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Topological Open Strings on Orbifolds

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Abstract: We use the remodeling approach to the B-model topological string in terms
of recursion relations to study open string amplitudes at orbifold points. To this end,
we clarify modular properties of the open amplitudes and rewrite them in a form that
makes their transformation properties under the modular group manifest. We exemplify
this procedure for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold point of local $\mathbb{P}^2$, where we present results for
topological string amplitudes for genus zero and up to three holes, and for the one-holed
torus. These amplitudes can be understood as generating functions for either open orbi-
fold Gromov–Witten invariants of $\mathbb{C}^3/\mathbb{Z}_3$, or correlation functions in the orbifold CFT
involving insertions of both bulk and boundary operators.

1. Introduction

Topological string theory on Calabi–Yau threefolds has played a crucial role in our
understanding of string theory and Gromov–Witten theory. One of the most fascinating
aspects of this topological sector of string theory is that very often amplitudes can be
computed exactly, and their dependence on the moduli can be studied in detail. This has
led to very rich pictures of the moduli space of the theory, involving different phases
which exhibit different physics [10,56].

Modular and analytic properties of the amplitudes connect the different phases of
the Calabi–Yau moduli space in a very precise way. Each phase of the moduli space
is characterized by a set of "good coordinates," and different good coordinates corre-
sponding to different phases are related by a transformation in the modular group of
the theory. As explained in [1], topological string amplitudes are modular objects with
specific transformation properties under this group, and as one goes from one phase
to the other, the amplitudes have to be transformed accordingly. For example, when
expanded at the large radius limit in moduli space, topological string amplitudes are
generating functions of Gromov–Witten invariants. As one moves away from this point
towards different regions in moduli space, the large radius expansion eventually ceases

to converge, but after suitable modular transformations and analytic continuations, the topological string amplitudes can be re-expanded in terms of the good variables of the new phase. In particular, when going to orbifold points of the moduli space, the amplitudes become generating functions for *orbifold* Gromov–Witten invariants. A detailed understanding of the modular transformation properties of the amplitudes makes it then possible to relate Gromov–Witten invariants to orbifold Gromov–Witten invariants, in the spirit of the crepant resolution conjecture [19,24,53]. In [1] this was used to calculate generating functions of orbifold Gromov–Witten invariants in the case of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, which corresponds to a phase in the moduli space of local $\mathbb{P}^2$, its crepant resolution. The predictions obtained in this way have been later verified mathematically in orbifold Gromov–Witten theory [11,17,20,25], and other examples have been recently calculated [18,26].

A crucial ingredient in the approach of [1] is the ability to obtain exact results for the topological string amplitudes on the whole of moduli space, so that they can be expanded in different phases. These exact expressions are typically calculated by using the B-model and mirror symmetry. On top of that, it is extremely useful to write these exact results in a way that makes the transformation properties manifest. For local Calabi–Yau threefolds, the mirror manifold reduces to an algebraic curve and the modular group is essentially the symplectic group acting on the homology of the surface. Topological string amplitudes can then be written in terms of modular forms with respect to this group, and when the curve has genus one, as in the case of the mirror to local $\mathbb{P}^2$, one can write them in terms of elliptic functions [1].

The results of [1] were obtained for *closed* string amplitudes, and it is natural to ask how one could extend these results to *open* topological string amplitudes. As in the closed case, we first need a formalism to compute open topological string amplitudes exactly on the whole closed and open moduli space. For the case of toric Calabi–Yau threefolds, this formalism has been proposed in [16,49] and it is based on a recursion relation first obtained in the context of matrix models [32,34]. One advantage of the framework developed in [16,49], as compared to the holomorphic anomaly equations of [12,54], is that the amplitudes are completely fixed by the recursion. It is then natural to use this formalism in order to understand the properties of open string amplitudes as one moves in the open and closed moduli space of toric Calabi–Yau threefolds, and in particular to extract information about the open counterparts of orbifold Gromov–Witten invariants (which so far have not been defined in the mathematical literature).

In [16] some steps were taken in this direction. In particular we discussed how to find "good coordinates" for the open moduli at the orbifold point, and we made a preliminary analysis of the disk amplitude. In this paper we present a detailed study of open topological string amplitudes at the orbifold point, focusing on the case of $\mathbb{C}^3/\mathbb{Z}_3$. First of all, we clarify the transformation properties of the string amplitudes in the open sector, and we present expressions for them which make their modular transformation properties manifest. Since the recursion of [16,34,49] is based on the Bergman kernel of the mirror curve, our first step is to write it (for a curve of genus one) in terms of elliptic functions. One can then plug the resulting expression in the recursion to find modular expressions for all the open string amplitudes. This leads to considerable improvements in terms of computional efficiency of the recursion relations. As a consequence of this refinement of the formalism of [16], we are able to calculate open orbifold string amplitudes at high order, and we present explicit expressions for amplitudes with $(g, h) = (0, 2)$, $(0, 3)$ and $(1, 1)$. These expressions give generating functions for open orbifold Gromov–Witten invariants, and from the CFT point of view they compute correlation functions of arbitrary

insertions of both bulk operators, associated with twist fields, and boundary operators, associated with deformation modes of the D-brane open moduli.

The organization of the paper is as follows. We start by reviewing the remodeling approach to the B-model using recursion relations in Sects. 2.1 to 2.3. In Sect. 2.4, we study modularity of the open amplitudes, which we rewrite in a form that makes their transformation properties explicit in Sect. 2.5. Section 3 is then devoted to the study of topological open string amplitudes at the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold point in the moduli space of local $\mathbb{P}^2$, using the formalism presented in Sect. 2. We also briefly comment on the calculation of the open amplitudes at the conifold point in the moduli space of local $\mathbb{P}^2$ in Sect. 3.5.

2. Open B-model on Mirrors of Toric Calabi-Yau Threefolds

2.1. The geometry. Consider the A-twisted sigma model on a (noncompact) toric Calabi-Yau threefold X. A-branes are objects in the "derived Fukaya category" of X; roughly speaking, they correspond to Lagrangian submanifolds of X with bundles on them. We consider a simple class of A-branes, given by noncompact special Lagrangian submanifolds $L \subset X$ with trivial bundle, with topology $\mathbb{R}^2 \times S^1$; those were constructed in [4,5,39] — see also [16] for a detailed description.

The mirror theory is a B-twisted sigma model[1] on a family $\pi : Y \to \mathcal{M}$ of noncompact Calabi-Yau threefolds, where $\mathcal{M}$ is the moduli space of the closed B-model. Let $z = (z_1, \ldots, z_k)$ be coordinates on $\mathcal{M}$ centered at a point of maximally unipotent monodromy. The fiber $Y_z = \pi^{-1}(z_1, \ldots, z_k)$ of the family has the form

$$Y_z = \{ww' = H(x, y; z)\} \subset (\mathbb{C})^2 \times (\mathbb{C}^*)^2, \tag{2.1}$$

where $H(x, y; z)$ is a Laurent polynomial in $x, y \in \mathbb{C}^*$ of degree 1. The precise form of $H(x, y; z)$ is dictated by the toric data of the mirror X. Y_z is a quadric fibration over $(\mathbb{C}^*)^2$, with degeneration locus the Riemann surface

$$\Sigma_z = \{H(x, y; z) = 0\} \subset (\mathbb{C}^*)^2. \tag{2.2}$$

B-branes are objects in the derived category of coherent sheaves, some of which correspond to holomorphic submanifolds of Y_z with bundles on them. The B-branes mirror to the simple A-branes considered above can be described as wrapping a holomorphic curve in Y_z, with trivial bundle on it. More precisely, fix a point $p_0 \in \Sigma_z$ parameterized by (x_0, y_0), and denote by $C_z(p_0)$ the holomorphic submanifold of Y_z defined by

$$w' = 0 = H(x_0, y_0; z). \tag{2.3}$$

It is given by the line parameterized by w over the point $p_0 \in \Sigma_z$. This is the holomorphic curve which is wrapped by the B-brane. The open moduli space corresponds to deformations of the B-brane $C_z(p_0)$ in Y_z, which are parameterized by the point $p_0 \in \Sigma_z$. As a result, the moduli space of the open B-model on (Y, C) is given by the family of Riemann surfaces $\Sigma \to \mathcal{M}$, with fiber (2.2).

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[1] The mirror is generally presented as a Landau-Ginzburg model; we explain the correspondence between the Landau-Ginzburg model and the sigma model in Appendix A.

Example 2.1. The main example that we will study is the mirror to local $\mathbb{P}^2$. Let $X = K_{\mathbb{P}^2}$ be the total space of the canonical bundle over $\mathbb{P}^2$. Its mirror is the family of Calabi-Yau threefolds $Y \to \mathcal{M}$, where the closed moduli space $\mathcal{M}$ is one-dimensional, whose fibers Y_z are given by (2.1) with

$$H(x, y; z) = 1 + x + y + \frac{z}{xy}. \tag{2.4}$$

The family of Riemann surfaces $\Sigma \to \mathcal{M}$ has fibers Σ_z (2.2), which are elliptic curves with three punctures.

2.2. Disk amplitude. In this paper we focus on the open amplitudes of the B-model. Let us start with the simplest amplitude, the disk amplitude (genus 0, 1 hole). Roughly speaking, it is the open analog of the genus 0 closed amplitude, which corresponds to the prepotential of special geometry of the closed moduli space $\mathcal{M}$. The disk amplitude on (Y, C) similarly admits a simple definition as follows.

Recall that the moduli space of the open B-model consists in a family of Riemann surfaces (with punctures) $\Sigma \to \mathcal{M}$. Choose an embedding[2] of the fibers Σ_z in $(\mathbb{C}^*)^2$,

$$\Sigma_z = \{H(x, y; z) = 0\} \subset (\mathbb{C}^*)^2, \tag{2.5}$$

and define the one-form

$$\begin{aligned}
\omega(p) &= \log y(x(p)) \frac{dx(p)}{x(p)} \\
&= \log y(x) \frac{dx}{x}
\end{aligned} \tag{2.6}$$

on Σ_z, where $p \in \Sigma_z$ and x is chosen as local coordinate.

Remark 2.1. Note that in the following we will always omit the dependence on z to simplify the notation. But since $\Sigma \to \mathcal{M}$ is a family of curves, all the objects we define on the fiber Σ_z will have an implicit dependence on z.

The main conjecture of [4,5], which comes from dimensional reduction of the holomorphic Chern-Simons theory on the brane C, goes as follows.

Conjecture 2.1 ([4,5]). The "Abel-Jacobi" map

$$F^{(0,1)} = \int_\gamma \omega(p), \tag{2.7}$$

where γ is the chain $[q^*, q]$ and $q^* \in \Sigma_z$ is a reference point, gives the B-model disk amplitude, up to classical terms. $F^{(0,1)}$ should be understood as a series expansion in the local coordinate x near $x = 0$, where x corresponds to the open modulus associated to the brane.

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[2] The choice of embedding of Σ_z in $(\mathbb{C}^*)^2$ corresponds to a choice of phase and framing of the mirror brane. This was considered in detail in [16].

The Abel-Jacobi map is defined on the Jacobian, that is only up to addition of integrals of $\omega(p)$ over one-cycles. But here we will only be interested in the series expansion of the amplitude in the open modulus, and so the ambiguity is irrelevant. Note that this conjecture is the local analog of the result of [50], where the disk amplitude is computed in terms of normal functions.

This formula has been verified in many examples, by expanding the disk amplitude near a point of maximally unipotent monodromy in the closed moduli space, and comparing with open A-model amplitudes on the toric mirror. It requires an explicit knowledge of the closed and open mirror maps, which can be understood as solutions of an extended Picard-Fuchs system (the latter was derived in the language of mixed Hodge structures and relative cohomology in [48]).

2.3. General formalism. We now move on to the general amplitudes $F^{(g,h)}$ with genus g and h holes. As for the closed amplitudes $F^{(g)}$, the physical B-model open amplitudes are generally non-holomorphic, and satisfy an open analog of the holomorphic anomaly equations of [12]. However, to compare with the A-model Gromov-Witten generating functions, one needs to consider the holomorphic limit of the physical B-model amplitudes expanded near a special point in the moduli space. The $F^{(g,h)}$ that we consider here are these holomorphic objects, rather than the physical B-model amplitudes. Stated from a modularity point of view, what we construct here are the quasi-modular forms, rather than the almost holomorphic modular forms [1]. We will discuss this point in more detail in the next subsection.

In [16,49] a general recursive formalism for computing B-model genus g, h hole open amplitudes $F^{(g,h)}$ on (Y, C) was proposed. From a mathematical point of view, since the open B-model is not really well understood, this can be taken as a proposal for a definition of the open B-model on these geometries.

Consider again the following data:

– A family of (punctured) Riemann surfaces $\Sigma \to \mathcal{M}$ (the open B-model moduli space);
– A choice of embedding of the fibers Σ_z in $(\mathbb{C}^*)^2$,

$$\Sigma_z = \{H(x, y; z) = 0\} \subset (\mathbb{C}^*)^2. \tag{2.8}$$

We claim that these data fully characterize the open B-model on (Y, C), with arbitrary genus and number of holes.

By projecting onto the x-axis we may see Σ_z as a branched cover of $\mathbb{C}^*$. Denote by $q_i \in \Sigma_z$ the ramification points of the projection map, such that $dx(q_i) = 0$. Let $\lambda_i := x(q_i) \in \mathbb{C}^*$ be the branch points. We assume that they all have branching order two. Then, near q_i, there are two points $q, \bar{q} \in \Sigma_z$ with the same projection $x(q) = x(\bar{q})$ (those are defined only locally near q_i). As before, define the one-form $\omega(p)$, which reads in local coordinates

$$\omega(p) = \log y(x) \frac{dx}{x}. \tag{2.9}$$

Definition 2.1. *The **Bergman kernel** $B(p, q)$ is the unique bilinear differential on Σ_z with a double pole at $p = q$ with no residue, and no other pole. It is normalized by*

$$\oint_{A^I} B(p, q) = 0, \tag{2.10}$$

where (A^I, B_I) is a symplectic basis of cycles on Σ_z.

