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Proof of the Projective Lichnerowicz Conjecture for Pseudo-Riemannian Metrics with Degree of Mobility Greater than Two

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Abstract: Degree of mobility of a (pseudo-Riemannian) metric is the dimension of the space of metrics geodesically equivalent to it. We prove that complete metrics on $(n \geq 3)-$dimensional manifolds with degree of mobility ≥ 3 do not admit complete metrics that are geodesically equivalent to them, but not affinely equivalent to them. As the main application we prove an important special case of the pseudo-Riemannian version of the projective Lichnerowicz conjecture stating that a complete manifold admitting an essential group of projective transformations is the standard round sphere (up to a finite cover and multiplication of the metric by a constant).

1. Introduction

1.1. Definitions and result. Let M be a connected manifold of dimension $n \geq 3$, let g be a (Riemannian or pseudo-Riemannian) metric on it. We say that a metric $\bar{g}$ on the same manifold M is *geodesically equivalent* to g, if every g-geodesic is a reparametrized $\bar{g}$-geodesic. We say that they are *affine equivalent*, if their Levi-Civita connections coincide.

As we recall in Sect. 2.1, the set of metrics geodesically equivalent to a given one (say, g) is in one-to-one correspondence with the nondegenerate solutions of Eq. (9). Since Eq. (9) is linear, the space of its solutions is a linear vector space. Its dimension is called the *degree of mobility* of g. Locally, the degree of mobility of g coincides with the dimension of the set (equipped with its natural topology) of metrics geodesically equivalent to g.

The degree of mobility is at least one (since $\mathrm{const} \cdot g$ is always geodesically equivalent to g) and is at most $(n+1)(n+2)/2$, which is the degree of mobility of simply-connected spaces of constant sectional curvature.

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Our main result is:

Theorem 1. *Let g be a complete Riemannian or pseudo-Riemannian metric on a connected M^n of dimension $n \geq 3$. Assume that for every constant $c \neq 0$ the metric $c \cdot g$ is not the Riemannian metric of constant curvature* $+1$.

If the degree of mobility of the metric is ≥ 3, then every complete metric $\bar{g}$ geodesically equivalent to g is affine equivalent to g.

The assumption that the metrics are complete is important: the examples constructed by Solodovnikov [70,71] show the existence of complete metrics with a big degree of mobility (all metrics geodesically equivalent to such metrics are not complete).

Theorem 2. *Let g be a complete Riemannian or pseudo-Riemannian metric on a closed (= compact, without boundary) connected manifold M^n of dimension $n \geq 3$. Assume that for every constant $c \neq 0$ the metric $c \cdot g$ is not the Riemannian metric of constant curvature* $+1$. *Then, at least one the following possibilities holds*:

- *the degree of mobility of g is at most two, or*
- *every metric g geodesically equivalent to $\bar{g}$ is affine equivalent to g.*

Remark 1. In the Riemannian case, Theorem 1 was proved in [57, Th. 16] and in [56]. The proof uses observations which are wrong in the pseudo-Riemannian situation; we comment on them in Sect. 1.2. Our proof for the pseudo-Riemannian case is also not applicable in the Riemannian case, since it uses lightlike geodesics in an essential way. In Sect. 2.5, we give a new, shorter (modulo results of our paper) proof of Theorem 1 for the Riemannian metrics as well.

Remark 2. In the Riemannian case, Theorem 2 follows from Theorem 1, since every Riemannian metric on a closed manifold is complete. In the pseudo-Riemannian case, Theorem 2 is a separate statement.

Remark 3. Moreover, the assumptions that the metric is complete and the dimension is ≥ 3 could be removed from Theorem 2: by [60, Cor. 5.2] and [61, Cor. 1], *if the degree of mobility of g on a closed $(n \geq 2)-$dimensional manifold is at least three, then for a certain constant $c \neq 0$ the metric $c \cdot g$ is the Riemannian metric of curvature* 1, *or every metric geodesically equivalent to g is affine equivalent to g.*

The proofs in [60] and [61] are nontrivial; the proof of [60, Cor. 5.2] is in particular based on the results of Sect. 2.3.5 of the present paper.

1.2. Motivation I: Projective Lichnerowicz conjecture. Recall that a *projective transformation* of the manifold (M, g) is a diffeomorphism of the manifold that takes (unparametrized) geodesics to geodesics. The infinitesimal generators of the group of projective transformations are complete *projective vector fields*, i.e., complete vector fields whose flows take (unparameterized) geodesics to geodesics.

Theorem 1 allows us to prove an important special case of the following conjecture, which folklore attributes (see [57] for discussion) to Lichnerowicz and Obata (the latter assumed in addition that the manifold is closed, see, for example, [26,63,77]):

Projective Lichnerowicz Conjecture. *Let a connected Lie group G act on a complete connected pseudo-Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by affine transformations, or for a certain $c \in \mathbb{R} \setminus \{0\}$ the metric $c \cdot g$ is the Riemannian metric of constant positive sectional curvature* $+1$.

We see that Theorem 1 implies

Corollary 1. *The projective Lichnerowicz Conjecture is true under the additional assumption that the dimension $n \geq 3$ and that the degree of mobility of the metric g is ≥ 3.*

Indeed, the pullback of the (complete) metric g under the projective transformation is a complete metric geodesically equivalent to g. Then, by Theorem 1, it is affine equivalent to g, i.e., the projective transformation is actually an affine transformation, as it is stated in Corollary 1.

Corollary 1 is thought to be the most complicated part of the solution of the projective Lichnerowicz conjecture for pseudo-Riemannian metrics. We do not know yet whether the Lichnerowicz conjecture is true (for pseudo-Riemannian metrics), but we expect that its solution (= proof or counterexample) will require no new additional ideas beyond those from the Riemannian case.

To support this optimistic expectation, let us recall that the projective Lichnerowicz conjecture was recently proved for Riemannian metrics [51,57]. The proof contained three parts:

(i) proof for the metrics with the degree of mobility 2 ([57, Th. 15], [51, Chap. 4]),
(ii) proof under the assumption $dim(M) \geq 3$ for the metrics with the degree of mobility ≥ 3 ([57, Th. 16]),
(iii) proof under the assumption $dim(M) = 2$ for the metrics with the degree of mobility ≥ 3, [51, Cor. 5 and Th. 7].

The most complicated (= lengthy; it is spread over [57, §§3.2–3.5, 4.2]) part was the proof under the additional assumptions (ii).

The proof was based on the Levi-Civita description of geodesically equivalent metrics, on the calculation of the curvature tensor for Levi-Civita metrics with degree of mobility ≥ 3 due to Solodovnikov [70,71], and on global ordering of eigenvalues of $a_i^j := a_{ip}g^{pj}$, where a_{ij} is a solution of (9), due to [6,54,74]. This proof can not be generalized to the pseudo-Riemannian metrics. More precisely, a pseudo-Riemannian analog of the Levi-Civita theorem is much more complicated, calculations of Solodovnikov essentially use positive-definiteness of the metric, and, as examples show, the global ordering of eigenvalues of a_i^j is simply wrong for pseudo-Riemannian metrics.

Thus, Theorem 1 and Corollary 1 close the a priori most difficult part of the solution of the Lichnerowicz conjecture for the pseudo-Riemannian metrics.

Let us now comment on (i), (iii), from the viewpoint of the possible generalization of the Riemannian proof to the pseudo-Riemannian case. We expect that this is possible. More precisely, the proof of (i) is based on a trick invented by Fubini [18] and Solodovnikov [70], see also [48,50,51]. The trick uses the assumption that the degree of mobility is two to double the number of PDEs (for a vector field v to be projective for the metric g), and to lower the order of this equation (the initial equations have order 2, the equations that we get after applying the trick have order 1). This of course makes everything much easier; moreover, in the Riemannian case, one can explicitly solve this system [18,64,70]. After doing this, one has to analyze whether the metrics and the projective field are complete; in the Riemannian case it was possible to do.

The trick survives in the pseudo-Riemannian setting. The obtained system of PDE was solved for the simplest situations (for small dimensions [11,58], or under the additional assumption that the minimal polynomial of a^i_j coincides with the characteristic polynomial). We expect that the other part of the program could be realized for pseudo-Riemannian metrics as well, though of course it will require a lot of work.

Now let us comment on the proof under the assumptions (iii): $dim(M) = 2$, degree of mobility is ≥ 3. The initial proof of [51] uses the description of quadratic integrals of geodesic flows of complete Riemannian metrics due to [28]. This description has no analog for pseudo-Riemannian metrics. Fortunately, one actually does not need this description anymore: in [11,58] a complete list of 2-dimensional pseudo-Riemannian metrics admitting a projective vector field was constructed; the degree of mobility for all these metrics has been calculated. The metrics that are interesting for the setting (iii) are the metrics (2a, 2b, 2c) of [11, Th. 1] and (3d) of [58, Th. 1], because all other metrics admitting projective vector fields have constant curvature or degree of mobility equal to 2. All these metrics are given by relatively simple formulas using only elementary functions. In order to prove the projective Lichnerowicz conjecture in the setting (iii), one needs to understand which metrics from this list could be extended to a bigger domain; it does not seem to be too complicated. For the metrics (2a, 2b,2c) of [11, Theorem 1] it was already done in [38]. Under the additional assumption that M^2 is closed, two dimensional pseudo-Riemannian version of the Licherowicz conjecture was proved in [61, Theorem 6].

As a consequence of Theorem 1, we obtain the following simpler version of the Lichnerowicz conjecture.

Corollary 2. *Let Proj$_o$ (respectively, Aff$_o$) be the connected component of the Lie group of projective transformations (respectively, affine transformations) of a complete connected pseudo-Riemannian manifold* (M^n, g) *of dimension* $n \geq 3$. *Assume that for no constant* $c \in \mathbb{R}\backslash\{0\}$ *the metric* $c \cdot g$ *is the Riemannian metric of constant positive curvature* $+1$. *Then, the codimension of Aff$_o$ in Proj$_o$ is at most one.*

Indeed, it is well known (see, for example [57], or more classical sources acknowelged therein) that a vector field is projective if the $(0, 2)-$tensor

$$a := L_v g - \frac{1}{n+1}\mathrm{trace}(g^{-1}L_v g) \cdot g \tag{1}$$

is a solution of (9), where L_v is the Lie derivative with respect to v. Moreover, the projective vector field is affine, iff the function (10) constructed by a_{ij} given by (1) is constant.

Now, let us take two infinitesimal generators of the Lie group $Proj_o$, i.e., two complete projective vector fields v and $\bar{v}$. In order to show that the codimension of Aff_o in $Proj_o$ is at most one, it is sufficient to show that a linear combination of these vector fields is an affine vector field. We consider the solutions $a := L_v g - \frac{1}{n+1}\mathrm{trace}(g^{-1}L_v g) \cdot g$ and $\bar{a} := L_{\bar{v}} g - \frac{1}{n+1}\mathrm{trace}(g^{-1}L_{\bar{v}} g) \cdot g$ of (9).

If $a, \bar{a}$, and g are linearly independent, the degree of mobility of the metric is ≥ 3. Then, Corollary 1 implies $Proj_o = Aff_o$.

Thus, $a, \bar{a},\ g$ are linearly dependent. Since the function $\lambda := \frac{1}{2} g_{pq} g^{pq}$, i.e., the function (10) constructed by $a = g$, is evidently constant, there exists a nontrivial linear combination $\hat{a}$ of $a, \bar{a}$ such that the corresponding $\hat{\lambda}$ given by (10) is constant. Since the mapping

$$v \mapsto a := L_v g - \frac{1}{n+1}\mathrm{trace}(g^{-1}L_v g) \cdot g$$

is linear, the linear combination of $v, \bar{v}$ with the same coefficients is an affine vector field. $\square$

1.3. Motivation II: New methods for investigation of global behavior of geodesically equivalent metrics. The theory of geodesically equivalent metrics has a long and fascinating history. First non-trivial examples were discovered by Lagrange [35]. Geodesically equivalent metrics were studied by Beltrami [5], Levi-Civita [36], Painlevé [65] and other classics. One can find more historical details in the surveys [3,62] and in the introduction to the papers [42,43,46,47,53,56,57,74].

The success of general relativity made it necessary to study geodesically equivalent pseudo-Riemannian metrics. The textbooks [15,23,66,67] on pseudo-Riemannian metrics have chapters on geodesically equivalent metrics. In the popular paper [76], Weyl stated a few interesting open problems on geodesic equivalence of pseudo-Riemannian metrics. Recent references (on the connection between geodesically equivalent metrics and general relativity) include Ehlers et al [16,17], Hall and Lonie [20,24,25], Hall [21,22].

