

Since $L \geq 0$, we have $|h^0|_\nu^2 = 1$ which implies $h^0 = \mathbf{P}h^0$. On the other hand since $h^n = \{\mathbf{I} - \mathbf{P}\}h^n$ the weak convergence implies $h^0 = \{\mathbf{I} - \mathbf{P}\}h^0$. This is a contradiction to $|h^0|_\nu^2 = 1$.

We now establish the *claim*. For any small $\eta > 0$, we split $K = K_c + K_s$ as in Lemma 3.3. Then $|\langle K_s h^n, h^n \rangle| \leq \eta$. Also K_c is a compact operator in L^2_ν so that

$$\lim_{n \to \infty} |K_c h^n - K_c h^0|_\nu = 0.$$

We conclude by first choosing η small and then sending $n \to \infty$. □

We are now ready to prove Lemma 3.5. We point out that similar estimates, but with strong Sobolev norms, have been established in recent years [28,29,31,51] via the macroscopic equations for the coefficients a, b and c. We will use the approach from [33], which exploits the hyperbolic nature of the transport operator, to prove our Lemma 3.5 in the low regularity L^2 setting.

Proof of Lemma 3.5. We use the method of contradiction, if Lemma 3.5 is not valid then for any $k \geq 1$ we can find a sequence of normalized solutions to (2.11) which we denote by f_k that satisfy

$$\int_0^1 ds \, \|\{\mathbf{I} - \mathbf{P}\} f_k\|_\nu^2(s) \leq \frac{1}{k} \int_0^1 ds \, \|\mathbf{P} f_k\|_\nu^2(s).$$

Equivalently the normalized function

$$Z_k(t, x, p) \stackrel{\text{def}}{=} \frac{f_k(t, x, p)}{\sqrt{\int_0^1 ds \, \|\mathbf{P} f_k\|_\nu^2(s)}},$$

satisfies

$$\int_0^1 ds\ \|\mathbf{P}Z_k\|_\nu^2(s) = 1,$$

and

$$\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}Z_k\|_\nu^2(s) \le \frac{1}{k}. \tag{3.30}$$

Moreover, from (2.6) the following integrated conservation laws hold:

$$\int_0^1 ds \int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp \begin{pmatrix} 1 \\ p \\ p_0 \end{pmatrix} \sqrt{J(p)}\ Z_k(s,x,p) = 0. \tag{3.31}$$

Furthermore, since f_k satisfies (2.11), so does Z_k. Clearly

$$\sup_{k\ge 1} \int_0^1 ds\ \|Z_k\|_\nu^2(s) \lesssim 1. \tag{3.32}$$

Hence there exists $Z(t,x,p)$ such that

$$Z_k(t,x,p) \rightharpoonup Z(t,x,p), \quad \text{as } k\to\infty,$$

weakly with respect to the inner product $\int_0^1 ds\ (\cdot,\cdot)_\nu$ of the norm $\int_0^1 ds\ \|\cdot\|_\nu^2$. Furthermore, from (3.30) we know that

$$\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}Z_k\|_\nu^2(s) \to 0. \tag{3.33}$$

We conclude that $\{\mathbf{I}-\mathbf{P}\}Z_k \to \{\mathbf{I}-\mathbf{P}\}Z$ and $\{\mathbf{I}-\mathbf{P}\}Z = 0$ from (3.33). It is then straightforward to verify that

$$\mathbf{P}Z_k \to \mathbf{P}Z \text{ weakly in } \int_0^1 ds\ \|\cdot\|_\nu^2.$$

Hence

$$Z(t,x,p) = \mathbf{P}Z = \{a(t,x) + p\cdot b(t,x) + p_0 c(t,x)\}\sqrt{J}. \tag{3.34}$$

At the same time notice that $LZ_k = L\{\mathbf{I}-\mathbf{P}\}Z_k$ and we have (3.33). Send $k\to\infty$ in (2.11) for Z_k to obtain, in the sense of distributions, that

$$\partial_t Z + \hat{p}\cdot\nabla_x Z = 0. \tag{3.35}$$

At this point our main strategy is to show, on the one hand, Z has to be zero from (3.33), the periodic boundary conditions, and the hyperbolic transport equation (3.35), and (3.31). On the other hand, Z_k will be shown to converge strongly to Z in $\int_0^1 ds\|\cdot\|_\nu^2$ with the help of the averaging lemma [12] in the relativistic formulation [47] and $\int_0^1 ds\ \|Z\|_\nu^2(s) > 0$. This would be a contradiction.

Strong convergence. We begin by proving the strong convergence, and then later we will prove that the limit is zero. Split $Z_k(t,x,p)$ as

$$Z_k(t,x,p) = \mathbf{P}Z_k + \{\mathbf{I}-\mathbf{P}\}Z_k = \sum_{j=1}^{5}\langle Z_k(t,x,\cdot), e_j\rangle e_j(p) + \{\mathbf{I}-\mathbf{P}\}Z_k,$$

where $e_j(p)$ are an orthonormal basis for (3.3) in $\|\cdot\|_\nu$.

To prove the strong convergence in $\int_0^1 ds\ \|\cdot\|_\nu^2$, recalling (3.33), we will show

$$\sum_{1\le j\le 5}\int_0^1 ds\ \|\langle Z_k, e_j\rangle e_j - \langle Z, e_j\rangle e_j\|_\nu^2(s) \to 0.$$

Since $e_j(p)$ are smooth with exponential decay when $p\to\infty$, it suffices to prove

$$\int_0^1 ds \int_{\mathbb{T}^3} dx\ |\langle Z_k, e_j\rangle - \langle Z, e_j\rangle|^2 \to 0. \tag{3.36}$$

We will now establish (3.36) using the averaging lemma.

Choose any small $\eta>0$ and a smooth cut off function $\chi_1(t,x,p)$ in $(0,1)\times\mathbb{T}^3\times\mathbb{R}^3$ such that $\chi_1(t,x,p)\equiv 1$ in $[\eta, 1-\eta]\times\mathbb{T}^3\times\left\{|p|\le\frac{1}{\eta}\right\}$ and $\chi_1(t,x,p)\equiv 0$ outside $[\eta/2, 1-\eta/2]\times\mathbb{T}^3\times\left\{|p|\le\frac{2}{\eta}\right\}$. Split

$$\langle Z_k(t,x,\cdot), e_j\rangle = \langle (1-\chi_1)\, Z_k(t,x,\cdot), e_j\rangle + \langle \chi_1 Z_k(t,x,\cdot), e_j\rangle. \tag{3.37}$$

For the first term above, notice that

$$\begin{aligned}
&\int_0^1 ds \int_{\mathbb{T}^3} dx\ |\langle (1-\chi_1)\,|Z_k - Z|\,, e_j\rangle|^2 \\
&\quad\lesssim \int_0^1\int_{\mathbb{T}^3\times\mathbb{R}^3} (1-\chi_1)^2\,|Z_k|^2|e_j|^2 + \int_0^1\int_{\mathbb{T}^3\times\mathbb{R}^3} (1-\chi_1)^2\,|Z|^2|e_j|^2 \\
&\quad\lesssim \int_{0\le s\le\eta} + \int_{1-\eta\le s\le 1} + \int_{|p|\ge 1/\eta}.
\end{aligned}$$

Since $e_j = e_j(p)$ has exponential decay in $|p|$ we have the crude bound

$$|e_j(p)| \le C\,\eta, \quad \text{for} \quad |p|\ge 1/\eta.$$

Thus all three integrals above can be bounded by

$$C\eta \sup_{0\le s\le 1}\left(\|Z_k\|_2^2(s) + \|Z\|_2^2(s)\right) \le C\left(\|Z_k(0)\|_2^2 + \|Z(0)\|_2^2\right)\eta \le C\eta, \tag{3.38}$$

which will hold for Z_k uniformly in k. These bounds follow from (2.11) and (3.35).

The second term in (3.37), $\langle \chi_1 Z_k(t,x,\cdot), e_j\rangle$, is actually uniformly bounded in $H^{1/4}([0,1]\times\mathbb{T}^3)$. To prove this, notice that (2.11) implies that $\chi_1 Z_k$ satisfies

$$[\partial_t + \hat{p}\cdot\nabla_x]\,(\chi_1 Z_k) = -\chi_1 L[Z_k] + Z_k[\partial_t + \hat{p}\cdot\nabla_x]\chi_1. \tag{3.39}$$

The goal is to show that each term on the right-hand side of (3.39) is uniformly bounded in $L^2([0,1]\times\mathbb{T}^3\times\mathbb{R}^3)$. This would imply the $H^{1/4}$ bound by the averaging lemma. It clearly follows from (3.32) that

$$Z_k[\partial_t+\hat{p}\cdot\nabla_x]\chi_1\in L^2([0,1]\times\mathbb{T}^3\times\mathbb{R}^3).$$

Furthermore, it follows from Lemma 3.1 and Lemma 3.3 with $\ell=0$ that

$$\int_{\mathbb{R}^3}dp\ |\chi_1 L[Z_k]|^2\le\int_{\mathbb{R}^3}dp\ |\chi_1\nu(p)Z_k|^2+\int_{\mathbb{R}^3}dp\ |\chi_1 K(Z_k)|^2\lesssim|Z_k(t,x)|_\nu^2.$$

Thus the right-hand side of (3.39) is uniformly bounded in $L^2([0,1]\times\mathbb{T}^3\times\mathbb{R}^3)$. By the averaging lemma [12,47] it follows that

$$\langle\chi_1 Z_k(t,x,\cdot),e_j\rangle=\int_{\mathbb{R}^3}dp\ \chi_1(t,x,p)Z_k(t,x,p)e_j(p)\in H^{1/4}([0,1]\times\mathbb{T}^3).$$

This holds uniformly in k, which implies up to a subsequence that

$$\langle\chi_1 Z_k(t,x,\cdot),e_j\rangle\to\langle\chi_1 Z(t,x,\cdot),e_j\rangle\quad\text{in } L^2([0,1]\times\mathbb{T}^3).$$

Combining this last convergence with (3.38) concludes the proof of (3.36).

As a consequence of this strong convergence we have

$$\int_0^1 ds\ \|Z_k-Z\|_\nu^2(s)\to 0,$$

which implies that

$$\int_0^1 ds\ \|\mathbf{P}Z\|_\nu^2(s)=1.$$

Now if we can show that at the same time $\mathbf{P}Z=0$, then we have a contradiction.

The limit function $Z(t,x,p)=0$. By analysing the equations satisfied by Z, we will show that Z must be trivial. We will now derive the macroscopic equations for $\mathbf{P}Z$'s coefficients a, b and c. Since $\{\mathbf{I}-\mathbf{P}\}Z=0$, we see that $\mathbf{P}Z$ solves (3.35). We plug the expression for $\mathbf{P}Z$ in (3.34) into Eq. (3.35), and expand in the basis (3.3) to obtain

$$\left\{\partial^0 a+\frac{p_j}{p_0}\left\{\partial^j a\right\}+\frac{p_j p_i}{p_0}\left\{\partial^i b_j\right\}+p_j\left\{\partial^0 b_j+\partial^j c\right\}+p_0\left\{\partial^0 c\right\}\right\}J^{1/2}(p)=0,$$

where $\partial^0=\partial_t$ and $\partial^j=\partial_{x_j}$. By a comparison of coefficients, we obtain the important relativistic macroscopic equations for $a(t,x)$, $b_i(t,x)$ and $c(t,x)$:

$$\partial^0 c=0, \tag{3.40}$$
$$\partial^i c+\partial^0 b_i=0, \tag{3.41}$$
$$(1-\delta_{ij})\partial^i b_j+\partial^j b_i=0, \tag{3.42}$$
$$\partial^i a=0, \tag{3.43}$$
$$\partial^0 a=0, \tag{3.44}$$

which hold in the sense of distributions.

We will show that these Eqs. (3.40)–(3.44), combined with the periodic boundary conditions imply that any solution to (3.40)–(3.44) is a constant. Then the conservation laws (3.31) will imply that the constant can only be zero.

We deduce from (3.44) and (3.43) that

$$a(t,x) = a(0,x), \quad a.e.\ x, t,$$
$$a(s,x_1) = a(s,x_2), \quad a.e.\ s, x_1, x_2.$$

Thus a is a constant for almost every (t,x). From (3.40), we have $c(t,x) = c(x)$ for a.e. t. Then from (3.41) for some spatially dependent function $\tilde{b}_i(x)$ we have

$$b_i(t,x) = \partial^i c(x) t + \tilde{b}_i(x).$$

From (3.42) and the above

$$0 = \partial^i b_i(t,x) = \partial^i \partial^i c(x) t + \partial^i \tilde{b}_i(x),$$

which implies

$$\partial^i \partial^i c(x) = 0, \quad \partial^i \tilde{b}_i(x) = 0.$$

Similarly if $i \neq j$ we have

$$0 = \partial^j b_i(t,x) + \partial^i b_j(t,x) = \left(\partial^j \partial^i c(x) + \partial^i \partial^j c(x)\right) t + \partial^j \tilde{b}_i(x) + \partial^i \tilde{b}_j(x),$$

so that

$$\partial^j \partial^i c(x) = -\partial^i \partial^j c(x), \quad \partial^j \tilde{b}_i(x) = -\partial^i \tilde{b}_j(x),$$

which implies $\partial^i c(x) = c_i$, and $c(x)$ is a polynomial. By the periodic boundary conditions $c(x) = \tilde{c} \in \mathbb{R}$.

We further observe that $b_i(t,x) = b_i(x)$ is a constant in time a.e. from (3.41) and the above. From (3.42) again $\partial^i b_i = 0$ so that trivially $\partial^i \partial^i b_i = 0$. Moreover (3.42) further implies that $\partial^j \partial^j b_i = 0$. Thus for each i, $b_i(x)$ is a periodic polynomial, which must be a constant: $b_i(x) = b_i \in \mathbb{R}$.

We compute from (3.34) that

$$\int_{\mathbb{R}^3} dp\ p_i J^{1/2}(p)\ Z(t,x,p) = b_i \int_{\mathbb{R}^3} dp\ p_i^2 J(p), \quad i = 1,2,3,$$
$$\int_{\mathbb{R}^3} dp\ J^{1/2}(p)\ Z(t,x,p) = a \int_{\mathbb{R}^3} dp\ J(p) + c \int_{\mathbb{R}^3} dp\ p_0 J(p),$$
$$\int_{\mathbb{R}^3} dp\ p_0 J^{1/2}(p)\ Z(t,x,p) = a \int_{\mathbb{R}^3} dp\ p_0 J(p) + c \int_{\mathbb{R}^3} dp\ p_0^2 J(p).$$

As in [51] we define

$$\rho_1 = \int_{\mathbb{R}^3} J(p) dp = 1, \quad \rho_0 = \int_{\mathbb{R}^3} p_0 J(p) dp, \quad \rho_2 = \int_{\mathbb{R}^3} p_0^2 J(p) dp.$$

Now the matrix given by $\begin{pmatrix} \rho_1 & \rho_0 \\ \rho_0 & \rho_2 \end{pmatrix}$ is invertible because $\rho_0^2 < \rho_1 \rho_2$. It then follows from the conservation law (3.31) which is satisfied by the limit function $Z(t,x,p)$ that the constants a, b_i, c must indeed be zero. □

We have now completed all of the L^2 energy estimates for the linearized relativistic Boltzmann equation (2.11). We will now use (3.5), Lemma 3.4, Lemma 3.5, and Lemma 3.6 to prove Theorem 3.7. This will be the final proof in this section.

Proof of Theorem 3.7. For $k \geq 0$, we define the time weight function by

$$P_k(t) \overset{\text{def}}{=} (1+t)^k. \tag{3.45}$$

For a solution $f(t,x,p)$ to the linear Boltzmann equation (2.11), with $P_k(t)$ from (3.45), $P_k(t)f(t)$ satisfies the equation

$$\left(\partial_t + \hat{p}\cdot\nabla_x + L\right)(P_k(t)f(t)) - kP_{k-1}(t)f(t) = 0. \tag{3.46}$$

For the moment, suppose that $t = m$ and $m \in \{1,2,3,\dots\}$. For the time interval $[0,m]$, we multiply $P_k(t)f(t)$ with (3.46) and take the L^2 energy estimate over $0 \leq s \leq m$ to obtain

$$\begin{aligned} &P_{2k}(m)\|f\|_2^2(m) + \int_0^m ds\ P_{2k}(s)(Lf,f) \\ &-k\int_0^m ds\ P_{2k-1}(s)\|f\|_2^2(s) = \|f_0\|_2^2. \end{aligned} \tag{3.47}$$

We divide the time interval into $\cup_{j=0}^{m-1}[j,j+1)$ and also $f_j(s,x,p) \overset{\text{def}}{=} f(j+s,x,p)$ for $j \in \{0,1,2,\dots,m-1\}$. We have

$$\begin{aligned} &P_{2k}(m)\|f(m)\|_2^2 + \sum_{j=0}^{m-1}\int_0^1 ds\ \left\{P_{2k}(j+s)(Lf_j,f_j) - kP_{2k-1}(j+s)\|f_j\|_2^2(s)\right\} \\ &= \|f_0\|_2^2. \end{aligned}$$

Clearly $f_j(s,x,p)$ satisfies the same linearized Boltzmann equation (2.11) on the interval $0 \leq s \leq 1$. Notice that on this time interval

$$P_{2k}(j+s) \geq P_{2k}(j), \quad P_{2k-1}(j+s) \leq \tilde{C}_k P_{2k-1}(j), \quad \forall k \geq 1/2, \quad s \in [0,1]. \tag{3.48}$$

These estimates are uniform in j, so that

$$\begin{aligned} &P_{2k}(m)\|f(m)\|_2^2 + \sum_{j=0}^{m-1}\int_0^1 ds\ \left\{P_{2k}(j)(Lf_j,f_j)(s) - \tilde{C}_k k P_{2k-1}(j)\|f_j\|_2^2(s)\right\} \\ &\leq \|f_0\|_2^2. \end{aligned}$$

Moreover, by Lemma 3.4, we have

$$(Lf_j,f_j) \geq \delta_0\|\{\mathbf{I}-\mathbf{P}\}f_j\|_\nu^2 = \frac{\delta_0}{2}\|\{\mathbf{I}-\mathbf{P}\}f_j\|_\nu^2 + \frac{\delta_0}{2}\|\{\mathbf{I}-\mathbf{P}\}f_j\|_\nu^2.$$

Furthermore, with Lemma 3.5 applied to each $f_j(s,x,p)$ we obtain

$$\frac{\delta_0}{2}\sum_{j=0}^{m-1}P_{2k}(j)\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}f_j\|_\nu^2 \geq \frac{\delta_0\delta_\nu}{2}\sum_{j=0}^{m-1}P_{2k}(j)\int_0^1 ds\ \|\mathbf{P}f_j\|_\nu^2(s). \tag{3.49}$$

We combine (3.49) with the estimate above it to conclude

$$\sum_{j=0}^{m-1} P_{2k}(j)\int_0^1 ds\ (Lf_j, f_j)(s) \geq \frac{\delta_0}{2}\sum_{j=0}^{m-1} P_{2k}(j)\int_0^1 ds\ \|\{\mathbf{I}-\mathbf{P}\}f_j\|_\nu^2(s)$$
$$+\frac{\delta_0\delta_\nu}{2}\sum_{j=0}^{m-1} P_{2k}(j)\int_0^1 ds\ \|\mathbf{P}f_j\|_\nu^2(s)$$
$$\geq \tilde{\delta}\sum_{j=0}^{m-1} P_{2k}(j)\int_0^1 ds\ \|f_j\|_\nu^2(s),$$

where $\tilde{\delta} = \frac{1}{2}\min\left\{\frac{\delta_0\delta_\nu}{2}, \frac{\delta_0}{2}\right\}$. Define $C_k = \tilde{C}_k k$. With this lower bound

$$P_{2k}(m)\|f\|_2^2(m) + \sum_{j=0}^{m-1}\int_0^1 ds\left\{\tilde{\delta}P_{2k}(j)\|f_j\|_\nu^2 - C_k P_{2k-1}(j)\|f_j\|_2^2\right\}(s) \leq \|f_0\|_2^2.$$