Note that the Bergman kernel is defined on the Riemann surface itself, and does not depend on the embedding in $(\mathbb{C}^*)^2$. Its definition however requires a choice of symplectic basis of cycles on Σ_z.

Definition 2.2. *Near $q_i \in \Sigma_z$, define the one-form*

$$dE_{q,\bar{q}}(p) = \frac{1}{2} \int_{\xi=q}^{\bar{q}} B(p, \xi), \tag{2.11}$$

where the integration is in a neighborhood of q_i. Note that this is defined only locally near q_i.

We are now ready to state the recursion, which was first derived in the context of matrix models in [22,32,34].

Definition 2.3. *Let $\tilde{W}^{(g,h)}(p_1, p_2, \ldots, p_h)$, $g, h \in \mathbb{Z}$, $g \geq 0$, $h \geq 1$, be multilinear differentials on Σ_z. Fix the initial conditions*

$$\tilde{W}^{(0,1)}(p_1) = 0, \qquad \tilde{W}^{(0,2)}(p_1, p_2) = B(p_1, p_2). \tag{2.12}$$

Define the remaining differentials by the recursion[3]

$$\tilde{W}^{(g,h)}(p_1, p_2, \ldots, p_h) = \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{dE_{q,\bar{q}}(p_1)}{\omega(q) - \omega(\bar{q})} \Bigg(\tilde{W}^{(g-1,h+1)}(q, \bar{q}, p_2, \ldots, p_h)$$

$$+ \sum_{l=0}^{g} \sum_{J \subseteq H} \tilde{W}^{(g-l,|J|+1)}(q, p_J) \tilde{W}^{(l,|H|-|J|+1)}(\bar{q}, p_{H \setminus J}) \Bigg), \tag{2.13}$$

where we used the notation $H = \{2, 3, \ldots, h\}$, and given any subset $J = \{i_1, \ldots, i_j\} \subseteq H$ we defined $p_J = \{p_{i_1}, \ldots, p_{i_j}\}$.

There is a second recursion which reads as follows.

Definition 2.4. *Let $F^{(g)}$, $g \in \mathbb{Z}$, $g \geq 2$, be functions on Σ_z defined by*

$$F^{(g)} = \frac{1}{2g-2} \sum_{q_i} \operatorname*{Res}_{q=q_i} \theta(q) \tilde{W}^{(g,1)}(q), \tag{2.14}$$

where $\theta(q)$ is any primitive of $\omega(q)$, i.e. $d\theta(q) = \omega(q)$.

We define the function $F^{(1)}$ by:

Definition 2.5. *Define*

$$F^{(1)} = -\frac{1}{2} \log \tau_B - \frac{1}{24} \log \prod_i \omega'(q_i), \tag{2.15}$$

where τ_B is the Bergman tau-function and

$$\omega'(q_i) = \frac{1}{dz_i(p)} d\left(\frac{\log y(x)}{x}\right)\Bigg|_{p=q_i}, \qquad z_i(p) = \sqrt{x(p) - x(q_i)}. \tag{2.16}$$

We refer the reader to [34] for more details.

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[3] Note that the integrand in the right-hand side is only defined locally near the ramification points q_i.

The main conjecture of [16,49], which relates the objects defined above recursively to the B-model amplitudes, could be stated as follows.

Conjecture 2.2. Let $F^{(0)}$ be the prepotential of special geometry, $F^{(1)}$ be as in Definition 2.5, and the $F^{(g)}$'s for $g \geq 2$ be as in Definition 2.4.

For $g \geq 0$, $h \geq 1$, and $(g, h) \neq (0, 1), (0, 2)$, define the multilinear differentials

$$W^{(g,h)}(p_1, \ldots, p_h) = \tilde{W}^{(g,h)}(p_1, \ldots, p_h), \tag{2.17}$$

using Definition 2.3. Let

$$W^{(0,2)}(p_1, p_2) = B(p_1, p_2) - \frac{\mathrm{d}p_1 \mathrm{d}p_2}{(p_1 - p_2)^2}, \tag{2.18}$$

and

$$W^{(0,1)}(p) = \omega(p). \tag{2.19}$$

Define

$$F^{(g,h)} = \int_{\gamma_1} \cdots \int_{\gamma_h} W^{(g,h)}(p_1, \ldots, p_h), \tag{2.20}$$

where the γ_i's are the chains $[q_i^*, q_i]$, with the $q_i^* \in \Sigma_z$ reference points.

The $F^{(g)}$ constructed above are the genus g closed B-model amplitudes on Y, and the $F^{(g,h)}$ are the genus g, h hole open B-model amplitudes on (Y, C). The $F^{(g,h)}$ should be understood as series expansions in the local coordinates $x_i := x(p_i)$, which correspond to the open moduli associated to the branes.

Note that as for the disk amplitude $F^{(0,1)}$, the $F^{(g,h)}$ are only defined modulo integration over closed cycles; but again, this ambiguity will be irrelevant since we only consider instanton expansions of the amplitudes.

Remark 2.2. Note that the conjecture for $F^{(1)}$ should probably follow from the result of Dubrovin and Zhang for the G-function associated to Frobenius manifolds [31], which was also studied by Givental [36]. It can also be understood from a topological field theory point of view as in [13].

There are various arguments behind this conjecture. First, a strong piece of evidence comes from direct calculation. In [16,49], various amplitudes for the mirrors of $\mathbb{C}^3$, local $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, F_1, F_2 were computed. By expanding the amplitudes near a point of maximally unipotent monodromy and plugging in the open and closed mirror maps, it was shown that one recovers the open A-model amplitudes on the toric mirrors. This however only tests the conjecture at large radius; in [16] the conjecture was also tested at the orbifold point of local $\mathbb{P}^1 \times \mathbb{P}^1$, by comparing with perturbative Chern-Simons theory on the lens space $S^3/\mathbb{Z}_2$.

A more conceptual argument for the conjecture goes as follows. The recursions were derived by [22,34] in the context of matrix models. When Σ is the spectral curve of a matrix model, the recursions (2.13) and (2.14) respectively generate the correlation functions and free energies of the matrix model. For some B-model geometries, using large N dualities on the mirror side, we can find matrix model representations with spectral curve Σ_z, which justifies the conjecture. However, in general no matrix model representation is known; but it was argued first in [49], and then in much more detail in

[30], that the B-model amplitudes should indeed satisfy the recursions (2.13) and (2.14), whether there is a matrix model representation or not. This involves a detailed analysis of the B-model, understood in the chiral boson picture developed in [2]. One can also show that the amplitudes obtained through the recursion, after restoring non-holomorphicity using modularity as in [1], satisfy the holomorphic anomaly equations (and their open analogs) [33]. In any case, in the following we will take this conjecture for granted and explore some of its consequences.

2.4. Modularity. In the previous subsection we introduced a recursive formalism to compute open and closed B-model amplitudes. While the formalism is very elegant conceptually, and provides a complete solution to the B-model on these geometries, it turns out to be rather complicated computationally. One reason is that the formalism makes no explicit use of the modular properties of the amplitudes; on the contrary, the intermediate step of taking residues at the branch points destroys the symmetry of the amplitudes. Indeed, the branch points are in general complicated functions of z, since the projection $\Sigma_z \to \mathbb{C}^*$ is a branched cover. But the final amplitudes are simple functions of z; in fact, only symmetric combinations of the branch points, which are simple rational functions of z, appear in the final amplitudes.

It thus seems desirable to recast the recursion in a different form, bypassing the intermediate step of taking residues at the branch points. However, one problem is that the integrand in the right hand side of (2.13) is only defined locally near the branch points. Hence, one cannot simply deform the contour integral to pick up residues at the other poles in a straightforward way. Indeed, localizing the integrand at the branch points turns out to be crucial in the derivation of the recursion in the matrix model context (see for instance pp.14-15 of [22]), in order to get rid of unfixed polynomials. So it seems that the intermediate step plays a more important role than one would have expected at first sight.[4]

Even though it seems difficult to reformulate the recursion in a more computationally effective way, what we can do is use our knowledge of the modular properties of the amplitudes to rewrite the amplitudes *a posteriori*. That is, using modularity and regularity of the amplitudes we write down a general ansatz for the amplitudes, either in terms of modular forms, or as functionals of solutions of the Picard-Fuchs equations. At each genus g and number of holes h, the ansatz involves rational functions in the open and closed moduli comprising a finite number of unknown parameters. The latter can be fixed by comparing the ansatz with the result obtained from the recursion (2.13). Alternatively, the parameters can be fixed by comparing with a mirror calculation at large radius using the topological vertex formalism [3].

These formulae prove to be very useful in studying the amplitudes at various points in the moduli space, as we will do in the next section. However, the rational functions become rather involved and increasingly difficult to determine for higher genus and a larger number of holes.

2.4.1. Picard-Fuchs equations, monodromy and modularity of the closed amplitudes
Given a family of Calabi-Yau threefolds $Y \to \mathcal{M}$, it is standard to associate a system of differential equations, called the *Picard-Fuchs equations*, which annihilate periods of the holomorphic volume form Ω_z on the fiber Y_z. In the noncompact setting, the

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[4] It is tempting to speculate that the process of localizing at the branch points is a B-model mirror analog to the process of using localization with respect to a torus action in Gromov-Witten theory.

Picard-Fuchs system can be extracted either by taking the limit of a compact three-fold [23], or by considering the equivalent Landau-Ginzburg setting [36,41]. Solutions to the Picard-Fuchs equations provide a set of flat coordinates on $\mathcal{M}$.

When Y is of the form studied previously, it can be shown that the geometry "reduces" to the family of curves $\Sigma \to \mathcal{M}$, and the Picard-Fuchs equations annihilate periods of the one-form $\omega(p)$ over one-cycles on the Riemann surface Σ_z. From now on, we focus on the case where Σ_z is a genus one curve. Let (A, B) be a canonical basis of one-cycles on the genus one curve Σ_z. Apart from a constant solution, there are two more solutions to the Picard-Fuchs equations, which provide a basis of dual periods:

$$T = \oint_A \omega(p), \qquad T_D = \oint_B \omega(p). \tag{2.21}$$

The Picard-Fuchs differential equations have regular singular points, around which the periods have monodromy. The monodromy group is a finite index subgroup of $SL(2, \mathbb{Z})$.

A natural question is to study modularity of the B-model amplitudes with respect to the monodromy group. This question was approached for the closed amplitudes from physical principles in [1]. The physical closed B-model amplitudes $\mathcal{F}^{(g)}$ are invariant under the monodromy group — indeed, this is required for consistency of the physical theory all over the moduli space $\mathcal{M}$ — but they are non-holomorphic. This can be reformulated in terms of modularity with respect to the modular parameter of Σ_z:

$$\tau = \frac{\partial T_D}{\partial T} = \frac{\partial^2 \mathcal{F}^{(0)}}{\partial T^2}, \tag{2.22}$$

where $\mathcal{F}^{(0)}$ is the prepotential of special geometry giving the genus 0 closed B-model amplitude. In this language, the statement becomes that for $g \geq 2$, the physical amplitudes $\mathcal{F}^{(g)}$ are *almost holomorphic modular forms* with respect to the monodromy group [1]. However, there is a canonical isomorphism between the ring of almost holomorphic modular forms and the ring of *quasi-modular forms* — forms that transform with a shift [44]. This is given by "taking the holomorphic limit" of $\mathcal{F}^{(g)}$, which breaks the modular invariance by keeping only the constant term in the finite expansion in $\mathrm{Im}(\tau)^{-1}$. We thus obtain the holomorphic closed B-model amplitudes $F^{(g)}$, which are quasi-modular with respect to the monodromy group. Those are the amplitudes that were constructed through the recursion.

2.4.2. Modularity of the open amplitudes We now want to understand the modular properties of the open amplitudes $F^{(g,h)}$, which are the holomorphic limits of the monodromy invariant physical amplitudes $\mathcal{F}^{(g,h)}$. This was studied in [33,34] using the recursion.

Let τ be the modular parameter of Σ_z, which parameterizes the upper half plane. Let

$$\tilde{\tau} = \frac{A\tau + B}{C\tau + D}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \subset SL(2, \mathbb{Z}) \tag{2.23}$$

be a symplectic transformation of the periods in the monodromy group Γ.

Under (2.23), the Bergman kernel transforms as

$$\tilde{B}(p, q) = B(p, q) - 2\pi \mathrm{i}\, u(p)(C\tau + D)^{-1} C u(q), \tag{2.24}$$

where $u(p)$ is the holomorphic differential. The shift makes the Bergman kernel a quasi-modular form of weight 0. Through the recursion (2.13), this induces quasi-modular properties for all the open amplitudes $W^{(g,h)}$. One can compute the explicit transformation properties of the differentials $W^{(g,h)}$ by plugging in the transformation properties of the Bergman kernel directly in the recursion, as was done in [33,34]; we will not repeat the analysis here. Instead, what we are doing next is to use our knowledge of modularity to write down explicit expressions for the (low genus and number of hole) amplitudes in terms of modular forms, and as functionals of solutions of the Picard-Fuchs equations (the periods).

2.5. Modular forms and functionals.

As we have just seen, the multilinear differentials $W^{(g,h)}$ are quasi-modular forms of weight 0 with respect to the monodromy group. They are obtained by taking the holomorphic limit of the non-holomorphic differentials $\mathcal{W}^{(g,h)}$, which correspond to the physical amplitudes, therefore are monodromy invariant. As a consequence, the holomorphic amplitudes $W^{(g,h)}$ can be universally written as functionals of the periods and their derivatives, where the periods are functions of some local coordinates on the moduli space.

