AdS_5 Solutions of Type IIB Supergravity and Generalized Complex Geometry

Maxime Gabella[1]**, Jerome P. Gauntlett**[2,3]**, Eran Palti**[1]**, James Sparks**[4]**,
Daniel Waldram**[2,3]

[1] Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, U.K.
[2] Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.
[3] The Institute for Mathematical Sciences, Imperial College, London SW7 2PE, U.K.
[4] Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, U.K.
   E-mail: sparks@maths.ox.ac.uk

Abstract: We use the formalism of generalized geometry to study the generic supersymmetric AdS_5 solutions of type IIB supergravity that are dual to $\mathcal{N} = 1$ superconformal field theories (SCFTs) in $d = 4$. Such solutions have an associated six-dimensional generalized complex cone geometry that is an extension of Calabi-Yau cone geometry. We identify generalized vector fields dual to the dilatation and R-symmetry of the dual SCFT and show that they are generalized holomorphic on the cone. We carry out a generalized reduction of the cone to a transverse four-dimensional space and show that this is also a generalized complex geometry, which is an extension of Kähler-Einstein geometry. Remarkably, provided the five-form flux is non-vanishing, the cone is symplectic. The symplectic structure can be used to obtain Duistermaat-Heckman type integrals for the central charge of the dual SCFT and the conformal dimensions of operators dual to BPS wrapped D3-branes. We illustrate these results using the Pilch-Warner solution.

1. Introduction

Supersymmetric $AdS_5 \times Y$ solutions of type IIB supergravity, where Y is a compact Riemannian manifold, are dual to supersymmetric conformal field theories (SCFTs) in $d = 4$ spacetime dimensions with (at least) $\mathcal{N} = 1$ supersymmetry. An important special subclass is when Y is a five-dimensional Sasaki-Einstein manifold SE_5, and the only non-trivial flux is the self-dual five-form. Recall that, by definition, the six-dimensional cone metric with base given by the SE_5 space is a Calabi-Yau cone, and that the dual SCFT arises from D3-branes located at the apex of this cone. There has been much progress in understanding the AdS/CFT correspondence in this setting. For example, there are rich sets of explicit SE_5 metrics [1–3], and there are also powerful constructions using toric geometry. Moreover, for the toric case, the corresponding dual SCFTs have been identified, *e.g.* [4–7].

A key aspect of this progress has been the appreciation that the abelian R-symmetry, which all $\mathcal{N} = 1$ SCFTs in $d = 4$ possess, contains important information about the SCFT. For example, the a central charge is fixed by the R-symmetry, as are the anomalous dimensions of (anti-)chiral primary operators [8]. It is also known that the R-symmetry can be identified via the procedure of a-maximization, which, roughly, says that the correct R-symmetry is the one that maximizes the value of a over all possible admissible R-symmetries [9]. For the solutions of type IIB supergravity with $Y = SE_5$, the R-symmetry manifests itself as a canonical Killing vector ξ on SE_5. This defines a Killing vector on the Calabi-Yau cone, also denoted by ξ, which is a real holomorphic vector field. The Calabi-Yau cone is Kähler, and hence symplectic, and Y admits a corresponding contact structure for which ξ is the Reeb vector. When $Y = SE_5$ the a central charge is inversely proportional to the volume of SE_5, and in [10,11] several geometric formulae for a in terms of ξ were derived. Analogous geometric formulae for the conformal dimension of the chiral primary operator dual to a D3-brane wrapped on a supersymmetric submanifold $\Sigma_3 \subset Y$ were also presented. Of particular interest here are the formulae that show that using symplectic geometry these quantities can be written as Duistermaat-Heckman integrals on the cone and hence can be evaluated by localization. In addition to providing a geometrical interpretation of a-maximization, these formulae and others in [10,11] also provide practical methods for calculating quantities of physical interest without needing the full explicit Sasaki-Einstein metric (which, apart from some special classes of solution, remains out of reach).

The focus of this paper is on $AdS_5 \times Y$ solutions with Y more general than SE_5. Most known solutions are actually part of continuous families of solutions containing a Sasaki-Einstein solution and correspond to exactly marginal deformations of the corresponding SCFT. For example, starting with a toric SE_5 solution one can construct new β-deformed solutions using the techniques of [12]. There is also the "Pilch-Warner solution" explicitly constructed in [13] (based on [14]). It has been shown numerically in [15] that the $\mathbb{Z}_2$ orbifold of the Pilch-Warner solution is part of a continuous family of solutions that includes the Sasaki-Einstein $AdS_5 \times T^{1,1}$ solution. Using the results of [16,17] this should be part of a larger family of continuous solutions that are yet to be found. Similarly, in addition to the β-deformations of the $AdS_5 \times S^5$ solution there are additional deformations [16] that are also not yet constructed (a perturbative analysis was studied in [18]). Having a better understanding of the geometry underlying general $AdS_5 \times Y$ solutions could be useful for finding these deformed solutions but more generally could be useful in constructing new solutions that are not connected with Sasaki-Einstein geometry at all.

The first detailed analysis of supersymmetric $AdS_5 \times Y$ solutions of type IIB supergravity, for general Y with all fluxes activated, was carried out in [19]. The conditions for supersymmetry boil down to a set of Killing spinor equations on Y for two spinors (when $Y = SE_5$ there is only one such spinor). By analysing these equations a set of necessary and sufficient conditions for supersymmetry were established. In light of the progress summarized above for the Sasaki-Einstein case, it is natural to investigate the associated geometry of the cone over Y, and that is the principal purpose of this paper.

As we shall discuss in detail, the cone X admits a specific kind of generalized complex geometry. Aspects of this geometry, restricting to a class of $SU(2)$-structures, were first studied in [20,21]. By viewing $AdS_5 \times Y$ as a supersymmetric warped product $\mathbb{R}^{3,1} \times X$, one sees immediately that the cone admits two compatible generalized almost complex structures [22,23], or equivalently two compatible pure spinors, $\Omega_\pm$. In fact $d\Omega_- = 0$, so that Ω_- defines an integrable generalized complex geometry, while $d\Omega_+$ is

related to the RR flux. The cone is thus generalized Hermitian, and it is also generalized Calabi-Yau in the sense of [24].

Here, we will identify a generalized vector field ξ on the cone that is dual to the R-symmetry and another that is dual to the dilatation symmetry of the dual SCFT and show that they are both generalized holomorphic vector fields on the cone (with respect to the integrable generalized complex structure). This precisely generalizes known results for the Sasaki-Einstein case. We also note that all supersymmetric $AdS_5 \times Y$ solutions satisfy the condition of [20] that there is an $SU(2)$-structure on the cone.

In the Sasaki-Einstein case, one can carry out a symplectic reduction of the Calabi-Yau cone to obtain a four-dimensional transverse Kähler-Einstein space which, in general, is only locally defined. Constructing locally defined Kähler-Einstein spaces has been a profitable way to construct Sasaki-Einstein manifolds, *e.g.* [25]. Here we will show, using the formalism of [26,27], that for general Y there is an analogous reduction of the corresponding six-dimensional generalized Calabi-Yau cone geometry to a four-dimensional space, which again is only locally defined in general, that is generalized Hermitian. More precisely, the four-dimensional geometry admits two compatible generalized almost complex structures, one of which is integrable.

We present explicit expressions for the pure spinors $\Omega_\pm$ associated with the six-dimensional cone in terms of the Killing spinor bilinears presented in [19]. We shall comment upon how Ω_-, associated with the integrable generalized complex structure, contains information on the mesonic moduli space of the dual SCFT and also, briefly, on some relations connected with generalized holomorphic objects and dual BPS operators.

By analysing the pure spinor Ω_+, associated with the non-integrable complex structure, and focusing on the case when the five-form flux is non-vanishing, we show that, perhaps somewhat surprisingly, the cone is symplectic. We shall see that Y is a contact manifold and that the vector part, ξ_v, of the generalized vector ξ, which also defines a Killing vector on Y, is the Reeb vector field associated with the contact structure. We show that the symplectic structure can be used to obtain Duistermaat-Heckman type integrals for the central charge a of the dual SCFT and also for the conformal dimensions of operators dual to wrapped BPS D3-branes. Once again these results precisely generalize those for the Sasaki-Einstein case. Some of these results were first presented in [28]; here we will provide additional details and also show how they are related to the generalized geometry on the cone.

Finally, we will illustrate some of our results using the Pilch-Warner solution. The paper begins with a review of generalized geometry and it ends with three appendices containing some details about our conventions, some technical derivations, and a brief discussion of the Sasaki-Einstein case.

2. Generalized Geometry

We begin by reviewing some aspects of generalized complex geometry [24], to fix conventions and notation. For further details see, for example, [29].

2.1. The generalized tangent and spinor bundles. Generalized geometry starts with the generalized tangent bundle E over a manifold X, which is a particular extension of TX by T^*X obtained by twisting with a *gerbe*. A gerbe is simply a higher degree version of a $U(1)$ bundle with unitary connection. Just as topologically a $U(1)$ bundle is determined by its first Chern class, the topology of a gerbe is determined by a class in

$H^3(X, \mathbb{Z})$. To define a gerbe [30], one begins with an open cover $\{U_i\}$ of X together with a set of functions $g_{ijk} : U_i \cap U_j \cap U_j \to U(1)$ defined on triple overlaps. These are required to satisfy $g_{ijk} = g_{jik}^{-1} = g_{ikj}^{-1} = g_{kji}^{-1}$, together with the cocycle condition $g_{jkl}g_{ikl}^{-1}g_{ijl}g_{ijk}^{-1} = 1$ on quadruple overlaps. A *connective structure* [30] on a gerbe is a collection of one-forms $\Lambda_{(ij)}$ defined on double overlaps $U_i \cap U_j$ satisfying $\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = -(2\pi i l_s^2)g_{ijk}^{-1}\mathrm{d}g_{ijk}$ on triple overlaps. In turn, a *curving* is a collection of two-forms $B_{(i)}$ on U_i satisfying

$$B_{(j)} - B_{(i)} = \mathrm{d}\Lambda_{(ij)}. \tag{2.1}$$

It follows that $\mathrm{d}B_{(j)} = \mathrm{d}B_{(i)} = H$ is a closed global three-form on X, called the *curvature*, and, in cohomology, $\frac{1}{(2\pi l_s)^2}H \in H^3(X, \mathbb{Z})$ (in the normalization that we shall use in this paper). In string theory, the collection of two-forms $B_{(i)}$, which we write simply as B, is the NS B-field and H is its curvature.

The generalized tangent bundle E is an extension of TX by T^*X,

$$0 \longrightarrow T^*X \longrightarrow E \xrightarrow{\pi} TX \longrightarrow 0. \tag{2.2}$$

Locally, sections of E, which we refer to as generalized tangent vectors, may be written as $V = x + \lambda$, where $x \in \Gamma(TX)$ and $\lambda \in \Gamma(T^*X)$. More precisely, in going from one coordinate patch U_i to another U_j the extension is defined by the connective structure

$$x_{(i)} + \lambda_{(i)} = x_{(j)} + \left(\lambda_{(j)} - i_{x_{(j)}}\mathrm{d}\Lambda_{(ij)}\right). \tag{2.3}$$

The bundle E is in fact isomorphic to $TX \oplus T^*X$. However, the isomorphism is not canonical but depends on a choice of splitting, defined by a two-form curving B satisfying (2.1). It follows that

$$x + (\lambda - i_x B) \in \Gamma(TX \oplus T^*X). \tag{2.4}$$

Thus the definition (2.3) of E can be viewed as encoding the patching of a class of two-form curvings B.

Writing $d = \dim_{\mathbb{R}} X$, there is a natural $O(d, d)$-invariant metric $\langle \cdot, \cdot \rangle$ on E, given by

$$\langle V, W \rangle = \tfrac{1}{2}(i_x\mu + i_y\lambda), \tag{2.5}$$

where $V = x + \lambda$, $W = y + \mu$, or in two-component notation,

$$\langle V, W \rangle = (x \quad \lambda) \begin{pmatrix} 0 & \tfrac{1}{2} \\ \tfrac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} y \\ \mu \end{pmatrix}. \tag{2.6}$$

This metric is invariant under $O(d, d)$ transformations acting on the fibres of E, defining a canonical $O(d, d)$-structure. A general element $O \in O(d, d)$ may be written in terms of $d \times d$ matrices a, b, c, and d as

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.7}$$

under which a general element $V \in E$ transforms by

$$V = \begin{pmatrix} x \\ \lambda \end{pmatrix} \mapsto OV = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}. \tag{2.8}$$

The requirement that $\langle OV, OV \rangle = \langle V, V \rangle$ implies $a^T c + c^T a = 0$, $b^T d + d^T b = 0$ and $a^T d + c^T b = 1$. Note that the $GL(d)$ action on the fibres of TX and T^*X embeds as a subgroup of $O(d, d)$. Concretely it maps

$$V \mapsto V' = \begin{pmatrix} a & 0 \\ 0 & a^{-T} \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}, \tag{2.9}$$

where $a \in GL(d)$. Given a two-form ω, one also has the abelian subgroup

$$e^\omega = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \qquad \text{such that} \qquad V = x + \lambda \mapsto V' = x + (\lambda - i_x \omega). \tag{2.10}$$

This is usually referred to as a B-transform. Given a bivector β one can similarly define another abelian subgroup of β-transforms

$$e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \qquad \text{such that} \qquad V = x + \lambda \mapsto V' = (x + i_\lambda \beta) + \lambda. \tag{2.11}$$

Note that the patching (2.3) corresponds to a B-transform with $\omega = \mathrm{d}\Lambda_{(ij)}$. Similarly, the splitting isomorphism between E and $TX \oplus T^*X$ defined by B is also a B-transform

$$E \underset{e^{-B}}{\overset{e^B}{\rightleftarrows}} TX \oplus T^*X. \tag{2.12}$$

There is a natural bracket on generalized vectors known as the Courant bracket, which encodes the differentiable structure of E. It is defined as

$$[V, W] = [x + \lambda, y + \mu] = [x, y]_{\mathrm{Lie}} + \mathcal{L}_x \mu - \mathcal{L}_y \lambda - \tfrac{1}{2}\mathrm{d}\left(i_x \mu - i_y \lambda\right), \tag{2.13}$$

where $[x, y]_{\mathrm{Lie}}$ is the usual Lie bracket between vectors and $\mathcal{L}_x$ is the Lie derivative along x. The Courant bracket is invariant under the action of diffeomorphisms and B-shifts ω that are closed, $\mathrm{d}\omega = 0$, giving an automorphism group which is a semi-direct product $\mathrm{Diff}(X) \ltimes \Omega^2_{\mathrm{closed}}(X)$. Note, however, that in string theory only B-shifts by the curvature of a unitary line bundle on X are gauge symmetries, as opposed to shifts by arbitrary closed two-forms, leading to a smaller automorphism group. Under an infinitesimal diffeomorphism generated by a vector field x and a B-shift with $\omega = \mathrm{d}\lambda$, one has the generalized Lie derivative by $V = x + \lambda$ on a generalized vector field $W = y + \mu$,

$$\delta W \equiv \mathbb{L}_V W = [x, y]_{\mathrm{Lie}} + (\mathcal{L}_x \mu - i_y \mathrm{d}\lambda). \tag{2.14}$$

This is also known as the Dorfman bracket $[V, W]_D$, the anti-symmetrization of which gives the Courant bracket (2.13). Note that since the metric $\langle \cdot, \cdot \rangle$ is invariant under $O(d, d)$ transformations its generalized Lie derivative vanishes. Given a particular choice of splitting (2.1) defined by B, the Courant bracket on E defines a Courant bracket on $TX \oplus T^*X$, known as the *twisted* Courant bracket. It is given by

$$\begin{aligned} [x + \lambda, y + \mu]_H &= e^B [e^{-B}(x + \lambda), e^{-B}(y + \mu)] \\ &= [x + \lambda, y + \mu] + i_y i_x H, \end{aligned} \tag{2.15}$$

where by an abuse of notation we are writing $x + \lambda$ and $y + \mu$ for sections of $TX \oplus T^*X$, whereas above they were sections of E.

Given the metric $\langle \cdot, \cdot \rangle$, one can define $Spin(d, d)$ spinors in the usual way. Since the volume element in $\mathrm{Cliff}(d, d)$ squares to one, one can define two helicity spin bundles

$S_\pm(E)$ as the ± 1 eigenspaces, and thus take spinors to be Majorana-Weyl. A section of $S_\pm(E)$ on U_i can be identified with a even- or odd-degree polyform $\Omega_\pm \in \Omega^{\text{even/odd}}(X)$ restricted to U_i, with the Clifford action of $V \in \Gamma(E)$ given by

$$V \cdot \Omega_\pm = i_x \Omega_\pm + \lambda \wedge \Omega_\pm. \tag{2.16}$$

It is easy to see that

$$(V \cdot W + W \cdot V) \cdot \Omega_\pm = 2\langle V, W \rangle \, \Omega_\pm, \tag{2.17}$$

as required. Using this Clifford action the B-transform (2.10) on spinors is given by

$$\Omega_\pm \mapsto e^\omega \Omega_\pm, \tag{2.18}$$

where the exponentiated action is by wedge product. The patching (2.3) of E then implies that

$$\Omega_\pm^{(i)} = e^{d\Lambda_{(ij)}} \Omega_\pm^{(j)}. \tag{2.19}$$

Furthermore a splitting B also induces an isomorphism between $S_\pm(E)$ and $S_\pm(TX \oplus TX^*)$,

$$S_\pm(E) \underset{e^{-B}}{\overset{e^B}{\rightleftarrows}} S_\pm(TX \oplus T^*X), \tag{2.20}$$

again by the action of the exponentiated wedge product. If $\Omega_\pm$ is a section of $S_\pm(E)$, we will sometimes write $\Omega_\pm^B \equiv e^B \Omega_\pm$ for the corresponding section of $S_\pm(TX \oplus T^*X)$ defined by the splitting B. The real $Spin(d, d)$-invariant spinor bilinear on sections of $S_\pm(E)$ is a top form given by the *Mukai pairing*

$$\langle \Omega_\pm, \Psi_\pm \rangle \equiv (\Omega_\pm \wedge \lambda(\Psi_\pm))_{\text{top}}, \tag{2.21}$$

where one defines the operator λ

$$\lambda(\Psi_m^\pm) \equiv (-1)^{\text{Int}[m/2]} \Psi_m^\pm, \tag{2.22}$$

with Ψ_m the degree m form in $\Psi_\pm$. The Mukai pairing is invariant under B-transforms: $\langle e^\omega \Omega_\pm, e^\omega \Psi_\pm \rangle = \langle \Omega_\pm, \Psi_\pm \rangle$. For $d = 6$ the bilinear is anti-symmetric. The usual action of the exterior derivative on the component forms of $\Omega_\pm$ is compatible with the patching (2.19) and defines an action

$$d : S_\pm(E) \to S_\mp(E), \tag{2.23}$$

while the generalized Lie derivative on spinors is given by

$$\mathbb{L}_V \Omega_\pm = \mathcal{L}_x \Omega_\pm + d\lambda \wedge \Omega_\pm = d(V \cdot \Omega_\pm) + V \cdot d\Omega_\pm. \tag{2.24}$$

Note that given a splitting B the operator on $\Omega_\pm^B \in S_\pm(TX \oplus T^*X)$ corresponding to d is d_H defined by

$$d_H \Omega_\pm^B \equiv e^B d(e^{-B} \Omega_\pm^B) = (d - H \wedge) \Omega_\pm^B, \tag{2.25}$$

where $H = dB$. Furthermore one has

$$\mathbb{L}_V \Omega = e^{-B} \left(\mathbb{L}_{V^B} - i_x H \wedge \right) \Omega^B, \tag{2.26}$$

where $V^B = e^B V = x + (\lambda - i_x B)$.

