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Generators of KMS Symmetric Markov Semigroups on $\mathcal{B}(\mathsf{h})$ Symmetry and Quantum Detailed Balance

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Abstract: We find the structure of generators of norm-continuous quantum Markov semigroups on $\mathcal{B}(\mathsf{h})$ that are symmetric with respect to the scalar product tr $(\rho^{1/2}x^*\rho^{1/2}y)$ induced by a faithful normal invariant state ρ and satisfy two quantum generalisations of the classical detailed balance condition related with this non-commutative notion of symmetry: the so-called standard detailed balance condition and the standard detailed balance condition with an antiunitary time reversal.

1. Introduction

Symmetric Markov semigroups have been extensively studied in classical stochastic analysis (Fukushima et al. [13] and the references therein) because their generators and associated Dirichlet forms are very well tractable by Hilbert space and probabilistic methods.

Their non-commutative counterpart has also been deeply investigated (Albeverio and Goswami [1], Cipriani [6], Davies and Lindsay [8], Goldstein and Lindsay [15], Guido, Isola and Scarlatti [17], Park [23], Sauvageot [26] and the references therein).

The classical notion of symmetry with respect to a measure, however, admits several non-commutative generalisations. Here we shall consider the so-called KMS-symmetry that seems more natural from a mathematical point of view (see e.g. Accardi and Mohari [3], Cipriani [6,7], Goldstein and Lindsay [14], Petz [25]) and find the structure of generators of norm-continuous quantum Markov semigroups (QMS) on the von Neumann algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on a complex separable Hilbert space h that are symmetric or satisfy quantum detailed balance conditions associated with KMS-symmetry or generalising it.

We consider QMS on $\mathcal{B}(\mathsf{h})$, i.e. weak*-continuous semigroups of normal, completely positive, identity preserving maps $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{B}(\mathsf{h})$, with a faithful normal invariant state ρ. This defines pre-scalar products on $\mathcal{B}(\mathsf{h})$ by $(x, y)_s = \text{tr}\ (\rho^{1-s}x^*\rho^s y)$ for $s \in [0, 1]$ and allows one to define the s-dual semigroup $\mathcal{T}'$ on $\mathcal{B}(\mathsf{h})$ satisfying tr $(\rho^{1-s}x^*\rho^s\mathcal{T}_t(y)) = \text{tr}\,(\rho^{1-s}\mathcal{T}'_t(x)^*\rho^s y)$ for all $x, y \in \mathcal{B}(\mathsf{h})$. The above scalar products coincide on an Abelian von Neumann algebra; the notion of symmetry $\mathcal{T} = \mathcal{T}'$, however, clearly depends on the choice of the parameter s.

The most studied cases are $s = 0$ and $s = 1/2$. Denoting $\mathcal{T}_*$ the predual semigroup, a simple computation yields $\mathcal{T}'_t(x) = \rho^{-(1-s)}\mathcal{T}_{*t}(\rho^{1-s}x\rho^s)\rho^{-s}$, and shows that for $s = 1/2$ the maps $\mathcal{T}'_t$ are positive but, for $s \neq 1/2$ this may not be the case. Indeed, it is well-known that, for $s \neq 1/2$, the maps $\mathcal{T}'_t$ are positive if and only if the maps $\mathcal{T}_t$ commute with the modular group $(\sigma_t)_{t\in\mathbb{R}}$, $\sigma_t(x) = \rho^{it}x\rho^{-it}$ (see e.g. [18] Prop. 2.1, p. 98, [22] Th. 6, p. 7985, for $s = 0$, [11] Th. 3.1, p. 341, Prop. 8.1, p. 362 for $s \neq 1/2$). This quite restrictive condition implies that the generator has a very special form that makes simpler the mathematical study of symmetry but imposes strong structural constraints (see e.g. [18 and 12]).

Here we shall consider the most natural choice $s = 1/2$ whose consequences are not so stringent and say that $\mathcal{T}$ is *KMS-symmetric* if it coincides with its dual $\mathcal{T}'$. KMS-symmetric QMS were introduced by Cipriani [6] and Goldstein and Lindsay [14]; we refer to [7] for a discussion of the connection with the KMS condition justifying this terminology.

All quantum versions of the classical principle of detailed balance (Agarwal [4], Alicki [5], Frigerio, Gorini, Kossakowski and Verri [18], Majewski [20,21]), which is at the basis of equilibrium physics, are formulated prescribing a certain relationship between $\mathcal{T}$ and $\mathcal{T}'$ or between their generators, therefore they depend on the underlying notion of symmetry. This work clarifies the structure of generators of QMS that are KMS-symmetric or satisfy a quantum detailed balance condition involving the above scalar product with $s = 1/2$ and is a key step towards understanding which is the most natural and flexible in view of the study of their generalisations for quantum systems out of equilibrium as, for instance, the *dynamical* detailed balance condition introduced by Accardi and Imafuku [2].

The generator $\mathcal{L}$ of a norm-continuous QMS can be written in the standard Gorini-Kossakowski-Sudarshan [16] and Lindblad [19] (GKSL) form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2}\sum_{\ell\geq 1}\left(L_\ell^*L_\ell x - 2L_\ell^* x L_\ell + xL_\ell^*L_\ell\right), \tag{1}$$

where $H, L_\ell \in \mathcal{B}(\mathsf{h})$ with $H = H^*$ and the series $\sum_{\ell\geq 1} L_\ell^* L_\ell$ is strongly convergent. The operators L_ℓ, H in (1) are not uniquely determined by $\mathcal{L}$, however, under a natural minimality condition (Theorem 2 below) and a zero-mean condition tr $(\rho L_\ell) = 0$ for all $\ell \geq 1$, H is determined up to a scalar multiple of the identity operator and the $(L_\ell)_{\ell\geq 1}$ up to a unitary transformation of the multiplicity space of the completely positive part of $\mathcal{L}$. We shall call *special* a GKSL representation of $\mathcal{L}$ by operators H, L_ℓ satisfying these conditions.

As a result, by the remark following Theorem 2, in a special GKSL representation of $\mathcal{L}$, the operator $G = -2^{-1}\sum_{\ell\geq 1} L_\ell^* L_\ell - iH$, is uniquely determined by $\mathcal{L}$ up to a purely imaginary multiple of the identity operator and allows us to write $\mathcal{L}$ in the form

$$\mathcal{L}(x) = G^*x + \sum_{\ell\geq 1} L_\ell^* x L_\ell + xG. \tag{2}$$

Our characterisations of QMS that are KMS-symmetric or satisfy a quantum detailed balance condition generalising related with KMS-symmetry are given in terms of the operators G, L_ℓ (or, in an equivalent way H, L_ℓ) of a special GKSL representation.

Theorem 7 shows that a QMS is KMS-symmetric if and only if the operators G, L_ℓ of a special GKSL representation of its generator satisfy $\rho^{1/2}G^* = G\rho^{1/2} + ic\rho^{1/2}$ for some $c \in \mathbb{R}$ and $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}$ for all k and some unitary $(u_{k\ell})$ on the multiplicity space of the completely positive part of $\mathcal{L}$ coinciding with its transpose, i.e. such that $u_{k\ell} = u_{\ell k}$ for all k, ℓ.