The functional point of view provides a very useful way of computing modular transformations of the amplitudes, since changing the period in the functional directly implements the symplectic transformation between the periods. In other words, the choice of period in the functional corresponds to a choice of modular parameter, or equivalently to a choice of canonical basis of cycles in the definition of the Bergman kernel. This approach renders the computation of the amplitudes everywhere in the moduli space straightforward.

To see how it goes, let us start by deriving a general expression for the annulus amplitude (or the Bergman kernel) in terms of modular forms, which is the main ingredient in the recursion relation, and induces the quasi-modular properties of the amplitudes. We then explain how it can be written generally as a functional; we will propose an exact form for the functional in the next section when we specialize to the local $\mathbb{P}^2$ geometry. Finally we propose a general ansatz for the higher order amplitudes, which we use in the next section to derive functional expressions and compute the amplitudes at the orbifold point of local $\mathbb{P}^2$.

2.5.1. The annulus amplitude

As usual, we start with a family of (punctured) Riemann surfaces $\Sigma \to \mathcal{M}$ (the open B-model moduli space), and a choice of embedding of the fibers Σ_z in $(\mathbb{C}^*)^2$,

$$\Sigma_z = \{H(x, y; z) = 0\} \subset (\mathbb{C}^*)^2. \tag{2.25}$$

We specialize to the case where Σ_z is a genus one curve. Denote by $q_i \in \Sigma_z$ the ramification points of the projection map onto the x-axis, and by $\lambda_i := x(q_i) \in \mathbb{C}^*$ the branch points.

When Σ_z has genus one, the annulus amplitude $W^{(0,2)}$ can be written in terms of the Weierstrass elliptic function, using uniformization parameters for the elliptic curve. Alternatively, when Σ_z has four distinct branch points λ_i, $i = 1, \ldots, 4$, one can work directly on the $\mathbb{C}^*$ which is the image of the x-projection. In terms of x-projected variables $x_1, x_2 \in \mathbb{C}^*$ (i.e. local coordinates $x_1 := x(p_1)$ and $x_2 := x(p_2)$), Akemann derived in [6] a formula for the annulus amplitude, which reads

$$W^{(0,2)}(p_1, p_2) = \frac{dx_1 dx_2}{4\sqrt{\sigma(x_1)\sigma(x_2)}} \left(\frac{M(x_1, x_2) + M(x_2, x_1)}{(x_1 - x_2)^2} \right.$$
$$\left. -(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)\frac{E(k)}{K(k)} \right) - \frac{dx_1 dx_2}{2(x_1 - x_2)^2}, \quad (2.26)$$

where

$$M(x_1, x_2) = (x_1 - \lambda_1)(x_1 - \lambda_2)(x_2 - \lambda_3)(x_2 - \lambda_4), \quad (2.27)$$

$$\sigma(x) = \prod_{i=1}^{4}(x - \lambda_i), \quad (2.28)$$

and $K(k)$ and $E(k)$ are elliptic integrals with modulus

$$k^2 = \frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \quad (2.29)$$

Note that the amplitude depends on a choice of ordering of the branch points, which corresponds to a choice of canonical basis of cycles on Σ_z.

Let us start by rewriting the amplitude in terms of modular forms.

Proposition 2.1. *Let*

$$S_k = \sum_{1 \le j_1 < j_2 < ... < j_k \le 4} \lambda_{j_1} \cdots \lambda_{j_k} \quad (2.30)$$

be the elementary symmetric polynomials in the four branch points, and let

$$u(x)dx = i\frac{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{4\sqrt{\sigma(x)}K(k)}dx \quad (2.31)$$

be the holomorphic differential. The annulus amplitude can be written as

$$W^{(0,2)}(p_1, p_2) = \left(-\frac{1}{2(x_1 - x_2)^2} + \frac{f_0^{(0,2)}(x_1, x_2)}{4\sqrt{\sigma(x_1)\sigma(x_2)}} \right.$$
$$\left. +\frac{\pi^2}{3}u(x_1)E_2(\tau)u(x_2) \right) dx_1 dx_2, \quad (2.32)$$

where τ is the modular parameter, $E_2(\tau)$ is the second Eisenstein series, and the rational function $f_0^{(0,2)}(x_1, x_2)$ reads[5]

$$f_0^{(0,2)}(x_1, x_2) = \frac{1}{3(x_1 - x_2)^2}\left(6x_1^2 x_2^2 - 3x_1 x_2(x_1 + x_2)S_1 + (x_1^2 + 4x_1 x_2 + x_2^2)S_2 \right.$$
$$\left. -3(x_1 + x_2)S_3 + 6S_4 \right). \quad (2.33)$$

---

[5] Recall from Remark 2.1 that we do not write explicitly the dependence on z for simplicity.

Proof. We start with Akemann's formula (2.26). Let us introduce

$$\omega_1 = \frac{2i}{\pi} \frac{K(k)}{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}, \qquad e_3 = \frac{1}{12}\left(S_2 - 3(\lambda_1\lambda_2 + \lambda_3\lambda_4)\right). \quad (2.34)$$

e_3 is one of the three roots of the elliptic curve in Weierstrass form. Then, manipulating some of Akhiezer's identities for elliptic integrals [7], we obtain the identity

$$E(k)K(k) = \pi^2 \left(\frac{1}{12}E_2(\tau) + \omega_1^2 e_3\right). \quad (2.35)$$

From this we rewrite the second term in (2.26) as

$$-\frac{1}{4\sqrt{\sigma(x_1)\sigma(x_2)}}(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)\frac{E(k)}{K(k)}$$
$$= \frac{\pi^2}{3}u(x_1)E_2(\tau)u(x_2) + \frac{e_3}{\sqrt{\sigma(x_1)\sigma(x_2)}}, \quad (2.36)$$

using the definition of the holomorphic differential above. By expanding the function $M(x_1, x_2)$ and combining with the e_3 term, we can rewrite the other terms of (2.26) in terms of elementary symmetric polynomials of the branch points, and we obtain (2.32).

Remark 2.3. In (2.32), the only term which is not quite modular invariant is the term with $E_2(\tau)$. Since $E_2(\tau)$ is a quasi-modular form of weight 2, and the holomorphic differentials are modular of weight -1, we see explicitly that the annulus amplitude is a quasi-modular form of weight 0, as it should. The shift in the modular transformation of the annulus amplitude comes, as in the closed case [1], from the shift in the modular transformation of the second Eisenstein series $E_2(\tau)$.

Remark 2.4. Note that the function $f_0^{(0,2)}(x_1, x_2)$ is also rational in z — hence manifestly modular invariant — since it involves only symmetric combinations of the branch points, which are necessarily rational functions of z. The function $f_0^{(0,2)}(x_1, x_2)$ corresponds to the "holomorphic ambiguity" in the integration of the holomorphic anomaly equation for the open amplitudes.

Let us now define

$$G(\tau) = \frac{E_2(\tau)}{3\omega_1^2}, \quad (2.37)$$

which is a function of z through the definition of ω_1, and depends on a choice of modular parameter τ, but does not depend on the open string variables x_1 and x_2. The annulus can now be rewritten as

$$W^{(0,2)}(p_1, p_2) = -\frac{dx_1 dx_2}{2(x_1 - x_2)^2} + \frac{f_0^{(0,2)}(x_1, x_2) + G(\tau)}{4\sqrt{\sigma(x_1)\sigma(x_2)}}dx_1 dx_2. \quad (2.38)$$

The Bergman kernel $B(p_1, p_2)$ is obtained by changing the sign in front of the first term.

$G(\tau)$ plays an important role in the following, since it encodes the quasi-modular properties of the amplitudes. As a result, $G(\tau)$ can be expressed as a *functional* of the period T and its derivative; in which case we will denote it as $G[T; z]$. The choice of period T corresponds to the choice of modular parameter τ. The exact form of $G[T; z]$ depends on the curve Σ_z; we will present it for the mirror of local $\mathbb{P}^2$ in the next section.

To summarize, we now have an expression for the annulus amplitude in terms of modular forms, which can be rewritten as a functional of the period and its derivatives, using $G[T; z]$. Let us now study the higher order amplitudes.

2.5.2. Higher amplitudes Now that we have a functional expression for the annulus amplitude (2.38), which is the main ingredient of the recursion, we can derive the principal functional form of the higher genus amplitudes from (2.13).

Lemma 2.1. *For $g \geq 0$, $h \geq 1$, and $(g, h) \neq (0, 1), (0, 2)$, the general form of the amplitudes is*

$$W^{(g,h)}(p_1, \ldots, p_h) = \frac{dx_1 \cdots dx_h}{\Delta^{2g-2+h} \prod_{i=1}^{h} \sqrt{\sigma(x_i)}} \sum_{i=0}^{3g-3+2h} G^i[T; z] f_i^{(g,h)}(x_1, \ldots, x_h),$$

(2.39)

where

$$\Delta = \prod_{i<j} (\lambda_i - \lambda_j)^2$$

(2.40)

is the discriminant of the curve. The functions $f_i^{(g,h)}(x_1, \ldots, x_h)$ are rational in their arguments and in the closed parameter z. Moreover, they have the form

$$f_i^{(g,h)}(x_1, \ldots, x_h) = \frac{Q_i^{(g,h)}(x_1, \ldots, x_h)}{\left(\prod_{j=1}^{h} \sigma(x_j)\right)^{3g-2+h}},$$

(2.41)

where the $Q_i^{(g,h)}(x_1, \ldots, x_h)$ are polynomials of finite degree in their arguments and in z.

Proof Sketch of the proof. We obtain this general form by close inspection of the recursion (2.13), and using the functional formula (2.38) for the annulus amplitude. Let us simply sketch the main lines of the argument.

Let

$$W^{(g,h)}(p_1, \ldots, p_h) = w^{(g,h)}(x_1, \ldots, x_h) dx_1 \cdots dx_h.$$

(2.42)

First, it is clear from the definition that the functions

$$\sqrt{\sigma(x_1)} \cdots \sqrt{\sigma(x_h)} w^{(g,h)}(x_1, \ldots, x_h)$$

(2.43)

are rational in the x_i's, since multiplying by the square roots amounts to cancelling the branch cuts.

Second, by pushing down the recursion (2.13) in order to obtain the analogs of Feynman rules, as in Definition 4.5 of [34], we see that each amplitude is represented by a graph with $3g - 3 + 2h$ edges. Each edge gives a factor of either $B(p, q)$ or $\frac{dE_{q,\bar{q}}(p)}{\omega(q) - \omega(\bar{q})}$. Since both of these factors are polynomials of degree 1 in $G[T; z]$, we obtain that $w^{(g,h)}$ must be a polynomial of order $3g - 3 + 2h$ in $G[T; z]$. So what we know so far is that

$$w^{(g,h)}(x_1, \ldots, x_h) = \frac{1}{\prod_{i=1}^{h} \sqrt{\sigma(x_i)}} \sum_{i=0}^{3g-3+2h} G^i[T; z] \tilde{f}_i^{(g,h)}(x_1, \ldots, x_h),$$

(2.44)

where the $\tilde{f}_i^{(g,h)}(x_1, \ldots, x_h)$ are rational in the x_i's. It is also clear that the $\tilde{f}_i^{(g,h)}$ are rational in z, since we are summing over branch points, hence the $\tilde{f}_i^{(g,h)}$ can be expressed in terms of elementary symmetric polynomials in the branch points, which must be rational functions of z.

Finally, the denominators of the functions $\tilde{f}_i^{(g,h)}(x_1, \ldots, x_h)$ can be obtained from the pole structure of the integrand in the recursion (2.13). The analysis is rather subtle, and we leave the details to the reader. Roughly speaking, after taking residues and summing over branch points, each pole of the form $\sigma(x)^{-k}$ contributes a factor of Δ^k in the denominator, and the double poles of the Bergman kernels combine to give the factors of $\sigma(x)$ in the denominator.

For a particular geometry, by comparing the generic form of the amplitudes (2.39) with the explicit result obtained with the recursion, we can determine the functions $f_i^{(g,h)}(x_1, \ldots, x_h)$ at each genus and number of holes. Once this is done, the main advantage of the functional form of the amplitudes is that the computation of the amplitudes at various points in the moduli space simply amounts to inserting the right period T in the functional. We exemplify this procedure in detail in the next section by studying the mirror of local $\mathbb{P}^2$ at the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold point.

Note that the general form of the amplitudes (2.39) was obtained directly by inspection of the recursion and using the functional formula for the annulus amplitude. Alternatively, it could have been obtained through direct integration of the open version of the holomorphic anomaly equation, in which case the functions $f_0^{(g,h)}(x_1, \ldots, x_h)$ would correspond to the holomorphic ambiguities. This complementary approach sheds new light on the structural constaints of the amplitudes coming directly from modularity; we hope to report on that in future work.

3. Open Orbifold Gromov-Witten Invariants of $\mathbb{C}^3/\mathbb{Z}_3$

In this section we apply our formalism to the study of the mirror to local $\mathbb{P}^2$ at the orbifold point in moduli space.

3.1. Geometry. We consider the geometry described in Example 2.1. $\Sigma \to \mathcal{M}$ is the one-parameter family of genus one Riemann surfaces with three punctures. We choose the following embedding for the fibers,

$$\Sigma_z = \left\{ y^2 + y(1+x) + zx^3 = 0 \right\} \subset (\mathbb{C}^*)^2. \tag{3.1}$$

We consider the B-model on this geometry, with B-branes wrapping the curve $C \subset Y$ as usual. The mirror theory is the A-model on the target space $X = K_{\mathbb{P}^2}$, with a non-compact A-brane wrapping a special Lagragian submanifold of topology $\mathbb{R}^2 \times S^1$ (see [5,4,16] for a detailed description of these branes). The parameterization of the curve Σ_z above corresponds on the A-model side to an "outer brane with zero framing", in the nomenclature of [4,16]. Unless specified, all our calculations in this section will be in this parameterization.