In many cases, local statements about Riemannian metrics could be generalised for the pseudo-Riemannian setting, though sometimes this generalisation is difficult. As a rule, it is very difficult to generalize global statements about Riemannian metrics to the pseudo-Riemannian setting. The theory of geodesically equivalent metrics is not an exception: most local results could be generalized. For example, the most classical results: the Dini/Levi-Civita description of geodesically equivalent metrics [12,36] and the Fubini Theorem [18] were generalised in [2,7–10].

Up to now, no global (if the manifold is closed or complete) methods for investigation of geodesically equivalent metrics were generalized for the pseudo-Riemannian setting. More precisely, virtually every global result on geodesically equivalent Riemannian metrics was obtained by combining the following methods.

- "Bochner technique". This is a group of methods combining local differential geometry and the Stokes theorem. In the theory of geodesically equivalent metrics, the most standard (de-facto, the only) way to use the Bochner technique was to use tensor calculus to canonically obtain a nonconstant function f such as $\Delta_g f = \text{const} \cdot f$, where const ≥ 0, which of course can not exist on closed Riemannian manifolds.
  An example could be derived from our paper: from Eq. (55) it follows that $(\Delta_g \lambda)_{,k} = 2(n+1)B\lambda_{,k}$. Thus, for a certain const $\in \mathbb{R}$ we have $(\Delta_g(\lambda + \text{const})) = 2(n+1)B(\lambda + \text{const})$. If B is positive, g is Riemannian, and M is closed, this implies that the function λ is constant, which is equivalent to the statement that the metrics are affine equivalent.
  The first application of this technique in the theory of geodesically equivalent metrics is due to the Japan geometry school of Yano, Tanno, and Obata, see for example [27]. Also, the mathematical schools of Odessa and Kazan were extremely strong in this group of methods, see the review papers [3,62], and the references inside these papers. Of course, since for pseudo-Riemannian metrics the equation $\Delta_g f = \text{const} \cdot f$ could have solutions for const ≥ 0, this technique completely fails in the pseudo-Riemannian case.
- "Volume and curvature estimations". For geodesically equivalent metrics g and $\bar{g}$, the repametrisation of geodesics is controlled by a function ϕ given by (5). This function also controls the difference between Ricci curvatures of g and $\bar{g}$. Playing with this, one can obtain obstructions for the existence of positively definite geodesically equivalent metrics with negatively definite Ricci-curvature (assuming the manifold is closed, or complete with finite volume). Recent references include [29,68].
  This method essentially uses the positive definiteness of the metrics.
- "Global ordering of eigenvalues of a^i_j". The existence of a metric $\bar{g}$ geodesically equivalent to g implies the existence of integrals of special form (we recall one of

the integrals in Lemma 1) for the geodesic flow of the metric g [39,42,43]. In the Riemannian case, analysis of the integrals implies global ordering of the eigenvalues of the tensor $a^i_j := \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} \bar g^{ip} g_{pj}$, where $\bar g^{ip}$ is the tensor dual to $\bar g_{ij}$, see [6,54,74]. Combining it with the Levi-Civita description of geodesically equivalent metrics, one could describe topology of closed manifolds admitting geodesically equivalent Riemannian metrics [33,40,41,44–47,49,52].
Though the integrability survives in the pseudo-Riemannian setting [6,73], the global ordering of the eigenvalues is not valid anymore (there exist counterexamples), so this method also is not applicable to the pseudo-Riemannian metrics.

Our proofs (we explain the scheme of the proofs in the beginning of Sect. 2) use essentially new methods. We would like to emphasize here once more that the last step of the proof, which uses the local results to obtain global statements, is based on the existence of lightlike geodesics, and, therefore, is essentially pseudo-Riemannian.

A similar idea was used in [30], where it was proved that complete Einstein metrics are geodesically rigid: every complete metric geodesically equivalent to a complete Einstein metric is affine equivalent to it.

We expect further application of these new methods in the theory of geodesically equivalent metrics.

1.4. Additional motivation: Superintegrable metrics. Recall that a metric is called *superintegrable*, if the number of independent integrals of special form is greater than the dimension of the manifold. Superintegrable systems are nowadays a hot topic in mathematical physics, probably because almost all exactly solvable systems are superintegrable. There are different possibilities for the special form of integrals; de facto the most standard special form of the integrals is the so-called Benenti integrals, which are essentially the same as geodesically equivalent metrics, see [4,6,34]. Theorem 2 of our paper shows that complete Benenti-superintegrable metrics of nonconstant curvature cannot exist on closed manifolds, which was a folklore conjecture.

2. Proof of Theorems 1, 2

In Sect. 2.1, we recall standard facts about geodesically equivalent metrics and fix the notation. In Sect. 2.2, we will prove Lemma 2, which is a purely linear algebraic statement. Given two solutions of Eq. (11), it gives us Eq. (27). The coefficients in the equation are a priori functions. We will work with this equation for a while: In Sect. 2.3.1, we prove (Lemma 5) that (under the assumptions of Theorem 1) one of the coefficients of (27) is actually a constant. Later, we will show (Lemma 8) that the metric g determines the constant uniquely.

Equation (27) will be used in Sect. 2.3.6. The main result of this section is Corollary 8. This corollary gives us (under assumptions of Theorem 1) an ODE that must be fulfilled along every lightlike geodesic, and that controls the reparameterization that produces g-geodesics from $\bar g$-geodesics. The ODE is relatively simple and could be solved explicitly (Sect. 2.4). Analyzing the solutions, we will see that the geodesic is complete with respect to both metrics iff the function controlling the reparametrization of the geodesics is a constant, which implies that the metrics are affine equivalent. This proves Theorem 1 provided lightlike geodesics exist. As we mentioned in the Introduction, Theorem 1 was already proved [45,57] for Riemannian metrics. Nevertheless, for self-containedness, in

Sect. 2.5 we give a new proof for Riemannian metrics as well, which is much shorter than the original proof from [45,57].

The proof of Theorem 2 will be done in Sect. 2.6. The idea is similar: we analyze a certain ODE along lightlike geodesics (this ODE will easily follow from Eq. (55), which is an easy corollary of Eq. (27)), and show that the assumption that the manifold is closed implies that the solution of the ODE coming from the metric $\bar g$ is constant, which implies that g and $\bar g$ are geodesically equivalent.

2.1. Standard formulas we will use. We work in tensor notation with the background metric g. That means, we sum with respect to repeating indexes, use g for raising and lowering indexes (unless we explicitly say otherwise), and use the Levi-Civita connection of g for covariant differentiation.

As it was known already for Levi-Civita [36], two connections $\Gamma = \Gamma^i_{jk}$ and $\bar\Gamma = \bar\Gamma^i_{jk}$ have the same unparameterized geodesics, if and only if their difference is a pure trace: there exists a (0, 1)-tensor ϕ such that

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_k\phi_j + \delta^i_j\phi_k. \tag{2}$$

The reparametrizations of the geodesics for Γ and $\bar\Gamma$ connected by (2) are done according to the following rule: for a parametrized geodesic $\gamma(\tau)$ of $\bar\Gamma$, the curve $\gamma(\tau(t))$ is a parametrized geodesic of Γ, if and only if the parameter transformation $\tau(t)$ satisfies the following ODE:

$$\phi_p\dot\gamma^p = \frac{1}{2}\frac{d}{dt}\left(\log\left(\left|\frac{d\tau}{dt}\right|\right)\right). \tag{3}$$

(We denote by $\dot\gamma$ the velocity vector of γ with respect to the parameter t, and assume summation with respect to repeating index p.)

If Γ and $\bar\Gamma$ related by (2) are Levi-Civita connections of metrics g and $\bar g$, then one can find explicitly (following Levi-Civita [36]) a function ϕ on the manifold such that its differential $\phi_{,i}$ coincides with the covector ϕ_i: indeed, contracting (2) with respect to i and j, we obtain $\bar\Gamma^p_{pi} = \Gamma^p_{pi} + (n+1)\phi_i$. On the other hand, for the Levi-Civita connection Γ of a metric g we have $\Gamma^p_{pk} = \frac{1}{2}\frac{\partial\log(|det(g)|)}{\partial x_k}$. Thus,

$$\phi_i = \frac{1}{2(n+1)}\frac{\partial}{\partial x_i}\log\left(\left|\frac{\det(\bar g)}{\det(g)}\right|\right) = \phi_{,i} \tag{4}$$

for the function $\phi : M \to \mathbb{R}$ given by

$$\phi := \frac{1}{2(n+1)}\log\left(\left|\frac{\det(\bar g)}{\det(g)}\right|\right). \tag{5}$$

In particular, the derivative of ϕ_i is symmetric, i.e., $\phi_{i,j} = \phi_{j,i}$.

The formula (2) implies that two metrics g and $\bar g$ are geodesically equivalent if and only if for a certain ϕ_i (which is, as we explained above, the differential of ϕ given by (5)) we have

$$\bar g_{ij,k} - 2\bar g_{ij}\phi_k - \bar g_{ik}\phi_j - \bar g_{jk}\phi_i = 0, \tag{6}$$

where "comma" denotes the covariant derivative with respect to the connection Γ. Indeed, the left-hand side of this equation is the covariant derivative with respect to $\bar{\Gamma}$, and vanishes if and only if $\bar{\Gamma}$ is the Levi-Civita connection for $\bar{g}$.

Equations (6) can be linearized by a clever substitution: consider a_{ij} and λ_i given by

$$a_{ij} = e^{2\phi}\bar{g}^{pq}g_{pi}g_{qj}, \tag{7}$$

$$\lambda_i = -e^{2\phi}\phi_p\bar{g}^{pq}g_{qi}, \tag{8}$$

where $\bar{g}^{pq}$ is the tensor dual to $\bar{g}_{pq}$: $\bar{g}^{pi}\bar{g}_{pj} = \delta^i_j$. It is an easy exercise to show that the following linear equations for the symmetric (0, 2)-tensor a_{ij} and (0, 1)-tensor λ_i are equivalent to (6).

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{9}$$

Remark 4. For dimension 2, the substitution (7,8) was already known to R. Liouville [37] and Dini [12], see [11, Sect. 2.4] for details and a conceptual explanation. For arbitrary dimension, the substitution (7,8) and Eq. (9) are due to Sinjukov [69]. The underlying geometry is explained in [13,14].

Note that it is possible to find a function λ whose differential is precisely the (0, 1)-tensor λ_i: indeed, multiplying (9) by g^{ij} and summing with respect to repeating indexes i, j we obtain $(g^{ij}a_{ij})_{,k} = 2\lambda_k$. Thus, λ_i is the differential of the function

$$\lambda := \frac{1}{2}g^{pq}a_{pq}. \tag{10}$$

In particular, the covariant derivative of λ_i is symmetric: $\lambda_{i,j} = \lambda_{j,i}$.

We see that Eqs. (9) are linear. Thus the space of the solutions is a linear vector space. Its dimension is called the *degree of mobility* of the metric g.

We will also need integrability conditions for Eq. (9) (one obtains them substituting the derivatives of a_{ij} given by (9) in the formula $a_{ij,lk} - a_{ij,kl} = a_{ip}R^p_{jkl} + a_{pj}R^p_{ikl}$, which is true for every (0, 2)−tensor a_{ij})

$$a_{ip}R^p_{jkl} + a_{pj}R^p_{ikl} = \lambda_{l,i}g_{jk} + \lambda_{l,j}g_{ik} - \lambda_{k,i}g_{jl} - \lambda_{k,j}g_{il}. \tag{11}$$

The integrability condition in this form was obtained by Sinjukov [69]; in equivalent form, it was known to Solodovnikov [70].

As a consequence of these integrability conditions, we obtain that every solution a_{ij} of (9) must commute with the Ricci tensor R_{ij}:

$$a^p_i R_{pj} = a^p_j R_{ip}. \tag{12}$$

To show this, we "cycle" Eq. (11) with respect to i, k, l, i.e., we sum it with itself after renaming the indexes according to $(i \mapsto k \mapsto l \mapsto i)$ and with itself after renaming the indexes according to $(i \mapsto l \mapsto k \mapsto i)$. The first term at the left-hand side of the equation will disappear because of the Bianchi equality $R^p_{ikl} + R^p_{kli} + R^p_{lik} = 0$, the right-hand side vanishes completely, and we obtain

$$a_{pi}R^p_{jkl} + a_{pk}R^p_{jli} + a_{pl}R^p_{jik} = 0. \tag{13}$$

Multiplying with g^{jk}, using symmetries of the curvature tensor, and summing over the repeating indexes we obtain $a_{pi}R^p_l - a_{pl}R^p_i = 0$, i.e., (12).