Next, for $\lambda > 0$ sufficiently small we introduce the following splitting:

$$E_{\lambda,j} = \left\{p \,\middle|\, p_0^{b/2} < \lambda(1+j)\right\}, \quad E_{\lambda,j}^c = \left\{p \,\middle|\, p_0^{b/2} \geq \lambda(1+j)\right\}. \tag{3.50}$$

We incorporate this splitting into our energy inequality as follows:

$$P_{2k}(m)\|f\|_2^2(m) + \frac{\tilde{\delta}}{2}\sum_{j=0}^{m-1} P_{2k}(j)\int_0^1 ds\ \|f_j\|_\nu^2(s)$$
$$+\sum_{j=0}^{m-1}\int_0^1 ds\ \left\{\frac{\tilde{\delta}}{2}\|\sqrt{P_{2k}(j)}f_j\mathbf{1}_{E_{\lambda,j}}\|_\nu^2(s) - C_k\|\sqrt{P_{2k-1}(j)}f_j\mathbf{1}_{E_{\lambda,j}}\|_2^2(s)\right\}$$
$$\leq \|f_0\|_2^2 + \sum_{j=0}^{m-1}\int_0^1 ds\ C_k\|\sqrt{P_{2k-1}(j)}f_j\mathbf{1}_{E_{\lambda,j}^c}\|_2^2(s),$$

where $\mathbf{1}_{E_{\lambda,j}}$ is the usual indicator function of the set $E_{\lambda,j}$. On $E_{\lambda,j}$, with the help of Lemma 3.1, we have

$$-1 \geq -\lambda(1+j)\,p_0^{-b/2} \geq -C_\nu\lambda(1+j)\,\nu(p), \quad C_\nu > 0.$$

Hence

$$\frac{\tilde{\delta}}{2}\|\sqrt{P_{2k}(j)}f_j\mathbf{1}_{E_{\lambda,j}}\|_\nu^2(s) - C_k\|\sqrt{P_{2k-1}(j)}f_j\mathbf{1}_{E_{\lambda,j}}\|_2^2(s)$$
$$\geq \left(\frac{\tilde{\delta}}{2} - C_kC_\nu\lambda\right)\|\sqrt{P_{2k}(j)}f_j\mathbf{1}_{E_{\lambda,j}}\|_\nu^2(s).$$

We choose $\lambda > 0$ small enough such that

$$C_\lambda \overset{\text{def}}{=} \frac{\tilde{\delta}}{2} - C_kC_\nu\lambda > 0.$$

Then we have the following useful energy inequality:

$$P_{2k}(m)\|f\|_2^2(m) + \frac{\tilde{\delta}}{2}\sum_{j=0}^{m-1} P_{2k}(j)\int_0^1 ds\ \|f_j\|_\nu^2(s)$$

$$+ C_\lambda \sum_{j=0}^{m-1}\int_0^1 ds\ \|\sqrt{P_{2k}(j)}f_j \mathbf{1}_{E_{\lambda,j}}\|_\nu^2(s)$$

$$\leq \|f_0\|_2^2 + \sum_{j=0}^{m-1}\int_0^1 ds\ C_k\|\sqrt{P_{2k-1}(j)}f_j \mathbf{1}_{E^c_{\lambda,j}}\|_2^2(s).$$

The last term on the left side of the inequality is positive and we discard it from the energy inequality. For the right side of the energy inequality, on the complementary set $E^c_{\lambda,j}$, using Lemma 3.1 again, we have

$$P_{2k-1}(j) \leq \left(\frac{p_0^{b/2}}{\lambda}\right)^{2k-1} \leq \frac{C_\nu}{\lambda^{2k-1}}\ \nu(p)\ w_{2k}(p).$$

Thus we bound the time weights with velocity weights and the dissipation norm

$$\sum_{j=0}^{m-1}\int_0^1 ds\ C_k\|\sqrt{P_{2k-1}(j)}f_j \mathbf{1}_{E^c_{\lambda,j}}\|_2^2(s) \leq C\sum_{j=0}^{m-1}\int_0^1 ds\ \|f_j \mathbf{1}_{E^c_{\lambda,j}}\|_{\nu,k}^2(s).$$

We switch back to $f_j(t,x,p) = f(t+j,x,p)$ and use (3.48) to deduce

$$P_{2k}(m)\|f\|_2^2(m) + \delta_k\int_0^m ds\ P_{2k}(s)\|f\|_\nu^2(s) \leq \|f_0\|_2^2 + C\int_0^m ds\ \|f\|_{\nu,k}^2(s).$$

We can obtain an upper bound for the right side above using the regular energy inequality in (3.5) to achieve

$$P_{2k}(m)\|f\|_2^2(m) + \delta_k\int_0^m ds\ P_{2k}(s)\|f\|_\nu^2(s) \leq C_k\|f_0\|_{2,k}^2.$$

We have thus established our desired energy inequality from Theorem 3.7 for any $m \in \{0,1,2,\dots\}$ and $\ell = 0$. For an arbitrary $t > 0$, we choose $m \in \{0,1,2,\dots\}$ such that $m \leq t \leq m+1$. We then split the time integral as $[0,t] = [0,m] \cup [m,t]$.

For the time interval $[m,t]$, we have the L^2 energy estimate as in (3.6). Since $L \geq 0$ by Lemma 3.4, we may use (3.48) and (3.6) to see that

$$P_{2k}(t)\|f(t)\|_2^2 \leq C_k P_{2k}(m)\|f(m)\|_2^2, \quad \forall t \in [m, m+1].$$

Since (3.47) holds for any time t (not necessarily an integer), we can use the estimate above together with (3.47), as in (3.7), using Lemma 3.6, for any $t > 0$ to obtain

$$P_{2k}(t)\|f\|_2^2(t) + \delta_k\int_0^t ds\ P_{2k}(s)\|f\|_\nu^2(s) \leq C_k\|f_0\|_{2,k}^2. \tag{3.51}$$

This proves our time decay theorem for $\ell = 0$. For general $\ell > 0$ this estimate can be proven in exactly the same way, except in this case we use Lemma 3.6 in the place of Lemma 3.4 and Lemma 3.5 as we did in the proof of (3.5). □

This concludes our discussion of L^2 estimates for the linear Boltzmann equation. In the next section we use these L^2 estimates to prove L^∞ estimates.

4. Linear L^∞ Bounds and Slow Decay

In this section we will prove global in time uniform bounds for solutions to the linearized equation (2.11) in $L^\infty([0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)$, and slow polynomial decay in time. We express solutions, $f(t,x,p)$, to (2.11) with the semigroup $U(t)$ as

$$f(t,x,p) = \{U(t)f_0\}(x,p), \tag{4.1}$$

with initial data given by

$$\{U(0)f_0\}(x,p) = f_0(x,p).$$

Our goal in this section will be to prove the following.

Theorem 4.1. *Given $\ell > 3/b$ and $k \in [0,1]$. Suppose that $f_0 \in L^\infty_{\ell+k}(\mathbb{T}^3\times\mathbb{R}^3)$ satisfies (2.6) initially, then under (2.7) the semi-group satisfies*

$$\|\{U(t)f_0\}\|_{\infty,\ell} \le C(1+t)^{-k}\|f_0\|_{\infty,\ell+k}.$$

Above the positive constant $C = C_{\ell,k}$ only depends on ℓ and k.

The first step towards proving Theorem 4.1 is an appropriate decomposition. Initially we consider solutions to the linearization of (2.1) with the compact operator K removed from (2.11). This equation is given by

$$\left(\partial_t + \hat{p}\cdot\nabla_x + \nu(p)\right) f = 0, \quad f(0,x,p) = f_0(x,p). \tag{4.2}$$

Let the semigroup $G(t)f_0$ denote the solution to this system (4.2). Explicitly

$$G(t)f_0(x,p) \overset{\text{def}}{=} e^{-\nu(p)t} f_0(x-\hat{p}t,p).$$

For soft potentials (2.7), with Lemma 3.1, this formula does not imply exponential decay in L^∞ for high momentum values. However, as we will see in Lemma 4.2 below, this formula does imply that one can trade between arbitrarily high polynomial decay rates and additional polynomial momentum weights on the initial data.

More generally we consider solutions to the full linearized system (2.11), which are expressed with the semi-group (4.1). By the Duhamel formula

$$\{U(t)f_0\}(x,p) = G(t)f_0(x,p) + \int_0^t ds_1\, G(t-s_1)K\{U(s_1)f_0\}(x,p).$$

We employ the splitting $K = K^{1-\chi} + K^\chi$ which is defined with the cut-off function (3.1) and (3.2). We then further expand out

$$\{U(t)f_0\}(x,p) = G(t)f_0(x,p) + \int_0^t ds_1\, G(t-s_1)K^{1-\chi}\{U(s_1)f_0\}(x,p)$$
$$+ \int_0^t ds_1\, G(t-s_1)K^{\chi}\{U(s_1)f_0\}(x,p).$$

We further iterate the Duhamel formula of the last term, as did Vidav [56]:

$$U(s_1) = G(s_1) + \int_0^{s_1} ds_2\, G(s_1-s_2)KU(s_2). \tag{4.3}$$

This will grant the so-called A-Smoothing property. Notice below that we only iterate on the K^χ term, which is different from Vidav. Plugging this Duhamel formula into the previous expression yields a more elaborate formula

$$\begin{aligned}\{U(t)f_0\}(x,p) = G(t)f_0(x,p) &+ \int_0^t ds_1\, G(t-s_1)K^{1-\chi}\{U(s_1)f_0\}(x,p)\\ &+ \int_0^t ds_1\, G(t-s_1)K^\chi G(s_1)f_0(x,p)\\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2\, G(t-s_1)K^\chi G(s_1-s_2)K\{U(s_2)f_0\}(x,p).\end{aligned}$$

However this is not quite yet in the form we want. To get the final form, we once again split the compact operator $K = K^{1-\chi} + K^\chi$ in the last term to obtain

$$\begin{aligned}\{U(t)f_0\}(x,p) = G(t)f_0(x,p) &+ \int_0^t ds_1\, G(t-s_1)K^{1-\chi}\{U(s_1)f_0\}(x,p)\\ &+ \int_0^t ds_1\, G(t-s_1)K^\chi G(s_1)f_0(x,p)\\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2\, G(t-s_1)K^\chi G(s_1-s_2)K^{1-\chi}\{U(s_2)f_0\}(x,p)\\ &+ \int_0^t ds_1 \int_0^{s_1} ds_2\, G(t-s_1)K^\chi G(s_1-s_2)K^\chi\{U(s_2)f_0\}(x,p)\\ &\stackrel{\text{def}}{=} H_1(t,x,p)+H_2(t,x,p)+H_3(t,x,p)+H_4(t,x,p)+H_5(t,x,p),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}H_1(t,x,p) &\stackrel{\text{def}}{=} e^{-\nu(p)t} f_0(x-\hat{p}t,p),\\ H_2(t,x,p) &\stackrel{\text{def}}{=} \int_0^t ds_1\, e^{-\nu(p)(t-s_1)}K^{1-\chi}\{U(s_1)f_0\}(y_1,p),\\ H_3(t,x,p) &\stackrel{\text{def}}{=} \int_0^t ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^\chi(p,q_1)\, e^{-\nu(q_1)s_1} f_0(y_1-\hat{q}_1 s_1, q_1).\end{aligned}$$

Just above and below we will be using the following short-hand notation:

$$\begin{aligned}y_1 &\stackrel{\text{def}}{=} x-\hat{p}(t-s_1),\\ y_2 &\stackrel{\text{def}}{=} y_1-\hat{q}_1(s_1-s_2) = x-\hat{p}(t-s_1)-\hat{q}_1(s_1-s_2).\end{aligned} \tag{4.5}$$

We are also using the notation $q_{10} = \sqrt{1+|q_1|^2}$ and $q_1 = (q_{11}, q_{12}, q_{13}) \in \mathbb{R}^3$ with $\hat{q}_1 = q_1/q_{10}$. Furthermore the next term is

$$\begin{aligned}H_4(t,x,p) \stackrel{\text{def}}{=} &\int_{\mathbb{R}^3} dq_1\, k^\chi(p,q_1) \int_0^t ds_1 \int_0^{s_1} ds_2\, e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)(s_1-s_2)}\\ &\times K^{1-\chi}\{U(s_2)f_0\}(y_2,q_1).\end{aligned}$$

Lastly, we may also expand out the fifth component as

$$H_5(t,x,p) = \int_{\mathbb{R}^3} dq_1\, k^\chi(p,q_1) \int_{\mathbb{R}^3} dq_2\, k^\chi(q_1,q_2) \int_0^t ds_1\, e^{-\nu(p)(t-s_1)} \times \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \{U(s_2)f_0\}(y_2,q_2). \tag{4.6}$$

We will estimate each of these five terms individually. In Lemma 4.2 below we will show that the first and third term exhibit rapid polynomial decay. Then after that, in Lemma 4.3, we show that the second and fourth terms can be bounded by the time decay norm (2.10) multiplied by an arbitrarily small constant. For the last term, in Lemma 4.4, we will show that H_5 can be estimated by (2.10) times a small constant plus the L^2_{-j} norm of the semi-group (for any $j > 0$) multiplied by a large constant. After stating each of these lemmas, we will put these estimates together to prove our key decay estimate on the semi-group for solutions to (2.11) in L^∞_ℓ as stated above in Theorem 4.1. Once this is complete we give the proofs of the three key Lemmas 4.2, 4.3 and 4.4 at the end of this section.

Lemma 4.2. *Given $\ell \geq 0$, for any $k \geq 0$ we have*

$$|w_\ell(p)H_1(t,x,p)| + |w_\ell(p)H_3(t,x,p)| \leq C_{\ell,k}(1+t)^{-k}\|f_0\|_{\infty,\ell+k}.$$

Next

Lemma 4.3. *Fix $\ell \geq 0$. For any small $\eta > 0$, which relies upon the small $\epsilon > 0$ from (3.1), and any $k \geq 0$ we have*

$$|w_\ell(p)H_2(t,x,p)| + |w_\ell(p)H_4(t,x,p)| \leq \eta(1+t)^{-k}|||f|||_{k,\ell}.$$

The estimates in the lemma above will be used to obtain upper bounds for the the L^∞_ℓ norm of the semi-group. The final lemma in this series is below

Lemma 4.4. *Fix $\ell \geq 0$, choose any (possibly large) $j > 0$. For any small $\eta > 0$, which depends upon (3.1), and any $k \geq 0$ we have the estimate*

$$|w_\ell(p)H_5(t,x,p)| \leq \eta(1+t)^{-k}|||f|||_{k,\ell} + C_\eta \int_0^t ds\, e^{-\eta(t-s)}\|f\|_{2,-j}(s) + w_\ell(p)\,|R_1(f)(t)|\,.$$

By the L^2 decay theory from Theorem 3.7, and also Proposition 4.5, we have

$$\int_0^t ds\, e^{-\eta(t-s)}\|f\|_{2,-j}(s) \leq C_\eta(1+t)^{-k}\|f_0\|_{2,k} \leq C_\eta(1+t)^{-k}\|f_0\|_{\infty,\ell+k}.$$

The above estimates hold for any $k \geq 0$ and $\ell > 3/b$ (as in (4.7) just below). On the other hand, for the last term if we restrict $k \in [0,1]$ then $\forall \eta > 0$ we have

$$w_\ell(p)\,|R_1(f)| \leq \eta(1+t)^{-k}|||f|||_{k,\ell}.$$

Above R_1 is defined in (4.15) during the course of the proof.

These estimates would imply almost exponential decay except for the problematic term $R_1(f)(t)$, which only appears to decay to first order. This will be discussed in more detail in Sect. 6, where it is shown that this term can decay to any polynomial order by performing a new high order expansion.

We now show that the above lemmas grant a uniform bound and slow decay for solutions to (2.11) in L^∞_ℓ. We are using the semi-group notation $f(t) = \{U(t)f_0\}$. Lemmas 4.2, 4.3, and 4.4 together imply that for any $\eta > 0$ and $k \in [0, 1]$ we have

$$\|f\|_{\infty,\ell}(t) \le C_{\ell,k}(1+t)^{-k}\|f_0\|_{\infty,\ell+k} + \eta(1+t)^{-k}|||f|||_{k,\ell} + C_\eta(1+t)^{-k}\|f_0\|_{2,k}.$$

Equivalently

$$|||f|||_{k,\ell} \le C_{\ell,k}\|f_0\|_{\infty,\ell+k} + C_{1/2}\|f_0\|_{2,k} \le C_{\ell,k}\|f_0\|_{\infty,\ell+k}.$$

The last estimate holds when we choose $\ell > 3/b$, with (2.9), as follows

$$\|f_0\|_{2,k} = \sqrt{\int_{\mathbb{T}^3} dx \int_{\mathbb{R}^3} dp\, w_k^2(p)|f_0(p)|^2} \le \|f_0\|_{\infty,\ell+k}\sqrt{\int_{\mathbb{R}^3} dp\, w_{-\ell}^2(p)}$$
$$\lesssim \|f_0\|_{\infty,\ell+k}. \tag{4.7}$$

With this inequality, we have the desired decay rate for the L^∞_ℓ norm of solutions to the linear equation (2.11), which proves Theorem 4.1 subject to Lemmas 4.2, 4.3, and 4.4. We now prove those lemmas.

Along this course we will repeatedly use the following basic decay estimate

Proposition 4.5. *Suppose without loss of generality that $\lambda \ge \mu \ge 0$. Then*

$$\int_0^t \frac{ds}{(1+t-s)^\lambda(1+s)^\mu} \le \frac{C_{\lambda,\mu}(t)}{(1+t)^\rho},$$

where $\rho = \rho(\lambda,\mu) = \min\{\lambda+\mu-1, \mu\}$ and

$$C_{\lambda,\mu}(t) = C\begin{cases} 1 & \text{if } \lambda \ne 1, \\ \log(2+t) & \text{if } \lambda = 1. \end{cases}$$

Furthermore, we will use the following basic estimate from the Calculus:

$$e^{-ay}(1+y)^k \le \max\{1, e^{a-k}k^k a^{-k}\}, \quad a, y, k \ge 0. \tag{4.8}$$

We will now write an elementary proof of this basic time decay estimate in Proposition 4.5. This result is not difficult and known, however we provide a short proof for the sake of completeness and because we have not seen a proof in the literature.

Proof of Proposition 4.5. We will consider the cases $\mu = 0$ and $\mu = \lambda$ separately. Then the general result will then be established by interpolation.

Case 1. $\mu = 0$. If $\lambda \ne 1$ we have

$$\int_0^t \frac{ds}{(1+t-s)^\lambda} = \int_0^t \frac{ds}{(1+s)^\lambda} = \frac{1}{\lambda-1}\left\{1-(1+t)^{1-\lambda}\right\} \le C(1+t)^{-\rho(\lambda,0)}.$$

Note $\rho = 0$ if $\lambda > 1$ and $\rho = \lambda - 1 < 0$ otherwise. Alternatively if $\lambda = 1$,

$$\int_0^t \frac{ds}{1+s} = \log(1+t).$$

This completes our study of the first case $\mu = 0$.

Case 2. $\mu = \lambda$. We split the integral as

$$\int_0^t \frac{ds}{(1+t-s)^\lambda (1+s)^\lambda} = \int_0^{t/2} + \int_{t/2}^t .$$

For the first integral

$$\int_0^{t/2} \frac{ds}{(1+t-s)^\lambda (1+s)^\lambda} \le (1+t/2)^{-\lambda} \int_0^{t/2} \frac{ds}{(1+s)^\lambda}.$$

Now from Case 1, we can estimate the remaining integral as

$$\int_0^{t/2} \frac{ds}{(1+s)^\lambda} \le C_{\lambda,\lambda}(t)(1+t)^{\max\{0,1-\lambda\}},$$

which conforms with the claimed decay. The second half of the integral can be estimated in exactly the same way as the first.