Finally, we note that there is actually a slight subtlety in the relation between generalized spinors and polyforms. Given the embedding (2.9) in $O(d, d)$ of the $GL(d)$ action on the fibres of TX one actually finds that the Clifford action (2.16) implies that on U_i we can identify $S_\pm(E)$ with $|\Lambda^d T^*X|^{-1/2} \otimes \Lambda^{\text{even/odd}} T^*X$; that is, there is an additional factor of the determinant bundle $|\Lambda^d T^*X|$. (This factor is the source, for instance, of the fact that the Mukai pairing is a top form, rather than a scalar.) This bundle is trivial, so generalized spinors can indeed be written as polyforms patched by (2.19), but there is no natural isomorphism to make this identification. The simplest solution, and one which will also allow us to incorporate the dilaton in a natural way, is to extend the $O(d, d)$ action to a conformal action $O(d, d) \times \mathbb{R}^+$. One can then define a family of spinor bundles $S_\pm^k(E)$ transforming with weight k under the conformal factor $\mathbb{R}^+$; that is, with sections transforming as $\Omega_\pm \to \rho^k \Omega_\pm$, where $\rho \in \mathbb{R}^+$. If one embeds the $GL(d)$ action on TX in $O(d, d)$ as in (2.9) and, in addition, makes a conformal scaling by $\rho = \det a$ then sections of $S_\pm^{-1/2}(E)$ can be directly identified with polyforms patched by (2.19).

2.2. Generalized metrics and complex structures. A *generalized metric G* on E is the generalized geometrical equivalent of a Riemannian metric on TX. We have seen that there is a natural $O(d, d)$ structure on E defined by the metric $\langle \cdot, \cdot \rangle$ (2.5). The generalized metric G defines an $O(d) \times O(d)$ substructure. It splits $E = C_+ \oplus C_-$ such that the metric $\langle \cdot, \cdot \rangle$ gives a positive-definite metric on C_+ and a negative-definite metric on C_-, corresponding to the two $O(d)$ structure groups. One can define G as a product structure on E; that is, $G : E \to E$ with $G^2 = 1$ and $\langle GU, GV \rangle = \langle U, V \rangle$, so that $\frac{1}{2}(1 \pm G)$ project onto $C_\pm$. In general G has the form

$$G = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \quad (2.27)$$

where g is a metric on X and B is a two-form. The patching of E implies B satisfies (2.1), so that B may be identified with the curving of the gerbe used in the twisting of E. Thus the generalized metric G defines a particular splitting of E. In particular, we see from (2.27) that $G = e^{-B} G_0 e^B$, where G_0 is a generalized metric on $TX \oplus T^*X$ defined by g.

The generalized metric G naturally encodes the NS fields g and B as the coset space $O(d, d)/O(d) \times O(d)$. The dilaton ϕ appears when one considers the conformal group $O(d, d) \times \mathbb{R}^+$, used to define the generalized spinors as true polyforms. To define a $O(d) \times O(d)$ substructure in $O(d, d) \times \mathbb{R}^+$, in addition to G which gives the embedding in the $O(d, d)$ factor, one must give the embedding ρ in the conformal factor $\rho \in \mathbb{R}^+$. Recall that under diffeomorphisms ρ transforms as a section of $\Lambda^d TX$. Given the metric g we can define the generic embedding by $\rho = e^{2\phi}/\sqrt{g}$ for some positive function $e^{2\phi}$, which we identify as the dilaton. Note that ρ is by definition invariant under $O(d, d)$ and so one finds the conventional T-duality transformation of the dilaton under $O(d, d)$.

Under the generalized Lie derivative, $\mathbb{L}_V G = 0$ implies [31]

$$\mathcal{L}_x g = 0, \qquad \mathcal{L}_x B - d\lambda = 0, \quad (2.28)$$

so that $\mathcal{L}_x H = 0$, where $H = dB$. Such a V is called a *generalized Killing vector*.

Given G we may decompose generalized spinors in $Spin(d, d)$ under $Spin(d) \times Spin(d)$. In fact one can go further. Using the projection $\pi : E \to TX$ the two $Spin(d)$

groups can be identified and the generalized spinors may be decomposed as bispinors of *Spin*(*d*):

$$\Omega_\pm = e^{-\phi} e^{-B} \Phi^{\text{even/odd}}. \tag{2.29}$$

In this expression, one first uses the Clifford map to identify the bispinors with a generalized spinor $\Phi^{\text{even/odd}}$ of $S_\pm(TX \oplus T^*X) \cong \Lambda^{\text{even/odd}} T^*X$ and then uses the splitting B to map to a spinor of $S_\pm(E)$. The factor of $e^{-\phi}$ appears because the polyforms are really sections of $S_\pm^{-1/2}(E)$ transforming with weight $-\frac{1}{2}$ under conformal rescalings. Explicitly, if Φ is a bispinor, $\Phi \in \Omega^*(X)$ a polyform, and γ^i are *Spin*(*d*) gamma matrices, the Clifford map is

$$\Phi = \sum_k \frac{1}{k!} \Phi_{i_1 \cdots i_k} \gamma^{i_1 \cdots i_k} \quad \longleftrightarrow \quad \Phi = \sum_k \frac{1}{k!} \Phi_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \tag{2.30}$$

The Cliff(d, d) action is realized via left and right multiplication by the gamma matrices γ^i. For the chiral spinors $\Omega_\pm$ the sum is over k even/odd respectively. We also note here the Fierz identity

$$\Phi = \frac{1}{n_d} \sum_k \frac{1}{k!} \text{Tr} \left(\Phi \gamma_{i_k \cdots i_1} \right) \gamma^{i_1 \cdots i_k}, \tag{2.31}$$

where the γ^i are $n_d \times n_d$ matrices. Finally, the generalized metric also defines an action $\star_G$ on generalized spinors which is the analogue of the Hodge star. It is given by

$$\Omega_\pm \mapsto \star_G \Omega_\pm = e^{-B} \star \lambda(e^B \Omega_\pm), \tag{2.32}$$

where λ is the operator defined in (2.22) and $\star$ denotes the ordinary Hodge star for the metric g.

If $d = 2n$ one can also introduce a *generalized almost complex structure* on E. This is a map $\mathcal{J} : E \to E$ with $\mathcal{J}^2 = -1$ and $\langle \mathcal{J}U, \mathcal{J}V \rangle = \langle U, V \rangle$ and gives a decomposition

$$E_{\mathbb{C}} = L \oplus \bar{L}, \tag{2.33}$$

where L denotes the $+i$ eigenspace of $\mathcal{J}$. Note that L is maximally isotropic: $\langle U, V \rangle = \langle \mathcal{J}U, \mathcal{J}V \rangle = \langle iU, iV \rangle = -\langle U, V \rangle = 0$. This defines a $U(n, n) \subset O(2n, 2n)$ structure on E. By definition $\langle U, \mathcal{J}V \rangle + \langle \mathcal{J}U, V \rangle = 0$, so $\mathcal{J}$ can be viewed either as an element of $O(2n, 2n)$ or of the Lie algebra $o(2n, 2n)$. A generic $\mathcal{J}$ can be written locally as

$$\mathcal{J} = \begin{pmatrix} I & P \\ Q & -I^* \end{pmatrix}, \tag{2.34}$$

where I^* is the linear map on T^*X dual to the map I on TX, P is a bivector and Q is a two-form. If the twisting (2.3) is trivial, so $E = TX \oplus T^*X$, there are two canonical examples of generalized almost complex structures. The first is an ordinary almost complex structure I on TX, for which[1]

$$\mathcal{J}_1 = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}. \tag{2.35}$$

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[1] Note that we have chosen the opposite sign in (2.35) compared with [29]. This is so that the $+i$ eigenspace is identified with $T^{(1,0)} \oplus T^{*(0,1)}$.

The second is a non-degenerate (stable) two-form ω, for which

$$\mathcal{J}_2 = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \tag{2.36}$$

If $d\omega = 0$ this corresponds to a symplectic structure.

More generally, a generalized almost complex structure $\mathcal{J}$ is *integrable* if L is closed under the Courant bracket. That is, given $U, V \in \Gamma(L)$ then $[U, V] \in \Gamma(L)$. In the above two cases (2.35), (2.36), this reduces to integrability of I and the closure of ω, respectively. A generalized almost complex structure is equivalent to (the conformal class of) a *pure spinor* Ω, which simply means a chiral complex generalized spinor such that the annihilator

$$L_\Omega = \{U \in E_{\mathbb{C}} : U \cdot \Omega = 0\} \tag{2.37}$$

is maximal isotropic. The sub-bundle L defined by $\mathcal{J}$ is then identified with L_Ω. Notice that L_Ω is invariant under conformal rescalings $\Omega \mapsto f\Omega$, for any function f. A generalized almost complex structure is therefore more precisely equivalent to the *pure spinor line bundle* generated by Ω. Integrability of $\mathcal{J}$ can be expressed as the condition $d\Omega = V \cdot \Omega$ for some $V \in \Gamma(E)$. If one can find a nowhere vanishing globally defined Ω then one has an $SU(n, n)$ structure and if in addition $d\Omega = 0$ then one has a *generalized Calabi-Yau* structure in the sense[2] of [24]. For example, in the complex structure case (2.35) one has $\Omega = c\bar{\Omega}_{(n,0)}$, where $\Omega_{(n,0)}$ is the holomorphic $(n, 0)$-form and c is a non-zero constant (the reason why $\bar{\Omega}_{(n,0)}$ appears, rather than $\Omega_{(n,0)}$, is directly related to the comment in footnote 1).

A generalized vector $V = x + \lambda$ is called (real) *generalized holomorphic* if $\mathbb{L}_V \mathcal{J} = 0$. Equivalently, $\mathbb{L}_V$ preserves the spinor line bundle generated by the corresponding pure spinor Ω; that is, $\mathbb{L}_V \Omega = f\Omega$ for some function f.

Given a splitting B, one can define the corresponding generalized complex objects on $TX \oplus T^*X$. In particular, if $\mathcal{J}$ is the generalized almost complex structure for a pure spinor Ω, then the corresponding generalized almost complex structure on $TX \oplus T^*X$ is defined in terms of the annihilator of

$$\Omega^B = e^B\Omega \tag{2.38}$$

and is given by

$$\mathcal{J}^B \equiv e^B \mathcal{J} e^{-B}. \tag{2.39}$$

In particular, integrability of $\mathcal{J}$ is equivalent to integrability of $\mathcal{J}^B$ using the twisted Courant bracket (2.15), or equivalently $d_H \Omega^B = V \cdot \Omega^B$.

Viewing $\mathcal{J}$ as a Lie algebra element one can define its action on generalized spinors via the Clifford action [32]. Explicitly, one has

$$\mathcal{J} \cdot = \frac{1}{2} \left(Q_{mn} dx^m \wedge dx^n \wedge + I^m{}_n [i_{\partial_m}, dx^n \wedge] + P^{mn} i_{\partial_m} i_{\partial_n} \right). \tag{2.40}$$

Note that for any generalized vector V one has, under the Clifford action, $[\mathcal{J} \cdot, V \cdot] = (\mathcal{J} V) \cdot$. One can also define the operator $\mathcal{J}_h : S_\pm(E) \to S_\pm(E)$,

$$\mathcal{J}_h \equiv e^{\frac{1}{2}\pi \mathcal{J} \cdot}, \tag{2.41}$$

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[2] Note that a different definition is used in [29].

which is the spinor space representation of $\mathcal{J}$ as an element of the group $Spin(d, d)$. If n is even and the pure spinor is a section of $S_\pm(E)$, then $\mathcal{J}_h$ defines a complex structure on $S_\mp(E)$, while if n is odd it defines a complex structure on $S_\pm(E)$. Observe that for any generalized vector V we have the Clifford action identity $\mathcal{J}_h \cdot V \cdot \mathcal{J}_h^{-1} \cdot = (\mathcal{J} V) \cdot$.

Finally, a pair of generalized almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are said to be *compatible* if

$$[\mathcal{J}_1, \mathcal{J}_2] = 0, \tag{2.42}$$

and the combination

$$G = -\mathcal{J}_1 \mathcal{J}_2 \tag{2.43}$$

is a generalized metric. If Ω_1 and Ω_2 are the corresponding pure spinors, (2.42) is equivalent to $\mathcal{J}_1 \cdot \Omega_2 = \mathcal{J}_2 \cdot \Omega_1 = 0$. An example of a pair of compatible pure spinors is (2.35), (2.36), with the compatibility condition being that $I_i^k \omega_{jk} = g_{ij}$ is positive definite. Note this is $\omega_{ij} = -g_{ik} I_j^k$, this mathematics convention differing by a sign to the usual physics convention. A pair of compatible almost complex structures defines an $SU(n) \times SU(n)$ structure. A *generalized Kähler* structure is an $SU(n) \times SU(n)$ structure where both generalized almost complex structures are integrable, while for a *generalized Hermitian* structure only *one* need be integrable.

Note that an $SU(n) \times SU(n)$ structure can equivalently be specified by a generalized metric and a pair of chiral $Spin(2n)$ spinors. For example, for $d = 6$ a pair of chiral spinors η_+^1, η_+^2 can be used to construct an $SU(3) \times SU(3)$ structure given by

$$\Omega_+ = e^{-\phi} e^{-B} \eta_+^1 \bar\eta_+^2, \qquad \Omega_- = e^{-\phi} e^{-B} \eta_+^1 \bar\eta_-^2, \tag{2.44}$$

with $\eta_-^2 \equiv (\eta_+^2)^c$. This will play a central role in the following sections. Similarly, for $d = 4$ a pair of chiral spinors η_+^1, η_+^2 give rise to an $SU(2) \times SU(2)$ structure specified by two compatible pure spinors, but both of them consist of sums of even forms, since now $(\eta_+^2)^c$ is a positive chirality spinor. We will see such an $SU(2) \times SU(2)$ structure in Sect. 4. That the spinors have the same chirality is necessary for them to be compatible in four dimensions [33].

3. AdS_5 Backgrounds as Generalized Complex Geometries

3.1. Supersymmetric AdS_5 backgrounds. Our starting point is the most general class of supersymmetric AdS_5 solutions of type IIB supergravity, as studied in [19]. The ten-dimensional metric in Einstein frame is

$$g_E = e^{2\Delta} (g_{AdS} + g_Y), \tag{3.1}$$

where g_Y is a Riemannian metric on the compact five-manifold Y, and Δ is a real function on Y. The AdS_5 metric g_{AdS} is normalized to have unit radius, so that

$$\text{Ric}_{g_{AdS}} = -4 \, g_{AdS}. \tag{3.2}$$

The ten-dimensional string frame metric is defined to be $g_\sigma \equiv e^{\phi/2} g_E$. In addition to the metric, there is the dilaton ϕ and NS three-form $H \equiv dB$ in the NS sector, and the forms

$F \equiv F_1 + F_3 + F_5$ in the RR sector. The RR fluxes F_n are related to the RR potentials C_n via

$$F_1 = dC_0, \tag{3.3}$$
$$F_3 = dC_2 - HC_0, \tag{3.4}$$
$$F_5 = dC_4 - H \wedge C_2. \tag{3.5}$$

These are all taken to be forms on Y, so as to preserve the $SO(4, 2)$ symmetry, with the exception of the self-dual five-form F_5 which necessarily takes the form

$$F_5 = f \left(\text{vol}_{AdS} - \widetilde{\text{vol}}_Y \right), \tag{3.6}$$

where f is a constant. Here $\widetilde{\text{vol}}_Y$ denotes a volume form for (Y, g_Y). It is related by $\widetilde{\text{vol}}_Y = -\text{vol}_5$ to the volume form of [19], where the latter was given in terms of an orthonormal frame as $\text{vol}_5 = e^{12345}$ and used to define, for instance, the Hodge star. In turns out that, in the Sasaki-Einstein limit, the conventional volume form is $\widetilde{\text{vol}}_Y$ rather than vol_5 and so here we will use the former throughout. In particular, it is the orientation that we will use when defining integrals over Y.

In [19] the conditions for a supersymmetric AdS_5 background were written in terms of two five-dimensional spinors ξ_1, ξ_2 on Y, giving the system of equations reproduced here in (A.1)–(A.6) of Appendix A. Various spinor bilinears involving ξ_1 and ξ_2 were also introduced, and used to determine the necessary and sufficient conditions for supersymmetry. For example, it was shown that

$$A \equiv \tfrac{1}{2} \left(\bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2 \right) = 1,$$
$$Z \equiv \bar{\xi}_2 \xi_1 = 0. \tag{3.7}$$

It will be useful for later in this paper to recall the definitions of the following scalar bilinears:

$$\sin \zeta \equiv \tfrac{1}{2} \left(\bar{\xi}_1 \xi_1 - \bar{\xi}_2 \xi_2 \right),$$
$$S \equiv \bar{\xi}_2^c \xi_1, \tag{3.8}$$

the one-form bilinears:

$$K \equiv \bar{\xi}_1^c \beta_{(1)} \xi_2,$$
$$K_3 \equiv \bar{\xi}_2 \beta_{(1)} \xi_1,$$
$$K_4 \equiv \tfrac{1}{2} \left(\bar{\xi}_1 \beta_{(1)} \xi_1 - \bar{\xi}_2 \beta_{(1)} \xi_2 \right),$$
$$K_5 \equiv \tfrac{1}{2} \left(\bar{\xi}_1 \beta_{(1)} \xi_1 + \bar{\xi}_2 \beta_{(1)} \xi_2 \right), \tag{3.9}$$

and the two-form bilinears:

$$V = -\frac{i}{2}(\bar{\xi}_1 \beta_{(2)} \xi_1 - \bar{\xi}_2 \beta_{(2)} \xi_2),$$
$$W = -\bar{\xi}_2 \beta_{(2)} \xi_1. \tag{3.10}$$

Here the β_m generate the Clifford algebra for g_Y, so $\{\beta_m, \beta_n\} = 2g_{Ymn}$. Equivalently, with respect to any orthonormal frame, we write $\hat{\beta}_m$ with $\{\hat{\beta}_m, \hat{\beta}_n\} = 2\delta_{mn}$. We have also introduced the notation $\beta_{(k)} \equiv \frac{1}{k!} \beta_{m_1 \cdots m_k} dx^{m_1} \wedge \cdots \wedge dx^{m_k}$.