Remark 5. For further use, let us recall that Eqs. (9) are of finite type (they close after two differentiations [14,62,69]). Since they are linear, and since in view of (10) they could be viewed as equations on a_{ij} only, linear independence of the solutions on the whole connected manifold implies linear independence of the restriction of the solutions to every neighborhood. Thus, the assumption that the degree of mobility of g (on a connected M) is ≥ 3 implies that the degree of mobility of the restriction of g to every neighborhood is also ≥ 3.

We will also need the following statement from [39,74]. We denote by $\mathrm{co}(a)^i_j$ the classical comatrix (adjugate matrix) of the $(1,1)$-tensor a^i_j viewed as an $n \times n$-matrix. $\mathrm{co}(a)^i_j$ is also a $(1,1)$-tensor.

Lemma 1 ([39,74]). *If the* $(0,2)$*-tensor* a_{ij} *satisfies* (9)*, then the function*

$$I : TM \to \mathbb{R}, \quad (\underbrace{x}_{\in M}, \underbrace{\xi}_{\in T_xM}) \mapsto g_{pq}\, co(a)^p_\gamma \xi^\gamma \xi^q \tag{14}$$

is an integral of the geodesic flow of g.

Recall that a function is an *integral* of the geodesic flow of g, if it is constant along the orbits of the geodesic flow of g, i.e., if for every parametrized geodesic $\gamma(t)$ the function $I(\gamma(t), \dot\gamma(t))$ does not depend on t.

Remark 6. If the tensor a_{ij} comes from a geodesically equivalent metric $\bar g$ by formula (7), the integral (14) is

$$I(x,\xi) = \left|\frac{\det(g)}{\det(\bar g)}\right|^{2/(n+1)} \bar g(\xi,\xi).$$

In this form, Lemma 1 was already known to Painlevé [65].

2.2. An algebraic lemma.

Lemma 2. *Assume symmetric* $(0,2)$ *tensors* a_{ij}*,* A_{ij}*,* λ_{ij} *and* Λ_{ij} *satisfy*

$$\begin{aligned} a_{ip}R^p_{jkl} + a_{pj}R^p_{ikl} &= \lambda_{li}g_{jk} + \lambda_{lj}g_{ik} - \lambda_{ki}g_{jl} - \lambda_{kj}g_{il}, \\ A_{ip}R^p_{jkl} + A_{pj}R^p_{ikl} &= \Lambda_{li}g_{jk} + \Lambda_{lj}g_{ik} - \Lambda_{ki}g_{jl} - \Lambda_{kj}g_{il}, \end{aligned} \tag{15}$$

where g_{ij} *is a metric and* R^i_{jkl} *is its curvature tensor. Assume* a_{ij}*,* A_{ij}*, and* g_{ij} *are linearly independent at the point* p*. Then, at the point,* λ_{ij} *is a linear combination of* a_{ij} *and* g_{ij}*.*

Remark 7. We would like to emphasize here that, though the lemma is formulated in the tensor notation, it is a purely algebraic statement (in the proof we will not use differentiation, and, as we see, no differential condition on a, A is required). Moreover, we can replace R^i_{jkl} by any (1,3)-tensor having the same algebraic symmetries (with respect to g) as the curvature tensor, so that for example the fact that the first equation of (15) coincides with (11) will not be used in the proof (but of course this will be used in the applications of Lemma 2). The underlying algebraic structure of the lemma is explained in the last section of [9].

Proof. First observe that Eqs. (15) are unaffected by replacing

$$a_{ij} \mapsto a_{ij} + a \cdot g_{ij}\ ,\ \lambda_{ij} \mapsto \lambda_{ij} + \lambda \cdot g_{ij}\ ,\ A_{ij} \mapsto A_{ij} + A \cdot g_{ij}\ ,\ \Lambda_{ij} \mapsto \Lambda_{ij} + \Lambda \cdot g_{ij}$$

for arbitrary $a, \lambda, A, \Lambda \in \mathbb{R}$. Therefore we may suppose, without loss of generality, that $a_{ij}, \lambda_{ij}, A_{ij}, \Lambda_{ij}$ are trace-free, i.e.,

$$a_{ij} g^{ij} = \lambda_{ij} g^{ij} = A_{ij} g^{ij} = \Lambda_{ij} g^{ij} = 0. \tag{16}$$

Our assumptions become that a_{ij} and A_{ij} are linearly independent and our aim is to show that $\lambda_{ij} = \text{const} \cdot a_{ij}$.

We multiply the first equation of (15) by $A^{l}_{l'}$ and sum over l. After renaming $l' \mapsto l$, we obtain

$$a_{ip} R^{p}_{jkq} A^{q}_{l} + a_{pj} R^{p}_{ikq} A^{q}_{l} = \lambda_{pi} A^{p}_{l} g_{jk} + \lambda_{pj} A^{p}_{l} g_{ik} - \lambda_{ki} A_{jl} - \lambda_{kj} A_{il}. \tag{17}$$

We use symmetries of the Riemann tensor to obtain $a^{p}_{i} R_{pjkq} A^{q}_{l} = a^{p}_{i} R_{qkjp} A^{q}_{l} = a^{p}_{i} A_{ql} R^{q}_{kjp}$. After substituting this in (17), we get

$$a^{p}_{i} A_{ql} R^{q}_{kjp} + a^{p}_{j} A_{ql} R^{q}_{kip} = \lambda_{pi} A^{p}_{l} g_{jk} + \lambda_{pj} A^{p}_{l} g_{ik} - \lambda_{ki} A_{jl} - \lambda_{kj} A_{il}. \tag{18}$$

Let us now symmetrize (18) by l, k,

$$\begin{aligned} &a^{p}_{i} \left(A_{ql} R^{q}_{kjp} + A_{qk} R^{q}_{ljp} \right) + a^{p}_{j} \left(A_{qk} R^{q}_{lip} + A_{ql} R^{q}_{kip} \right) \\ &\quad = \lambda_{pi} A^{p}_{l} g_{jk} + \lambda_{pj} A^{p}_{l} g_{ik} - \lambda_{ki} A_{jl} - \lambda_{kj} A_{il} + \lambda_{pi} A^{p}_{k} g_{jl} + \lambda_{pj} A^{p}_{k} g_{il} - \lambda_{li} A_{jk} - \lambda_{lj} A_{ik}. \end{aligned} \tag{19}$$

We see that the components in brackets are the left-hand side of the second equation of (15) with other indexes. Substituting (15) in (19), we obtain

$$\begin{aligned} &a^{p}_{i} \Lambda_{pl} g_{jk} + a^{p}_{i} \Lambda_{pk} g_{jl} - \Lambda_{jl} a_{ik} - \Lambda_{jk} a_{il} + a^{p}_{j} \Lambda_{pl} g_{ik} + a^{p}_{j} \Lambda_{pk} g_{il} - \Lambda_{il} a_{jk} - \Lambda_{ik} a_{jl} \\ &\quad = \lambda_{pi} A^{p}_{l} g_{jk} + \lambda_{pj} A^{p}_{l} g_{ik} - \lambda_{ki} A_{jl} - \lambda_{kj} A_{il} + \lambda_{pi} A^{p}_{k} g_{jl} + \lambda_{pj} A^{p}_{k} g_{il} - \lambda_{li} A_{jk} - \lambda_{lj} A_{ik}. \end{aligned} \tag{20}$$

Collecting the terms by g, we see that (20) is can be written as

$$\begin{aligned} &\left(a^{p}_{i} \Lambda_{pl} - \lambda_{pi} A^{p}_{l}\right) g_{jk} + \left(a^{p}_{i} \Lambda_{pk} - \lambda_{pi} A^{p}_{k}\right) g_{jl} \\ &\quad + \left(a^{p}_{j} \Lambda_{pl} - \lambda_{pj} A^{p}_{l}\right) g_{ik} + \left(a^{p}_{j} \Lambda_{pk} - \lambda_{pj} A^{p}_{k}\right) g_{il} \\ &\quad = \Lambda_{jl} a_{ik} + \Lambda_{jk} a_{il} + \Lambda_{il} a_{jk} + \Lambda_{ik} a_{jl} - \lambda_{ki} A_{jl} - \lambda_{kj} A_{il} - \lambda_{li} A_{jk} - \lambda_{lj} A_{ik}. \end{aligned} \tag{21}$$

After denoting

$$\tau_{il} := a^{p}_{i} \Lambda_{pl} - A^{p}_{l} \lambda_{pi}, \tag{22}$$

Eq. (21) can be written as

$$\begin{aligned} &\tau_{il} g_{jk} + \tau_{ik} g_{jl} + \tau_{jl} g_{ik} + \tau_{jk} g_{il} \\ &\quad = \Lambda_{jl} a_{ik} + \Lambda_{jk} a_{il} + \Lambda_{il} a_{jk} + \Lambda_{ik} a_{jl} - \lambda_{ki} A_{jl} - \lambda_{kj} A_{il} - \lambda_{li} A_{jk} - \lambda_{lj} A_{ik}. \end{aligned} \tag{23}$$

Multiplying (23) by g^{jk}, contracting with respect to j, k, and using (16), we obtain

$$(n+2)\tau_{il} + \left(\tau_{jk}g^{jk}\right) g_{il} = \Lambda_{pl}a_i^p + \Lambda_{ip}a_l^p - \lambda_{pi}A_l^p - \lambda_{lp}A_i^p \stackrel{(22)}{=} \tau_{il} + \tau_{li}. \tag{24}$$

We see that the right-hand side is symmetric with respect to i, l. Then, so should be the left-hand-side implying $\tau_{il} = \tau_{li}$. Then, Eq. (24) implies $n\tau_{il} + \left(\tau_{jk}g^{jk}\right) g_{il} = 0$ implying $\tau_{il} = 0$. Then, Eq. (23) reads

$$0 = \Lambda_{jl}a_{ik} + \Lambda_{jk}a_{il} + \Lambda_{il}a_{jk} + \Lambda_{ik}a_{jl} - \lambda_{ki}A_{jl} - \lambda_{kj}A_{il} - \lambda_{li}A_{jk} - \lambda_{lj}A_{ik}. \tag{25}$$

We alternate (25) with respect to j, k to obtain

$$0 = \Lambda_{jl}a_{ik} + \Lambda_{ik}a_{jl} - \lambda_{ki}A_{jl} - \lambda_{lj}A_{ik} - \Lambda_{kl}a_{ij} - \Lambda_{ij}a_{kl} + \lambda_{ji}A_{kl} + \lambda_{lk}A_{ij}. \tag{26}$$

Let us now rename $i \leftrightarrow k$ in (26) and add the result with (25). We obtain

$$\Lambda_{jl}a_{ik} + \Lambda_{ik}a_{jl} - \lambda_{ki}A_{jl} - \lambda_{lj}A_{ik} = 0.$$

In other words, $\Lambda_\alpha a_\beta + \Lambda_\beta a_\alpha = \lambda_\beta A_\alpha + \lambda_\alpha A_\beta$, where α and β stand for the symmetric indices jl and ik, respectively.

But it is easy to check that a non-zero simple symmetric tensor $X_{\alpha\beta} = P_\alpha Q_\beta + P_\beta Q_\alpha$ determines its factors P_α and Q_β up to scale and order (it is sufficient to check, for example, by taking P_α and Q_β to be basis vectors). Since a_{ij} and A_{ij} are supposed to be linearly independent, it follows that $\lambda_{ij} = \text{const} \cdot a_{ij}$, as required. □

2.3. Local results. Within this section, we assume that (M, g) is a connected Riemannian or pseudo-Riemannian manifold of dimension $n \geq 3$. Recall that the degree of mobility of a metric g is the dimension of the space of the solutions of (9).

Lemma 3. *Suppose that the degree of mobility of g is ≥ 3. Then for every solution a_{ij} of (9), where λ_i is the differential of the function λ given by (10), there exists an open dense subset N of M each of whose points admits an open neighborhood U, a constant B, and a function μ on U, such that the hessian of λ satisfies on U the equation*

$$\lambda_{,ij} = \mu g_{ij} + B a_{ij}. \tag{27}$$

Proof. If $a = \text{const}\cdot g$, then λ is constant and the lemma holds with $N = M$, $\mu \equiv B = 0$. Otherwise there exists a solution A of (9) such that a, A, g are linearly independent. We denote by Λ_i the (0, 1)-tensor from Eq. (8) corresponding to A, i.e., $\Lambda_i = \Lambda_{,i}$ for $\Lambda := \frac{1}{2}A_{pq}g^{pq}$.