Case 3. $0 < \mu < \lambda$. By Hölder's inequality, we have

$$\int_0^t \frac{ds}{(1+t-s)^\lambda (1+s)^\mu} = \int_0^t \frac{ds}{(1+t-s)^{\frac{\lambda}{p}+\frac{\lambda}{q}} (1+s)^\mu}$$
$$\le \left(\int_0^t \frac{ds}{(1+t-s)^\lambda}\right)^{1/p} \left(\int_0^t \frac{ds}{(1+t-s)^\lambda (1+s)^{q\mu}}\right)^{1/q}.$$

Above $q = \frac{\lambda}{\mu}$ and $p = \frac{\lambda}{\lambda-\mu}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Therefore the above is

$$\le C \left(\int_0^t \frac{ds}{(1+t-s)^\lambda}\right)^{(\lambda-\mu)/\lambda} \left(\int_0^t \frac{ds}{(1+t-s)^\lambda (1+s)^\lambda}\right)^{\mu/\lambda}.$$

By the previous two cases, this is

$$\le \left(C_{\lambda,0}(t)(1+t)^{-\rho(\lambda,0)}\right)^{(\lambda-\mu)/\lambda} \left(C_{\lambda,\lambda}(t)(1+t)^{-\rho(\lambda,\lambda)}\right)^{\mu/\lambda}$$
$$= C_{\lambda,0}(t)^{(\lambda-\mu)/\lambda} C_{\lambda,\lambda}(t)^{\mu/\lambda}(1+t)^{-\rho(\lambda,0)(\lambda-\mu)/\lambda-\rho(\lambda,\lambda)\mu/\lambda}.$$

The proposition follows by adding the exponents. □

We are now ready to proceed to the

Proof of Lemma 4.2. We start with H_1. From Lemma 3.1 and (4.8) we have

$$e^{-\nu(p)t} \le C_k p_0^{kb/2}(1+t)^{-k} \le C_k w_k(p)(1+t)^{-k}, \quad \forall t, k > 0. \tag{4.9}$$

Here we use the notation from (2.9). This procedure grants high polynomial time decay on the solution if we admit similar high polynomial momentum decay on the initial data. In particular we have shown

$$|w_\ell(p) H_1(t,x,p)| = \left| w_\ell(p)\, e^{-\nu(p)t} f_0(x - \hat{p}t, p)\right| \le C(1+t)^{-k} \| f_0\|_{\infty,\ell+k},$$

which is the desired estimate for H_1.

We finish off this lemma by estimating H_3. Notice that we trivially have

$$e^{-\nu(p)(t-s_1)}e^{-\nu(q)s_1} \le e^{-\nu(\max\{|p|,|q|\})t},$$

where $\nu(\max\{|p|, |q|\})$ is ν evaluated at $\max\{|p|, |q|\}$. We have

$$|w_\ell(p)H_3(t, x, p)| \le w_\ell(p)\int_0^t ds_1\, e^{-\nu(p)t}\int_{|p|\ge|q_1|} dq_1 \left|k^\chi(p, q_1)\right| \sup_{y\in\mathbb{T}^3}|f_0(y, q_1)|$$
$$+ w_\ell(p)\int_0^t ds_1 \int_{|p|<|q_1|} dq_1 \left|k^\chi(p, q_1)\right| e^{-\nu(q_1)t} \sup_{y\in\mathbb{T}^3}|f_0(y, q_1)|.$$

We will estimate the second term, and we remark that the first term can be handled in exactly the same way. As in the previous estimate for H_1, we use (4.9) to obtain

$$e^{-\nu(q_1)t}\sup_{y\in\mathbb{T}^3}|f_0(y, q_1)| \le C_k(1+t)^{-k-1}w_{k+1}(q_1)\sup_{y\in\mathbb{T}^3}|f_0(y, q_1)|.$$

Next we use the estimate for $k^\chi(p, q)$ from Lemma 3.2. When using this estimate we may suppose $|q_1| \le 2|p|$. For otherwise, if say $|q_1| \ge 2|p|$, then as in (3.21) we have $|p - q_1| \ge |q_1|/2$ which leads directly to

$$w_{k+1}(q_1)w_\ell(p)e^{-c|p-q_1|/2} \le C w_{k+1+\ell}(q_1)e^{-c|q_1|/4} \le C. \tag{4.10}$$

In this case we easily obtain an estimate better than (4.11) below. In particular

$$\int_0^t ds_1\, w_\ell(p)\int_{|p|<|q_1|} dq_1\, \mathbf{1}_{|q_1|\ge 2|p|} \left|k^\chi(p, q_1)\right| e^{-\nu(q_1)t}\sup_{y\in\mathbb{T}^3}|f_0(y, q_1)|$$
$$\lesssim (1+t)^{-k}\|f_0\|_{\infty,-j}, \quad \forall k > 0,\ j > 0.$$

Thus in the following we assume $|p| < |q_1| \le 2|p|$. On this region we may plug in the last few estimates including (4.9) to obtain

$$\int_0^t ds_1\, w_\ell(p)\int_{|p|<|q_1|\le 2|p|} dq_1 \left|k^\chi(p, q_1)\right| e^{-\nu(q_1)t}\sup_{y\in\mathbb{T}^3}|f_0(y, q_1)|$$
$$\le C_k\frac{t}{(1+t)^{k+1}}\int_{|p|<|q_1|\le 2|p|} dq_1\, w_\ell(q_1)\, w_{k+1}(q_1) \left|k^\chi(p, q_1)\right| \sup_{y\in\mathbb{T}^3}|f_0(y, q_1)|$$
$$\le C_k(1+t)^{-k}\|f_0\|_{\infty,\ell+k}\, w_1(p)\int_{|p|<|q_1|\le 2|p|} dq_1 \left|k^\chi(p, q_1)\right|. \tag{4.11}$$

Then from Lemma 3.2 we clearly have the following bound:

$$w_1(p)\int_{|p|<|q_1|\le 2|p|} dq_1 \left|k^\chi(p, q_1)\right| \le C.$$

This completes the time decay estimate for H_3 and our proof of the lemma. □

Our next aim is to prove Lemma 4.3. To do this we will use the following

Lemma 4.6. *Fix any $\ell \geq 0$ and any $j > 0$. Then given any small $\eta > 0$, which depends upon χ in (3.1), the following estimate holds*:

$$\left| w_\ell(p) K^{1-\chi}(h)(p) \right| \leq \eta e^{-cp_0} \|h\|_{\infty,-j}.$$

Above the constant $c > 0$ is independent of η.

Proof of Lemma 4.6. We consider $K^{1-\chi}(h)$ as defined in (3.2). From (3.29), for ϵ in (3.1) chosen sufficiently small (ϵ smaller than $1/4$ is sufficient), we see that

$$\sqrt{J(q)J(p')} + \sqrt{J(q)J(q')} \lesssim e^{-cq_0-cp_0}, \quad \exists c > 0.$$

With that estimate, and additionally the conservation of energy (1.6), we have

$$\begin{aligned}
\left| w_\ell(p) K_2^{1-\chi}(h)(p) \right| &\lesssim \int_{\mathbb{R}^3 \times \mathbb{S}^2} d\omega dq \ (1-\chi(g)) \ v_\text{ø} \, \sigma(g,\theta) e^{-cq_0-cp_0} \\
&\quad \times \left(\left| h(p') \right| + \left| h(q') \right| \right) \\
&\lesssim \int_{\mathbb{R}^3 \times \mathbb{S}^2} d\omega dq \ (1-\chi(g)) \ v_\text{ø} \, \sigma \ e^{-\frac{c}{2}q_0 - \frac{c}{2}p_0} \\
&\quad \times \left(e^{-\frac{c}{2}p_0'} \left| h(p') \right| + e^{-\frac{c}{2}q_0'} \left| h(q') \right| \right) \\
&\leq \eta e^{-\frac{c}{4}p_0} \|h\|_{\infty,-j}.
\end{aligned}$$

In the last inequality we have used the following estimate for any small $\eta > 0$:

$$\int_{\mathbb{R}^3 \times \mathbb{S}^2} d\omega dq \ (1-\chi(g)) \ v_\text{ø} \, \sigma(g,\theta) \ e^{-\frac{c}{2}q_0 - \frac{c}{2}p_0} \leq \eta \ e^{-\frac{c}{4}p_0}. \tag{4.12}$$

The proof of this bound uses (3.25), but also the exact strategy used in the paragraph containing (3.25) and the paragraph just below it. The idea is to use the splitting (3.23) inside the integral in (4.12):

$$1 = \phi_R(p,q) + (1 - \phi_R(p,q)).$$

First when $|p| + |q|$ is large, this is the term containing $(1 - \phi_R(p,q))$, we have a bound for the integral in (4.12) which is of the form Ce^{-cR}, where the constant $C, c > 0$ are independent of both R and ϵ similar to (3.28). On the other hand, when $|p| + |q| \leq R$, this is the term containing $\phi_R(p,q)$, we have a bound which is of the form $C_R \epsilon^p$. Here $p = p(b)$ can be chosen to be $p = 3$ if $b \in [0,1]$ and $p = 4 - b$ if $b \in (1,4)$. This second estimate is similar to (3.27). Since this strategy is already performed in detail nearby (3.25), we will not re-write the details.

Since $p > 0$ we can first choose $R >> 1$ sufficiently large, and then choose $\epsilon > 0$ sufficiently small so that the constant $\eta > 0$ in (4.12) can be chosen arbitrarily small. This yields the desired estimate. We remark that the estimate for $K_1^{1-\chi}$ can be shown in the same exact way; it is in fact slightly easier. □

With Lemma 4.6 and Proposition 4.5 in hand, we proceed to the

Proof of Lemma 4.3. We begin with the estimate for H_2. Using Lemma 4.6, for any small $\eta' > 0$ we have

$$\begin{aligned} |w_\ell(p) H_2(t,x,p)| &= w_\ell(p) \left| \int_0^t ds_1 \, e^{-\nu(p)(t-s_1)} K^{1-\chi} \{U(s_1) f_0\} (y_1, p) \right| \\ &\leq \eta' \, e^{-cp_0} \int_0^t ds_1 \, e^{-\nu(p)(t-s_1)} \, \| \{U(s_1) f_0\} \|_{\infty,-j}, \quad \forall j > 0 \\ &\leq \eta' \, e^{-cp_0} |||f|||_{k,\ell} \int_0^t ds_1 \, e^{-\nu(p)(t-s_1)} (1+s_1)^{-k}. \end{aligned}$$

The norm is from (2.10) for $k \geq 0$. As in (4.9) for any $\lambda > \max\{1, k\}$ we have

$$\begin{aligned} |w_\ell(p) H_2(t,x,p)| &\leq \eta' \, w_\lambda(p) e^{-cp_0} |||f|||_{k,\ell} \int_0^t ds \, (1+t-s)^{-\lambda} (1+s)^{-k} \\ &\leq \eta \, (1+t)^{-k} |||f|||_{k,\ell}, \end{aligned}$$

which follows from Proposition 4.5. This is the desired estimate for H_2.

For H_4 we once again use Lemma 4.6, for any small $\eta' > 0$, to obtain

$$\begin{aligned} |w_\ell(p) H_4(t,x,p)| &\leq \int_{\mathbb{R}^3} dq_1 \, \left| k^\chi(p, q_1) \right| \int_0^t ds_1 \int_0^{s_1} ds_2 \, e^{-\nu(p)(t-s_1)} \\ &\quad \times e^{-\nu(q_1)(s_1-s_2)} w_\ell(p) \left| K^{1-\chi} (\{U(s_2) f_0\}) (y_2, q_1) \right| \\ &\leq \eta' |||f|||_{k,\ell} \int_{\mathbb{R}^3} dq_1 \, \left| k^\chi(p, q_1) \right| \frac{w_\ell(p)}{w_\ell(q_1)} e^{-cq_{10}} \\ &\quad \times \int_0^t ds_1 \int_0^{s_1} ds_2 \, e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)(s_1-s_2)} (1+s_2)^{-k}. \end{aligned}$$

We recall that $q_{10} = \sqrt{1+|q_1|^2}$. For the time decay, from Proposition 4.5 with (4.8) as in (4.9) we notice that

$$\begin{aligned} &\int_0^t ds_1 \int_0^{s_1} ds_2 \, e^{-\nu(p)(t-s_1)} e^{-\nu(q_1)(s_1-s_2)} (1+s_2)^{-k} \\ &\lesssim w_\lambda(p) w_\lambda(q_1) \int_0^t ds_1 \, (1+t-s_1)^{-\lambda} \int_0^{s_1} ds_2 \, (1+s_1-s_2)^{-\lambda} (1+s_2)^{-k} \\ &\lesssim w_\lambda(p) w_\lambda(q_1) (1+t)^{-k}. \end{aligned}$$

Above we have taken $\lambda > \max\{1, k\}$. Combining these estimates yields

$$|w_\ell(p) H_4| \lesssim \eta' (1+t)^{-k} |||f|||_{k,\ell} \int_{\mathbb{R}^3} dq_1 \, \left| k^\chi(p, q_1) \right| w_{\ell+\lambda}(p) w_{-\ell+\lambda}(q_1) e^{-cq_{10}}.$$

To estimate the remaining integral and weights we split into three cases. If either $2|q_1| \leq |p|$, or $|q_1| \geq 2|p|$, then we bound all the weights and the remaining momentum integral by a constant as in (4.10). Alternatively if $\frac{1}{2}|q_1| \leq |p| \leq 2|q_1|$, then the desired estimate is obvious since we have strong exponential decay in both p and q_1. In either of these cases we have the estimate for H_4. □

We will finish this section with a proof of the crucial Lemma 4.4.

Proof of Lemma 4.4. We now turn to the proof of our estimate for H_5. Recall the definition of H_5 from (4.6) with y_2 defined in (4.5). We will utilize rather extensively the estimate for k^χ from Lemma 3.2. We now further split

$$H_5(t,x,p) = H_5^{high}(t,x,p) + H_5^{low}(t,x,p), \tag{4.13}$$

and estimate each term on the right individually. For $M >> 1$ we define

$$\mathbf{1}_{high} \stackrel{\text{def}}{=} \mathbf{1}_{|p|>M}\mathbf{1}_{|q_1|\le M} + \mathbf{1}_{|q_1|>M}. \tag{4.14}$$

Notice $\mathbf{1}_{high} + \mathbf{1}_{|p|\le M}\mathbf{1}_{|q_1|\le M} = 1$. Now the first term in the expansion is

$$H_5^{high}(t,x,p) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dq_1\, k^\chi(p,q_1) \int_{\mathbb{R}^3} dq_2\, k^\chi(q_1,q_2)\, \mathbf{1}_{high} \int_0^t ds_1\, e^{-\nu(p)(t-s_1)} \times \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \{U(s_2)f_0\}(y_2,q_2).$$

We use (4.8), as in (4.9), and Lemma 3.1 to see that for any $\lambda \ge 0$ we have

$$\int_0^t ds_1\, e^{-\nu(p)(t-s_1)} \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \le C_\lambda w_\lambda(p) w_\lambda(q_1) \int_0^t ds_1 \int_0^{s_1} ds_2\, (1+(t-s_1))^{-\lambda}(1+(s_1-s_2))^{-\lambda}.$$

When either $|p| > M$ or $|q_1| > M$, by Lemma 3.2, we have the bound

$$\left|k^\chi(p,q_1)\right| \le C M^{-\zeta}\, (p_0+q_{10})^{-b/2}\, e^{-c|p-q_1|}.$$

If either $|p| \ge 2|q_1|$ or $|q_1| \ge 2|p|$ then as in (4.10) we have

$$w_{\ell+\lambda}(p) w_\lambda(q_1) e^{-c|p-q_1|} \le C.$$

Thus by combining the last few estimates we have

$$\begin{aligned} &w_\ell(p) \int_{\mathbb{R}^3} dq_1\, \left|k^\chi(p,q_1)\right| \int_{\mathbb{R}^3} dq_2\, \left|k^\chi(q_1,q_2)\right|\, \mathbf{1}_{high}\, \left(\mathbf{1}_{|p|\ge 2|q_1|} + \mathbf{1}_{|p|\le \frac{1}{2}|q_1|}\right) \\ &\quad \times \int_0^t ds_1\, e^{-\nu(p)(t-s_1)} \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \left|\{U(s_2)f_0\}(y_2,q_2)\right| \\ &\le \frac{C_\lambda}{M^{\zeta+b/2}} |||f|||_{k,\ell} \int_{\mathbb{R}^3} dq_1\, e^{-c|p-q_1|} \int_{\mathbb{R}^3} dq_2\, e^{-c|q_1-q_2|} \\ &\quad \times \int_0^t ds_1 \int_0^{s_1} ds_2\, \frac{(1+s_2)^{-k}}{(1+(t-s_1))^\lambda (1+(s_1-s_2))^\lambda}. \end{aligned}$$

With Proposition 4.5, for any $k \ge 0$ and $\lambda > \max\{k,1\}$ the previous term is bounded from above by

$$\le \frac{C_{k,\lambda}}{M^{\zeta+b/2}}\, (1+t)^{-k}\, |||f|||_{k,\ell}.$$

This is the desired estimate for $M >> 1$ chosen sufficiently large.

We now consider the remaining part of H_5^{high}. As in the previous estimates and (4.10), if either $|q_2| \geq 2|q_1|$ or $|q_1| \geq 2|q_2|$ then for any $k \geq 0$ we have

$$
\begin{aligned}
w_\ell(p) &\int_{\mathbb{R}^3} dq_1 \, \left|k^\chi(p, q_1)\right| \int_{\mathbb{R}^3} dq_2 \, \left|k^\chi(q_1, q_2)\right| \left(\mathbf{1}_{|q_2|\geq 2|q_1|} + \mathbf{1}_{|q_2|\leq \frac{1}{2}|q_1|}\right) \\
&\times \mathbf{1}_{\frac{1}{2}|p|\leq|q_1|\leq 2|p|} \mathbf{1}_{high} \int_0^t ds_1 \, e^{-\nu(p)(t-s_1)} \int_0^{s_1} ds_2 \, e^{-\nu(q_1)(s_1-s_2)} \\
&\times |\{U(s_2) f_0\}(y_2, q_2)| \leq \frac{C_{k,\lambda}}{M^{2\zeta+b}} |||f|||_{k,\ell} \int_{\mathbb{R}^3} dq_1 \, e^{-c|p-q_1|} \int_{\mathbb{R}^3} dq_2 \, e^{-c|q_1-q_2|} \\
&\times \int_0^t ds_1 \int_0^{s_1} ds_2 \, \frac{(1+s_2)^{-k}}{(1+(t-s_1))^\lambda (1+(s_1-s_2))^\lambda} \\
\leq & \frac{C_{k,\lambda}}{M^{2\zeta+b/2}} (1+t)^{-k} \, |||f|||_{k,\ell}.
\end{aligned}
$$

Above we have used exactly the same estimates as in the prior case. Both of the last two terms have a suitably small constant in front if M is sufficiently large.

Thus the remaining part of H_5^{high} to estimate is

$$
\begin{aligned}
R_1(f)(t) \stackrel{\text{def}}{=} &\int_{\mathbb{R}^3} dq_1 \, k^\chi(p, q_1) \int_{\mathbb{R}^3} dq_2 \, k^\chi(q_1, q_2) \, \mathbf{1}_{\frac{1}{2}|p|\leq|q_1|\leq 2|p|} \mathbf{1}_{\frac{1}{2}|q_1|\leq|q_2|\leq 2|q_1|} \\
&\times \mathbf{1}_{high} \int_0^t ds_1 \, e^{-\nu(p)(t-s_1)} \int_0^{s_1} ds_2 \, e^{-\nu(q_1)(s_1-s_2)} \{U(s_2) f_0\}(y_2, q_2).
\end{aligned}
\tag{4.15}
$$

It is only this term which slows down the time decay rate. In this current proof we will only make a basic argument to show that this term can naively exhibit first order decay. Since all the momentum variables are comparable, we have

$$
\begin{aligned}
|R_1(f)(t)| \leq &\int_{\mathbb{R}^3} dq_1 \, \left|k^\chi(p, q_1)\right| \int_{\mathbb{R}^3} dq_2 \, \left|k^\chi(q_1, q_2)\right| \\
&\times \mathbf{1}_{\frac{1}{2}|p|\leq|q_1|\leq 2|p|} \mathbf{1}_{\frac{1}{2}|q_1|\leq|q_2|\leq 2|q_1|} \mathbf{1}_{high} \\
&\times \int_0^t ds_1 \, e^{-c\nu(q_1)(t-s_1)} \int_0^{s_1} ds_2 \, e^{-c\nu(q_1)(s_1-s_2)} \, |\{U(s_2) f_0\}(y_2, q_2)|.
\end{aligned}
$$

Next using similar techniques as in the previous two estimates, including Proposition 4.5 twice, we obtain the following upper bound for any $k \in [0, 1]$:

$$
\begin{aligned}
w_\ell(p) \, |R_1(f)(t)| \leq &\frac{C_{k,\lambda}}{M^{2\zeta}} |||f|||_{k,\ell} \int dq_1 \, q_{10}^{-b/2-\zeta} e^{-c|p-q_1|} \\
&\times \int dq_2 \, q_{10}^{-b/2-\zeta} e^{-c|q_1-q_2|} \, w_{2+2\delta}(q_1) \int_0^t \frac{ds_1}{(1+(t-s_1))^{1+\delta}} \\
&\times \int_0^{s_1} \frac{ds_2}{(1+(s_1-s_2))^{1+\delta}(1+s_2)^k} \leq \frac{C_{k,\lambda}}{M^{2\zeta}} (1+t)^{-k} |||f|||_{k,\ell}.
\end{aligned}
$$

In the last line we used the fact that we have chosen $\delta > 0$ to satisfy $\delta < 2\zeta/b$, where $\zeta > 0$ is defined in the statement of Lemma 3.2. In Sect. 6 we will examine this term at length to show that (4.15) actually decays “almost exponentially.”