In order to describe our results on the structure of generators of QMS satisfying a quantum detailed balance condition we first recall some basic definitions. The best known is due to Alicki [5] and Frigerio-Gorini-Kossakowski-Verri [18]: a norm-continuous QMS $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{B}(\mathsf{h})$ satisfies the *Quantum Detailed Balance* (QDB) condition if there exists an operator $\widetilde{\mathcal{L}}$ on $\mathcal{B}(\mathsf{h})$ and a self-adjoint operator K on h such that tr $(\rho\widetilde{\mathcal{L}}(x)y) =$ tr $(\rho x\mathcal{L}(y))$ and $\mathcal{L}(x) - \widetilde{\mathcal{L}}(x) = 2i[K, x]$ for all $x, y \in \mathcal{B}(\mathsf{h})$. Roughly speaking we can say that $\mathcal{L}$ satisfies the QDB condition if the difference of $\mathcal{L}$ and its adjoint $\widetilde{\mathcal{L}}$ with respect to the pre-scalar product on $\mathcal{B}(\mathsf{h})$ given by tr $(\rho a^* b)$ is a derivation.

This QDB implies that the operator $\widetilde{\mathcal{L}} = \mathcal{L} - 2i[K, \cdot\,]$ can be written in the form (2) replacing G by $G + 2iK$ and then generates a QMS $\widetilde{\mathcal{T}}$. Therefore $\mathcal{L}$ and the maps $\mathcal{T}_t$ commute with the modular group. This restriction does not follow if the dual QMS is defined with respect to the symmetric pre-scalar product with $s = 1/2$.

The QDB can be readily reformulated replacing $\widetilde{\mathcal{L}}$ with the adjoint $\mathcal{L}'$ defined via the symmetric scalar product; the resulting condition will be called the *Standard Quantum Detailed Balance* condition (SQDB) (see e.g. [9]).

Theorem 5 characterises generators $\mathcal{L}$ satisfying the SQDB and extends previous partial results by Park [23] and the authors [11]: the SQDB holds if and only if there exists a unitary matrix $(u_{k\ell})$, coinciding with its transpose, i.e. $u_{k\ell} = u_{\ell k}$ for all k, ℓ, such that $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}$. This shows, in particular, that the SQDB depends only on the L_ℓ's and does not involve directly H and G. Moreover, we find explicitly the unitary $(u_{k\ell})_{k\ell}$ providing also a geometrical characterisation of the SQDB (Theorem 6) in terms of the operators $L_\ell\rho^{1/2}$ and their adjoints as Hilbert-Schmidt operators on h.

We also consider (Definition 3) another notion of quantum detailed balance, inspired by Agarwal's original notion (see [4], Majewski [20,21], Talkner [27]) involving an antiunitary *time reversal* operator θ which does not play any role in the Alicki et al. definition. Time reversal appears to keep into account the parity of quantum observables; position and energy, for instance, are even, i.e. invariant under time reversal, momentum are odd, i.e. change sign under time reversal. Agarwal's original definition, however, depends on the $s = 0$ pre-scalar product and implies then, that a QMS satisfying this quantum detailed balance condition must commute with the modular automorphism. Here we study the modified version (Definition 3) involving the symmetric $s = 1/2$ pre-scalar product that we call the SQDB-θ condition.

Theorem 8 shows that $\mathcal{L}$ satisfies the SQDB-θ condition if and only if there exists a special GKSL representation of $\mathcal{L}$ by means of operators H, L_ℓ such that $G\rho^{1/2} = \rho^{1/2}\theta G^*\theta$ and a unitary self-adjoint $(u_{k\ell})_{k\ell}$ such that $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}\theta L_\ell\theta\rho^{1/2}$ for all k. Here again $(u_{k\ell})_{k\ell}$ is explicitly determined by the operators $L_\ell\rho^{1/2}$ (Theorem 9).

We think that these results show that the SQDB condition is somewhat weaker than the SQDB-θ condition because the first does not involve directly the operators H, G. Moreover, the unitary operator in the linear relationship between $L_\ell\rho^{1/2}$ and their adjoints is transpose symmetric and any point of the unit disk could be in its spectrum while, for generators satisfying the SQDB-θ, it is self-adjoint and its spectrum is contained in $\{-1, 1\}$. Therefore, by the spectral theorem, it is possible in principle to find a standard form for the generators of QMSs satisfying the SQDB-θ generalising the standard form of generators satisfying the usual QDB condition (that commute with the modular group) as illustrated in the case of QMSs on $M_2(\mathbb{C})$ studied in the last section. This classification must be much more complex for generators of QMSs satisfying the SQDB.

The above arguments and the fact that the SQDB-θ condition can be formulated in a simple way both on the QMS or on its generator (this is not the case for the QDB when $\mathcal{L}$ and its Hamiltonian part $i[H, \cdot]$ do not commute), lead us to the conclusion that the SQDB-θ is the more natural non-commutative version of the classical detailed balance condition.

The paper is organised as follows. In Sect. 2 we construct the dual QMS $\mathcal{T}'$ and recall the quantum detailed balance conditions we investigate, then we study the relationship between the generators of a QMS and its adjoint in Sect. 3. Our main results on the structure of generators are proved in Sects. 4 (QDB without time reversal) and 5 (with time reversal).

2. The Dual QMS, KMS-Symmetry and Quantum Detailed Balance

We start this section by constructing the dual semigroup of a norm-continuous QMS with respect to the $(\cdot, \cdot)_{1/2}$ pre-scalar product on $\mathcal{B}(\mathsf{h})$ defined by an invariant state ρ and prove some properties that will be useful in the sequel. Although this result may be known, the presentation given here leads in a simple and direct way to the dual QMS avoiding non-commutative L^p-spaces techniques.

Proposition 1. *Let Φ be a positive unital normal map on $\mathcal{B}(\mathsf{h})$ with a faithful normal invariant state ρ. There exists a unique positive unital normal map Φ' on $\mathcal{B}(\mathsf{h})$ such that*

$$tr\left(\rho^{1/2}\Phi'(x)\rho^{1/2}y\right) = tr\left(\rho^{1/2}x\rho^{1/2}\Phi(y)\right)$$

for all $x, y \in \mathcal{B}(\mathsf{h})$. If Φ is completely positive, then Φ' is also completely positive.

Proof. Let Φ_* be the predual map on the Banach space of trace class operators on h and let $Rk(\rho^{1/2})$ denote the range of the operator $\rho^{1/2}$. This is clearly dense in h because ρ is faithful and coincides with the domain of the unbounded self-adjoint operator $\rho^{-1/2}$.

For all self-adjoint $x \in \mathcal{B}(\mathsf{h})$ consider the sesquilinear form on the domain $Rk(\rho^{1/2}) \times Rk(\rho^{1/2})$,

$$F(v, u) = \langle \rho^{-1/2}v, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u\rangle.$$

By the invariance of ρ and positivity of Φ_* we have

$$-\|x\|\rho = -\|x\|\Phi_*(\rho) \le \Phi_*(\rho^{1/2}x\rho^{1/2}) \le \|x\|\Phi_*(\rho) = \|x\|\rho.$$

Therefore $|F(u, u)| \le \|x\| \cdot \|v\| \cdot \|u\|$. Thus sesquilinear form is bounded and there exists a unique bounded operator y such that, for all $u, v \in Rk(\rho^{1/2})$,

$$\langle v, yu\rangle = \langle \rho^{-1/2}v, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u\rangle.$$

Note that, Φ being a *-map, and x self-adjoint

$$\begin{aligned}
\langle v, y^*u\rangle &= \overline{\langle y^*u, v\rangle} \\
&= \overline{\langle \rho^{-1/2}u, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}v\rangle} \\
&= \overline{\langle \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u, \rho^{-1/2}v\rangle} \\
&= \langle \rho^{-1/2}v, \Phi_*(\rho^{1/2}x\rho^{1/2})\rho^{-1/2}u\rangle.
\end{aligned}$$

This shows that y is self-adjoint. Defining $\Phi'(x) := y$, we find a real-linear map on self-adjoint operators on $\mathcal{B}(\mathsf{h})$ that can be extended to a linear map on $\mathcal{B}(\mathsf{h})$ decomposing each self-adjoint operator as the sum of its self-adjoint and anti self-adjoint parts.

