

Fluctuations of the Nodal Length of Random Spherical Harmonics

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Abstract: Using the multiplicities of the Laplace eigenspace on the sphere (the space of spherical harmonics) we endow the space with Gaussian probability measure. This induces a notion of random Gaussian spherical harmonics of degree n having Laplace eigenvalue $E = n(n + 1)$. We study the length distribution of the nodal lines of random spherical harmonics.

It is known that the expected length is of order n . It is natural to conjecture that the variance should be of order n , due to the natural scaling. Our principal result is that, due to an unexpected cancelation, the variance of the nodal length of random spherical harmonics is of order $\log n$. This behaviour is consistent with the one predicted by Berry for nodal lines on chaotic billiards (Random Wave Model). In addition we find that a similar result is applicable for “generic” linear statistics of the nodal lines.

1. Introduction

Nodal patterns (first described by Ernest Chladni in the 18th century) appear in many problems in engineering, physics and natural sciences: they describe the sets that remain stationary during vibrations. Hence, their importance in such diverse areas as musical instruments, mechanical structures, earthquake study and other areas. They also arise in the study of wave propagation and in astrophysics; this is a very active and rapidly developing research area. Let (\mathcal{M}, g) be a compact manifold and $f : \mathcal{M} \rightarrow \mathbb{R}$ be a real valued function. The nodal set of f is its zero set $f^{-1}(0) = \{x \in \mathcal{M} : f(x) = 0\}$.

The most important or fundamental case is that of f being the eigenfunction of the Laplace-Beltrami operator on \mathcal{M} ,

$$\Delta_g f + Ef = 0, \tag{1}$$

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with $E \geq 0$. In this case it is known [8], that generically, the nodal sets are smooth submanifolds of \mathcal{M} of codimension 1. For example, if \mathcal{M} is a surface, the nodal sets are smooth curves, also called the nodal *lines*. One is interested in studying their volume (i.e. the length of the nodal line for the 2-dimensional case) and other properties for highly excited eigenstates. Yau conjectured [25,26] that the volume of the nodal set is commensurable to \sqrt{E} in the sense that there exist constants $c_{\mathcal{M}}, C_{\mathcal{M}} > 0$ such that if f satisfies (1) then

$$c_{\mathcal{M}}\sqrt{E} \leq \text{Vol}(f^{-1}(0)) \leq C_{\mathcal{M}}\sqrt{E}. \tag{2}$$

The lower bound was proved by Bruning and Gromes [7] and Bruning [6] for the planar case. Donnelly and Fefferman [10] finally settled Yau’s conjecture for real analytic metrics. However, the general case of a smooth manifold is still open.

1.1. Spherical Harmonics. In this paper, we will concentrate on the nodal sets on the sphere. It is well known that the eigenvalues E of the Laplace equation

$$\Delta f + Ef = 0$$

on the m -dimensional sphere S^m are all the numbers of the form

$$E_n^m = n(n + m - 1), \tag{3}$$

where n is an integer. Given a number E_n^m , the corresponding eigenspace is the space \mathcal{E}_n^m of spherical harmonics of degree n . Its dimension is given by

$$\mathcal{N} = \mathcal{N}_n^m = \frac{2n + m - 1}{n + m - 1} \binom{n + m - 1}{m - 1} \sim \frac{2}{(m - 1)!} n^{m-1}. \tag{4}$$

Given an integral number n , we fix an $L^2(S^m)$ orthonormal basis of \mathcal{E}_n^m ,

$$\eta_1^{n;m}(x), \eta_2^{n;m}(x), \dots, \eta_{\mathcal{N}_n^m}^{n;m}(x),$$

giving an identification $\mathcal{E}_n^m \cong \mathbb{R}^{\mathcal{N}_n^m}$. For further reading on the spherical harmonics we refer the reader to [1], Chap. 9.

1.2. Random model. We consider a *random eigenfunction*

$$f_n^m(x) = \sqrt{\frac{|\mathcal{S}^m|}{\mathcal{N}_n^m}} \sum_{k=1}^{\mathcal{N}_n^m} a_k \eta_k^{n;m}(x), \tag{5}$$

where a_k are standard Gaussian $N(0, 1)$ i.i.d. That is, we use the identification

$$\mathcal{E}_n^m \cong \mathbb{R}^{\mathcal{N}_n^m}$$

to endow the space \mathcal{E}_n^m with Gaussian probability measure ν as

$$d\nu(f_n^m) = e^{-\frac{1}{2}\|\mathbf{a}\|^2} \frac{da_1 \cdot \dots \cdot da_{\mathcal{N}_n^m}}{(2\pi)^{\mathcal{N}_n^m/2}},$$

where $\mathbf{a} = (a_i) \in \mathbb{R}^{\mathcal{N}_n^m}$ are as in (5).

Note that ν is invariant with respect to the orthonormal basis for \mathcal{E}_n^m . Moreover, the Gaussian random field f_n^m is isotropic in the sense that for every $x_1, \dots, x_l \in \mathcal{S}^m$ and every orthogonal $R \in O(m + 1)$,

$$(f_n(Rx_1), \dots, f_n(Rx_l)) \stackrel{d}{=} (f_n(x_1), \dots, f_n(x_l)). \tag{6}$$

As usual, for any random variable X , we denote its expectation $\mathbb{E}X$. For example, with the normalization factor in (5), for every $m \geq 2, n$ and fixed point $x \in \mathcal{S}^m$, one has

$$\mathbb{E}[f_n^m(x)^2] = \frac{\mathcal{S}^m}{\mathcal{N}_n^m} \sum_{k=1}^{\mathcal{N}_n^m} \eta_k^{n;m}(x)^2 = 1, \tag{7}$$

a simple corollary from the addition theorem (see [1], or (18) for $m = 2$).

Any characteristic $X(L)$ of the nodal set

$$L = L(f_n^m) = \{x \in \mathcal{S}^2 : f_n^m(x) = 0\}$$

is a random variable. The most natural characteristic of the nodal set $L_{f_n^m}$ of f_n^m is, of course, its $(m - 1)$ -dimensional volume $\mathcal{Z}(f_n^m)$. The main goal of the present paper is the study of the distribution of the random variable $\mathcal{Z}(f_n^m)$ for a random Gaussian $f_n \in \mathcal{E}_n$.

1.3. Some Conventions. Throughout the paper, the letters x, y will denote either points on the sphere \mathcal{S}^m or spherical variables. For $x, y \in \mathcal{S}^m, d(x, y)$ will stand for the spherical distance between x and y . Given a set $F \subseteq \mathcal{S}^m$, we denote its area by $|F|$; $\text{len}(\mathcal{C})$ will stand for the length of a smooth curve $\mathcal{C} \subset \mathcal{S}^m$. For example,

$$|\mathcal{S}^2| = 4\pi,$$

and

$$\mathcal{Z}(f_n^2) = \text{len}(\{f_n^2(x) = 0\}).$$

In this paper we are mainly concerned with the 2-dimensional case. Therefore, to simplify the notation we will use $f_n(x) := f_n^2(x)$, and accordingly $\mathcal{E}_n := \mathcal{E}_n^2, E_n := E_n^2, \mathcal{N}_n := \mathcal{N}_n^2, \eta_k^n := \eta_k^{n;2}$.

In this manuscript, we will use the notations $A \ll B$ and $A = O(B)$ interchangeably. If necessary, the constant involved will depend on the parameters written in the subscript. For example, O_φ or \ll_φ means that the constants involved depend on the function φ .

1.4. Nodal length and related subjects. It is widely believed that for generic chaotic billiards, one can model the nodal lines for eigenfunctions of eigenvalue of order $\approx E$ with nodal lines of isotropic, monochromatic random waves of wavenumber \sqrt{E} (this is called Berry’s Random Wave Model or RWM). Berry [3] found that the expected length (per unit area) of the nodal lines for the RWM is of size approximately \sqrt{E} , and he argued that the variance should be of order $\log E$.

Berard [2] proved that for every $m \geq 2$,

$$\mathbb{E}[\mathcal{Z}(f_n^m)] = c_m \cdot \sqrt{E_n^m}, \tag{8}$$

where

$$c_m = \frac{2\pi^{m/2}}{\sqrt{m}\Gamma\left(\frac{m}{2}\right)},$$

(see also [15 and 23]). Furthermore, Neuheisel [15] established an asymptotic upper bound for the variance of the form

$$\text{Var}(\mathcal{Z}(f_n^m)) = O\left(\frac{E_n^m}{n^{\frac{(m-1)^2}{3m+1}}}\right) = O\left(\frac{E_n^m}{\mathcal{N}^{\frac{m-1}{3m+1}}}\right) \tag{9}$$

and in our previous work [23], we improved the latter to be

$$\text{Var}(\mathcal{Z}(f_n^m)) = O\left(\frac{E_n^m}{\sqrt{\mathcal{N}_n^m}}\right).$$

Either of the bounds implies that the variance of the length, *normalized* so that its expected value is 1, vanishes with prescribed rate,

$$\text{Var}\left(\frac{\mathcal{Z}(f_n)}{\mathbb{E}[\mathcal{Z}(f_n)]}\right) = O\left(\frac{1}{\sqrt{\mathcal{N}_n}}\right)$$

for the latter bound. This means that the constants $c_{\mathcal{S}^2}$ and $C_{\mathcal{S}^2}$ guaranteed by Donnelly-Fefferman (2) may be taken as essentially equal for “generic” eigenfunctions $f_n^m \in \mathcal{E}_n^m$, where n is large.

The volume of the nodal line of a random eigenfunction on the torus

$$\mathcal{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$$

was studied by Rudnick and Wigman [16] and subsequently by Krishnapur and Wigman [14]. In this case, it is not difficult to see that the expectation is again $\mathbb{E}[\mathcal{Z}(f^{\mathcal{T}^2})] = \text{const} \cdot \sqrt{E}$. Their principal result is that as the eigenspace dimension \mathcal{N} grows to infinity, the variance is bounded by

$$\text{Var}(\mathcal{Z}(f^{\mathcal{T}^2})) = O\left(\frac{E}{\mathcal{N}^2}\right),$$

and it is likely that it is asymptotic to $\text{Var}(\mathcal{Z}(f^{\mathcal{T}^2})) \sim * \frac{E}{\mathcal{N}^2}$ for a “generic” sequence of eigenvalues.

For generic manifolds, one does not expect the Laplacian to have any multiplicities, so that we cannot introduce a Gaussian ensemble on the eigenspace. Let E_j be the eigenvalues and ϕ_j the corresponding eigenfunctions. It is well known that the E_j are discrete, $E_j \rightarrow \infty$ and $L^2(\mathcal{M}) = \text{span}\{\phi_j\}$.

In this case, rather than considering random eigenfunctions, one considers random *combinations* of eigenfunctions with growing energy window of either type

$$f^L(x) = \sum_{E_j \in [0, E]} a_j \phi_j(x)$$

(called the long range), or

$$f^S(x) = \sum_{\sqrt{E_j} \in [\sqrt{E}, \sqrt{E+1}]} a_j \phi_j(x),$$

(called the short range), as $E \rightarrow \infty$. Berard [2] and Zelditch [27] found that

$$\mathbb{E}Z(f^L) \sim \tilde{C}_{\mathcal{M}} \cdot \sqrt{E}$$

and recently Zelditch [27] proved that

$$\mathbb{E}Z(f^S) \sim \tilde{C}_{\mathcal{M}} \cdot \sqrt{E},$$

notably with the same constant $\tilde{C}_{\mathcal{M}} > 0$ for both the long and the short ranges.

For billiards (i.e. surfaces with piecewise smooth boundary), one is interested in the number of intersections of the nodal line with the boundary, or, equivalently, the number of open nodal components. Toth and Wigman [21] studied the number of boundary intersections for random combinations of eigenfunctions $f^L(x)$ and $f^S(x)$ on generic billiards, defined precisely as above. They found that the expected number of the intersections is of order \sqrt{E} .

In the first part of this paper, we resolve the high energy asymptotic behaviour for the variance of the nodal length for random 2-dimensional spherical harmonics

$$f_n = f_n^2 : \mathcal{S}^2 \rightarrow \mathbb{R}.$$

Theorem 1.1. *One has*

$$\text{Var}(Z(f_n)) = \frac{65}{32} \log n + O(1), \tag{10}$$

asymptotically as $n \rightarrow \infty$.

For the higher dimensional sphere $\mathcal{S}^m \subseteq \mathbb{R}^{m+1}$ with $m \geq 3$, it is possible to prove [24] that

$$\text{Var}(Z(f_n^m)) = O\left(\frac{1}{n^{m-2}}\right) = O\left(\frac{E_n^m}{n\mathcal{N}_n^m}\right),$$

and it is likely that

$$\text{Var}(Z(f_n^m)) \sim \frac{c}{n^{m-2}}$$

for some constant $c > 0$. We intend to address the question of precise asymptotics for the higher dimensional case in the future.

1.5. *Smooth linear statistics.*¹ Rather than considering the volume of the full nodal line one may choose a nice submanifold $F \subseteq S^m$ of the sphere and consider the nodal volume

$$\mathcal{Z}^F(f_n^m) := \text{Vol}(\{f_n^m = 0\} \cap F)$$

inside F . More generally, let $\varphi : S^m \rightarrow \mathbb{R}$ be a function. One then defines

$$\mathcal{Z}^\varphi(f_n^m) := \int_{f_n^{m-1}(0)} \varphi(x) d \text{Vol}_{f_n^{m-1}(0)}(x).$$

The random variable $\mathcal{Z}^\varphi(f_n^m)$ is called a (smooth) *linear statistic* of the nodal set. A priori, this definition makes sense only for the *continuous* test function $\varphi \in C(S^m)$, so that the restriction $\varphi|_{f_n^{m-1}(0)} \in C(f_n^{m-1}(0))$ is defined. Unfortunately, the class $C(S^2)$ of continuous functions does not contain the characteristic functions of smooth sets. However, it is known [13] that for a smooth $(m - 1)$ -dimensional hypersurface C one can define the *trace* $\text{tr}_C(\varphi) \in L^1(C)$ of φ for some wider classes of functions such as $W^{1,1}(S^m)$, the class of integrable functions with integrable *weak* derivatives, even though the values of $\varphi \in W^{1,1}(S^m)$ are defined up to measure zero spherical sets. To define the trace, one exploits the values of φ in a small tubular neighbourhood of C .

Unfortunately again, the class $W^{1,1}$ does not contain the family of characteristic functions of nice sets. As an example, let us consider the 2-dimensional spherical disc $F = B(N, \frac{\pi}{4}) \subsetneq S^2$ centered at the north pole of radius $\frac{\pi}{4}$, $C = \partial F$ its boundary, and $\varphi = \chi_F$. Then the definition of $\text{tr}_C(\chi_F)$ is ambiguous since one may define it as either $0 \in L^1(C)$ or $1 \in L^1(C)$. This phenomenon (i.e. the jump in f occurring precisely on C) is typical to the class $BV(S^m)$ of functions of bounded variation; it is known [13], that for any characteristic function χ_F of a submanifold F with C^2 boundary, $\chi_F \in BV(S^m)$, and, in addition, $W^{1,1}(S^m) \subsetneq BV(S^m)$. It turns out that, despite this subtlety, one can still extend the notion of *average* trace

$$\varphi_C^\pm = \text{tr}_C^\pm(\varphi) \in L^1(C)$$

to the full class $\varphi \in BV(S^m)$ (see Appendix C for more details). For instance, in our previous example, $\text{tr}_{\partial F}^\pm(\chi_F) \equiv \frac{1}{2}$. It is then natural to define

$$\mathcal{Z}^\varphi(f_n^m) := \int_{f_n^{m-1}(0)} \varphi_C^\pm(x) d \text{Vol}_{f_n^{m-1}(0)}(x).$$

It is easy to compute the expected value of a “generic” linear statistic, following along the lines of the proof of [23], Prop. 1.4, starting from (121).