Then the integrability conditions (11) for the solutions a and A are given by (15) (with $\lambda_{ij} = \lambda_{,ij}$ and $\Lambda_{ij} = \Lambda_{,ij}$).

Let N be the set of all $x \in M$ which admit a neighborhood on which a, A, g are either pointwise linearly independent or pointwise linearly dependent. Being a union of open sets, N is open. N is also dense in M: every nonempty open set $U \subset M$ either consists only of points where a, A, g are linearly dependent, then $U \subset N$; or it contains a point where a, A, g are linearly independent and which is therefore contained in N.

By definition every point in N has an open connected neighborhood U on which one of two possibilities holds:

(a) a, A, g are pointwise linearly independent. Then, by Lemma 2, $\lambda_{,ij} = \mu g_{ij} + B a_{ij}$, where μ and B are functions; they are unique and smooth because of linear independence. Our goal is to show that B is actually a constant, this will be done in Sect. 2.3.3.

(b) a, A, g are pointwise linearly dependent. Then there exist a nonempty open connected subset U' of U and (smooth) functions $\overset{1}{c}, \overset{2}{c}$ on U' such that on U', we have $a + \overset{1}{c} A + \overset{2}{c} g \equiv 0$ or $A + \overset{1}{c} a + \overset{2}{c} g \equiv 0$. (To see that $\overset{1}{c}, \overset{2}{c}$ can be chosen to be smooth, distinguish three cases: the span of a, A, g has on U pointwise dimension 1; or A, g are linearly independent somewhere; or a, g are linearly independent somewhere.) We will prove in Sect. 2.3.1 that $\overset{1}{c}, \overset{2}{c}$ are actually constants. (Lemma 5 can be applied here because if a or A had the form $\text{const} \cdot g$ on U', then also on M, in contradiction to linear independence.) Thus a, A, g are linearly dependent on U' and therefore on M. This contradiction rules out case (b).

2.3.1. Linear dependence of three solutions over functions implies their linear dependence over numbers. We will use the following statement (essentially due to Weyl [75]); its proof can be found for example in [74], see also [9, Lemma 1 in Sect. 2.4].

Lemma 4. *Suppose a_{ij} and A_{ij} are solutions of (9). Assume $a = f \cdot A$, where f is a function. Then f is actually a constant.*

Our main goal is the following lemma, which settles the case (b) of the proof of Lemma 3.

Lemma 5. *Suppose for certain functions $\overset{1}{c}, \overset{2}{c}$ the solutions a, A (of (9) on a connected manifold $(M^{n\geq 3}, g)$) satisfy*

$$a_{ij} = \overset{1}{c}\, g_{ij} + \overset{2}{c}\, A_{ij}. \tag{28}$$

We assume in addition that A is not const $\cdot\, g$*. Then the functions $\overset{1}{c}, \overset{2}{c}$ are constants.*

Remark 8. Though we will use that the dimension of the manifold is at least three, the statement is true in dimension two as well provided the curvature of g is not constant, see [33].

Proof of Lemma 5. We assume that $\overset{1}{c}_{,k}$ or $\overset{2}{c}_{,k}$ is not zero everywhere, and find a contradiction.

Differentiating (28) and substituting (9) and its analog for the solution A, we obtain

$$\lambda_i g_{jk} + \lambda_j g_{ik} = \overset{1}{c}_{,k}\, g_{ij} + \overset{2}{c}\, \Lambda_i g_{jk} + \overset{2}{c}\, \Lambda_j g_{ik} + \overset{2}{c}_{,k}\, A_{ij}, \tag{29}$$

which is evidently equivalent to

$$\tau_i g_{jk} + \tau_j g_{ik} = \overset{1}{c}_{,k}\, g_{ij} + \overset{2}{c}_{,k}\, A_{ij}, \tag{30}$$

where $\tau_i = \lambda_i - \overset{2}{c}\, \Lambda_i$. We see that for every fixed k the left-hand side is a symmetric matrix of the form $\tau_i v_j + \tau_j v_i$. If $\overset{1}{c}_{,k}$ is not proportional to $\overset{2}{c}_{,k}$ at some point

$x \in M$, this will imply that g_{ij} also is of the form $\tau_i v_j + \tau_j v_i$ at x, which contradicts the nondegeneracy of g. Thus there exists a function f with

$$\overset{1}{c}_{,k} = f \cdot \overset{2}{c}_{,k}\ . \tag{31}$$

At each point x there exists a nonzero vector $\xi = (\xi^k) \in T_x M$ such that $\xi^k \overset{2}{c}_{,k} = 0$. Multiplying (30) with ξ^k and summing with respect to k, we see that the right-hand side vanishes, and obtain the equation $\tau_i v_j + \tau_j v_i = 0$, where $v_i := \xi^k g_{ik}$. Since $v_i \neq 0$, we obtain $\tau_i = 0$ at x; hence Eq. (30) reads $f \cdot \overset{2}{c}_{,k}\ g_{ij} = -\overset{2}{c}_{,k}\ A_{ij}$ everywhere on M. Since the covector field $\overset{2}{c}_{,k}$ is pointwise nonzero on some nonempty connected open subset U of M, this equation implies $f \cdot g_{ij} = -A_{ij}$ on U. By Lemma 4, f is constant on U. By Remark 4, it is constant globally, which contradicts the assumptions. □

2.3.2. In dimension 3*, only metrics of constant curvature can have the degree of mobility* ≥ 3

Lemma 6. *Assume that the conformal Weyl tensor C^h_{ijk} of the metric g on* (*a connected*) *$M^{n\geq 3}$ vanishes. If the curvature of the metric is not constant, the degree of mobility of g is at most two.*

Since the conformal Weyl tensor C^h_{ijk} of any metric on a 3-dimensional manifold vanishes, a special case of Lemma 6 is

Corollary 3. *The degree of mobility of each metric g of nonconstant curvature on M^3 is at most two.*

Proof of Lemma 6. It is well-known that the curvature tensor of spaces with $C^h_{ijk} = 0$ has the form

$$R^h_{ijk} = P^h_k g_{ij} - P^h_j g_{ik} + \delta^h_k P_{ij} - \delta^h_j P_{ik}, \tag{32}$$

where $P_{ij} := \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)} g_{ij}\right)$ (and therefore $P^h_k = P_{pk} g^{ph}$). We denote by P the trace of P^h_k; easy calculations give us $P = \frac{R}{2(n-1)}$.

Substituting Eqs. (32) in the integrability conditions (11), we obtain

$$\begin{aligned} a_{pi} P^p_l g_{jk} - a_{pi} P^p_k g_{jl} + a_{li} P_{jk} - a_{ki} P_{jl} + a_{pj} P^p_l g_{ik} - a_{pj} P^p_k g_{il} + a_{lj} P_{ik} - a_{kj} P_{il} \\ = \lambda_{l,i} g_{jk} + \lambda_{l,j} g_{ik} - \lambda_{k,i} g_{jl} - \lambda_{k,j} g_{il}. \end{aligned} \tag{33}$$

Multiplying (33) with g^{jk} and summing with respect to repeating indexes, and using the symmetry of P_{ij} due to (12), we obtain

$$a_{pi} P^p_l = \lambda_{l,i} - \frac{P}{n} a_{li} + \frac{\hat{P}}{n} g_{li} + \frac{2\lambda}{n} P_{il}, \tag{34}$$

where $\hat{P} = g^{q\gamma} a_{pq} P^p_\gamma - \lambda^p_{\ ,p}$. Substituting (34) in (33), we obtain

$$\begin{aligned} 0 =& \frac{2\lambda}{n} P_{il} g_{jk} - \frac{2\lambda}{n} P_{ik} g_{jl} + \frac{2\lambda}{n} P_{jl} g_{ik} - \frac{2\lambda}{n} P_{jk} g_{il} \\ &+ a_{li} P_{jk} - a_{ki} P_{jl} + a_{lj} P_{ik} - a_{kj} P_{il} - \frac{P}{n} a_{il} g_{jk} + \frac{P}{n} a_{ik} g_{jl} - \frac{P}{n} a_{jl} g_{ik} + \frac{P}{n} a_{jk} g_{il}. \end{aligned} \tag{35}$$

Alternating Eq. (35) with respect to j, k, renaming $i \longleftrightarrow k$, and adding the result to (35), we obtain

$$\frac{2\lambda}{n} P_{il} g_{jk} - \frac{2\lambda}{n} P_{jk} g_{il} + a_{li} P_{jk} - a_{kj} P_{il} - \frac{P}{n} a_{il} g_{jk} + \frac{P}{n} a_{jk} g_{il} = 0, \tag{36}$$

which is evidently equivalent to

$$\frac{2\lambda}{n} P_{il} g_{jk} - \frac{2\lambda}{n} P_{jk} g_{il} + a_{li} \left(P_{jk} - \frac{P}{n} g_{jk} \right) - a_{kj} \left(P_{il} - \frac{P}{n} g_{il} \right) = 0. \tag{37}$$

Hence (in view of $P_{jk} - \frac{P}{n} g_{jk} \neq 0$ because by assumption the curvature of g is not constant) there exists a nonempty open set U such that every solution a_{ij} of (9) is on U a smooth linear combination of g_{ij} and P_{ij}. Thus every three solutions $g, a, \hat{a}$ of (9) are on U linearly dependent over functions. By Lemma 5, they are on U, and therefore everywhere, linearly dependent over numbers. □

2.3.3. Case (a) of Lemma 3: Proof that $B =$ const. We consider a neighborhood $U \subseteq M^{n \geq 3}$ such that a, A, g are linearly independent at every point of the neighborhood; by Lemma 5, almost every point has such neighborhood.

Remark 9. Within the whole paper we understand "almost everywhere" and "almost every" in the topological sense: a condition is fulfilled almost everywhere (or in almost every point) if and only if it the set of the points where it is fulfilled is everywhere dense.

In the beginning of the proof of Lemma 3, we explained that at every point of the neighborhood Eq. (27) holds for certain smooth functions μ and B. Our goal is to show that B is actually a constant (on U).

Because of Corollary 3, we can assume $n = \dim(M) \geq 4$. Indeed, otherwise by Corollary 3 the curvature of the metrics is constant, and the metric is Einstein. Then, by [30, Cor. 1], Eq. (27) holds.

Within the proof, we will use the following equations, the first one is (9), the second follows from Lemma 3:

$$\begin{cases} a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \\ \lambda_{,ij} = \mu g_{ij} + B a_{ij}. \end{cases} \tag{38}$$

Our goal will be to show that B is constant. We assume that it is not the case and show that for a certain covector field u_i and functions α, β on the manifold we have $a_{ij} = \alpha g_{ij} + \beta u_i u_j$. Later we will show that this gives a contradiction with the assumption that the degree of mobility is three.

We consider the equation $\lambda_{i,j} = \mu g_{ij} + B a_{ij}$. Taking the covariant derivative ∇_k, we obtain

$$\lambda_{i,jk} = \mu_{,k} g_{ij} + B_{,k} a_{ij} + B a_{ij,k} \overset{(9)}{=} \mu_{,k} g_{ij} + B_{,k} a_{ij} + B \lambda_i g_{jk} + B \lambda_j g_{ik}. \tag{39}$$

By definition of the Riemannian curvature, we have $\lambda_{i,jk} - \lambda_{i,kj} = \lambda_p R^p_{ijk}$. Substituting (39) in this equation, we obtain

$$\lambda_p R^p_{ijk} = \mu_{,k} g_{ij} + B_{,k} a_{ij} - \mu_{,j} g_{ik} - B_{,j} a_{ik} + B \lambda_j g_{ik} - B \lambda_k g_{ij}. \tag{40}$$

Now, substituting the second equation of (38) in (11), we obtain

$$a_{pi} R^p_{jkl} + a_{pj} R^p_{ikl} = B\left(a_{li} g_{jk} + a_{lj} g_{ik} - a_{ki} g_{jl} - a_{kj} g_{il}\right). \tag{41}$$

We multiply this equation by λ^l and sum over l. Using that $a_{pi} R^p_{jkq} \lambda^q$ is evidently equal to $a^p_i R^q_{kjp} \lambda_q$, we obtain

$$a^p_i R^q_{kjp} \lambda_q + a^p_j R^q_{kip} \lambda_q = B\left(a_{iq} \lambda^q g_{jk} + a_{jq} \lambda^q g_{ik} - a_{ki} \lambda_j - a_{kj} \lambda_i\right). \tag{42}$$

Substituting the expressions for $R^q_{kjp}\lambda_q$ and $R^q_{kip}\lambda_q$, we obtain

$$\overset{1}{\tau}_i\ a_{jk} + \overset{1}{\tau}_j\ a_{ik} + \overset{2}{\tau}_i\ g_{jk} + \overset{2}{\tau}_j\ g_{ki} - B_{,j} a^p_i a_{pk} - B_{,i} a^p_j a_{pk} = 0, \tag{43}$$

where $\overset{1}{\tau}_i := a^p_i B_{,p} - \mu_{,i} + 2B\lambda_i$ and $\overset{2}{\tau}_i := a^p_i \mu_{,p} - 2B\lambda_p a^p_i$.