Clearly Φ' is positive because $\rho^{1/2}\Phi'(x^*x)\rho^{1/2} = \Phi_*(\rho^{1/2}x^*x\rho^{1/2})$ and Φ_* is positive. Moreover, by the above construction $\Phi'(\mathbb{1}) = \mathbb{1}$, i.e. Φ' is unital. Therefore Φ' is a norm-one contraction.

If Φ is completely positive, then Φ_* is also and formula $\rho^{1/2}\Phi'(x)\rho^{1/2} = \Phi_*(\rho^{1/2} x\rho^{1/2})$ shows that Φ' is completely positive.

Finally we show that Φ' is normal. Let $(x_\alpha)_\alpha$ be a net of positive operators on $\mathcal{B}(\mathsf{h})$ with least upper bound $x \in \mathcal{B}(\mathsf{h})$. For all $u \in \mathsf{h}$ we have then

$$\begin{aligned}\sup_\alpha\langle \rho^{1/2}u, \Phi'(x_\alpha)\rho^{1/2}u\rangle &= \sup_\alpha\langle u, \Phi_*(\rho^{1/2}x_\alpha\rho^{1/2})u\rangle \\ &= \langle u, \Phi_*(\rho^{1/2}x\rho^{1/2})u\rangle = \langle \rho^{1/2}u, \Phi'(x)\rho^{1/2}u\rangle.\end{aligned}$$

Now if $u \in \mathsf{h}$, for every $\varepsilon > 0$, we can find a $u_\varepsilon \in Rk(\rho^{1/2})$ such that $\|u - u_\varepsilon\| < \varepsilon$ by the density of the range of $\rho^{1/2}$. We have then

$$\begin{aligned}\left|\langle u, \left(\Phi'(x_\alpha) - \Phi'(x)\right) u\rangle\right| &\le \varepsilon \left\| \Phi'(x_\alpha) - \Phi'(x)\right\| (\|u\| + \|u_\varepsilon\|) \\ &\quad + \left|\langle u_\varepsilon, \left(\Phi'(x_\alpha) - \Phi'(x)\right) u_\varepsilon\rangle\right|\end{aligned}$$

for all α. The conclusion follows from the arbitrarity of ε and the uniform boundedness of $\| \Phi'(x_\alpha) - \Phi'(x)\|$ and $\|u_\varepsilon\|$. □

Theorem 1. *Let $\mathcal{T}$ be a QMS on $\mathcal{B}(\mathsf{h})$ with a faithful normal invariant state ρ. There exists a QMS $\mathcal{T}'$ on $\mathcal{B}(\mathsf{h})$ such that*

$$\rho^{1/2}\mathcal{T}_t'(x)\rho^{1/2} = \mathcal{T}_{*t}(\rho^{1/2}x\rho^{1/2}) \tag{3}$$

for all $x \in \mathcal{B}(\mathsf{h})$ and all $t \ge 0$.

Proof. By Proposition 1, for each $t \ge 0$, there exists a unique completely positive normal and unital contraction $\mathcal{T}_t'$ on $\mathcal{B}(\mathsf{h})$ satisfying (3). The semigroup property follows from the algebraic computation

$$\begin{aligned}\rho^{1/2}\mathcal{T}_{t+s}'(x)\rho^{1/2} &= \mathcal{T}_{*t}\left(\mathcal{T}_{*s}(\rho^{1/2}x\rho^{1/2})\right) \\ &= \mathcal{T}_{*t}\left(\rho^{1/2}\mathcal{T}_s'(x)\rho^{1/2}\right) = \rho^{1/2}\mathcal{T}_t'\left(\mathcal{T}_s'(x)\right)\rho^{1/2}.\end{aligned}$$

Since the map $t \to \langle \rho^{1/2}v, \mathcal{T}_t'(x)\rho^{1/2}u\rangle$ is continuous by the identity (3) for all $u, v \in \mathsf{h}$, and $\|\mathcal{T}_t'(x)\| \le \|x\|$ for all $t \ge 0$, a 2ε approximation argument shows that $t \to \mathcal{T}_t'(x)$ is continuous for the weak*-operator topology on $\mathcal{B}(\mathsf{h})$. It follows that $\mathcal{T}' = (\mathcal{T}_t')_{t\ge 0}$ is a QMS on $\mathcal{B}(\mathsf{h})$. □

Definition 1. *The quantum Markov semigroup $\mathcal{T}'$ is called the* **dual semigroup** *of $\mathcal{T}$ with respect to the invariant state ρ.*

It is easy to see, using (3), that ρ is an invariant state also for $\mathcal{T}'$.

Remark 1. When $\mathcal{T}$ is norm-continuous it is not clear whether also $\mathcal{T}'$ is norm-continuous. Here, however, we are interested in generators of symmetric or detailed balance QMS. We shall see that these additional properties of $\mathcal{T}$ imply that also $\mathcal{T}'$ is norm continuous. Therefore we proceed studying norm-continuous QMSs whose dual is also norm-continuous.

The quantum detailed balance condition of Alicki, Frigerio, Gorini, Kossakowski and Verri modified by considering the pre-scalar product $(\cdot,\cdot)_{1/2}$ on $\mathcal{B}(\mathsf{h})$, usually called *standard* (see e.g. [9]) because of multiplications by $\rho^{1/2}$ as in the standard representation of $\mathcal{B}(\mathsf{h})$, is defined as follows.

Definition 2. *The QMS $\mathcal{T}$ generated by $\mathcal{L}$ satisfies the* **standard quantum detailed balance condition** *(SQDB) if there exists an operator $\mathcal{L}'$ on $\mathcal{B}(\mathsf{h})$ and a self-adjoint operator K on h such that*

$$tr\,(\rho^{1/2}x\rho^{1/2}\mathcal{L}(y)) = tr\,(\rho^{1/2}\mathcal{L}'(x)\rho^{1/2}y), \qquad \mathcal{L}(x) - \mathcal{L}'(x) = 2i[K,x] \tag{4}$$

for all $x \in \mathcal{B}(\mathsf{h})$.

The operator $\mathcal{L}'$ in the above definition must be norm-bounded because it is everywhere defined and norm closed. To see this consider a sequence $(x_n)_{n\geq 1}$ in $\mathcal{B}(\mathsf{h})$ converging in norm to a $x \in \mathcal{B}(\mathsf{h})$ such that $(\mathcal{L}(x_n))_{n\geq 1}$ converges in norm to $b \in \mathcal{B}(\mathsf{h})$ and note that

$$\begin{aligned}\mathrm{tr}\left(\rho^{1/2}\mathcal{L}'(x)\rho^{1/2}y\right) &= \lim_{n\to\infty}\mathrm{tr}\left(\rho^{1/2}x_n\rho^{1/2}\mathcal{L}(y)\right)\\ &= \lim_{n\to\infty}\mathrm{tr}\left(\rho^{1/2}\mathcal{L}'(x_n)\rho^{1/2}y\right) = \mathrm{tr}\left(\rho^{1/2}b\rho^{1/2}y\right)\end{aligned}$$

for all $y \in \mathcal{B}(\mathsf{h})$. The elements $\rho^{1/2}y\rho^{1/2}$, with $y \in \mathcal{B}(\mathsf{h})$, are dense in the Banach space of trace class operators on h because ρ is faithful. Therefore it shows that $\mathcal{L}'(x) = b$ and $\mathcal{L}'$ is closed.