A key result of [19] is that $K_5^{\#}$, the vector dual to the one-form K_5, defines a Killing vector that preserves all of the fluxes. This was identified as corresponding to the R-symmetry in the dual SCFT. Another important result was

$$e^{-4\Delta} f = 4 \sin \zeta. \tag{3.11}$$

The Killing spinors ξ_1, ξ_2 were used to introduce a canonical five-dimensional orthonormal frame in Appendix B of [19], which is convenient for certain calculations. We will refer to that paper for further details. Finally, we note that Eq. (A.22) of Appendix A may be used to obtain expressions for the two-form potentials B and C_2 in terms of the bilinear W introduced in (3.10):

$$B = -\frac{4}{f} e^{6\Delta+\phi/2} \operatorname{Re} W + b_2, \tag{3.12}$$

$$C_2 = -\frac{4}{f} e^{6\Delta+\phi/2} C_0 \operatorname{Re} W - \frac{4}{f} e^{6\Delta-\phi/2} \operatorname{Im} W + c_2. \tag{3.13}$$

Here b_2 and c_2 are real closed two-forms. Notice that the first term in B in (3.12) is a globally defined two-form, and thus $H = dB$ is exact. It follows that $[H] = 0 \in H^3(Y, \mathbb{R})$, although notice that b_2 may be taken to be globally defined if and only if the *torsion class* of H is zero in $H^3(Y, \mathbb{Z})$ (which for simplicity we shall assume). Similar remarks apply to C_2 (up to large gauge transformations of C_0).

3.2. Reformulation as generalized complex geometries. The supersymmetric AdS_5 geometry described above can be simply reformulated in terms of generalized complex geometry, as in the discussion of [20,21]. The basic observation is simply that these solutions can be viewed as warped products of flat four-dimensional space with a six-dimensional manifold X, satisfying a set of supersymmetry conditions that imply the existence of a particular generalized complex geometry [22,23]. As we shall explain in more detail below, combining this structure with the existence of the Killing vector $K_5^{\#}$ precisely generalizes the correspondence between Sasaki-Einstein geometry and Calabi-Yau cone geometry. In the following we analyze this reformulation in detail. We find in particular that all supersymmetric AdS_5 solutions necessarily satisfy the condition of [20] that there is an $SU(2)$-structure on X.

One begins by rewriting the unit AdS_5 metric in a Poincaré patch as

$$g_{AdS} = \frac{dr^2}{r^2} + r^2 g_{\mathbb{R}^{3,1}}. \tag{3.14}$$

Switching to the string frame, we can consider (3.1) as a special case of a warped supersymmetric $\mathbb{R}^{3,1}$ solution of the form

$$g_\sigma = e^{2A} g_{\mathbb{R}^{3,1}} + g_6, \tag{3.15}$$

where the warp factor is given by

$$e^{2A} = e^{2\Delta+\phi/2} r^2, \tag{3.16}$$

and the six-dimensional metric is given by

$$g_6 = e^{2\Delta+\phi/2} \left(\frac{dr^2}{r^2} + g_Y \right). \tag{3.17}$$

We also define the six-dimensional volume form as

$$\text{vol}_6 \equiv e^{12\Delta + 3\phi} r^5 dr \wedge \widetilde{\text{vol}}_Y. \tag{3.18}$$

Notice that the six-dimensional manifold X, on which g_6 is a metric, is a product $\mathbb{R}^+ \times Y$, where r may be interpreted as a coordinate on $\mathbb{R}^+$. In particular, X is non-compact. It thus follows that supersymmetric AdS_5 solutions are special cases of supersymmetric $\mathbb{R}^{3,1}$ solutions.

In [23] the general conditions for an $\mathcal{N} = 1$ supersymmetric $\mathbb{R}^{3,1}$ background, in the string frame metric (3.15), were written in terms of two chiral six-dimensional spinors η_+^1, η_+^2 on X, namely the system of equations given here in (A.17)–(A.20). The relation between the two sets of Killing spinors, for AdS_5 solutions, is given by first decomposing $\text{Cliff}(6)$ into $\text{Cliff}(5)$ via

$$\begin{aligned} \hat{\gamma}_m &= \hat{\beta}_m \otimes \sigma_3, \qquad m = 1, \ldots, 5, \\ \hat{\gamma}_6 &= 1 \otimes \sigma_1, \end{aligned} \tag{3.19}$$

where $\hat{\gamma}_i$, $i = 1, \ldots, 6$, generate $\text{Cliff}(6)$ and σ_α, $\alpha = 1, 2, 3$, denote the Pauli matrices. Changing basis to $\xi_1 = \chi_1 + i\chi_2$, $\xi_2 = \chi_1 - i\chi_2$, we then have

$$\begin{aligned} \eta_+^1 &= e^{A/2} \begin{pmatrix} \chi_1 \\ i\chi_1 \end{pmatrix}, \qquad \eta_-^1 = e^{A/2} \begin{pmatrix} -\chi_1^c \\ i\chi_1^c \end{pmatrix}, \\ \eta_+^2 &= e^{A/2} \begin{pmatrix} -\chi_2 \\ -i\chi_2 \end{pmatrix}, \qquad \eta_-^2 = e^{A/2} \begin{pmatrix} \chi_2^c \\ -i\chi_2^c \end{pmatrix}, \end{aligned} \tag{3.20}$$

where $\chi_i^c \equiv \tilde{D}_5 \chi_i^*$ denotes 5D charge conjugation, and correspondingly $\eta_-^i \equiv (\eta_+^i)^c \equiv D_6(\eta_+^i)^*$ where $D_6 = \tilde{D}_5 \otimes \sigma_2$. For further details, see Appendix A. Using the two chiral spinors η_1^+, η_2^+ we may define the bispinors

$$\Phi_+ \equiv \eta_+^1 \otimes \bar{\eta}_+^2, \qquad \Phi_- \equiv \eta_+^1 \otimes \bar{\eta}_-^2. \tag{3.21}$$

Notice that, in the conventions of Appendix A, we have $A_6 = 1$, so that $\bar{\eta} \equiv \eta^\dagger$ is just the Hermitian conjugate. Via the Clifford map (2.30) the bispinors for $Spin(6)$ in (3.21) may also be viewed as elements of $\Omega^*(X, \mathbb{C})$. We will mainly tend to think of $\Phi_\pm$ as complex differential forms of mixed degree. These are then $Spin(6, 6)$ spinors, as explained in Sect. 2.2. In fact $\Phi_\pm$ in (3.21) are both *pure* spinors, and also *compatible*. They then define an $SU(3) \times SU(3)$ structure on $TX \oplus T^*X$.

In terms of (3.21), the Killing spinor equations for a general supersymmetric $\mathbb{R}^{3,1}$ solution (*i.e.* not necessarily associated with an AdS_5 solution, but with vanishing four-dimensional cosmological constant) may be rewritten as [34] (see also [22])

$$d_H \left(e^{2A-\phi} \Phi_- \right) = 0, \tag{3.22}$$

$$d_H \left(e^{2A-\phi} \Phi_+ \right) = e^{2A-\phi} dA \wedge \bar{\Phi}_+$$
$$+ \frac{1}{16} e^{2A} \left[(|a|^2 - |b|^2) F + i(|a|^2 + |b|^2) \star \lambda(F) \right]. \tag{3.23}$$

Here recall that $F = F_1 + F_3 + F_5$ is the sum of RR fields and from (2.22),

$$\lambda(F) = F_1 - F_3 + F_5. \tag{3.24}$$

Note that the Hodge star is with respect to the metric g_6, with positive orientation given by $\mathrm{d}r \wedge \mathrm{vol}_Y$. The remaining Bianchi identities and equations of motion are (*cf.* [34] Eq. (4.9)–(4.10))

$$\mathrm{d}H = 0 , \qquad \mathrm{d}_H F = \delta_{\text{source}}, \tag{3.25}$$

$$\mathrm{d}(\mathrm{e}^{4A-2\phi} \star H) - \mathrm{e}^{4A} F_n \wedge \star F_{n+2} = 0, \tag{3.26}$$

$$(\mathrm{d} + H \wedge)(\mathrm{e}^{4A} \star F) = 0. \tag{3.27}$$

The equation of motion for F can also be written as

$$\mathrm{d}\left[\mathrm{e}^{4A}\mathrm{e}^{-B} \star \lambda(F) \right] = 0, \tag{3.28}$$

and follows from the supersymmetry equations. In fact, for AdS_5 solutions it was shown in [19] that supersymmetry implies *all* of the equations of motion and Bianchi identities. We have also introduced the spinor norms

$$|a|^2 = |\eta_1^+|^2, \qquad |b|^2 = |\eta_2^+|^2, \tag{3.29}$$

which for a supersymmetric $\mathbb{R}^{3,1}$ background must satisfy

$$|a|^2 + |b|^2 = \mathrm{e}^A c_+, \qquad |a|^2 - |b|^2 = \mathrm{e}^{-A} c_-, \tag{3.30}$$

where $c_\pm$ are constants. Upon squaring and subtracting the equations one obtains

$$|\Phi_\pm|^2 = \frac{1}{8} |a|^2|b|^2 = \frac{1}{32} \left(\mathrm{e}^{2A} c_+^2 - \mathrm{e}^{-2A} c_-^2 \right). \tag{3.31}$$

As we now show, for the particular case of AdS_5 solutions the above equations simplify somewhat. In this case it is possible to fix the constant c_- in (3.31) by the scaling of $\Phi_\pm$ with r which, using (3.16), implies that $c_- = 0$ and hence $|\eta_1^+| = |\eta_2^+|$. This is consistent with the equation $Z = 0$ in (3.7), since from (3.20) we see that $|\eta_\pm^1| = |\eta_\pm^2|$ is equivalent to Re $Z = 0$. Notice that $c_- = 0$ is also a necessary condition in order to have supersymmetric probe branes [35]. The normalization that was used in [19] implies $|a|^2 = |b|^2 = \mathrm{e}^A$ and hence $c_+ = 2$. One can actually go a little further. In [20] it was assumed that there was an $SU(2)$-structure on the cone. In terms of the spinors η_i^+ this is equivalent to the condition that, in addition to $c_- = 0$, one has

$$\bar{\eta}_1^+ \eta_2^+ + \bar{\eta}_2^+ \eta_1^+ = 0. \tag{3.32}$$

However it is easy to see that this is equivalent to Im $Z = 0$, which again is required by supersymmetry on Y. Thus in fact all supersymmetric AdS_5 solutions necessarily satisfy the $SU(2)$ condition of [20].

We now define the pure spinor

$$\Omega_-^B \equiv \mathrm{e}^{2A-\phi}\Phi_-, \tag{3.33}$$

which by (3.22) is d_H closed, $\mathrm{d}_H\Omega_-^B = 0$. The associated generalized almost complex structure $\mathcal{J}_-^B$ is then integrable with respect to the twisted Courant bracket (2.15). We also define

$$\Omega_- \equiv \mathrm{e}^{-B}\Omega_-^B = \mathrm{e}^{-B}\mathrm{e}^{2A-\phi}\Phi_-, \tag{3.34}$$

which is closed under the usual exterior derivative:

$$d\Omega_- = 0. \tag{3.35}$$

The associated generalized almost complex structure, which we denote by $\mathcal{J}_-$,[3] is then integrable. Combined with the fact that the norm of Φ_-, and hence of Ω_-, is nowhere vanishing, this means, in particular, that we have a generalized Calabi-Yau manifold in the sense of [24].

We similarly define

$$\Omega_+ \equiv e^{-B}e^{2A-\phi}\Phi_+. \tag{3.36}$$

However, the corresponding generalized almost complex structure $\mathcal{J}_+$ is not integrable in general, its integrability being obstructed by the RR fields in (3.23). If it *were* integrable, we would have a generalized Kähler manifold. With these definitions we can write the supersymmetry equations for AdS_5 solutions as

$$\begin{aligned} d\Omega_- &= 0, \\ d\Omega_+ &= dA \wedge \bar{\Omega}_+ + \frac{i}{8}e^{3A}e^{-B} \star \lambda(F). \end{aligned} \tag{3.37}$$

It is worth noting that the latter equation may also be written as

$$d\left(e^{-A}\mathrm{Re}\,\Omega_+\right) = 0, \tag{3.38}$$

$$d\left(e^{A}\mathrm{Im}\,\Omega_+\right) = \frac{1}{8}e^{4A}e^{-B} \star \lambda(F), \tag{3.39}$$

and that in turn Eq. (3.39) can be written as [36]

$$e^{-B}F = 8\mathcal{J}_- \cdot d\left(e^{-3A}\mathrm{Im}\,\Omega_+\right) = 8d^{\mathcal{J}_-}\left(e^{-3A}\mathrm{Im}\,\Omega_+\right), \tag{3.40}$$

where $d^{\mathcal{J}_-} \equiv -[d, \mathcal{J}_- \cdot]$. For most of the paper we will demand that $F_5 \neq 0$, or equivalently $f \neq 0$. Physically this corresponds to having non-vanishing D3-brane charge. It would be interesting to know whether or not all supersymmetric AdS_5 solutions of type IIB supergravity have this property.

3.3. Canonical vector fields. In this section we examine the geometric properties of the generalized vector fields $r\partial_r$, $\xi \equiv \mathcal{J}_-(r\partial_r)$ and $\eta \equiv \mathcal{J}_-(d\log r)$. As in the Sasaki-Einstein case, $r\partial_r$ and ξ correspond respectively to the dilatation symmetry and the R-symmetry in the dual SCFT (while η is related to a contact structure on Y, as we shall show later in Sect. 6).

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[3] The generalized complex structures $\mathcal{J}_-^B$ and $\mathcal{J}_-$ are related by (2.39).

3.3.1. Dilatation symmetry. We begin with the dilatation vector field $r\partial_r$. It immediately follows from (3.16), (3.20) and (3.21) that

$$\mathcal{L}_{r\partial_r}\Phi_\pm = \Phi_\pm, \tag{3.41}$$

and therefore

$$\mathcal{L}_{r\partial_r}\Omega_\pm = 3\Omega_\pm. \tag{3.42}$$

This follows since e^{2A} has scaling dimension 2 (3.16), and both the *B*-field and the dilaton ϕ are pull-backs from Y. Notice that Eq. (3.42) may also be trivially rewritten in terms of the generalized Lie derivative (2.24):

$$\mathbb{L}_{r\partial_r}\Omega_\pm = 3\Omega_\pm. \tag{3.43}$$

This implies that

$$\mathbb{L}_{r\partial_r}\mathcal{J}_\pm = 0. \tag{3.44}$$

To see this, recall that $\mathcal{J}_\pm$ is defined by saying that its $+i$ eigenspace is equal to the annihilator $L_{\Omega_\pm}$ of $\Omega_\pm$, and the latter is clearly preserved under the one-parameter family of (generalized) diffeomorphisms generated by $r\partial_r$. It further follows that $\mathbb{L}_{r\partial_r}G = 0$, where G is the generalized metric $G = -\mathcal{J}_+\mathcal{J}_- = -\mathcal{J}_+\mathcal{J}_-$, so that $r\partial_r$ is generalized Killing. Equation (3.44) says that $r\partial_r$ is a (real) *generalized holomorphic* vector field for the integrable generalized complex structure $\mathcal{J}_-$. We shall not use this terminology for $\mathcal{J}_+$, since the latter is not in general integrable. Clearly, this generalizes the Sasaki-Einstein result where the cone is Calabi-Yau and the dilatation vector $r\partial_r$ is holomorphic.

3.3.2. R-symmetry. We next define the generalized vectors

$$\xi \equiv \mathcal{J}_-(r\partial_r), \tag{3.45}$$
$$\eta \equiv \mathcal{J}_-(\mathrm{d}\log r), \tag{3.46}$$

which are, in general, a mixture of vectors and one-forms. Recall that the generalized almost complex structures $\mathcal{J}_\pm$ are related to the generalized metric via $G = -\mathcal{J}_+\mathcal{J}_- = -\mathcal{J}_-\mathcal{J}_+$. The conical form (3.17) of the metric g_6 and the fact that B has no component along $\mathrm{d}r$ implies that $G\,\mathrm{d}\log r = e^{-2\Delta-\frac{\phi}{2}}r\partial_r$, $G\,r\partial_r = e^{2\Delta+\frac{\phi}{2}}\mathrm{d}\log r$, and hence in addition to (3.45) and (3.46) we may also write

$$\xi = e^{2\Delta+\frac{\phi}{2}}\mathcal{J}_+(\mathrm{d}\log r),$$
$$\eta = e^{-2\Delta-\frac{\phi}{2}}\mathcal{J}_+(r\partial_r). \tag{3.47}$$

We may split ξ and η into a vector part and a one-form part, in a fixed splitting of E,

$$\xi = \xi_v + \xi_f, \tag{3.48}$$
$$\eta = \eta_v + \eta_f. \tag{3.49}$$

By carrying out a calculation, presented in Appendix B, we may then write these as bilinears constructed from the five-dimensional Killing spinors (3.9):

$$\xi_v = K_5^\#,$$
$$\xi_f = i_{\xi_v} b_2,$$
$$\eta_v = e^{-2\Delta - \phi/2} \operatorname{Re} K_3^\#, \tag{3.50}$$
$$\eta_f = \frac{4}{f} e^{4\Delta} K_4 + i_{\eta_v} b_2.$$

As discussed in Appendix B, it is the B-transform, ξ^B, of the generalized vector ξ that is naturally related to the bilinears of [19]. We have obtained (3.50) by performing an inverse B-transform using the expression for the B-field given in terms of bilinears presented in (3.12). In particular, this is where the closed two-form b_2 appears. Since the B-transform of b_2 by an exact form is a generalized diffeomorphism, and a gauge symmetry of string theory, we see that the physical information in b_2 is represented by its cohomology class in $H^2(X, \mathbb{R})$. More precisely, large gauge transformations of the B-field, which correspond to tensoring the underlying gerbe by a unitary line bundle on X, lead to the torus $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$ (with suitable normalization). Turning on the two-form b_2 corresponds to giving vacuum expectation values to moduli (of the NS field B) and so is a symmetry of the supersymmetry equations. It is therefore left undetermined. In the field theory dual, the cohomology class of b_2 thus corresponds to a marginal deformation.

In [19] it was shown that $K_5^\#$ is a Killing vector that preserved all of the fluxes, and thus $K_5^\#$ was identified as being dual to the R-symmetry in the SCFT. In the generalized geometry we can show the stronger conditions that

$$\mathbb{L}_\xi \mathcal{J}_\pm = 0, \tag{3.51}$$

and hence ξ is a generalized holomorphic Killing vector field. In fact it is straightforward to show $\mathbb{L}_\xi \Omega_- = -3i\Omega_-$ and hence $\mathbb{L}_\xi \mathcal{J}_- = 0$. Indeed since $d\Omega_- = 0$ and $r\partial_r - i\xi \in L_{\Omega_-}$ annihilates Ω_-, using (2.24) and (3.43) we have

$$\mathbb{L}_\xi \Omega_- = d(\xi \cdot \Omega_-) = -id(r\partial_r \cdot \Omega_-) = -i\mathbb{L}_{r\partial_r}\Omega_- = -3i\Omega_-. \tag{3.52}$$

In Appendix B we show that $\mathbb{L}_\xi \Omega_+ = 0$ and hence $\mathbb{L}_\xi \mathcal{J}_+ = 0$. There we also show that

$$\mathbb{L}_\xi(e^{-B} F) = 0. \tag{3.53}$$

Thus, we have established that $\xi \equiv \mathcal{J}_-(r\partial_r)$ is a generalized holomorphic vector field, which moreover is generalized Killing for the generalized metric $G = -\mathcal{J}_-\mathcal{J}_+$, and also preserves the RR fluxes. Again, this clearly generalizes the Sasaki-Einstein result, where $\xi = I(r\partial_r)$ is a holomorphic Killing vector field for the Calabi-Yau cone.