We are ready to define the second term in our splitting of H_5. It must be

$$H_5^{low}(t,x,p) \stackrel{\text{def}}{=} \mathbf{1}_{|p|\le M} \int_{|q_1|\le M} dq_1\, k^\chi(p,q_1) \int_{\mathbb{R}^3} dq_2\, k^\chi(q_1,q_2) \times \int_0^t ds_1\, e^{-\nu(p)(t-s_1)} \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \{U(s_2)f_0\}(y_2,q_2).$$

For any small $\kappa > 0$, we further split this term into two terms, one of which is

$$\begin{aligned} H_5^{low,\kappa}(t,x,p) \stackrel{\text{def}}{=} & \int_{|q_1|\le M} dq_1\, k^\chi(p,q_1) \int_{\mathbb{R}^3} dq_2\, k^\chi(q_1,q_2) \int_0^\kappa ds_1\, e^{-\nu(p)(t-s_1)} \\ & \times \mathbf{1}_{|p|\le M} \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \{U(s_2)f_0\}(y_2,q_2) \\ & + \int_{|q_1|\le M} dq_1\, k^\chi(p,q_1) \int_{\mathbb{R}^3} dq_2\, k^\chi(q_1,q_2) \int_\kappa^t ds_1\, e^{-\nu(p)(t-s_1)} \\ & \times \mathbf{1}_{|p|\le M} \int_{s_1-\kappa}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \{U(s_2)f_0\}(y_2,q_2). \end{aligned}$$

The other term in this latest splitting is defined just below as $H_5^{low,2}$. On this temporal integration domain $s_1 - s_2 \le \kappa$. Since we are proving uniform bounds, it is safe to assume when proving decay that $t \ge 1$ for instance.

Since p and q_1 are both bounded by M, from Lemma 3.1 we have

$$\mathbf{1}_{|p|\le M}\mathbf{1}_{|q_1|\le M}\, e^{-\nu(p)(t-s_1)-\nu(q_1)(s_1-s_2)} \le e^{-C(t-s_2)/M^{b/2}}. \tag{4.16}$$

Then for the first term in $H_5^{low,\kappa}$ above multiplied by $w_\ell(p)$ we have the bound

$$\begin{aligned} & w_\ell(p) \int_{|q_1|\le M} dq_1\, \left|k^\chi(p,q_1)\right| \int_{\mathbb{R}^3} dq_2\, \left|k^\chi(q_1,q_2)\right| \int_0^\kappa ds_1\, e^{-\nu(p)(t-s_1)} \\ & \quad \times \mathbf{1}_{|p|\le M} \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \left|\{U(s_2)f_0\}(y_2,q_2)\right| \\ & \le C_M |||f|||_{0,\ell} \int_0^\kappa ds_1 \int_0^{s_1} ds_2\, e^{-C(t-s_2)/M^{b/2}} \\ & \le C_M \kappa^2 |||f|||_{0,\ell}\, e^{-Ct/M^{b/2}} e^{C\kappa/M^{b/2}} \le C_M\kappa^2 (1+t)^{-k} |||f|||_{0,\ell}. \end{aligned}$$

We have just used (4.8). We obtain the desired estimate for the above terms by first choosing M large, and second choosing $\kappa = \kappa(M) > 0$ sufficiently small.

For the second term in $H_5^{low,\kappa}$ multiplied by $w_\ell(p)$ for any $k \ge 0$ we have

$$\begin{aligned} & w_\ell(p) \int_{|q_1|\le M} dq_1\, \left|k^\chi(p,q_1)\right| \int_{\mathbb{R}^3} dq_2\, \left|k^\chi(q_1,q_2)\right| \int_\kappa^t ds_1\, e^{-\nu(p)(t-s_1)} \\ & \quad \times \mathbf{1}_{|p|\le M} \int_{s_1-\kappa}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \left|\{U(s_2)f_0\}(y_2,q_2)\right| \\ & \le C_M |||f|||_{k,\ell} \int_\kappa^t ds_1 \int_{s_1-\kappa}^{s_1} ds_2\, e^{-C(t-s_1)/M^{b/2}} e^{-C(s_1-s_2)/M^{b/2}} (1+s_2)^{-k}. \end{aligned}$$

Since $s_2 \in [s_1 - \kappa, s_1]$ and $\kappa \in (0, 1/2)$, then $(1+s_2) \geq \left(\frac{1}{2} + s_1\right)$. We have

$$\leq C_M \kappa |||f|||_{k,\ell} \int_\kappa^t ds_1 \, e^{-C(t-s_1)/M^{b/2}} (1+s_1)^{-k}$$
$$\leq C_M \kappa (1+t)^{-k} |||f|||_{k,\ell}.$$

In the last step we have used (4.8) and Proposition 4.5. We conclude the desired estimate for $H_5^{low,\kappa}$ by first choosing M large, and then $\kappa > 0$ sufficiently small.

The only remaining part of $H_5^{low}(t, x, p)$ to be estimated is given by

$$H_5^{low,2}(t,x,p) \overset{\text{def}}{=} \int_{|q_1| \leq M} dq_1 \, k^\chi(p, q_1) \int_{\mathbb{R}^3} dq_2 \, k^\chi(q_1, q_2) \int_\kappa^t ds_1 \, e^{-\nu(p)(t-s_1)}$$
$$\times \mathbf{1}_{|p| \leq M} \int_0^{s_1 - \kappa} ds_2 \, e^{-\nu(q_1)(s_1 - s_2)} \{U(s_2) f_0\}(y_2, q_2).$$

With all of the H_5^{high} and H_5^{low} terms defined above, we remark that (4.13) holds.

We now estimate $H_5^{low,2}$. Since p and q_1 are both bounded by M, from Lemma 3.1 we still have (4.16). For any $j \geq 0$, Lemma 3.2 implies the following bound:

$$\int_{|q_1| \leq M} dq_1 \int_{\mathbb{R}^3} dq_2 \, \left| w_j(q_2) k^\chi(p, q_1) k^\chi(q_1, q_2) \right|^2 \leq C_M.$$

Indeed if $|q_2| \geq 2|q_1|$ then as in (4.10) we can prove this bound. Alternatively if $|q_2| \leq 2|q_1|$ then $w_j(q_2) \leq C w_j(q_1) \leq C M^{jb/2}$ and the bound above also holds.

We use the above and Cauchy-Schwartz to estimate the momentum integrals:

$$\left| \int_{|q_1| \leq M} dq_1 \, k^\chi(p, q_1) \int_{\mathbb{R}^3} dq_2 \, k^\chi(q_1, q_2) \, \{U(s_2) f_0\}(y_2, q_2) \right|$$
$$\lesssim \left(\int_{|q_1| \leq M} dq_1 \int_{\mathbb{R}^3} dq_2 \, \left| w_j(q_2) k^\chi(p, q_1) k^\chi(q_1, q_2) \right|^2 \right)^{1/2}$$
$$\times \left(\int_{|q_1| \leq M} dq_1 \int_{\mathbb{R}^3} dq_2 \, \left| w_{-j}(q_2) \{U(s_2) f_0\}(y_2, q_2) \right|^2 \right)^{1/2}$$
$$\leq C_M \left(\int_{|q_1| \leq M} dq_1 \int_{\mathbb{R}^3} dq_2 \, \left| w_{-j}(q_2) \{U(s_2) f_0\}(y_2, q_2) \right|^2 \right)^{1/2}.$$

That step was significant to yield rapid polynomial momentum decay. We change variables $q_1 \to y_2$ on the dq_1 integration with y_2 given by (4.5). Then

$$\left(\frac{dy_2}{dq_1} \right)_{mn} = -(s_1 - s_2) \left(\frac{\delta_{mn} q_{10}^2 - q_{1m} q_{1n}}{q_{10}^3} \right). \tag{4.17}$$

This is a 3×3 matrix with two eigenvalues equal to $-\frac{(s_1 - s_2)}{q_{10}}$, and a third eigenvalue given by $-(s_1 - s_2) \frac{q_{10}^2 - |q_1|^2}{q_{10}^3} = \frac{-(s_1 - s_2)}{q_{10}^3}$. Thus the Jacobian is

$$\left| \frac{dy_2}{dq_1} \right| = \frac{|(s_1 - s_2)|^3}{q_{10}^5} \geq C \frac{\kappa^3}{M^5}.$$

This lower bound holds on the set $|q_1| \le M$, $s_2 \in [0, s_1 - \kappa]$ and $s_1 \in [\kappa, t]$ so that $s_1 - s_2 \ge \kappa$. After application of this change of variables we have

$$\left(\int_{|q_1|\le M} dq_1 \int_{\mathbb{R}^3} dq_2 \, \left|w_{-j}(q_2)\{U(s_2)f_0\}(y_2,q_2)\right|^2\right)^{1/2}$$
$$\lesssim \left(\frac{M^5}{\kappa^3}\right)^{1/2} \left(\int_{|y_2-x|\le C(t-s_2)} dy_2 \int_{\mathbb{R}^3} dq_2 \, \left|w_{-j}(q_2)\{U(s_2)f_0\}(y_2,q_2)\right|^2\right)^{1/2}$$
$$\lesssim \left(\frac{M^5}{\kappa^3}\right)^{1/2} \{1+(t-s_2)^{3/2}\} \left(\int_{\mathbb{T}^3} dy_2 \int_{\mathbb{R}^3} dq_2 \, \left|w_{-j}(q_2)\{U(s_2)f_0\}(y_2,q_2)\right|^2\right)^{1/2}$$
$$= C(M,\kappa)\{1+(t-s_2)^{3/2}\}\| \{U(s_2)f_0\} \|_{2,-j}.$$

Putting together all of these estimates, in particular using (4.16), we have shown

$$\left|w_\ell(p) H_5^{low,2}\right| \le C_{\kappa,M} \int_\kappa^t ds_1 \int_0^{s_1-\kappa} ds_2 \, e^{-C(t-s_1)/M^{b/2}} e^{-C(s_1-s_2)/M^{b/2}}$$
$$\times \{1+(t-s_2)^{3/2}\}\| \{U(s_2)f_0\} \|_{2,-j}$$
$$\le C_{\kappa,M} \int_0^t ds_1 \, e^{-\frac{C}{2}(t-s_1)/M^{b/2}} \int_0^t ds_2 \, e^{-\frac{C}{2}(t-s_1)/M^{b/2}} e^{-\frac{C}{2}(s_1-s_2)/M^{b/2}}$$
$$\times \{1+(t-s_2)^{3/2}\}\| \{U(s_2)f_0\} \|_{2,-j}.$$

Notice that the first exponential controls the s_1 time integral, and the second and third exponential control the remaining time integral as follows:

$$\left|w_\ell(p) H_5^{low,2}\right| \le C_{\kappa,M} \int_0^t ds_2 \, e^{-\frac{C}{2}(t-s_2)/M^{b/2}} \{1+(t-s_2)^{3/2}\}\| \{U(s_2)f_0\} \|_{2,-j}$$
$$\le C_{\kappa,M} \int_0^t ds_2 \, e^{-\frac{C}{4}(t-s_2)/M^{b/2}} \| \{U(s_2)f_0\} \|_{2,-j}.$$

That was the last case. Adding up the individual estimates for H_5^{high} and H_5^{low} in (4.13) completes our proof after first choosing M large enough and then second choosing κ sufficiently small. □

5. Nonlinear L^∞ Bounds and Slow Decay

Suppose $f = f(t,x,p)$ solves (2.1) with initial condition $f(0,x,p) = f_0(x,p)$. We may express mild solutions to this problem (2.1) in the form

$$f(t,x,p) = \{U(t)f_0\}(x,p) + N[f,f](t,x,p), \tag{5.1}$$

where we have used the notation

$$N[f_1,f_2](t,x,p) \overset{\text{def}}{=} \int_0^t ds \, \{U(t-s)\Gamma[f_1(s), f_2(s)]\}(x,p).$$

Here as usual $U(t)$ is the semi-group (4.1) which represents solutions to the linear problem (2.11). The main result of this section is to prove the following

Theorem 5.1. *Choose $\ell > 3/b$, $k \in (1/2, 1]$. Consider the following initial data $f_0 = f_0(x,p) \in L^\infty_{\ell+k}(\mathbb{T}^3 \times \mathbb{R}^3)$ which satisfies (2.6) initially. There is an $\eta > 0$ such that if $\|f_0\|_{\infty,\ell+k} \leq \eta$, then there exists a unique global in time mild solution (5.1), $f = f(t,x,p)$, to Eq. (2.1) which satisfies*

$$\|f\|_{\infty,\ell}(t) \leq C_{\ell,k}(1+t)^{-k}\|f_0\|_{\infty,\ell+k}.$$

These solutions are continuous if it is so initially. We further have positivity, i.e. $F = \mu + \sqrt{\mu} f \geq 0$, if $F_0 = \mu + \sqrt{\mu} f_0 \geq 0$.

In Theorem 7.1, which is proven in Sect. 7, we will show that these solutions exhibit rapid polynomial decay to any order. Notice that our main Theorem 2.1 will follow directly from Theorem 5.1 and Theorem 7.1. To prove this current Theorem 5.1 we will use the following non-linear estimate.

Lemma 5.2. *Considering the non-linear operator defined in (2.5) with (2.7), we have the following pointwise estimates*:

$$|w_\ell(p)\Gamma(h_1,h_2)(p)| \lesssim \nu(p)\|h_1\|_{\infty,\ell}\|h_2\|_{\infty,\ell}.$$

These hold for any $\ell \geq 0$. Furthermore, $\|\Gamma(h_1,h_2)\|_{\infty,\ell+1} \lesssim \|h_1\|_{\infty,\ell}\|h_2\|_{\infty,\ell}$.

The lemma above combined with Proposition 4.5 will be important tools in our proof of Theorem 5.1. We now give a simple proof.

Proof of Lemma 5.2. We recall (2.3), (2.5), and (2.9). For $\ell \geq 0$, it follows from (1.6) that

$$w_\ell(p) \lesssim p_0^{\ell b/2} \lesssim (p_0')^{\ell b/2}(q_0')^{\ell b/2} \lesssim w_\ell(p')w_\ell(q').$$

A proof of this estimate above was given in [24, Lemma 2.2]. Thus

$$\begin{aligned} w_\ell(p)\,|\Gamma(h_1,h_2)| &\lesssim \int_{\mathbb{R}^3\times\mathbb{S}^2} d\omega dq\; v_\o\, \sigma(g,\theta)\,\sqrt{J(q)}\; w_\ell(p')w_\ell(q')\left|h_1(p')h_2(q')\right| \\ &\quad + \int_{\mathbb{R}^3\times\mathbb{S}^2} d\omega dq\; v_\o\, \sigma(g,\theta)\,\sqrt{J(q)}\; w_\ell(p)\,|h_1(p)h_2(q)| \\ &\lesssim \|h_1\|_{\infty,\ell}\|h_2\|_{\infty,\ell}\int_{\mathbb{R}^3\times\mathbb{S}^2} d\omega dq\; v_\o\, \sigma(g,\theta)\; J^{1/2}(q) \\ &\lesssim \nu(p)\|h_1\|_{\infty,\ell}\|h_2\|_{\infty,\ell}. \end{aligned}$$

The last inequality above follows directly from Lemma 3.1 since both the integral and $\nu(p)$ have the same asymptotic behavior at infinity. That yields the first estimate. For the second estimate we notice from the first estimate that $w_{\ell+1}(p)\,|\Gamma(h_1,h_2)| \lesssim w_1(p)\nu(p)\|h_1\|_{\infty,\ell}\|h_2\|_{\infty,\ell}$. But $w_1(p)\nu(p) \lesssim 1$ from Lemma 3.1 and (2.9). This completes the proof. □

We now proceed to the

Proof of Theorem 5.1. We will prove Theorem 5.1 in three steps. The first step gives existence, uniqueness and slow decay via the contraction mapping argument. The second step will establish continuity, and the last step shows positivity.

Step 1. Existence and uniqueness. When proving existence of mild solutions to (5.1) it is natural to consider the mapping

$$M[f] \overset{\text{def}}{=} \{U(t) f_0\}(x, p) + N[f, f](t, x, p).$$

With the norm (2.10), we will show that this is a contraction mapping on the space

$$M^R_{k,\ell} \overset{\text{def}}{=} \{f \in L^\infty([0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3) : |||f|||_{k,\ell} \le R\}, \quad R > 0.$$

We first estimate the non-linear term $N[f, f]$ defined in the equation display below (5.1). We apply Theorem 4.1, with $\ell > 3/b$, and $k \in (1/2, 1]$ to obtain

$$\begin{aligned} w_\ell(p)\,|N[f_1, f_2](t, x, p)| &\lesssim \int_0^t ds\; w_\ell(p)\,|\{U(t-s)\Gamma[f_1(s), f_2(s)]\}(x, p)| \\ &\lesssim \int_0^t \frac{ds}{(1+t-s)^k}\, \|\Gamma[f_1(s), f_2(s)]\|_{\infty,\ell+k}\,. \end{aligned}$$

Next Lemma 5.2 allows us to bound the above by

$$\lesssim \int_0^t \frac{ds}{(1+t-s)^k}\, \|f_1(s)\|_{\infty,\ell+k-1}\, \|f_2(s)\|_{\infty,\ell+k-1}\,.$$

From Proposition 4.5 and the decay norm (2.10) we see that the last line is

$$\begin{aligned} &\lesssim |||f_1|||_{k,\ell}|||f_2|||_{k,\ell} \int_0^t \frac{ds}{(1+t-s)^k(1+s)^{2k}} \\ &\lesssim (1+t)^{-k}|||f_1|||_{k,\ell}|||f_2|||_{k,\ell}. \end{aligned}$$

We have shown

$$|||N[f_1, f_2]|||_{k,\ell} \lesssim |||f_1|||_{k,\ell}|||f_2|||_{k,\ell}.$$

To handle the linear semigroup, $U(t)$, we again use Theorem 4.1 to obtain

$$|||M[f]|||_{k,\ell} \le C_{\ell,k}\left(\|f_0\|_{\infty,\ell+k} + |||f|||^2_{k,\ell}\right).$$

We conclude that $M[\cdot]$ maps $M^R_{k,\ell}$ into itself for $0 < R$ chosen sufficiently small and e.g. $\|f_0\|_{\infty,\ell+k} \le \frac{R}{2C_{\ell,k}}$. To obtain a contraction, we consider the difference

$$M[f_1] - M[f_2] = N[f_1 - f_2, f_1] + N[f_2, f_1 - f_2].$$

Then as in the previous estimates we have

$$|||M[f_1] - M[f_2]|||_{k,\ell} \le C^*_{\ell,k}\left(|||f_1|||_{k,\ell} + |||f_2|||_{k,\ell}\right)|||f_1 - f_2|||_{k,\ell}.$$

With these estimates, the existence and uniqueness of solutions to (2.1) follows from the contraction mapping principle on $M^R_{k,\ell}$ when $R > 0$ is suitably small.

Step 2. Continuity. We perform the estimates from Step 1 on the space

$$M^{R,0}_{k,\ell} \overset{\text{def}}{=} \{f \in C^0([0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3) : |||f|||_{k,\ell} \le R\}, \quad R > 0.$$

As in Step 1, we have a uniform in time contraction mapping on $M^{R,0}_{k,\ell}$ for suitable R. Furthermore $M[f]$ is continuous if $f \in M^{R,0}_{k,\ell}$ and f_0 is continuous. Since the convergence is uniform, the limit will be continuous globally in time. This argument is standard and we refer for instance to [21,24,33] for full details.