Lemma 1.2. *For*

$$\varphi \in BV(S^2) \cap L^\infty(S^2)$$

we have

$$\mathbb{E}[\mathcal{Z}^\varphi(f_n)] = \frac{\int \varphi(x) dx}{2^{3/2}} \sqrt{E_n}. \tag{11}$$

¹ The author wishes to thank Steve Zelditch for his suggestion to consider the smooth linear statistics as a measure of the “stability” for the result obtained for the length and “Berry’s cancellation phenomenon” (discussed in Sect. 1.6.1 below)

Remark 1.3. Note that f_n is odd or even if n is odd or even respectively, so that in particular the nodal lines are symmetric w.r.t. the involution $x \mapsto -x$. Therefore if φ is *odd* then $\mathcal{Z}^\varphi(f_n)$ vanishes identically in either case. Moreover,

$$\mathcal{Z}^\varphi(f_n) = \mathcal{Z}^{\varphi^{ev}}(f_n),$$

where the even part of φ

$$\varphi^{ev}(x) := \frac{\varphi(x) + \varphi(-x)}{2}$$

does not vanish identically, if and only if, φ is *not odd*. Therefore, we may assume that φ is *even* in the first place, and we will assume so throughout the rest of this paper.

Under the assumption of continuous differentiability we have the following result for the variance of \mathcal{Z}^φ for 2-dimensional spherical harmonics.

Theorem 1.4. *Let $\varphi : \mathcal{S}^2 \rightarrow \mathbb{R}$ be a continuously differentiable even function, which does not vanish identically. Then as $n \rightarrow \infty$,*

$$\text{Var}(\mathcal{Z}^\varphi(f_n)) = c(\varphi) \cdot \log n + O_{\|\varphi\|_\infty, V(\varphi)}(1), \tag{12}$$

where

$$c(\varphi) := 65 \frac{\|\varphi\|_{L^2(\mathcal{S}^2)}^2}{128\pi} > 0, \tag{13}$$

i.e. the constant involved in the “O”-notation depends only on the L^∞ norm $\|\varphi\|_\infty$ and the total variation $V(\varphi)$ of φ , and moreover, this dependency is monotone increasing.

Unfortunately, Theorem 1.4 does not cover the characteristic functions of nice submanifolds. For this, we have Theorem 1.5; the main idea of its proof is approximating a function $\varphi \in BV(\mathcal{S}^2)$ with C^∞ functions φ_i , for which we apply Theorem 1.4. We control the error term in (12) applied to φ_i using its L^∞ norm and variation, which is why we included this technical statement in the formulation of Theorem 1.4 in the first place.

Theorem 1.5. *Let*

$$\varphi \in BV(\mathcal{S}^2) \cap L^\infty(\mathcal{S}^2)$$

be a not identically vanishing even function. Then as $n \rightarrow \infty$,

$$\text{Var}(\mathcal{Z}^\varphi(f_n)) = c(\varphi) \cdot \log n + O_\varphi(1), \tag{14}$$

where

$$c(\varphi) := 65 \frac{\|\varphi\|_{L^2(\mathcal{S}^2)}^2}{128\pi} > 0.$$

The characteristic function χ_F of a subsurface $F \subseteq \mathcal{S}^2$ with C^2 boundary is of bounded variation, i.e. $\chi_F \in BV(\mathcal{S}^2)$, [13] Example 1.4. Therefore, in this case the statement of Theorem 1.5 is valid for \mathcal{Z}^F , as the following corollary states.

Corollary 1.6. *Let $F \subseteq S^2$ be a subsurface of the sphere with C^2 boundary. Then as $n \rightarrow \infty$,*

$$\begin{aligned} \text{Var}(\mathcal{Z}^F(f_n)) &= c \cdot \log n + O_F(1), \\ c = c(F) &:= 65 \frac{|F|}{128\pi} > 0. \end{aligned} \tag{15}$$

Remark 1.7. One may observe from the proof of Theorem 1.5, that the constant involved in the “ O ”-notation in (14) depends only on $\|\varphi\|_\infty$ and the total variation $V(\varphi)$. In particular, the constant involved in the “ O ”-notation (15) depends only on the length of the boundary ∂F .

1.6. Discussion.

1.6.1. “Berry’s Cancellation Phenomenon”. Originally, it was conjectured that the variance $\text{Var}(\mathcal{Z}(f_n))$ should be asymptotic to $c \cdot n$, where $c > 0$ is a constant, due to the natural scaling; however, it turned out that c vanishes, precisely as predicted by Berry [3] for the RWM. The reason for this phenomenon, which we refer to as “Berry’s cancellation phenomenon”, is that the leading nonconstant term in the long range asymptotics of the 2-point correlation function is purely oscillating (see the Key Proposition 3.5), so that it does not contribute to the variance. The non-oscillating leading terms cancel (which is, according to Michael Berry [3], “obscure”). It seems that “Berry’s cancellation phenomenon” is of general nature: it also occurs on the torus [14], and it is likely to hold for random combinations of eigenfunctions on a generic manifold [22].

1.6.2. Spherical Harmonics vs. RWM The principal result of the present paper shows that the behaviour of the nodal lines of 2-dimensional spherical harmonics of eigenvalue E is consistent with the RWM of wavenumber \sqrt{E} , predicted for nodal lines of generic chaotic systems. In both cases, the expected nodal length is of order \sqrt{E} and variance of order $\log E$. More precisely, Berry [3] argued that for a billiard of area A , the variance of the nodal length should be asymptotic to

$$\frac{A}{512\pi} \log E$$

in the high energy limit. Taking into account the symmetry of the nodal lines on S^2 , its “effective” area is 2π , and therefore, according to the RWM, the variance should be asymptotic to $\frac{1}{256} \log E$, which differs from the statement of Theorem 1.1 by a constant.

There is a direct relation between the random spherical harmonics and the RWM. Kolmogorov’s theorem implies that a random centered Gaussian ensemble of functions is determined by its covariance function (see Sect. 2.1). The covariance function for the RWM is

$$r_{RWM}(x, y) = J_0(\sqrt{E}|x - y|),$$

$x, y \in \mathbb{R}^2$, and for the random spherical harmonics is

$$r_n(x, y) = P_n(\cos(d(x, y))),$$

where P_n are the Legendre polynomials. The Legendre polynomials admit Hilb’s asymptotics,

$$P_n(\cos(\phi)) \approx \sqrt{\frac{\phi}{\sin \phi}} J_0(\phi(n + 1/2)),$$

i.e. almost identical to RWM, up to the “correction factor” $\sqrt{\frac{\phi}{\sin \phi}}$. This factor seems to “know” about the geometry of the sphere; it is one of the underlying factors responsible for the difference in the constants in the variance asymptotics. The geometry of the sphere occurs in some other places as well.

1.6.3. Nodal Set vs. Level Sets. Interestingly, the behaviour of the level curves $f_n^{-1}(L)$ for $L > 0$ is very different. Let $\mathcal{Z}^L(f_n)$ be the length of the level curve of f_n . The expected length is [24]

$$\mathbb{E}[\mathcal{Z}^L(f_n)] = c_1 e^{-L^2/2} \sqrt{E_n},$$

consistent with the nodal case $L = 0$. However, unlike the nodal lines, the variance of the level curves length is asymptotic to [24]

$$\text{Var}(\mathcal{Z}^L(f_n)) \sim c_2 L^4 e^{-L^2} \cdot n.$$

*1.6.4. Real vs. Complex Zeros.*² The behaviour of the zeros of *complex analytic* functions was studied extensively in recent years and it is interesting to learn that their behaviour is very different from our case of *real valued* spherical harmonics. Sodin and Tsirelson [20] considered 3 different models of random complex analytic functions $\psi_L : \mathcal{M} \rightarrow \mathbb{C}$, all parametrized by an integer $L \rightarrow \infty$, roughly corresponding to the degree of the harmonic polynomials n . Here \mathcal{M} is the natural domain corresponding to the model with G -invariant measure m^* , where G is a group of symmetries; \mathcal{M} is either the sphere $\mathbb{C} \cup \{\infty\}$, the complex plain \mathbb{C} or the unit disc $\{|z| < 1\}$. In this case the set of zeros is almost surely finite. The authors establish the asymptotic Gaussianity for *smooth* linear statistics $h : \mathcal{M} \rightarrow \mathbb{R}$, $h \in C_c^2(\mathcal{M})$,

$$\mathcal{Z}^h(\psi_L) := \sum_{z:\psi_L(z)=0} h(z),$$

where the expected value is given by

$$\mathbb{E}[\mathcal{Z}^h(\psi_L)] = L \cdot \frac{1}{\pi} \int_{\mathcal{M}} h dm^*,$$

for each of the models considered, consistent with (11). However the variance is of order

$$\text{Var}(\mathcal{Z}^h(\psi_L)) \sim \frac{\kappa}{L} \|\Delta^* h\|_{L^2(m^*)}^2 \tag{16}$$

decaying with $L \rightarrow \infty$; here $\kappa > 0$ is a universal constant, and Δ^* is the invariant Laplacian. Note also that here the dependency on the test function h is via the L^2 norm

² The author wishes to thank Mikhail Sodin and Steve Zelditch for pointing out the unexpected differences between the real and the complex analytic cases.

of a second order differential operator acting on h (namely the invariant Laplacian), whereas in the real valued spherical harmonics case it depends on the L^2 norm of φ itself (i.e. the operator is the identity, see (12) and (13)).

For $h = \chi_U$ the characteristic function of a smooth domain $U \subseteq \mathcal{M}$ (i.e. \mathcal{Z}^h is the number of zeros in U), while the expected value of $\mathcal{Z}^h(\psi_L)$ is still proportional to $area(U) \cdot L$ the variance is of different shape (cf. Corollary 1.6 in the spherical harmonics case). Namely it is known [12] that the variance is asymptotically proportional to $\sqrt{L} \cdot len(\partial U)$, different from Corollary 1.6 both in the power of L and the dependency on the test function. This reflects the high frequency oscillations of the zeros smoothed out by a smooth test function.

Shiffman and Zelditch [18, 19] considered a more general situation of random independent Gaussian sections $s_1 = s_1^{\mathcal{L},L}, \dots, s_k = s_k^{\mathcal{L},L} \in \Gamma(\mathcal{L}^L, \mathcal{M})$ of high powers $\mathcal{L}^L, L \rightarrow \infty$ of holomorphic line bundles \mathcal{L} on an m -dimensional Kähler manifold \mathcal{M} , where $1 \leq k \leq m$. They considered the volume of the intersection of the zero sets of s_i ,

$$\mathcal{Z}^U(s_1, \dots, s_k) = Vol_{2m-2k} \left((s_1, \dots, s_k)^{-1}(0) \right),$$

and its smooth linear statistics

$$\mathcal{Z}^h(s_1, \dots, s_k) = \int_{(s_1, \dots, s_k)^{-1}(0)} h(z) d Vol \left((s_1, \dots, s_k)^{-1}(0) \right)$$

(here in case the system is full $k = m$, the volume is the number of points, and the integral is a sum).

In both cases the expected value is asymptotic to

$$\mathbb{E} \left[\mathcal{Z}^U(s_1^L, \dots, s_k^L) \right], \mathbb{E} \left[\mathcal{Z}^h(s_1^L, \dots, s_k^L) \right] \sim cL^k,$$

where as earlier, $c > 0$ is proportional to either $Vol(U)$ or the mass of h . For the “sharp” random variable they obtained [18] the asymptotic

$$Var(\mathcal{Z}^U(s_1^L, \dots, s_k^L)) \sim c_{mk} L^{2k-m-1/2} \cdot Vol(\partial U),$$

where c_{mk} are some universal constants, extending Forrester-Honner [12], whereas for the smooth statistics they established a Central Limit Theorem with variance

$$Var(\mathcal{Z}^h(s_1^L, \dots, s_k^L)) \sim c_h L^{2k-m-2},$$

where, as in case of Sodin-Tsirelson (16), c_h involves a certain 2nd order differential operator acting on h .

1.7. On the proof of the main results. The proof of Theorem 1.1 involves some geometric as well as some probabilistic aspects; we improve upon both in comparison with our previous paper. We employ the Kac-Rice formula, which reduces the computation of the length variance to the 2-point correlation function, given in terms of distribution of the values $f_n(x)$ as well as their gradients $\nabla f_n(x) \in T_x(S^2)$, for all $x \in S^2$.

Thanks to the isotropicity of the model, it is sufficient to evaluate the 2-point correlation function only on the arc $\{\theta = 0\}$ (in the usual spherical coordinates); this reduces the problem to an essentially 1-dimensional one. One then has to identify the spaces $T_x(S^2)$

via a family of isometries ϕ_x , smooth w.r.t. x , for x on the arc only, which is natural in the spherical coordinates. Scaling the arc we find out that for typical $x, y \in \mathcal{S}^2$, the distribution of the values and the gradients is a small perturbation of standard Gaussian i.i.d random variables $N(0, I)$, the latter recovering the square of the expected value of the nodal length to be canceled. We then expand the 2-point correlation function into a Taylor polynomial around the asymptotic one; to do so we use Berry’s elegant method.

It turns out that the long range behaviour of the two-point correlation function given is also sufficient to extend the result to continuously differentiable linear statistics (i.e. Theorem 1.4). In the course of generalizing the proof to include this case we naturally encounter an auxiliary function $W^\varphi : [0, \pi] \rightarrow \mathbb{R}$. To conclude the proof of Theorem 1.4 we will have to understand its behaviour at the origin.

To prove Theorem 1.5, we apply a standard density argument, approximating φ with C^∞ functions, to which we apply Theorem 1.4. To this end we use the full strength of the statement of Theorem 1.4 applied to φ_i , which enables to uniformly control the error term in (12). For a more detailed explanation see Sect. 5.1.

1.8. Plan of the paper. The goal of Sect. 2 is to give a formula for the length variance, explicit as possible, starting from the classical Kac-Rice formula. In Sect. 3, we use the formula obtained to analyze the variance, asymptotically for high energy (i.e. prove Theorem 1.1). In Sects. 4 and 5, we give the proofs for Theorem 1.4 and Theorem 1.5, respectively.

Appendix A will carry on a certain technical computation we will encounter in this paper, namely, that of the covariance matrix of a random vector involving values and gradients of f_n . Appendix B will be devoted to the Legendre polynomials and some of their basic properties. The goal of Appendix C is to give the definition and some properties of the class $BV(\mathcal{S}^2)$ of functions of bounded variation, including their traces on smooth curves.

2. An Explicit Integral Formula for the Variance

In this section, culminating in Proposition 2.7, we derive an “explicit” integral formula for the variance. First we need to introduce the covariance function.

2.1. Covariance function. The *covariance* function (sometimes also referred to as two-point function) is defined as

$$u_n(x, y) := \mathbb{E}[f_n(x)f_n(y)] = \frac{|\mathcal{S}^2|}{\mathcal{N}_n} \sum_{k=1}^{\mathcal{N}_n} \eta_k^n(x)\eta_k^n(y). \tag{17}$$

It follows from the Kolmogorov theorem [9], that, in principle $u_n(x, y)$ *determines* the centered Gaussian random field f_n , so that one can compute any property of f_n in terms of u_n and its derivatives. By the addition theorem [1], p. 456, Theorem 9.6.3, $u_n(x, y)$ has an explicit expression as

$$u_n(x, y) = P_n(\cos d(x, y)), \tag{18}$$

where

$$P_n : [-1, 1] \rightarrow \mathbb{R}$$

is the Legendre polynomial of degree n (see e.g. [17]). Recall that $d(x, y)$ is the spherical distance so that

$$\cos d(x, y) = \langle x, y \rangle,$$

thinking of \mathcal{S}^2 as being embedded into \mathbb{R}^3 .

The orthogonal invariance (6) is then equivalent to the corresponding property of the covariance function, namely

$$u_n(Rx, Ry) = u_n(x, y) \tag{19}$$

for every orthogonal $R \in O(3)$. In case y is not specified, we take it to be the northern pole $N \in \mathcal{S}^2$, that is

$$u_n(x) := u_n(x, N). \tag{20}$$

For every $t \in [-1, 1]$, $|P_n(t)| \leq 1$ and $|P_n(t)| = 1$, if and only if $t = \pm 1$. Therefore

$$(u_n(x, y) = \pm 1) \Leftrightarrow (x = \pm y), \tag{21}$$

and

$$(u_n(x) = \pm 1) \Leftrightarrow (x \in \{N, S\}), \tag{22}$$

where N and S are the northern and the southern poles respectively.

2.2. Kac-Rice formulas for moments of length. In this section we express the first couple of moments of $\mathcal{Z}(f_n)$ via the Kac-Rice formula. The most general version due to Bleher-Shiffman-Zelditch [4,5] gives an integral expression for all the moments $k \geq 1$ of the $(m - l)$ -dimensional volume of $\{\mathcal{F} = 0\}$ for “generic” smooth vector valued random field

$$\mathcal{F} = (\mathcal{F}_i)_{1 \leq i \leq l} : \mathcal{M} \rightarrow \mathbb{R}^l$$

defined on an m -dimensional smooth manifold \mathcal{M} , $1 \leq l \leq m$. In our previous paper [23] we gave an independent elementary proof for the Kac-Rice formula in the particular case of our interest $\mathcal{M} = \mathcal{S}^m$, $\mathcal{F} = f_n^m$, $k = 1, 2$.