To conclude this section we note that when $f \neq 0$ the vector field $\xi_v = K_5^\#$ is nowhere vanishing on $Y = \{r = 1\}$. One can see this from the formula

$$|K_5^\#|^2 = \sin^2 \zeta + |S|^2, \tag{3.54}$$

and using (3.11). Thus for $f \neq 0$, ξ_v acts locally freely on Y and hence the orbits of ξ_v define a corresponding one-dimensional foliation of Y. This is again precisely as in the Sasaki-Einstein case (although in the Sasaki-Einstein case the norm of ξ_v is constant).

4. Generalized Reduction of AdS_5 Backgrounds

Recall that in the Sasaki-Einstein case one can consider the symplectic reduction of the Calabi-Yau cone metric with respect to the R-symmetry Killing vector ξ (or equivalently a holomorphic quotient with respect to $r\partial_r - \mathrm{i}\xi$). Generically ξ does not define a $U(1)$ fibration and the four-dimensional reduced space is not a manifold. Nonetheless, locally one can consider the geometry on the transversal section to the foliation formed by the orbits of ξ in the Sasaki-Einstein space. The result of the reduction is that this four-dimensional geometry is Kähler-Einstein. Thus locally one can always write the Sasaki-Einstein metric as

$$g_Y = \eta \otimes \eta + g_{\text{KE}}, \tag{4.1}$$

where g_{KE} is a Kähler-Einstein metric.

   The existence of the generalized holomorphic vectors ξ and $r\partial_r$ in the generic case suggests one can make an analogous generalized reduction to four dimensions. In this section, we show that this is indeed the case following the theory of generalized quotients developed in [26,27]. We first review the formalism and then apply it to our particular case, showing that there is a generalized Hermitian structure on the local transversal section, giving the conditions satisfied by the corresponding reduced pure spinors.

4.1. Generalized reductions. We will follow the description of generalized quotients given in [27][4]. These include both symplectic reductions and complex quotients as special cases. One first needs to introduce the *reduction data*.

   In conventional geometry, the action of a Lie group G on M is generated infinitesimally by a set of vector fields, defined by a map from the Lie algebra $\psi : \mathfrak{g} \to \Gamma(TM)$. Given a vector field $x \in \Gamma(TM)$, the infinitesimal action of $u \in \mathfrak{g}$ is then just the Lie derivative (or in this case Lie bracket)

$$\delta x = \mathcal{L}_{\psi(u)} x = [\psi(u), x]. \tag{4.2}$$

One requires that given $u, v \in \mathfrak{g}$, one has $[\psi(u), \psi(v)] = \psi([u, v])$ so that

$$\mathcal{L}_{\psi(u)} \mathcal{L}_{\psi(v)} - \mathcal{L}_{\psi(v)} \mathcal{L}_{\psi(u)} = \mathcal{L}_{[\psi(u),\psi(v)]} = \mathcal{L}_{\psi([u,v])}, \tag{4.3}$$

and thus there is a Lie algebra homomorphism between $\mathfrak{g}$ and the algebra of vector fields under the Lie bracket.

   In generalized geometry, we have a larger group of symmetries, diffeomorphisms and B-shifts, which are generated infinitesimally by the generalized Lie derivative (2.14). Thus given an action of G on M, it is natural to consider the infinitesimal "lifted action" of G on E defined by the map $\tilde{\psi} : \mathfrak{g} \to \Gamma(E)$, such that for any $V \in \Gamma(E)$ and $u \in \mathfrak{g}$ we have

$$\delta V = \mathbb{L}_{\tilde{\psi}(u)} V, \tag{4.4}$$

and under the projection $\pi : E \to TM$ we simply get the vector fields $\psi(u)$, that is

$$\pi \tilde{\psi}(u) = \psi(u). \tag{4.5}$$

---

[4] Note that the bracket $[\![,]\!]$ used in [27] is the Dorfman bracket or generalized Lie derivative $[\![V, W]\!] = \mathbb{L}_V W$ and is not anti-symmetric.

Such transformations are infinitesimal automorphisms of E that have the property that they preserve both the metric $\langle \cdot, \cdot \rangle$ on E and the Courant bracket (2.13). If we again assume that

$$\mathbb{L}_{\tilde{\psi}(u)} \mathbb{L}_{\tilde{\psi}(v)} - \mathbb{L}_{\tilde{\psi}(v)} \mathbb{L}_{\tilde{\psi}(u)} = \mathbb{L}_{\tilde{\psi}([u,v])}, \tag{4.6}$$

then $\tilde{\psi}$ defines an equivariant structure on E. (Note that this is equivalent to the Courant bracket condition $[\tilde{\psi}(u), \tilde{\psi}(v)] = \tilde{\psi}([u, v])$.) In what follows it will also be assumed that $\tilde{\psi}$ is isotropic, that is

$$\langle \tilde{\psi}(u_1), \tilde{\psi}(u_2) \rangle = 0 \tag{4.7}$$

for all $u_1, u_2 \in \mathfrak{g}$.

One can actually define a more general action on E which is a homomorphism between algebras with Courant brackets rather than Lie algebras. One starts by extending $\mathfrak{g}$ to a larger algebra. The construction considered in [27] which is relevant for us is as follows. Let $\mathfrak{h}$ be a vector space on which there is some representation of $\mathfrak{g}$. Then we can form a "Courant algebra" $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ with Courant bracket[5] given $u_i \in \mathfrak{g}$ and $w_i \in \mathfrak{h}$,

$$[(u_1, w_1), (u_2, w_2)] = ([u_1, u_2], \tfrac{1}{2}(u_1 \cdot w_2 - u_2 \cdot w_1)), \tag{4.8}$$

where $u \cdot w$ is the action of $\mathfrak{g}$ on $\mathfrak{h}$. Suppose in addition $\mu : M \to \mathfrak{h}^*$ is a $\mathfrak{g}$-equivariant map, meaning $\mathcal{L}_{\psi(u)} \mu(w) = \mu(u \cdot w)$ for all $u \in \mathfrak{g}$ and $w \in \mathfrak{h}$. Then, given some isotropic lifted action $\tilde{\psi}$ of $\mathfrak{g}$, one can then define the *extended action*

$$\begin{aligned} \Psi : \mathfrak{g} \oplus \mathfrak{h} &\to \Gamma(E), \\ (u, w) &\mapsto \tilde{\psi}(u) + \mathrm{d}\mu(w), \end{aligned} \tag{4.9}$$

which, it is easy to show, has the property that

$$[\Psi(u_1, w_1), \Psi(u_2, w_2)] = \Psi([u_1, u_2], \tfrac{1}{2}(u_1 \cdot w_2 - u_1 \cdot w_2)), \tag{4.10}$$

and hence Ψ defines a homomorphism of Courant algebras as opposed to Lie algebras as in (4.3). Note that the extra factor $\mathrm{d}\mu(w)$ in (4.9) corresponds to a trivial B-shift, and thus $\mathbb{L}_{\Psi(u,v)} = \mathbb{L}_{\tilde{\psi}(u)}$. Note also that μ will play the role of a moment map. In the conventional case of symplectic reductions one has $\mu : M \to \mathfrak{g}^*$, whereas here $\mathfrak{h}$ can be any representation space. The triple $(\tilde{\psi}, \mathfrak{h}, \mu)$ is known as the *reduction data*.

This reduction data can then be used to define a reduced generalized tangent bundle E^{red}. First one makes the usual assumptions about μ and the G action on M so that $M^{\mathrm{red}} = \mu^{-1}(0)/G$ is a manifold. (This requires that 0 is a regular point of μ and that the G action on $\mu^{-1}(0)$ is free and proper.) Then define the sub-bundle K which is the image of the bundle map $\mathfrak{a} \times M$ associated to Ψ, that is

$$K = \left\{ \tilde{\psi}(u) + \mathrm{d}\mu(w), \ u \in \mathfrak{g}, \ w \in \mathfrak{h} \right\} \subseteq E, \tag{4.11}$$

and also the orthogonal bundle $K^{\perp}$, the fibres of which are orthogonal to K with respect to the $O(d, d)$ metric $\langle \cdot, \cdot \rangle$. One can then construct the generalized tangent space on M^{red},

$$E^{\mathrm{red}} = \left. \frac{K^{\perp}|_{\mu^{-1}(0)}}{K|_{\mu^{-1}(0)}} \right/ G. \tag{4.12}$$

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[5] Note we take a slightly different definition of the bracket to that in [27] in to order to match the Courant bracket (2.13) on E.

The main results of [26,27] are then to show how various geometrical structures can be transported from E to E^{red}. The case of particular interest to us is that of generalized Hermitian reduction.

As discussed in Sect. 2.2 a generalized Hermitian manifold is a generalized complex manifold with a compatible generalized metric (or equivalently a second, compatible, generalized almost complex structure). Let $\mathcal{J}$ be the integrable generalized complex structure and G be the generalized metric. Given some reduction data $(\tilde{\psi}, \mathfrak{h}, \mu)$, recalling $\mathbb{L}_{\Psi(u,v)} = \mathbb{L}_{\tilde{\psi}(u)}$, the structures are G-invariant if

$$\mathbb{L}_{\tilde{\psi}(u)} G = \mathbb{L}_{\tilde{\psi}(u)} \mathcal{J} = 0, \tag{4.13}$$

for all $u \in \mathfrak{g}$. One can also define the sub-bundles

$$K^G = GK^\perp \cap K^\perp, \tag{4.14}$$

which is the sub-bundle of $K^\perp$ the fibres of which are orthogonal to K, with respect to G, and

$$E_K = K \oplus GK, \tag{4.15}$$

which is the G-orthogonal complement to K^G. Theorem 4.4 of [27] then states

Theorem 1 (Generalized Hermitian reduction [27]). *Let E be a generalized tangent space over M with reduction data $(\tilde{\psi}, \mathfrak{h}, \mu)$. Suppose E is equipped with a G-invariant generalized Hermitian structure $(\mathcal{J}, G)$. If over $\mu^{-1}(0)$, $\mathcal{J}K^G = K^G$, or equivalently $\mathcal{J}E_K = E_K$, then $\mathcal{J}$ and G can be reduced to E^{red} where they define a generalized Hermitian structure.*

Even if the group action is such that the reduced space is not a manifold, one can still define a generalized Hermitian structure on the transversal section to the foliation.

4.2. Generalized reduction from ξ. We now use the reduction formalism to show that the generalized Calabi-Yau geometry on the cone X reduces to a generalized Hermitian geometry in four dimensions. This is the analogue of the reduced Kähler-Einstein geometry in the Sasaki-Einstein case.

First we note that there is a group action on the cone X generated by the vectors $r\partial_r$ and ξ_v. These commute and the corresponding Lie algebra is simply $\mathbb{R} \oplus \mathbb{R}$. If the orbits of ξ_v form a $U(1)$ action then together $r\partial_r$ and ξ_v integrate to a $\mathbb{C}^*$ action, but this need not be the case. The generalized vectors $r\partial_r$ and ξ give a lifted action of $\mathbb{R} \oplus \mathbb{R}$ on E, so that, if $u = (a, b) \in \mathbb{R} \oplus \mathbb{R}$,

$$\tilde{\psi}(u) = ar\partial_r + b\xi. \tag{4.16}$$

By definition we have $\pi\tilde{\psi}(u) = ar\partial_r + b\xi_v$. Under the Courant bracket, given the expressions (3.50) we see that $[r\partial_r, \xi] = 0$, and hence

$$[\tilde{\psi}(u_1), \tilde{\psi}(u_2)] = 0 = \tilde{\psi}([u_1, u_2]) \tag{4.17}$$

for all u_1 and u_2, as required for a lifted action. Furthermore, from (B.5) we see that $\tilde{\psi}$ is isotropic.

We also have a generalized Hermitian structure on X given by $\mathcal{J}_\pm$. The generalized complex structure $\mathcal{J}_-$ is integrable, and we have the compatible generalized metric

$G = -\mathcal{J}_+\mathcal{J}_-$. In Sect. 3 we showed $\mathbb{L}_{r\partial_r}\mathcal{J}_\pm = \mathbb{L}_\xi\mathcal{J}_\pm = 0$, and hence the Hermitian structure is invariant under both group actions.

There are then two different ways we can view the generalized reduction, mirroring the symplectic reduction and the complex quotient in the Sasaki-Einstein case. In the first reduction, we take $\mathfrak{g} = \mathbb{R}$ generated by ξ_v, and in the reduction data we take $\mathfrak{h} = \mathfrak{g}$ and $\mu = \log r$. This is the same moment map one takes in the symplectic reduction. In the second case we take the complex Lie algebra $\mathfrak{g} = \mathbb{C}$ generated by $r\partial_r - i\xi_v$ and $\mathfrak{h} = 0$ so there is no moment map. The reduction is then analogous to a complex quotient. We now discuss these in turn. As in the Sasaki-Einstein case, both lead to the same reduced structure.

4.2.1. $\mathfrak{g} = \mathbb{R}$ reduction. In this case, the reduction data is

$$\tilde{\psi}(u) = u\xi, \qquad \mathfrak{h} = \mathbb{R}, \qquad \mu = \log r. \tag{4.18}$$

We have already seen that $\tilde{\psi}$ is an isotropic lifted action. It is also clear that μ is $\mathfrak{g}$-equivariant since, from (B.5), $i_{\xi_v}d\mu = 0$. Thus $(\tilde{\psi}, \mathfrak{h}, \mu)$ are suitable reduction data. Furthermore, $\mathcal{J}_-$ and G are both invariant under $\tilde{\psi}(u)$. We have

$$\mu^{-1}(0) = Y, \tag{4.19}$$

and given $u \in \mathfrak{g}$ and $v \in \mathfrak{h}$,

$$K = \{u\xi + v\,d\log r\}. \tag{4.20}$$

Using

$$G\xi = e^{2\Delta+\phi/2}\eta, \qquad G\,d\log r = e^{-2\Delta-\phi/2}r\partial_r \tag{4.21}$$

we have

$$GK = \{u'\eta + v'r\partial_r\}, \tag{4.22}$$

and hence

$$E_K \equiv K \oplus GK = \{u\xi + v\,d\log r + u'\eta + v'r\partial_r\}. \tag{4.23}$$

Using the definitions (3.46) we immediately see that $\mathcal{J}_-E_K = E_K$. Hence, assuming the action of ξ_v on Y gives a $U(1)$ fibration, using the generalized Hermitian reduction theorem, we see that we have a generalized Hermitian structure on E^{red} over the four-dimensional space $M^{\text{red}} = Y/U(1)$. More generally, we get a generalized Hermitian structure on the transversal section to the ξ_v orbits.

4.2.2. $\mathfrak{g} = \mathbb{C}$ reduction. In this case, the reduction data is

$$\tilde{\psi}(u) = u(r\partial_r - i\xi), \qquad \mathfrak{h} = 0, \qquad \mu = 0. \tag{4.24}$$

Given $\mathfrak{h}$ is trivial, we have

$$K = \{u(r\partial_r - i\xi)\}. \tag{4.25}$$

As before, we have already seen that $\tilde{\psi}$ is an isotropic lifted action and so $(\tilde{\psi}, \mathfrak{h}, \mu)$ are suitable reduction data. $\mathcal{J}_-$ and G are both invariant under $\tilde{\psi}(u)$ and finally, using (4.21), we now have

$$GK = \{u'(d\log r - i\eta)\}, \tag{4.26}$$

and hence

$$E_K = \{u(r\partial_r - i\xi) + u'(d\log r - i\eta)\}. \tag{4.27}$$

Again we immediately see that $\mathcal{J}_- E_K = E_K$. Hence, again assuming the action of ξ_v on Y gives a $U(1)$ fibration, using the generalized Hermitian reduction theorem, we see that we have a generalized Hermitian structure on E^{red} over the four-dimensional space $M^{\text{red}} = X/\mathbb{C}^* = Y/U(1)$, or more generally, we get a generalized Hermitian structure on the transversal section to the $r\partial_r - i\xi_v$ orbits.

Note that in both cases the reduced manifold M^{red} is the same. Furthermore, the (complexified) spaces E_K, and hence the G-orthogonal complements K^G, also agree. As discussed in [27], K^G is a model for the reduced bundle E^{red}. Thus the two reductions give identical generalized Hermitian structures on M^{red}.

4.3. The reduced pure spinors. We now calculate the conditions on the reduced generalized Hermitian structure implied by supersymmetry. The reduced structure can be defined by a pair of commuting generalized almost complex structures: $\tilde{\mathcal{J}}_1$ which is integrable and is the reduction of $\mathcal{J}_-$, and a non-integrable structure $\tilde{\mathcal{J}}_2$, defined such that $-\tilde{\mathcal{J}}_1 \tilde{\mathcal{J}}_2$ is the reduced generalized metric. Equivalently, the structures are defined as a pair of pure spinors $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$. It is the differential conditions on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ implied by supersymmetry that we will derive.

In order to construct the reduced pure spinors, first note that the reduction gives a splitting of the generalized tangent space

$$E = E_K \oplus K^G \tag{4.28}$$

such that, in general, the $O(d, d)$ metric $\langle \cdot, \cdot \rangle$ factors into an $O(p, p)$ metric on K^G and an $O(d - p, d - p)$ metric on E_K. Thus we can similarly decompose sections of the spinor bundles $S^{\pm}(E)$ into spinors of $Spin(d - p, d - p) \times Spin(p, p)$. In particular, generic sections $\Omega_{\pm} \in S^{\pm}(E)$ can be written as

$$\Omega_{\pm} = \Theta_{\pm} \otimes \tilde{\Omega}_+ \oplus \Theta_{\mp} \otimes \tilde{\Omega}_-. \tag{4.29}$$

It is then the spinor components of $\tilde{\Omega}_{\pm}$ in $S^{\pm}(K_G)$ which correspond to the reduced pure spinors. For the case in hand the relevant decomposition is under $Spin(2, 2) \times Spin(4, 4) \subset Spin(6, 6)$. As we will see below, the reduction is such that the pure spinors defining the supersymmetric background decompose as

$$\begin{aligned} \Omega_- &= \Theta_- \otimes \tilde{\Omega}_1, \\ \Omega_+ &= \Theta_+ \otimes \tilde{\Omega}_2. \end{aligned} \tag{4.30}$$

Thus the reduced spinors $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are both positive helicity in $Spin(4, 4)$.

To make this explicit we need a basis for the *Spin*(6, 6) gamma matrices reflecting the decomposition (4.28). We first introduce coordinates adapted to the reduction. We write the *R*-symmetry Killing vector as[6]

$$\xi_v = K_5^{\#} = \partial_{\psi}. \tag{4.31}$$

Let y^m be coordinates on the transversal section to the *R*-symmetry foliation. This means that $i_{\xi_v} dy^m = 0$ and, in particular, the metric decomposes as

$$g_Y = K_5 \otimes K_5 + g_{mn}^{\text{red}} dy^m dy^n, \tag{4.32}$$

in analogy to (4.1). The reduction structure already defines a natural basis on E_K given by

$$\begin{aligned}
\hat{f}_1 &= r\partial_r, \quad f^1 = d\log r, \\
\hat{f}_2 &= \xi, \quad\ \ f^2 = \eta,
\end{aligned} \tag{4.33}$$

and satisfying $\langle f^i, \hat{f}_j \rangle = \frac{1}{2}\delta^i{}_j$ and $\langle f^i, f^j \rangle = \langle \hat{f}_i, \hat{f}_j \rangle = 0$. We can then define an orthogonal basis on K^G given by

$$\hat{e}_m = e^{-b_2}\partial_{y^m} - \tilde{\eta}_m \xi, \quad e^m = dy^m - \eta^m \xi, \tag{4.34}$$

where $\tilde{\eta}_m = 2\langle \eta, e^{-b_2}\partial_{y^m} \rangle$ and $\eta^m = 2\langle \eta, dy^m \rangle = i_{\eta_v} dy^m$. This basis again satisfies $\langle e^m, \hat{e}_n \rangle = \frac{1}{2}\delta^m{}_n$ and $\langle e^m, e^n \rangle = \langle \hat{e}_m, \hat{e}_n \rangle = 0$.