Now let us work with (43): we alternate the equation with respect to i, k to obtain:

$$\overset{1}{\tau}_i\ a_{jk} + \overset{2}{\tau}_i\ g_{jk} - B_{,i} a^p_j a_{pk} - \overset{1}{\tau}_k\ a_{ji} - \overset{2}{\tau}_k\ g_{ji} + B_{,k} a^p_j a_{pi} = 0. \tag{44}$$

We rename $j \leftrightarrow k$ and add the result to (43): we obtain

$$\overset{1}{\tau}_i\ a_{jk} + \overset{2}{\tau}_i\ g_{jk} = B_{,i} a^p_j a_{pk}. \tag{45}$$

Remark 10. If $B = \text{const}$ on U, then $\overset{1}{\tau}_i\ a_{jk} + \overset{2}{\tau}_i\ g_{jk} = 0$. Since by Lemma 4 a_{jk} is not proportional to g_{jk}, we have $\overset{1}{\tau}_i = 0$, which implies that $\mu_{,i} = 2B\lambda_i$.

The condition (45) implies that under the assumption $B \neq \text{const}$ the covectors $\overset{1}{\tau}_i, \overset{2}{\tau}_i$ and $B_{,i}$ are collinear: Moreover, for certain functions $\overset{1}{c}, \overset{2}{c}$,

$$\overset{1}{c}\ B_{,i} = \overset{1}{\tau}_i, \quad \overset{2}{c}\ B_{,i} = \overset{2}{\tau}_i, \quad \overset{1}{c}\ a_{jk} + \overset{2}{c}\ g_{jk} = a^p_j a_{pk}. \tag{46}$$

Taking the ∇_k derivative of the last formula of (46), we obtain

$$\lambda_p a^p_j g_{ik} + \lambda_i a_{jk} + \lambda_p a^p_i g_{jk} + \lambda_j a_{ik} = \overset{1}{c}_{,k}\ a_{ij} + \overset{2}{c}_{,k}\ g_{ij} + \overset{1}{c}\ \lambda_i g_{jk} + \overset{1}{c}\ \lambda_j g_{ik}.$$

Alternating the last formula with respect to i and k, we obtain:

$$\overset{3}{\tau}_i\ a_{jk} - \overset{3}{\tau}_k\ a_{ij} + \overset{4}{\tau}_i\ g_{jk} - \overset{4}{\tau}_k\ g_{ij} = 0, \tag{47}$$

where $\overset{3}{\tau}_i = \lambda_i + \overset{1}{c}_{,i}$, $\overset{4}{\tau}_i = \lambda_p a^p{}_i - \overset{1}{c}\ \lambda_i + \overset{2}{c}_{,i}$. Let us explain that this equation implies either $a_{ij} = \alpha g_{ij} + \beta u_i u_j$ (which was our goal), or $\overset{3}{\tau} = \overset{4}{\tau} = 0$.

We fix a point $x \in U$ and assume that $\overset{3}{\tau}_i \neq 0$ at the point. Then, $\overset{4}{\tau}_i \neq 0$ as well. For every vector $\xi \in T_x M$ we multiply (47) by ξ^j and sum with respect to j. Denoting $A(\xi)_k := a_{jk}\xi^j$ and $G(\xi)_k := g_{jk}\xi^j$, we obtain

$$\overset{3}{\tau}_i\ A(\xi)_k - \overset{3}{\tau}_k\ A(\xi)_i + \overset{4}{\tau}_i\ G(\xi)_i - \overset{4}{\tau}_k\ G(\xi)_i = 0. \tag{48}$$

Then, the (at most two-dimensional) subspaces of T_x^*M generated by $\{\overset{3}{\tau}_i, A(\xi)_i\}$ and by $\{\overset{4}{\tau}_i, G(\xi)_i\}$ coincide. Since the metric g is nondegenerate, varying ξ we obtain all possible elements of T_x^*M as $G(\xi)_i$, so the subspaces generated by $\{\overset{4}{\tau}_i, G(\xi)_i\}$ are all possible at most two-dimensional subspaces containing $\overset{4}{\tau}_i$, and the subspace generated by $\{\overset{4}{\tau}_i\}$ is the intersection of all such subspaces. Similarly, the subspace generated by $\{\overset{3}{\tau}_i\}$ is the intersection of subspaces generated by $\{\overset{3}{\tau}_i, A(\xi)_i\}$. Thus, $\overset{3}{\tau}_i = -\alpha \overset{4}{\tau}_i$ for a certain constant α, and Eq. (47) looks like

$$\overset{3}{\tau}_i (a_{jk} - \alpha g_{jk}) - \overset{3}{\tau}_k (a_{ij} - \alpha g_{jk}) = 0. \tag{49}$$

We take $\eta \in T_xM$ such that $\eta^k \overset{3}{\tau}_k = 0$, multiply (49) by η^k and sum over k. We obtain that $A(\eta) = \alpha G(\eta)$ for all such η. Thus, for a certain const β we have $a_{ij} = \alpha g_{ij} + \beta \overset{3}{\tau}_i \overset{3}{\tau}_j$ as we claimed.

In the case where $\overset{3}{\tau}$ and $\overset{4}{\tau}$ vanish identically on U', using (46), (9) and the definition of $\overset{3}{\tau}$ and $\overset{4}{\tau}$, we obtain $\lambda_\alpha a_i^\alpha = \frac{(n+2)\overset{1}{c} - 2\lambda}{n+4} \lambda_i$, i.e., that λ_α is an eigenvector of a_i^j. Differentiating this equation and substituting (38), (46), (9), and $\overset{3}{\tau} = 0$, we obtain

$$\left(\mu + \overset{1}{c} B - \frac{(n+2)\overset{1}{c} - 2\lambda}{n+4} B\right) a_{ij} = \left(\frac{(n+2)\overset{1}{c} - 2\lambda}{n+4}\mu - \lambda^p \lambda_p - \overset{2}{c} B\right) g_{ij} - 2\lambda_i \lambda_j.$$

Assume that the coefficient of a_{ij} vanishes identically on U'. Since g_{ij} has rank ≥ 4 and $\lambda_i \lambda_j$ has rank ≤ 1, the coefficient of g_{ij} vanishes identically on U', and thus the covector field λ_i vanishes identically on U'. Differentiating $\lambda_i = 0$, and using $\lambda_{ij} = \mu g_{ij} + B a_{ij}$ and Lemma 5, we see that either $a = \text{const} \cdot g$ on U' and therefore everywhere, in contradiction to our linear independence assumption; or $B \equiv 0$ on U', in contradiction to the choice of U'. This shows that also in the case $\overset{3}{\tau} = \overset{4}{\tau} \equiv 0$ there exist a nonempty open subset U'' of U' and functions α, β on U'' and a covector field u on U'' with $a_{ij} = \alpha g_{ij} + \beta u_i u_j$.

Let us now explain that if a_{ij} is not proportional to g and $a_{ij} = \alpha(x) g_{ij} + \beta(x) u_i u_j$ for every point x of some neighborhood, then α is a smooth function, and β (resp. u_i) can be chosen to be smooth function (resp. smooth covector field), probably in a smaller neighborhood. Indeed, under these assumptions α is the eigenvalue of a_i^j of (algebraic and geometric) multiplicity precisely $n-1$. Then, it is a smooth function. Then, $\beta u_i u_j$ is a smooth $(0,2)$-tensor field. Since a_{ij} and g_{ij} are not proportional, $\beta u_i u_j$ is not zero and we can choose $\beta = \pm 1$. Then, we have precisely two choices for the covector $u_i(x)$ at every point x and in a small neighborhood we can choose $u_i(x)$ smoothly.

Thus, under the assumptions of this section, for every solution a_{ij} of (9), we have (for certain functions α_1, α_2 and a covector field u_i)

$$a_{ij} = \alpha_1 g_{ij} + \alpha_2 u_i u_j. \tag{50}$$

For the solution A_{ij} an analog of Eq. (50) holds so (in a possible smaller neighborhood) we also have (for certain functions β_1, β_2 and a covector field v_i)

$$A_{ij} = \beta_1 g_{ij} + \beta_2 v_i v_j. \tag{51}$$

Without loss of generality, we can assume that $a_{ij} + A_{ij}$ (which is certainly a solution of (9)) is also not proportional to g_{ij}, otherwise we replace A_{ij} by $\frac{1}{2}A_{ij}$. Then,

$$a_{ij} + A_{ij} = \gamma_1 g_{ij} + \gamma_2 w_i w_j. \tag{52}$$

Subtracting (52) from the sum of (50) and (51), we obtain

$$(\gamma_1 - \alpha_1 - \beta_1) g_{ij} = \alpha_2 u_i u_j + \beta_2 v_i v_j - \gamma_2 w_i w_j. \tag{53}$$

Since the tensor g_{ij} is nondegenerate, its rank coincides with the dimension of M that is at least 4. The rank of the tensor $\alpha_2 u_i u_j + \beta_2 v_i v_j - \gamma_2 w_i w_j$ is at most three. Thus the coefficient $(\gamma_1 - \alpha_1 - \beta_1)$ must vanish, which implies that

$$\alpha_2 u_i u_j + \beta_2 v_i v_j = \gamma_2 w_i w_j. \tag{54}$$

We see that the rank of $\alpha_2 u_i u_j + \beta_2 v_i v_j$ is at most one, which implies that u_i is proportional to v_i (the coefficient of the proportionality is a function). Thus (54) implies that w_i is proportional to u_i as well. Thus a_{ij}, A_{ij}, and g_{ij} are linearly dependent over functions, which implies by Lemma 5 that they are linearly dependent over numbers. This is a contradiction to the assumptions, which proves the remaining part of Lemma 3.

2.3.4. The constant B is universal. Let $(M^{n\geq 3}, g)$ be a connected pseudo-Riemannian manifold. Assume the degree of mobility of g is ≥ 3, let (a_{ij}, λ_i) be a solution of Eqs. (9) such that $a_{ij} \neq \text{const} \cdot g_{ij}$ for every const $\in \mathbb{R}$. Then, in a neighborhood of almost every point there exist a constant B and a function μ such that Eqs. (38) hold. Note that the constant B determines the function μ: indeed, multiplying (27) by g^{ij} and summing with respect to i, j we obtain $\lambda^{i}{}_{,i} = n\mu - 2B\lambda$.

Our goal is to prove the statement announced in the title of the section: we would like to show that the constant B is the same in all such neighborhoods (which in particular implies that Eqs. (38) hold at all points with one universal constant B and one universal function μ). We will need the following

Corollary 4. *Let a_{ij}, λ_i satisfy Eqs.* (38) *in a neighborhood $U \subseteq (M, g)$ with a certain constant B and a smooth function μ. Then the function λ given by* (10) *satisfies the equation*

$$\lambda_{,ijk} - B\left(2\lambda_{,k} g_{ij} + \lambda_{,j} g_{ik} + \lambda_{,i} g_{jk}\right) = 0. \tag{55}$$

Remark 11. This equation is a famous one; it naturally appeared in different parts of differential geometry. Obata and Tanno used this equation trying to understand the connection between the eigenvalues of the laplacian Δ_g and the geometry and topology of the manifold. They observed [64,72] that the eigenfunctions corresponding to the second eigenvalue of the Laplacian of the metrics of constant positive curvature $-B$ on the sphere satisfy Eq. (55).

Tanno [72] and Hiramatu [27] related the equations to projective vector fields. Tanno has shown that for every solution λ of this equation the vector field $\lambda_{,}{}^{i}$ is a projective vector field (assuming $B \neq 0$), Hiramatu proved the reciprocal statement under certain additional assumptions.

As it was shown by Gallot [19], see also [1,59,60], decomposability of the holonomy group of the cone over a manifold implies the existence of a nonconstant solution of Eq. (55) on the manifold.