Step 3. Positivity. We use the standard alternative approximating formula

$$\left(\partial_t + \hat{p} \cdot \nabla_x\right) F^{n+1} + R(F^n) F^{n+1} = \mathcal{Q}_+(F^n, F^n),$$

with the same initial conditions $F^{n+1}\big|_{t=0} = F_0 = J + \sqrt{J} f_0$, for $n \geq 1$ and for instance $F^1 \overset{\text{def}}{=} J + \sqrt{J} f_0$. Here we have used the standard decomposition of $\mathcal{Q} = \mathcal{Q}_+ - \mathcal{Q}_-$ into gain and loss terms with

$$\mathcal{Q}_-(F^{n+1}, F^n) = R(F^n) F^{n+1},$$

and $R(F^n) \overset{\text{def}}{=} \mathcal{Q}_-(1, F^n)$. If we consider $F^{n+1}(t,x,p) = J + \sqrt{J} f^{n+1}(t,x,p)$, then related to Step 1 we may show that $f^{n+1}(t,x,p)$ is convergent in L^∞_ℓ on a local time interval $[0, T]$, where T will generally depend upon the size of the initial data. In particular $f^{n+1}(t,x,p) = \frac{F^{n+1} - J}{\sqrt{J}}$ satisfies the equation

$$\left(\partial_t + \hat{p} \cdot \nabla_x + \nu(p)\right) f^{n+1} = K(f^n) + \Gamma_+(f^n, f^n) - \Gamma_-(f^{n+1}, f^n).$$

We rewrite this equation using the solution formula to the system (4.2) as

$$f^{n+1} = G(t) f_0 + \mathcal{L}(f^{n+1}, f^n).$$

This solution formula $G(t)$ is defined just below (4.2). Furthermore

$$\mathcal{L}(f^{n+1}, f^n) \overset{\text{def}}{=} \int_0^t ds\, G(t-s) K(f^n) + \int_0^t ds\, G(t-s)\Gamma_+(f^n, f^n) - G(t-s)\Gamma_-(f^{n+1}, f^n).$$

For given $T > 0$ and $R > 0$ we consider the space $M^R_{k,\ell}([0,T])$ defined by

$$\left\{ f \in L^\infty([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3) : \operatorname{ess\,sup}_{0 \leq t \leq T} (1+t)^k \| f(t) \|_{\infty,\ell} \leq R \right\}.$$

Now given $f^n \in M^R_{k,\ell}([0,T])$ and $\|f_0\|_{\infty,\ell+k} \leq \frac{R}{2C_{k,\ell}}$ with $R > 0$ chosen sufficiently small, as in Step 1, we can prove the existence of $f^{n+1} \in M^R_{k,\ell}([0,T])$.

With the estimates established in this paper, it is now not hard to show that

$$\sup_{0 \leq t \leq T} \| f^{n+1} - f^n \|_{\infty,\ell}(t) \leq CT \sup_{0 \leq t \leq T} \| f^n - f^{n-1} \|_{\infty,\ell}(t).$$

Here $T > 0$ is sufficiently small, and the constant $C > 0$ can be chosen independent of any small T. Therefore there exists a $T^* > 0$ such that $f^n \to f$ uniformly in L^∞_ℓ on $[0, T^*]$. This will be sufficient to prove the positivity globally in time.

Indeed if $F^n \geq 0$, then so is $\mathcal{Q}_+(F^n, F^n) \geq 0$. With the representation formula

$$F^{n+1}(t, x, p) = e^{-\int_0^t ds R(F^n)(s, x-\hat{p}(t-s), p)} F_0(x - \hat{p}t, p) + \int_0^t ds\, e^{-\int_s^t d\tau R(F^n)(\tau, x-\hat{p}(t-\tau), p)} \mathcal{Q}_+(F^n, F^n)(s, x - \hat{p}(t-s), p).$$

Induction shows $F^{n+1}(t, x, p) \geq 0$ for all $n \geq 0$ if $F_0 \geq 0$, which implies in the limit $n \to \infty$ that $F(t, x, p) = J + \sqrt{J} f(t, x, p) \geq 0$. Using our L^∞_ℓ uniqueness, this is the same F as the one from Step 1 on the time interval $[0, T^*]$. We extend this positivity for all time intervals $[0, T^*] + T^* k$ for any $k \geq 1$ by repeating this procedure and using the global uniform bound in L^∞_ℓ from Step 1. □

6. Linear L^∞ Rapid Decay

In this section we prove that the linear semi-group (4.1) exhibits rapid polynomial decay in L^∞_ℓ. For any $k \geq 1$ we will discover a $k' = k'(k) \geq k$ such that

$$\|\{U(t) f_0\}\|_{\infty,\ell} \leq C_{\ell,k}(1 + t)^{-k} \|f_0\|_{\infty,\ell+k'}. \tag{6.1}$$

The main obstruction to proving such rapid decay in this low regularity L^∞_ℓ framework was the term (4.15) which came up during the course of the proof of Lemma 4.4. In this section we perform a new high-order expansion of this remainder which allows one to prove rapid decay as follows.

Proposition 6.1. *Consider $R_1(f)(t)$ defined in (4.15). Choose $\ell > 3/b$. For any small $\eta > 0$, and any $k \geq 1$, there exists a $k' = k'(k) \geq k$ such that*

$$w_\ell(p) |R_1(f)(t)| \leq \eta(1 + t)^{-k} |||f|||_{k,\ell} + C_{\ell,k',\eta}(1 + t)^{-k} \|f_0\|_{\infty,\ell+k'}.$$

The power k' can be explicitly computed from the proof.

The crucial difficulty with proving rapid decay for the soft potentials is caused by the high momentum values, for which the time decay is diluted by the momentum decay. This causes the generation of weights on the initial data, typically one weight for each order of time decay. In the proof below we are able to overcome this apparent obstruction by performing a new high order expansion for $R_1(f)$ which is explained in detail at the beginning of the proof.

We will first show that Proposition 6.1 implies (6.1). We use the expansion (4.4) and the semi-group notation $f(t) = \{U(t) f_0\}$. We now see that Lemmas 4.2, 4.3, and 4.4 together imply that for any $\eta > 0$ and $k \geq 1/2$ we have

$$\|f\|_{\infty,\ell}(t) \leq C_{\ell,k}(1 + t)^{-k} \|f_0\|_{\infty,\ell+k} + \frac{\eta}{2}(1 + t)^{-k} |||f|||_{k,\ell} + w_\ell(p) |R_1(f)(t)|.$$

Here we use (2.10). Then Proposition 6.1 further implies for some $k' \geq k$ that

$$w_\ell(p) |R_1(f)(t)| \leq C_{\ell,k,\eta}(1 + t)^{-k} \|f_0\|_{\infty,\ell+k'} + \frac{\eta}{2}(1 + t)^{-k} |||f|||_{k,\ell}.$$

Equivalently

$$|||f|||_{k,\ell} \leq C_{\ell,k'} \|f_0\|_{\infty,\ell+k'}.$$

This is the desired decay rate for the L^∞_ℓ norm of mild solutions to the linear equation (2.11), which proves (6.1) subject to Proposition 6.1. In the rest of this section we prove this crucial new proposition.

Proof of Proposition 6.1. We will prove this proposition with a new high order expansion of (4.15) by iterating the semi-group (4.3). For ease of exposition we write

$$k^{\chi}_{\approx}(p,q_1) \stackrel{\text{def}}{=} k^{\chi}(p,q_1)\mathbf{1}_{\frac{1}{2}|p|\le|q_1|\le 2|p|}.$$

Recall that $\mathbf{1}_{\frac{1}{2}|p|\le|q_1|\le 2|p|}$ is the function which is one when $\frac{1}{2}|p|\le|q_1|\le 2|p|$ and zero elsewhere. We will use similar expressions for $k^{\chi}_{\approx}$ with different arguments. Then we may split (4.15) as

$$R_1(f) = S_1(f) + L_1(f) + R_2(f).$$

For any small $\kappa > 0$ we choose $\kappa_1 = \kappa$ and $\kappa_2 = \kappa_1/2$ so that

$$S_1(f) \stackrel{\text{def}}{=} \int_0^{\kappa_1} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\,\mathbf{1}_{high} \int_0^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2)\,\{U(s_2)f_0\}(y_2,q_2),$$

$$L_1(f) \stackrel{\text{def}}{=} \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\,\mathbf{1}_{high} \int_0^{s_1-\kappa_2} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2)\,\{U(s_2)f_0\}(y_2,q_2).$$

Then we may define the remainder term as

$$R_2(f) \stackrel{\text{def}}{=} \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\,\mathbf{1}_{high} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2)\,\{U(s_2)f_0\}(y_2,q_2).$$

Our notation above is from the proof of Lemma 4.4, in particular (4.14). We will show that the first term $S_1(f)$ exhibits rapid decay in L^∞_ℓ. The last term $L_1(f)$ is bounded in L^2 which further has rapid decay as in Theorem 3.7.

We will notice first of all that the term $R_2(f)$ naively exhibits second order polynomial decay. However if we continue the expansion then we can obtain higher and higher order decay rates as follows. We may expand

$$R_2(f) = G_2(f) + D_2(f) + N_2(f) + L_2(f) + R_3(f).$$

Now each of the terms $G_2(f)$, $D_2(f)$, $N_2(f)$, and $L_2(f)$– to be defined below–will exhibit (for different reasons) high order polynomial decay right away again at a cost of momentum weights on the initial data. The term $R_3(f)$ will clearly exhibit third order polynomial decay, however we may continue this expansion at each level so that at level k we can again expand

$$R_k(f) = G_k(f) + D_k(f) + N_k(f) + L_k(f) + R_{k+1}(f).$$

As in the initial case each of the terms $G_k(f)$, $D_k(f)$, $N_k(f)$, and $L_k(f)$–which are defined recursively–will exhibit high order polynomial decay. The last term $R_{k+1}(f)$ will have $k+1$ order polynomial decay. This expansion is well defined and can be continued to any order, which yields rapid decay.

We define the 2nd order terms by plugging the iteration (4.3) into $R_2(f)$, using the expansion of $K = K^{1-\chi} + K^{\chi}$ with (3.1), and splitting the remaining time and momentum integrals in the following useful way. For $\kappa_3 = \kappa_2/2 = \kappa_1/2^2$, we define

$$
\begin{aligned}
G_2(f) &\stackrel{\text{def}}{=} \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\, \mathbf{1}_{high} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \\
&\quad \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2)\, \{G(s_2) f_0\}(y_2,q_2), \\
D_2(f) &\stackrel{\text{def}}{=} \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\, \mathbf{1}_{high} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \\
&\quad \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2) \int_0^{s_2} ds_3\, e^{-\nu(q_2)(s_2-s_3)} K^{1-\chi}\, \{U(s_3) f_0\}(y_3,q_2).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
N_2(f) &\stackrel{\text{def}}{=} \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\, \mathbf{1}_{high} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \\
&\quad \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2) \int_0^{s_2} ds_3\, e^{-\nu(q_2)(s_2-s_3)} \\
&\quad \times \int_{\mathbb{R}^3} dq_3\, k^{\chi}_{\not\approx}(q_2,q_3)\, \{U(s_3) f_0\}(y_3,q_3), \\
L_2(f) &\stackrel{\text{def}}{=} \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\, \mathbf{1}_{high} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \\
&\quad \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2) \int_0^{s_2-\kappa_3} ds_3\, e^{-\nu(q_2)(s_2-s_3)} \\
&\quad \times \int_{\mathbb{R}^3} dq_3\, k^{\chi}_{\approx}(q_2,q_3)\, \{U(s_3) f_0\}(y_3,q_3).
\end{aligned}
$$

And the remainder is given by

$$
\begin{aligned}
R_3(f) &\stackrel{\text{def}}{=} \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^{\chi}_{\approx}(p,q_1)\, \mathbf{1}_{high} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \\
&\quad \times \int_{\mathbb{R}^3} dq_2\, k^{\chi}_{\approx}(q_1,q_2) \int_{s_2-\kappa_3}^{s_2} ds_3\, e^{-\nu(q_2)(s_2-s_3)} \\
&\quad \times \int_{\mathbb{R}^3} dq_3\, k^{\chi}_{\approx}(q_2,q_3)\, \{U(s_3) f_0\}(y_3,q_3),
\end{aligned}
$$

where above y_1 and y_2 are defined in (4.5), and more generally

$$
\begin{aligned}
y_{i+1} &\stackrel{\text{def}}{=} y_1 - \hat{q}_1(s_1-s_2) - \cdots - \hat{q}_i(s_i - s_{i+1}) \\
&= x - \hat{p}(t-s_1) - \hat{q}_1(s_1-s_2) - \cdots - \hat{q}_i(s_i-s_{i+1}). \qquad (6.2)
\end{aligned}
$$

So that in general for $i \geq 1$ we have

$$
y_{i+1} = y_i - \hat{q}_i(s_i - s_{i+1}).
$$

Furthermore $k^\chi(p,q) = k^\chi_{\neq}(p,q) + k^\chi_{\approx}(p,q)$ with the notation

$$k^\chi_{\neq}(p,q) \stackrel{\text{def}}{=} k^\chi(p,q)\left(\mathbf{1}_{|p|\geq 2|q|} + \mathbf{1}_{|q|\geq 2|p|}\right).$$

Now we will develop a collection of notations in order to put this expansion into a general framework and appropriately define the high order terms. We consider the sequence $\{\kappa\}$, where for $i \geq 1$ we define $\kappa_{i+1} = \kappa_i/2$ with a small $\kappa_1 = \kappa > 0$ as above so that $\kappa_i = \kappa/2^{i-1}$. For $i \geq 1$ we can now define

$$A(f)(t,x,p,\{\kappa\}) \stackrel{\text{def}}{=} \int_{\kappa_1}^t ds_1\, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1\, k^\chi_{\approx}(p,q_1)\, \mathbf{1}_{high}\, f(s_1,y_1,q_1),$$

$$\begin{aligned} B_{i+1}(f)(s_1,y_1,q_1,\{\kappa\}) \stackrel{\text{def}}{=} & \int_{\mathbb{R}^3} dq_2\, k^\chi_{\approx}(q_1,q_2)\cdots \int_{\mathbb{R}^3} dq_{i+1}\, k^\chi_{\approx}(q_i,q_{i+1}) \\ & \times \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \dots \\ & \times \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1}\, e^{-\nu(q_i)(s_i-s_{i+1})} f(s_{i+1},y_{i+1},q_{i+1}). \end{aligned}$$

To finish off our expansion we further define

$$\begin{aligned} \tilde{G}(f)(s,y,q) &\stackrel{\text{def}}{=} G(s) f_0(y,q), \\ D(f)(s_{i+1},y_{i+1},q_{i+1}) &\stackrel{\text{def}}{=} \int_0^{s_{i+1}} ds_{i+2}\, e^{-\nu(q_{i+1})(s_{i+1}-s_{i+2})} \\ &\quad \times K^{1-\chi}\left\{U(s_{i+2})f_0\right\}(y_{i+2},q_{i+1}). \end{aligned}$$

Above we recall that $G(s)$ is defined just below (4.2). Additionally

$$\begin{aligned} N(f)(s_{i+1},y_{i+1},q_{i+1}) &\stackrel{\text{def}}{=} \int_0^{s_{i+1}} ds_{i+2}\, e^{-\nu(q_{i+1})(s_{i+1}-s_{i+2})} \\ &\quad \times \int_{\mathbb{R}^3} dq_{i+2}\, k^\chi_{\neq}(q_{i+1},q_{i+2})\left\{U(s_{i+2})f_0\right\}(y_{i+2},q_{i+2}), \\ \tilde{L}(f)(s_{i+1},y_{i+1},q_{i+1},\{\kappa\}) &\stackrel{\text{def}}{=} \int_0^{s_{i+1}-\kappa_{i+2}} ds_{i+2}\, e^{-\nu(q_{i+1})(s_{i+1}-s_{i+2})} \\ &\quad \times \int_{\mathbb{R}^3} dq_{i+2}\, k^\chi_{\approx}(q_{i+1},q_{i+2})\ \left\{U(s_{i+2})f_0\right\}(y_{i+2},q_{i+2}). \end{aligned}$$

Then we may use this notation to write

$$\begin{gathered} G_2(f) \stackrel{\text{def}}{=} A(B_2(\tilde{G}(f)))(t,x,p,\{\kappa\}), \quad D_2(f) \stackrel{\text{def}}{=} A(B_2(D(f)))(t,x,p,\{\kappa\}), \\ N_2(f) \stackrel{\text{def}}{=} A(B_2(N(f)))(t,x,p,\{\kappa\}), \quad L_2(f) \stackrel{\text{def}}{=} A(B_2(\tilde{L}(f)))(t,x,p,\{\kappa\}), \\ R_3(f) \stackrel{\text{def}}{=} A(B_3(f))(t,x,p,\{\kappa\}). \end{gathered}$$

We iteratively define the higher order terms of this expansion for $i \geq 2$ as

$$\begin{gathered} G_i(f) \stackrel{\text{def}}{=} A(B_i(\tilde{G}(f)))(t,x,p,\{\kappa\}), \quad D_i(f) \stackrel{\text{def}}{=} A(B_i(D(f)))(t,x,p,\{\kappa\}), \\ N_i(f) \stackrel{\text{def}}{=} A(B_i(N(f)))(t,x,p,\{\kappa\}), \quad L_i(f) \stackrel{\text{def}}{=} A(B_i(\tilde{L}(f)))(t,x,p,\{\kappa\}), \\ R_{i+1}(f) \stackrel{\text{def}}{=} A(B_{i+1}(f))(t,x,p,\{\kappa\}). \end{gathered}$$

This expansion works to high order using (4.3) which implies

$$B_i = B_i \tilde{G} + B_i D + B_i N + B_i \tilde{L} + B_{i+1}, \quad i \geq 2.$$

This completes our general discussion of the expansion formula, and our strategy for obtaining the desired decay. In the following we prove the claimed time decay estimates for each term in a general framework.

We initially estimate the main term, R_{k+1}, with $k \geq 1$. We *claim* that

$$w_\ell(q_1) \left| B_{k+1}(f)(s_1, y_1, q_1, \{\kappa\}) \right| \lesssim w_{-k(1+\delta)}(q_1) \, |||f|||_{k+1,\ell} \int_{s_1-\kappa_2}^{s_1} ds_2 \, e^{-c\nu(q_1)(s_1-s_2)} \, (1+s_2)^{-k-1}. \tag{6.3}$$

Here we have chosen $\delta > 0$ to satisfy $\delta < 2\zeta/b$, for ζ from Lemma 3.2. This claim (6.3) would imply with Lemma 3.2 that

$$\begin{aligned} w_\ell(p) \left| R_{k+1}(f)(t, x, p, \{\kappa\}) \right| &= w_\ell(p) \left| A(B_{k+1}(f))(t, x, p, \{\kappa\}) \right| \\ &\lesssim \int_{\kappa_1}^{t} ds_1 \, e^{-\nu(p)(t-s_1)} \int_{\mathbb{R}^3} dq_1 \, \left| k_{\approx}^{\chi}(p, q_1) \right| \, \mathbf{1}_{high} \, w_{-k(1+\delta)}(q_1) \, |||f|||_{k+1,\ell} \\ &\quad \times \int_{s_1-\kappa_2}^{s_1} ds_2 \, e^{-c\nu(q_1)(s_1-s_2)} \, (1+s_2)^{-k-1} \\ &\lesssim \frac{w_{-(k+1)(1+\delta)}(p)}{M^\zeta} |||f|||_{k+1,\ell} \int_0^t ds_1 \, e^{-c\nu(p)(t-s_1)} \, (1+s_1)^{-k-1}. \end{aligned}$$

We have used that $s_1 \approx s_2$ for κ sufficiently small, and also $e^{-c\nu(q_1)(s_1-s_2)} \lesssim 1$ since $s_1 - s_2 \geq 0$. We have additionally used the fact that the momentum variables are comparable because of the support condition for $k_{\approx}^{\chi}$. The large M comes from the support of $\mathbf{1}_{high}$ in (4.14) and Lemma 3.2. Furthermore

$$\begin{aligned} &\lesssim \frac{1}{M^\zeta} \, |||f|||_{k+1,\ell} \int_0^t ds_1 \, (1+t-s_1)^{-k-1-\delta} \, (1+s_1)^{-k-1} \\ &\lesssim \frac{1}{M^\zeta} \, (1+t)^{-k-1} \, |||f|||_{k+1,\ell}. \end{aligned}$$

We have additionally used (4.8) and then Proposition 4.5. For $M \gg 1$ chosen sufficiently large, this is the desired estimate for R_{k+1}.