To present the Kac-Rice formula in our case we will need some notation. For $x, y \in \mathcal{S}^2$ we define the following random vectors:

$$Z_1^{n;x} = (f_n(x), \nabla f_n(x)) \in \mathbb{R} \times T_x(\mathcal{S}^2)$$

and

$$Z_2^{n;x,y} = (f_n(x), f_n(y), \nabla f_n(x), \nabla f_n(y)) \in \mathbb{R}^2 \times T_x(\mathcal{S}^2) \times T_y(\mathcal{S}^2). \tag{23}$$

More generally, for $k \geq 1$ and $x_1, \dots, x_k \in \mathcal{S}^2$ one may define

$$Z_k = Z_k^{n;x_1, \dots, x_k} \in \mathbb{R}^k \times \prod_{i=1}^k T_{x_i}(\mathcal{S}^2)$$

in similar fashion. The vectors Z_k are all centered Gaussian in the sense that for every fixed $x_1, \dots, x_k \in \mathcal{S}^2$ any linear functional of Z_k is mean zero Gaussian. Let

$$D_k^{n;x_1, \dots, x_k}(v_1, \dots, v_k, \xi_1, \dots, \xi_k)$$

be the (mean zero Gaussian) probability density function of $Z_k^{n;x_1, \dots, x_k}$. The Kac-Rice formula expresses the k^{th} moment of $\mathcal{Z}(f_n)$ in terms of the distributions of Z_k only (see Lemma 2.1), namely $D_k^{n;*}$. Therefore to express the variance (and the expected value) of $\mathcal{Z}(f_n)$ we will only need to study $D_1^{n;*}$ and $D_2^{n;*}$.

Lemma 2.1. ([4] Th. 2.2; [5] Th. 4.3; [23] Prop. 3.3) *The first two moments of the nodal length of the spherical harmonics are given by the following formulas:*

(1) *Expectation:*

$$\mathbb{E}[\mathcal{Z}(f_n)] = \int_{\mathcal{S}^2} \tilde{P}_n(x) dx, \tag{24}$$

where the density of the zero set $\tilde{P}_n(x)$ is given by

$$\tilde{P}_n(x) = \int_{T_x(\mathcal{S}^2)} \|\xi\| D_1^{n;x}(0, \xi) d\xi.$$

(2) *Second moment:*

$$\mathbb{E}[\mathcal{Z}(f_n)^2] = \iint_{\mathcal{S}^2 \times \mathcal{S}^2} \tilde{K}_n(x, y) dx dy, \tag{25}$$

where the 2-point correlation function \tilde{K}_n is given by

$$\tilde{K}_n(x, y) = \frac{1}{2\pi \sqrt{1 - u_n(x, y)^2}} \iint_{T_x(\mathcal{S}^2) \times T_y(\mathcal{S}^2)} \|\xi^x\| \cdot \|\xi^y\| D_2^n(0, 0, \xi^x, \xi^y) d\xi^x d\xi^y. \tag{26}$$

Neuheisel [15] (see also [23]) noticed that for every $x \in \mathcal{S}^2$, under any isometry $T_x(\mathcal{S}^m) \cong \mathbb{R}^2$ (i.e. any choice of orthonormal basis of $T_x(\mathcal{S}^2)$), the distribution of $Z_1^{n;x}$ is mean zero Gaussian with the diagonal covariance matrix

$$\begin{pmatrix} 1 & \\ & \frac{E_n}{2} I_2 \end{pmatrix},$$

where I_2 is the 2×2 identity matrix. It is then clear that \tilde{P}_n is x -independent (this also follows from the rotational independence), and

$$\tilde{P}_n(x) \equiv \frac{1}{\sqrt{2\pi}},$$

by a standard computation. This, together with (24), yields (8) for $m = 2$ and finishes the treatment of the expectation in this case³. Moreover, slightly modifying the proof of (24), we obtain

$$\mathbb{E}[\mathcal{Z}^\varphi(f_n)] = \int_{\mathcal{S}^2} \varphi(x) \tilde{P}_n(x) dx,$$

and (11) follows.

The goal of the remaining part of the present section, culminating with Proposition 2.7, is to make the formula (25) for the second moment “explicit” and suitable for asymptotic analysis. The rotational invariance (6) of our model implies that \tilde{K}_n depends only on the spherical distance $d(x, y)$ between x and y , i.e. (with a slight abuse of notations)

$$\tilde{K}_n(x, y) = \tilde{K}_n(d(x, y)), \tag{27}$$

which will be used later.

Remark 2.2. One may define $\tilde{K}_n(x, y)$ intrinsically as

$$\tilde{K}_n(x, y) = \frac{1}{(2\pi)\sqrt{1 - u_n(x, y)^2}} \mathbb{E}[\|\nabla f(x)\| \cdot \|\nabla f(y)\| | f(x) = f(y) = 0],$$

the expectation being one of the product of gradients conditioned on f vanishing at x and y .

Remark 2.3. It is important to note that the symmetry of the nodal lines w.r.t. the involution $x \mapsto -x$ (see Remark 1.3) implies that

$$\tilde{K}_n(x, y) = \tilde{K}_n(x, -y). \tag{28}$$

The main disadvantage of the formula (26) is that one has to work with probability densities defined on the tangent planes $T_x(\mathcal{S}^2)$ which depend on the point $x \in \mathcal{S}^2$. In principle one may consider the tangent planes being embedded in \mathbb{R}^3 . This, however, is highly inadvisable since that would result in working with singular Gaussians supported on a plane corresponding to $T_x(\mathcal{S}^2)$. It is thus desired to identify for every $x \in \mathcal{S}^2$,

$$T_x(\mathcal{S}^2) \cong \mathbb{R}^2,$$

via an isometry, i.e. fix an orthonormal basis B_x varying smoothly for $x \in \mathcal{S}^2$ (i.e. an orthonormal frame). Unfortunately it is impossible to choose a global orthonormal frame on \mathcal{S}^2 ; however one can still get around that by noting that in fact all we need is a local choice for given $x, y \in \mathcal{S}^2$.

In general, the orthonormal frame chosen will affect the probability density function $D_2^{n,*}$ of $Z_2^{n,*}$ induced on \mathbb{R}^6 (though $D_1^{n,*}$ will stay invariant); in Sect. 2.3 we show how to compute $D_2^{n,*}$ for a given choice of local orthonormal frames. In Sect. 2.4 we will show how to choose the orthonormal frames to simplify the computations; we will use this construction while evaluating the two-point correlation function (26).

³ The same computation gives the result for every $m \geq 2$.

2.3. *Kac-Rice formula in coordinate system.* Given $x, y \in \mathcal{S}^2$, we consider two local orthonormal frames $F^x(z) = \{e_1^x, e_2^x\}$ and $F^y(z) = \{e_1^y, e_2^y\}$, defined in some neighbourhood of x and y respectively. This gives rise to (local) identifications

$$T_x(\mathcal{S}^2) \cong \mathbb{R}^2 \cong T_y(\mathcal{S}^2), \tag{29}$$

which are isometries.

Under the identification (29) the random vector (23) is a \mathbb{R}^6 mean zero Gaussian with covariance matrix

$$\Sigma = \Sigma(x, y) = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

where

$$A_{2 \times 2} = A_n(x, y) = \begin{pmatrix} 1 & u_n(x, y) \\ u_n(x, y) & 1 \end{pmatrix}, \tag{30}$$

$$B_{2 \times 4} = B_n(x, y) = \begin{pmatrix} \mathbf{0} & \nabla_y u_n(x, y) \\ \nabla_x u_n(x, y) & \mathbf{0} \end{pmatrix} \tag{31}$$

and

$$C_{4 \times 4} = C_n(x, y) = \begin{pmatrix} \frac{E_n}{2} I_2 & H \\ H^t & \frac{E_n}{2} I_2 \end{pmatrix} \tag{32}$$

with ‘‘pseudo-Hessian’’

$$H_{2,2}(x, y) = (\nabla_x \otimes \nabla_y) u_n(x, y), \tag{33}$$

i.e. $H = (h_{jk})_{j,k=1,2}$ with entries given by

$$h_{jk} = \frac{\partial^2}{\partial e_j^x \partial e_k^y} u_n(x, y).$$

The covariance matrix of the Gaussian distribution of Z_2 in (23) conditioned upon $f_n(x) = f_n(y) = 0$ is given by⁴

$$\Omega_n(x, y) = C - B^t A^{-1} B. \tag{34}$$

We then have a frame-dependent formula for the two-point correlation function (26),

$$\begin{aligned} \tilde{K}_n(x, y) &= \frac{1}{\sqrt{1 - u_n(x, y)^2}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \|w_1\| \cdot \|w_2\| \\ &\times \exp\left(-\frac{1}{2}(w_1, w_2)\Omega_n(x, y)^{-1}(w_1, w_2)^t\right) \frac{dw_1 dw_2}{(2\pi)^3 \sqrt{\det \Omega_n(x, y)}}, \end{aligned}$$

where $\Omega_n(x, y)$ is given by (34).

Remark 2.4. Note that even though \tilde{K}_n is rotational invariant (i.e. $\tilde{K}_n(x, y)$ depends only on the spherical distance $d(x, y)$), the same is, in general, false for the covariance matrices Σ_n (and Ω_n).

⁴ This is the inverse of the lower right corner of Σ^{-1} .

Let ϕ, θ be the standard spherical coordinates on S^2 . Using the rotational invariance (27) of the 2-point correlation function we obtain

$$\begin{aligned} \mathbb{E} \left[\mathcal{Z}(f_n)^2 \right] &= \iint_{S^2 \times S^2} \tilde{K}_n(x, y) dx dy = |S^2| \int_{S^2} \tilde{K}_n(N, x) dx \\ &= 2\pi |S^2| \int_0^\pi \tilde{K}_n(N, x(\phi)) \sin \phi d\phi, \end{aligned}$$

where $x(\phi) \in S^2$ is the point corresponding to the spherical coordinates $(\phi, \theta = 0)$. Note that $\tilde{K}(N, x(\phi)) = \tilde{K}(x, y)$ for any $x, y \in S^2$ with $d(x, y) = \phi$. We therefore have the following corollary.

Corollary 2.5. *One has*

$$\mathbb{E} \left[\mathcal{Z}(f_n)^2 \right] = 2\pi |S^2| \int_0^\pi \tilde{K}_n(\phi) \sin \phi d\phi, \tag{35}$$

where

$$\tilde{K}_n(\phi) = \tilde{K}_n(x, y),$$

$x, y \in S^2$ being any pair of points with $d(x, y) = \phi$.

The main goal of the present paper is to understand the asymptotic behaviour of the function $\tilde{K}_n(\phi)$. To this end we will have to provide a more explicit formula for \tilde{K}_n by choosing concrete orthonormal frames in Sect. 2.4. It also turns out that it is more natural to *scale* the parameter ϕ by essentially n ; this will be done in Sect. 2.5.

2.4. Choosing orthonormal frames. Corollary 2.5 implies that it is sufficient to provide a choice $x, y \in S^2$ with $d(x, y) = \phi$ for any given $\phi \in (0, \pi)$, and for the choice made, provide local frames around x and y . Hence we may restrict ourselves only to points on the half circular arc

$$\check{N}S = \{\theta = 0\}.$$

Let

$$F = \left\{ e_1 = \frac{\partial}{\partial \phi}, e_2 = \frac{1}{\sin \phi} \frac{\partial}{\partial \theta} \right\} \tag{36}$$

be the orthonormal frame defined on $S^2 \setminus \{N, S\}$.

Given $\phi \in (0, \pi)$ we choose any pair of points $x, y \in \check{N}S \setminus \{N, S\}$ with $d(x, y) = \phi$ and set $F^x := F$ and $F^y := F$ locally in the neighbourhood of x and y respectively. An explicit computation shows that in this case the covariance matrix $\Sigma_n(x, y)$ depends

only on ϕ rather than on x, y , and, thus so does $\Omega_n(\phi) = \Omega_n(x, y)$ of our interest. We compute in Appendix A the conditional distribution covariance matrix explicitly to be

$$\Omega_n(\phi) = \begin{pmatrix} \frac{E_n}{2} + \tilde{a} & 0 & \tilde{b} & 0 \\ 0 & \frac{E_n}{2} & 0 & \tilde{c} \\ \tilde{b} & 0 & \frac{E_n}{2} + \tilde{a} & 0 \\ 0 & \tilde{c} & 0 & \frac{E_n}{2} \end{pmatrix}, \tag{37}$$

whose entries are given by

$$\tilde{a} = \tilde{a}_n(\phi) = -\frac{1}{1 - P_n(\cos \phi)^2} \cdot P'_n(\cos \phi)^2 (\sin \phi)^2, \tag{38}$$

$$\begin{aligned} \tilde{b} = \tilde{b}_n(\phi) &= P'_n(\cos \phi) \cos \phi - P''_n(\cos \phi) (\sin \phi)^2 \\ &+ \frac{P_n(\cos \phi)}{1 - P_n(\cos \phi)^2} \cdot P'_n(\cos \phi)^2 (\sin \phi)^2 \end{aligned} \tag{39}$$

and

$$\tilde{c} = \tilde{c}_n(\phi) = P'_n(\cos \phi). \tag{40}$$

We then have

$$\begin{aligned} \tilde{K}_n(\phi) &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{\sqrt{1 - u(x)^2}} \|w_1\| \cdot \|w_2\| \\ &\times \exp\left(-\frac{1}{2}(w_1, w_2)\Omega_n(\phi)^{-1}(w_1, w_2)^t\right) \frac{dw_1 dw_2}{(2\pi)^3 \sqrt{\det \Omega_n(\phi)}}, \end{aligned}$$

with the covariance matrix Ω_n given by (37).

Remark 2.6. We choose to work with points on the arc $\{\theta = 0\}$ since here the covariance matrix $\Omega_n(\phi)$ is relatively simple. This corresponds to Berry’s [3] choice of points on the x axis while dealing with random waves on \mathbb{R}^2 , which takes advantage of the fact that for two points $x, y \in \mathbb{R}^2$ on the x axis the canonical orthonormal bases for $T_x(\mathbb{R}^2)$ and $T_y(\mathbb{R}^2)$ coincide under the natural identification $T_x(\mathbb{R}^2) \cong T_y(\mathbb{R}^2)$. Rather than working with the canonical bases, one may of course choose to work with such orthonormal bases for any two points on the plane; this approach results in the same computation as on the x axis.

2.5. Scaling the integral formula. As pointed out earlier, the two-point correlation function \tilde{K}_n is expressible in terms of the covariance function u_n and a couple of its derivatives, which in turn are expressible in terms of degree n Legendre polynomial and its derivatives. The high energy asymptotics $n \rightarrow \infty$ of \tilde{K} is then intimately related to the behaviour of $P_n(\cos d)$ for large n . It is known from the Hilb’s asymptotics (see Appendix B) that

$$P_n(\cos \phi) \approx \sqrt{\frac{\phi}{\sin \phi}} J_0(\phi(n + 1/2)).$$

It is thus only natural to introduce a new parameter ψ related to ϕ by

$$\phi = \frac{\psi}{m},$$

where from this point and throughout the rest of the paper we denote

$$m := n + \frac{1}{2}. \tag{41}$$

We will rewrite the formula (35) in terms of ψ rather than ϕ , in the hope to simplify the subsequent computations.