Proof of Corollary 4. Covariantly differentiating (27) and replacing the covariant derivative of a_{ij} by (9) we obtain (55) from Remark 10 if $a \neq \text{const} \cdot g$. If $a = \text{const} \cdot g$, we have $\lambda_{,i} = 0$, thus (55) holds as well. □

Corollary 5. *Let the degree of mobility of a metric g on a connected ($n > 3$)-dimensional M be ≥ 3. Assume (a_{ij}, λ_i) is a solution of (9). Then, if $\lambda_i \neq 0$ at a point, then the set of the points such that $\lambda_i \neq 0$ is everywhere dense.*

Remark 12. The assumption that the degree of mobility of g is ≥ 3 is important: Levi-Civita's description of geodesically equivalent metrics [36] immediately gives counterexamples.

Proof of Corollary 5. Combining Lemma 3, Remark 10, and Corollary 4, we obtain that in a neighborhood of almost every point λ given by (8) satisfies (55). By [72, Prop. 2.1], the vector field λ^i is a projective vector field (almost everywhere, and, therefore, everywhere) on (M, g). As it was shown for example in [23, Th. 21.1(ii)], if it is not zero at a point, then it is not zero at almost every point. □

Corollary 6. *Let a_{ij}, λ_i satisfy Eqs. (38) in a neighborhood U with a certain constant B and a smooth function μ. Let λ be the function constructed by (10). Then for every geodesic $\gamma(t)$ the following equation holds (at every $t \in \gamma^{-1}(U)$):*

$$\frac{d^3}{dt^3}\lambda(\gamma(t)) = 4Bg(\dot\gamma(t), \dot\gamma(t)) \cdot \frac{d}{dt}\lambda(\gamma(t)), \tag{56}$$

where $\dot\gamma$ denotes the velocity vector of the geodesic γ, and $g(\dot\gamma(t), \dot\gamma(t)) := g_{ij}\dot\gamma^i\dot\gamma^j$.

Proof. Multiplying (55) by $\dot\gamma^i\dot\gamma^j\dot\gamma^k$ and summing with respect to i, j, k we obtain (56). □

Lemma 7. *Let $(M^{n\geq 3}, g)$ be a connected manifold and (a_{ij}, λ_i) be a solution of (9). Assume almost every point has a neighborhood such that in this neighborhood there exists a constant B and a smooth function μ such that Eq. (27) is fulfilled. Then the constant B is the same in all such neighborhoods.*

Proof. It is sufficient to prove this statement locally, in a sufficiently small neighborhood of arbitrary point. We take a small neighborhood U, two points $p_0, p_1 \in U$, and two neighborhoods $U(p_0) \subset U$, $U(p_1) \subset U$ of these points. We assume that our neighborhoods are small enough and that we can connect every point of $U(p_0)$ with every point of $U(p_1)$ by a unique geodesic lying in U. We assume that Eq. (27) holds in $U(p_i)$ with the constant $B := B_i$; our goal is to show that $B_0 = B_1$.

Suppose it is not the case. We consider all geodesics γ_{p,p_0} lying in U connecting all points $p \in U(p_1)$ with p_0, see Fig. 1. We will think that $\gamma(0) = p_0$ and $\gamma(1) \in U(p_1)$.

For every such geodesic $\gamma_{p,p_0}(t)$ there exists a point $q_{p,p_0} := \gamma_{p,p_0}(t_{p,p_0})$ on this geodesic such that for all $t \in [0, t_{p,p_0})$ the following conditions are fulfilled:

1. Equations (38) are fulfilled with $B = B_0$ in a small neighborhood of $\gamma(t)$, and
2. for no neighborhood of $\gamma_{p,p_0}(t_{p,p_0})$ Eqs. (38) are fulfilled with $B = B_0$.

Then, at every such point $\gamma_{p,p_0}(t_{p,p_0})$ we have that $a_{ij} = \frac{2}{n}\lambda g_{ij}$. Indeed, the trace-free version of (27) is

$$\lambda_{,ij} - \frac{1}{n}\lambda_{,k}^{\ k} = B\left(a_{ij} - \frac{2}{n}\lambda g_{ij}\right), \tag{57}$$

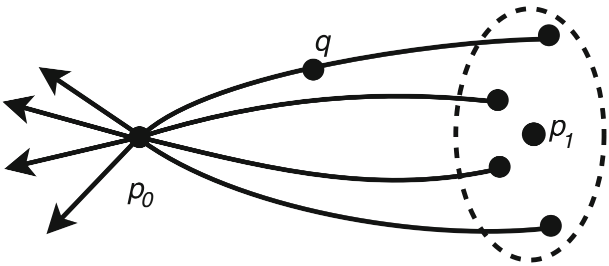

Fig. 1. The geodesics γ_{p,p_0}, their velocity vectors at p_0, and the point $q_{p,p_0} = \gamma_{p,p_0}(t_{p,p_0})$ on one of these geodesics

implying that B is the coefficient of proportionality of two smooth tensors. If $a_{ij} \neq \frac{2}{n}\lambda g_{ij}$ at $\gamma_{p,p_0}(t_{p,p_0})$, we have $a_{ij} - \frac{2}{n}\lambda g_{ij} \neq 0$, and B can be prolonged to a smooth function in a small neighborhood of $\gamma_{p,p_0}(t_{p,p_0})$. Since it is locally-constant, it is (the same) constant at all points of the neighborhood of $\gamma_{p,p_0}(t_{p,p_0})$ contradicting the conditions 1, 2.

Moreover, at every such point $\gamma_{p,p_0}(t_{p,p_0})$ we have $\lambda_i = 0$. Indeed, otherwise we multiply (55) by g^{ij} and sum with respect to i, j. We obtain $\lambda^i{}_{,ik} = 2(n+1)B\lambda_k$. We again have that B is the coefficient of proportionality of two smooth tensors. Arguing as above we obtain that $\lambda_i = 0$ at every point $\gamma_{p,p_0}(t_{p,p_0})$.

Since at every point $\gamma_{p,p_0}(t_{p,p_0})$ we have $\lambda_i = 0$, we have that $\frac{d}{dt}\lambda(\gamma_{p,p_0}(t))_{|t=t_{p,p_0}} = 0$. Then, the set of all such $\gamma_{p,p_0}(t_{p,p_0})$ contains a smooth (connected) hypersurface (because the set of zeros of the derivatives of the solutions of Eq. (56) depends smoothly on the initial data and on $g(\dot{\gamma}, \dot{\gamma})$). We denote this hypersurface by H. Since $\lambda_i = 0$ at every point of H, the function λ is constant (we denote it by $\tilde{\lambda} \in \mathbb{R}$) on H.

Now let us return to the geodesics γ_{p,p_0} connecting points $p \in U(p_1)$ with p_0. We consider the integral I given by (14). Direct calculations show that at every point q where $a_{ij} = c \cdot g_{ij}$ the integral is given by

$$I(\xi) = c^{n-1} g(\xi, \xi) \tag{58}$$

(for every tangent vector $\xi \in T_q M$). As we explained above, every such geodesic passing through a point of H has a point such that $a_{ij} = c \cdot g_{ij}$, where $c = \frac{2}{n}\tilde{\lambda}$ is a constant. Since the integral is constant on the orbits, we have that $I\left(\dot{\gamma}_{p,p_0}(0)\right) = c^{n-1} \cdot g\left(\dot{\gamma}_{p,p_0}(0), \dot{\gamma}_{p,p_0}(0)\right)$. Then, the measure of the subset

$$\{\xi \in T_{p_0}M \mid I(\xi) = c^{n-1} \cdot g(\xi, \xi)\} \subseteq T_{p_0}M$$

is not zero. Since this set is given by an algebraic equation, it must coincide with the whole $T_{p_0}M$. Then, $a_{ij} = c \cdot g_{ij}$ at the point p_0. Since we can replace p_0 by every point of its neighborhood $U(p_0)$, we obtain that $a_{ij} = c^{n-1} \cdot g_{ij}$ at every point of $U(p_0)$. By Remark 5, $a = c^{n-1} \cdot g$ on the whole manifold. □

2.3.5. The metric g uniquely determines B. By Lemma 3, under the assumption that the degree of mobility is ≥ 3, for every solution a of (9) there exists a constant B such that Eq. (27) holds on a suitable open set. In this chapter we show that the constant B is the same for all (nontrivial) solutions a_{ij}, i.e., the metric determines it uniquely.

Lemma 8. *Suppose two nonconstant functions $f, F : M^n \to \mathbb{R}$ on a connected manifold (M^n, g) of dimension $n > 1$ satisfy*

$$f_{,ijk} - b\left(2f_{,k}g_{ij} + f_{,j}g_{ik} + f_{,i}g_{jk}\right) = 0,$$
$$F_{,ijk} - B\left(2F_{,k}g_{ij} + F_{,j}g_{ik} + F_{,i}g_{jk}\right) = 0, \tag{59}$$

where b and B are constants. Assume that there exists a point where the derivative of f is nonzero and a point where the derivative of F is nonzero. Then, $b = B$.

Proof. By definition of the curvature, for every function f, we have $f_{,ijk} - f_{,ikj} = f_p R^p_{ijk}$; replacing $f_{,ijk}$ by the right-hand side of the first equation of (59) we obtain.

$$f_{,p} R^p_{ijk} = b\left(f_{,k}g_{ij} - f_{,j}g_{ik}\right). \tag{60}$$

The same is true for the second equation of (59):

$$F_{,p} R^p_{ijk} = B\left(F_{,k}g_{ij} - F_{,j}g_{ik}\right). \tag{61}$$

Multiplying (60) by $F_{,}^{\ k}$, summing with respect to repeating indexes and using (61) we obtain

$$B\left(F_{,p}f_{,}^{\ p}g_{ij} - F_{,j}f_{,i}\right) = b\left(F_{,p}f_{,}^{\ p}g_{ij} - F_{,i}f_{,j}\right). \tag{62}$$

Multiplying by g^{ij} and summing with respect to repeating indexes, we obtain $B(n-1)F_{,p}f_{,}^{\ p} = b(n-1)F_{,p}f_{,}^{\ p}$. If $F_{,p}f_{,}^{\ p} \neq 0$ we are done: $B = b$. Assume $F_{,p}f_{,}^{\ p} = 0$. Then, (62) reads $BF_{,j}f_{,i} = bF_{,i}f_{,j}$. Since by Corollary 5 there exists a point where $F_{,j}$ and $f_{,i}$ are both nonzero, we obtain again $B = b$. Then, $f_{,i}$ is proportional to $F_{,j}$. Hence, $B = b$. □

2.3.6. An ODE along geodesics.

Lemma 9. *Let g be a metric on a connected $M^{n\geq 3}$ of degree of mobility ≥ 3. For a metric $\bar g$ geodesically equivalent to g, let us consider a_{ij}, λ_i, and ϕ given by (7, 8, 5). Then, there exist constants B, $\bar B$ such that the following formula holds*:

$$\phi_{i,j} - \phi_i\phi_j = -Bg_{ij} + \bar B\bar g_{ij}. \tag{63}$$

Proof. We covariantly differentiate (8) (the index of differentiation is "j"); then we substitute the expression (6) for $\bar g_{ij,k}$ to obtain

$$\begin{aligned}\lambda_{i,j} &= -2e^{2\phi}\phi_j\phi_p\bar g^{pq}g_{qi} - e^{2\phi}\phi_{p,j}\bar g^{pq}g_{qi} + e^{2\phi}\phi_p\bar g^{ps}\bar g_{sl,j}\bar g^{lq}g_{qi}\\ &= -e^{2\phi}\phi_{p,j}\bar g^{pq}g_{qi} + e^{2\phi}\phi_p\phi_s\bar g^{ps}g_{ij} + e^{2\phi}\phi_j\phi_l\bar g^{lq}g_{qi},\end{aligned} \tag{64}$$

where $\bar g^{pq}$ is the tensor dual to $\bar g_{pq}$, i.e., $\bar g^{pi}\bar g_{pj} = \delta^i_j$. We now substitute $\lambda_{i,j}$ from (27), use that a_{ij} is given by (7), and divide by $e^{2\phi}$ for cosmetic reasons to obtain

$$e^{-2\phi}\mu g_{ij} + B\bar g^{pq}g_{pj}g_{qi} = -\phi_{p,j}\bar g^{pq}g_{qi} + \phi_p\phi_s\bar g^{ps}\bar g_{ij} + \phi_j\phi_l\bar g^{lq}g_{qi}. \tag{65}$$

Multiplying with $g^{i\xi}\bar g_{\xi k}$, we obtain

$$\phi_{k,j} - \phi_k\phi_j = \underbrace{\left(\phi_p\phi_q\bar g^{pq} - e^{-2\phi}\mu\right)}_{\bar b}\bar g_{kj} - Bg_{kj}. \tag{66}$$