Given such a basis we can then write a generic *Spin*(6, 6) spinor using the standard raising and lowering operator construction. Consider the polyform $\Omega^{(0)} = e^{-b_2} \in S^+(E)$. It is easy to see that we have the Clifford actions

$$\hat{f}_i \cdot \Omega^{(0)} = \hat{e}_m \cdot \Omega^{(0)} = 0, \tag{4.35}$$

for all i and m. Thus we can regard $\Omega^{(0)}$ as a ground state for the lowering operators $(\hat{f}_i, \hat{e}_m)$. A generic spinor is then given by acting with the anti-commuting raising operators (f^i, e^m). Acting with the e^m first, we see that a generic (non-chiral) spinor has the form

$$\Omega = e^{-b_2}\tilde{\Omega}_0 + f^1 \cdot e^{-b_2}\tilde{\Omega}_1 + f^2 \cdot e^{-b_2}\tilde{\Omega}_2 + f^1 \cdot f^2 \cdot e^{-b_2}\tilde{\Omega}_3, \tag{4.36}$$

where $\tilde{\Omega}_i$ are polyforms in dy^m, and $e^{-b_2}\tilde{\Omega}_i$ transform as a *Spin*(4, 4) spinor under the Clifford action of $(e^m, \hat{e}_m)$.

We can now write the supersymmetry pure spinors $\Omega_{\pm}$ in the form (4.36). Requiring that $r\partial_r - i\xi$ and $d\log r - i\eta$ annihilate Ω_- while $r\partial_r - ie^{2\Delta+\phi/2}\eta$ and $d\log r - ie^{-2\Delta-\phi/2}\xi$ annihilate Ω_+, one finds the only possibility is

$$\begin{aligned}
\Omega_- &= (d\log r - i\eta) \cdot r^3 e^{-3i\psi} e^{-b_2}\tilde{\Omega}_1, \\
\Omega_+ &= \left(1 + ie^{2\Delta+\phi/2} d\log r \cdot \eta\right) \cdot r^3 e^{-b_2}\tilde{\Omega}_2,
\end{aligned}$$

where $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are both even polyforms in dy^m, as claimed in (4.30). We have introduced factors of r^3 and $e^{-3i\psi}$ so that $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are independent of the r and ψ coordinates. In general, they are then only locally defined.

---

[6] Note that this, more conventional, normalization of ψ differs from the corresponding coordinate in [19] by a factor of three.

One can then derive the conditions on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, reduced to the transversal section, implied by supersymmetry. From $d\Omega_- = 0$ one finds

$$d\tilde{\Omega}_1 = -3i\tilde{\eta} \cdot \tilde{\Omega}_1, \tag{4.37}$$

where $\tilde{\eta}$ is a generalized vector on the transversal section defined by

$$\eta = d\psi + e^{-b_2}\tilde{\eta}. \tag{4.38}$$

This is means that $\tilde{\Omega}_1$ defines an integrable generalized complex structure on the transverse section, as expected from the reduction theorem.

For the second pure spinor, the condition (3.38) on $\mathrm{Re}\,\Omega_+$ is equivalent to

$$d\left(e^{\Delta+\phi/4}\tilde{\eta} \cdot \mathrm{Im}\,\tilde{\Omega}_2\right) = -2e^{-\Delta-\phi/4}\,\mathrm{Re}\,\tilde{\Omega}_2,$$
$$d\left(e^{\Delta+\phi/4}\,\mathrm{Im}\,\tilde{\Omega}_2\right) = 0. \tag{4.39}$$

Finally, since $i_{r\partial_r}F = 0$, following (4.36), we can decompose the flux as

$$e^{-B}F = e^{-b_2}\tilde{F} + \eta \cdot e^{-b_2}\tilde{G}. \tag{4.40}$$

The final condition (3.40) is then equivalent to

$$d\left(e^{-\Delta-\phi/4}\tilde{\eta} \cdot \mathrm{Re}\,\tilde{\Omega}_2\right) = -\tfrac{1}{8}\tilde{G},$$
$$\left[\tilde{\mathcal{J}}_1\,d(4\Delta+\phi)\right] \cdot \mathrm{Im}\,\tilde{\Omega}_2 = -\tfrac{1}{8}e^{3\Delta+3\phi/4}\tilde{F}, \tag{4.41}$$

and to

$$\tilde{\mathcal{J}}_1 \cdot d\left(e^{-\Delta-\phi/4}\tilde{\eta} \cdot \mathrm{Re}\,\tilde{\Omega}_2\right) = 0, \tag{4.42}$$

where $\tilde{\mathcal{J}}_1$ is the reduction to the transverse section of the generalized complex structure $\mathcal{J}_-^{b_2} = e^{b_2}\mathcal{J}_-e^{-b_2}$, and we have used the compatibility relation $\tilde{\mathcal{J}}_1 \cdot \tilde{\Omega}_2 = 0$.[7]

The conditions (4.39) and (4.42), which do not involve the flux, can be viewed as a generalization of the usual Kähler-Einstein conditions. Given an $\tilde{\Omega}_2$ satisfying these conditions, the flux is then determined by (4.41).

5. The Pure Spinor Ω_-

The closed pure spinor Ω_- is associated with the integrable generalized complex-structure $\mathcal{J}_-$. The latter in turn holds information regarding BPS operators in the dual field theory. In this section we explore two aspects of this duality. The first is the mesonic moduli space of the dual theory, which is known to correspond to the subspace for which the polyform Ω_- reduces to a three-form. The second is the connection between generalized holomorphic objects and dual BPS operators.

---

[7] Note that in the language defined in Sect. 5.3 below, the condition (4.42) states that $d\left(e^{-\Delta-\phi/4}\tilde{\eta} \cdot \mathrm{Re}\,\tilde{\Omega}_2\right)$ (and hence $\tilde{G}$) is an element of $U^0_{\tilde{\mathcal{J}}_1}$.

5.1. The general form of Ω_-. Recall that the most general pure spinor takes the form [29]

$$\Omega = \alpha \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_k \wedge e^{-b+i\omega^0}, \tag{5.1}$$

where α is some complex function, θ_i are complex one-forms, while b and ω^0 are both real two-forms. The integer k is called the *type* of the pure spinor, which can change along various subspaces of X.

Using the definition of $\Omega_\pm$, the Fierz identity (2.31), and the results of Sect. 3.1 and Appendix A, one can find expressions for $\Omega_\pm$ in terms of spinor bilinears introduced in [19]. We find that in general Ω_- is of type one with

$$\Omega_- = \theta \wedge e^{-b_-+i\omega_-}, \tag{5.2}$$

where

$$\theta = -\frac{r^3}{8}e^{4\Delta}(iK + S d\log r),$$

$$\omega_- = \frac{4e^{6\Delta+\phi/2}}{f(\sin 2\bar\phi)^2}\left(K_5 \wedge \mathrm{Im}(K_3) - \cos 2\bar\theta \cos 2\bar\phi \,\mathrm{Re}(K_3) \wedge d\log r\right), \tag{5.3}$$

$$b_- = -\frac{4e^{6\Delta+\phi/2}}{f(\sin 2\bar\phi)^2}\left(K_4 \wedge \mathrm{Re}(K_3) + (\cos 2\bar\phi)^2 \,\mathrm{Im}(K_3) \wedge d\log r\right) + b_2.$$

Note that ω_- and b_- are not uniquely defined since we can add two-forms that vanish when wedged with θ. Here the angles $\bar\theta$ and $\bar\phi$, which appear in Appendix B of [19] without bars, are functions on the link Y that are related to the scalar spinor bilinears through

$$\sin\zeta = \cos 2\bar\theta \cos 2\bar\phi, \tag{5.4}$$

$$|S| = -\sin 2\bar\theta \cos 2\bar\phi. \tag{5.5}$$

Using the results of [19], we have the important result that θ is exact[8]

$$\theta = d\left[-\frac{1}{24}e^{4\Delta}r^3 S\right] \equiv d\left(r^3\theta_0\right). \tag{5.6}$$

Alternatively, from the supersymmetry equation $d\Omega_- = 0$ and the definite scaling dimension $\mathcal{L}_{r\partial_r}\Omega_- = 3\Omega_-$, we immediately obtain

$$\Omega_- = \tfrac{1}{3}d(r\partial_r \lrcorner \Omega_-), \tag{5.7}$$

the one-form part of which reduces to (5.6).

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[8] The fact that θ is closed was essentially observed in [37], and it was also shown to be exact in the special cases of the Pilch-Warner and Lunin-Maldacena solutions in [20].

5.2. Type change of Ω_- and the mesonic moduli space. The pure spinor Ω_- has the property that its type can jump from type one to type three on the locus $\theta = 0$. This locus can be neatly parameterized through the angles $\bar{\theta}$ and $\bar{\phi}$. Assuming $f \neq 0$, we have from (3.11) that $\sin \zeta$ is nowhere zero and then (5.4) implies that both $\cos 2\bar{\phi}$ and $\cos 2\bar{\theta}$ are nowhere zero. Using the expression for K in Appendix B of [19], we see that when $f \neq 0$,

$$\sin 2\bar{\theta} = 0 \iff \theta_0 = 0, \tag{5.8}$$

$$\sin 2\bar{\theta} = \sin 2\bar{\phi} = 0 \iff \theta = 0. \tag{5.9}$$

The locus $\theta = 0$ is thus a sublocus of $\theta_0 = 0$. Notice that, where $\theta = 0$, Ω_- is not identically zero, as one might have naively expected from (5.2), but instead reduces to a finite, non-zero three-form. Indeed, the powers of $\sin 2\bar{\phi}$ in the denominator of b_- and ω_- are cancelled by those in K, K_3 and K_4.

The locus $\theta = 0$ is precisely where a probe pointlike D3-brane in X is supersymmetric. This follows from [37] where it was shown that the pull-back of θ to the D3-brane worldvolume is equal to the F-term of the worldvolume theory. The supersymmetric locus of such a pointlike D3-brane is naturally interpreted as the mesonic moduli space.

5.3. BPS operators and generalized holomorphic spinors. In the Sasaki-Einstein case, holomorphic functions on the Calabi-Yau cone with a definite scaling weight λ under the action of $r\partial_r$ also have a charge λ under the action of ξ. This stems from the intimate connection (via Kaluza-Klein reduction on the Sasaki-Einstein manifold) between holomorphic functions on the cone and BPS operators in the dual CFT, in fact (anti-)chiral primary operators. For general AdS_5 solutions we might expect that the holomorphic functions should be replaced by polyforms and that the BPS condition of matching charges should be with respect to the generalized Lie derivative $\mathbb{L}$ discussed in Sect. 2. We now derive such a result, leaving the detailed connection with Kaluza-Klein reduction on the internal space Y to future work.

We first recall that a generalized almost complex structure $\mathcal{J}$ defines a grading on generalized spinors, or equivalently differential forms. If $\Omega \in \Gamma(S_\pm(E))$ is a pure spinor corresponding to $\mathcal{J}$, one defines the canonical pure spinor line bundle $U_{\mathcal{J}}^n \subset S_\pm(E)$ as sections of the form $\varphi = f\Omega$ for some function f. One can then define

$$U_{\mathcal{J}}^{(n-k)} = \wedge^k \bar{L} \otimes U_{\mathcal{J}}^n. \tag{5.10}$$

Elements of $U_{\mathcal{J}}^k$ have eigenvalues $\mathrm{i}k$ under the Lie algebra action of $\mathcal{J}$ given in (2.40). These bundles then give a grading of the spinor bundles $S_\pm(E)$. A generalized vector $V \in \Gamma(E)$ acting on an element of $U_{\mathcal{J}}^k$ gives an element of $U_{\mathcal{J}}^{k+1} \oplus U_{\mathcal{J}}^{k-1}$. In particular an annihilator of Ω acts by purely raising the level by one. If the generalized complex structure $\mathcal{J}$ is also integrable then the exterior derivative splits into the sum

$$\mathrm{d} = \partial_{\mathcal{J}} + \bar{\partial}_{\mathcal{J}}, \tag{5.11}$$

where

$$C^\infty\left(U_{\mathcal{J}}^k\right) \underset{\partial_{\mathcal{J}}}{\overset{\bar{\partial}_{\mathcal{J}}}{\leftrightarrows}} C^\infty\left(U_{\mathcal{J}}^{k-1}\right). \tag{5.12}$$

Consider now a spinor ψ satisfying

$$\psi \in U^k_{\mathcal{J}_-},$$
$$\mathbb{L}_{r\partial_r}\psi = \lambda\psi, \tag{5.13}$$

for some k and λ. Then imposing in addition

$$\bar\partial_{\mathcal{J}_-}\psi = 0, \quad (r\partial_r + \mathrm{i}\xi)\cdot\psi = 0 \quad \text{implies} \quad \mathbb{L}_\xi\psi = \mathrm{i}\mathbb{L}_{r\partial_r}\psi. \tag{5.14}$$

In other words, subject to the constraints (5.13), a spinor is BPS if it is generalized holomorphic and is annihilated by $r\partial_r + \mathrm{i}\xi$. To see this result, we first write $r\partial_r = (1/2)(r\partial_r + \mathrm{i}\xi) + (1/2)(r\partial_r - \mathrm{i}\xi)$ and use (5.13) to deduce that

$$\partial_{\mathcal{J}_-}[(r\partial_r + \mathrm{i}\xi)\cdot\psi] + (r\partial_r + \mathrm{i}\xi)\cdot\partial_{\mathcal{J}_-}\psi = 0,$$
$$\bar\partial_{\mathcal{J}_-}[(\partial_r - \mathrm{i}\xi)\cdot\psi] + (r\partial_r - \mathrm{i}\xi)\cdot\bar\partial_{\mathcal{J}_-}\psi = 0. \tag{5.15}$$

In obtaining this we used the fact that since $r\partial_r - \mathrm{i}\xi$ is an annihilator of Ω_- it raises the level of ψ and similarly $r\partial_r + \mathrm{i}\xi$ lowers the level. We then compute

$$\mathbb{L}_\xi\psi = \mathrm{i}\mathbb{L}_{r\partial_r}\psi - \mathrm{i}\{\mathrm{d}\,[(r\partial_r + \mathrm{i}\xi)\cdot\psi] + (r\partial_r + \mathrm{i}\xi)\cdot\mathrm{d}\psi\}$$
$$= \mathrm{i}\mathbb{L}_{r\partial_r}\psi - \mathrm{i}\{\bar\partial_{\mathcal{J}_-}[(r\partial_r + \mathrm{i}\xi)\cdot\psi] + (r\partial_r + \mathrm{i}\xi)\cdot\bar\partial_{\mathcal{J}_-}\psi\}. \tag{5.16}$$

In a similar way, given (5.13) we also have

$$\partial_{\mathcal{J}_-}\psi = 0, \quad (r\partial_r - \mathrm{i}\xi)\cdot\psi = 0 \quad \text{implies} \quad \mathbb{L}_\xi\psi = -\mathrm{i}\mathbb{L}_{r\partial_r}\psi. \tag{5.17}$$

6. The Pure Spinor Ω_+

6.1. The general form of Ω_+. One can see immediately from the supersymmetry equation (3.39) that if we assume $F_5 \neq 0$, which we shall do, then $\mathrm{Im}\,\Omega_+$ must have a scalar component and hence Ω_+ is of type 0:

$$\Omega_+ = \alpha_+ e^{-b_+ + \mathrm{i}\omega^0_+}. \tag{6.1}$$

Using the same procedure as in the last section, we may again express these quantities in terms of the bilinears of [19]. After defining the rescaled two-form

$$\omega = e^{-2A}r^4\omega^0_+, \tag{6.2}$$

we find

$$\alpha_+ = -\frac{\mathrm{i}}{32}fe^{-A}r^4,$$
$$\omega = -\frac{4r^2}{f}e^{4\Delta}\left(V + K_4 \wedge \mathrm{d}\log r\right), \tag{6.3}$$
$$b_+ = e^{6\Delta+\phi/2}\frac{4}{f}\,\mathrm{Im}\,K_3 \wedge \mathrm{d}\log r + b_2,$$

where b_2 appears in (3.12).

6.2. A canonical symplectic structure. The rescaling (6.2) is motivated by the fact that ω defines a canonical symplectic structure. To see this, we first observe that Y admits a contact structure defined by the one-form

$$\sigma \equiv \frac{4}{f} e^{4\Delta} K_4. \tag{6.4}$$

Recall that for a one-form σ to be contact, the top-degree form $\sigma \wedge d\sigma \wedge d\sigma$ must be nowhere vanishing. Using (3.19) of [19], and results in Appendix B of [19], one can easily show that

$$\sigma \wedge d\sigma \wedge d\sigma = \frac{128}{f^2} e^{8\Delta} \widetilde{\text{vol}}_Y = \frac{8}{\sin^2 \zeta} \widetilde{\text{vol}}_Y, \tag{6.5}$$

where recall $\widetilde{\text{vol}}_Y = -e^{12345}$ (using the orthonormal frame in Appendix B of [19]). We then observe, using (3.19) of [19], that

$$\omega = \tfrac{1}{2} d(r^2 \sigma), \tag{6.6}$$

which shows that ω is closed and non-degenerate, and hence defines a symplectic structure on the cone $X = \mathbb{R}^+ \times Y$. Alternatively, one can see the formula (6.6) for ω directly from the supersymmetry equation (3.38) on noting that $e^{-A} \Omega_+$ has scaling dimension 2 under $r \partial_r$. Furthermore, again using the results of Appendix B of [19], we have

$$1 = \xi_v \lrcorner \sigma, \qquad 0 = \xi_v \lrcorner d\sigma, \tag{6.7}$$

which shows that ξ_v is also the unique "Reeb vector field" associated with the contact structure. Notice also that (6.6) implies that $\mathcal{H} = r^2/2$ is precisely the Hamiltonian function for the Hamiltonian vector field ξ_v, *i.e.* $d\mathcal{H} = -i_{\xi_v} \omega$. It is remarkable that these features, which are well-known in the Sasaki-Einstein case, are valid for all supersymmetric AdS_5 solutions with non-vanishing five-form flux.