We now prove the *claim* from (6.3). Since all of the momentum variables are comparable in this operator we have the following iterated estimate:

$$e^{-\nu(q_1)(s_1-s_2)} \cdots e^{-\nu(q_k)(s_k-s_{k+1})} \leq e^{-C\nu(q_1)(s_1-s_{k+1})}. \tag{6.4}$$

This uses in particular Lemma 3.1. We use (6.4) to obtain

$$\begin{aligned} &w_\ell(q_1) \left| B_{k+1}(f)(s_1, y_1, q_1, \{\kappa\}) \right| \\ &\lesssim |||f|||_{k+1,\ell} \int_{\mathbb{R}^3} dq_2 \, \left| k_{\approx}^{\chi}(q_1, q_2) \right| \cdots \int_{\mathbb{R}^3} dq_{k+1} \, \left| k_{\approx}^{\chi}(q_k, q_{k+1}) \right| \\ &\quad \times \int_{s_1-\kappa_2}^{s_1} ds_2 \, e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_k-\kappa_{k+1}}^{s_k} ds_{k+1} \, e^{-\nu(q_k)(s_k-s_{k+1})} (1+s_{k+1})^{-k-1} \\ &\lesssim |||f|||_{k+1,\ell} \int_{\mathbb{R}^3} dq_2 \, \left| k_{\approx}^{\chi}(q_1, q_2) \right| \cdots \int_{\mathbb{R}^3} dq_{k+1} \, \left| k_{\approx}^{\chi}(q_k, q_{k+1}) \right| \\ &\quad \times \int_{s_1-\kappa_2}^{s_1} ds_2 \, \cdots \int_{s_k-\kappa_{k+1}}^{s_k} ds_{k+1} \, e^{-C\nu(q_1)(s_1-s_{k+1})} (1+s_{k+1})^{-k-1}. \end{aligned}$$

We *sub-claim* that Lemma 3.2 can be used to control the momentum integrals as

$$\int_{\mathbb{R}^3} dq_2 \, \left|k^\chi_\approx(q_1,q_2)\right| \cdots \int_{\mathbb{R}^3} dq_{k+1} \, \left|k^\chi_\approx(q_k,q_{k+1})\right| \lesssim w_{-k(1+\delta)}(q_1). \tag{6.5}$$

The trick used in this estimate is Lemma 3.2 combined with $q_1 - q_2 \to q_2$ to obtain

$$\int_{\mathbb{R}^3} dq_2 \, \left|k^\chi_\approx(q_1,q_2)\right| \lesssim w_{-(1+\delta)}(q_1) \int_{\mathbb{R}^3} dq_2 \, e^{-c|q_2|}.$$

We can do that k times starting with the dq_{k+1} integral and iterating backwards (using the essential point that all the momentum variables are comparable) to obtain the *sub-claim*. Now by the definition of the sequence $\{\kappa\}$ we can say that $s_{k+1} \le s_2$ for $k \ge 1$ and more generally (on the integration region of B_{k+1})

$$\begin{aligned} s_{k+1} &\ge s_k - \kappa_{k+1} \ge s_2 - \kappa_3 - \cdots - \kappa_{k+1} \\ &= s_2 - \kappa\left(\frac{1}{4} + \cdots + \frac{1}{2^k}\right) \ge s_2 - \frac{\kappa}{2} \ge s_2 - \frac{1}{4}. \end{aligned}$$

Thus for $\kappa \le 1/2$ we use these estimates above to obtain

$$\begin{aligned} &\int_{s_1-\kappa_2}^{s_1} ds_2 \cdots \int_{s_k-\kappa_{k+1}}^{s_k} ds_{k+1} \, e^{-c\nu(q_1)(s_1-s_{k+1})}(1+s_{k+1})^{-k-1} \\ &\lesssim \kappa_{k+1}\cdots\kappa_3 \int_{s_1-\kappa_2}^{s_1} ds_2 \, e^{-c\nu(q_1)(s_1-s_2)} \left(\frac{1}{2} + s_2\right)^{-k-1}. \end{aligned}$$

Collecting the estimates above proves the claim (6.3).

To estimate the first term above, S_1, we obtain

$$\begin{aligned} w_\ell(p)\,|S_1(f)(t)| \lesssim w_\ell(p) &\int_{\mathbb{R}^3} dq_1 \, \left|k^\chi_\approx(p,q_1)\right| \int_{\mathbb{R}^3} dq_2 \, \left|k^\chi_\approx(q_1,q_2)\right| \, \mathbf{1}_{high} \\ &\times \int_0^{\kappa_1} ds_1 \, e^{-\nu(p)(t-s_1)} \int_0^{\kappa_1} ds_2 \, e^{-\nu(q_1)(s_1-s_2)} \left|\{U(s_2)f_0\}(y_2,q_2)\right|. \end{aligned}$$

Now we use (4.8), and Lemma 3.1, to observe that

$$e^{-\nu(p)t} \lesssim w_k(p)(1+t)^{-k}. \tag{6.6}$$

Furthermore $\int_0^{\kappa_1} ds_1 \, e^{\nu(p)s_1} \int_0^{\kappa_1} ds_2 \, e^{-\nu(q_1)(s_1-s_2)} \lesssim \kappa_1^2$. We thus have

$$\begin{aligned} w_\ell(p)\,|S_1(f)(t)| &\lesssim \kappa_1^2(1+t)^{-k} |||f|||_{0,\ell+k} \int_{\mathbb{R}^3} dq_1 \, \left|k^\chi_\approx(p,q_1)\right| \\ &\quad\times \int_{\mathbb{R}^3} dq_2 \, \left|k^\chi_\approx(q_1,q_2)\right| \, \frac{w_k(p)\mathbf{1}_{high}}{w_k(q_2)} \\ &\lesssim \frac{\kappa_1^2(1+t)^{-k}}{M^{2\zeta}} \|f_0\|_{\infty,\ell+k}. \end{aligned}$$

We have used the uniform bound from Theorem 4.1 with no decay and the bound for $|k^\chi|$ from Lemma 3.2 and (4.14). This is the desired estimate for S_1.

We continue with an estimate for $L_i(f)$ with $i \ge 1$. For all of the terms below we switch from the notation of k to the notation of i to indicate that the decay of each of these terms will not depend upon the index of the term, which is contrary to the decay of the R_k terms above. We estimate from above

$$w_\ell(p)\,|L_i(f)| \lesssim w_\ell(p) \int_{\mathbb{R}^3} dq_1\; k^\chi_\approx(p,q_1)\cdots \int_{\mathbb{R}^3} dq_{i+1}\; k^\chi_\approx(q_i,q_{i+1})$$
$$\times \int_{\kappa_1}^t ds_1 e^{-\nu(p)(t-s_1)} \left(\int_{s_1-\kappa_2}^{s_1} ds_2 e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_{i-1}-\kappa_i}^{s_{i-1}} ds_i e^{-\nu(q_{i-1})(s_{i-1}-s_i)} \right)$$
$$\times \mathbf{1}_{high} \int_0^{s_i-\kappa_{i+1}} ds_{i+1}\; e^{-\nu(q_i)(s_i-s_{i+1})}\, |\{U(s_{i+1}) f_0\}(y_{i+1},q_{i+1})|\,.$$

The term in parenthesis above would be simply unity in the case of L_1. Since all the momentum variables are comparable, we control the time integrals as

$$\int_{\kappa_1}^t ds_1 e^{-\nu(p)(t-s_1)} \int_{s_1-\kappa_2}^{s_1} ds_2\; e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_{i-1}-\kappa_i}^{s_{i-1}} ds_i\; e^{-\nu(q_{i-1})(s_{i-1}-s_i)}$$
$$\times \int_0^{s_i-\kappa_{i+1}} ds_{i+1}\; e^{-\nu(q_i)(s_i-s_{i+1})}$$
$$\lesssim (\kappa_2\cdots\kappa_i) \int_{\kappa_1}^t ds_1 \int_0^{t-\kappa_{i+1}} ds_{i+1}\; e^{-C\nu(p)(t-s_{i+1})}.$$

We have used $\nu(q_j) \geq C q_{j0}^{-b/2}$ from Lemma 3.1, $\nu(p) \approx p_0^{-b/2}$, and $q_{j0}^{-b/2} \geq C p_0^{-b/2}$; these estimates hold for any $j \in \{1,\ldots,i\}$. We have then used an estimate analogous to (6.4). Furthermore

$$(\kappa_2\cdots\kappa_i) \int_{\kappa_1}^t ds_1 \int_0^{t-\kappa_{i+1}} ds_{i+1}\; e^{-C\nu(p)(t-s_{i+1})}$$
$$\lesssim \kappa^{i-1}\,(1+t) \int_0^{t-\kappa_{i+1}} ds_{i+1}\; e^{-C\nu(p)(t-s_{i+1})}$$
$$\lesssim \kappa^{i-1}\,(1+t)\, w_j(p) \int_0^{t-\kappa_{i+1}} ds_{i+1}\;\; (1+t-s_{i+1})^{-j}\,.$$

These estimates follow from the definition of the sequence $\{\kappa\}$ as well as Lemma 3.1 together with (4.8) in the form (6.6) for any $j \geq 0$.

Next we use Cauchy-Schwartz to estimate the following two integrals:

$$w_{\ell+j}(q_{i-1}) \int_{\mathbb{R}^3} dq_i\; \left|k^\chi_\approx(q_{i-1},q_i)\right|$$
$$\times \int_{\mathbb{R}^3} dq_{i+1}\; \left|k^\chi_\approx(q_i,q_{i+1})\right|\, |\{U(s_{i+1}) f_0\}(y_{i+1},q_{i+1})|$$
$$\lesssim \left(\int_{\mathbb{R}^3} dq_i \int_{\mathbb{R}^3} dq_{i+1}\; \left|w_2(q_{i+1}) k^\chi_\approx(q_{i-1},q_i) k^\chi_\approx(q_i,q_{i+1})\right|^2 \right)^{1/2}$$
$$\times \left(\int_{\mathbb{R}^3} dq_i \int_{Z_i} dq_{i+1}\; \left|w_{\ell+j-2}(q_{i+1})\, \{U(s_{i+1}) f_0\}(y_{i+1},q_{i+1})\right|^2 \right)^{1/2}. \quad (6.7)$$

Above $Z_i \overset{\text{def}}{=} \{q_{i+1} : \frac{1}{2}|q_i| \leq |q_{i+1}| \leq 2|q_i|\}$. Also in the case $i = 1$ we consider $q_{i-1} = p$. For now we focus on the second set of integrals involving the semi-group.

We apply the change of variables $q_i \to y_{i+1}$ on the dq_i integration with y_{i+1} given by (6.2). Notice that similar to (4.17) the 3×3 matrix Jacobian is

$$\left(\frac{dy_{i+1}}{dq_i}\right)_{mn} = -(s_i - s_{i+1})\left(\frac{\delta_{mn}q_{i0}^2 - q_{im}q_{in}}{q_{i0}^3}\right).$$

We recall $q_i = (q_{i1}, q_{i2}, q_{i3})$ with $q_{i0} = \sqrt{1+|q_i|^2}$. This Jacobian matrix has two eigenvalues equal to $-\frac{(s_i - s_{i+1})}{q_{i0}}$, and a third eigenvalue which is given by $-(s_i - s_{i+1})\frac{q_{i0}^2 - |q_i|^2}{q_{i0}^3} = -(s_i - s_{i+1})\frac{1}{q_{i0}^3}$. Therefore the Jacobian determinant is

$$\left|\frac{dy_{i+1}}{dq_i}\right| = \frac{|(s_i - s_{i+1})|^3}{q_{i0}^5} \geq \frac{\kappa_{i+1}^3}{2^5 q_{(i+1)0}^5} = \frac{\kappa^3}{2^{5+3i} q_{(i+1)0}^5}.$$

This lower bound holds on the region $q_{i0} \leq 2q_{(i+1)0}$, $s_{i+1} \in [0, s_i - \kappa_{i+1}]$. Furthermore we have used that $s_i - s_{i+1} \geq \kappa_{i+1}$; this temporal estimate holds on the integration region of L_i. These estimates explain the lower bound for the Jacobian. Notice while the old variable q_i occupies the whole space, the new variable y_{i+1} satisfies the estimate

$$\begin{aligned} |y_{i+1} - x| &\leq \left|\hat{p}(t - s_1) + \hat{q}_1(s_1 - s_2) + \cdots + \hat{q}_i(s_i - s_{i+1})\right| \\ &\leq C\left((t - s_1) + (s_1 - s_2) + \cdots + (s_i - s_{i+1})\right) \leq C\,(t - s_{i+1}). \end{aligned}$$

We remark that this procedure would not hold in the non-relativistic situation, since in that case we do not have bounded velocities. In particular, because of relativity, the mapping $q_i \to y_{i+1}$ sends $\mathbb{R}^3$ into a bounded domain (for any finite t). After application of this change of variables, denoting $y_{i+1} = y$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} dq_i \int_{Z_i} dq_{i+1} \left|w_{\ell+j-2}(q_{i+1}) \{U(s_{i+1}) f_0\}(y_{i+1}, q_{i+1})\right|^2 \\ &\lesssim \int_{|y-x|\leq C(t-s_{i+1})} dy \left|\frac{dq_i}{dy}\right| \int_{Z_i} dq_{i+1} \left|w_{\ell+j-2}(q_{i+1}) \{U(s_{i+1}) f_0\}(y, q_{i+1})\right|^2 \\ &\lesssim \frac{(1+t-s_{i+1})^3}{\kappa^3} \int_{\mathbb{T}^3} dy \int_{\mathbb{R}^3} dq_{i+1}\, w_{2\ell+2j-4+10/b}(q_{i+1}) \left|\{U(s_{i+1}) f_0\}(y, q_{i+1})\right|^2 \\ &= C(\kappa)\,(1+t-s_{i+1})^3 \,\|\{U(s_{i+1}) f_0\}\|_{2,\ell+j-2+5/b}^2. \end{aligned}$$

We have used that $q_{(i+1)0}^5 = w_{10/b}(q_{i+1})$. This estimate above is the main one for the $L_i(f)$ terms which allows us to deduce high order decay.

Since we have used (6.7), for the momentum integrals in L_i, we are left to control the iteration of kernels. We *claim* the following estimate:

$$\begin{aligned} &\left(\int_{\mathbb{R}^3} dq_1 \, \left|k_{\approx}^{\chi}(p, q_1)\right| \cdots \int_{\mathbb{R}^3} dq_{i-1} \, \left|k_{\approx}^{\chi}(q_{i-2}, q_{i-1})\right|\right) \\ &\times \left(\int_{\mathbb{R}^3} dq_i \int_{\mathbb{R}^3} dq_{i+1} \left|w_2(q_{i+1}) k_{\approx}^{\chi}(q_{i-1}, q_i) k_{\approx}^{\chi}(q_i, q_{i+1})\right|^2 \mathbf{1}_{high}\right)^{1/2} \lesssim \frac{1}{M^{\zeta}}. \end{aligned}$$

Note that if $i = 1$ then the first term in parenthesis above is simply unity. Firstly, from Lemma 3.2 we have

$$w_2(q_{i+1}) k_{\approx}^{\chi}(q_{i-1}, q_i) k_{\approx}^{\chi}(q_i, q_{i+1}) \mathbf{1}_{high} \lesssim \frac{1}{M^{\zeta}} e^{-c|q_{i-1} - q_i|} e^{-c|q_i - q_{i+1}|}.$$

This also uses (4.14) and the fact that all the momentum variables are comparable. They key point is then to employ the following series of changes of variables which begins with $q_i - q_{i+1} \to q_{i+1}$ on dq_{i+1}, $q_{i-1} - q_i \to q_i$ on dq_i, and ends with $p - q_1 \to q_1$ on dq_1. The end result, with Lemma 3.2, is that

$$\left(\int_{\mathbb{R}^3} dq_1 \, \left|k^\chi_{\approx}(p, q_1)\right| \cdots \int_{\mathbb{R}^3} dq_{i-1} \, \left|k^\chi_{\approx}(q_{i-2}, q_{i-1})\right|\right)$$
$$\times \left(\int_{\mathbb{R}^3} dq_i \int_{\mathbb{R}^3} dq_{i+1} \left|e^{-c|q_{i-1}-q_i|}e^{-c|q_i-q_{i+1}|}\right|^2\right)^{1/2}$$
$$\lesssim \int_{\mathbb{R}^3} dq_1 \, e^{-c|q_1|} \cdots \int_{\mathbb{R}^3} dq_{i-1} \, e^{-c|q_{i-1}|} \left(\int_{\mathbb{R}^3} dq_i \int_{\mathbb{R}^3} dq_{i+1} e^{-2c|q_i|}e^{-2c|q_{i+1}|}\right)^{1/2}$$
$$\lesssim 1.$$

Collecting the estimates in this paragraph establishes the *claim*.

We gather all of the estimates for $L_i(f)$ to obtain

$$w_\ell(p) \, |L_i(f)| \lesssim (1+t) \int_0^t ds_{i+1} \, (1+t-s_{i+1})^{-j+3/2}$$
$$\times \| \{U(s_{i+1}) f_0\} \|_{2,\ell+j-2+5/b}$$
$$\lesssim \| f_0 \|_{2,\ell+j-2+5/b+k} \, (1+t) \int_0^t ds_{i+1} \, (1+t-s_{i+1})^{-j+3/2} \, (1+s_{i+1})^{-k-1} .$$

Above we have used the decay of the linear solutions to (2.11) from Theorem 3.7; these solutions are represented by (4.1). Then for $j \geq k+1+3/2$ and $k' = j-2+5/b+k+\ell'$ (for any $\ell' > 3/b$) we use Proposition 4.5 to show that

$$w_\ell(p) \, |L_i(f)| \lesssim \| f_0 \|_{2,\ell+k'-\ell'} \, (1+t)^{-k} \lesssim \| f_0 \|_{\infty,\ell+k'} \, (1+t)^{-k} .$$

The last inequality above follows as in (4.7). This is the desired estimate for $L_i(f)(t)$ which holds for any $i \geq 1$ and $k \geq 0$.

It remains to estimate $G_{i+1}(f)$, $D_{i+1}(f)$, and $N_{i+1}(f)$ for $i \geq 1$. First

$$w_\ell(p) \, |G_{i+1}(f)| \lesssim \int_{\mathbb{R}^3} dq_1 \left|k^\chi_{\approx}(p, q_1)\right| \mathbf{1}_{high} \int_{\kappa_1}^t ds_1 \, e^{-\nu(p)(t-s_1)}$$
$$\times \int_{\mathbb{R}^3} dq_2 \left|k^\chi_{\approx}(q_1, q_2)\right| \cdots \int_{\mathbb{R}^3} dq_{i+1} \left|k^\chi_{\approx}(q_i, q_{i+1})\right|$$
$$\times \int_{s_1-\kappa_2}^{s_1} ds_2 \, e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1} \, e^{-\nu(q_i)(s_i-s_{i+1})}$$
$$\times w_\ell(q_{i+1}) e^{-\nu(q_{i+1})s_{i+1}} f_0(y_{i+1} - \hat{q}_{i+1}s_{i+1}, q_{i+1}).$$

We have used that all the momentum variables are comparable and the trick from (6.4) to conclude that the upper bound above is further bounded as

$$\lesssim \int_{\mathbb{R}^3} dq_1 \left|k^\chi_{\approx}(p, q_1)\right| \mathbf{1}_{high} \int_{\mathbb{R}^3} dq_2 \left|k^\chi_{\approx}(q_1, q_2)\right| \cdots \int_{\mathbb{R}^3} dq_{i+1} \left|k^\chi_{\approx}(q_i, q_{i+1})\right|$$
$$\times e^{-C\nu(p)t} \frac{\| f_0 \|_{\infty,\ell+k}}{w_k(q_{i+1})} \int_{\kappa_1}^t ds_1 \int_{s_1-\kappa_2}^{s_1} ds_2 \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1}.$$

Notice that using the definition of $\kappa_i = \kappa/2^{i-1}$ we have

$$\int_{\kappa_1}^{t} ds_1 \int_{s_1-\kappa_2}^{s_1} ds_2 \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1} = \kappa^i 2^{-i(i+1)/2} \int_{\kappa_1}^{t} ds_1 \lesssim (1+t).$$

Furthermore, as in (6.5) and Lemma 3.2 with (4.14) we have

$$\int_{\mathbb{R}^3} dq_1 \left|k_{\approx}^{\chi}(p,q_1)\right| \mathbf{1}_{high} \times \int_{\mathbb{R}^3} dq_2 \left|k_{\approx}^{\chi}(q_1,q_2)\right| \cdots \int_{\mathbb{R}^3} dq_{i+1} \left|k_{\approx}^{\chi}(q_i,q_{i+1})\right| \frac{1}{w_k(q_{i+1})} \lesssim \frac{1}{M^{2\zeta}} \frac{1}{w_{k+i+1}(p)} \lesssim \frac{1}{w_{k+2}(p)}.$$

Moreover, for any $k \geq 0$, by (4.8) as in (6.6) we have

$$e^{-C\nu(p)t} \lesssim w_{k+2}(p)(1+t)^{-k-2}.$$

Collecting the last few estimates we obtain

$$w_\ell(p)\left|G_{i+1}(f)\right| \lesssim (1+t)^{-k-1} \|f_0\|_{\infty,\ell+k}.$$

This is the desired estimate for $G_{i+1}(f)$.