The same holds with the roles of g and $\bar{g}$ exchanged (the function (5) constructed by the interchanged pair $\bar{g}$, g is evidently equal to $-\phi$). We obtain

$$-\phi_{k;j} - \phi_k\phi_j = \underbrace{(\phi_p\phi_q g^{pq} - e^{2\phi}\bar{\mu})}_{b} g_{kj} - \bar{B}\bar{g}_{kj}, \tag{67}$$

where $\phi_{i;j}$ denotes the covariant derivative of ϕ_i with respect to the Levi-Civita connection of the metric $\bar{g}$. Since the Levi-Civita connections of g and of $\bar{g}$ are related by the formula (2), we have

$$-\phi_{k;j} - \phi_k\phi_j = \underbrace{-\phi_{k,j} + 2\phi_k\phi_j}_{-\phi_{k;j}} - \phi_k\phi_j = -(\phi_{k,j} - \phi_k\phi_j).$$

We see that the left hand side of (66) is equal to minus the left hand side of (67). Thus, $b \cdot g_{ij} - \bar{B} \cdot \bar{g}_{ij} = B \cdot g_{ij} - \bar{b} \cdot \bar{g}_{ij}$ holds on U. Since the metrics g and $\bar{g}$ are not proportional on U by assumption, $\bar{b} = \bar{B}$, and the formula (66) coincides with (63). □

Corollary 7. *Let g, $\bar{g}$ be geodesically equivalent metrics on a connected $M^{n\geq 3}$ such that the degree of mobility of g is ≥ 3. We consider a (parametrized) geodesic $\gamma(t)$ of the metric g, and denote by $\dot{\phi}$, $\ddot{\phi}$ and $\dddot{\phi}$ the first, second and third derivatives of the function ϕ given by (5) along the geodesic. Then, there exists a constant B such that for every geodesic γ the following ordinary differential equation holds:*

$$\dddot{\phi} = 4Bg(\dot{\gamma}, \dot{\gamma})\dot{\phi} + 6\dot{\phi}\ddot{\phi} - 4(\dot{\phi})^3, \tag{68}$$

where $g(\dot{\gamma}, \dot{\gamma}) := g_{ij}\dot{\gamma}^i\dot{\gamma}^j$.

Since lightlike geodesics have $g(\dot{\gamma}, \dot{\gamma}) = 0$ at every point, a partial case of Corollary 7 is

Corollary 8. *Let g, $\bar{g}$ be geodesically equivalent metrics on a connected $M^{n\geq 3}$ such that the degree of mobility of g is ≥ 3. Consider a (parametrized) lightlike geodesic $\gamma(t)$ of the metric g, and denote by $\dot{\phi}$, $\ddot{\phi}$ and $\dddot{\phi}$ the first, second and third derivatives of the function ϕ given by (5) along the geodesic. Then, along the geodesic, the following ordinary differential equation holds:*

$$\dddot{\phi} = 6\dot{\phi}\ddot{\phi} - 4(\dot{\phi})^3. \tag{69}$$

Proof of Corollary 7. If $\phi \equiv 0$ in a neighborhood U, the equation is automatically fulfilled. Then, it is sufficient to prove Corollary 7 assuming ϕ_i is not constant.

The formula (63) is evidently equivalent to

$$\phi_{i,j} = \bar{B}\bar{g}_{ij} - Bg_{ij} + \phi_i\phi_j. \tag{70}$$

Taking the covariant derivative of (70), we obtain

$$\phi_{i,jk} = \bar{B}\bar{g}_{ij,k} + \phi_{i,k}\phi_j + \phi_{j,k}\phi_i. \tag{71}$$

Substituting the expression for $\bar{g}_{ij,k}$ from (6), and substituting $\bar{B}\bar{g}_{ij}$ given by (63), we obtain

$$\begin{aligned}\phi_{i,jk} &= \bar{B}(2\bar{g}_{ij}\phi_k + \bar{g}_{ik}\phi_j + \bar{g}_{jk}\phi_i) + \phi_{i,k}\phi_j + \phi_{j,k}\phi_i \\ &= B(2g_{ij}\phi_k + g_{ik}\phi_j + g_{jk}\phi_i) + 2(\phi_k\phi_{i,j} + \phi_i\phi_{j,k} + \phi_j\phi_{k,i}) - 4\phi_i\phi_j\phi_k.\end{aligned} \tag{72}$$

Contracting with $\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k$ and using that ϕ_i is the differential of the function (5) we obtain the desired ODE (68). □

2.4. Proof of Theorem 1 for pseudo-Riemannian metrics. Let g be a metric on a connected $M^{n\geq 3}$. Assume that for no constant $c \neq 0$ the metric $c \cdot g$ is Riemannian, which in particular implies the existence of lightlike geodesics.

Let $\bar{g}$ be geodesically equivalent to g. Assume both metrics are complete. Our goal is to show that ϕ given by (5) is constant, because in view of (2) this implies that the metrics are affine equivalent.

Consider a parameterized lightlike geodesic $\gamma(t)$ of g. Since the metrics are geodesically equivalent, for a certain function $\tau : \mathbb{R} \to \mathbb{R}$ the curve $\gamma(\tau)$ is a geodesic of $\bar{g}$. Since the metrics are complete, the reparameterization $\tau(t)$ is a diffeomorphism $\tau : \mathbb{R} \to \mathbb{R}$. Without loss of generality we can think that $\dot{\tau} := \frac{d}{dt}\tau$ is positive, otherwise we replace t by $-t$. Then, Eq. (3) along the geodesic reads

$$\phi(t) = \frac{1}{2}\log(\dot{\tau}(t)) + \mathrm{const}_0. \tag{73}$$

Now let us consider Eq. (69). Substituting

$$\phi(t) = -\frac{1}{2}\log(p(t)) + \mathrm{const}_0 \tag{74}$$

in it (since $\dot{\tau} > 0$, the substitution is global), we obtain

$$\dddot{p} = 0. \tag{75}$$

The solution of (75) is $p(t) = C_2 t^2 + C_1 t + C_0$. Combining (74) with (73), we see that $\dot{\tau} = \frac{1}{C_2 t^2 + C_1 t + C_0}$. Then

$$\tau(t) = \int_{t_0}^{t} \frac{d\xi}{C_2\xi^2 + C_1\xi + C_0} + \mathrm{const}. \tag{76}$$

We see that if the polynomial $C_2 t^2 + C_1 t + C_0$ has real roots (which is always the case if $C_2 = 0$, $C_1 \neq 0$), then the integral explodes in finite time. If the polynomial has no real roots, but $C_2 \neq 0$, the function τ is bounded. Thus, the only possibility for τ to be a diffeomorphism is $C_2 = C_1 = 0$ implying $\tau(t) = \frac{1}{C_0}t + \mathrm{const}_1$, implying $\dot{\tau} = \frac{1}{C_0}$, implying ϕ is constant along the geodesic.

Since every two points of a connected pseudo-Riemannian manifold such that for no constant c the metric $c \cdot g$ is Riemannian can be connected by a sequence of lightlike geodesics, ϕ is a constant, so that $\phi_i \equiv 0$, and the metrics are affine equivalent by (2). □

2.5. Proof of Theorem 1 for Riemannian metrics. As we already mentioned in the Introduction and at the beginning of Sect. 2, Theorem 1 was proved for Riemannian metrics in [45,57]. We present an alternative proof, which is much shorter (modulo the results of the previous sections and a nontrivial result of Tanno [72]).

We assume that g is a complete Riemannian metric on a connected manifold such that its degree of mobility is ≥ 3. Then, by Corollary 4, the function λ is a solution of (55). If the metrics are not affine equivalent, λ is not identically constant.

Let us first assume that the constant B in Eq. (55) is negative. Under this assumption, Eq. (55) was studied by Obata [64], Tanno [72], and Gallot [19]. Tanno [72] and Gallot [19] proved that a complete Riemannian g such that there exists a nonconstant function

λ satisfying (55) must have a constant positive sectional curvature. Applying this result in our situation, we obtain the claim.

Now, let us suppose $B \geq 0$. Then, one can slightly modify the proof from Sect. 2.4 to obtain the claim. More precisely, substituting (74) in (68), we obtain the following analog of Eq. (75):

$$\dddot{p} = 4Bg(\dot{\gamma}, \dot{\gamma})\dot{p}. \tag{77}$$

If $B = 0$, the equation coincides with (75). Arguing as in Sect. 2.4, we obtain that ϕ is constant along the geodesic.

If $B > 0$, the general solution of Eq. (77) is

$$C + C_+ e^{2\sqrt{Bg(\dot{\gamma},\dot{\gamma})}\cdot t} + C_- e^{-2\sqrt{Bg(\dot{\gamma},\dot{\gamma})}\cdot t}. \tag{78}$$

Then, the function τ satisfies the ODE $\dot{\tau} = \frac{1}{C+C_+e^{2\sqrt{Bg(\dot{\gamma},\dot{\gamma})}\cdot t}+C_-e^{-2\sqrt{Bg(\dot{\gamma},\dot{\gamma})}\cdot t}}$ implying

$$\tau(t) = \int_{t_0}^{t} \frac{d\xi}{C + C_+ e^{2\sqrt{Bg(\dot{\gamma},\dot{\gamma})}\cdot \xi} + C_- e^{-2\sqrt{Bg(\dot{\gamma},\dot{\gamma})}\cdot \xi}} + \text{const}. \tag{79}$$

If one of the constants C_+, C_- is not zero, the integral (79) is bounded from one side, or explodes in finite time. In both cases, τ is not a diffeomorphism of $\mathbb{R}$ on itself, i.e., one of the metrics is not complete. The only possibility for τ to be a diffeomorphism of $\mathbb{R}$ on itself is $C_+ = C_- = 0$. Finally, ϕ is a constant along the geodesic γ.

Since every two points of a connected complete Riemannian manifold can be connected by a geodesic, ϕ is a constant, so that $\phi_i \equiv 0$, and the metrics are affine equivalent by (2). □

Remark 13. A similar idea (contracting the equation with lightlike geodesic and investigating the obtained ODE along the geodesic) was recently used in [31,59]

2.6. Proof of Theorem 2. Let g be a complete pseudo-Riemannian metric on a connected closed manifold M^n such that for no const $\neq 0$ the metric const $\cdot\, g$ is Riemannian (if g is Riemannian, Theorem 2 follows from Theorem 1). We assume that the degree of mobility of g is ≥ 3. Our goal is to show that every metric $\bar{g}$ geodesically equivalent to g is actually affine equivalent to g.

We consider the function λ constructed by (10) for the solution a_{ij} of (9) given by (7). We consider a lightlike geodesic $\gamma(t)$ of the metric g, and the function $\lambda(\gamma(t))$. By Corollary 6, the function $\lambda(\gamma(t))$ satisfies the ODE $\frac{d^3}{dt^3}\lambda(\gamma(t)) = 0$. Hence $\lambda(\gamma(t)) = C_2t^2 + C_1t + C_0$. If $C_2 \neq 0$, or $C_1 \neq 0$, then the function λ is not bounded; that contradicts the compactness of the manifold. Thus $\lambda(\gamma(t))$ is constant along every lightlike geodesic. Since every two points can be connected by a sequence of lightlike geodesics, λ is constant. Thus $\lambda_i = 0$, implying in view of (8) that $\phi_i = 0$, implying in view of (6) that the metrics are affine equivalent. □