Since both $\mathcal{L}$ and $\mathcal{L}'$ are bounded, also K is bounded.

We now introduce another definition of quantum detailed balance, due to Agarwal [4] with the $s = 0$ pre-scalar product, that involves a *time reversal* θ. This is an antiunitary operator on h, i.e. $\langle\theta u, \theta v\rangle = \langle v, u\rangle$ for all $u, v \in \mathsf{h}$, such that $\theta^2 = \mathbb{1}$ and $\theta^{-1} = \theta^* = \theta$.

Recall that θ is antilinear, i.e. $\theta z u = \bar{z}u$ for all $u \in \mathsf{h}$, $z \in \mathbb{C}$, and its adjoint θ^* satisfies $\langle u, \theta v\rangle = \langle v, \theta^* u\rangle$ for all $u, v \in \mathsf{h}$. Moreover $\theta\, x\, \theta$ belongs to $\mathcal{B}(\mathsf{h})$ (linearity is re-established) and $\mathrm{tr}\,(\theta\, x\theta) = \mathrm{tr}\,(x^*)$ for every trace-class operator x ([10] Prop. 4), indeed, taking an orthonormal basis of h, we have

$$\begin{aligned}\mathrm{tr}\,(\theta x\theta) = \sum_j\langle e_j, \theta x\theta e_j\rangle &= \sum_j\langle x\theta e_j, \theta^* e_j\rangle\\ &= \sum_j\langle\theta e_j, x^*\theta^* e_j\rangle = \mathrm{tr}(x^*).\end{aligned}$$

It is worth noticing that the cyclic property of the trace does not hold for θ, since $\mathrm{tr}\,(\theta\, x\theta) = \mathrm{tr}\,(x^*)$ may not be equal to $\mathrm{tr}\,(x)$ for non-self-adjoint x.

Definition 3. *The QMS $\mathcal{T}$ generated by $\mathcal{L}$ satisfies the standard quantum detailed balance condition with respect to the time reversal θ (SQDB-θ) if*

$$tr\,(\rho^{1/2}x\rho^{1/2}\mathcal{L}(y)) = tr\,(\rho^{1/2}\theta y^*\theta\rho^{1/2}\mathcal{L}(\theta x^*\theta)), \tag{5}$$

for all $x, y \in \mathcal{B}(\mathsf{h})$.

The operator θ is used to keep into account parity of the observables under time reversal. Indeed, a self-adjoint operator $x \in \mathcal{B}(\mathsf{h})$ is called *even* (resp. *odd*) if $\theta x \theta = x$ (resp. $\theta x \theta = -x$). The typical example of antilinear time reversal is a conjugation (with respect to some orthonormal basis of h).

This condition is usually stated ([20,21,27]) for the QMS $\mathcal{T}$ as

$$\mathrm{tr}\,(\rho^{1/2} x \rho^{1/2} \mathcal{T}_t(y)) = \mathrm{tr}\,(\rho^{1/2} \theta y^* \theta \rho^{1/2} \mathcal{T}_t(\theta x^* \theta)), \tag{6}$$

for all $t \geq 0$, $x, y \in \mathcal{B}(\mathsf{h})$. In particular, for $t = 0$ we find that this identity holds if and only if ρ and θ commute, i.e. ρ is an even observable. This is the case, for instance, when ρ is a function of the energy.

Lemma 1. *The following conditions are equivalent:*

(i) *θ and ρ commute,*
(ii) *$tr\,(\rho^{1/2} x \rho^{1/2} y) = tr\,(\rho^{1/2} \theta y^* \theta \rho^{1/2} \theta x^* \theta)$ for all $x, y \in \mathcal{B}(\mathsf{h})$.*

Proof. If ρ and θ commute, from $\mathrm{tr}\,(\theta a \theta) = \mathrm{tr}\,(a^*)$, we have

$$\mathrm{tr}\,(\rho^{1/2} \theta y^* \theta \rho^{1/2} \theta x^* \theta) = \mathrm{tr}\,(\theta(\rho^{1/2} y^* \rho^{1/2} x^*)\theta) = \mathrm{tr}\,(x \rho^{1/2} y \rho^{1/2})$$

and (ii) follows cycling $\rho^{1/2}$. Conversely, if (ii) holds, taking $x = \mathbb{1}$, we have

$$\mathrm{tr}\,(\rho y) = \mathrm{tr}\,(\rho \theta y^* \theta) = \mathrm{tr}\left(\theta(\theta y^* \theta)^* \rho \theta\right) = \mathrm{tr}\,(y \theta \rho \theta) = \mathrm{tr}\,(\theta \rho \theta y),$$

for all $y \in \mathcal{B}(\mathsf{h})$, and $\rho = \theta \rho \theta$. □

Proposition 2. *If ρ and θ commute then (5) and (6) are equivalent.*

Proof. Clearly (5) follows from (6) differentiating at $t = 0$.

Conversely, putting $\alpha(x) = \theta x \theta$ and denoting $\mathcal{L}_*$ the predual of $\mathcal{L}$ we can write (5) as

$$\mathrm{tr}\,(\mathcal{L}_*(\rho^{1/2} x \rho^{1/2}) y) = \mathrm{tr}\left(\rho^{1/2} \alpha(y^*) \rho^{1/2} \mathcal{L}(\alpha(x^*))\right) = \mathrm{tr}\left(\rho^{1/2} \alpha(\mathcal{L}(\alpha(x))) \rho^{1/2} y\right),$$

for all $y \in \mathcal{B}(\mathsf{h})$, because $\mathrm{tr}\,(\alpha(a)) = \mathrm{tr}\,(a^*)$. Therefore we have

$$\mathcal{L}_*(\rho^{1/2} x \rho^{1/2}) = \rho^{1/2} \alpha(\mathcal{L}(\alpha(x))) \rho^{1/2}$$

and, iterating, $\mathcal{L}_*^n(\rho^{1/2} x \rho^{1/2}) = \rho^{1/2} \alpha(\mathcal{L}^n(\alpha(x))) \rho^{1/2}$ for all $n \geq 1$. It follows that (5) holds for all powers $\mathcal{L}^n$ with $n \geq 1$. Since ρ and θ commute, it is true also for $n = 0$ and we find (6) by the exponentiation formula $\mathcal{T}_t = \sum_{n \geq 0} t^n \mathcal{L}^n / n!$. □

We do not know whether the SQDB condition (4) of Definition 2 has a simple explicit formulation in terms of the maps $\mathcal{T}_t$ if $\mathcal{L}$ and $\mathcal{L}'$ do not commute.

Remark 2. The SQDB condition (5), by $\mathrm{tr}\,(\theta a \theta) = \mathrm{tr}\,(a^*)$, reads

$$\mathrm{tr}\,(\rho^{1/2} x \rho^{1/2} \mathcal{L}(y)) = \mathrm{tr}\,(\rho^{1/2}(\theta \mathcal{L}(\theta x \theta)\theta) \rho^{1/2} x),$$

for all $x, y \in \mathcal{B}(\mathsf{h})$, i.e. $\mathcal{L}'(x) = \theta \mathcal{L}(\theta x \theta) \theta$.