By mirror symmetry, the closed A-model moduli space is isomorphic to $\mathcal{M}$. It has two patches, which correspond to two phases of the A-model. In each phase, there is a limit

point near which the A-model amplitudes have a convergent expansion, and become the amplitudes of a non-linear sigma model (coupled to two-dimensional gravity). In the first patch, the limit point is the *large radius point*, which is located at $z = 0$. The amplitudes expanded near this point become generating functions of Gromov-Witten invariants of $X = K_{\mathbb{P}^2}$. In the second patch, the limit point is the *orbifold point*, located at $z = \infty$; a good local coordinate is $\psi = z^{-1/3}$. The amplitudes expanded near this point become generating functions of orbifold Gromov-Witten invariants of $X' = \mathbb{C}^3/\mathbb{Z}_3$. As a result, moving from one patch to the other in $\mathcal{M}$ induces a topologically-changing transition of the target space.

This analysis also extends to the open sector. In the large radius patch, the amplitudes $F^{(g,h)}$ expanded near the large radius point become generating functions for open Gromov-Witten invariants of (X, L), where L is the special Lagrangian submanifold mirror to C. Open Gromov-Witten invariants are defined in terms of stable maps from bordered Riemann surfaces with Lagrangian boundary conditions [45] — see also [37]. If X admits a $U(1)$ action which fixes L, then the $U(1)$ acts naturally on the space of stable maps, and one can use localization to compute open Gromov-Witten invariants [45].

In the orbifold patch, one expects a similar story to hold, and the amplitudes expanded near the orbifold point should be generating functions for open orbifold Gromov-Witten invariants of (X', L'). Here, L' is a Lagrangian submanifold of $\mathbb{C}^3$ which is fixed by the $\mathbb{Z}_3$ action, hence descends to a Lagrangian submanifold of the orbifold. This Lagrangian L' corresponds to the original Lagrangian L at the large radius point, and should exist as a consequence of the A-version of the McKay correspondence for derived categories. Therefore, one can consider stable maps from bordered Riemann surfaces to the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, in such a way that the boundaries are mapped to L', and construct the corresponding open orbifold Gromov–Witten invariants. One could then follow the approach of [45] in the context of orbifolds, and use localization with respect to a $U(1)$ action to compute open orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_3$. Such open orbifold Gromov-Witten invariants have not been defined in the mathematical literature yet. However, Renzo Cavalieri informed us that he is presently working on this [21]. In particular, he has managed to compute disk orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_3$ using localization of $\mathbb{Z}_3$-Hodge integrals. His calculation matches perfectly with the results we present in Subsect. 3.3.1, as we explain there.

Another useful point of view on these open orbifold Gromov–Witten invariants is to consider the topological string theory near the orbifold point as a perturbed $\mathcal{N} = 2$ orbifold conformal field theory (CFT) coupled to gravity. From this point of view, the open topological string amplitude $F^{(g,h)}$ is a generating function of arbitrary insertions of bulk and boundary operators of the orbifold CFT. In the case of $\mathbb{C}^3/\mathbb{Z}_3$ there is only one bulk operator $\mathcal{O}$. This is a twist operator which corresponds to a blow-up mode of the orbifold singularity, *i.e.* to a deformation mode of the closed string modulus. In the presence of Lagrangian boundary conditions specified by L', one also has boundary preserving operators. These operators correspond to the insertion of open string states on the boundaries of the Riemann surface which maps to L', and they are in one-to-one correspondence with $H^1(L, \mathrm{End}(E))$, where E is an appropriate vector bundle on L [9,55]. Since we have open strings with h boundaries, the most general configuration can be obtained by considering h branes wrapping L'. In our case $b_1(L') = 1$, therefore there will be h (integrated) boundary operators Ψ_ℓ, $\ell = 1, \ldots, h$, corresponding to the h branes wrapping L'. We then have

$$F^{(g,h)} = \left\langle \exp\left(T_{orb}\mathcal{O} + \sum_{\ell=1}^{h} X_\ell \Psi_\ell \right) \right\rangle \Big|_{g,h}$$

$$= \sum_{j,i_1,\ldots,i_h \geq 0} \frac{1}{j!} N^{(g,h)}_{(i_1,\ldots,i_h),j} T^j_{orb} X_1^{i_1} \cdots X_h^{i_h}, \tag{3.2}$$

where

$$N^{(g,h)}_{(i_1,\ldots,i_h),j} = \frac{1}{i_1! \cdots i_h!} \langle \mathcal{O}^j \Psi_1^{i_1} \cdots \Psi_h^{i_h} \rangle_{g,h}, \tag{3.3}$$

and the vevs are calculated for the twisted $\mathcal{N} = 2$ SCFT of the orbifold coupled to gravity on a Riemann surface $\Sigma_{g,h}$. The numbers $N^{(g,h)}_{(i_1,\ldots,i_h),j}$ should be identified with the open orbifold Gromov–Witten invariants. The combinatorial factor $i_1! \cdots i_h!$ is included in the invariant in order to agree with the conventions of Cavalieri for the open orbifold Gromov–Witten invariants to which we will compare our results later on.

Our goal in this section is to use mirror symmetry and our B-model recursive formalism to compute generating functions of open orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_3$. This can be done in two ways; either by extracting the B-model amplitudes at the orbifold point from the large radius ones using the quasi-modular properties of the amplitudes, or by generating the amplitudes directly at the orbifold point using the functional expressions derived in the previous section. But before doing that, we need to understand the open and closed mirror maps near the orbifold point, in order to map the B-model amplitudes to the A-model amplitudes.

3.2. Open and closed mirror maps.

3.2.1. Closed mirror map The closed mirror map provides a local isomorphism between the closed A- and B-model moduli spaces. One needs to compute the flat coordinate near a given point of $\mathcal{M}$, which is given by a solution of the associated Picard-Fuchs system. Inversion of the flat coordinate gives the closed mirror map.

For the case under consideration, one obtains a single Picard-Fuchs equation, which reads

$$(\Theta + 3z(3\Theta + 2)(3\Theta + 1))\,\Theta f = 0, \tag{3.4}$$

with the logarithmic derivative $\Theta = z\frac{\partial}{\partial z}$.

The constant function $f = 1$ is always a solution of (3.4). Near $z = 0$, the two other solutions are

$$T(z) = \log z - 6z + 45z^2 - 560z^3 + \cdots := \log z + \sigma(z),$$

$$T_D(z) = (\log z)^2 + 2\sigma(z)\log z - 18z + \frac{423}{2}z^2 - 2972z^3 + \cdots, \tag{3.5}$$

$T(z)$ is the flat coordinate in the large radius patch. At $z = 0$, $T(0) \to -\infty$, and the expansion parameter is set to $Q = e^T$. The closed mirror map in this patch, which expresses z in terms of the flat parameter T, is then given by

$$z(Q) = Q + 6Q^2 + 9Q^3 + 56Q^4 + \cdots. \tag{3.6}$$

In the orbifold patch, the two non-trivial solutions to (3.4) read

$$B_k(\psi) = \frac{(-1)^{k+1}\psi^k}{k} \, {}_3F_2\left(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}; \frac{2k}{3}, 1 + \frac{k}{3}; \left(-\frac{\psi}{3}\right)^3\right), \tag{3.7}$$

with $k = 1, 2$, and we used the local coordinate $\psi = z^{-1/3}$. Using the explicit expansion of the hypergeometric system we get

$$B_k(\psi) = \sum_{n \geq 0} \frac{(-1)^{3n+k+1}\psi^{3n+k}}{(3n+k)!} \left(\frac{\Gamma\left(n + \frac{k}{3}\right)}{\Gamma\left(\frac{k}{3}\right)}\right)^3. \tag{3.8}$$

The flat parameter in this patch reads [1]

$$T_{orb}(\psi) = B_1(\psi), \tag{3.9}$$

and the dual period is $T_{orb,D}(\psi) = B_2(\psi)$. At $\psi = 0$, we get $T_{orb}(0) = 0$, hence T_{orb} itself is a good expansion parameter. The closed mirror map reads

$$\psi(T_{orb}) = T_{orb} + \frac{1}{648}T_{orb}^4 - \frac{29}{3674160}T_{orb}^7 + \frac{6607}{71425670400}T_{orb}^{10} + \cdots. \tag{3.10}$$

3.2.2. The open mirror map The open mirror map extends the isomorphism to the open sector, which in the case under consideration is the fiber of the moduli space $\Sigma \to \mathcal{M}$. Again, one needs to determine the open flat coordinate, which is a solution of the extended Picard-Fuchs system, as derived in [47,48]. The open mirror map is given by inverting this open flat coordinate. We refer the reader to [16,47,48] for a detailed explanation of the extended Picard-Fuchs system.

In the large radius patch, it was shown in [4,47] that the open flat coordinate is given by

$$X(x, z) = x e^{\frac{1}{3}(\log z - T(z))}, \tag{3.11}$$

where x is the local coordinate x on Σ_z, and $T(z)$ is the closed flat coordinate. Note that $X(x, z)$ is monodromy-invariant under $z \mapsto e^{2\pi i}z$. At $(x, z) = (0, 0)$, we have $X \to 0$, hence it is a good expansion parameter. The open mirror map becomes

$$x(Q, X) = X(1 - 2Q + 5Q^2 - 32Q^3 + \ldots). \tag{3.12}$$

In the orbifold patch, we argued in [16] — by requiring that the disk amplitude, when expressed in flat coordinates, be monodromy-invariant under the $\mathbb{Z}_3$ orbifold monodromy $\psi \mapsto e^{2\pi i/3}\psi$, which fixes the open flat coordinate uniquely, up to scale — that the open flat coordinate must be given by

$$X_{orb}(x, \psi) = xz^{1/3} = x\psi^{-1}. \tag{3.13}$$

The open mirror map simply becomes

$$x(X_{orb}, T_{orb}) = X_{orb}\psi(T_{orb}), \tag{3.14}$$

where $\psi(T_{orb})$ is the closed mirror map.

Table 3.1. Some invariants $N_{i,j}^{(0,1)}$ for the orbifold disk amplitude of $\mathbb{C}^3/\mathbb{Z}_3$ at zero framing

					i					
j	1	2	3	4	5	6	7	8	9	10
0	0	0	$-\frac{1}{3}$	0	0	$-\frac{1}{4}$	0	0	$-\frac{10}{27}$	0
1	1	0	0	$\frac{1}{2}$	0	0	$\frac{6}{7}$	0	0	2
2	0	$-\frac{1}{2}$	0	0	$-\frac{6}{5}$	0	0	$-\frac{15}{4}$	0	0
3	0	0	$\frac{2}{3}$	0	0	4	0	0	20	0
4	$\frac{1}{27}$	0	0	$-\frac{40}{27}$	0	0	$-\frac{154}{9}$	0	0	$-\frac{3400}{27}$
5	0	$-\frac{5}{54}$	0	0	$\frac{206}{45}$	0	0	$\frac{3215}{36}$	0	0
6	0	0	$\frac{10}{27}$	0	0	$-\frac{160}{9}$	0	0	$-\frac{4940}{9}$	0
7	$-\frac{29}{729}$	0	0	$-\frac{1432}{729}$	0	0	$\frac{19586}{243}$	0	0	$\frac{2820200}{729}$
8	0	$\frac{197}{1458}$	0	0	$\frac{15514}{1215}$	0	0	$-\frac{384575}{972}$	0	0
9	0	0	$-\frac{2}{3}$	0	0	$-\frac{292}{3}$	0	0	$\frac{5540}{3}$	0
10	$\frac{6607}{19683}$	0	0	$\frac{80456}{19683}$	0	0	$\frac{5544602}{6561}$	0	0	$-\frac{90503800}{19683}$
11	0	$-\frac{63107}{39366}$	0	0	$-\frac{945934}{32805}$	0	0	$-\frac{214690135}{26244}$	0	0
12	0	0	$\frac{8074}{729}$	0	0	$\frac{53768}{243}$	0	0	$\frac{21092500}{243}$	0
13	$-\frac{4736087}{531441}$	0	0	$-\frac{51705832}{531441}$	0	0	$-\frac{307254682}{177147}$	0	0	$-\frac{528718078600}{531441}$
14	0	$\frac{58248455}{1062882}$	0	0	$\frac{906117742}{885735}$	0	0	$\frac{8720423035}{708588}$	0	0

3.3. Quasi-modular transformations. Let us start by computing the amplitudes explicitly, using the quasi-modular transformation of the amplitudes from large radius to the orbifold point.

3.3.1. Disk amplitude For completeness, let us review the calculation of the orbifold disk amplitude, which was done in [16]. Recall that the disk amplitude is simply given by the Abel-Jacobi map

$$F^{(0,1)} = \int \log(y(x)) \frac{dx}{x}, \tag{3.15}$$

up to classical terms. $y(x)$ is obtained by solving the curve Σ_z and keeping the relevant branch:

$$y(x) = \frac{1}{2}\left(1 + x + \sqrt{(1+x)^2 - 4zx^3}\right). \tag{3.16}$$

We want to expand the Abel-Jacobi map at the orbifold point. Remark that the open mirror map (3.14) is linear in ψ. Hence we must plug in the open mirror map before expanding in the closed coordinate to get a meaningful expansion. This being done, we get the orbifold disk amplitude

$$F_{orb}^{(0,1)} = \sum_{i,j} \frac{1}{j!} N_{i,j}^{(0,1)} X_{orb}^i T_{orb}^j, \tag{3.17}$$

with the invariants given in Table 3.1.

3.3.2. Incorporating framing As we mentioned earlier, the calculation above was done for an outer brane with zero framing. However, for the orbifold disk amplitude the calculation can be easily generalized to arbitrary framing. From a Gromov-Witten point of view, framing corresponds to a choice of torus action in the localization process.