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References

1. Alekseevsky, D.V., Cortes, V., Galaev, A.S., Leistner, T.: Cones over pseudo-Riemannian manifolds and their holonomy. J. Reine Angew. Math. (Crelle's journal), **2009**(635), 23–69 (2009)
2. Aminova, A.V.: Pseudo-Riemannian manifolds with general geodesics. Russ. Math. Surv. **48**(2), 105–160 (1993)
3. Aminova, A.V.: Projective transformations of pseudo-Riemannian manifolds. Geometry, 9. J. Math. Sci. (N. Y.) **113**(3), 367–470 (2003)
4. Benenti, S.: Special symmetric two-tensors, equivalent dynamical systems, cofactor and bi-cofactor systems. Acta Appl. Math. **87**(1-3), 33–91 (2005)
5. Beltrami, E.: Risoluzione del problema: riportare i punti di una superficie sopra un piano in modo che le linee geodetische vengano rappresentante da linee rette. Ann. di Mat. **1**(7), 185–204 (1865)
6. Bolsinov, A.V., Matveev, V.S.: Geometrical interpretation of Benenti's systems. J. Geom. Phys. **44**, 489–506 (2003)
7. Bolsinov, A.V., Matveev, V.S., Pucacco, G.: Dini theorem for pseudo-Riemannian metrics. Appendix to [58], to appear in Math. Ann., http://arxiv.org/abs/0802.2346v1 [math.DG], 2008
8. Bolsinov, A.V., Matveev, V.S., Pucacco, G.: Normal forms for pseudo-Riemannian 2-dimensional metrics whose geodesic flows admit integrals quadratic in momenta. J. Geom. Phys. **59**(7), 1048–1062 (2009)
9. Bolsinov, A.V., Kiosak, V., Matveev, V.S.: A Fubini theorem for pseudo-Riemannian geodesically equivalent metrics. J. London Math. Soc. **80**(2), 341–356 (2009)
10. Bolsinov, A.V., Matveev, V.S.: Splitting and gluing lemmas for geodesically equivalent pseudo-Riemannian metrics, to appear Trans. Am. Math. Soc. http://arxiv.org/abs/0904.0535v1 [math.DG], 2009
11. Bryant, R.L., Manno, G., Matveev, V.S.: A solution of a problem of Sophus Lie: Normal forms of 2-dim metrics admitting two projective vector fields. Math. Ann. **340**(2), 437–463 (2008)
12. Dini, U.: Sopra un problema che si presenta nella theoria generale delle rappresetazioni geografice di una superficie su un'altra. Ann. Mat., Ser. 2 **3**, 269–293 (1869)
13. Eastwood, M.: Notes on projective differential geometry. In: Symmetries and Overdetermined Systems of Partial Differential Equations (Minneapolis, MN, 2006), IMA Vol. Math. Appl. **144**, New York: Springer, (2007), pp. 41–61
14. Eastwood, M., Matveev, V.S.: Metric connections in projective differential geometry. In: Symmetries and Overdetermined Systems of Partial Differential Equations (Minneapolis, MN, 2006), 339–351, IMA Vol. Math. Appl. **144**, New York: Springer, 2007, pp. 339–351
15. Eisenhart, L.P.: Non-Riemannian Geometry. American Mathematical Society Colloquium Publications **VIII**) NewYork: Dover, 1927
16. Ehlers, J., Pirani, F., Schild, A.: The geometry of free fall and light propagation. In: "General relativity" (papers in honour of J. L. Synge). Oxford: Clarendon Press, (1972). pp. 63–84
17. Ehlers, J., Schild, A.: Geometry in a manifold with projective structure. Commun. Math. Phys. **32**, 119–146 (1973)
18. Fubini, G.: Sui gruppi transformazioni geodetiche. Mem. Acc. Torino **53**, 261–313 (1903)
19. Gallot, S.: Équations différentielles caractéristiques de la sphère. Ann. Sci. École Norm. Sup. (4) **12**(2), 235–267 (1979)
20. Hall, G.S., Lonie, D.P.: Projective collineations in spacetimes. Class. Quant. Grav. **12**(4), 1007–1020 (1995)
21. Hall, G.S.: Some remarks on symmetries and transformation groups in general relativity. Gen. Rel. Grav. **30**(7), 1099–1110 (1998)
22. Hall, G.S.: Projective symmetry in FRW spacetimes. Class. Quant. Grav. **17**(22), 4637–4644 (2000)
23. Hall, G.S.: Symmetries and curvature structure in general relativity. World Scientific Lecture Notes in Physics **46**. River Edge, NJ: World Scientific Publishing Co., Inc., 2004
24. Hall, G.S., Lonie, D.P.: The principle of equivalence and projective structure in spacetimes. Class. Quant. Grav. **24**(14), 3617–3636 (2007)
25. Hall, G.S., Lonie, D.P.: The principle of equivalence and cosmological metrics. J. Math. Phys. **49**(2), 022502 (2008)
26. Hasegawa, I., Yamauchi, K.: Infinitesimal projective transformations on tangent bundles with lift connections. Sci. Math. Jpn. **57**(3), 469–483 (2003)
27. Hiramatu, H.: Riemannian manifolds admitting a projective vector field. Kodai Math. J. **3**(3), 397–406 (1980)
28. Igarashi, M., Kiyohara, K., Sugahara, K.: Noncompact Liouville surfaces. J. Math. Soc. Japan **45**(3), 459–479 (1993)
29. Kim, S.: Volume and projective equivalence between Riemannian manifolds. Ann. Global Anal. Geom. **27**(1), 47–52 (2005)
30. Kiosak, V., Matveev, V.S.: Complete Einstein metrics are geodesically rigid. Commun. Math. Phys. **289**(1), 383–400 (2009)

31. Kiosak, V., Matveev, V.S.: There are no conformal Einstein rescalings of complete pseudo-Riemannian Einstein metrics. C. R. Acad. Sci. Paris, Ser. I **347**, 1067–1069 (2009)
32. Koenigs, G.: Sur les géodesiques a intégrales quadratiques. Note II from "Lecons sur la théorie générale des surfaces," Vol. **4**, New York: Chelsea Publishing, 1896
33. Kruglikov, B.S., Matveev, V.S.: Strictly non-proportional geodesically equivalent metrics have $h_{\text{top}}(g) = 0$. Erg. Th. Dyn. Syst. **26**(1), 247–266 (2006)
34. Kruglikov, B.S., Matveev, V.S.: Vanishing of the entropy pseudonorm for certain integrable systems. Electron. Res. Announc. Amer. Math. Soc. **12**, 19–28 (2006)
35. Lagrange, J.-L.: Sur la construction des cartes géographiques. Novéaux Mémoires de l'Académie des Sciences et Bell-Lettres de Berlin, 1779
36. Levi-Civita, T.: Sulle trasformazioni delle equazioni dinamiche. Ann. di Mat., Serie 2^a **24**, 255–300 (1896)
37. Liouville, R.: Sur les invariants de certaines équations différentielles et sur leurs applications. Journal de l'École Polytechnique **59**, 7–76 (1889)
38. Manno, G., Matveev, V.S.: 2-dim metrics admitting two projective vector fields near the points where the vector fields are linearly dependent. In preparation
39. Matveev, V.S., Topalov, P.J.: Trajectory equivalence and corresponding integrals. Reg. and Chaotic Dyn. **3**(2), 30–45 (1998)
40. Matveev, V.S., Topalov, P.J.: Geodesic equivalence of metrics on surfaces, and their integrability. Dokl. Math. **60**(1), 112–114 (1999)
41. Matveev, V.S., Topalov, P.J.: Metric with ergodic geodesic flow is completely determined by unparameterized geodesics. ERA-AMS **6**, 98–104 (2000)
42. Matveev, V.S., Topalov, P.J.: Quantum integrability for the Beltrami-Laplace operator as geodesic equivalence. Math. Z. **238**, 833–866 (2001)
43. Matveev, V.S., Topalov, P.J.: Integrability in theory of geodesically equivalent metrics. J. Phys. A. **34**, 2415–2433 (2001)
44. Matveev, V.S.: Geschlossene hyperbolische 3-Mannigfaltigkeiten sind geodätisch starr. Manuscripta Math. **105**(3), 343–352 (2001)
45. Matveev, V.S.: Low-dimensional manifolds admitting metrics with the same geodesics. In: Contemporary Mathematics, Providence, RI: Amer. Math. Soc., **308** 2002, pp. 229–243
46. Matveev, V.S.: Three-dimensional manifolds having metrics with the same geodesics. Topology **42**(6), 1371–1395 (2003)
47. Matveev, V.S.: Hyperbolic manifolds are geodesically rigid. Invent. Math. **151**, 579–609 (2003)
48. Matveev, V.S.: Die Vermutung von Obata für Dimension 2. Arch. Math. **82**, 273–281 (2004)
49. Matveev, V.S.: Projectively equivalent metrics on the torus. Diff. Geom. Appl. **20**, 251–265 (2004)
50. Matveev, V.S.: Solodovnikov's theorem in dimension two. Dokl. Math. **69**(3), 338–341 (2004)
51. Matveev, V.S.: Lichnerowicz-Obata conjecture in dimension two. Commun. Math. Helv. **81**(3), 541–570 (2005)
52. Matveev, V.S.: Closed manifolds admitting metrics with the same geodesics. Proceedings of SPT2004 (Cala Gonone). River Edge, NJ: World Scientific, 2005, pp. 198–209
53. Matveev, V.S.: Beltrami problem, Lichnerowicz-Obata conjecture and applications of integrable systems in differential geometry. Tr. Semin. Vektorn. Tenzorn. Anal **26**, 214–238 (2005)
54. Matveev, V.S.: The eigenvalues of Sinjukov's operator are globally ordered. Mathematical Notes **77**(3–4), 380–390 (2005)
55. Matveev, V.S.: Geometric explanation of Beltrami theorem. Int. J. Geom. Methods Mod. Phys. **3**(3), 623–629 (2006)
56. Matveev, V.S.: On degree of mobility of complete metrics. Adv. Stud. Pure Math. **43**, 221–250 (2006)
57. Matveev, V.S.: Proof of projective Lichnerowicz-Obata conjecture. J. Diff. Geom. **75**, 459–502 (2007)
58. Matveev, V.S.: Two-dimensional metrics admitting precisely one projective vector field. Math. Ann., accepted, http://arxiv.org/abs/0802.2344v2 [math.DG], 2010
59. Matveev, V.S.: Gallot-Tanno theorem for pseudo-Riemannian metrics and a proof that decomposable cones over closed complete pseudo-Riemannian manifolds do not exist. J. Diff. Geom. and Its Appl. **28**(2), 236–240 (2010)
60. Matveev, V.S., Mounoud, P.: Gallot-Tanno Theorem for closed incomplete pseudo-Riemannian manifolds and applications. Ann. Glob. Anal. Geom. doi:10.1007/s10455-010-9211-7 (2010)
61. Matveev, V.S.: Pseudo-Riemannian metrics on closed surfaces whose geodesic flows admit nontrivial integrals quadratic in momenta, and proof of the projective Obata conjecture for two-dimensional pseudo-Riemannian metrics, http://arxiv.org/abs/1002.3934 [math.DG], 2010
62. Mikes, J.: Geodesic mappings of affine-connected and Riemannian spaces. Geometry, 2. J. Math. Sci. **78**(3), 311–333 (1996)
63. Nagano, T., Ochiai, T.: On compact Riemannian manifolds admitting essential projective transformations. J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **33**, 233–246 (1986)

64. Obata, M.: Riemannian manifolds admitting a solution of a certain system of differential equations. Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965), Tokyo: Nippon Hyoronsha, 1966, pp. 101–114
65. Painlevé, P.: Sur les intégrale quadratiques des équations de la Dynamique. Compt.Rend. **124**, 221–224 (1897)
66. Petrov, A.Z.: Einstein spaces. London: Pergamon Press. XIII, 1969
67. Petrov, A.Z.: New methods in the general theory of relativity. (in Russian). Moscow: Izdat. "Nauka", 1966
68. Shen, Z.: On projectively related Einstein metrics in Riemann-Finsler geometry. Math. Ann. **320**(4), 625–647 (2001)
69. Sinjukov, N.S.: Geodesic mappings of Riemannian spaces (in Russian). Moscow: "Nauka", 1979
70. Solodovnikov, A.S.: Projective transformations of Riemannian spaces. Uspehi Mat. Nauk (N.S.) **11**, no. 4(70), 45–116 (1956)
71. Solodovnikov, A.S.: Geometric description of all possible representations of a Riemannian metric in Levi-Civita form. Trudy Sem. Vektor. Tenzor. Anal. **12**, 131–173 (1963)
72. Tanno, S.: Some differential equations on Riemannian manifolds. J. Math. Soc. Japan **30**(3), 509–531 (1978)
73. Topalov, P.: Geodesic hierarchies and involutivity. J. Math. Phys. **42**(8), 3898–3914 (2001)
74. Topalov, P.J., Matveev, V.S.: Geodesic equivalence via integrability. Geometriae Dedicata **96**, 91–115 (2003)
75. Weyl, H.: Zur Infinitisimalgeometrie: Einordnung der projektiven und der konformen Auffasung. Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1921; "Selecta Hermann Weyl", Basel Stuttgart: Birkhäuser Verlag, 1956
76. Weyl, H.: Geometrie und Physik. Die Naturwissenschaftler **19**, 49–58 (1931); "Hermann Weyl Gesammelte Abhandlungen", Band **3**, Berlin-Heidelberg: Springer-Verlag, 1968
77. Yamauchi, K.: On infinitesimal projective transformations. Hokkaido Math. J. **3**, 262–270 (1974)

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