Write $\mathcal{L}$ in a special GKSL form as in (1) and decompose the generator $\mathcal{L} = \mathcal{L}_0 + i[H, \cdot\,]$ into the sum of its dissipative part $\mathcal{L}_0$ and derivation part $i[H, \cdot\,]$. If H commutes with θ, by the antilinearity of θ, we find $\mathcal{L}'(x) = \theta \mathcal{L}_0(\theta x \theta)\theta - i[H, x]$. Therefore, if the dissipative part is time reversal invariant, i.e. $\mathcal{L}_0(x) = \theta \mathcal{L}_0(\theta x \theta)\theta$, we end up with $\mathcal{L}' = \mathcal{L} - 2i[H, \cdot\,]$.

The relationship with Definition 2 of SQDB, in this case, is then clear. The SQDB conditions of Definition 2 and 3, however, in general are not comparable.

3. The Generator of a QMS and its Dual

We shall always consider *special* GKSL representations of the generator of a norm-continuous QMS by means of operators L_ℓ, H. These are described by the following theorem (we refer to [24] Theorem 30.16 for the proof).

Theorem 2. *Let $\mathcal{L}$ be the generator of a norm-continuous QMS on $\mathcal{B}(\mathsf{h})$ and let ρ be a normal state on $\mathcal{B}(\mathsf{h})$. There exists a bounded self-adjoint operator H and a finite or infinite sequence $(L_\ell)_{\ell\geq 1}$ of elements of $\mathcal{B}(\mathsf{h})$ such that:*

(i) $\mathrm{tr}(\rho L_\ell) = 0$ *for each* $\ell \geq 1$,
(ii) $\sum_{\ell\geq 1} L_\ell^* L_\ell$ *is a strongly convergent sum,*
(iii) *if* $\sum_{\ell\geq 0} |c_\ell|^2 < \infty$ *and* $c_0 + \sum_{\ell\geq 1} c_\ell L_\ell = 0$ *for complex scalars* $(c_k)_{k\geq 0}$ *then* $c_k = 0$ *for every* $k \geq 0$,
(iv) *the GKSL representation* (1) *holds.*

If H', $(L'_\ell)_{\ell\geq 1}$ is another family of bounded operators in $\mathcal{B}(\mathsf{h})$ with H' self-adjoint and the sequence $(L'_\ell)_{\ell\geq 1}$ is finite or infinite then the conditions (i)–(iv) *are fulfilled with H, $(L_\ell)_{\ell\geq 1}$ replaced by H', $(L'_\ell)_{\ell\geq 1}$ respectively if and only if the lengths of the sequences $(L_\ell)_{\ell\geq 1}$, $(L'_\ell)_{\ell\geq 1}$ are equal and for some scalar $c \in \mathbb{R}$ and a unitary matrix $(u_{\ell j})_{\ell,j}$ we have*

$$H' = H + c, \qquad L'_\ell = \sum_j u_{\ell j} L_j.$$

As an immediate consequence of the uniqueness (up to a scalar) of the Hamiltonian H, the decomposition of $\mathcal{L}$ as the sum of the derivation $i[H,\cdot]$ and a dissipative part $\mathcal{L}_0 = \mathcal{L} - i[H,\cdot\,]$ determined by special GKSL representations of $\mathcal{L}$ is unique. Moreover, since $(u_{\ell j})$ is unitary, we have

$$\sum_{\ell\geq 1} \left(L'_\ell\right)^* L'_\ell = \sum_{\ell,k,j\geq 1} \overline{u}_{\ell k} u_{\ell j} L_k^* L_j = \sum_{k,j\geq 1} \left(\sum_{\ell\geq 1} \overline{u}_{\ell k} u_{\ell j}\right) L_k^* L_j = \sum_{k\geq 1} L_k^* L_k.$$

Therefore, putting $G = -2^{-1}\sum_{\ell\geq 1} L_\ell^* L_\ell - iH$, we can write $\mathcal{L}$ in the form (2), where G is uniquely determined by $\mathcal{L}$ up to a purely imaginary multiple of the identity operator.

Theorem 2 can be restated in the index free form ([24] Thm. 30.12).

Theorem 3. *Let $\mathcal{L}$ be the generator of a norm continuous QMS on $\mathcal{B}(\mathsf{h})$, then there exist an Hilbert space k, a bounded linear operator $L : \mathsf{h} \to \mathsf{h}\otimes\mathsf{k}$ and a bounded self-adjoint operator H on h satisfying the following:*

1. $\mathcal{L}(x) = i[H, x] - \frac{1}{2}\left(L^* L x - 2L^*(x \otimes \mathbb{1}_{\mathsf{k}})L + xL^* L\right)$ *for all* $x \in \mathcal{B}(\mathsf{h})$;
2. *the set* $\{(x \otimes \mathbb{1}_{\mathsf{k}})Lu : x \in \mathcal{B}(\mathsf{h}),\ u \in \mathsf{h}\}$ *is total in* $\mathsf{h}\otimes\mathsf{k}$.

Proof. Let k be a Hilbert space with Hilbertian dimension equal to the length of the sequence $(L_k)_k$ and let (f_k) be an orthonormal basis of k. Defining $Lu = \sum_k L_k u \otimes f_k$, where the L_k are as in Theorem 2, a simple calculation shows that 1 is fulfilled.

Suppose that there exists a non-zero vector ξ orthogonal to the set of $(x \otimes \mathbb{1}_{\mathsf{k}})Lu$ with $x \in \mathcal{B}(\mathsf{h}),\ u \in \mathsf{h}$; then $\xi = \sum_k v_k \otimes f_k$ with $v_k \in \mathsf{h}$ and

$$0 = \langle \xi, (x \otimes \mathbb{1}_{\mathsf{k}})Lu\rangle = \sum_k \langle v_k, xL_k u\rangle = \sum_k \langle L_k^* x^* v_k, u\rangle$$

for all $x \in \mathcal{B}(\mathsf{h}), u \in \mathsf{h}$. Hence, $\sum_k L_k^* x^* v_k = 0$. Since $\xi \neq 0$, we can suppose $\|v_1\| = 1$; then, putting $p = |v_1\rangle\langle v_1|$ and $x = py^*$, $y \in \mathcal{B}(\mathsf{h})$, we get

$$0 = L_1^* y v_1 + \sum_{k \geq 2} \langle v_1, v_k \rangle L_k^* y v_1 = \left(L_1^* + \sum_{k \geq 2} \langle v_1, v_k \rangle L_k^* \right) y v_1. \tag{7}$$

Since $y \in \mathcal{B}(\mathsf{h})$ is arbitrary, Eq. (7) contradicts the linear independence (see Theorem 2 (iii)) of the L_k's. Therefore the set in (2) must be total. □

The Hilbert space k is called the *multiplicity space* of the completely positive part of $\mathcal{L}$. A unitary matrix $(u_{\ell j})_{\ell, j \geq 1}$, in the above basis $(f_k)_{k \geq 1}$, clearly defines a unitary operator on k. From now on we shall identify such matrices with operators on k.

We end this section by establishing the relationship between the operators G, L_ℓ and G', L'_ℓ in two special GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$ when these generators are both bounded.