Proposition 2.7. *The variance of the nodal length is given by*

$$\text{Var}(\mathcal{Z}(f_n)) = 4\pi^2 \frac{E_n}{n + 1/2} I_n, \tag{42}$$

where

$$I_n = \int_0^{\pi m} \left(K_n(\psi) - \frac{1}{4} \right) \sin(\psi/m) d\psi, \tag{43}$$

the scaled two-point correlation function

$$K_n(\psi) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{\sqrt{1 - u(x)^2}} \|w_1\| \cdot \|w_2\| \times \exp\left(-\frac{1}{2}(w_1, w_2)\Delta_n(\psi)^{-1}(w_1, w_2)^t\right) \frac{dw_1 dw_2}{(2\pi)^3 \sqrt{\det \Delta_n(\psi)}} \tag{44}$$

with scaled covariance matrix

$$\Delta_n(\psi) = \frac{\Omega(\psi/m)}{E_n/2} = \begin{pmatrix} 1 + 2a & 0 & 2b & 0 \\ 0 & 1 & 0 & 2c \\ 2b & 0 & 1 + 2a & 0 \\ 0 & 2c & 0 & 1 \end{pmatrix}, \tag{45}$$

whose entries are explicitly given by

$$a = a_n(\psi) = \frac{1}{E_n} \tilde{a}_n(\psi/m) = -\frac{1}{E_n} \frac{1}{1 - P_n(\cos(\psi/m))^2} \cdot P'_n(\cos(\psi/m))^2 \sin(\psi/m)^2, \tag{46}$$

$$b = b_n(\psi) = \frac{1}{E_n} \tilde{b}_n(\psi/m) = \frac{1}{E_n} \left[P'_n(\cos(\psi/m)) \cos(\psi/m) - P''_n(\cos(\psi/m)) \sin(\psi/m)^2 + \frac{P_n(\cos(\psi/m))}{1 - P_n(\cos(\psi/m))^2} \cdot P'_n(\cos(\psi/m))^2 \sin(\psi/m)^2 \right] \tag{47}$$

and

$$c = c_n(\psi) = \frac{1}{E_n} \tilde{c}_n(\psi/m) = \frac{1}{E_n} P'_n(\cos(\psi/m)). \tag{48}$$

Remark 2.8. Using the Cauchy-Schwartz inequality one can easily check that $|b_n(\psi)|, |c_n(\psi)| \leq \frac{1}{2}$. The inequality $|a_n(\psi)| \leq \frac{1}{2}$ is obvious.

Remark 2.9. We rewrite (28) as

$$\tilde{K}_n(\phi) = \tilde{K}_n(\pi - \phi). \tag{49}$$

Remark 2.10. One can express $K_n(\psi)$ in probabilistic language as

$$K_n(\psi) = \frac{1}{(2\pi)\sqrt{1 - P_n(\cos \psi/m)^2}} \mathbb{E}[\|U\| \cdot \|V\|],$$

where (U, V) are mean zero Gaussian random variables with covariance matrix $\Delta_n(\psi)$. We will find this expression useful later, when we will study the asymptotic behaviour of K_n for large ψ (see Proposition 3.5).

3. Asymptotics for the Variance

In this section we establish the asymptotics for the variance of the nodal length, i.e. prove Theorem 1.1. Recall that the variance of the nodal length is given by (42). Thus Theorem 1.1 is equivalent to the following proposition.

Proposition 3.1. *As $n \rightarrow \infty$ one has*

$$I_n = \frac{65}{128\pi^2} \frac{\log n}{n} + O\left(\frac{1}{n}\right).$$

The rest of the present section is dedicated to the proof of Proposition 3.1.

3.1. Asymptotics for I_n . Recall the definition

$$I_n = \int_0^{\pi m} \left(K_n(\psi) - \frac{1}{4} \right) \sin(\psi/m) d\psi$$

of I_n , where K_n is given by (44) or, equivalently, (61), and m is related to n via (41).

We note that the scaled version of (49) is

$$K_n(\psi) = K_n(\pi m - \psi). \tag{50}$$

Thus, by the definition (43) of I_n and (50), we have

$$I_n = 2 \cdot \int_0^{\pi m/2} \left(K_n(\psi) - \frac{1}{4} \right) \sin(\psi/m) d\psi. \tag{51}$$

Therefore we may concentrate ourselves on $[0, \pi m/2]$ rather than the full interval.

Ideally, to evaluate I_n , one would hope to obtain an explicit formula for $K_n(\psi)$. Unfortunately, to the best knowledge of the author of this paper, no such formula exists. However we will still be able to give an asymptotic expression for $K_n(\psi)$ for large values of ψ uniformly w.r.t. n and ψ .

For *small* values of ψ the behaviour of K_n is very different, due to the fact that as $\psi \rightarrow 0+$, $P_n(\cos(\psi/m))$ approaches 1, which results in the singularity of $\frac{1}{\sqrt{1-P_n(\cos(\psi/m))^2}}$ and of the covariance matrix $\Delta_n(\psi)$ at the origin. Nevertheless, we will see that this “singular” contribution is negligible, so that a relatively soft upper bound already obtained in [23] will suffice (see Lemma 3.2).

More precisely, we choose a constant $C > 0$, which is kept fixed throughout the rest of the computations, and write

$$\int_0^{\pi m/2} = \int_0^C + \int_C^{\pi m/2}. \tag{52}$$

The main contribution to the integral will come from the second (“nonsingular”) integral in (52) i.e. outside the origin. Our first task is then to bound the first (“singular”) integral of (52). A satisfactory bound was already given in [23].

Lemma 3.2. (Restatement of Lemma 4.2 from [23])⁵ *For any constant $C > 0$ we have as $n \rightarrow \infty$,*

$$\int_0^C \left| K_n(\psi) - \frac{1}{4} \right| \sin(\psi/m) d\psi = O\left(\frac{1}{n}\right).$$

Lemma 3.2 together with (52) and (51) yield the following lemma.

Lemma 3.3. *For any choice of the constant $C > 0$ we have as $n \rightarrow \infty$,*

$$I_n = 2\tilde{I}_n + O\left(\frac{1}{n}\right), \tag{53}$$

where

$$\tilde{I}_n = \int_C^{\pi m/2} \left(K_n(\psi) - \frac{1}{4} \right) \sin(\psi/m) d\psi. \tag{54}$$

Therefore, to understand the asymptotic behaviour of I_n it is sufficient to understand the asymptotic behavior of \tilde{I}_n . Proposition 3.4 resolves the latter. The proof of Proposition 3.4 is given throughout the rest of the present section.

Proposition 3.4. *For any choice of the constant $C > 0$ in the definition (54) of \tilde{I}_n , we have as $n \rightarrow \infty$,*

$$\tilde{I}_n = \frac{65}{256\pi^2} \frac{\log n}{n} + O\left(\frac{1}{n}\right). \tag{55}$$

Proof of Proposition 3.1 assuming Proposition 3.4. Just use Proposition 3.4 together with (53). \square

⁵ Note that in [23], Lemma 4.2 was given without scaling, i.e. in terms of ϕ rather than ψ .

3.2. *Asymptotics for the 2-point correlation function.* Recall that $K_n(\psi)$ is given by (44). One may notice that

$$K_n(\psi) = \frac{1}{2\pi\sqrt{1 - P_n(\cos(\psi/m))^2}} F(a_n(\psi), b_n(\psi), c_n(\psi)),$$

where $F(\alpha, \beta, \gamma)$ is a smooth function independent of n , defined on some neighbourhood of the origin $(\alpha, \beta, \gamma) = (0, 0, 0)$. Its arguments $a_n(\psi)$, $b_n(\psi)$ and $c_n(\psi)$ are uniformly small for $\psi > C$ (see Lemma 3.9).

An easy explicit computation shows that

$$F(0, 0, 0) = \frac{1}{4},$$

which cancels out with the constant term in (43) which corresponds to $(\mathbb{E}\mathcal{Z})^2$. This is not a coincidence, since the origin $\alpha = \beta = \gamma = 0$ corresponds to the covariance matrix Δ_n being the identity matrix $\Delta_n = I_n$; in this case the probability density function factors.

We choose to expand $F(\alpha, \beta, \gamma)$ into a finite Taylor polynomial around the origin.⁶ We note that the matrix elements are of different order or decay rate, so that we may cut the smaller terms earlier than the larger ones. The decay rate of a_n, b_n and c_n prescribed by Lemma 3.9 implies that it is sufficient to expand a_n, b_n and c_n up to 2nd, 4th and 1st degrees respectively.

The following is the Key Proposition for the whole paper. We will reuse it while proving Theorem 1.4 (see Sect. 4) for smooth linear statistics of the nodal line, (see also Remark 3.7).

Proposition 3.5. (Key Proposition) *For any choice of $C > 0$, as $n \rightarrow \infty$, one has*

$$K_n(\psi) = \frac{1}{4} + \frac{1}{2} \frac{\sin(2\psi)}{\pi n \sin(\psi/m)} + \frac{65}{256} \frac{1}{\pi^2 n \sin(\psi/m)\psi} + \frac{9}{32} \frac{\cos(2\psi)}{\pi n \psi \sin(\psi/m)} + \frac{\frac{27}{64} \sin(2\psi) - \frac{11}{256} \cos(4\psi)}{\pi^2 n \psi \sin(\psi/m)} + O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right) \tag{56}$$

uniformly for $C < \psi < \pi m/2$.

Remark 3.6. It is important to notice that the leading nonconstant term $\frac{1}{2} \frac{\sin(2\psi)}{\pi n \sin(\psi/m)}$ is oscillating, and we will see that it does not contribute to the variance (see the proof of Proposition 3.1). We observe this obscure ‘‘Berry’s cancelation phenomenon’’, which is responsible for the variance being surprisingly small, in some other situations, such as Berry’s original work [3], and on the torus [14]. This suggests that this phenomenon is of a more general nature, and we expect it to occur on a ‘‘generic’’ surface [22].

Remark 3.7. As another application of Proposition 3.5, one may exploit it to study the morphology of the nodal lines for n -dependent linear statistics $\varphi = \varphi_n$. It is most efficient for φ_n whose support is not shrinking too rapidly (relatively to the scaling $\psi \approx n\phi$ we introduced earlier), for example φ_n a characteristic function of a spherical disc of radius a_n , where

$$a_n \cdot n \rightarrow \infty.$$

⁶ Intuitively, the origin $a = b = c = 0$ corresponds to $\psi = \infty$ (see the decay at infinity in Lemma 3.9), hence this expansion should be good for large values of ψ .

For finer “local” statistics of the nodal lines, such as studying the nodal length inside a spherical disc of radius $\approx \frac{1}{n}$, one needs to expand $K_n(\psi)$ around the origin, where the behaviour is very different from the one at infinity. We may want to do so in the future.

The asymptotic evaluation (56) in Proposition 3.5 is done in two steps. Lemma 3.8 provides an approximation for the two-point correlation function $K_n(\psi)$ with a polynomial in $P_n(\cos(\psi/m))$, $a_n(\psi)$, $b_n(\psi)$ and $c_n(\psi)$ i.e. the Taylor expansion of K_n as a function of the above expressions. In the second step, performed in Lemma 3.9, we evaluate each of the terms appearing in the Taylor expansion obtained in the first step using the high degree asymptotics of the Legendre polynomials and its derivatives (Hilb asymptotics). Lemmas 3.8 and 3.9 are proved in Sects. 3.4.1 and 3.4.2 respectively.

Lemma 3.8. *For $C > 0$ large enough, one has the following expansion on $[C, \pi m/2]$:*

$$\begin{aligned}
 K_n(\psi) &= \frac{1}{4} + \frac{1}{4} \cdot a_n(\psi) + \frac{1}{8} \cdot b_n(\psi)^2 - \frac{1}{32} \cdot a_n(\psi)^2 - \frac{3}{16} \cdot a_n(\psi)b_n(\psi)^2 \\
 &\quad + \frac{3}{128} \cdot b_n(\psi)^4 + \frac{1}{8} \cdot P_n(\cos(\psi/m))^2 + \frac{1}{8} \cdot a_n(\psi)P_n(\cos(\psi/m))^2 \\
 &\quad + \frac{1}{16} \cdot b_n(\psi)^2P_n(\cos(\psi/m))^2 + \frac{3}{32} \cdot P_n(\cos(\psi/m))^4 \\
 &\quad + O\left(P_n(\cos(\psi/m))^6 + a_n(\psi)^3 + b_n(\psi)^5 + c_n(\psi)^2\right), \tag{57}
 \end{aligned}$$

where the constants involved in the “ O ”-notation depend only on C .

Lemma 3.9. *For $n \geq 1$, $C < \psi < \pi m/2$ we have the following estimates for a , b , c and $P_n(\cos(\psi/m))$:*

(1)

$$\begin{aligned}
 P_n(\cos(\psi/m))^2 &= \frac{1 + \sin(2\psi)}{\pi n \sin(\psi/m)} - \frac{\cos(2\psi)}{4\pi n \psi \sin(\psi/m)} + O\left(\frac{1}{\psi^3} + \frac{1}{\psi n}\right) \\
 &= \frac{1 + \sin(2\psi)}{\pi n \sin(\psi/m)} + O\left(\frac{1}{\psi^2}\right). \tag{58}
 \end{aligned}$$

(2)

$$\begin{aligned}
 a_n(\psi) &= -\frac{1 - \sin(2\psi)}{\pi n \sin(\psi/m)} + \frac{3 \cos(2\psi)}{4\pi n^2 \sin(\psi/m)^2} - \frac{1 + \cos(4\psi)}{2\pi^2 n^2 \sin(\psi/m)^2} \\
 &\quad + O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right) \\
 &= -\frac{1 - \sin(2\psi)}{\pi n \sin(\psi/m)} + O\left(\frac{1}{\psi^2}\right).
 \end{aligned}$$

(3)

$$b_n(\psi)^2 = \frac{1 + \sin(2\psi)}{\pi n \sin(\psi/m)} + \frac{7 \cos(2\psi)}{4\pi n \sin(\psi/m)\psi} + \frac{1 + \cos(4\psi)}{\pi^2 n \sin(\psi/m)\psi} + O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right).$$

(4)

$$|c_n(\psi)| = O\left(\frac{1}{\psi^{3/2}}\right).$$

(5)

$$P_n(\cos(\psi/m))^4 = \frac{\frac{3}{2} + 2 \sin(2\psi) - \frac{1}{2} \cos(4\psi)}{\pi^2 n^2 \sin(\psi/m)^2} + O\left(\frac{1}{\psi^3}\right).$$

(6)

$$a_n(\psi)^2 = \frac{\frac{3}{2} - 2 \sin(2\psi) - \frac{1}{2} \cos(4\psi)}{\pi^2 n^2 \sin(\psi/m)^2} + O\left(\frac{1}{\psi^3}\right).$$

(7)

$$b_n(\psi)^4 = \frac{\frac{3}{2} + 2 \sin(2\psi) - \frac{1}{2} \cos(4\psi)}{\pi^2 n^2 \sin(\psi/m)^2} + O\left(\frac{1}{\psi^3}\right).$$

(8)

$$P_n(\cos(\psi/m))^2 a_n(\psi) = -\frac{1 + \cos(4\psi)}{2\pi^2 n^2 \sin(\psi/m)^2} + O\left(\frac{1}{\psi^3}\right).$$

(9)

$$P_n(\cos(\psi/m))^2 b_n(\psi)^2 = \frac{3/2 + 2 \sin(2\psi) - \frac{1}{2} \cos(4\psi)}{\pi^2 n^2 \sin(\psi/m)^2} + O\left(\frac{1}{\psi^3}\right).$$

(10)

$$a_n(\psi)b_n(\psi)^2 = -\frac{1 + \cos(4\psi)}{2\pi^2 n^2 \sin(\psi/m)^2} + O\left(\frac{1}{\psi^3}\right).$$

Proof of Proposition 3.5. Substituting all the various estimates in Lemma 3.9 into (57) we obtain, after collecting similar terms together and some reorganization (replacing $\frac{1}{n^2 \sin(\psi/m)^2}$ by $\frac{1}{n \sin(\psi/m)\psi}$ whenever necessary)

$$\begin{aligned} K_n(\psi) &= \frac{1}{4} + \frac{1}{\pi n \sin(\psi/m)} \left(\frac{1}{4} \cdot \sin(2\psi) + \frac{1}{8} \sin(2\psi) + \frac{1}{8} \sin(2\psi) \right) \\ &+ \frac{1}{\pi^2 n \sin(\psi/m)\psi} \left(-\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{8} \cdot 1 - \frac{1}{32} \cdot \frac{3}{2} + \frac{3}{16} \cdot \frac{1}{2} + \frac{3}{128} \cdot \frac{3}{2} \right. \\ &+ \left. \frac{1}{8} \cdot 0 - \frac{1}{8} \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{3}{2} + \frac{3}{32} \cdot \frac{3}{2} \right) \\ &+ \frac{1}{\pi n \psi \sin(\psi/m)} \left(\frac{1}{4} \cdot \frac{3}{4} \cos(2\psi) + \frac{1}{8} \cdot \frac{7}{4} \cos(2\psi) - \frac{1}{8} \cdot \frac{1}{4} \cos(2\psi) \right) \\ &+ \frac{1}{\pi^2 n \psi \sin(\psi/m)} \left(-\frac{1}{4} \cdot \frac{1}{2} \cos(4\psi) + \frac{1}{8} \cdot \cos(4\psi) + \frac{1}{32} (2 \sin(2\psi) \right. \\ &+ \left. \frac{1}{2} \cos(4\psi)) + \frac{3}{16} \cdot \frac{1}{2} \cos(4\psi) + \frac{3}{128} (2 \sin(2\psi) - \frac{1}{2} \cos(4\psi)) - \frac{1}{8} \cdot \frac{1}{2} \right. \\ &\times \left. \cos(4\psi) + \frac{1}{16} (2 \sin(2\psi) - \frac{1}{2} \cos(4\psi)) + \frac{3}{32} \cdot (2 \sin(2\psi) - \frac{1}{2} \cos(4\psi)) \right) \end{aligned}$$

$$\begin{aligned}
 &+O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right) \\
 = &\frac{1}{4} + \frac{1}{2} \frac{\sin(2\psi)}{\pi n \sin(\psi/m)} + \frac{65}{256} \frac{1}{\pi^2 n \sin(\psi/m)\psi} + \frac{3}{8} \frac{\cos(2\psi)}{\pi n \psi \sin(\psi/m)} \\
 &+ \frac{\frac{27}{64} \sin(2\psi) - \frac{11}{256} \cos(4\psi)}{\pi^2 n \psi \sin(\psi/m)} + O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right).
 \end{aligned}$$

□

3.3. *Concluding the proof of Proposition 3.4.* All the hard work establishing the asymptotics (56) of $K_n(\psi)$ at infinity finally pays off as the proof of Proposition 3.4 is now straightforward.