Although we have a symplectic structure, we do not quite have a Kähler structure, as in the Calabi-Yau case, but it is quite close. Using the last equation in (3.50) and the definition (6.4) we see that

$$\eta_f = \sigma + i_{\eta_v} b_2, \tag{6.8}$$

and thus $\left(e^{b_2} \eta \right) |_{1-\text{form}} = \sigma$. Since $e^{b_2}(d \log r) = d \log r$ manifestly, and by definition $\eta \equiv \mathcal{J}_-(d \log r)$, we have, using (2.34),

$$\sigma = \mathcal{J}_-^{b_2}(d \log r) |_{1-\text{form}} = -(I_-^{b_2})^*(d \log r). \tag{6.9}$$

Note this is precisely analogous to the formula for the contact form in the Sasakian case. We then have

$$d^{\mathcal{J}_-^{b_2}} r^2 = -r^2 \, d \left(Q_-^{b_2} + \frac{1}{2} \text{Tr} \, I_-^{b_2} \right) - (I_-^{b_2})^*(d(r^2)), \tag{6.10}$$

where here we recall that in general we define $d^{\mathcal{J}_-} \equiv -[d, \mathcal{J}_-]$, and we use (2.40) for the action on generalized spinors. From this it follows that

$$\omega = \frac{1}{4} dd^{\mathcal{J}_-^{b_2}} r^2 + \frac{1}{4} d(r^2) \wedge d \left(Q_-^{b_2} + \frac{1}{2} \text{Tr} \, I_-^{b_2} \right). \tag{6.11}$$

Thus r^2 is almost a Kähler potential, for the b_2-transformed complex structure $\mathcal{J}_-^{b_2} = e^{b_2} \mathcal{J}_- e^{-b_2}$, except for the last term.

6.3. The central charge as a Duistermaat-Heckman integral. Recall that in any four-dimensional CFT there are two central charges, usually called a and c, that are constant coefficients in the conformal anomaly

$$\langle T^\mu_\mu \rangle = \frac{1}{120(4\pi)^2} \left(c(\text{Weyl})^2 - \frac{a}{4}(\text{Euler}) \right). \tag{6.12}$$

Here $T_{\mu\nu}$ denotes the stress-energy tensor, and Weyl and Euler denote certain curvature invariants for the background four-dimensional metric. For SCFTs, both a and c are related to the R-symmetry [8] via

$$a = \frac{3}{32} \left(3\text{Tr}R^3 - \text{Tr}R \right), \qquad c = \frac{1}{32} \left(9\text{Tr}R^3 - 5\text{Tr}R \right). \tag{6.13}$$

Here the trace is over the fermions in the theory. For SCFTs with AdS_5 gravity duals, in fact $a = c$ holds necessarily in the large N limit [38]. The central charge of the SCFT is then inversely proportional to the dual five-dimensional Newton constant G_5 [38], obtained here by Kaluza-Klein reduction on Y. The Newton constant, in turn, was computed in Appendix E of [19], and is given by

$$G_5 = \frac{G_{10}}{V_5} = \frac{\kappa_{10}^2}{8\pi V_5}, \tag{6.14}$$

where G_{10} is the ten-dimensional Newton constant of type IIB supergravity, and we have defined

$$V_5 \equiv \int_Y e^{8\Delta} \widetilde{\text{vol}}_Y. \tag{6.15}$$

We may derive an alternative formula for G_5 as follows. We begin by rewriting

$$V_5 = \frac{f^2}{16} \int_Y \frac{1}{\sin^2 \zeta} \widetilde{\text{vol}}_Y, \tag{6.16}$$

where we have used the relation (3.11). Importantly, the constant f is quantized, being essentially the number of D3-branes N. Specifically, we have

$$N = \frac{1}{(2\pi l_s)^4 g_s} \int_Y dC_4 = \frac{1}{(2\pi l_s)^4 g_s} \int_Y (F_5 + H \wedge C_2). \tag{6.17}$$

Using the Bianchi identity $DG = -P \wedge G^*$ and the result (A.22), one derives that $d(H \wedge C_2) = -(2/f)d[e^{6\Delta}\text{Im}(W^* \wedge G)]$, and so we can also write

$$N = \frac{1}{(2\pi l_s)^4 g_s} \int_Y \left(F_5 - \frac{2e^{6\Delta}}{f} \text{Im}\left[W^* \wedge G \right] \right). \tag{6.18}$$

We may evaluate this expression in terms of the orthonormal basis of forms e^i introduced in Appendix B of [19], and after some calculation we find

$$N = -\frac{f}{(2\pi l_s)^4 g_s} \int_Y \frac{1}{\sin^2 \zeta} \widetilde{\text{vol}}_Y. \tag{6.19}$$

Combining these formulae and using $2\kappa_{10}^2 = (2\pi)^7 l_s^8 g_s^2$ leads to the result

$$G_5 = \frac{8V_5}{\pi^2 f^2 N^2}. \tag{6.20}$$

Consider now the integral

$$\mu = \frac{1}{(2\pi)^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!}. \tag{6.21}$$

This is the Duistermaat-Heckman integral for a symplectic manifold (X, ω) with Hamiltonian function $\mathcal{H} = r^2/2$, which we have shown is the Hamiltonian for the Reeb vector field ξ_v. Using (6.6) and (6.5) we may rewrite

$$\frac{\omega^3}{3!} = \frac{16}{f^2} e^{8\Delta} r^5 dr \wedge \widetilde{\text{vol}}_Y. \tag{6.22}$$

Performing the r-integral in (6.21) allows us to rewrite the five-dimensional Newton constant as

$$G_5 = \frac{\pi \mu}{2N^2}. \tag{6.23}$$

Since $\mu = 1$ for the round five-sphere solution, we thus obtain the ratio $\frac{G_5}{G_{S^5}} = \mu$. Recalling that this is, by AdS/CFT duality, the inverse ratio of central charges [38], we deduce the key result

$$\frac{a_{\mathcal{N}=4}}{a} = \frac{1}{(2\pi)^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!} = \frac{1}{(2\pi)^3} \int_Y \sigma \wedge d\sigma \wedge d\sigma. \tag{6.24}$$

Here $a_{\mathcal{N}=4} = N^2/4$ denotes the (large N) central charge of $\mathcal{N} = 4$ super-Yang-Mills theory.

The formula (6.24) implies that the central charge depends only on the symplectic structure of the cone (X, ω) and the Reeb vector field ξ_v. This is perhaps surprising: one might have anticipated that the quantum numbers of quantized fluxes would appear explicitly in the central charge formula. However, recall from formulae (3.12), (3.13) that the two-form potentials B and C_2 are globally defined. In particular, for example, the period of $H = dB$ through any three-cycle in Y is zero.

As discussed in [11], the Duistermaat-Heckman integral in (6.24) may be evaluated by localization. The integral localizes where $\xi_v = 0$, which is formally at the tip of the cone $r = 0$. Unless the differentiable and symplectic structure is smooth here (which is only the case when $X \cup \{r = 0\}$ is diffeomorphic to $\mathbb{R}^6$), one needs to equivariantly resolve the singularity in order to apply the localization formula. Notice here that since ξ_v preserves all the structure on the compact manifold (Y, g_Y, σ), the closure of its orbits defines a $U(1)^s$ action preserving all the structure, for some $s \geq 1$. Here we have used the fact that the isometry group of a compact Riemannian manifold is compact. Thus (X, ω) comes equipped with a $U(1)^s$ action.

Rather than attempt to describe this in general, we focus here on the special case where the solution is *toric*: that is, there is a $U(1)^3$ action on Y under which σ, and hence ω under the lift to X, is invariant. Notice that we do not necessarily require that the *full* supergravity solution is invariant under $U(1)^3$ – we shall illustrate this in the next section with the Pilch-Warner solution, where σ and the metric are invariant under

$U(1)^3$, but the G-flux is invariant only under a $U(1)^2$ subgroup. For the arguments that follow, it is only σ (and hence ω) that we need to be invariant under a maximal dimension torus $U(1)^3$. In fact any such symplectic toric cone is also an affine toric variety. This implies that there is a (compatible) complex structure on X, and that the $U(1)^3$ action complexifies to a holomorphic $(\mathbb{C}^*)^3$ action with a dense open orbit. There is then always a symplectic toric resolution (X', ω') of (X, ω), obtained by toric blow-up. In physics language, this is because one can realize (X, ω) as a gauged linear sigma model, and one obtains (X', ω') by simply turning on generic Fayet-Iliopoulos parameters. One can also describe this in terms of moment maps as follows. The image of a symplectic toric cone under the moment map $\mu : X \to \mathbb{R}^3$ is a strictly convex rational polyhedral cone (see [11]). Choosing a toric resolution (X', ω') then amounts to choosing any simplicial resolution $\mathcal{P}$ of this polyhedral cone. Here $\mathcal{P}$ is the image of $\mu' : X' \to \mathbb{R}^3$. Assuming the fixed points of ξ_v are all isolated, the localization formula is then simply [11]

$$\frac{1}{(2\pi)^3} \int_X e^{-r^2/2} \frac{\omega^3}{3!} = \sum_{\text{vertices } p \in \mathcal{P}} \prod_{i=1}^{3} \frac{1}{\langle \xi_v, u_i^p \rangle}. \tag{6.25}$$

Here u_i^p, $i = 1, 2, 3$, are the three edge vectors of the moment polytope $\mathcal{P}$ at the vertex point p, and $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product on $\mathbb{R}^3$ (where we regard ξ_v as being an element of the Lie algebra $\mathbb{R}^3$ of $U(1)^3$). The vertices of $\mathcal{P}$ precisely correspond to the $U(1)^3$ fixed points of the symplectic toric resolution $X' = X_{\mathcal{P}}$ of X. Thus, remarkably, these results of [11] hold in general, even when there are non-trivial fluxes turned on and X is not Calabi-Yau.

6.4. The conformal dimensions of BPS branes. A supersymmetric D3-brane wrapped on $\Sigma_3 \subset Y$ gives rise to a BPS particle in *AdS*₅. The quantum field Φ whose excitations give rise to this particle state then couples, in the usual way in AdS/CFT, to a dual chiral primary operator $\mathcal{O} = \mathcal{O}_{\Sigma_3}$ in the boundary SCFT. More precisely, there is an asymptotic expansion of Φ near the *AdS*₅ boundary

$$\Phi \sim \Phi_0 r^{\Delta-4} + A_\Phi r^{-\Delta}, \tag{6.26}$$

where Φ_0 acts as the source for $\mathcal{O}$ and $\Delta = \Delta(\mathcal{O})$ is the conformal dimension of $\mathcal{O}$. In [39], following [40], it was argued that the vacuum expectation value A_Φ of $\mathcal{O}$ in a given asymptotically *AdS*₅ background may be computed from e^{-S_E}, where S_E is the on-shell Euclidean action of the D3-brane wrapped on $\Sigma_4 = \mathbb{R}^+ \times \Sigma_3$, where $\mathbb{R}^+$ is the r-direction. In particular, via the second term in (6.26) this identifies the conformal dimension $\Delta = \Delta(\mathcal{O}_{\Sigma_3})$ with the coefficient of the logarithmically divergent part of the on-shell Euclidean action of the D3-brane wrapped on Σ_4. We refer to Sect. 2.3 of [39] for further details.

We are thus interested in the on-shell Euclidean action of a supersymmetric D3-brane wrapped on $\Sigma_4 = \mathbb{R}^+ \times \Sigma_3$. The condition of supersymmetry is equivalent to a generalized calibration condition, namely Eq. (3.16) of [35]. In our notation and conventions, this calibration condition reads

$$\text{Re}\left[-i\Phi_+ \wedge e^{\mathcal{F}} \right] |_{\Sigma_4} = \frac{|a|^2}{8} \sqrt{\det(h + \mathcal{F})} \, dx_1 \wedge \cdots \wedge dx_4. \tag{6.27}$$

Here h is the induced (string frame) metric on Σ_4, and $\mathcal{F} = F - B$ is the gauge-invariant worldvolume gauge field, satisfying

$$\mathrm{d}\mathcal{F} = -H \mid_{\Sigma_4}. \tag{6.28}$$

Recalling from Sect. 3.2 that $|a|^2 = \mathrm{e}^A$, we may then substitute for Φ_+ in terms of Ω_+ using (3.36) and (6.1) to obtain

$$\mathrm{Re}\left[-\mathrm{i}\Phi_+ \wedge \mathrm{e}^{\mathcal{F}}\right] \mid_{\Sigma_4} = \frac{f}{64}\mathrm{e}^{A+\phi}\mathrm{d}\log r \wedge \sigma \wedge \mathrm{d}\sigma \mid_{\Sigma_4} -\frac{f}{64}\mathrm{e}^{-3A+\phi}r^4(F - b_+)^2 \mid_{\Sigma_4}, \tag{6.29}$$

where, as in (6.3),

$$b_+ = \mathrm{e}^{6\Delta+\phi/2}\frac{4}{f}\,\mathrm{Im}\,K_3 \wedge \mathrm{d}\log r + b_2. \tag{6.30}$$

Here b_2 is a closed two-form, whose gauge-invariant information is contained in its cohomology class in $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$. In writing b_+ in (6.29) we have chosen a particular representative two-form for the class of b_2 in $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$. Then under any gauge transformation of b_+ (induced from a B-transform of Ω_+), the worldvolume gauge field F transforms by precisely the opposite gauge transformation restricted to Σ_4, so that the quantity $F - b_+$ is gauge invariant on Σ_4. We now choose the worldvolume gauge field F to be

$$F = b_2 \mid_{\Sigma_4}, \tag{6.31}$$

so that (6.29) becomes simply

$$\mathrm{Re}\left[-\mathrm{i}\Phi_+ \wedge \mathrm{e}^{\mathcal{F}}\right] \mid_{\Sigma_4} = \frac{f}{64}\mathrm{e}^{A+\phi}\mathrm{d}\log r \wedge \sigma \wedge \mathrm{d}\sigma \mid_{\Sigma_4}. \tag{6.32}$$

In fact, there is a slight subtlety in (6.31). If the cohomology class of $b_2/(2\pi l_s)^2 \mid_{\Sigma_4}$ in $H^2(\Sigma_4, \mathbb{R})$ is not integral, then the choice (6.31) is not possible as F is the curvature of a unitary line bundle. Having said this, notice $H^2(\Sigma_4, \mathbb{R}) \cong H^2(\Sigma_3, \mathbb{R})$, and thus in particular that if $H^2(\Sigma_3, \mathbb{R}) = 0$ then every closed $b_2 \mid_{\Sigma_4}$ is exact, and thus may be gauge transformed to zero on Σ_4. Then (6.31) simply sets $F = 0$. For every example of a supersymmetric Σ_3 that we are aware of, this is indeed the case. In any case, we shall assume henceforth that the choice (6.31) is possible.

The calibration condition (6.27) for a D3-brane with worldvolume Σ_4 and with gauge field (6.31) is thus

$$\frac{f}{8}\mathrm{d}\log r \wedge \sigma \wedge \mathrm{d}\sigma = \mathrm{e}^{-\phi}\sqrt{\det(h - B)}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_4. \tag{6.33}$$

Notice the right hand side is precisely the Dirac-Born-Infeld Lagrangian, up to the D3-brane tension $\tau_3 = 1/(2\pi)^3 l_s^4 g_s$. From (6.33), and the comments above on the scaling dimension $\Delta(\mathcal{O}(\Sigma_3))$ of the dual operator $\mathcal{O}(\Sigma_3)$, we thus deduce

$$\Delta(\mathcal{O}(\Sigma_3)) = -\frac{\tau_3 f}{8}\int_{\Sigma_3}\sigma \wedge \mathrm{d}\sigma. \tag{6.34}$$

(The sign just arising from a convenient choice of orientation.) Using (6.19) and (6.5) we have

$$f = -\frac{8(2\pi l_s)^4 g_s N}{\int_Y \sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma},$$

(6.35)

and hence

$$\Delta(\mathcal{O}(\Sigma_3)) = \frac{2\pi N \int_{\Sigma_3} \sigma \wedge \mathrm{d}\sigma}{\int_Y \sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma}.$$

(6.36)

This is our final formula for the conformal dimension of the chiral primary operator dual to a BPS D3-brane wrapped on Σ_3. Since we may write

$$\int_{\Sigma_3} \sigma \wedge \mathrm{d}\sigma = \int_{\Sigma_4} \mathrm{e}^{-r^2/2} \frac{\omega^2}{2!},$$

(6.37)

we again see that it depends only on the symplectic structure of (X, ω) and the Reeb vector field ξ_v. This again may be evaluated by localization, having appropriately resolved the tip of the cone Σ_4.

7. The Pilch-Warner Solution

In this section we illustrate the general results derived so far with the Pilch-Warner solution [13]. (Some aspects of the generalized complex geometry of this background have already been discussed in [20].) Recall that the Pilch-Warner solution is dual to a Leigh-Strassler fixed point theory [16] which is obtained by giving a mass to one of the three chiral superfields (in $\mathcal{N} = 1$ language) of $\mathcal{N} = 4\ SU(N)$ super-Yang-Mills theory, and following the resulting renormalization group flow to the IR fixed point theory. This latter theory is an $\mathcal{N} = 1\ SU(N)$ gauge theory with two adjoint fields Z_a, $a = 1, 2$, which form a doublet under an $SU(2)$ flavour symmetry, and a quartic superpotential. Since the superpotential has scaling dimension three, this fixes $\Delta(Z_a) = 3/4$, implying that the IR theory is strongly coupled. The mesonic moduli space is simply $\mathrm{Sym}^N \mathbb{C}^2$.

The Pilch-Warner supergravity solution [13] was rederived in [19], and we shall use some of the results from that reference also. We have $Y = S^5$ with non-trivial metric,

$$g_Y = \frac{1}{9} \left[6\mathrm{d}\vartheta^2 + \frac{6\cos^2\vartheta}{3 - \cos 2\vartheta}(\sigma_1^2 + \sigma_2^2) + \frac{6\sin^2 2\vartheta}{(3 - \cos 2\vartheta)^2}\sigma_3^2 \right.$$
$$\left. +4\left(\mathrm{d}\varphi + \frac{2\cos^2\vartheta}{3 - \cos 2\vartheta}\sigma_3\right)^2 \right],$$

(7.1)

where $0 \leq \vartheta \leq \frac{\pi}{2}$, $0 \leq \varphi \leq 2\pi$, and σ_i, $i = 1, 2, 3$, are left-invariant one-forms on $SU(2)$ (denoted with hats in [19]). The dilaton ϕ and axion C_0 are simply constant, while the warp factor is

$$\mathrm{e}^{4\Delta} = \frac{f}{8}(3 - \cos 2\vartheta).$$

(7.2)

There is also a non-trivial NS and RR three-form flux given by (see (A.7))

$$G = \frac{(2f)^{1/2}}{3^{3/2}}\mathrm{e}^{2\mathrm{i}\varphi}\cos\vartheta\left(\mathrm{d}\varphi \wedge \mathrm{d}\vartheta - \frac{\mathrm{i}\sin 2\vartheta}{3 - \cos 2\vartheta}\mathrm{d}\varphi \wedge \sigma_3\right.$$
$$\left. -\frac{4\cos^2\vartheta}{(3 - \cos 2\vartheta)^2}\mathrm{d}\vartheta \wedge \sigma_3\right) \wedge (\sigma_2 - \mathrm{i}\sigma_1).$$

(7.3)

We introduce the Euler angles (α, β, γ) on $SU(2)$ (as in [19]), so that

$$
\begin{aligned}
\sigma_1 &= -\sin \gamma \, \mathrm{d}\alpha - \cos \gamma \sin \alpha \, \mathrm{d}\beta, \\
\sigma_2 &= \cos \gamma \, \mathrm{d}\alpha - \sin \gamma \sin \alpha \, \mathrm{d}\beta, \\
\sigma_3 &= \mathrm{d}\gamma - \cos \alpha \, \mathrm{d}\beta.
\end{aligned}
\tag{7.4}
$$

In terms of these coordinates, the R-symmetry vector ξ_v is [19]

$$
\xi_v = \frac{3}{2} \partial_\varphi - 3 \partial_\gamma.
\tag{7.5}
$$

Using the explicit formulae in [19], it is easy to show that the contact form is

$$
\sigma = -\frac{2}{3} \left(\cos 2\vartheta \, \mathrm{d}\varphi + \cos^2 \vartheta \, \sigma_3 \right).
\tag{7.6}
$$

The solution is toric, in the sense that both σ and the metric are invariant under shifts of φ, β and γ. However, notice that the G-flux in (7.3) is not invariant under shifts of φ, thus breaking this $U(1)^3$ symmetry to only a $U(1)^2$ symmetry of the full supergravity solution. This is expected, since the dual field theory described above has only an $SU(2) \times U(1)_R$ global symmetry.