Table 3.2. Some invariants $N_{i,j}^{(0,1)}$ for the orbifold disk amplitude of $\mathbb{C}^3/\mathbb{Z}_3$ at general framing f

			i	
j	1	2	3	4
0	0	0	$-\frac{1}{3}$	0
1	1	0	0	$f + \frac{1}{2}$
2	0	$-f - \frac{1}{2}$	0	0
3	0	0	$3f^2 + 3f + \frac{2}{3}$	0
4	$\frac{1}{27}$	0	0	$-\frac{8}{27}\left(54f^3 + 81f^2 + 37f + 5\right)$
5	0	$-\frac{5}{54}(2f + 1)$	0	0
6	0	0	$\frac{5}{27}\left(9f^2 + 9f + 2\right)$	0
7	$-\frac{29}{729}$	0	0	$-\frac{8}{729}\left(1890f^3 + 2835f^2 + 1303f + 179\right)$
8	0	$\frac{197(2f+1)}{1458}$	0	0
9	0	0	$\frac{1}{3}\left(-9f^2 - 9f - 2\right)$	0
10	$\frac{6607}{19683}$	0	0	$\frac{8\left(102870f^3 + 154305f^2 + 71549f + 10057\right)}{19683}$
11	0	$-\frac{63107(2f+1)}{39366}$	0	0
12	0	0	$\frac{4037}{729}\left(9f^2 + 9f + 2\right)$	0
13	$-\frac{4736087}{531441}$	0	0	$-\frac{8\left(65783718f^3 + 98675577f^2 + 45818317f + 6463229\right)}{531441}$
14	0	$\frac{58248455(2f+1)}{1062882}$	0	0

Hence the calculation at arbitrary framing is relevant for comparison with localization computations in Gromov-Witten theory.

Recall from [16] that a framing transformation of the brane is given by reparameterizing the embedding of the fibers Σ_z in $(\mathbb{C}^*)^2$ by

$$(x_f, y_f) = (xy^f, y), \tag{3.18}$$

where (x_f, y_f) are the new coordinates, and $f \in \mathbb{Z}$ is the framing. In particular, the embedding of Σ_z becomes

$$\Sigma_z = \left\{ y_f^{3f+2} + y_f^{3f+1} + x_f y_f^{2f+1} + zx_f^3 = 0 \right\} \subset (\mathbb{C}^*)^2. \tag{3.19}$$

We compute the disk amplitude for this curve as

$$F_f^{(0,1)} = \int \log(y_f(x_f)) \frac{\mathrm{d}x_f}{x_f}, \tag{3.20}$$

where the function $y_f(x_f)$ is obtained by solving (3.19) for x_f (as a series expansion). Plugging in the mirror map, we obtain the invariants presented in Table 3.2, for general framing f.

As we mentioned already, the framing f is correlated to the choice of torus weights for localization of the Hodge integrals in Gromov–Witten theory. Renzo Cavalieri has implemented the Hodge integral calculation for the disk amplitude of $\mathbb{C}^3/\mathbb{Z}_3$ [21]. It turns out that the most natural choice of torus weights in Gromov–Witten theory does not correspond to $f = 0$, but rather to $f = -2/3$ (or $f = -1/3$). To ease comparisons, we present in Table 3.3 the disk invariants for $f = -2/3$. Rather amazingly, these invariants are precisely equal to the invariants computed by Cavalieri in orbifold Gromov–Witten theory! Since the Gromov–Witten calculation is done on the A-model side, this comparison also shows that our choice of open orbifold mirror map (3.14), which was argued in [16] from monodromy considerations, is correct, including the scale.

Table 3.3. Some invariants $N_{i,j}^{(0,1)}$ for the orbifold disk amplitude of $\mathbb{C}^3/\mathbb{Z}_3$ at framing $f = -2/3$

							i			
j	1	2	3	4	5	6	7	8	9	10
0	0	0	$-\frac{1}{3}$	0	0	$\frac{1}{12}$	0	0	$-\frac{1}{27}$	0
1	1	0	0	$-\frac{1}{6}$	0	0	$\frac{5}{63}$	0	0	$-\frac{4}{81}$
2	0	$\frac{1}{6}$	0	0	$-\frac{4}{45}$	0	0	$\frac{7}{108}$	0	0
3	0	0	0	0	0	0	0	0	0	0
4	$\frac{1}{27}$	0	0	$-\frac{8}{81}$	0	0	$\frac{35}{243}$	0	0	$-\frac{400}{2187}$
5	0	$\frac{5}{162}$	0	0	$-\frac{188}{1215}$	0	0	$\frac{875}{2916}$	0	0
6	0	0	0	0	0	0	0	0	0	0
7	$-\frac{29}{729}$	0	0	$-\frac{248}{2187}$	0	0	$\frac{5705}{6561}$	0	0	$-\frac{146800}{59049}$
8	0	$-\frac{197}{4374}$	0	0	$-\frac{10972}{32805}$	0	0	$\frac{221221}{78732}$	0	0
9	0	0	0	0	0	0	0	0	0	0
10	$\frac{6607}{19683}$	0	0	$\frac{10984}{59049}$	0	0	$\frac{889805}{177147}$	0	0	$-\frac{74714800}{1594323}$
11	0	$\frac{63107}{118098}$	0	0	$\frac{385132}{885735}$	0	0	$\frac{51307949}{2125764}$	0	0
12	0	0	0	0	0	0	0	0	0	0
13	$-\frac{4736087}{531441}$	0	0	$-\frac{6768584}{1594323}$	0	0	$\frac{17027675}{4782969}$	0	0	$-\frac{33798787600}{43046721}$
14	0	$-\frac{58248455}{3188646}$	0	0	$-\frac{381155716}{23914845}$	0	0	$\frac{3576521095}{57395628}$	0	0

It may seem however odd to assign a non-integral value to f; it would be interesting to understand this issue better. Presumably, the denominator of 3 comes from the orbifold $\mathbb{Z}_3$ action at the orbifold point — indeed, framing has so far only been interpreted from a large radius point of view in topological strings. Note however that non-integral framings have already been considered, although in a different context [27].

3.3.3. Annulus amplitude

We now want to compute the annulus amplitude, which is slightly more complicated, since it has non-trivial modular properties and transforms with a shift. More precisely, recall from (2.24) that the annulus transforms as

$$W_{orb}^{(0,2)}(p_1, p_2) = W^{(0,2)}(p_1, p_2) - 2\pi i\, u(p_1)(C\tau + D)^{-1} C u(p_2), \qquad (3.21)$$

where $W^{(0,2)}(p_1, p_2)$ is the large radius annulus amplitude and $(C\tau + D)^{-1}C$ comes from the modular transformation of the period matrix τ from large radius to the orbifold.

The first step consists then in computing the annulus amplitude at large radius, using Akemann's formula (2.26). This was done in [16,49], and we will not repeat the calculation here. What we need to do however is to analytically continue this result to the orbifold point, to obtain the first term on the right hand side of (3.21). The analytic continuation can be done directly in Akemann's formula, by expanding the branch points around $\psi = 0$. However, it is important to note that as for the disk amplitude, we must write things in terms of the open flat coordinates X_1 and X_2 — henceforth we will drop the subscript orb — before expanding in ψ, since the open mirror map is linear in ψ.

After using a few identities involving elliptic functions and Γ-functions, we obtain the following analytic continuation of the large radius annulus amplitude to the orbifold point, in orbifold flat coordinates X_1 and X_2:

$$W^{(0,2)} = dX_1 dX_2 \left(\frac{-9\sqrt{3}\,\Gamma(\frac{2}{3})^6}{8\,\pi^3} - \frac{81\,\psi\,\Gamma(\frac{2}{3})^{12}}{64\,\pi^6} + \psi^2 \left(\frac{1}{18} - \frac{243\sqrt{3}\,\Gamma(\frac{2}{3})^{18}}{512\,\pi^9} \right) \right.$$

$$+ \left(1 + \frac{9\sqrt{3}\,\psi\,\Gamma(\frac{2}{3})^6}{8\,\pi^3} + \frac{81\,\psi^2\,\Gamma(\frac{2}{3})^{12}}{64\,\pi^6} \right) X_1 + \left(-2\psi - \frac{9\sqrt{3}\,\psi^2\,\Gamma(\frac{2}{3})^6}{8\,\pi^3} \right) X_1{}^2$$

$$+ \left(1 + \frac{9\sqrt{3}\,\psi\,\Gamma(\frac{2}{3})^6}{8\,\pi^3} + \frac{81\,\psi^2\,\Gamma(\frac{2}{3})^{12}}{64\,\pi^6} \right.$$

$$+ \left(-3\psi - \frac{9\sqrt{3}\,\psi^2\,\Gamma(\frac{2}{3})^6}{8\,\pi^3} \right) X_1 + 5\,\psi^2\,X_1{}^2 \right) X_2$$

$$+ \left(-2\psi - \frac{9\sqrt{3}\,\psi^2\,\Gamma(\frac{2}{3})^6}{8\,\pi^3} + 5\,\psi^2\,X_1 + 3\,X_1{}^2 \right) X_2{}^2 + \cdots \right). \tag{3.22}$$

One can see that it is not rational, as expected; the non-rational terms should be cancelled by the shift in (3.21).

The next step is to compute the modular transformation between the large radius periods (T, T_D) and the orbifold periods $(T_{orb}, T_{orb,D})$. This can be done by standard analytic continuation, as in [1]. Define

$$c_1 = -\frac{1}{2\pi i}\frac{\Gamma(1/3)}{\Gamma(2/3)^2}, \qquad c_2 = \frac{1}{2\pi i}\frac{\Gamma(2/3)}{\Gamma(1/3)^2}, \qquad \beta = \frac{1}{(2\pi i)^3}, \qquad \omega = e^{2\pi i/3}. \tag{3.23}$$

We get the transformation

$$\begin{pmatrix} T_D \\ T \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\beta\omega^2}{c_1} & \frac{\beta\omega}{c_2} & \frac{1}{3} \\ -c_2 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_{orb,D} \\ T_{orb} \\ 1 \end{pmatrix}. \tag{3.24}$$

Note that this transformation is not quite symplectic, since its determinant is $-\beta$; that is, it changes the scale of the symplectic form. However, this can be taken into account by renormalizing the string coupling constant, as in [1].

Now the modular transformation that we want is given by the inverse of this matrix. We get that

$$C = -\frac{c_2}{\beta}, \qquad D = -\frac{\omega^2}{c_1}. \tag{3.25}$$

We also need the large radius period matrix τ, analytically continued around $\psi = 0$. By definition, it is given by

$$\tau(\psi) = \frac{\partial T_D}{\partial T} = \frac{\partial T_D/\partial\psi}{\partial T/\partial\psi}. \tag{3.26}$$

Using the transformation above between T_D, T and $T_{orb,D}, T_{orb}$, and expanding around $\psi = 0$, we get

$$\tau(\psi) = -\frac{(-1)^{\frac{1}{6}}}{\sqrt{3}} - \frac{i\,2^{\frac{1}{3}}\,\psi\,\Gamma(\frac{2}{3})^2}{\Gamma(\frac{1}{6})^2} - \frac{i\,\psi^2\,\Gamma(\frac{2}{3})^7}{2\pi\,\Gamma(\frac{1}{3})^5} + \mathcal{O}(\psi^3). \tag{3.27}$$

Table 3.4. Some invariants $N^{(0,2)}_{(i_1,i_2),j}$ for the orbifold annulus amplitude of $\mathbb{C}^3/\mathbb{Z}_3$

	(i_1, i_2)								
j	$(1,1)$	$(2,1)$	$(3,1)$	$(2,2)$	$(4,1)$	$(3,2)$	$(5,1)$	$(4,2)$	$(3,3)$
0	0	$\frac{1}{2}$	0	0	0	0	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{1}{3}$
1	0	0	$-\frac{2}{3}$	$-\frac{3}{4}$	0	0	0	0	0
2	$\frac{1}{9}$	0	0	0	$\frac{14}{9}$	$\frac{5}{3}$	0	0	0
3	0	$-\frac{1}{6}$	0	0	0	0	$-\frac{26}{5}$	$-\frac{16}{3}$	$-\frac{16}{3}$
4	0	0	$\frac{34}{81}$	$\frac{11}{36}$	0	0	0	0	0
5	$-\frac{1}{243}$	0	0	0	$-\frac{338}{243}$	$-\frac{65}{81}$	0	0	0
6	0	$-\frac{1}{54}$	0	0	0	0	$\frac{238}{45}$	$\frac{56}{27}$	$\frac{40}{27}$
7	0	0	$\frac{562}{2187}$	$\frac{197}{972}$	0	0	0	0	0
8	$\frac{391}{6561}$	0	0	0	$-\frac{17206}{6561}$	$-\frac{4261}{2187}$	0	0	0
9	0	$-\frac{29}{162}$	0	0	0	0	$\frac{3614}{135}$	$\frac{1552}{81}$	$\frac{160}{9}$
10	0	0	$\frac{31606}{59049}$	$\frac{8333}{26244}$	0	0	0	0	0
11	$-\frac{225595}{177147}$	0	0	0	$\frac{30802}{177147}$	$\frac{158125}{59049}$	0	0	0
12	0	$\frac{8455}{1458}$	0	0	0	0	$-\frac{44338}{1215}$	$-\frac{48104}{729}$	$-\frac{52712}{729}$
13	0	0	$-\frac{49954466}{1594323}$	$-\frac{15072793}{708588}$	0	0	0	0	0
14	$\frac{301065409}{4782969}$	0	0	0	$\frac{712334462}{4782969}$	$\frac{8347925}{1594323}$	0	0	0

Finally, we can compute the holomorphic differential $u(p)$ from the standard formula (2.31).

Putting all this together, and integrating, we obtain the orbifold annulus amplitude in flat orbifold coordinates X_1, X_2, T_{orb},

$$F^{(0,2)}_{orb} = \sum_{i_1,i_2,j} \frac{1}{j!} N^{(0,2)}_{(i_1,i_2),j} X_1^{i_1} X_2^{i_2} T_{orb}^j, \tag{3.28}$$

with the invariants $N^{(0,2)}_{(i_1,i_2),j}$ given in Table 3.4; the invariants are symmetric in (i_1, i_2).