Proof of Proposition 3.4. Recall that \tilde{I}_n is given by (54), where $K_n(\psi)$ for large ψ we may asymptotically expand $K_n(\psi)$ as (56). First note that the constant term $\frac{1}{4}$ in (56) cancels out in (54). Thus we have

$$\begin{aligned}
 \tilde{I}_n &= \int_C^{\pi m/2} \left[\frac{1}{2} \frac{\sin(2\psi)}{\pi n \sin(\psi/m)} + \frac{65}{256} \frac{1}{\pi^2 n \sin(\psi/m)\psi} + \frac{9}{32} \frac{\cos(2\psi)}{\pi n \psi \sin(\psi/m)} \right. \\
 &\quad \left. + \frac{\frac{27}{64} \sin(2\psi) - \frac{11}{256} \cos(4\psi)}{\pi^2 n \psi \sin(\psi/m)} \right] \sin(\psi/m) d\psi \\
 &\quad + O\left(\int_C^{\pi m/2} \left[\frac{1}{\psi^3} + \frac{1}{n\psi} \right] \sin(\psi/m) d\psi \right) \\
 &= \frac{1}{\pi n} \int_C^{\pi m/2} \left[\frac{1}{2} \sin(2\psi) + \frac{65}{256} \frac{1}{\pi \psi} + \frac{9}{32} \frac{\cos(2\psi)}{\psi} + \frac{\frac{27}{64} \sin(2\psi) - \frac{11}{256} \cos(4\psi)}{\pi \psi} \right] d\psi \\
 &\quad + O\left(\frac{1}{n}\right). \tag{59}
 \end{aligned}$$

The contribution of the first term in (59) is bounded by

$$\ll \frac{1}{n} \int_C^{\pi m/2} \sin(2\psi) d\psi = O\left(\frac{1}{n}\right).$$

The main contribution to (59) comes from the leading non-oscillatory term, i.e. the second term:

$$\frac{65}{256\pi^2 n} \int_C^{\pi m/2} \frac{d\psi}{\psi} = \frac{65}{256\pi^2} \cdot \frac{\log n}{n} + O\left(\frac{1}{n}\right). \tag{60}$$

Bounding the contribution of the other oscillatory terms using integration by parts, as well as bounding the error term in (59), is easy. The asymptotic expression (60) together with the bound for the contribution of the other terms in (56) yields the result (55) of the present proposition. □

3.4. Proofs of auxiliary lemmas.

3.4.1. Taylor expansion for $K_n(\psi)$ In principle, one may compute the coefficients of the multivariate Taylor expansion directly using the Leibnitz rule for differentiating under the integral sign. The following elegant method due to Berry [3] gives the necessary Taylor coefficients avoiding the long and tedious computations.

Proof of Lemma 3.8. We use Remark 2.10 to write

$$K_n(\psi) = \frac{1}{(2\pi)\sqrt{1 - P_n(\cos \psi/m)^2}} \mathbb{E}[\|U\| \|V\|], \tag{61}$$

where (U, V) is a mean zero multivariate Gaussian random vector with covariance matrix $\Delta = \Delta_n(\psi)$ given by (45). On $[C, \pi m/2]$ we may Taylor expand the first term in (61) as

$$\frac{1}{\sqrt{1 - P_n(\cos \psi/m)^2}} = 1 + \frac{1}{2} P_n(\cos \psi/m)^2 + \frac{3}{8} P_n(\cos \psi/m)^4 + O\left(P_n(\cos \psi/m)^6\right), \tag{62}$$

since by part 1 of Lemma 3.9, $|P_n(\cos(\psi/m))|$ is bounded away from 1, provided that C is large enough. It then remains to expand the remaining part of (61), i.e. $\mathbb{E}[\|U\| \|V\|]$ in terms of powers of a, b and c .

To this end we use the identity

$$\sqrt{\alpha} = \frac{1}{\sqrt{2\pi}} \int_0^\infty (1 - e^{-\frac{\alpha t}{2}}) \frac{dt}{t^{3/2}},$$

which implies

$$\begin{aligned} \mathbb{E}[\|U\| \|V\|] &= \mathbb{E} \left[\frac{1}{2\pi} \int_0^\infty \int_0^\infty \left(1 - e^{-\frac{t\|U\|^2}{2}}\right) \left(1 - e^{-\frac{t\|V\|^2}{2}}\right) \frac{dt ds}{(ts)^{3/2}} \right] \\ &= \frac{1}{2\pi} \int_0^\infty \int_0^\infty [f(0, 0) - f(t, 0) - f(0, s) + f(t, s)] \frac{dt ds}{(ts)^{3/2}}, \end{aligned} \tag{63}$$

where we define

$$\begin{aligned} f(t, s) &= f^{\Delta_n(\psi)}(t, s) := \mathbb{E} \left[e^{-\frac{t\|U\|^2 + s\|V\|^2}{2}} \right] \\ &= \frac{1}{(2\pi)^2 (\det \Delta_n(\psi))^{1/2}} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{-\frac{1}{2} W^t \left(\Delta_n(\psi)^{-1} + \begin{pmatrix} tI_2 & \\ & sI_2 \end{pmatrix} \right) W} dW \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(\det \Delta_n(\psi))^{1/2} \det \left(\Delta_n(\psi)^{-1} + \begin{pmatrix} tI_2 & \\ & sI_2 \end{pmatrix} \right)^{1/2}} \\
 &= \frac{1}{\det \left(I + \begin{pmatrix} tI_2 & \\ & sI_2 \end{pmatrix} \Delta_n(\psi) \right)^{1/2}}.
 \end{aligned} \tag{64}$$

Let

$$\Delta_n(\psi) = I + \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = A_n(\psi) = \begin{pmatrix} 2a & \\ & 0 \end{pmatrix}; \quad B = B_n(\psi) = \begin{pmatrix} 2b & \\ & 2c \end{pmatrix},$$

with entries defined by (46), (47) and (48). Thus we have

$$I + \begin{pmatrix} tI_2 & \\ & sI_2 \end{pmatrix} \Delta_n(\psi) = \begin{pmatrix} (1+t)I + tA & tB \\ sB & (1+s)I + sA \end{pmatrix},$$

so that

$$\begin{aligned}
 &\det \left(I + \begin{pmatrix} tI_2 & \\ & sI_2 \end{pmatrix} \Delta \right) \\
 &= \det ((1+t)I + tA) \det \left((1+s)I + sA - stB((1+t)I + tA)^{-1}B \right) \\
 &= (1+t)^2(1+s)^2 \det \left(I + \frac{t}{1+t}A \right) \\
 &\quad \times \det \left(I + \frac{s}{1+s}A - \frac{st}{(1+s)(1+t)}B \left(I + \frac{t}{1+t}A \right)^{-1}B \right),
 \end{aligned} \tag{65}$$

where we make use of the fact that both A and B are diagonal and hence commute.

We compute the first determinant explicitly as

$$\det \left(I + \frac{t}{1+t}A \right) = 1 + \frac{t}{1+t}2a. \tag{66}$$

Next we wish to compute the other determinant in (65). For this we write

$$\left(I + \frac{t}{1+t}A \right)^{-1} = I - \frac{t}{1+t}A + O(a^2),$$

where we understand the “ O ”-notation entry-wise. Therefore (taking advantage of the fact that all the matrices involved are diagonal), we have

$$\begin{aligned}
 &I + \frac{s}{1+s}A - \frac{st}{(1+s)(1+t)}B^2 \left(I + \frac{t}{1+t}A \right)^{-1} \\
 &= I + \frac{s}{1+s}A - \frac{st}{(1+s)(1+t)}B^2 + \frac{st^2}{(1+s)(1+t)^2}AB^2 + O(a^2b^2) \\
 &= \left(1 + \frac{2s}{1+s}a - \frac{4st}{(1+s)(1+t)}b^2 + \frac{8st^2}{(1+s)(1+t)^2}ab^2 \right) \left(1 - \frac{4st}{(1+s)(1+t)}c^2 \right) + O(a^2b^2).
 \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \det \left(I + \frac{s}{1+s} A - \frac{st}{(1+s)(1+t)} B^2 \left(I + \frac{t}{1+t} A \right)^{-1} \right) \\ &= \left(1 + \frac{2s}{1+s} a - \frac{4st}{(1+s)(1+t)} b^2 + \frac{8st^2}{(1+s)(1+t)^2} ab^2 \right) \left(1 - \frac{4st}{(1+s)(1+t)} c^2 \right) \\ & \quad + O \left((a^2 b^2 + c^2) \right) \\ &= \left(1 + \frac{2s}{1+s} a - \frac{4st}{(1+s)(1+t)} b^2 + \frac{8st^2}{(1+s)(1+t)^2} ab^2 \right) + O \left(a^2 b^2 + c^2 \right). \end{aligned}$$

Substituting (66) and (67) into (65) we obtain

$$\begin{aligned} \det \left(I + \begin{pmatrix} tI_2 & \\ & sI_2 \end{pmatrix} \Delta \right) &= (1+t)^2(1+s)^2 \left(1 + \frac{2t}{1+t} a \right) \tag{67} \\ & \times \left(1 + \frac{2s}{1+s} a - \frac{4st}{(1+s)(1+t)} b^2 + \frac{8st^2}{(1+s)(1+t)^2} ab^2 + O \left(a^2 b^2 + c^2 \right) \right) \\ &= (1+t)^2(1+s)^2 \\ & \times \left(1 + \frac{2(t+s+2st)}{(1+s)(1+t)} a + \frac{4st}{(1+s)(1+t)} (a^2 - b^2) + O \left(a^2 b^2 + c^2 \right) \right). \end{aligned}$$

Now we use the expansion

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + O(x^3)$$

to write

$$\begin{aligned} f^{\Delta_n(\psi)}(t, s) &= \frac{1}{\det \left(I + \begin{pmatrix} tI_2 & \\ & sI_2 \end{pmatrix} \Delta \right)^{1/2}} = \frac{1}{(1+t)(1+s)} \\ & \times \left(1 - \frac{(t+s+2st)}{(1+s)(1+t)} a + \frac{2st}{(1+s)(1+t)} b^2 \right. \\ & \quad + \left(\frac{3s^2}{2(1+s)^2} + \frac{3t^2}{2(1+t)^2} + \frac{st}{(1+s)(1+t)} \right) a^2 \\ & \quad \left. - \frac{6st}{(1+t)^2(1+s)^2} (t+s+2st) ab^2 + \frac{6t^2s^2}{(1+t)^2(1+s)^2} b^4 O(a^3 + b^5 + c^2) \right). \tag{68} \end{aligned}$$

The asymptotic expansion (68) implies, in particular,

$$f^{\Delta_n(\psi)}(0, 0) = 1, \tag{69}$$

$$f^{\Delta_n(\psi)}(t, 0) = \frac{1}{1+t} \left(1 - \frac{t}{1+t} a + \frac{3t^2}{2(1+t)^2} a^2 + O(a^3 + b^5 + c^2) \right), \tag{70}$$

and

$$f^{\Delta_n(\psi)}(0, s) = \frac{1}{1+s} \left(1 - \frac{s}{1+s}a + \frac{3s^2}{2(1+s)^2}a^2 + O(a^3 + b^5 + c^2) \right). \tag{71}$$

Define

$$F^{\Delta_n(\psi)}(t, s) := f^{\Delta_n(\psi)}(0, 0) - f^{\Delta_n(\psi)}(t, 0) - f^{\Delta_n(\psi)}(0, s) + f^{\Delta_n(\psi)}(t, s), \tag{72}$$

so that in the new notations (63) is

$$\mathbb{E}[\|U\| \|V\|] = \frac{1}{2\pi} \int_0^\infty \int_0^\infty F^{\Delta_n(\psi)}(t, s) \frac{dt ds}{(ts)^{3/2}}. \tag{73}$$

Plugging the estimates (68), (69), (70) and (71) into the definition (72) of $F^{\Delta_n(\psi)}$ yields

$$\begin{aligned} F^{\Delta_n(\psi)}(t, s) &= \frac{ts}{(1+t)(1+s)} + \frac{st(2+t+s)}{(1+t)^2(1+s)^2} \cdot a \\ &+ \frac{2st}{(1+s)^2(1+t)^2} \cdot b^2 - \frac{ts(t+10ts+s+3ts^2+3t^2s-2)}{2(1+t)^3(1+s)^3} \cdot a^2 \\ &- \frac{6st}{(1+t)^3(1+s)^3}(t+s+2st) \cdot ab^2 + \frac{6t^2s^2}{(1+t)^3(1+s)^3} \cdot b^4 \\ &+ O(a^3 + b^5 + c^2), \end{aligned} \tag{74}$$

where the constants involved in the ‘‘O’’-notation are *universal*. We wish to plug (74) into (73) and integrate with respect to t and s . The problem is that the integral $\int_0^\infty \frac{dt}{t^{3/2}}$ diverges at the origin so that the bound for the error term in (74) is not sufficient. To resolve this issue we notice that (72) implies that we have

$$F^{\Delta_n(\psi)}(t, s)|_{t=0} = F^{\Delta_n(\psi)}(t, s)|_{s=0} = 0, \tag{75}$$

and identify the expression (74) as the Taylor expansion of $F^{\Delta_n(\psi)}(t, s)$ considered as a function of (a, b, c) with fixed parameters t, s around the origin $(a, b, c) = (0, 0, 0)$. The vanishing property (75) implies that all the Taylor coefficients in the expansion (74) considered as a function of t, s , are divisible by ts , so that we may improve the error term in (74) as

$$\begin{aligned} F^{\Delta_n(\psi)}(t, s) &= \frac{ts}{(1+t)(1+s)} + \frac{st(2+t+s)}{(1+t)^2(1+s)^2} \cdot a \\ &+ \frac{2st}{(1+s)^2(1+t)^2} \cdot b^2 - \frac{ts(t+10ts+s+3ts^2+3t^2s-2)}{2(1+t)^3(1+s)^3} \cdot a^2 \\ &- \frac{6st}{(1+t)^3(1+s)^3}(t+s+2st) \cdot ab^2 + \frac{6t^2s^2}{(1+t)^3(1+s)^3} \cdot b^4 \\ &+ O\left(m(t, s) \cdot (a^3 + b^5 + c^2)\right), \end{aligned} \tag{76}$$

where we introduce the notations $m(t) := \min\{t, 1\}$ and

$$m(t, s) := m(t) \cdot m(s).$$

Plugging (76) into (73) and integrating term by term we obtain

$$\mathbb{E}[\|U\| \|V\|] = \frac{\pi}{2} + \frac{\pi}{2} \cdot a + \frac{\pi}{4} \cdot b^2 - \frac{\pi}{16} \cdot a^2 - \frac{3\pi}{8} \cdot ab^2 + \frac{3\pi}{64} b^4 + O(a^3 + b^5 + c^2), \tag{77}$$

where we used the standard integrals

$$\int_0^\infty \frac{dt}{\sqrt{t}(1+t)} = \pi, \quad \int_0^\infty \frac{dt}{\sqrt{t}(1+t)^2} = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sqrt{t}}{(1+t)^2} dt = \frac{\pi}{2},$$

$$\int_0^\infty \frac{t^{3/2}}{(1+t)^3} dt = \frac{3\pi}{8}, \quad \int_0^\infty \frac{\sqrt{t}}{(1+t)^3} dt = \frac{\pi}{8}, \quad \int_0^\infty \frac{dt}{\sqrt{t}(1+t)^3} = \frac{3\pi}{8}.$$

We finally plug the estimates (62) and (77) into (61) to obtain (57), that is the statement of the present lemma. \square

3.4.2. Some estimates related to the matrix elements In this section we evaluate the various expressions appearing in (57) asymptotically as $\psi \rightarrow \infty$, namely prove Lemma 3.9. To evaluate the matrix elements a, b, c we will need to deal with the asymptotic behaviour of the Legendre polynomials of high degree. The reader may find the necessary background on the Legendre polynomials as well as some basic asymptotic estimates in Appendix B (see Lemma B.3).