On $Y = S^5$ there are precisely three invariant circles under the $U(1)^3$ action, where two of the $U(1)$ actions degenerate, namely at $\{\vartheta = \frac{\pi}{2}\}$, $\{\vartheta = 0, \alpha = 0\}$, $\{\vartheta = 0, \alpha = \pi\}$. A set of 2π-period coordinates on $U(1)^3$ are

$$
\varphi_1 = \varphi, \quad \varphi_2 = -\frac{1}{2}(\varphi + \gamma - \beta), \quad \varphi_3 = -\frac{1}{2}(\varphi + \gamma + \beta).
\tag{7.7}
$$

These restrict to coordinates on the above three invariant circles, respectively. On $X \cong \mathbb{R}^6 \backslash 0$ we also have three corresponding moment maps:

$$
\mu_1 = \frac{r^2}{3} \sin^2 \vartheta, \quad \mu_2 = \frac{r^2}{3} \cos^2 \vartheta (1 + \cos \alpha), \quad \mu_3 = \frac{r^2}{3} \cos^2 \vartheta (1 - \cos \alpha),
\tag{7.8}
$$

so that $\omega = \frac{1}{2} \mathrm{d}(r^2 \sigma) = \sum_{i=1}^3 \mathrm{d}\mu_i \wedge \mathrm{d}\varphi_i$. It follows that the image of the moment map – the space spanned by the μ_i coordinates – is the cone $(\mathbb{R}_{\geq 0})^3$, where the three invariant circles map to the three generating rays $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (0, 0, 1)$. The Reeb vector (7.5) in this basis is then computed to be

$$
\xi = \frac{3}{2} \partial_\varphi - 3 \partial_\gamma = \frac{3}{2} \partial_{\varphi_1} + \frac{3}{4} \partial_{\varphi_2} + \frac{3}{4} \partial_{\varphi_3}.
\tag{7.9}
$$

Since the symplectic structure is smooth at $r = 0$, we may evaluate (6.25) by localization without having to resolve X at $r = 0$. In the case at hand, we have the single fixed point at $r = 0$, and from (7.9) one obtains the known result

$$
\frac{a_{\mathcal{N}=4}}{a_{\mathrm{PW}}} = \frac{1}{\xi_1 \xi_2 \xi_3} = \frac{32}{27}.
\tag{7.10}
$$

They key point about the above calculation is that we have computed this knowing only the symplectic structure of the solution and the Reeb vector field ξ_v.

We may similarly compute the conformal dimensions of the operators $\det Z_a$, using (6.36), by interpreting them as arising from a BPS D3-brane wrapped on the three-spheres at $\alpha = 0$ and $\alpha = \pi$, respectively. It is simple to check these indeed satisfy the

calibration condition (6.33) and are thus supersymmetric. Using (6.37) and localization at $r = 0$ implies that (6.37) is equal to $1/\xi_1\xi_2$, $1/\xi_1\xi_3$, respectively, which in both cases is 8/9. The formula (6.36) thus gives $\Delta(\det Z_a) = 3N/4$, or equivalently $\Delta(Z_a) = 3/4$, which is indeed the correct result.

Next recall that the complex one-form $\theta = \mathrm{d}(r^3\theta_0)$, where $\theta_0 = -\frac{1}{24}\mathrm{e}^{4\Delta}S$, and the mesonic moduli space should be the locus $\theta = 0$. As discussed in Sect. 5.2, this is the locus $\sin 2\bar{\theta} = \sin 2\bar{\phi} = 0$. For the Pilch-Warner solution, we may easily compute

$$\sin 2\bar{\theta} = -\frac{\sqrt{3}\sin^2\vartheta}{\sqrt{1+3\sin^4\vartheta}}, \quad \cos 2\bar{\phi} = \frac{\sqrt{1+3\sin^4\vartheta}}{1+\sin^2\vartheta}. \tag{7.11}$$

Thus, as discussed in [20], the mesonic moduli space $S = 0$ is equivalent to $\vartheta = 0$, which is a codimension two submanifold in $\mathbb{R}^6$ diffeomorphic to $\mathbb{R}^4$. Moreover, this is $\mathbb{C}^2$ in the induced complex structure, and we thus see explicit agreement with the field theory $N = 1$ mesonic moduli space.

Finally, although the Pilch-Warner solution is generalized complex, rather than complex, we note that one can nevertheless define a natural complex structure [41]. The relation between this integrable complex structure and the generalized geometry has been discussed in [20]. Let us conclude this section by elucidating this connection. One can introduce the following complex coordinates [20] in terms of the angular variables (7.7):

$$\begin{aligned} s_1 &= r^{3/2}\sin\vartheta\,\mathrm{e}^{-\mathrm{i}\varphi_1}, \\ s_2 &= r^{3/4}\cos\vartheta\cos\tfrac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\varphi_2}, \\ s_3 &= r^{3/4}\cos\vartheta\sin\tfrac{\alpha}{2}\,\mathrm{e}^{\mathrm{i}\varphi_3}. \end{aligned} \tag{7.12}$$

This makes $\mathbb{R}^6 \cong \mathbb{C}^3$. However, because of the minus sign in the first coordinate in (7.12), the corresponding integrable complex structure I_* is *not* the unique complex structure that is compatible with the toric structure of the solution: the latter instead has complex coordinates $\bar{s}_1, s_2, s_3$. Indeed, also the Reeb vector field ξ_v is *not* given by $I_*(r\partial_r)$. This makes the physical significance of this complex structure rather unclear. Nevertheless, one can show that I_* does in fact come from an $SU(3)$ structure defined by a Killing spinor. Following [20], we define

$$2\hat{a}\eta_* = \eta_+^1 + \mathrm{i}\eta_+^2 = \mathrm{e}^{A/2}\begin{pmatrix}\xi_2 \\ \mathrm{i}\xi_2\end{pmatrix}, \tag{7.13}$$

where by definition we require $\bar{\eta}_*\eta_* = 1$. It is then convenient to define $\hat{a} \equiv |\hat{a}|\mathrm{e}^{\mathrm{i}z}$, where $|\hat{a}|^2 = \frac{1}{2}\mathrm{e}^A|\xi_2|^2 = \frac{1}{2}\mathrm{e}^A(1-\sin\zeta)$. We then introduce the bilinears corresponding to the $SU(3)$ structure defined by η_*:

$$J_* \equiv -\mathrm{i}\bar{\eta}_*\gamma_{(2)}\eta_*, \tag{7.14}$$

$$\Omega_* \equiv \bar{\eta}_*^c\gamma_{(3)}\eta_*. \tag{7.15}$$

One computes that $\mathrm{d}\Omega_* = 0$, implying that the corresponding complex structure I_* is integrable, and moreover that

$$\mathrm{e}^{2\mathrm{i}z}\Omega_* = -\mathrm{e}^{2\mathrm{i}\alpha}\frac{\sqrt{2}f^{3/2}}{9\mathrm{e}^{3A}}\mathrm{d}s^1 \wedge \mathrm{d}s^2 \wedge \mathrm{d}s^3, \tag{7.16}$$

implying that (7.12) are indeed complex coordinates for this complex structure. We also compute

$$
J_* = -\frac{e^{2A}}{r^2}\left[d\log r \wedge \frac{2}{3}\left(d\varphi + \frac{\cos^2 \vartheta}{(1 + \sin^2 \vartheta)}\sigma_3 \right) + \frac{1}{3(1 + \sin^2 \vartheta)}\left(\sin 2\vartheta\,\sigma_3 \wedge d\vartheta \right.\right.
$$
$$
\left.\left. + \cos^2 \vartheta\,\sigma_1 \wedge \sigma_2 \right) \right]. \tag{7.17}
$$

8. Conclusion

In this paper we have initiated an analysis of the generalized cone geometry associated with supersymmetric $AdS_5 \times Y$ solutions of type IIB supergravity. The cone is generalized Hermitian and generalized Calabi-Yau and we have identified holomorphic generalized vector fields that are dual to the dilatation and R-symmetry of the dual SCFT. We identified a relationship between "BPS polyforms", *i.e.* polyforms with equal R-charge and scaling weight, and generalized holomorphic polyforms that should be worth exploring further. In particular, we would like to make a precise connection between such objects and the spectrum of chiral operators in the SCFT via Kaluza-Klein reduction on Y.

We also showed how one can carry out a generalized reduction of the six-dimensional cone to obtain a new four-dimensional transverse generalized Hermitian geometry. This generalizes the transverse Kähler-Einstein geometry in the Sasaki-Einstein case. By analogy with the Sasaki-Einstein case (*e.g.* [25]) this perspective could be useful for constructing new explicit solutions.

We also analysed the symplectic structure on the cone geometry, which exists providing that the five-form flux is non-vanishing. It would be interesting to know whether or not this includes all solutions. We obtained Duistermaat-Heckman type integrals for the central charge of the dual SCFT and the conformal dimensions of operators dual to BPS wrapped D3-branes. These formulae precisely generalize analogous formulae that were derived in [10,11] for the Sasaki-Einstein case. Other formulae for these quantities were also presented in [10,11] and we expect that these will also have precise generalizations in terms of generalized geometry. In particular, we expect a generalized geometric interpretation of a-maximization.

A. Conventions and 6D to 5D Map

We use exactly[9] the same conventions as in [19], up to some simple relabelling. Here we will explain how the results of that paper concerning the five-dimensional geometry with metric g_Y can be uplifted to six-dimensions. In particular, we will relate the five-dimensional Killing spinors discussed in [19] to the six-dimensional chiral spinors η^i that

---

[9] Although it will not be relevant in this paper we point out that there is a typo in [19]: the ρ_a matrices generating Cliff(4, 1) actually satisfy $\rho_{01234} = -i$.

define the bispinors $\Phi_\pm$. We first recall the Killing spinor equations in five-dimensions, related to the geometry g_Y, given in [19]. There are two differential conditions

$$(\nabla_m - \frac{i}{2}Q_m)\xi_1 + \frac{i}{4}\left(e^{-4\Delta}f - 2\right)\beta_m\xi_1 + \frac{1}{8}e^{-2\Delta}G_{mnp}\beta^{np}\xi_2 = 0, \quad (A.1)$$

$$(\nabla_m + \frac{i}{2}Q_m)\xi_2 - \frac{i}{4}\left(e^{-4\Delta}f + 2\right)\beta_m\xi_2 + \frac{1}{8}e^{-2\Delta}G^*_{mnp}\beta^{np}\xi_1 = 0, \quad (A.2)$$

and four algebraic conditions

$$\beta^m\partial_m\Delta\xi_1 - \frac{1}{48}e^{-2\Delta}\beta^{mnp}G_{mnp}\xi_2 - \frac{i}{4}\left(e^{-4\Delta}f - 4\right)\xi_1 = 0, \quad (A.3)$$

$$\beta^m\partial_m\Delta\xi_2 - \frac{1}{48}e^{-2\Delta}\beta^{mnp}G^*_{mnp}\xi_1 + \frac{i}{4}\left(e^{-4\Delta}f + 4\right)\xi_2 = 0, \quad (A.4)$$

$$\beta^m P_m\xi_2 + \frac{1}{24}e^{-2\Delta}\beta^{mnp}G_{mnp}\xi_1 = 0, \quad (A.5)$$

$$\beta^m P^*_m\xi_1 + \frac{1}{24}e^{-2\Delta}\beta^{mnp}G^*_{mnp}\xi_2 = 0. \quad (A.6)$$

Here[10] the β_m generate the Clifford algebra for g_Y, so $\{\beta_m, \beta_n\} = 2g_{Ymn}$. Equivalently, with respect to any orthonormal frame the corresponding $\hat{\beta}_m$ satisfy $\{\hat{\beta}_m, \hat{\beta}_n\} = 2\delta_{mn}$. We have chosen $\hat{\beta}_{12345} = +1$. In addition we have set the parameter m in [19] to be $m = 1$, consistent with (3.2). In the usual string theory variables we have

$$\begin{aligned} P &= \frac{1}{2}d\phi + \frac{i}{2}e^\phi F_1, \\ Q &= -\frac{1}{2}e^\phi F_1, \\ G &= -ie^{\phi/2}F_3 - e^{-\phi/2}H, \end{aligned} \quad (A.7)$$

where the RR field strengths F_n are defined by (3.3). We also note that the constant f appearing in the Killing spinor equations is related to the component of the self-dual five-form flux on Y (3.6) via

$$F_5|_Y = -f\widetilde{\text{vol}}_Y, \quad (A.8)$$

where the five-dimensional volume form is defined as $\widetilde{\text{vol}}_Y = -e^{12345}$ and e^i is the orthonormal frame introduced in Appendix B of [19].

We now provide a map between the five-dimensional spinors and Killing spinor equations (A.1)-(A.6) to six-dimensional quantities. We begin by using the Cliff(5) gamma matrices $\hat{\beta}_m$ to construct Cliff(6) gamma matrices $\hat{\gamma}_i$, $i = 1, \ldots, 6$, via

$$\begin{aligned} \hat{\gamma}_m &= \hat{\beta}_m \otimes \sigma_3, \quad m = 1, \ldots, 5, \\ \hat{\gamma}_6 &= 1 \otimes \sigma_1, \end{aligned} \quad (A.9)$$

where σ_α, $\alpha = 1, 2, 3$, are the Pauli matrices. These satisfy $\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}$. The corresponding gamma matrices for the six-dimensional metric g_6 will be denoted γ_i. We define the 6D chirality operator to be

$$\tilde{\gamma} \equiv -i\hat{\gamma}_{123456} = 1 \otimes \sigma_2. \quad (A.10)$$

---

[10] Notice we have relabelled $\gamma_i \mapsto \beta_m$ in [19], as in this paper we want to keep the notation γ_i for six-dimensional gamma matrices.

We may choose the D_6 intertwiner

$$D_6 = \tilde{D}_5 \otimes \sigma_2, \tag{A.11}$$

where $\tilde{D}_5 = D_5 = C_5$ is the intertwiner of Cliff(5) discussed in [19], and one checks $D_6^{-1} \gamma_i D_6 = -\gamma_i^*$. We also note that since in [19] the intertwiner $A_5 = 1$ we have $A_6 = 1$ and $\gamma_i^\dagger = \gamma_i$. If η_+ is a Weyl spinor, satisfying $\tilde{\gamma} \eta_+ = \eta_+$, then the conjugate spinor $\eta_- \equiv \eta_+^c \equiv D_6 \eta_+^*$ satisfies $\tilde{\gamma} \eta_- = -\eta_-$.

To construct the relevant 6D spinors we first write

$$\xi_1 = \chi_1 + i\chi_2, \qquad \xi_2 = \chi_1 - i\chi_2, \tag{A.12}$$

as in [19]. Given this, the normalization for ξ_i chosen in [19] implies that the χ_i are normalized as

$$\bar{\chi}_1 \chi_1 = \bar{\chi}_2 \chi_2 = \tfrac{1}{2}. \tag{A.13}$$

We then define

$$\begin{aligned}
\eta_+^1 &= e^{A/2} \begin{pmatrix} \chi_1 \\ i\chi_1 \end{pmatrix}, & \eta_-^1 &= e^{A/2} \begin{pmatrix} -\chi_1^c \\ i\chi_1^c \end{pmatrix}, \\
\eta_+^2 &= e^{A/2} \begin{pmatrix} -\chi_2 \\ -i\chi_2 \end{pmatrix}, & \eta_-^2 &= e^{A/2} \begin{pmatrix} \chi_2^c \\ -i\chi_2^c \end{pmatrix},
\end{aligned} \tag{A.14}$$

where recall from (3.16) that

$$e^{A/2} = r^{1/2} e^{\Delta/2 + \phi/8}, \tag{A.15}$$

and also from [19] that

$$\chi^c \equiv \tilde{D}_5 \chi^*. \tag{A.16}$$

In the conventions of [19] we have $\bar{\chi} = \chi^\dagger$.

After some detailed calculation one finds that the five-dimensional Killing spinor equations (A.1)–(A.6), using the five-dimensional metric g_Y, are equivalent to the six-dimensional Killing spinor equations, using the six-dimensional metric g_6 in (3.17) and volume form (3.18), given by

$$\left(D_i - \frac{1}{4} H_i \right) \eta_+^1 + \frac{e^\phi}{8} F \gamma_i \eta_+^2 = 0, \tag{A.17}$$

$$\frac{1}{2} e^A \slashed{\partial} A \, \eta_+^1 - \frac{1}{8} e^{A+\phi} F \eta_+^2 = 0, \tag{A.18}$$

$$\slashed{D} \eta_+^1 + \left(\slashed{\partial}(2A - \phi) - \frac{1}{4} \slashed{H} \right) \eta_+^1 = 0, \tag{A.19}$$

and additional equations obtained by applying the rule:

$$\eta^1 \leftrightarrow \eta^2, \quad F \to -F^\dagger, \quad H \to -H. \tag{A.20}$$

In these equations we are using the notation that, *e.g.*

$$H_i = \tfrac{1}{2} H_{ijk} \gamma^{jk}, \qquad F = F_{1i} \gamma^i + \tfrac{1}{3!} F_{3ijk} \gamma^{ijk} + \tfrac{1}{5!} F_{5ijklm} \gamma^{ijklm}. \tag{A.21}$$

These are precisely the same equations that were used in [34] (for zero four-dimensional cosmological constant).

Finally, we record the following equation of [19]:

$$D(e^{6\Delta}W) = -e^{6\Delta}P \wedge W^* + \frac{f}{4}G, \tag{A.22}$$

where W is the two-form bilinear defined in (3.10). Using this one can show that

$$i_{K_5^\#}\left(\frac{4}{f}e^{6\Delta+\phi/2}\,\mathrm{Re}\,W\right) = e^{2\Delta+\frac{\phi}{2}}\,\mathrm{Re}\,K_3, \tag{A.23}$$

and furthermore that

$$\mathrm{d}\left(e^{2\Delta+\frac{\phi}{2}}\,\mathrm{Re}\,K_3\right) = i_{\xi_v}H. \tag{A.24}$$

To see the latter one can derive an expression for the left hand side using, amongst other things, (3.18), (3.38) and (B.10) of [19], and an expression for the right hand side using Eq. (3.38) and (B.8) of [19]. Using these results we can deduce that

$$\begin{aligned}
\mathcal{L}_{K_5^\#}B &= \mathrm{d}(i_{K_5^\#}b_2), \\
\mathcal{L}_{K_5^\#}C_2 &= \mathrm{d}(i_{K_5^\#}c_2),
\end{aligned} \tag{A.25}$$

where b_2, c_2 were introduced in (3.12), (3.13), respectively.