The dual QMS $\mathcal{T}'$ clearly satisfies

$$\rho^{1/2} \mathcal{T}'_t(x) \rho^{1/2} = \mathcal{T}_{*t}(\rho^{1/2} x \rho^{1/2}),$$

where $\mathcal{T}_*$ denotes the predual semigroup of $\mathcal{T}$. Since $\mathcal{L}'$ is bounded, differentiating at $t = 0$, we find the relationship among the generator $\mathcal{L}'$ of $\mathcal{T}$ and $\mathcal{L}_*$ of the predual semigroup $\mathcal{T}_*$ of $\mathcal{T}$,

$$\rho^{1/2} \mathcal{L}'(x) \rho^{1/2} = \mathcal{L}_*(\rho^{1/2} x \rho^{1/2}). \tag{8}$$

Proposition 3. *Let $\mathcal{L}(a) = G^* a + aG + \sum_\ell L_\ell^* a L_\ell$ be a special GKSL representation of $\mathcal{L}$ with respect to a $\mathcal{T}$-invariant state $\rho = \sum_k \rho_k |e_k\rangle\langle e_k|$. Then*

$$G^* u = \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|) e_k - \mathrm{tr}(\rho G) u, \tag{9}$$

$$G v = \sum_{k \geq 1} \rho_k \mathcal{L}_*(|v\rangle\langle e_k|) e_k - \mathrm{tr}(\rho G^*) v \tag{10}$$

for every $u, v \in \mathsf{h}$.

Proof. Since $\mathcal{L}(|u\rangle\langle v|) = |G^* u\rangle\langle v| + |u\rangle\langle Gv| + \sum_\ell |L_\ell^* u\rangle\langle L_\ell^* v|$, putting $v = e_k$ we have $G^* u = |G^* u\rangle\langle e_k| e_k$ and

$$G^* u = \mathcal{L}(|u\rangle\langle e_k|) e_k - \sum_\ell \langle e_k, L_\ell e_k \rangle L_\ell^* u - \langle e_k, G e_k \rangle u.$$

Multiplying both sides by ρ_k and summing on k, we find then

$$\begin{aligned} G^* u &= \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|) e_k - \sum_{\ell, k} \rho_k \langle e_k, L_\ell e_k \rangle L_\ell^* u - \sum_{k \geq 1} \rho_k \langle e_k, G e_k \rangle u \\ &= \sum_{k \geq 1} \rho_k \mathcal{L}(|u\rangle\langle e_k|) e_k - \sum_\ell \mathrm{tr}(\rho L_\ell) L_\ell^* u - \mathrm{tr}(\rho G) u \end{aligned}$$

and (9) follows since $\mathrm{tr}(\rho L_j) = 0$. The identity (10) is now immediate computing the adjoint of G. □

Proposition 4. *Let $\mathcal{T}'$ be the dual of a QMS $\mathcal{T}$ generated by $\mathcal{L}$ with normal invariant state ρ. If G and G' are the operators (10) in two GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$ then*

$$G'\rho^{1/2} = \rho^{1/2}G^* + \left(\mathrm{tr}(\rho G) - \mathrm{tr}(\rho G')\right)\rho^{1/2}. \tag{11}$$

Moreover, we have $\mathrm{tr}(\rho G) - \mathrm{tr}(\rho G') = ic$ *for some* $c \in \mathbb{R}$.

Proof. The identities (10) and (8) yield

$$\begin{aligned} G'\rho^{1/2}v &= \sum_{k\geq 1} \mathcal{L}'_*(\rho^{1/2}\,|v\rangle\langle\rho_k^{1/2}e_k\,|)\rho_k^{1/2}e_k - \mathrm{tr}\,(\rho G'^*)\rho^{1/2}v \\ &= \sum_{k\geq 1} \mathcal{L}'_*(\rho^{1/2}(|v\rangle\langle e_k\,|)\rho^{1/2})\rho^{1/2}e_k - \mathrm{tr}\,(\rho G'^*)\rho^{1/2}v \\ &= \sum_{k\geq 1} \rho^{1/2}\mathcal{L}(|v\rangle\langle e_k\,|)\rho^{1/2}\rho^{1/2}e_k - \mathrm{tr}\,(\rho G'^*)\rho^{1/2}v \\ &= \rho^{1/2}G^*v + \left(\mathrm{tr}\,(\rho G) - \mathrm{tr}\,(\rho G'^*)\right)\rho^{1/2}v. \end{aligned}$$

Therefore, we obtain (11). Right multiplying this equation by $\rho^{1/2}$ we have $G'\rho = \rho^{1/2}G^*\rho^{1/2} + \left(\mathrm{tr}\,(\rho G) - \mathrm{tr}\,(\rho G'^*)\right)\rho$, and, taking the trace,

$$\begin{aligned} \mathrm{tr}\,(\rho G) - \mathrm{tr}\,(\rho G'^*) &= \mathrm{tr}\,(G'\rho) - \mathrm{tr}\,(\rho^{1/2}G^*\rho^{1/2}) \\ &= \mathrm{tr}\,(G'\rho) - \mathrm{tr}\,(G^*\rho) = -\overline{(\mathrm{tr}\,(\rho G) - \mathrm{tr}\,(\rho G'^*))}; \end{aligned}$$

this proves the last claim. □

We can now prove as in [11] Th. 7.2, p. 358 the following

Theorem 4. *For all special GKSL representations of $\mathcal{L}$ by means of operators G, L_ℓ as in (2) there exists a special GKSL representation of $\mathcal{L}'$ by means of operators G', L'_ℓ such that:*

1. $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ *for some* $c \in \mathbb{R}$,
2. $L'_\ell\rho^{1/2} = \rho^{1/2}L^*_\ell$ *for all* $\ell \geq 1$.

Proof. Since $\mathcal{L}'$ is bounded, it admits a special GKSL representation $\mathcal{L}'(a) = G'^*a + \sum_k L'^*_k a L'_k + aG'$. Moreover, by Proposition 4, we have $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$, $c \in \mathbb{R}$, and so (8) implies

$$\sum_k \rho^{1/2}L'^*_k x L'_k \rho^{1/2} = \sum_k L_k\rho^{1/2}x\rho^{1/2}L^*_k. \tag{12}$$

Let k (resp. k') be the multiplicity space of the completely positive part of $\mathcal{L}$ (resp. $\mathcal{L}'$), $(f_k)_k$ (resp. $(f'_k)_k$) an orthonormal basis of k (resp. k') and define a linear operator $X : \mathsf{h}\otimes\mathsf{k}' \to \mathsf{h}\otimes\mathsf{k}$,

$$X(x\otimes \mathbb{1}_{\mathsf{k}'})L'\rho^{1/2}u = (x\otimes \mathbb{1}_{\mathsf{k}})\sum_k \rho^{1/2}L^*_k u\otimes f_k$$

for all $x \in \mathcal{B}(\mathsf{h})$ and $u \in \mathsf{h}$, where $L : \mathsf{h} \to \mathsf{h} \otimes \mathsf{k}$, $Lu = \sum_k L_k u \otimes f_k$, $L' : \mathsf{h} \to \mathsf{h} \otimes \mathsf{k}'$, $L'u = \sum_k L'_k u \otimes f'_k$. Note that the right-hand side series is convergent for all $u \in \mathsf{h}$ because of (12), since

$$\left\| \sum_{k=m}^{n} \rho^{1/2} L_k^* u \otimes f_k \right\|^2 = \sum_{k=m}^{n} \left\| \rho^{1/2} L_k^* u \right\|^2 = \sum_{k=m}^{n} \left\langle u, L_k \rho L_k^* u \right\rangle,$$

and the right-hand side goes to 0 for n, m tending to infinity because ρ is an invariant state and the series $\sum_k L_k \rho L_k^* = -(G\rho + \rho G)$ is trace-norm convergent.