We will now study $D_{i+1}(f)$, which satisfies the following general estimate:

$$w_\ell(p)\left|D_{i+1}(f)\right| \lesssim \int_{\mathbb{R}^3} dq_1 \left|k_{\approx}^{\chi}(p,q_1)\right| \cdots \int_{\mathbb{R}^3} dq_{i+1} \left|k_{\approx}^{\chi}(q_i,q_{i+1})\right| \times \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1}\, e^{-\nu(q_i)(s_i-s_{i+1})} \times \mathbf{1}_{high} w_\ell(p) \int_0^{s_{i+1}} ds_{i+2}\, e^{-\nu(q_{i+1})(s_{i+1}-s_{i+2})} \left|K^{1-\chi}\left(\{U(s_{i+2})f_0\}\right)(y_{i+2},q_{i+1})\right|.$$

Since all the momentum variables are comparable, with Lemma 4.6, we have

$$\lesssim \int_{\mathbb{R}^3} dq_1 \left|k_{\approx}^{\chi}(p,q_1)\right| \mathbf{1}_{high} \int_{\mathbb{R}^3} dq_2 \left|k_{\approx}^{\chi}(q_1,q_2)\right| \cdots \int_{\mathbb{R}^3} dq_{i+1} \left|k_{\approx}^{\chi}(q_i,q_{i+1})\right| \times \int_{\kappa_1}^{t} ds_1\, e^{-\nu(p)(t-s_1)} \int_{s_1-\kappa_2}^{s_1} ds_2\, e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1}\, e^{-\nu(q_i)(s_i-s_{i+1})} \times e^{-cp_0} |||f|||_{k,\ell} \int_0^{s_{i+1}} ds_{i+2}\, e^{-\nu(q_{i+1})(s_{i+1}-s_{i+2})} (1+s_{i+2})^{-k}.$$

For this term Lemma 4.6 would allow a better estimate for the momentum weight on $|||f|||_{k,\ell}$. As in the estimate for G_{i+1} above, with (4.14), we have

$$w_\ell(p)\left|D_{i+1}(f)\right| \lesssim \frac{1}{M^{2\zeta}} e^{-cp_0} |||f|||_{k,\ell} \times \int_{\kappa_1}^{t} ds_1 e^{-C\nu(p)(t-s_1)} \int_{s_1-\kappa_2}^{s_1} ds_2 e^{-C\nu(p)(s_1-s_2)} \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1} e^{-C\nu(p)(s_i-s_{i+1})} \times \int_0^{s_{i+1}} ds_{i+2}\, e^{-C\nu(p)(s_{i+1}-s_{i+2})} (1+s_{i+2})^{-k}.$$

We have again used the crucial fact that all the momentum variables are comparable. Since we have exponential decay, we can iterate the estimates from (4.8) and Proposition 4.5 as in (6.6) to obtain

$$\begin{aligned}
&\int_{\kappa_1}^{t} ds_1\ e^{-C\nu(p)(t-s_1)} \int_{s_1-\kappa_2}^{s_1} ds_2\ e^{-C\nu(p)(s_1-s_2)} \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1}\ e^{-C\nu(p)(s_i-s_{i+1})} \\
&\quad \times \int_0^{s_{i+1}} ds_{i+2}\ e^{-C\nu(p)(s_{i+1}-s_{i+2})}\ (1+s_{i+2})^{-k} \\
&\le \int_0^{t} ds_1\ e^{-C\nu(p)(t-s_1)} \int_0^{s_1} ds_2\ e^{-C\nu(p)(s_1-s_2)} \cdots \int_0^{s_i} ds_{i+1}\ e^{-C\nu(p)(s_i-s_{i+1})} \\
&\quad \times \int_0^{s_{i+1}} ds_{i+2}\ e^{-C\nu(p)(s_{i+1}-s_{i+2})}\ (1+s_{i+2})^{-k} \\
&\lesssim \int_0^{t} ds_1\ (1+(t-s_1))^{-k-1} \prod_{j=1}^{i} \int_0^{s_j} ds_{j+1}\ (1+(s_j-s_{j+1}))^{-k-1} \\
&\quad \times w_{(k+1)(i+2)}(p) \int_0^{s_{i+1}} ds_{i+2}\ (1+(s_{i+1}-s_{i+2}))^{-k-1}\ (1+s_{i+2})^{-k}. \qquad (6.8)
\end{aligned}$$

After iteratively applying Proposition 4.5 we obtain an upper bound of

$$\lesssim w_{(k+1)(i+2)}(p)(1+t)^{-k}.$$

Plugging this into the previous estimate we have

$$\begin{aligned}
w_\ell(p)\,|D_{i+1}(f)| &\le \frac{C}{M^{2\zeta}}\ w_{(k+1)(i+2)}(p)e^{-cp_0}(1+t)^{-k}\ |||f|||_{k,\ell} \\
&\le \frac{C}{M^{2\zeta}}\ (1+t)^{-k}\ |||f|||_{k,\ell}.
\end{aligned}$$

This is the desired estimate for $D_{i+1}(f)$ when M is chosen sufficiently large.

The final term to estimate is $N_{i+1}(f)$. In this case we have the upper bound

$$\begin{aligned}
w_\ell(p)\,|N_{i+1}(f)| \lesssim w_\ell(p) &\int_{\mathbb{R}^3} dq_1\ \left|k^{\chi}_{\approx}(p,q_1)\right| \mathbf{1}_{high} \int_{\kappa_1}^{t} ds_1\ e^{-\nu(p)(t-s_1)} \\
&\times \int_{\mathbb{R}^3} dq_2\ \left|k^{\chi}_{\approx}(q_1,q_2)\right| \cdots \int_{\mathbb{R}^3} dq_{i+1}\ \left|k^{\chi}_{\approx}(q_i,q_{i+1})\right| \\
&\times \int_{s_1-\kappa_2}^{s_1} ds_2\ e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1}\ e^{-\nu(q_i)(s_i-s_{i+1})} \\
&\times \int_0^{s_{i+1}} ds_{i+2} e^{-\nu(q_{i+1})(s_{i+1}-s_{i+2})} \int_{\mathbb{R}^3} dq_{i+2}\ \left|k^{\chi}_{\neq}(q_{i+1},q_{i+2})\right| \\
&\times \left|\{U(s_{i+2})f_0\}(y_{i+2},q_{i+2})\right|.
\end{aligned}$$

We first estimate the time integrals above. Since the relevant momentum variables are all comparable, as in (6.8) and (6.6) we have

$$\begin{aligned}
&\int_{\kappa_1}^{t} ds_1\ e^{-\nu(p)(t-s_1)} \int_{s_1-\kappa_2}^{s_1} ds_2\ e^{-\nu(q_1)(s_1-s_2)} \cdots \int_{s_i-\kappa_{i+1}}^{s_i} ds_{i+1}\ e^{-\nu(q_i)(s_i-s_{i+1})} \\
&\quad \times \int_0^{s_{i+1}} ds_{i+2}\ e^{-\nu(q_{i+1})(s_{i+1}-s_{i+2})}(1+s_{i+2})^{-k} \\
&\lesssim w_{(k+1)(i+2)}(p)(1+t)^{-k}.
\end{aligned}$$

Now for the momentum integrals, from Lemma 3.2 and (4.14) we have

$$
\begin{aligned}
&w_\ell(p) w_{(k+1)(i+2)}(p) \left| \int_{\mathbb{R}^3} dq_1 k^\chi_{\approx}(p, q_1) \int_{\mathbb{R}^3} dq_2 k^\chi_{\approx}(q_1, q_2) \cdots \int_{\mathbb{R}^3} dq_{i+1} k^\chi_{\approx}(q_i, q_{i+1}) \right| \\
&\quad \times \mathbf{1}_{high} \int_{\mathbb{R}^3} dq_{i+2} \left| k^\chi_{\neq}(q_{i+1}, q_{i+2}) \right| \\
&\leq \frac{C}{M^{(i+2)\zeta}} \int_{\mathbb{R}^3} dq_1 \, e^{-c|p-q_1|} \, \mathbf{1}_{high} \int_{\mathbb{R}^3} dq_2 \, e^{-c|q_1-q_2|} \cdots \int_{\mathbb{R}^3} dq_{i+1} \, e^{-c|q_i - q_{i+1}|} \\
&\quad \times \int_{\mathbb{R}^3} dq_{i+2} \, e^{-c|q_{i+1}-q_{i+2}|} \left(\mathbf{1}_{|q_{i+2}| \geq 2|q_{i+1}|} + \mathbf{1}_{|q_{i+1}| \geq 2|q_{i+2}|} \right) \, w_{\ell + k(i+2)}(q_{i+1}) \\
&\leq \frac{C}{M^{(i+2)\zeta}}.
\end{aligned}
$$

In the last step we have used the following estimate:

$$
\left(\mathbf{1}_{|q_{i+2}| \geq 2|q_{i+1}|} + \mathbf{1}_{|q_{i+1}| \geq 2|q_{i+2}|} \right) e^{-\frac{c}{2}|q_{i+1}-q_{i+2}|} \, w_{\ell + k(i+2)}(q_{i+1}) \leq C.
$$

Indeed in either of the regions $|q_{i+1}| \geq 2|q_{i+2}|$ or $|q_{i+2}| \geq 2|q_{i+1}|$ we can use estimates such as those in (3.21) or (3.22) to directly establish this bound.

Now by collecting the estimates in this paragraph, we have shown

$$
w_\ell(p) \, |N_{i+1}(f)| \leq \frac{C}{M^{(i+2)\zeta}} \, (1+t)^{-k} \, |||f|||_{k,0}.
$$

Since $k \geq 0$ is arbitrary, we conclude our estimate and our proposition after choosing M sufficiently large in this last upper bound. □

This concludes our proof of rapid linear decay.

7. Nonlinear L^∞ Rapid Decay

In Sect. 5, we have proven the existence of mild solutions (5.1) to the non-linear relativistic Boltzmann equation (2.1) with the soft potentials. For $\ell > 3/b$ and $k \in (1/2, 1]$ we have shown in Theorem 5.1 that these solutions, $f = f(t, x, p)$, satisfy

$$
\|f\|_{\infty,\ell}(t) \leq C_{\ell,k}(1+t)^{-k} \|f_0\|_{\infty,\ell+k}.
$$

Then in Sect. 6 we prove high order "almost exponential" decay for the linear semi-group as in (6.1). From these estimates and the solution formula, (5.1), we can prove the following non-linear almost exponential decay.

Theorem 7.1. *Given any $\ell > 3/b$ and $k \geq 0$, there is a $k' = k'(k) \geq 0$ such that the solutions which were proven to exist in Thereom 5.1 further satisfy*

$$
\|f\|_{\infty,\ell}(t) \leq C_{\ell,k}(1+t)^{-k} \|f_0\|_{\infty,\ell+k'}.
$$

Proof of Theorem 7.1. We use an induction which allows one to continually improve the decay. The main point is to bound the non-linear term, since we already know this kind of rapid decay for the linear part of (5.1) from (6.1) which follows from the crucial

Proposition 6.1. In the first step we note that by Thereom 5.1, Theorem 7.1 is true for $k \in (1/2, 1]$. Then given any $j \geq 0$, from (6.1) we have

$$w_\ell(p)\,|N[f,f](t,x,p)| \lesssim \int_0^t ds\; w_\ell(p)\,|\{U(t-s)\Gamma[f(s),f(s)]\}(x,p)| \lesssim \int_0^t ds\;(1+t-s)^{-j}\,\|\Gamma[f(s),f(s)]\|_{\infty,\ell+j'}\,.$$

Above $j' \geq j$ is the number corresponding to j in (6.1). From Lemma 5.2 we have

$$\lesssim \int_0^t ds\;(1+t-s)^{-j}\,\|f(s)\|_{\infty,\ell+j'-1}\,\|f(s)\|_{\infty,\ell+j'-1}\,.$$

Next we use the non-linear decay from Theorem 5.1 to see

$$\lesssim \|f_0\|^2_{\infty,\ell+j'+i-1}\int_0^t ds\;(1+t-s)^{-j}(1+s)^{-2i} \lesssim (1+t)^{-\rho}\,\|f_0\|^2_{\infty,\ell+j'+i-1}\,.$$

The last estimate follows from Proposition 4.5 with $\rho = \min\{j+2i-1, \min\{j,2i\}\}$. In the above estimates we can choose $j \in (1,2]$ and then $i \in (1/2,1]$ such that $j = 2i > 1$. Then we have shown Theorem 7.1 for $k \in (1,2]$ by choosing $\rho = j = k$ and $k' = \max\{j', j'+i-1\} = j'$.

Next suppose the theorem is correct for some $k > 2$; we will show that we may go beyond this k. Indeed similar to the initial case we have

$$w_\ell(p)\,|N[f,f](t,x,p)| \leq C(1+t)^{-\rho}\,\|f_0\|^2_{\infty,\ell+j'+i'-1}\,,$$

with $\rho = \min\{j+2i-1, \min\{j,2i\}\} = \min\{j,2i\}$. Above j' corresponds to the power of the weight coming from decay level j in (6.1) and i' corresponds to the power of the weight generated by decay level $i \in (0,k]$ in this Theorem 7.1.

Choose $i \in (k/2, k]$ and $j = 2i \in (k, 2k]$. This is always possible. Then we have $\rho = j$ so that we have proven Theorem 7.1 for any $\tilde{k} \in (k, 2k]$ and the corresponding $\tilde{k}' = \max\{j', j'+i'-1\}$. We conclude by induction. □

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Appendix: Derivation of the Compact Operator

In this section, we give a complete exposition of the derivation of the Hilbert-Schmidt form (3.10) for the compact operator from (2.4). The linearized collision operator takes the form (2.2). In that formulation we have the multiplication operator as in (2.3). The remaining "compact" part of the linearized operator is given by (2.4) with $K = K_2 - K_1$ and in particular

$$K_2(h) \overset{\text{def}}{=} \frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q')\sqrt{J(q)}\left\{\sqrt{J(q')}h(p')\right\}$$
$$+\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q')\sqrt{J(q)}\left\{\sqrt{J(p')}h(q')\right\}.$$

We are using the original notation from the top of this paper, which includes the delta functions. We will outline in detail a procedure which is sketched in [9, p. 277] (see also [11]), that allows a nice reduction of the term K_2 as in (3.10). In particular we give the exact form of the Lorentz transformation. This reduction for the K_1 term can be reduced to the form (3.9) using much simpler methods than the ones we use below, see e.g. [9,49,50].

We recall the definition of the transition rate, W, from the top of this paper. We plug the definition of W into K_2 above to obtain

$$K_2(h) \stackrel{\text{def}}{=} \frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q'_0}\int_{\mathbb{R}^3}\frac{dp'}{p'_0} s\sigma(g,\theta)\delta^{(4)}(p^\mu+q^\mu-p^{\mu\prime}-q^{\mu\prime}) \times\sqrt{J(q)}\left\{\sqrt{J(q')}h(p')+\sqrt{J(p')}h(q')\right\}.$$

We will first reduce this to a Hilbert-Schmidt form and second carry out the delta function integrations in the kernel.

Recall the discussion at the beginning of Sect. 1.2 regarding our convention for raising and lowering indices and the Lorentz inner product. In preparation, we write down some invariant quantities. By (1.6) and (1.4) we obtain

$$\begin{aligned}(p^{\mu\prime}-q^{\mu\prime})(p_\mu-q_\mu) &= 2p^{\mu\prime}p_\mu+2q^{\mu\prime}q_\mu-(p^{\mu\prime}+q^{\mu\prime})(p_\mu+q_\mu)\\ &= 2p^{\mu\prime}p_\mu+2q^{\mu\prime}q_\mu+s.\end{aligned}$$

Further notice that (1.6) implies

$$(p^\mu-p^{\mu\prime})(p_\mu-p'_\mu)=(q^{\mu\prime}-q^\mu)(q'_\mu-q_\mu).$$

Expanding this we have

$$-2-2p^\mu p'_\mu=-2-2q^{\mu\prime}q_\mu.$$

We thus have another invariant $p^\mu p'_\mu=q^\mu q'_\mu$. Define $\bar{g}=g(p^\mu,p^{\mu\prime})$ as in (1.5). We will always use g without the bar to exclusively denote $g=g(p^\mu,q^\mu)$.

From (1.4) and (1.5) we know $s=g^2+4$. We may re-express θ from (1.7) as

$$\cos\theta=(p^\mu-q^\mu)(p'_\mu-q'_\mu)/g^2=1-2\left(\frac{\bar{g}}{g}\right)^2.$$

This follows from the invariant calculations in the previous paragraph and

$$(p^\mu-q^\mu)(p'_\mu-q'_\mu)=g^2+4+4p^\mu p'_\mu=g^2-2\bar{g}^2.$$

We further *claim* that

$$g^2=\bar{g}^2-\frac{1}{2}(p^\mu+p^{\mu\prime})(q_\mu+q'_\mu-p_\mu-p'_\mu). \tag{7.1}$$

Let $\bar{s}=s(p^\mu,p^{\mu\prime})=\bar{g}^2+4$. Then (7.1) is equivalent to

$$\begin{aligned}g^2&=\bar{g}^2-\frac{1}{2}\bar{s}-\frac{1}{2}(p^\mu+p^{\mu\prime})(q_\mu+q'_\mu)\\ &=\frac{1}{2}\bar{g}^2-2-\frac{1}{2}(p^\mu+p^{\mu\prime})(q_\mu+q'_\mu)\\ &=\frac{1}{2}\bar{g}^2+g^2+2p^\mu q_\mu-\frac{1}{2}(p^\mu+p^{\mu\prime})(q_\mu+q'_\mu).\end{aligned}$$

We thus prove (7.1) by showing that

$$\frac{1}{2}\bar{g}^2 + 2p^\mu q_\mu - \frac{1}{2}(p^\mu + p^{\mu\prime})(q_\mu + q'_\mu) = 0.$$

Expanding this expression we obtain

$$-p^\mu p'_\mu - 1 + 2p^\mu q_\mu - \frac{1}{2}p^\mu q_\mu - \frac{1}{2}p^\mu q'_\mu - \frac{1}{2}p^{\mu\prime} q_\mu - \frac{1}{2}p^{\mu\prime} q'_\mu.$$

Notice further that $p^\mu q_\mu = p^{\mu\prime} q'_\mu$ and $p^\mu p'_\mu = q^\mu q'_\mu$ as a result of (1.6) and (1.4). We thus obtain

$$p^\mu q_\mu - 1 - \frac{1}{2}p^\mu p'_\mu - \frac{1}{2}q^\mu q'_\mu - \frac{1}{2}p^\mu q'_\mu - \frac{1}{2}p^{\mu\prime} q_\mu,$$

which by (1.6) is

$$p^\mu q_\mu - 1 - \frac{1}{2}(p^\mu + q^\mu)(p'_\mu + q'_\mu) = p^\mu q_\mu - 1 + \frac{1}{2}s = 0.$$

This establishes the claim (7.1).