Proof of Lemma 3.9. It is easy to check that parts 5-10 of the present lemma follow directly from parts 1-4. Moreover, part 1 may be obtained by a straightforward application of (116), and part 4 is a direct consequence of the high degree asymptotics (117) for the derivatives of Legendre polynomials. It then remains to prove parts 2-3.

Recall that we assume that $C < \psi < \pi m/2$, so that $P_n(\cos(\psi/m))$ is bounded away from 1 by Hilb’s asymptotics (115). Hence we may write

$$a_n(\psi) = -\frac{1}{n^2} P'(\cos(\psi/m))^2 \sin(\psi/m)^2 - \frac{1}{n^2} P(\cos(\psi/m))^2 P'(\cos(\psi/m))^2 \sin(\psi/m)^2 + O\left(\frac{1}{n\psi} + \frac{1}{\psi^3}\right), \tag{78}$$

where to bound the error term we used the decay

$$|P'(\cos(\psi/m))| = O\left(\frac{n^2}{\psi^{3/2}}\right), \tag{79}$$

which follows from (117).

Now we use (117) to obtain

$$\begin{aligned}
 \frac{1}{n^2} P_n'(\cos(\psi/m))^2 \sin(\psi/m)^2 &= \frac{2}{\pi n \sin(\psi/m)^3} \left(\sin(\psi/m)^2 \sin\left(\psi - \frac{\pi}{4}\right)^2 \right. \\
 &\quad \left. - \frac{3}{8n} \sin(\psi/m) \cos(2\psi) + O\left(\frac{1}{n^2}\right) \right) \\
 &\quad + O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right) \\
 &= \frac{1 - \sin(2\psi)}{\pi n \sin(\psi/m)} - \frac{3 \cos(2\psi)}{4\pi n^2 \sin(\psi/m)^2} \\
 &\quad + O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right), \tag{80}
 \end{aligned}$$

and (58) together with (80) imply

$$\begin{aligned}
 \frac{1}{n^2} P_n(\cos(\psi/m))^2 P_n'(\cos(\psi/m))^2 \sin(\psi/m)^2 &= \frac{1 + \cos(4\psi)}{2\pi^2 n^2 \sin(\psi/m)^2} \\
 &\quad + O\left(\frac{1}{\psi^3} + \frac{1}{n\psi}\right). \tag{81}
 \end{aligned}$$

Substituting (80) and (81) into (78) we obtain part 2 of the present lemma.

It then remains to prove part 3 of the lemma, i.e. establish a two-term asymptotics for $b_n(\psi)^2$. To achieve that we first evaluate $b_n(\psi)$. From the definition (47) of $b_n(\psi)$ we have, using (79) to replace $E_n = n(n+1)$ by n^2 and

$$\cos(\psi/m) = 1 + O\left(\frac{\psi^2}{n^2}\right)$$

to replace $\cos(\psi/m)$ by 1,

$$\begin{aligned}
 b_n(\psi) &= \frac{1}{n^2} P_n'(\cos(\psi/m)) \cos(\psi/m) - \frac{1}{n^2} P_n''(\cos(\psi/m)) \sin(\psi/m)^2 \\
 &\quad + \frac{1}{n^2} P_n(\cos(\psi/m)) P_n'(\cos(\psi/m))^2 \sin(\psi/m)^2 + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{n\sqrt{\psi}}\right) \\
 &= \frac{1}{n^2} P_n'(\cos(\psi/m)) - \frac{1}{n^2} P_n''(\cos(\psi/m)) \sin(\psi/m)^2 \\
 &\quad + \frac{1}{n^2} P_n(\cos(\psi/m)) P_n'(\cos(\psi/m))^2 \sin(\psi/m)^2 + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{n\sqrt{\psi}}\right).
 \end{aligned}$$

Next we use the differential equation (113) satisfied by the Legendre polynomials to write

$$\begin{aligned}
 b_n(\psi) &= P_n(\cos(\psi/m)) - \frac{1}{n^2} P_n'(\cos(\psi/m)) \\
 &\quad + \frac{1}{n^2} P_n(\cos(\psi/m)) P_n'(\cos(\psi/m))^2 \sin(\psi/m)^2 + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{n\sqrt{\psi}}\right) \\
 &= \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \left(\sin\left(\psi + \frac{\pi}{4}\right) - \frac{1}{8} \frac{\cos\left(\psi + \frac{\pi}{4}\right)}{\psi} \right) - \sqrt{\frac{2}{\pi}} \frac{\sin\left(\psi - \frac{\pi}{4}\right)}{n^{3/2} \sin(\psi/m)^{3/2}} \\
 &\quad + \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \sin\left(\psi + \frac{\pi}{4}\right) \cdot \frac{1 - \sin(2\psi)}{\pi n \sin(\psi/m)} + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{n\sqrt{\psi}}\right),
 \end{aligned}$$

where we used (116), (117) and reused (80) once more to obtain the second equality. Reorganizing the terms in the last expression, we have

$$\begin{aligned}
 b_n(\psi) &= \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \sin\left(\psi + \frac{\pi}{4}\right) + \frac{7}{8} \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \frac{\cos\left(\psi + \frac{\pi}{4}\right)}{\psi} \\
 &\quad + \frac{\sqrt{2} \cdot \left(\sin\left(\psi + \frac{\pi}{4}\right) + \frac{1}{2} \cos\left(3\psi + \frac{\pi}{4}\right) - \frac{1}{2} \cos\left(\psi - \frac{\pi}{4}\right)\right)}{\pi^{3/2} n^{3/2} \sin(\psi/m)^{3/2}} \\
 &\quad + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{n\sqrt{\psi}}\right) \\
 &= \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \sin\left(\psi + \frac{\pi}{4}\right) + \frac{7}{8} \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \frac{\cos\left(\psi + \frac{\pi}{4}\right)}{\psi} \\
 &\quad + \frac{\left(\sin\left(\psi + \frac{\pi}{4}\right) + \cos\left(3\psi + \frac{\pi}{4}\right)\right)}{\sqrt{2} \pi^{3/2} n^{3/2} \sin(\psi/m)^{3/2}} + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{n\sqrt{\psi}}\right),
 \end{aligned}$$

and we obtain part 3 of the present lemma by squaring the last equality. \square

4. Proof of Theorem 1.4

In this section we assume that $\varphi : S^2 \rightarrow \mathbb{R}$ is a *continuously differentiable even* function. For the sake of proving Theorem 1.5, we will conduct the analysis of the error terms in terms of the L^∞ norm $\|\varphi\|_\infty$ and the total variation $V(\varphi)$ of the test function, as prescribed by Theorem 1.4.

Our first goal is to formulate an analogue of Proposition 2.7 for the variance of $\mathcal{Z}^\varphi(f_n)$. It turns out that a certain auxiliary function W^φ defined below comes out from a straightforward repetition of the steps we performed in Sect. 2.5, adapted to suit \mathcal{Z}^φ rather than \mathcal{Z} .

For $\varphi \in C^1(S^2)$ the analogue of (25) is

$$\mathbb{E} \left[\mathcal{Z}(f_n)^2 \right] = \iint_{S^2 \times S^2} \varphi(x)\varphi(y) \tilde{K}_n(x, y) dx dy,$$

where $\tilde{K}_n(x, y)$ is given again by (26). Since $\tilde{K}(x, y) = \tilde{K}(\phi)$, where $\phi = d(x, y)$, we may employ Fubini, to obtain (cf. (35))

$$\mathbb{E} \left[\mathcal{Z}^\varphi(f_n)^2 \right] = 2\pi |S^2| \int_0^\pi \tilde{K}_n(\phi) W^\varphi(\phi) d\phi,$$

where $W^\varphi : [0, \pi] \rightarrow \mathbb{R}$ is a continuously differentiable function defined by

$$W^\varphi(\phi) := \frac{1}{8\pi^2} \int_{d(x,y)=\phi} \varphi(x)\varphi(y)dx dy. \tag{82}$$

For example, for the constant function $\varphi \equiv 1$ we have

$$W^1(\phi) = \sin(\phi).$$

It is easy to check that, since $d(x, -y) = \pi - d(x, y)$, we have

$$W^\varphi(\pi - \phi) = W^\varphi(\phi), \tag{83}$$

as we assume that φ is even.

Scaling the integrand in the same manner exactly as in Sect. 2.5, we finally obtain the following lemma (cf. Proposition 2.7).

Lemma 4.1. *The variance of $\mathcal{Z}^\varphi(f_n)$ is given by*

$$\text{Var}(\mathcal{Z}^\varphi(f_n)) = 4\pi^2 \frac{E_n}{n + 1/2} I_n^\varphi, \tag{84}$$

where

$$I_n^\varphi = \int_0^{\pi m} \left(K_n(\psi) - \frac{1}{4} \right) W^\varphi(\psi/m) d\psi.$$

Remark 4.2. One deduces from (50) and (83) that

$$I_n^\varphi = 2 \int_0^{\pi m/2} \left(K_n(\psi) - \frac{1}{4} \right) W^\varphi(\psi/m) d\psi. \tag{85}$$

We will need some rather simple properties of W^φ . Writing the double integral (82) as an iterated integral and using the spherical coordinates with pole at x for each $x \in \mathcal{S}^2$ we obtain

$$W^\varphi(\phi) = \frac{1}{8\pi^2} \sin(\phi) W_0^\varphi(\phi), \tag{86}$$

where

$$W_0^\varphi(\phi) = \int_{\mathcal{S}^2} \varphi(x) dx \int_{ST_x(\mathcal{S}^2)} \varphi(\exp_x(d \cdot \eta)) d\eta$$

is a continuously differentiable function with

$$W_0^\varphi(0) = 2\pi \|\varphi\|_{L^2(\mathcal{S}^2)}^2, \tag{87}$$

whose values are uniformly bounded by

$$|W_0^\varphi(\phi)| \leq 2\pi \|\varphi\|_\infty \|\varphi\|_{L^1(\mathcal{S}^2)} \leq 8\pi^2 \|\varphi\|_\infty^2 \tag{88}$$

and derivative uniformly bounded by

$$|W_0^{\varphi'}(\phi)| \leq 2\pi \|\varphi\|_\infty V(\varphi). \tag{89}$$

Now we pursue the proof of Theorem 1.4. By (84), evaluating the variance of \mathcal{Z}^φ is equivalent to evaluating I_n^φ , and for notational convenience we choose to work with the expression (85). As in the proof of Theorem 1.1, we choose a constant $C > 0$, which remains fixed throughout the present section, and divide the interval $[0, \pi m/2] = [0, C] \cup [C, \pi m/2]$ (see Sect. 3.1),

We then have the following lemma (cf. Lemma 3.3); to prove it just use (86) and the bound (88) for W_0^φ together with Lemma 3.2.

Lemma 4.3. *For any constant $C > 0$, we have as $n \rightarrow \infty$,*

$$I_n^\varphi = 2\tilde{I}_n^\varphi + O_{\|\varphi\|_\infty} \left(\frac{1}{n} \right), \tag{90}$$

where

$$\tilde{I}_n^\varphi := \int_C^{\pi m/2} \left(K_n(\psi) - \frac{1}{4} \right) W^\varphi \left(\frac{\psi}{m} \right) d\psi. \tag{91}$$

Proof of Theorem 1.4. First we evaluate \tilde{I}_n^φ as defined in (91). Plugging (56) into (91), we have (cf. (59))

$$\begin{aligned} \tilde{I}_n^\varphi &= \int_C^{\pi m/2} \left[\frac{1}{2} \frac{\sin(2\psi)}{\pi n \sin(\psi/m)} + \frac{65}{256} \frac{1}{\pi^2 n \sin(\psi/m)\psi} + \frac{9}{32} \frac{\cos(2\psi)}{\pi n \psi \sin(\psi/m)} \right. \\ &\quad \left. + \frac{27}{64} \frac{\sin(2\psi) - \frac{11}{256} \cos(4\psi)}{\pi^2 n \psi \sin(\psi/m)} \right] W^\varphi \left(\frac{\psi}{m} \right) d\psi \\ &\quad + O \left(\int_C^{\pi m/2} \left[\frac{1}{\psi^3} + \frac{1}{n\psi} \right] W^\varphi \left(\frac{\psi}{m} \right) d\psi \right) \\ &= \frac{1}{16\pi^3 n} \int_C^{\pi m/2} \left[\sin(2\psi) + \frac{65}{128} \frac{1}{\pi\psi} + \frac{9}{16} \frac{\cos(2\psi)}{\psi} \right. \\ &\quad \left. + \frac{27}{32} \frac{\sin(2\psi) - \frac{11}{128} \cos(4\psi)}{\pi\psi} \right] W_0^\varphi \left(\frac{\psi}{m} \right) d\psi + O \left(\|\varphi\|_\infty^2 \frac{1}{n} \right), \tag{92} \end{aligned}$$

with constants involved in the “O”-notation universal. Here we used the identity (86); to effectively control the error term we use (88).

We integrate by parts the first oscillatory term in (92), using the continuous differentiability assumptions; this yields the bound for its contribution

$$\begin{aligned} &\ll \frac{1}{n} \int_C^{\pi m/2} \sin(2\psi) W_0^\varphi(\psi/m) d\psi \\ &\ll \frac{1}{n} \left| \cos(2\psi) \cdot W_0^\varphi(\psi/m) \right|_{\psi=\pi m/2}^C + \frac{1}{n^2} \left| \int_C^{\pi m/2} \cos(2\psi) W_0^{\varphi'}(\psi/m) d\psi \right| \\ &\ll \frac{\|\varphi\|_\infty^2}{n} + \frac{\|W_0^{\varphi'}\|_{L^1([0,\pi])}}{n} \ll (\|\varphi\|_\infty^2 + \|\varphi\|_\infty V(\varphi)) \cdot \frac{1}{n} \end{aligned}$$

with constants involved in the “ \ll ”-notation universal, by (89). It is easy to establish similar bounds for the remaining oscillatory terms in (92), i.e. the 3rd and the 4th terms.

To analyze the main contribution, which comes from the remaining second term in (92), we note that the continuous differentiability of W_0^φ implies

$$W_0(\phi) = 2\pi \|\phi\|_{L^2(\mathcal{S}^2)}^2 + O_{\|\phi\|_\infty, V(\phi)}(\phi),$$

by (87) and (89). The main contribution to (92) is then

$$\begin{aligned} \frac{65}{2048\pi^4} \frac{1}{n} \int_C^{\pi m/2} \frac{W_0^\varphi(\psi/m)}{\psi} d\psi &= \frac{65}{1024\pi^3} \|\varphi\|_{L^2(\mathcal{S}^2)}^2 \cdot \frac{1}{n} \int_C^{\pi m/2} \frac{d\psi}{\psi} \\ &\quad + O_{\|\varphi\|_\infty, V(\varphi)} \left(\frac{1}{n^2} \int_C^{\pi m/2} d\psi \right) \\ &= \frac{65}{1024\pi^3} \|\varphi\|_{L^2(\mathcal{S}^2)}^2 \cdot \frac{\log n}{n} + O_{\|\varphi\|_\infty, V(\varphi)} \left(\frac{1}{n} \right). \end{aligned}$$

All in all we evaluated \tilde{I}_n^φ as

$$\tilde{I}_n^\varphi = \frac{65}{1024\pi^3} \|\varphi\|_{L^2(\mathcal{S}^2)}^2 \cdot \frac{\log n}{n} + O_{\|\varphi\|_\infty, V(\varphi)} \left(\frac{1}{n} \right).$$

Plugging this into (90) yields

$$I_n^\varphi = \frac{65}{512\pi^3} \|\varphi\|_{L^2(\mathcal{S}^2)}^2 \cdot \frac{\log n}{n} + O_{\|\varphi\|_\infty, V(\varphi)} \left(\frac{1}{n} \right). \tag{93}$$

We finally obtain the statement of Theorem 1.4 by plugging (93) into (84). \square

5. Proof of Theorem 1.5

As implied by the formulation of Theorem 1.5, in this section we will deal with functions of bounded variation. The definition and some basic properties of the class $BV(\mathcal{S}^2)$ of functions of bounded variation is given in Appendix C.