The identity (12) yields

$$\langle X(x \otimes \mathbb{1}_{\mathsf{k}'}) L' \rho^{1/2} u, X(y \otimes \mathbb{1}_{\mathsf{k}'}) L' \rho^{1/2} v \rangle = \sum_k \langle u, \rho^{1/2} L_k'^* x^* y L'_k \rho^{1/2} v \rangle$$
$$= \langle (x \otimes \mathbb{1}_{\mathsf{k}'}) L' \rho^{1/2} u, (y \otimes \mathbb{1}_{\mathsf{k}'}) L' \rho^{1/2} v \rangle$$

for all $x, y \in \mathcal{B}(\mathsf{h})$ and $u, v \in \mathsf{h}$, i.e. X preserves the scalar product. Therefore, since the set $\{(x \otimes \mathbb{1}_{\mathsf{k}'}) L' \rho^{1/2} u \mid x \in \mathcal{B}(\mathsf{h}),\ u \in \mathsf{h}\}$ is total in $\mathsf{h} \otimes \mathsf{k}'$ (for $\rho^{1/2}(\mathsf{h})$ is dense in h and Theorem 3 holds), X is well defined and extends to an isometry from $\mathsf{h} \otimes \mathsf{k}'$ to $\mathsf{h} \otimes \mathsf{k}$.

The operator X is unitary because its range is dense in $\mathsf{h} \otimes \mathsf{k}$. Indeed, if we suppose that there exists a vector $\xi = \sum_k v_k \otimes f_k$, with $v_k \in \mathsf{h}$ and $\sum_k \|v_k\|^2 < \infty$, orthogonal to all $(x \otimes \mathbb{1}_{\mathsf{k}}) \sum_k \rho^{1/2} L_k^* u \otimes f_k$; then

$$0 = \langle \xi, (x \otimes \mathbb{1}_{\mathsf{k}}) \sum_k \rho^{1/2} L_k^* u \otimes f_k \rangle = \sum_k \langle v_k, x \rho^{1/2} L_k^* u \rangle = \sum_k \langle L_k \rho^{1/2} x^* v_k, u \rangle$$

for all $x \in \mathcal{B}(\mathsf{h})$, $u \in \mathsf{h}$. Taking $x = |w_1\rangle\langle w_2|$, by the arbitrarity of u, we have then $\sum_k \langle w_1, v_k \rangle L_k \rho^{1/2} w_2 = 0$. Since w_2 is arbitrary, the range of $\rho^{1/2}$ is dense in h and the sequence $(\langle w_1, v_k \rangle)_{k \geq 1}$ is square-summable we find $\sum_k \langle w_1, v_k \rangle L_k = 0$. The linear independence of the L_k, in the sense of Theorem 2 (iii), implies then $\langle w_1, v_k \rangle = 0$ for all k and all $w_1 \in \mathsf{h}$, i.e. $\xi = 0$.

As a consequence we have $X^* X = \mathbb{1}_{\mathsf{h} \otimes \mathsf{k}'}$ and $X X^* = \mathbb{1}_{\mathsf{h} \otimes \mathsf{k}}$.

Moreover, since $X(y \otimes \mathbb{1}_{\mathsf{k}'}) = (y \otimes \mathbb{1}_{\mathsf{k}'}) X$ for all $y \in \mathcal{B}(\mathsf{h})$, we can conclude that $X = \mathbb{1}_{\mathsf{h}} \otimes Y$ for some unitary map $Y : \mathsf{k}' \to \mathsf{k}'$.

The definition of X implies then

$$(\rho^{1/2} \otimes \mathbb{1}_{\mathsf{k}}) L^* = X L' \rho^{1/2} = (\mathbb{1}_{\mathsf{h}} \otimes Y) L' \rho^{1/2}.$$

This means that, replacing L' by $(\mathbb{1}_{\mathsf{h}} \otimes Y) L'$, or more precisely L'_k by $\sum_\ell u_{k\ell} L'_\ell$ for all k, we have

$$\rho^{1/2} L_k^* = L'_k \rho^{1/2}.$$

Since $\mathrm{tr}\,(\rho L'_k) = \mathrm{tr}\,(\rho L_k^*) = 0$ and, from $\mathcal{L}'(\mathbb{1}) = 0$, $G'^* + G' = -\sum_k L'^*_k L'_k$, the properties of a special GKSL representation follow. □

Remark 3. Condition 2 implies that the completely positive parts $\Phi(x) = \sum_\ell L_\ell^* x L_\ell$ and Φ' of the generators $\mathcal{L}$ and $\mathcal{L}'$, respectively are mutually adjoint, i.e.

$$\mathrm{tr}\,(\rho^{1/2} \Phi'(x) \rho^{1/2} y) = \mathrm{tr}\,(\rho^{1/2} x \rho^{1/2} \Phi(y)) \tag{13}$$

for all $x, y \in \mathcal{B}(\mathsf{h})$. As a consequence, also the maps $x \to G^* x + x G$ and $x \to (G')^* x + x G'$ are mutually adjoint.

4. Generators of Standard Detailed Balance QMSs

In this section we characterise the generators of norm-continuous QMSs satisfying the SQDB of Definition 2.

We start noting that, since ρ is invariant for $\mathcal{T}$ and $\mathcal{T}'$, i.e. $\mathcal{L}_*(\rho) = \mathcal{L}'_*(\rho) = 0$, the operator K commutes with ρ. Moreover, by comparing two special GKSL representations of $\mathcal{L}$ and $\mathcal{L}' + 2i[K, \cdot\,]$, we have immediately the following

Lemma 2. *A QMS $\mathcal{T}$ satisfies the SQDB $\mathcal{L} - \mathcal{L}' = 2i[K, \cdot\,]$ if and only if for all special GKSL representations of the generators $\mathcal{L}$ and $\mathcal{L}'$ by means of operators G, L_k and G', L'_k respectively, we have*

$$G = G' + 2iK + ic \qquad L'_k = \sum_j u_{kj} L_j$$

for some $c \in \mathbb{R}$ and some unitary $(u_{kj})_{kj}$ on k.

Since we know the relationship between the operators G', L'_k and G, L_k thanks to Theorem 4, we can now characterise generators of QMSs satisfying the SQDB. We emphasize the following definition of T*-symmetric* matrix (operator) on k in order to avoid confusion with the usual notion of symmetric operator X meaning that X^* is an extension of X.

Definition 4. *Let $Y = (y_{k\ell})_{k,\ell \geq 1}$ be a matrix with entries indexed by k, ℓ running on the set (finite or infinite) of indices of the sequence $(L_\ell)_{\ell \geq 1}$. We denote by Y^T the transpose matrix $Y^T = (y_{\ell k})_{k,\ell \geq 1}$. The matrix Y is called* ***T*-symmetric** *if $Y = Y^T$.*

Theorem 5. *$\mathcal{T}$ satisfies the SQDB if and only if for all special GKSL representation of the generator $\mathcal{L}$ by means of operators G, L_k there exists a T-symmetric unitary $(u_{m\ell})_{m\ell}$ on k such that, for all $k \geq 1$,*

$$\rho^{1/2} L_k^* = \sum_\ell u_{k\ell} L_\ell \rho^{1/2}. \tag{14}$$

Proof. Given a special GKSL representation of $\mathcal{L}$, adding a purely imaginary multiple of the identity operator to the anti-selfadjoint part of G' if necessary, Theorem 4 allows us to write the dual $\mathcal{L}'$ in a special GKSL representation by means of operators G', L'_k with

$$G'\rho^{1/2} = \rho^{1/2} G^*, \qquad L'_k \rho^{1/2} = \rho^{1/2} L_k^*. \tag{15}$$

Suppose first that $\mathcal{T}$ satisfies the SQDB. Since $L'_k = \sum_j u_{kj} L_j$ for some unitary $(u_{kj})_{kj}$ by Lemma 2, we can find (14) substituting L'_k with $\sum_j u_{kj} L_j$ in the second formula (15).