B. More on the Generalized Vectors ξ and η

In this appendix we derive an expression for the generalized vector ξ in terms of the bilinears introduced in [19]. We also use the results of [19] to show that $\mathbb{L}_\xi \mathcal{J}_\pm = 0$. The projections of ξ onto the vector and form parts (in a fixed trivialization of E) are denoted ξ_v, ξ_f, respectively. It will also be convenient to introduce $\xi^B \equiv e^B\xi$ whose form part is given by

$$\xi_f^B = \xi_f - i_{\xi_v}B, \tag{B.1}$$

and we recall that $\xi_v^B = \xi_v$. We next construct the following two generalized $(1,0)_-$ vectors, which, by definition, are in the $+i$ eigenspace of $\mathcal{J}_-$:

$$\begin{aligned}
Z_1^- &= r\partial_r - i\xi, \\
Z_2^- &= \mathrm{d}\log r - i\eta.
\end{aligned} \tag{B.2}$$

That is, both are in the annihilator of Ω_-. We may similarly also construct the $(1,0)_+$ vectors, with respect to $\mathcal{J}_+$:

$$\begin{aligned}
Z_1^+ &= e^{-\Delta-\phi/4}r\partial_r - ie^{\Delta+\phi/4}\eta, \\
Z_2^+ &= e^{\Delta+\phi/4}\mathrm{d}\log r - ie^{-\Delta-\phi/4}\xi.
\end{aligned} \tag{B.3}$$

Together $Z_i^\pm$ are four independent generalized vectors. We next note that since $Z_i^\pm$ live within null isotropic subspaces we have six relations of the form, using the notation of (2.5),

$$\langle Z_i^\pm, Z_j^\pm \rangle = 0. \tag{B.4}$$

Explicitly we have

$$i_{\xi_v}\xi_f = 0, \qquad i_{\xi_v}d\log r = 0, \qquad i_{r\partial_r}\xi_f = 0,$$
$$i_{\eta_v}\eta_f = 0, \qquad i_{\eta_v}d\log r = 0, \qquad i_{r\partial_r}\eta_f = 0, \qquad \langle\xi,\eta\rangle = \tfrac{1}{2}. \tag{B.5}$$

Since Z_1^- annihilates Ω_-, using the definition (3.34) we deduce that

$$i_{r\partial_r}\Phi_- = i\left(i_{\xi_v}\Phi_- + \xi_f^B \wedge \Phi_-\right). \tag{B.6}$$

To proceed we use (3.20) to write

$$\Phi_- \equiv \eta_+^1 \otimes \bar{\eta}_-^2 = e^A \chi_1 \bar{\chi}_2^c \otimes (\sigma_3 + i\sigma_1). \tag{B.7}$$

Since $\Phi_- = \sum_{\text{odd } n} \frac{1}{n!}\Phi_{i_1\ldots i_n}\gamma^{i_1\ldots i_n}$ we have

$$i_v\Phi_- = \tfrac{1}{2}\{v^i\gamma_i, \Phi_-\}, \qquad \omega\wedge\Phi_- = \tfrac{1}{2}[\omega_i\gamma^i, \Phi_-]. \tag{B.8}$$

Hence, using the Clifford algebra decomposition (3.19) and metric (3.17) we have

$$i_{r\partial_r}\Phi_- = \tfrac{1}{2}\{e^{\Delta+\phi/4}\hat{\gamma}_6, \Phi_-\} = \tfrac{1}{2}e^{A+\Delta+\phi/4}\chi_1\bar{\chi}_2^c \otimes \{\sigma_1, \sigma_3+i\sigma_1\} = ie^{A+\Delta+\phi/4}\chi_1\bar{\chi}_2^c\otimes 1. \tag{B.9}$$

On the other hand using (B.5) we have

$$i_{\xi_v}\Phi_- + \xi_f^B \wedge \Phi_- = \tfrac{1}{2}\{e^{\Delta+\phi/4}\xi_v^m\beta_m \otimes \sigma_3, \Phi_-\} + \tfrac{1}{2}[e^{-\Delta-\phi/4}\xi_{fm}^B\beta^m \otimes \sigma_3, \Phi_-]$$
$$= e^{\Delta+\phi/4}v_m^+\beta^m \otimes \sigma_3\Phi_- + e^{\Delta+\phi/4}v_m^-\Phi_-\beta^m \otimes \sigma_3$$
$$= e^{A+\Delta+\phi/4}\left(v_m^+\beta^m(\chi_1\bar{\chi}_2^c) + v_m^-(\chi_1\bar{\chi}_2^c)\beta^m\right) \otimes 1$$
$$- e^{A+\Delta+\phi/4}\left(v_m^+\beta^m(\chi_1\bar{\chi}_2^c) - v_m^-(\chi_1\bar{\chi}_2^c)\beta^m\right) \otimes \sigma_2, \tag{B.10}$$

where recall that $\{\beta_m, \beta_n\} = 2g_{Ymn}$ and we have defined

$$v_m^\pm = \tfrac{1}{2}(\xi_{vm} \pm e^{-2\Delta-\phi/2}\xi_{fm}^B). \tag{B.11}$$

To satisfy (B.6) we thus require

$$v_m^+\beta^m(\chi_1\bar{\chi}_2^c) = v_m^-(\chi_1\bar{\chi}_2^c)\beta^m = \tfrac{1}{2}\chi_1\bar{\chi}_2^c, \tag{B.12}$$

which implies

$$v_m^+\beta^m\chi_1 = \tfrac{1}{2}\chi_1, \qquad v_m^-\beta^m\chi_2 = \tfrac{1}{2}\chi_2, \tag{B.13}$$

or equivalently

$$v_m^+ = \frac{\bar{\chi}_1\beta_m\chi_1}{2\bar{\chi}_1\chi_1}, \qquad v_m^- = \frac{\bar{\chi}_2\beta_m\chi_2}{2\bar{\chi}_2\chi_2}. \tag{B.14}$$

Hence, given the normalizations (A.13) we deduce that, in terms of the bilinears defined in (3.9),

$$\xi_v = K_5^\#,$$
$$\xi_f^B = e^{2\Delta+\phi/2}\text{Re }K_3. \tag{B.15}$$

A similar calculation using

$$i_{r\partial_r} \Phi_+ = \mathrm{i}e^{2\Delta+\phi/2} \left(i_{\eta_v} \Phi_- + \eta_f^B \wedge \Phi_+ \right), \tag{B.16}$$

leads to

$$\eta_v = e^{-2\Delta-\phi/2} \operatorname{Re} K_3^{\#},$$
$$\eta_f^B = K_5. \tag{B.17}$$

Using the expression for the *B*-field given in (3.12) we obtain the expressions for ξ_f and η_f given in (3.50).

In [19] it was shown that K_5 is a Killing one-form, so that its dual vector field $K_5^{\#}$, with respect to the metric g_Y on Y, is a Killing vector field. In fact $K_5^{\#}$ generates a full symmetry of the supergravity solution, in that all bosonic fields (warp factor, dilaton, NS three-form H and RR fields) are preserved under the Lie derivative along $\xi_v = K_5^{\#}$. However, importantly, the Killing spinors ξ_1, ξ_2 are not invariant under ξ_v. In [19] it was shown that

$$\mathcal{L}_{\xi_v} S = -3\mathrm{i}S, \tag{B.18}$$

where $S \equiv \bar{\xi}_2^c \xi_1$. Notice that, since ξ_v preserves all of the bosonic fields, one may take the Lie derivative of the Killing spinor equations (A.1)-(A.6) for ξ_1, ξ_2 along ξ_v, showing that $\{\mathcal{L}_{\xi_v} \xi_i\}$ satisfy the same equations as the $\{\xi_i\}$. It thus follows that

$$\mathcal{L}_{\xi_v} \xi_i = \mathrm{i}\mu\xi_i, \tag{B.19}$$

where μ is a constant. Now (B.18) implies that $2\mu = -3$, and thus

$$\mathcal{L}_{\xi_v} \xi_i = -\frac{3\mathrm{i}}{2}\xi_i. \tag{B.20}$$

One can also derive this last equation directly from the Killing spinor equations (A.1)–(A.6) of [19]. It thus follows that

$$\mathcal{L}_{\xi_v} \Phi_+ = 0, \tag{B.21}$$
$$\mathcal{L}_{\xi_v} \Phi_- = -3\mathrm{i}\, \Phi_-. \tag{B.22}$$

From (A.24) we have $\mathrm{d}\xi_f^B = i_{\xi_v} H$ and we deduce that

$$\mathbb{L}_{\xi^B} \Phi_+ = i_{\xi_v} H \wedge \Phi_+,$$
$$\mathbb{L}_{\xi^B} \Phi_- = -3\mathrm{i}\, \Phi_- + i_{\xi_v} H \wedge \Phi_-. \tag{B.23}$$

Since (A.24) is also equivalent to $\mathrm{d}\xi_f = \mathcal{L}_{\xi_v} B$ we deduce that

$$\mathbb{L}_{\xi} \Omega_+ = 0,$$
$$\mathbb{L}_{\xi} \Omega_- = -3\mathrm{i}\Omega_-, \tag{B.24}$$

and hence $\mathbb{L}_{\xi} \mathcal{J}_{\pm} = 0$. It is also interesting to point out that

$$(\mathbb{L}_{\xi^B} - i_{\xi_v} H\wedge)F = 0, \quad \text{or equivalently,} \quad \mathbb{L}_{\xi}(e^{-B}F) = 0. \tag{B.25}$$

C. The Sasaki-Einstein Case

Here we discuss the special case in which the compact five-manifold Y is Sasaki-Einstein. Setting $G = P = Q = 0$, $f = 4e^{4\Delta}$ and $\xi_2 = 0$, the Killing spinor equations (A.1)-(A.6) reduce to

$$\nabla_m \xi_1 + \frac{i}{2} \beta_m \xi_1 = 0. \tag{C.1}$$

In terms of Appendix B of [19] we choose $\bar{\theta} = \bar{\phi} = 0$ and $e^{2i\bar{\alpha}} = -1$ (these angles had no bars on them in [19]). We then have the equalities

$$\eta = \frac{1}{2}\bar{\xi}_1 \beta_{(1)} \xi_1 = K_5 = e^1,$$

$$\omega_{KE} = \frac{i}{2}\bar{\xi}_1 \beta_{(2)} \xi_1 = -V = e^{25} + e^{43}, \tag{C.2}$$

$$\Omega_{KE} = \frac{1}{2}\bar{\xi}_1 \beta_{(2)} \xi_1^c = (e^2 + ie^5) \wedge (e^4 + ie^3),$$

and

$$d\eta = 2\omega_{KE},$$
$$d\Omega_{KE} = 3i\eta \wedge \Omega_{KE}. \tag{C.3}$$

Observe that

$$\eta \wedge \frac{1}{2!}\omega_{KE}^2 = -e^{12345} = \widetilde{\mathrm{vol}}_Y. \tag{C.4}$$

Next using the 5D-6D map (3.20), we obtain

$$i\bar{\eta}_+^1 \gamma_{(2)} \eta_+^1 = r(d\log r \wedge e^1 + \omega_{KE}) \equiv \frac{1}{r}\omega_{CY},$$

$$-i\bar{\eta}_+^{1c} \gamma_{(3)} \eta_+^1 = r(d\log r - ie^1)(e^2 - ie^5)(e^4 - ie^3) \equiv \frac{1}{r^2}\bar{\Omega}_{CY}. \tag{C.5}$$

It is worth noting that

$$\frac{1}{3!}\omega_{CY}^3 = r^6 e^{123456} = r^6 d\log r \wedge \eta \wedge \frac{1}{2!}\omega_{KE}^2, \tag{C.6}$$

where $e^6 = dr/r$. We also find, directly from (3.33), (3.36),

$$\Omega_- = \frac{1}{8}\bar{\Omega}_{CY},$$

$$\Omega_+ = -\frac{ir^3}{8} \exp\left(\frac{i}{r^2}\omega_{CY}\right). \tag{C.7}$$

A useful check is that these expressions agree with those obtained from the general expressions obtained in Sects. 5.1 and 6.1, respectively.

One can also write down the corresponding reduced structures $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, as defined in Sect. 4.3, on the Kähler-Einstein space. One finds

$$\tilde{\Omega}_1 = \frac{1}{8}e^{3i\psi}\bar{\Omega}_{KE}, \qquad \tilde{\Omega}_2 = -\frac{i}{8}e^{i\omega_{KE}}, \tag{C.8}$$

where ψ is the coordinate, defined such that $K_5^\# = \partial_\psi$, introduced in (4.31).

References

1. Gauntlett, J.P., Martelli, D., Sparks, J., Waldram, D.: Supersymmetric AdS_5 solutions of M-theory. Class. Quant. Grav. **21**, 4335 (2004)
2. Gauntlett, J.P., Martelli, D., Sparks, J., Waldram, D.: Sasaki-Einstein metrics on $S_2 \times S_3$. Adv. Theor. Math. Phys. **8**, 711 (2004)
3. Cvetic, M., Lu, H., Page, D.N., Pope, C.N.: New Einstein-Sasaki spaces in five and higher dimensions. Phys. Rev. Lett. **95**, 071101 (2005)
4. Benvenuti, S., Franco, S., Hanany, A., Martelli, D., Sparks, J.: An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals. JHEP **0506**, 064 (2005)
5. Hanany, A., Kennaway, K.D.: Dimer models and toric diagrams. http://arXiv.org/abs/hep-th/0503149v2, 2005
6. Franco, S., Hanany, A., Kennaway, K.D., Vegh, D., Wecht, B.: Brane Dimers and Quiver Gauge Theories. JHEP **0601**, 096 (2006)
7. Franco, S., Hanany, A., Martelli, D., Sparks, J., Vegh, D., Wecht, B.: Gauge theories from toric geometry and brane tilings. JHEP **0601**, 128 (2006)
8. Anselmi, D., Erlich, J., Freedman, D.Z., Johansen, A.A.: Positivity constraints on anomalies in supersymmetric gauge theories. Phys. Rev. D **57**, 7570 (1998)
9. Intriligator, K.A., Wecht, B.: The exact superconformal R-symmetry maximizes a. Nucl. Phys. B **667**, 183 (2003)
10. Martelli, D., Sparks, J., Yau, S.T.: The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds. Commun. Math. Phys. **268**, 39 (2006)
11. Martelli, D., Sparks, J., Yau, S.T.: Sasaki-Einstein manifolds and volume minimisation. Commun. Math. Phys. **280**, 611 (2008)
12. Lunin, O., Maldacena, J.M.: Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals. JHEP **0505**, 033 (2005)
13. Pilch, K., Warner, N.P.: A new supersymmetric compactification of chiral IIB supergravity. Phys. Lett. B **487**, 22 (2000)
14. Khavaev, A., Pilch, K., Warner, N.P.: New vacua of gauged $N = 8$ supergravity in five dimensions. Phys. Lett. B **487**, 14 (2000)
15. Halmagyi, N., Pilch, K., Romelsberger, C., Warner, N.P.: Holographic duals of a family of $N = 1$ fixed points. JHEP **0608**, 083 (2006)
16. Leigh, R.G., Strassler, M.J.: Exactly Marginal Operators And Duality In Four-Dimensional $N = 1$ Supersymmetric Gauge Theory. Nucl. Phys. B **447**, 95 (1995)
17. Benvenuti, S., Hanany, A.: Conformal manifolds for the conifold and other toric field theories. JHEP **0508**, 024 (2005)
18. Aharony, O., Kol, B., Yankielowicz, S.: On exactly marginal deformations of $N = 4$ SYM and type IIB supergravity on $AdS_5 \times S^5$. JHEP **0206**, 039 (2002)
19. Gauntlett, J.P., Martelli, D., Sparks, J., Waldram, D.: Supersymmetric AdS_5 solutions of type IIB supergravity. Class. Quant. Grav. **23**, 4693 (2006)
20. Minasian, R., Petrini, M., Zaffaroni, A.: Gravity duals to deformed SYM theories and generalized complex geometry. JHEP **0612**, 055 (2006)
21. Butti, A., Forcella, D., Martucci, L., Minasian, R., Petrini, M., Zaffaroni, A.: On the geometry and the moduli space of beta-deformed quiver gauge theories. JHEP **0807**, 053 (2008)
22. Grana, M., Minasian, R., Petrini, M., Tomasiello, A.: Supersymmetric backgrounds from generalized Calabi-Yau manifolds. JHEP **0408**, 046 (2004)
23. Grana, M., Minasian, R., Petrini, M., Tomasiello, A.: Generalized structures of $N = 1$ vacua. JHEP **0511**, 020 (2005)
24. Hitchin, N.: Generalized Calabi-Yau manifolds. Quart. J. Math. Oxford Ser. **54**, 281–308 (2003)
25. Gauntlett, J.P., Martelli, D., Sparks, J.F., Waldram, D.: A new infinite class of Sasaki-Einstein manifolds. Adv. Theor. Math. Phys. **8**, 987 (2006)
26. Bursztyn, H., Cavalcanti, G., Gualtieri, M.: Reduction of Courant algebroids and generalized complex structures. Adv. Math. **211**, 726 (2007)
27. Bursztyn, H., Cavalcanti, G., Gualtieri, M.: Generalized Kahler and hyper-Kahler quotients. In: Poisson geometry in mathematics and physics, Contemp. Math., 450, pp. 61–77. American Mathematics Society, Providence (2008)
28. Gabella, M., Gauntlett, J.P., Palti, E., Sparks, J., Waldram, D.: The central charge of supersymmetric AdS₅ solutions of type IIB supergravity. Phys. Rev. Lett. **103**, 051601 (2009)
29. Gualtieri, M.: Generalized complex geometry. http://arXiv.org/abs/math/0703298v2[math.DG], 2007
30. Hitchin, N.: Brackets, forms and invariant functionals. Asian J. Math. **10**(3), 541 (2006)
31. Grana, M., Minasian, R., Petrini, M., Waldram, D.: T-duality, Generalized Geometry and Non-Geometric Backgrounds. JHEP **0904**, 075 (2009)

32. Cavalcanti, G.R.: New aspects of the dd^c-lemma. Oxford Univ., D. Phil. thesis, http://arXiv.org/abs/math/0501406v1[math.DG], 2005
33. Gualtieri, M.: Generalized complex geometry. Oxford University DPhil thesis, http://arXiv.org/abs/math/0401221v1[math-DG], 2004
34. Grana, M., Minasian, R., Petrini, M., Tomasiello, A.: A scan for new $N = 1$ vacua on twisted tori. JHEP **0705**, 031 (2007)
35. Martucci, L., Smyth, P.: Supersymmetric D-branes and calibrations on general N = 1 backgrounds. JHEP **0511**, 048 (2005)
36. Tomasiello, A.: Reformulating Supersymmetry with a Generalized Dolbeault Operator. JHEP **0802**, 010 (2008)
37. Martucci, L.: D-branes on general $N = 1$ backgrounds: Superpotentials and D-terms. JHEP **0606**, 033 (2006)
38. Henningson, M., Skenderis, K.: The holographic Weyl anomaly. JHEP **9807**, 023 (1998)
39. Martelli, D., Sparks, J.: Baryonic branches and resolutions of Ricci-flat Kahler cones. JHEP **0804**, 067 (2008)
40. Klebanov, I.R., Murugan, A.: Gauge/Gravity Duality and Warped Resolved Conifold. JHEP **0703**, 042 (2007)
41. Halmagyi, N., Pilch, K., Romelsberger, C., Warner, N.P.: The complex geometry of holographic flows of quiver gauge theories. JHEP **0609**, 063 (2006)

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