5.1. *On the proof of Theorem 1.5.* To prove Theorem 1.5 one wishes to apply a standard approximation argument, approximating our test function φ of bounded variation with a sequence φ_i of C^∞ , for which we can apply Theorem 1.4. There are two major issues with this approach however.

On one hand, one needs to check that φ_i approximating φ implies the corresponding statement for the random variables $\mathcal{Z}^\varphi(f_n)$ and $\mathcal{Z}^{\varphi_i}(f_n)$, and, in particular, their variance. While it is easy to check that if $\varphi_i \rightarrow \varphi$ in L^1 then for every fixed n we also have

$$\mathbb{E}[\mathcal{Z}^{\varphi_i}(f_n)] \rightarrow \mathbb{E}[\mathcal{Z}^\varphi(f_n)],$$

the analogous statement for the variance is much less trivial (see Proposition 5.1⁷).

On the other hand, when applying Theorem 1.4 for φ_i , one needs to control the error term in (14), which may a priori depend on φ_i . To resolve the latter we take advantage⁸ of the fact that Theorem 1.4 allows us to control the dependency of the error term in (14) on the test function in terms of its L^∞ norm and total variation. Thus to resolve this issue it would be sufficient to require from φ_i to be essentially uniformly bounded and having uniformly bounded total variation.

Fortunately the standard symmetric mollifiers construction from [13] as given in Appendix C satisfy both the requirements above. Namely given a function $\varphi \in BV(\mathcal{S}^2)$ we obtain a sequence φ_i of the C^∞ function, that converge in L^1 to φ , $\|\varphi_i\|_\infty \leq \|\varphi\|_\infty$ and in addition

$$V(\varphi_i) \rightarrow V(\varphi).$$

5.2. *Continuity of the distribution of \mathcal{Z}^φ .* As pointed in Sect. 5.1, to prove Theorem 1.5 we will need to show that the distribution of \mathcal{Z}^φ depends continuously on φ . Proposition 5.1 makes this statement precise. We believe that it is of independent interest.

Proposition 5.1. *Let $\varphi \in BV(\mathcal{S}^2) \cap L^\infty(\mathcal{S}^2)$ be any test function. Then*

$$\mathbb{E} \left[\mathcal{Z}^\varphi(f_n)^2 \right] = O \left(n^2 \|\varphi\|_{L^1(\mathcal{S}^2)}^2 + \|\varphi\|_\infty \|\varphi\|_{L^1(\mathcal{S}^2)} \right), \tag{94}$$

where the constant involved in the “O”-notation are universal. In particular, if $F \subseteq \mathcal{S}^2$ has a C^2 boundary then

$$\mathbb{E} \left[\left(\mathcal{Z}^F(f_n) \right)^2 \right] = O(n^2 |F|^2 + |F|).$$

Proof. Recall that we defined W^φ as (82); the assumption $\varphi \in L^\infty(\mathcal{S}^2)$ saves us from dealing with the validity of this definition. Starting from (121), and repeating the steps in the proof of Lemma 2.1 from either [23] or [4,5], we may extend the validity of the Kac-Rice formula (84) with (85) for this class as well. Note that the constant term

⁷ Proposition 5.1 gives a stronger claim. First, it evaluates the second moment rather than the variance. Secondly, it gives a general bound for $E \left[(\mathcal{Z}^{\varphi_i}(f_n) - \mathcal{Z}^\varphi(f_n))^2 \right] = E \left[(\mathcal{Z}^{\varphi_i - \varphi}(f_n))^2 \right]$. It is easy to derive the result we need employing the triangle inequality.

⁸ This is by no means a lucky coincidence; it is precisely the proof of Theorem 1.5 that motivated the technical statement made in Theorem 1.4.

in (85) comes from the squared expectation, so that we need to omit it if we want to compute the second moment. We then have

$$\mathbb{E} \left[(\mathcal{Z}^\varphi(f_n))^2 \right] = 8\pi^2 \frac{E_n}{n + 1/2} J_n^\varphi, \tag{95}$$

where

$$J_n^\varphi = \int_0^{\pi m/2} K_n(\psi) W^\varphi \left(\frac{\psi}{m} \right) d\psi,$$

denoting as usual $m := n + 1/2$.

As usual while estimating this kind of integrals we remove the origin by choosing a constant $C > 0$ and writing

$$J_n^\varphi = J_{n,1}^\varphi + J_{n,2}^\varphi, \tag{96}$$

where

$$J_{n,1}^\varphi = \int_0^C K_n(\psi) W^\varphi \left(\frac{\psi}{m} \right) d\psi,$$

and

$$J_{n,2}^\varphi = \int_C^{\pi m/2} K_n(\psi) W^\varphi \left(\frac{\psi}{m} \right) d\psi.$$

First, for $C < \psi < \frac{\pi m}{2}$, $K_n(\psi)$ is bounded by a constant, which may depend only on C , i.e.

$$|K_n(\psi)| = O_C(1),$$

which follows directly from Proposition 3.5. Therefore we may bound $J_{n,2}^\varphi$ as

$$\begin{aligned} |J_{n,2}^\varphi| &\ll_C \int_C^{\pi m/2} \left| W^\varphi \left(\frac{\psi}{m} \right) \right| d\psi \leq \int_0^{\pi m/2} \left| W^\varphi \left(\frac{\psi}{m} \right) \right| d\psi \\ &= m \int_0^{\pi/2} |W^\varphi(\phi)| d\psi \ll n \|\varphi\|_{L^1(\mathcal{S}^2)}^2, \end{aligned} \tag{97}$$

as earlier.

We claim that for $0 < \psi < C$ we may bound K_n as

$$|K_n(\psi)| = O_C \left(\frac{1}{\psi} \right). \tag{98}$$

Before proving this estimate we will show how it helps us to bound $J_{n,1}^\varphi$. We have by the definition of $J_{n,1}^\varphi$,

$$\begin{aligned} |J_{n,1}^\varphi| &\ll \int_0^C \frac{1}{\psi} \left| W^\varphi \left(\frac{\psi}{m} \right) \right| d\psi \ll \frac{1}{n} \int_0^C \left| W_0^\varphi \left(\frac{\psi}{m} \right) \right| d\psi \\ &\ll \int_0^{C/n} |W_0^\varphi(\phi)| d\phi \ll_C \frac{1}{n} \|\varphi\|_\infty \|\varphi\|_{L^1(\mathcal{S}^2)}, \end{aligned} \tag{99}$$

by (86) and the first inequality of (88).

The statement of the present lemma now follows from plugging the estimates (97) and (99) into (96) and (95). We still have to prove (98) though.

To see (98) we use Remark 2.10 and the Cauchy-Schwartz inequality to write

$$K_n(\psi) = \frac{1}{(2\pi)\sqrt{1 - P_n(\cos \psi/m)^2}} \mathbb{E}[\|U\| \cdot \|V\|], \tag{100}$$

where U and V are 2-dimensional mean zero Gaussian vectors with covariance matrix (45), whose entries uniformly bounded by an absolute constant, whence

$$\mathbb{E}[\|U\| \cdot \|V\|] \leq \sqrt{\mathbb{E}[\|U\|^2] \mathbb{E}[\|V\|^2]} = O(1), \tag{101}$$

with the constant involved in the “ O ”-notation uniform. For the other term Lemma B.2 yields

$$\sqrt{1 - P_n(\cos(\psi/m))^2} \ll \frac{1}{\psi}, \tag{102}$$

so that we obtain the necessary bound (98) for $K_n(\psi)$ plugging the estimates (101) and (102) into (100). \square

5.3. *Proof of Theorem 1.5.* Now we are ready to give a proof of Theorem 1.5.

Proof of Theorem 1.5. Given a function $\varphi \in BV(\mathcal{S}^2)$, let $\varphi_i \in C^\infty$ be a sequence of smooth functions such that $\varphi_i \rightarrow \varphi$ in $L^1(\mathcal{S}^2)$,

$$V_i := V(\varphi_i) \rightarrow V(\varphi)$$

and

$$\|\varphi_i\|_\infty \leq \|\varphi\|_\infty. \tag{103}$$

(see Appendix C). Let $M_1 := \|\varphi\|_\infty$ and

$$M_2 := \max\{V_i\}_{i \geq 1} < \infty,$$

since V_i is convergent.

Theorem 1.4 applied on $\varphi_i \in C^\infty(\mathcal{S}^2)$ states that

$$\text{Var}(\mathcal{Z}^{\varphi_i}(f_n)) = c(\varphi_i) \cdot \log n + O_{M_1, M_2}(1), \tag{104}$$

where $c(\varphi_i)$ is given by

$$c(\varphi_i) := 65 \frac{\|\varphi_i\|_{L^2(\mathcal{S}^2)}^2}{128\pi} > 0.$$

Note that since φ_i and φ are uniformly bounded (103), $L^1(\mathcal{S}^2)$ convergence implies $L^2(\mathcal{S}^2)$ convergence, so that

$$c(\varphi_i) \rightarrow c(\varphi), \quad (105)$$

the latter being given by (13).

On the other hand we know from Proposition 5.1 that

$$\mathbb{E} \left[(\mathcal{Z}^{\varphi_i}(f_n) - \mathcal{Z}^\varphi(f_n))^2 \right] = \mathbb{E} \left[(\mathcal{Z}^{\varphi_i - \varphi}(f_n))^2 \right] \rightarrow 0,$$

using the uniform boundedness (103) again to ensure that (94) holds uniformly. This together with the triangle inequality implies that

$$\text{Var}(\mathcal{Z}^{\varphi_i}(f_n)) \rightarrow \text{Var}(\mathcal{Z}^\varphi(f_n)), \quad (106)$$

and we take the limit $i \rightarrow \infty$ in (104) to finally obtain the main statement of Theorem 1.5. \square

Remark 5.2. From the proof presented, it is easy to see that the constant in the ‘‘O’’-notation in the statement (14) of Theorem 1.5 could be made dependent only on $\|\varphi\|_\infty$ and $V(\varphi)$.

Appendix A. Computation of the Covariance Matrix

In this section we compute the matrix $\Omega_n(\phi)$ explicitly, as prescribed by (37). The matrix $\Omega_n(\phi)$ is the 4×4 covariance matrix of the mean zero Gaussian random vector Z_2 in (23) with $x \neq y \in \mathcal{S}^2$ any two points on the arc $\{\theta = 0\}$ with $d(x, y) = \phi$, conditioned upon $f(x) = f(y) = 0$. Recall that as such, $\Omega_n(\phi)$ is given by (34), where $A = A_n(x, y)$, $B = B_n(x, y)$ and $C = C_n(x, y)$ are given by (30), (31) and (32) respectively, and $x, y \in \mathcal{S}^2$ are any points on the arc $\{\theta = 0\}$ with $d(x, y) = \phi$. Here the gradients are given in the orthonormal frame (36) of the tangent planes $T_x(\mathcal{S}^2)$ associated to the spherical coordinates (see Sect. 2.4 for explanation).

Let x and y correspond to the spherical coordinates $(\phi_x, \theta_x = 0)$ and $(\phi_y, \theta_y = 0)$, and denote

$$\phi = d(x, y) = |\phi_x - \phi_y|.$$

Recall that

$$u_n(x, y) = P_n(\cos(d(x, y))) = P_n(\cos \phi).$$

First we compute the inverse of A in (30) as

$$A_n(\phi)^{-1} = \frac{1}{1 - P_n(\cos \phi)^2} \begin{pmatrix} 1 & -P_n(\cos \phi) \\ -P_n(\cos \phi) & 1 \end{pmatrix}. \quad (107)$$

It is easy to either see from the geometric picture or compute explicitly that

$$\nabla_x u_n(x, y) = -\nabla_y u_n(x, y) = \pm P'_n(\cos \phi) \sin(\phi)(1, 0), \tag{108}$$

depending on whether $\phi_x > \phi_y$ or $\phi_x < \phi_y$, so that

$$B_n(\phi) = \pm \begin{pmatrix} 0 & 0 & P'_n(\cos \phi) \sin \phi & 0 \\ -P'_n(\cos \phi) \sin \phi & 0 & 0 & 0 \end{pmatrix}. \tag{109}$$

Next we turn to the missing part of $C_n(\phi)$ defined in (32), i.e. the ‘‘pseudo-Hessian’’ $H_n(\phi)$ given by (33). By the chain rule

$$\begin{aligned} H_n(\phi) &= (\nabla_x \otimes \nabla_y) u_n(x, y) = \nabla_x \otimes P'_n(\cos(d(x, y))) \nabla_y \cos(d(x, y)) \\ &= P''_n(\cos \phi) \nabla_x \cos(d(x, y)) \otimes \nabla_y \cos(d(x, y)) \\ &\quad + P'_n(\cos \phi) (\nabla_x \otimes \nabla_y) \cos(d(x, y)). \end{aligned} \tag{110}$$

We denote

$$h(x, y) := \cos d(x, y) = \cos \phi_x \cos \phi_y + \sin \phi_x \sin \phi_y \cos(\theta_x - \theta_y),$$

and compute explicitly that for $\theta_x = \theta_y = 0$ we have

$$(\nabla_x \otimes \nabla_y) \cos(d(x, y)) = (\nabla_x \otimes \nabla_y) h(x, y) = \begin{pmatrix} \cos \phi & 0 \\ 0 & 1 \end{pmatrix}. \tag{111}$$

Plugging (108) and (111) into (110) we obtain

$$H = \begin{pmatrix} P'_n(\cos \phi) \cos \phi - P''_n(\cos \phi) \sin(\phi)^2 & 0 \\ 0 & P'_n(\cos \phi) \end{pmatrix}. \tag{112}$$

Finally plugging (112) into (32), and plugging that together with (107) and (109) into (34), we obtain an explicit expression for $\Omega_n(\phi)$ as prescribed by (37) with entries given by (38), (39) and (40).

Appendix B. Estimates for the Legendre Polynomials and Related Functions

The goal of this section is to give a brief introduction to the Legendre polynomials $P_n : [-1, 1] \rightarrow \mathbb{R}$ and give some relevant basic information necessary for the purposes of the present paper. The high degree asymptotic analysis of behaviour of P_n and its first two derivatives involves the Hilb’s asymptotics in Lemma B.1 together with the recursion (114) for the 1st derivative and the differential equation (113) for the second one. We refer the reader to [17] for more information.

The Legendre polynomials P_n are defined as the unique polynomials of degree n orthogonal w.r.t. the constant weight function $\omega(t) \equiv 1$ on $[-1, 1]$ with the normalization $P_n(1) = 1$. They satisfy the following second order differential equation:

$$P''_n(\cos(\psi/m)) = -\frac{n(n+1)}{\sin(\psi/m)^2} P_n(\cos(\psi/m)) + \frac{2 \cos(\psi/m)}{\sin(\psi/m)^2} P'_n(\cos(\psi/m)), \tag{113}$$

as well as the recursion

$$P'_n(\cos(\psi/m)) = (P_{n-1}(\cos(\psi/m)) - \cos(\psi/m) P_n(\cos(\psi/m))) \frac{n}{\sin(\psi/m)^2}. \tag{114}$$

The Hilb asymptotics gives the high degree asymptotic behaviour of P_n .

Lemma B.1. (Hilb Asymptotics (formula (8.21.17) on p. 197 of Szego [17]))

$$P_n(\cos \phi) = \left(\frac{\phi}{\sin \phi}\right)^{1/2} J_0((n + 1/2)\phi) + \delta(\phi), \tag{115}$$

uniformly for $0 \leq \phi \leq \pi/2$, J_0 is the Bessel J function of order 0 and the error term is

$$\delta(\phi) \ll \begin{cases} \phi^{1/2} O(n^{-3/2}), & Cn^{-1} < \phi < \pi/2 \\ \phi^{\alpha+2} O(n^\alpha), & 0 < \phi < Cn^{-1}, \end{cases}$$

where $C > 0$ is any constant and the constants involved in the “ O ”-notation depend on C only.