The invariants are rational, as they should be. Moreover, it is easy to see that the amplitude is invariant under the $\mathbb{Z}_3$ orbifold monodromy. Indeed, the orbifold monodromy is given by

$$(T_{orb}, X_1, X_2) \mapsto (\omega T_{orb}, \omega^2 X_1, \omega^2 X_2), \qquad \omega = e^{2\pi i/3}. \tag{3.29}$$

Thus all terms in the expansion above are monodromy invariant.

In Table 3.5 we also present some results for the corresponding framed invariants.

3.3.4. Higher amplitudes Computing the higher amplitudes directly using the modular shift is rather complicated, partially because of all the elliptic functions involved in the calculation. It is much simpler to use the functional expressions to compute the orbifold amplitudes. We have however checked that the genus 0, three-hole amplitude computed through the shift also matches the functional calculation, but we will not present the calculation here for brevity.

3.4. Calculation using the functionals. Let us now use the functional expressions for the amplitude derived in the previous section to compute the open orbifold amplitudes. First, we need to specify what the functional $G[T; z]$ is for the curve Σ_z given by (3.1).

Table 3.5. Some invariants $N^{(0,2)}_{(i_1,i_2),j}$ for the framed orbifold annulus amplitude of $\mathbb{C}^3/\mathbb{Z}_3$

	(i_1, i_2)			
j	(1,1)	(2,1)	(3,1)	(2,2)
0	0	$\frac{1}{2} + f$	0	0
1	0	0	$-\frac{2}{3} + f(2f-1)$	$-\frac{3}{4} + f(2f-1)$
2	$\frac{1}{9}$ $+f(f+1)$	0	0	0
3	0	$-\frac{1}{6}$ $-\frac{1}{3}f\left(12f^2 + 18f + 7\right)$	0	0
4	0	0	$\frac{34}{81}$ $-\frac{f(2646f^3+2592f^2+538f-53)}{27}$	$\frac{11}{36}$ $-\frac{f(2727f^3+2754f^2+637f-35)}{27}$
5	$-\frac{1}{243} + \frac{5f(1+f)}{27}$	0	0	0
6	0	$-\frac{1}{54}$ $-\frac{f(31+90f+60f^2)}{27}$	0	0

3.4.1. Generalities First, from the embedding of the elliptic curve (3.1), we obtain

$$\sigma(x) = (x + 1)^2 - 4x^3 z, \tag{3.30}$$

and the discriminant

$$\Delta = 1 + 27z. \tag{3.31}$$

We claim that the functional $G[T; z]$ reads

$$G[T; z] = -\frac{1}{z^2 C_{zzz}} \frac{\partial}{\partial z} \left(4 \log \frac{\partial T}{\partial z} + \frac{1}{3} \log \Delta + 5 \log z \right), \tag{3.32}$$

where

$$C_{zzz} = \frac{\partial^3 F^{(0)}}{\partial z^3} = \frac{3}{z^3 \Delta} \tag{3.33}$$

is the Yukawa coupling in the local variable z. Let us sketch the derivation of this functional formula.

The genus one amplitude $F^{(1)}$ was defined in Definition 2.5. In our context, one can show that (2.15) becomes, up to a constant term,

$$F^{(1)} = -\log \eta(\tau) - \frac{1}{24} \log \tilde{\Delta}(\psi), \tag{3.34}$$

where $\tilde{\Delta}(\psi) = 27 + \psi^3$ is the discriminant in terms of $\psi = z^{-1/3}$, and $\eta(\tau)$ is the Dedekind η-function. Alternatively, $F^{(1)}$ can also be expressed as [13]

$$F^{(1)} = -\frac{1}{2} \log \frac{\partial T}{\partial \psi} - \frac{1}{12} \log \tilde{\Delta}(\psi). \tag{3.35}$$

Combining the two formulae, we obtain

$$\log \eta(\tau) = \frac{1}{2} \log \frac{\partial T}{\partial \psi} + \frac{1}{24} \log \tilde{\Delta}(\psi). \tag{3.36}$$

Now the second Eisenstein series $E_2(\tau)$ is related to the Dedekind η-function by:

$$E_2(\tau) = 24 \frac{d}{d\tau} \log \eta(\tau). \qquad (3.37)$$

As a result, we get

$$E_2(\tau) = \frac{\partial}{\partial \tau} \left(12 \log \frac{\partial T}{\partial \psi} + \log \tilde{\Delta}(\psi) \right). \qquad (3.38)$$

Using the fact that

$$\tau = \frac{\partial^2 F^{(0)}}{\partial T^2}, \qquad \tilde{\Delta}(z^{-1/3}) = \frac{\Delta}{z}, \qquad (3.39)$$

we obtain

$$E_2(\tau) = \left(\frac{\partial T}{\partial z} \right)^2 \frac{1}{C_{zzz}} \frac{\partial}{\partial z} \left(12 \log \frac{\partial T}{\partial z} + \log \Delta + 15 \log z \right). \qquad (3.40)$$

Finally, recall that $G[T; z]$ is defined by

$$G[T; z] = \frac{E_2(\tau)}{3\omega_1^2}. \qquad (3.41)$$

By direct computation, we can write ω_1 as a functional of T and z,

$$\omega_1 = \mathrm{i}z \frac{\partial T}{\partial z}, \qquad (3.42)$$

and we obtain the final formula for $G[T; z]$ given in (3.32).

  With this explicit formula for the functional $G[T; z]$, we can proceed with the calculation of the higher amplitudes, using our ansatz (2.39). As explained previously, to compute the amplitudes at the orbifold point, all that we need to do is to input the period T_{orb} corresponding to the flat parameter at the orbifold point in the functional.

3.4.2. Annulus amplitude The functional expression for the annulus amplitude was obtained in (2.38), using the expression (3.32) for $G[T; z]$. For the curve Σ_z under consideration, the rational function $f_0^{(0,2)}(x_1, x_2)$ can be computed, and reads

$$f_0^{(0,2)}(x_1, x_2) = \frac{6 + 6x_2 + x_2^2 + x_1^2(1 - 12x_2 z) + x_1(6 + 4x_2 - 12x_2^2 z)}{3(x_1 - x_2)^2}. \qquad (3.43)$$

  All that one needs to do to obtain the orbifold annulus amplitude, is to do the change of variable $z = \psi^{-3}$, replace the open moduli x_1 and x_2 by the open orbifold mirror map $x_{1,2} = X_{1,2}\psi$, where X_1 and X_2 are the open flat coordinates, and insert the closed flat orbifold coordinate $T = T_{orb}(\psi)$ in the functional. Then, we plug in the closed mirror map in the result and expand in X_1, X_2 and T_{orb} to obtain the orbifold annulus amplitude. It is easy to show that we obtain precisely (3.28) with the invariants of Table 3.4; note however how much simpler the calculation was.

Table 3.6. Some invariants $N_{i,j}^{(1,1)}$ for the genus 1, 1 hole orbifold amplitude of $\mathbb{C}^3/\mathbb{Z}_3$

					i				
j	1	2	3	4	5	6	7	8	9
0	0	0	$\frac{5}{24}$	0	0	$\frac{11}{8}$	0	0	$\frac{85}{12}$
1	$\frac{1}{72}$	0	0	$-\frac{5}{9}$	0	0	$-\frac{77}{12}$	0	0
2	0	$-\frac{1}{36}$	0	0	$\frac{25}{12}$	0	0	$\frac{110}{3}$	0
3	0	0	$\frac{1}{12}$	0	0	-10	0	0	$-\frac{495}{2}$
4	$\frac{1}{1944}$	0	0	$-\frac{86}{243}$	0	0	$\frac{18823}{324}$	0	0
5	0	$-\frac{11}{972}$	0	0	$\frac{3301}{1620}$	0	0	$-\frac{127415}{324}$	0
6	0	0	$\frac{31}{324}$	0	0	$-\frac{412}{27}$	0	0	$\frac{162755}{54}$
7	$\frac{475}{52488}$	0	0	$-\frac{5210}{6561}$	0	0	$\frac{1237285}{8748}$	0	0
8	0	$-\frac{223}{26244}$	0	0	$\frac{307847}{43740}$	0	0	$-\frac{6757145}{4374}$	0
9	0	0	$-\frac{1}{12}$	0	0	$-\frac{610}{9}$	0	0	$\frac{344095}{18}$
10	$-\frac{395585}{1417176}$	0	$\frac{172678}{177147}$	0	0	0	$\frac{168774025}{236196}$	0	0
11	0	$\frac{712639}{708588}$	0	0	$-\frac{5242661}{1180980}$	0	0	$-\frac{1966276115}{236196}$	0
12	0	0	$-\frac{38945}{8748}$	0	0	$-\frac{62488}{729}$	0	0	$\frac{158337275}{1458}$
13	$\frac{640118305}{38263752}$	0	0	$\frac{133378114}{4782969}$	0	0	$\frac{23152439695}{6377292}$	0	0
14	0	$-\frac{1726238977}{19131876}$	0	0	$-\frac{11317800859}{31886460}$	0	0	$-\frac{152933889775}{1594323}$	0

3.4.3. Genus 1*, one-hole* The amplitude has the form predicted by the ansatz (2.39). By comparing with the result obtained through the recursion, we can fix the functions $f_i^{(1,1)}(x)$. We obtain

$$W^{(1,1)} = \frac{dx}{\sqrt{\sigma(x)}\Delta}\left(\frac{9}{32}G^2[T;z] + f_1^{(1,1)}(x)G[T;z] + f_0^{(1,1)}(x)\right), \qquad (3.44)$$

with the functions:

$$f_1^{(1,1)}(x) = \frac{x(1+x)\Delta}{8\sigma(x)},$$

$$f_0^{(1,1)}(x) = \frac{1}{96\sigma(x)^2}(1 + 36z + 4x(1+36z) + 16x^6z^2(1+36z)$$
$$+ 6x^2(1 + 46z + 270z^2) + x^4(1 + 56z + 396z^2)$$
$$+ 4x^3(1 + 55z + 495z^2) + 4x^5z(1 + 57z + 1296z^2)). \qquad (3.45)$$

Doing the transformations as above to go to the orbifold point, we obtain the amplitude

$$F_{orb}^{(1,1)} = \sum_{i,j}\frac{1}{j!}N_{i,j}^{(1,1)}X^i T_{orb}^j, \qquad (3.46)$$

with the invariants given in Table 3.6.

3.4.4. Genus 0*, three-hole* The amplitude has again the form predicted by the ansatz (2.39). We can fix the functions $f_i^{(0,3)}(x_1, x_2, x_3)$ by comparing with the recursion, and we obtain

$$W^{(0,3)} = \frac{dx_1 dx_2 dx_3}{\sqrt{\sigma(x_1)\sigma(x_2)\sigma(x_3)}\Delta}\left(\frac{9}{64}G^3[T;z] + f_2^{(0,3)}(x_1, x_2, x_3)G^2[T;z]\right.$$
$$\left. + f_1^{(0,3)}(x_1, x_2, x_3)G[T;z] + f_0^{(0,3)}(x_1, x_2, x_3)\right). \qquad (3.47)$$

Table 3.7. Some invariants $N^{(0,3)}_{(i_1,i_2,i_3),j}$ for the genus 0, 3 hole orbifold amplitude of $\mathbb{C}^3/\mathbb{Z}_3$

				(i_1, i_2, i_3)			
j	$(1, 1, 1)$	$(2, 1, 1)$	$(3, 1, 1)$	$(2, 2, 1)$	$(4, 1, 1)$	$(3, 2, 1)$	$(2, 2, 2)$
0	$\frac{2}{3}$	0	0	0	$\frac{4}{3}$	1	$\frac{9}{8}$
1	0	$-\frac{5}{6}$	0	0	0	0	0
2	0	0	$\frac{52}{27}$	$\frac{23}{12}$	0	0	0
3	$-\frac{1}{27}$	0	0	0	$-\frac{176}{27}$	$-\frac{19}{3}$	$-\frac{49}{8}$
4	0	$\frac{13}{162}$	0	0	0	0	0
5	0	0	$-\frac{124}{729}$	$-\frac{11}{324}$	0	0	0
6	$-\frac{1}{81}$	0	0	0	$-\frac{32}{81}$	$-\frac{37}{27}$	$-\frac{133}{72}$
7	0	$\frac{397}{4374}$	0	0	0	0	0
8	0	0	$-\frac{17972}{19683}$	$-\frac{7303}{8748}$	0	0	0
9	$\frac{37}{243}$	0	0	0	$\frac{2480}{243}$	$\frac{709}{81}$	$\frac{1771}{216}$
10	0	$-\frac{92273}{118098}$	0	0	0	0	0
11	0	0	$\frac{3393164}{531441}$	$\frac{1584895}{236196}$	0	0	0
12	$-\frac{42703}{6561}$	0	0	0	$-\frac{475424}{6561}$	$-\frac{57857}{729}$	$-\frac{172613}{1944}$
13	0	$\frac{120276571}{3188646}$	0	0	0	0	0
14	0	0	$-\frac{4470350924}{14348907}$	$-\frac{1939962841}{6377292}$	0	0	0

The functions $f_i^{(0,3)}(x_1, x_2, x_3)$ are rather complicated; we present them in Appendix B.

Doing the transformations as above to go to the orbifold point, we obtain the amplitude

$$F_{orb}^{(0,3)} = \sum_{i_1,i_2,i_3,j} \frac{1}{j!} N^{(0,3)}_{(i_1,i_2,i_3),j} X_1^{i_1} X_2^{i_2} X_3^{i_3} T_{orb}^j, \tag{3.48}$$

with the invariants given in Table 3.7. The invariants are symmetric in (i_1, i_2, i_3).