Now we establish the Hilbert-Schmidt form. First consider

$$\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q'_0}\int_{\mathbb{R}^3}\frac{dp'}{p'_0} s\sigma(g,\theta)\delta^{(4)}(p^\mu + q^\mu - p^{\mu\prime} - q^{\mu\prime})\sqrt{J(q)}\sqrt{J(q')}h(p').$$

Exchanging q with p' the integral above is equal to

$$\frac{1}{p_0}\int\frac{dq}{q_0}h(q)\left\{\int\frac{dq'}{q'_0}\int\frac{dp'}{p'_0}\bar{s}\sigma(\bar{g},\theta)\delta^{(4)}(p^\mu + p^{\mu\prime} - q^\mu - q^{\mu\prime})\sqrt{J(p')}\sqrt{J(q')}\right\},$$

where θ is now defined by

$$\cos\theta = 1 - 2\left(\frac{g}{\bar{g}}\right)^2, \tag{7.2}$$

and from (7.1), with the new argument in the delta function above, we have

$$\bar{g}^2 = g^2 + \frac{1}{2}(p^\mu + q^\mu)(p_\mu + q_\mu - p'_\mu - q'_\mu), \tag{7.3}$$

and further $\bar{s}$ is defined by $\bar{s} = \bar{g}^2 + 4$. We do a similar calculation for the second term in $K_2 h$, e.g. exchange q with q' and then swap the q' and p' notation. The result is that we can define

$$k_2(p,q) \stackrel{\text{def}}{=} \frac{2}{p_0 q_0}\int\frac{dq'}{q'_0}\int\frac{dp'}{p'_0}\bar{s}\sigma(\bar{g},\theta)\delta^{(4)}(p^\mu + p^{\mu\prime} - q^\mu - q^{\mu\prime})\sqrt{J(p')}\sqrt{J(q')}. \tag{7.4}$$

We now write the Hilbert-Schmidt form $K_2(h) = \int k_2(p,q)h(q)dq$. We will carry out the delta function integrations in $k_2(p,q)$ using a special Lorentz matrix.

We first translate (7.4) into an expression involving the total and relative momentum variables, $p^{\mu\prime}+q^{\mu\prime}$ and $p^{\mu\prime}-q^{\mu\prime}$ respectively. Define u by $u(r)=0$ if $r<0$ and $u(r)=1$ if $r\geq 0$. Let $\underline{g}=g(p^{\mu\prime},q^{\mu\prime})$ and $\underline{s}=s(p^{\mu\prime},q^{\mu\prime})$. We *claim* that

$$\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}G(p^\mu,q^\mu,p^{\mu\prime},q^{\mu\prime})=\frac{1}{16}\int_{\mathbb{R}^4\times\mathbb{R}^4}d\Theta(p^{\mu\prime},q^{\mu\prime})G(p^\mu,q^\mu,p^{\mu\prime},q^{\mu\prime}), \tag{7.5}$$

where we are now integrating over the eight vector $(p^{\mu\prime},q^{\mu\prime})$ and

$$d\Theta(p^{\mu\prime},q^{\mu\prime})=dp^{\mu\prime}dq^{\mu\prime}u(p_0'+q_0')u(\underline{s}-4)\delta(\underline{s}-\underline{g}^2-4)\delta((p^{\mu\prime}+q^{\mu\prime})(p_\mu'-q_\mu')).$$

To establish the claim, first notice that

$$\begin{aligned}-(p^{\mu\prime}+q^{\mu\prime})(p_\mu'-q_\mu')&=-p^{\mu\prime}p_\mu'+q^{\mu\prime}q_\mu'\\&=(p_0')^2-|p'|^2-(q_0')^2+|q'|^2=A_p-A_q,\end{aligned}$$

where now $p^{0\prime}$ and $q^{0\prime}$ are integration variables and we have defined

$$A_p=(p_0')^2-(|p'|^2+1),\quad A_q=(q_0')^2-(|q'|^2+1).$$

Integrating first over $dp^{\mu\prime}$, we see that alternatively

$$\begin{aligned}-(p^{\mu\prime}+q^{\mu\prime})(p_\mu'-q_\mu')&=(p_0')^2-(|p'|^2+1+A_q)\\&=\left\{p_0'-\sqrt{|p'|^2+1+A_q}\right\}\left\{p_0'+\sqrt{|p'|^2+1+A_q}\right\}.\end{aligned}$$

Furthermore, by (1.4) and (1.5) we have

$$\begin{aligned}\underline{s}-\underline{g}^2-4&=-(p^{\mu\prime}+q^{\mu\prime})(p_\mu'+q_\mu')-(p^{\mu\prime}-q^{\mu\prime})(p_\mu'-q_\mu')-4\\&=-2p^{\mu\prime}p_\mu'-2q^{\mu\prime}q_\mu'-4\\&=2A_p+2A_q.\end{aligned}$$

Then similarly

$$\begin{aligned}\underline{s}-\underline{g}^2-4&=2(q_0')^2-2[|q'|^2+1-A_p]\\&=2\left\{q_0'-\sqrt{|q'|^2+1-A_p}\right\}\left\{q_0'+\sqrt{|q'|^2+1-A_p}\right\}.\end{aligned}$$

Further note that $p_0'+q_0'\geq 0$ and $\underline{s}-4\geq 0$ together imply $p_0'\geq 0$ and $q_0'\geq 0$. With these expressions and standard calculations we establish (7.5).

We thus conclude that

$$k_2(p,q)=\frac{1}{p_0q_0}\frac{1}{8}\int_{\mathbb{R}^4\times\mathbb{R}^4}d\Theta(p^{\mu\prime},q^{\mu\prime})\bar{s}\sigma(\bar{g},\theta)\delta^{(4)}(p^\mu+p^{\mu\prime}-q^\mu-q^{\mu\prime})\sqrt{J(q')J(p')}.$$

Now apply the change of variables

$$\bar{p}^\mu=p^{\mu\prime}+q^{\mu\prime},\quad \bar{q}^\mu=p^{\mu\prime}-q^{\mu\prime}.$$

This transformation has Jacobian = 16 and inverse tranformation as

$$p^{\mu\prime} = \frac{1}{2}\bar{p}^{\mu} + \frac{1}{2}\bar{q}^{\mu}, \quad q^{\mu\prime} = \frac{1}{2}\bar{p}^{\mu} - \frac{1}{2}\bar{q}^{\mu}.$$

With this change of variable, for some $c' > 0$, the integral becomes

$$k_2(p,q) = \frac{c'}{p_0 q_0}\int_{\mathbb{R}^4\times\mathbb{R}^4} d\Theta(\bar{p}^{\mu},\bar{q}^{\mu})\bar{s}\sigma(\bar{g},\theta)\delta^{(4)}(p^{\mu}-q^{\mu}+\bar{q}^{\mu})\sqrt{J(\bar{p})},$$

with $\sqrt{J(\bar{p})} = e^{-\bar{p}_0/2}$ (ignoring constants). Above the measure is now

$$d\Theta(\bar{p}^{\mu},\bar{q}^{\mu}) = d\bar{p}^{\mu}\, d\bar{q}^{\mu}\, u(\bar{p}_0)u(-\bar{p}^{\mu}\bar{p}_{\mu}-4)\delta(-\bar{p}^{\mu}\bar{p}_{\mu}-\bar{q}^{\mu}\bar{q}_{\mu}-4)\delta(\bar{p}^{\mu}\bar{q}_{\mu}).$$

Also $\bar{g} \geq 0$ from (7.3) is now given by

$$\bar{g}^2 = g^2 + \frac{1}{2}(p^{\mu}+q^{\mu})(p_{\mu}+q_{\mu}-\bar{p}_{\mu}),$$

and θ and $\bar{s}$ can be defined through the new $\bar{g}$ with (7.2).

We next carry out the delta function argument for $\delta^{(4)}(p^{\mu}-q^{\mu}+\bar{q}^{\mu})$ to obtain

$$k_2(p,q) = \frac{c'}{p_0 q_0}\int_{\mathbb{R}^4} d\Theta(\bar{p}^{\mu})\bar{s}\sigma(\bar{g},\theta)e^{-\bar{p}_0/2}, \quad \exists c' > 0,$$

where the measure is

$$d\Theta(\bar{p}^{\mu}) = d\bar{p}^{\mu}\, u(\bar{p}_0)\, u(-\bar{p}^{\mu}\bar{p}_{\mu}-4)\,\delta(-\bar{p}^{\mu}\bar{p}_{\mu}-g^2-4)\delta(\bar{p}^{\mu}(q_{\mu}-p_{\mu})).$$

Since $s = g^2+4$ we have

$$\begin{aligned} u(\bar{p}_0)\delta(-\bar{p}^{\mu}\bar{p}_{\mu}-g^2-4) &= u(\bar{p}_0)\delta(-\bar{p}^{\mu}\bar{p}_{\mu}-s) \\ &= u(\bar{p}_0)\delta((\bar{p}_0)^2-|\bar{p}|^2-s) \\ &= \frac{\delta(\bar{p}_0-\sqrt{|\bar{p}|^2+s})}{2\sqrt{|\bar{p}|^2+s}}. \end{aligned}$$

We then carry out one integration using the delta function to get

$$k_2(p,q) = \frac{c'}{p_0 q_0}\int_{\mathbb{R}^3} \frac{d\bar{p}}{\bar{p}_0} u(-\bar{p}^{\mu}\bar{p}_{\mu}-4)\delta(\bar{p}^{\mu}(q_{\mu}-p_{\mu}))\bar{s}\sigma(\bar{g},\theta)e^{-\bar{p}_0/2},$$

with $\bar{p}_0 \overset{\text{def}}{=} \sqrt{|\bar{p}|^2+s}$. Using $s = g^2+4$ we have

$$-\bar{p}^{\mu}\bar{p}_{\mu}-4 = s-4 = g^2 \geq 0.$$

So always $u(-\bar{p}^{\mu}\bar{p}_{\mu}-4) = 1$ and the integral reduces to

$$k_2(p,q) = \frac{c'}{p_0 q_0}\int_{\mathbb{R}^3} \frac{d\bar{p}}{\bar{p}_0}\delta(\bar{p}^{\mu}(q_{\mu}-p_{\mu}))\bar{s}\sigma(\bar{g},\theta)e^{-\bar{p}^{\mu}\bar{U}_{\mu}/2},$$

where $\bar{U}^{\mu} = (1,0,0,0)$, $\bar{U}_{\mu} = (-1,0,0,0)$ and $e^{-\bar{p}_0/2} = e^{-\bar{p}^{\mu}\bar{U}_{\mu}/2}$.

We finish off our reduction by moving to a new Lorentz frame. We consider a Lorentz transformation Λ which maps into the center-of-momentum system as

$$A_\nu \stackrel{\text{def}}{=} \Lambda^{\mu\nu}(p_\mu + q_\mu) = (\sqrt{s}, 0, 0, 0), \quad B^\nu \stackrel{\text{def}}{=} -\Lambda^{\mu\nu}(p_\mu - q_\mu) = (0, 0, 0, g).$$

We recall our notation for raising and lowering indices from the beginning of Sect. 1.2 as $p_\mu = g_{\mu\nu}p^\nu$, where $\left(g_{\mu\nu}\right) \stackrel{\text{def}}{=} \text{diag}(-1\ 1\ 1\ 1)$. Also recall $p^\mu = (p^0, p)$ with $p_0 > 0$. Then we use the Einstein summation convention as $\Lambda^{\mu\nu}p_\mu = \sum_{\mu=0}^{3} \Lambda^{\mu\nu}p_\mu$. From this information, we have derived in [50] and exposited in [49] that

$$\Lambda = \left(\Lambda^{\mu\nu}\right) = \begin{pmatrix} \frac{p_0+q_0}{\sqrt{s}} & -\frac{p_1+q_1}{\sqrt{s}} & -\frac{p_2+q_2}{\sqrt{s}} & -\frac{p_3+q_3}{\sqrt{s}} \\ \Lambda^{01} & \Lambda^{11} & \Lambda^{21} & \Lambda^{31} \\ 0 & \frac{(p\times q)_1}{|p\times q|} & \frac{(p\times q)_2}{|p\times q|} & \frac{(p\times q)_3}{|p\times q|} \\ \frac{p_0-q_0}{g} & -\frac{p_1-q_1}{g} & -\frac{p_2-q_2}{g} & -\frac{p_3-q_3}{g} \end{pmatrix},$$

with the second row given by

$$\Lambda^{01} = \frac{2|p \times q|}{g\sqrt{s}},$$

and

$$\Lambda^{i1} = \frac{2\left(p_i\left\{p_0 + q_0 p^\mu q_\mu\right\} + q_i\left\{q_0 + p_0 p^\mu q_\mu\right\}\right)}{g\sqrt{s}|p \times q|} \quad (i = 1, 2, 3).$$

We have a complete description of this Lorentz transformation in terms of p, q.

Define $U_\mu = \Lambda^{\nu\mu}\bar{U}_\nu$, notice that

$$U^\mu = \left(\frac{p_0 + q_0}{\sqrt{s}}, \frac{2|p \times q|}{g\sqrt{s}}, 0, \frac{p_0 - q_0}{g}\right).$$

Then

$$\int \frac{d\bar{p}}{\bar{p}_0}\bar{s}\sigma(\bar{g}, \theta)e^{-\bar{p}_0/2}\delta(\bar{p}^\mu(q_\mu - p_\mu)) = \int \frac{d\bar{p}}{\bar{p}_0}\bar{s}_\Lambda\sigma(\bar{g}_\Lambda, \theta_\Lambda)e^{-\frac{1}{2}\bar{p}^\mu U_\mu}\delta\left(\bar{p}^\mu B_\mu\right),$$

where $\bar{g}_\Lambda, \bar{s}_\Lambda \geq 0$ are now given by

$$\begin{aligned} \bar{g}_\Lambda^2 &= g^2 + \frac{1}{2}A^\mu\left(A_\mu - \bar{p}_\mu\right) = g^2 + \frac{1}{2}\sqrt{s}\left(\bar{p}_0 - \sqrt{s}\right), \\ \bar{s}_\Lambda &= 4 + \bar{g}_\Lambda^2, \\ \cos\theta_\Lambda &= 1 - 2\left(\frac{g}{\bar{g}_\Lambda}\right)^2. \end{aligned} \tag{7.6}$$

The equality of the two integrals holds because $d\bar{p}/\bar{p}_0$ is a Lorentz invariant.

We work with the integral on the right-hand side above. Now

$$\bar{p}^\mu B_\mu = \bar{p}_3 g.$$

We switch to polar coordinates in the form

$$d\bar{p} = |\bar{p}|^2 d|\bar{p}| \sin\psi d\psi d\varphi, \quad \bar{p} \equiv |\bar{p}|(\sin\psi\cos\varphi, \sin\psi\sin\varphi, \cos\psi).$$

Then we can write $k_2(p,q)$ as

$$\frac{c'}{p_0 q_0}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\psi d\psi \int_0^{\infty} \frac{|\bar{p}|^2 d|\bar{p}|}{\bar{p}_0} \bar{s}_\Lambda\, \sigma(\bar{g}_\Lambda, \theta_\Lambda)\, e^{-\bar{p}^\mu U_\mu/2}\, \delta(|\bar{p}| g\cos\psi).$$

We evaluate the last delta function at $\psi = \pi/2$ to write $k_2(p,q)$ as

$$\frac{c'}{g p_0 q_0}\int_0^{2\pi} d\varphi \int_0^{\infty} \frac{|\bar{p}| d|\bar{p}|}{\bar{p}_0} \bar{s}_\Lambda \sigma(\bar{g}_\Lambda, \theta_\Lambda) e^{-\bar{p}_0 \frac{p_0+q_0}{2\sqrt{s}}} e^{\frac{|p\times q|}{g\sqrt{s}}|\bar{p}|\cos\varphi}. \tag{7.7}$$

This is already a useful reduced form for $k_2(p,q)$.

We recall, for $I_0\left(\frac{|p\times q|}{g\sqrt{s}}|\bar{p}|\right)$, the modified Bessel function of index zero given in (3.11). We further re-label the integration as $|\bar{p}| = y$. Notice that (7.6) implies

$$g = \bar{g}_\Lambda \sqrt{\frac{1-\cos\theta_\Lambda}{2}} = \bar{g}_\Lambda \sin\frac{\theta_\Lambda}{2},$$

with

$$\sin\frac{\theta_\Lambda}{2} = \frac{g}{\bar{g}_\Lambda} = \frac{g}{\sqrt{g^2 - s/2 + \frac{\sqrt{s}}{2}\sqrt{y^2+s}}} = \frac{\sqrt{2}g}{\sqrt{g^2 - 4 + s\sqrt{y^2/s+1}}}.$$

We may rewrite (7.7) as

$$k_2(p,q) = \frac{c'}{g p_0 q_0}\int_0^{\infty} \frac{y dy}{\sqrt{y^2+s}} \bar{s}_\Lambda \sigma\left(\frac{g}{\sin\frac{\theta_\Lambda}{2}}, \theta_\Lambda\right) e^{-\frac{p_0+q_0}{2\sqrt{s}}\sqrt{y^2+s}} I_0\left(\frac{|p\times q|}{g\sqrt{s}} y\right).$$

From (7.6) we have

$$\bar{s}_\Lambda = 4 + g^2 + \frac{1}{2}\sqrt{s}\{\sqrt{y^2+s} - \sqrt{s}\} = \frac{1}{2}s + \frac{1}{2}s\sqrt{y^2/s+1}.$$

We apply the change of variables $y \to y/\sqrt{s}$ to obtain that $k_2(p,q)$ is given by

$$\frac{c' s^{3/2}}{g p_0 q_0}\int_0^{\infty} \frac{y\left(1+\sqrt{y^2+1}\right) dy}{\sqrt{y^2+1}} \sigma\left(\frac{g}{\sin\frac{\psi}{2}}, \psi\right) e^{-\frac{p_0+q_0}{2}\sqrt{y^2+1}} I_0\left(\frac{|p\times q|}{g} y\right),$$

where $\sin\frac{\psi}{2}$ is given by (3.12). This is the expression from (3.10) once we incorporate the cut-off function (3.1) which is insignificant for the purposes of this calculation.

Significant simplifications can be performed in the case of the "hard ball" cross section where $\sigma = constant$. The relevant integral is then a Laplace transform and a known integral, which can be calculated exactly via a taylor expansion [39, p. 134]. For instance, it is well known that for any $R > r \geq 0$,

$$\int_0^{\infty} \frac{e^{-R\sqrt{1+y^2}} y I_0(ry)}{\sqrt{1+y^2}} dy = \frac{e^{-\sqrt{R^2-r^2}}}{\sqrt{R^2-r^2}},$$

$$\int_0^{\infty} e^{-R\sqrt{1+y^2}} y I_0(ry) dy = \frac{R}{R^2-r^2}\left\{1 + \frac{1}{\sqrt{R^2-r^2}}\right\} e^{-\sqrt{R^2-r^2}}.$$

See for instance [46,45, p. 383, or 24, p. 322].

Using these formulas we can express the integral as

$$k_2(p,q) = \frac{c' s^{3/2}}{g p_0 q_0} \tilde{U}_1(p,q) \exp\left(-\tilde{U}_2(p,q)\right),$$

where $\tilde{U}_2(p,q) \stackrel{\text{def}}{=} \sqrt{\{(p_0+q_0)/2\}^2 - (|p \times q|/g)^2}$ and

$$\tilde{U}_1(p,q) \stackrel{\text{def}}{=} \left(1 + \frac{p_0+q_0}{2}\left(\tilde{U}_2(p,q)\right)^{-1} + \frac{p_0+q_0}{2}\left(\tilde{U}_2(p,q)\right)^{-2}\right)\left(\tilde{U}_2(p,q)\right)^{-1}.$$

Further,

$$\tilde{U}_2(p,q) = \frac{\sqrt{s}}{2g}|p-q| = |p-q|\sqrt{\frac{g^2+4}{4g^2}}.$$

Therefore, $\tilde{U}_2(p,q) \geq \frac{1}{2}|p-q| + 1$. This completes our discussion of the Hilbert-Schmidt form for the linearized collision operator.

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