We have the following rough estimate for the behaviour of the Legendre polynomials at ± 1 , which follows directly from Hilb’s asymptotic.

Lemma B.2. For $0 < \phi < \frac{\pi}{2}$ one has

$$1 - P_n(\cos(\phi))^2 \gg n^2 \phi^2,$$

where the constant in the “ \gg ”-notation is universal.

Lemma B.3. The Legendre polynomials P_n and its couple of derivatives satisfy uniformly for $n \geq 1$, $\psi > C$:

(1)

$$P_n(\cos(\psi/m)) = \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \left(\sin\left(\psi + \frac{\pi}{4}\right) - \frac{1}{8} \frac{\cos\left(\psi + \frac{\pi}{4}\right)}{\psi} \right) + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{\sqrt{\psi n}}\right), \tag{116}$$

(2)

$$P'_n(\cos(\psi/m)) = \sqrt{\frac{2}{\pi}} \frac{\sqrt{n}}{\sin(\psi/m)^{5/2}} \left(\sin(\psi/m) \sin\left(\psi - \frac{\pi}{4}\right) + \frac{3}{8n} \sin\left(\psi + \frac{\pi}{4}\right) \right) + O\left(\frac{n^2}{\psi^{7/2}} + \frac{n}{\psi^{3/2}}\right), \tag{117}$$

(3)

$$P''_n(\cos(\psi/m)) = -\frac{n^2}{\sin(\psi/m)^2} P_n(\cos(\psi/m)) + \frac{2}{\sin(\psi/m)^2} P'_n(\cos(\psi/m)) + O\left(\frac{n^3}{\psi^{5/2}}\right). \tag{118}$$

Proof. By Lemma B.1 and the standard asymptotics for the Bessel functions we obtain

$$\begin{aligned}
 P_n(\cos(\psi/m)) &= \frac{\sqrt{\psi/m}}{\sqrt{\sin(\psi/m)}} J_0(\psi) + O\left(\frac{\sqrt{\psi}}{n^2}\right) \\
 &= \sqrt{\frac{2}{\pi}} \frac{\sqrt{\psi/m}}{\sqrt{\sin(\psi/m)}} \left(\frac{\sin(\psi + \frac{\pi}{4})}{\sqrt{\psi}} - \frac{1}{8} \frac{\cos(\psi + \frac{\pi}{4})}{\psi^{3/2}} \right) \\
 &\quad + O\left(\frac{1}{\psi^{5/2}} + \frac{\sqrt{\psi}}{n^2}\right) \\
 &= \sqrt{\frac{2}{\pi n \sin(\psi/m)}} \left(\sin(\psi + \frac{\pi}{4}) - \frac{1}{8} \frac{\cos(\psi + \frac{\pi}{4})}{\psi} \right) \\
 &\quad + O\left(\frac{1}{\psi^{5/2}} + \frac{1}{\sqrt{\psi}n}\right),
 \end{aligned}$$

which is (116).

To obtain (117) we employ the recursive formula (114), evaluating the Legendre polynomials appearing there using (116). Finally we obtain a simple approximate differential equation (118), replacing $n(n+1)$ by n^2 and $\cos(\psi/m)$ by 1 in the differential equation (113) satisfied by the Legendre polynomials. To do so we use the decay

$$|P_n(\cos(\psi/m))| = O\left(\frac{1}{\sqrt{\psi}}\right)$$

of P_n , which follows directly from (116), as well as (79) of its derivative. \square

Appendix C. Functions of Bounded Variation

In this section we give the definition and some basic properties on the functions of bounded variation. For more information we refer the reader to [13].

Classically, the *variation* of a function $\eta : [a, b] \rightarrow \mathbb{R}$ on $[a, x]$ is defined as

$$V(\eta; x) := \sup_{\lambda: t_1=a < t_2 < \dots < t_k=x} \sum_{i=1}^{k-1} |\eta(t_{i+1}) - \eta(t_i)|,$$

where the supremum is over all the partitions λ of $[a, x]$. We denote $I := [a, b]$. If $\eta \in C^1(I)$ then the variation is

$$V(\eta; x) = \int_a^x |\eta'(t)| dt.$$

In fact, the last inequality holds even for $\eta \in W^{1,1}(I)$, where for this class of functions the derivative η' is the *weak derivative*.

This definition has two major disadvantages. First, one wishes to identify functions

$$\eta_1 \sim \eta_2, \quad \text{if } \eta_1(x) = \eta_2(x) \quad \text{for almost all } x \in I. \tag{119}$$

However, altering the values of η on a measure zero set does impact its variation. Secondly, one cannot extend this definition for the multivariate case.

We then need to find a better definition. Fortunately, the following definition eliminates the disadvantages of the previous one. Let

$$V(\eta; x) := \sup_g \int_0^x \eta(t)g'(t)dt,$$

where the supremum is over all the continuously differentiable functions $g : [a, x] \rightarrow \mathbb{R}$ with $|g(t)| \leq 1$ for all $t \in [a, x]$. The number $V(\eta) := V(\eta; I)$ is called the *total variation* of η on I . We define the space $BV(I)$ to be the equivalence classes of functions η with finite total variation, i.e.

$$BV(I) := \{\eta \in L^1(I) : V(\eta) < \infty\} / \sim,$$

where the equivalence relation is given by (119). It is known [13] that

$$W^{1,1}(I) \subsetneq BV(I).$$

We may extend the latter definition quite naturally for the multivariate case. Of our interest is the case of the sphere. Let $\varphi \in L^1(\mathcal{S}^2)$ be an integrable function. We define its variation on an open subset $\Omega \subseteq \mathcal{S}^2$ as

$$V(\varphi; \Omega) := \sup_g \int_{\Omega} \varphi(x) \operatorname{div} g(x) dx,$$

where the supremum is over the continuously differentiable compactly supported vector fields

$$g \in C_c^1(\Omega, T\Omega)$$

with $|g(x)| \leq 1$ for all $x \in \Omega$. We define the *total variation* as

$$V(\varphi) := V(\varphi; \mathcal{S}^2).$$

The space $BV(\Omega)$ is defined as the equivalence class of functions φ with $V(\varphi) < \infty$, with the equivalence relation (119) adapted to the sphere. Again, for a smooth (and $W^{1,1}(\mathcal{S}^2)$) function $\varphi \in C^1(\mathcal{S}^2)$ we have

$$V(\varphi) = \int_{\mathcal{S}^2} \|\nabla\varphi(x)\| dx,$$

and

$$W^{1,1}(\Omega) \subsetneq BV(\Omega).$$

For a function $\varphi \in BV(\mathcal{S}^2)$ [13], Theorem 1.17 gives a construction⁹ of a sequence $\varphi_i \in C^\infty$ of smooth test functions such that $\varphi_i \rightarrow \varphi$ in $L^1(\mathcal{S}^2)$ as well as

$$V(\varphi_i) \rightarrow V(\varphi).$$

⁹ This book gives only the theory of functions of bounded variation on \mathbb{R}^m . One can obtain a similar theory for the sphere only slightly modifying the one given.

Moreover, part (b) of that theorem implies that

$$\|\varphi_i\|_\infty \leq \|\varphi\|_\infty.$$

We are interested in the linear statistics of the nodal sets of smooth functions, where the test functions are of bounded variation. The definition of the linear statistics is natural for continuous test functions $\varphi : \mathcal{S}^2 \rightarrow \mathbb{R}$ as

$$\mathcal{Z}^\varphi(f) = \int_{f^{-1}(0)} \varphi(x) dx,$$

i.e. integrating the restriction of φ on the nodal line. However, things become more complicated as one drops the continuity assumption; since the values of $\varphi \in BV(\mathcal{S}^2)$ (or $\varphi \in L^1(\mathcal{S}^2)$) are only defined up to measure zero sets, there is no meaning to restricting φ on curves. In general, one cannot define linear statistics corresponding to integrable functions, and to define a notion of *trace* of φ on a smooth curve C , we will have to exploit the values of φ in a tubular neighbourhood around C . Such a construction is known for the functions belonging to the class $W^{1,1}(\mathcal{S}^2)$, i.e. for every smooth curve $C \subseteq \Omega$ there exists a map

$$\text{tr}_C : W^{1,1}(\Omega) \rightarrow L^1(C)$$

satisfying the natural properties.

The situation is more involved in the BV -case, which is essential to us, since $W^{1,1}$ does not contain the characteristic functions of nice spherical subsets. A smooth curve divides the sphere and a tubular neighbourhood around it into two parts. One may then define [13], chap. 2, *two* traces $\varphi^+ = \text{tr}_C^+ \varphi$ and $\varphi^- = \text{tr}_C^- \varphi$ both belonging to $L^1(C)$, corresponding to the values of φ on the different parts. The traces φ^+ and φ^- may in general be different¹⁰, and moreover, one cannot canonically distinguish between the traces. For instance, if $F \subseteq \mathcal{S}^2$ is a nice subset, and χ_F is its characteristic function, then $\text{tr}_{\partial F}(\chi_F)$ might be defined as either 1 or 0, depending on whether we approach the circle from inside or outside the disc respectively. Accordingly, the corresponding linear statistic might be $\text{len}(\partial F)$ or 0.

We define the *average* trace of φ on a smooth curve $C \subseteq \mathcal{S}^2$ as

$$\varphi^\pm := \frac{1}{2}\varphi^+ + \frac{1}{2}\varphi^-, \tag{120}$$

and this is the notion that appears in the formulation of Theorem 1.5 and throughout the present paper. For $\varphi \in L^\infty(\mathcal{S}^2)$ we have

$$\|\varphi^\pm\|_\infty \leq \|\varphi\|_\infty.$$

¹⁰ Intuitively, the traces φ^+ and φ^- will be different precisely if the jump of φ occurs on a subset of C , as follows from [13], Proposition 2.8. It is plausible that with probability 1 this situation will not happen for the nodal lines of spherical harmonics; we believe that this is a minor issue and of little interest to the present paper. This situation is almost surely impossible for the characteristic functions of nice sets, which are the main motivation for considering the class BV .

Following the approach of [13], (2.10) and Federer’s co-area formula [11], one may obtain the inequality

$$\begin{aligned} & \frac{1}{\epsilon} \int_0^\epsilon \left| \int_{f^{-1}(t)} \varphi(x) dx - \int_{f^{-1}(0)} \varphi^+(x) dx \right| dt \\ &= O_f \left(V(\varphi; f^{-1}((0, \epsilon))) + \sup_{0 < t < \epsilon} \left| \text{len}(f^{-1}(t)) - \text{len}(f^{-1}(0)) \right| \right). \end{aligned}$$

As $\beta \rightarrow 0$, the right-hand side of the last inequality vanishes. Therefore we have the following Kac-Rice type formula:

$$\int_{f^{-1}(0)} \varphi^+(x) dx = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{0 < f(x) < \epsilon} \|\nabla f(x)\| \varphi(x) dx,$$

and similarly

$$\int_{f^{-1}(0)} \varphi^-(x) dx = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\epsilon < f(x) < 0} \|\nabla f(x)\| \varphi(x) dx.$$

Combining the last two formulas we obtain

$$\int_{f^{-1}(0)} \varphi^\pm(x) dx = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_{|f(x)| < \epsilon} \|\nabla f(x)\| \varphi(x) dx. \tag{121}$$

We employ (121) to extend the validity of the Kac-Rice formula for the second moment for $\varphi \in BV(\mathcal{S}^2)$ (see (95)).

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References

1. Andrews, G.E., Askey, R., Roy, R.: *Special functions Encyclopedia of Mathematics and its Applications* **71**. Cambridge: Cambridge University Press, 1999
2. Bérard, P.: *Volume des ensembles nodaux des fonctions propres du laplacien*. Bony-Sjostrand-Meyer seminar, 1984–1985, Exp. No. 14, 10 pp., École Polytech., Palaiseau, 1985
3. Berry, M.V.: Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. *J. Phys. A* **35**, 3025–3038 (2002)
4. Bleher, P., Shiffman, B., Zelditch, S.: Universality and scaling of correlations between zeros on complex manifolds. *Invent. Math.* **142**(2), 351–395 (2000)
5. Bleher, P., Shiffman, B., Zelditch, S.: Universality and scaling of zeros on symplectic manifolds. In: *Random Matrix Models and their Applications*, Math. Sci. Res. Inst. Publ. **40**, Cambridge: Cambridge Univ. Press, 2001, pp. 31–69
6. Brüning, J.: Über Knoten Eigenfunktionen des Laplace-Beltrami Operators. *Math. Z.* **158**, 15–21 (1978)

7. Brüning, J., Gromes, D.: Über die Länge der Knotenlinien schwingender Membranen. *Math. Z.* **124**, 79–82 (1972)
8. Cheng, S.Y.: Eigenfunctions and nodal sets. *Comm. Math. Helv.* **51**, 43–55 (1976)
9. Cramér, H., Leadbetter, M.R.: *Stationary and Related Stochastic Processes. Sample Function Properties and Their Applications*. Reprint of the 1967 original. Mineola, NY: Dover Publications, Inc., 2004
10. Donnelly, H., Fefferman, C.: Nodal sets of eigenfunctions on Riemannian manifolds. *Invent. Math.* **93**, 161–183 (1988)
11. Federer, H.: Curvature measures. *Trans. Amer. Math. Soc.* **93**, 418–491 (1959)
12. Forrester, P.J., Honner, G.: Exact statistical properties of the zeros of complex random polynomials. *J. Phys. A* **32**(16), 2961–2981 (1999)
13. Giusti, E.: *Minimal Surfaces and Functions of Bounded Variation*. Monographs in Mathematics, **80**. Basel: Birkhäuser Verlag, 1984
14. Krishnapur, M., Wigman, I.: *Fluctuations of the Nodal Length of Random Eigenfunctions of the Laplacian on the Torus*. In preparation
15. Neuheisel, J.: *The Asymptotic Distribution of Nodal Sets on Spheres*. Johns Hopkins Ph.D. thesis, 2000
16. Rudnick, Z., Wigman, I.: On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. *Ann. Henri Poincaré* **9**(1), 109–130 (2008)
17. Szegő, G.: *Orthogonal Polynomials*. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. **XXIII**. Providence, RI: Amer. Math. Soc., 1975
18. Shiffman, B., Zelditch, S.: Number variance of random zeros on complex manifolds. *Geom. Funct. Anal.* **18**(4), 1422–1475 (2008)
19. Shiffman, B., Zelditch, S.: *Number variance of random zeros on complex manifolds, II: smooth statistics*. available online [http://arxiv.org/abs/0711.1840v1\[math.CV\]](http://arxiv.org/abs/0711.1840v1[math.CV]), 2007
20. Sodin, M., Tsirelson, B.: Random complex zeroes. I. Asymptotic normality. *Israel J. Math.* **144**, 125–149 (2004)
21. Toth, J.A., Wigman, I.: Counting open nodal lines of random waves on planar domains. *IMRN* **2009**, 3337–3365 (2009)
22. Toth, J.A., Wigman, I.: *Universality of length distribution of nodal lines of random waves on generic surfaces*. In progress (2009)
23. Wigman, I.: On the distribution of the nodal sets of random spherical harmonics. *J. Math. Phys.* **50**, 013521 (2009)
24. Wigman, I.: *Volume fluctuations of the nodal sets of random Gaussian subordinated spherical harmonics*. In preparation
25. Yau, S.T.: Survey on partial differential equations in differential geometry. In: *Seminar on Differential Geometry*, Ann. of Math. Stud. **102**, Princeton, NJ: Princeton Univ. Press, 1982, pp. 3–71
26. Yau, S.T.: Open problems in geometry. In: *Differential Geometry: Partial Differential Equations on Manifolds* (Los Angeles, CA, 1990), Proc. Sympos. Pure Math. **54**, Part I, Providence, RI: Amer. Math. Soc., 1993, pp. 1–28
27. Zelditch, S.: *Real and Complex zeros of Riemannian Random Waves*. To appear in the Proceedings of the Conference, “Spectral Analysis in Geometry and Number Theory on the occasion of Toshikazu Sunada’s 60th birthday”, to appear in the Contemp. Math. Series, available online [http://arxiv.org/abs/0803.4334v1\[math.Sp\]](http://arxiv.org/abs/0803.4334v1[math.Sp]), 2008

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