3.5. Conifold point. So far we considered the A- and B-model amplitudes in the two distinct phases of $\mathcal{M}$, namely the large radius phase and the orbifold one. There is however a third point around which the amplitudes have an interesting expansion, which is the *conifold point*. This is not a limit point of a phase of $\mathcal{M}$; rather, it is a singular point of the moduli space, where the target space of the A-model develops a conifold singularity.[6] This point is located at $z = -\frac{1}{27}$.

It is generally interesting to expand the amplitudes near the conifold point. For instance, the leading behavior of the closed amplitudes $F_{con}^{(g)}$ expanded at the conifold point can be understood as the amplitudes of the non-critical $c = 1$ string at the self-dual radius [35]. Moreover, the amplitudes $F_{con}^{(g)}$ seem to possess a universal gap, as discovered in [42]. That is, the leading behavior of the closed amplitudes is of the form:

$$F_{con}^{(g)} = \frac{B_{2g}}{2g(2g-2)T_{con}^{2g-2}} + k_1^{(g)} T_{con} + \mathcal{O}(T_{con}^2), \tag{3.49}$$

where the B_n are the Bernoulli numbers and T_{con} is the vanishing period at the conifold. This feature is rather striking, and very useful computationally. Indeed, one of the most

---

[6] In the gauged linear sigma model description of the A-model, at the conifold point new massless modes appear, which defines a new branch of vacua.

effective approaches for computing closed amplitudes is by directly integrating [38] the holomorphic anomaly equation of [12], using the polynomial structure of the amplitudes proposed in [57]. However, the holomorphic anomaly equation is not complete; at each genus one needs to fix a finite number of constants (the holomorphic ambiguity) using extra data. In conjunction with the leading behavior of the amplitudes, the gap behavior at the conifold point — more precisely the absence of the $2g - 3$ subleading negative powers in the T_{con} expansion — imposes $2g - 2$ such extra conditions, which have been shown to completely fix the holomorphic ambiguity in many local geometries. In the compact setting, they allow computation of closed amplitudes to very high genus [43].

One may wonder if this approach has an open counterpart. So far, we relied entirely on the recursion formalism to compute open amplitudes. As we have seen, while this formalism is very satisfactory conceptually, it is rather cumbersome computationally. Direct integration of the open holomorphic anomaly equations — recently derived in [33] in the local setting — would provide an alternative method to compute the open amplitudes. In particular, one could hope that a gap behavior exists for the open amplitudes expanded at the conifold point, providing sufficient boundary conditions to fix the holomorphic ambiguity.

This is surely enough motivation for studying in more detail the open amplitudes near the conifold point. In what follows we present general properties of the amplitudes; technical and computational aspects are relegated to Appendix C.

Consider as usual the moduli space $\Sigma \to \mathcal{M}$. As mentioned before, the conifold point in the closed moduli space $\mathcal{M}$ is located at $z = -1/27$; a good local coordinate is

$$w = 27z + 1. \tag{3.50}$$

Since we are computing open amplitudes, we must also specify where we expand the amplitudes in the open moduli space Σ_w. At the conifold point $w = 0$, it turns out that two of the branch points of the x-projection of the curve Σ_w collapse to the same value, $x = -1/3$. Instead of expanding the open amplitudes near $x = 0$, we will now expand the amplitudes near this critical point $x = -1/3$, using a new local coordinate centered at this point:[7]

$$p = \frac{1}{x} + 3. \tag{3.51}$$

On the mirror A-model side, expanding the amplitudes near this point should correspond to considering branes located near a vertex of the toric diagram.

The open B-model amplitudes expanded near this critical point should correspond, at leading order, to $c = 1$ string amplitudes at the self-dual radius,[8] which are in turn equivalent to Gaussian matrix model amplitudes [29]. More precisely, the expected leading behavior of the open amplitudes near the critical point is [14]:

$$F^{(g,h)} \sim T_{con}^{2-2g-h} \tilde{F}^{(g,h)}, \tag{3.52}$$

---

[7] Note however that this critical point is a singular limit of the curve Σ_w, hence one has to choose an appropriate set of coordinates to smooth out the singularity. In particular, one must consider a double–scaling limit, where the open coordinate p is rescaled with the closed modulus, as explained in [34]. We will come back to that in the explicit computations in Appendix C.

[8] Note that such critical points have already been considered in the context of matrix models. In [28] it was proposed that $c = 1$ amplitudes can be obtained in a two-cut matrix model by considering the critical limit where the two cuts touch each other.

where T_{con} is the closed flat coordinate near the conifold point $w = 0$, which corresponds to the vanishing period. The amplitudes $\tilde{F}^{(g,h)}$, which are independent of T_{con}, are to be identified with the amplitudes for FZZT branes in the $c = 1$ string at the self-dual radius. Indeed, as noticed in a similar context in [49], in the critical limit toric branes should become FZZT branes.

The leading behavior (3.52) is the open analog to the leading behavior for the closed amplitudes, as presented in (3.49). Recalling the discussion above for the closed amplitudes, the open amplitudes would possess a gap if the subleading terms in T_{con} with negative exponents vanished.

We can use the recursion and the formalism developed in Sect. 2 to compute the B-model amplitudes explicitly at the critical point; we report this calculation in Appendix C. The calculation shows that the amplitudes indeed possess the expected leading behavior in T_{con}. However, the subleading terms in T_{con} are not vanishing, in contrast with the closed amplitudes. As a result, we conclude that in the open case, there is no simple gap behavior at the conifold. This renders the use of the direct integration of the holomorphic anomaly equations as a method to solve for the amplitudes rather limited in the open case, since one lacks the boundary conditions provided by the gap behavior and required to fix the holomorphic ambiguity.

A word of caution to end this section; as we discuss in Appendix C, it is not clear to us how to fix the open flat coordinate near the critical point $(w, p) = (0, 0)$. This prevents us from providing unambiguous results for the open amplitudes near the conifold point. It would be interesting to clarify these issues further.

A. Landau-Ginzburg vs Sigma Model

In this Appendix we explain the relation between the standard Landau-Ginzburg mirrors to toric threefolds and the sigma models described previously. We follow the argument presented by Hori, Iqbal and Vafa in p.93 of [40].

Consider the A-twisted sigma model on a (noncompact) toric Calabi-Yau threefold X defined by the toric charge vectors Q^a, $a = 1, \ldots, k$. Its mirror [36,41] — see also [24] for a clear explanation — is a B-twisted Landau-Ginzburg model on the family of algebraic tori $\pi : V \to \mathcal{M}$, with $V = (\mathbb{C}^*)^{3+k}$, and $\mathcal{M} = (\mathbb{C}^*)^k$, with projection map

$$\pi : (y_0, \ldots, y_{k+2}) \mapsto (z_1, \ldots, z_k) = \left(\prod_{i=0}^{k+2} y_i^{Q_i^1}, \ldots, \prod_{i=0}^{k+2} y_i^{Q_i^k} \right). \qquad (A.1)$$

The Landau-Ginzburg superpotential $W : V \to \mathbb{C}$ reads

$$W = \sum_{i=0}^{k+2} y_i. \qquad (A.2)$$

Choose local coordinates y_0, y_1, y_2 on the fiber $V_z = \pi^{-1}(z_1, \ldots, z_k)$, and use the projection map π to rewrite the superpotential as

$$W_z := W\big|_{V_z} = G(y_0, y_1, y_2; z), \qquad (A.3)$$

where $G(y_0, y_1, y_2; z)$ is a homogeneous Laurent polynomial in (y_0, y_1, y_2) of degree 1.

Consider now the Landau-Ginzburg model on $V'_z = (\mathbb{C}^*)^3 \times (\mathbb{C}^2)$, with superpotential

$$W'_z = G(y_0, y_1, y_2; z) - ww'. \tag{A.4}$$

By Knörrer periodicity, the category of B-branes in the Landau-Ginzburg model (V_z, W_z) is equivalent to the category of B-branes in the Landau-Ginzburg model (V'_z, W'_z) [51]. The "periods" of the Landau-Ginzburg model consists in integrals of the form

$$\int e^{G(y_0, y_1, y_2; z) - ww'} \mathrm{d} \log y_0 \wedge \mathrm{d} \log y_1 \wedge \mathrm{d} \log y_2 \wedge \mathrm{d}w \wedge \mathrm{d}w'. \tag{A.5}$$

Since y_0 is a $\mathbb{C}^*$-coordinate, we can define new coordinates $\tilde{y}_i = y_i/y_0$, $i = 1, 2$, and $\tilde{w} = w/y_0$. The superpotential becomes

$$W'_z = y_0(G(1, \tilde{y}_1, \tilde{y}_2; z) - \tilde{w}w'), \tag{A.6}$$

and the periods now take the form

$$\int e^{y_0(G(1, \tilde{y}_1, \tilde{y}_2; z) - \tilde{w}w')} \mathrm{d}y_0 \wedge \mathrm{d} \log \tilde{y}_1 \wedge \mathrm{d} \log \tilde{y}_2 \wedge \mathrm{d}\tilde{w} \wedge \mathrm{d}w'. \tag{A.7}$$

Note that $\mathrm{d} \log y_0$ has become $\mathrm{d}y_0$, due to the rescaling of w. As a result, we can "integrate out" y_0, and we obtain a delta function

$$\delta(G(1, \tilde{y}_1, \tilde{y}_2; z) - \tilde{w}w'). \tag{A.8}$$

In other words, the B-twisted Landau-Ginzburg model "localizes" on the B-twisted sigma model on the family of noncompact threefolds $Y \to \mathcal{M}$ with fiber

$$Y_z = \{ww' = H(x, y; z)\} \subset (\mathbb{C}^*)^2 \times (\mathbb{C}^2), \tag{A.9}$$

where we redefined $x = \tilde{y}_1$, $y = \tilde{y}_2$, $w = \tilde{w}$ and $H(x, y; z) := G(1, x, y; z)$.

Note however that we have only shown equivalence of the period integrals. What one would need to show is the equivalence of the category of B-branes for both models, as in the first step involving Knörrer periodicity. In other words, there should be an equivalence between the category of B-branes of the Landau-Ginzburg model (V'_z, W'_z), which is generally understood as the category of matrix factorizations, and the derived category of coherent sheaves of Y_z. It would be very interesting to understand this relation better, in the spirit of the Landau-Ginzburg/Calabi-Yau correspondence which was derived in [52].

B. Functions $f_i^{(0,3)}(x_1, x_2, x_3)$ for the Genus 0, 3 Hole Amplitude

We present in this Appendix the functions $f_i^{(0,3)}(x_1, x_2, x_3)$, $i = 0, 1, 2$ entering into the expression for the amplitude $W^{(0,3)}$ (3.47):

$$
\begin{aligned}
f_2^{(0,3)} = {} & \frac{1}{64\sigma(x_1)\sigma(x_2)\sigma(x_3)}(4z(4z(12zx_3^3 + (108z+1)x_3^2 + 2(54z-1)x_3 - 3)x_2^3 + (4z(108z+1)x_3^3 \\
& -(216z+5)x_3^2 - 6(54z+1)x_3 - 108z-1)x_2^2 + 2(4z(54z-1)x_3^3 - 3(54z+1)x_3^2 \\
& -2(108z+1)x_3 - 54z+1)x_2 - 12zx_3^3 - (108z+1)x_3^2 + (2-108z)x_3 + 3)x_1^3 \\
& +(4z(4z(108z+1)x_3^3 - (216z+5)x_3^2 - 6(54z+1)x_3 - 108z-1)x_2^3 \\
& +(-4z(216z+5)x_3^3 + 9(36z+1)x_3^2 + 2(270z+7)x_3 + 216z+5)x_2^2 + (-24z(54z+1)x_3^3 \\
& +2(270z+7)x_3^2 + 4(216z+5)x_3 + 324z+6)x_2 - 4z(108z+1)x_3^3 + (216z+5)x_3^2 + 108z \\
& +6(54z+1)x_3 + 1)x_1^2 + 2(4z(4z(54z-1)x_3^3 - 3(54z+1)x_3^2 - 2(108z+1)x_3 - 54z+1)x_2^3 \\
& +(-12z(54z+1)x_3^3 + (270z+7)x_3^2 + 2(216z+5)x_3 + 162z+3)x_2^2 + (-8z(108z+1)x_3^3 \\
& +2(216z+5)x_3^2 + 12(54z+1)x_3 + 216z+2)x_2 + 4(1-54z)zx_3^3 + 3(54z+1)x_3^2 + 54z \\
& +(216z+2)x_3 - 1)x_1 + 12zx_3^3 + 108zx_3^2 + x_3^2 + 108zx_3 - 2x_3 \\
& -4zx_2^3(12zx_3^3 + (108z+1)x_3^2 + 2(54z-1)x_3 - 3) + x_2(8(1-54z)zx_3^3 + 6(54z+1)x_3^2 \\
& +(432z+4)x_3 + 108z-2) + x_2^2(-4z(108z+1)x_3^3 + (216z+5)x_3^2 \\
& +6(54z+1)x_3 + 108z+1) - 3), \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(B.1)}
\end{aligned}
$$

$$
\begin{aligned}
f_1^{(0,3)} = {} & \frac{1}{192\sigma(x_1)\sigma(x_2)\sigma(x_3)}(-4z(4z(12z(180z+7)x_3^3 + (2592z^2 - 12z - 5)x_3^2 - 2x_3 + 108z+3)x_2^3 \\
& +(4z(2592z^2 - 12z - 5)x_3^3 + (1296z^2+48z+1)x_3^2 + 6(648z^2 - 12z - 1)x_3 + 1296z^2 - 168z - 7)x_2^2 \\
& -2(4zx_3^3 + (-1944z^2 + 36z + 3)x_3^2 + (-1944z^2 + 360z + 14)x_3 + 324z+11)x_2 \\
& +12z(36z+1)x_3^3 + (1296z^2 - 168z - 7)x_3^2 - 3(144z+5) - 2(324z+11)x_3)x_1^3 \\
& +(-4z(4z(2592z^2 - 12z - 5)x_3^3 + (1296z^2 + 48z + 1)x_3^2 \\
& +6(648z^2 - 12z - 1)x_3 + 1296z^2 - 168z - 7)x_2^2 + (-4z(1296z^2 + 48z + 1)x_3^3 \\
& +2(10368z^2 + 492z + 5)x_3 + 9072z^2 + 9(36zx_3 + x_3)^2 + 336z+1)x_2^2 \\
& +2(12z(-648z^2 + 12z + 1)x_3^3 + (10368z^2 + 492z + 5)x_3^2 + 2(7452z^2 + 276z + 1)x_3 \\
& +5832z^2 + 108z - 3)x_2 + 4z(-1296z^2 + 168z + 7)x_3^3 + 5184z^2 + (9072z^2 + 336z + 1)x_3^2 \\
& -24z + 6(1944z^2 + 36z - 1)x_3 - 7)x_1^2 + 2(4z(4zx_3^3 + (-1944z^2 + 36z + 3)x_3^2 \\
& +(-1944z^2 + 360z + 14)x_3 + 324z+11)x_2^3 + (12z(-648z^2 + 12z + 1)x_3^3 \\
& +(10368z^2 + 492z + 5)x_3^2 + 2(7452z^2 + 276z + 1)x_3 + 5832z^2 + 108z - 3)x_2^2 \\
& -2(4z(972z^2 - 180z - 7)x_3^3 - (7452z^2 + 276z + 1)x_3^2 + (6 - 5832z^2)x_3 - 972z^2 + 180z + 7)x_2 \\
& +4z(324z+11)x_3^3 + 3(1944z^2+36z-1)x_3^2 - 324z + 2(972z^2 - 180z - 7)x_3 - 11)x_1 + 1728z^2x_3^3 \\
& +60zx_3^3 + 5184z^2x_3^2 - 24zx_3^2 - 7x_3^2 - 432z - 648zx_3 - 22x_3 \\
& +4zx_2^3(-12z(36z+1)x_3^3 + (-1296z^2 + 168z + 7)x_3^2 + (648z + 22)x_3 + 432z + 15) \\
& +2x_2(4z(324z+11)x_3^3 + 3(1944z^2 + 36z - 1)x_3^2 + 2(972z^2 - 180z - 7)x_3 - 324z - 11) \\
& +x_2^2(4z(-1296z^2 + 168z + 7)x_3^3 + (9072z^2 + 336z + 1)x_3^2 + 6(1944z^2 + 36z - 1)x_3 \\
& +5184z^2 - 24z - 7) - 15), \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(B.2)}
\end{aligned}
$$

$$f_0^{(0,3)} = \frac{1}{576\sigma(x_1)\sigma(x_2)\sigma(x_3)}(4z(4z(36z(144z^2 + 32z + 1)x_3^3 + (7776z^2 + 480z + 7)x_3^2$$

$$+2(4536z^2 + 306z + 5)x_3 + 3(864z^2 + 60z + 1))x_2^3 + (4z(7776z^2 + 480z + 7)x_3^3 + (108864z^3$$

$$+7920z^2 + 168z + 1)x_3^2 + 6(7776z^3 + 1368z^2 + 66z + 1)x_3 + 3888z^2 + 276z + 5)x_2^2$$

$$+2(4z(4536z^2 + 306z + 5)x_3^3 + 3(7776z^3 + 1368z^2 + 66z + 1)x_3^2$$

$$+2(4212z^2 + 288z + 5)x_3 + 5184z^2 + 378z + 7)x_2 + 12z(864z^2 + 60z + 1)x_3^3 + 9(24z + 1)^2$$

$$+(3888z^2 + 276z + 5)x_3^2 + 2(5184z^2 + 378z + 7)x_3)x_1^3$$

$$+(4z(4z(7776z^2 + 480z + 7)x_3^3 + (108864z^3 + 7920z^2 + 168z + 1)x_3^2$$

$$+6(7776z^3 + 1368z^2 + 66z + 1)x_3 + 3888z^2 + 276z + 5)x_2^3 + (124416z^3 + 8496z^2$$

$$+4(108864z^3 + 7920z^2 + 168z + 1)x_3^3 z + 120z + 3(186624z^4 + 53568z^3 + 3888z^2 + 108z + 1)x_3^2$$

$$+(264384z^3 + 20160z^2 + 444z + 2)x_3 - 1)x_2^2 + 2(46656z^3 + 2376z^2$$

$$+12(7776z^3 + 1368z^2 + 66z + 1)x_3^3 z - 54z + (132192z^3 + 10080z^2 + 222z + 1)x_3^2$$

$$+2(81648z^3 + 5292z^2 + 60z - 1)x_3 - 3)x_2 + 4z(3888z^2 + 276z + 5)x_3^3$$

$$+(124416z^3 + 8496z^2 + 120z - 1)x_3^2 - 132z + 6(15552z^3 + 792z^2 - 18z - 1)x_3 - 5)x_1^2$$

$$+2(4z(4z(4536z^2 + 306z + 5)x_3^3 + 3(7776z^3 + 1368z^2 + 66z + 1)x_3^2$$

$$+2(4212z^2 + 288z + 5)x_3 + 5184z^2 + 378z + 7)x_2^3 + (46656z^3 + 2376z^2$$

$$+12(7776z^3 + 1368z^2 + 66z + 1)x_3^3 z - 54z + (132192z^3 + 10080z^2 + 222z + 1)x_3^2$$

$$+2(81648z^3 + 5292z^2 + 60z - 1)x_3 - 3)x_2^2 + 2(4z(4212z^2 + 288z + 5)x_3^3$$

$$+(81648z^3 + 5292z^2 + 60z - 1)x_3^2 + 6(5832z^3 + 108z^2 - 30z - 1)x_3 - 324z^2 - 144z - 5)x_2$$

$$+4z(5184z^2 + 378z + 7)x_3^3 + 2592z^2 + 3(15552z^3 + 792z^2 - 18z - 1)x_3^2 - 90z$$

$$-2(324z^2 + 144z + 5)x_3 - 7)x_1 + 20736z^3 x_3^3 + 1728z^2 x_3^3 + 36zx_3^3 + 10368z^2 - 132zx_3^2$$

$$-5x_3^2 + 144z + 5184z^2 x_3 - 180zx_3 - 14x_3 + 4zx_2^3(12z(864z^2 + 60z + 1)x_3^3$$

$$+(3888z^2 + 276z + 5)x_3^2 + 2(5184z^2 + 378z + 7)x_3 + 9(24z + 1)^2)$$

$$+x_2^2(4z(3888z^2 + 276z + 5)x_3^3 + (124416z^3 + 8496z^2 + 120z - 1)x_3^2$$

$$+6(15552z^3 + 792z^2 - 18z - 1)x_3 - 132z - 5) + 2x_2(4z(5184z^2 + 378z + 7)x_3^3$$

$$+3(15552z^3 + 792z^2 - 18z - 1)x_3^2 - 2(324z^2 + 144z + 5)x_3 + 2592z^2 - 90z - 7) - 9). \quad \text{(B.3)}$$

C. Conifold Expansion

In this Appendix we provide detailed calculations supporting the discussion of the open amplitudes near the conifold point in Subsect. 3.5.

Before computing the amplitudes, one needs to fix the open and closed mirror maps near the critical point $(w, p) = (0, 0)$ in the moduli space. First, the closed mirror map

can be easily obtained by performing analytic continuation of the large radius periods to the conifold point $w = 0$. One obtains

$$
\begin{aligned}
T_{con} &= w + \frac{11w^2}{18} + \frac{109w^3}{243} + \frac{9389w^4}{26244} + \cdots, \\
T_{con}^D &= w\log(w) + \left(\frac{11\log(w)}{18} + \frac{7}{12}\right)w^2 + \left(\frac{109\log(w)}{243} + \frac{877}{1458}\right)w^3 + \cdots.
\end{aligned}
$$
(C.1)

The vanishing period T_{con} at $w = 0$ gives the closed flat coordinate at the conifold point, and the closed mirror map is obtained as usual by inverting the series.

The open mirror map is much more delicate. As explained in [16], it should be given by a linear combination of solutions of the extended Picard-Fuchs system. That is, by a linear combination of the constant solution, the closed periods (C.1), the solution (see [16]):

$$
u = \log(p - 3) + \frac{1}{3}\log\left(\frac{w - 1}{27}\right),
$$
(C.2)

and the other relevant solution which is given by the disk amplitude.[9] At the critical point $(w, p) = (0, 0)$, the latter reads

$$
\begin{aligned}
F^{(0,1)} = i\sqrt{3}\Bigg[&\frac{p^2}{36} + \frac{11p^3}{972} + \frac{47p^4}{11664} + \cdots \\
&+ w\left(-\frac{1}{3}\log(p) - \frac{p}{54} + \frac{p^2}{324} + \frac{23p^3}{13122} + \cdots\right) \\
&+ w^2\left(-\frac{11}{54}\log(p) - \frac{1}{2p} + \frac{1}{2p^2} - \frac{29p}{2916} + \frac{31p^2}{17496} + \cdots\right) + \cdots\Bigg].
\end{aligned}
$$
(C.3)

Therefore, generically, the open flat coordinate should be of the form:

$$
P = A\,u + B\,F^{(0,1)} + C\,T_{con} + D\,T_{con}^D + G.
$$
(C.4)

We can directly set B and D to zero, as both $F^{(0,1)}$ and T_{con}^D contain a logarithm which would then introduce non-trivial monodromy in the physical disk amplitude. We further decide to fix $A = -1$ and $G = \frac{4\pi i}{3}$, for the following reasons. First, fixing A just fixes the overall scale of the map. For instance, for $A = -1$ we get that

$$
P(p, w) = \frac{p}{3} + \frac{w}{3} + \cdots + C\,T_{con} + G - \frac{4\pi i}{3}.
$$
(C.5)

Then, we fix $G = \frac{4i\pi}{3}$ to cancel the constant term in the p, w expansion, as we want the flat coordinate to vanish at $(p, w) = (0, 0)$. We then obtain

$$
P(p, w) = \frac{p}{3} + \frac{p^2}{18} + \cdots + \frac{w}{3} + \frac{w^2}{6} + \cdots + C\,T_{con},
$$
(C.6)

---

[9] By disk amplitude here we mean its completion with classical terms such that it is a solution of the extended Picard-Fuchs system.

and the inverse mirror map reads:

$$p = -(3C + 1)T_{con} - \frac{1}{18}\left(27C^2 + 18C + 1\right)T_{con}^2 + \cdots$$

$$+ \left(3 + (3C + 1)T_{con} + \frac{1}{18}\left(27C^2 + 18C + 1\right)T_{con}^2 + \cdots\right)P + \cdots. \quad \text{(C.7)}$$

We did not however find any argument to fix the constant C in the open mirror map. As a result, we are left with a one-parameter family of open mirror maps at the conifold, parameterized by C.

Now, as we already mentioned in footnote 7, the conifold point is a singular limit for the mirror curve, and one needs to choose an appropriate coordinate on the resolution. To smooth out the singularity, as in [34] we introduce the rescaled open flat coordinate:

$$X = P\sqrt{T_{con}}. \quad \text{(C.8)}$$

We will then expand the open amplitudes in the flat coordinates X and T_{con}.

Conifold amplitudes can be easily obtained with the method developed in Sect. 2. As for the orbifold point studied in Sect. 3, basically we only need to input the flat coordinate T_{con} in the functionals $W^{(g,h)}$, and expand the result in the flat coordinates T_{con} and X at the conifold point. We obtain the following results.

The disk amplitude at the conifold point reads:

$$F^{(0,1)} = \left[X + \frac{3X^2}{4} + \cdots\right]T_{con} + \left[-\frac{3CX}{2} + \left(\frac{3C}{8} + \frac{1}{8}\right)X^2 + \cdots\right]T_{con}^{3/2}$$

$$+ \left[-\left(\frac{C}{4} + \frac{1}{72} + \frac{3C^2}{8}\right)X + \cdots\right]T_{con}^2 + \dots. \quad \text{(C.9)}$$

The annulus amplitude:

$$F^{(0,2)} = \frac{3X_1X_2}{16} + \frac{9X_1^2X_2^2}{512} + \frac{9(X_1^3X_2 + X_1X_2^3)}{256} + \cdots$$

$$+ \left[\frac{1}{64}\left(-9C + 1\right)\left(X_2X_1^2 + X_1X_2^2\right) + \cdots\right]\sqrt{T_{con}}$$

$$+ \left[\left(\frac{1}{24} - \frac{C}{16} + \frac{9C^2}{32}\right)X_1X_2 + \cdots\right]T_{con} + \cdots, \quad \text{(C.10)}$$

and the genus 1, one-hole amplitude:

$$F^{(1,1)} = \left(\frac{3}{32}X + \frac{15}{256}X^3 + \cdots\right)\frac{1}{T_{con}} + \left[\frac{1}{256}\left(1 - 45C\right)X^2 + \dots\right]\frac{1}{\sqrt{T_{con}}}$$

$$+ \left(\frac{45C^2}{256} - \frac{C}{128} - \frac{77}{2304}\right)X + \cdots + \left[-\frac{15C^3}{256} + \frac{C^2}{256} + \frac{77C}{2304} + \frac{83}{6912}\right.$$

$$+ \left(\frac{135C^2}{256} + \frac{595C}{3072} + \frac{35}{9216}\right)X^2 + \cdots\right]\sqrt{T_{con}} + \cdots. \quad \text{(C.11)}$$

These amplitudes have indeed the expected leading behavior T_{con}^{2-2g-h}, as explained in (3.52). However the subleading terms are not vanishing (for any value of C), and so there is no simple gap behavior.

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