Finally we show that the unitary matrix $u = (u_{m\ell})_{m\ell}$ is T-symmetric. Indeed, taking the adjoint of (14) we find $L_\ell \rho^{1/2} = \sum_m \bar{u}_{\ell m} \rho^{1/2} L_m^*$. Writing $\rho^{1/2} L_m^*$ as in (14) we have then

$$L_\ell \rho^{1/2} = \sum_{m,k} \bar{u}_{\ell m} u_{mk} L_k \rho^{1/2} = \sum_k \left((u^*)^T u \right)_{\ell k} L_k \rho^{1/2}.$$

The operators $L_\ell \rho^{1/2}$ are linearly independent by property (iii) Theorem 2 of a special GKSL representation, therefore $(u^*)^T u$ is the identity operator on k. Since u is also unitary, we have also $u^* u = (u^*)^T u$, namely $u^* = (u^*)^T$ and $u = u^T$.

Conversely, if (14) holds, by (15), we have $L'_k \rho^{1/2} = \sum_\ell u_{k\ell} L_\ell \rho^{1/2}$, so that $L'_k = \sum_\ell u_{k\ell} L_\ell$ for all k and for some unitary $(u_{kj})_{kj}$. Therefore, thanks to Lemma 2, to conclude it is enough to prove that $G = G' + i(2K + c)$ namely, that $G - G'$ is anti self-adjoint.

To this end note that, since ρ is an invariant state, we have

$$0 = \rho G^* + \sum_k L_k \rho L_k^* + G\rho, \tag{16}$$

with

$$\begin{aligned}\sum_k L_k \rho L_k^* &= \sum_k (L_k \rho^{1/2})(\rho^{1/2} L_k^*) = \sum_k \sum_{\ell,j} \overline{u}_{k\ell} u_{kj} \rho^{1/2} L_\ell^* L_j \rho^{1/2} \\ &= \sum_\ell \rho^{1/2} L_\ell^* L_\ell \rho^{1/2} = -\rho^{1/2}(G + G^*)\rho^{1/2},\end{aligned}$$

(for condition (14) holds) and so, by substituting in Eq. (16) we get

$$\begin{aligned}0 = \rho G^* - \rho^{1/2} G \rho^{1/2} - \rho^{1/2} G^* \rho^{1/2} + G\rho = \rho^{1/2}\left(\rho^{1/2} G^* - G\rho^{1/2}\right) \\ - \left(\rho^{1/2} G^* - G\rho^{1/2}\right)\rho^{1/2} = [G\rho^{1/2} - \rho^{1/2} G^*, \rho^{1/2}],\end{aligned}$$

i.e. $G\rho^{1/2} - \rho^{1/2} G^*$ commutes with $\rho^{1/2}$.

We can now prove that $G - G'$ is anti self-adjoint. Clearly, it suffices to show that $\rho^{1/2} G \rho^{1/2} - \rho^{1/2} G' \rho^{1/2}$ is anti self-adjoint. Indeed, by (15), we have

$$\begin{aligned}\left(\rho^{1/2} G \rho^{1/2} - \rho^{1/2} G' \rho^{1/2}\right)^* &= \left(\rho^{1/2} G \rho^{1/2} - \rho G^*\right)^* \\ &= \left(\rho^{1/2}\left(G\rho^{1/2} - \rho^{1/2} G^*\right)\right)^* \\ &= \left(\left(G\rho^{1/2} - \rho^{1/2} G^*\right)\rho^{1/2}\right)^* \\ &= \rho G^* - \rho^{1/2} G \rho^{1/2} = \rho^{1/2} G' \rho^{1/2} - \rho^{1/2} G \rho^{1/2},\end{aligned}$$

because $G\rho^{1/2} - \rho^{1/2} G^*$ commutes with $\rho^{1/2}$. This completes the proof. □

It is worth noticing that, as in Remark 3, $\mathcal{T}$ satisfies the SQDB if and only if the completely positive part Φ of the generator $\mathcal{L}$ is symmetric. This improves our previous result, Thm. 7.3 [11], where we gave $G\rho^{1/2} = \rho^{1/2} G^* - (2iK + ic)\,\rho^{1/2}$ for some $c \in \mathbb{R}$ as an additional condition. Here we showed that it follows from (14) and the invariance of ρ.

Remark 4. Note that (14) holds for the operators L_ℓ of a special GKSL representation of $\mathcal{L}$ if and only if it is true for *all* special GKSL representations because of the second part of Theorem 2. Therefore the conclusion of Theorem 5 holds true also if and only if we can find a single special GKSL representation of $\mathcal{L}$ satisfying (14).

The T-symmetric unitary $(u_{m\ell})_{m\ell}$ is determined by the L_ℓ's because they are linearly independent. We shall now exploit this fact to give a more geometrical characterisation of SQDB.

When the SQDB holds, the matrices $(b_{kj})_{k,j\geq 1}$ and $(c_{kj})_{k,j\geq 1}$ with

$$b_{kj} = \mathrm{tr}\left(\rho^{1/2} L_k^* \rho^{1/2} L_j^*\right), \quad \text{and} \quad c_{kj} = \mathrm{tr}\left(\rho L_k^* L_j\right) \tag{17}$$

define two trace class operators B and C on k by Lemma 3 (see the Appendix); B is T-symmetric and C is self-adjoint. Moreover, it admits a self-adjoint inverse C^{-1} because ρ is faithful. When k is infinite dimensional, C^{-1} is unbounded and its domain coincides with the range of C.

We can now give the following characterisation of QMS satisfying the SQDB condition which is more direct because the unitary $(u_{k\ell})_{k\ell}$ in Theorem 5 is explicitly given by $C^{-1}B$.

Theorem 6. *$\mathcal{T}$ satisfies the SQDB if and only if the operators G, L_k of a special GKSL representation of the generator $\mathcal{L}$ satisfy the following conditions:*

(i) *the closed linear span of $\left\{\rho^{1/2} L_\ell^* \mid \ell \geq 1\right\}$ and $\left\{L_\ell \rho^{1/2} \mid \ell \geq 1\right\}$ in the Hilbert space of Hilbert-Schmidt operators on h coincide,*
(ii) *the trace-class operators B, C defined by (17) satisfy $CB = BC^T$ and $C^{-1}B$ is unitary T-symmetric.*

Proof. If $\mathcal{T}$ satisfies the SQDB then, by Theorem 5, the identity (14) holds. The series in the right-hand side of (14) is convergent with respect to the Hilbert-Schmidt norm because

$$\begin{aligned}\left\| \sum_{m+1\leq \ell\leq n} u_{k\ell} L_\ell \rho^{1/2} \right\|_{HS}^2 &= \sum_{m+1\leq \ell,\ell'\leq n} \bar{u}_{k\ell'} u_{k\ell} \mathrm{tr}\left(\rho L_{\ell'}^* L_\ell\right) \\ &\leq \frac{1}{2} \sum_{m+1\leq \ell,\ell'\leq n} |u_{k\ell'}|^2 \, |u_{k\ell}|^2 + \frac{1}{2} \sum_{m+1\leq \ell,\ell'\leq n} |c_{\ell'\ell}|^2 \\ &\leq \frac{1}{2} \left(\sum_{m+1\leq \ell\leq n} |u_{k\ell}|^2 \right)^2 + \frac{1}{2} \sum_{m+1\leq \ell,\ell'\leq n} |c_{\ell'\ell}|^2,\end{aligned}$$

and the right-hand side vanishes as n, m go to infinity because the operator C is trace-class by Lemma 3 and the columns of $U = (u_{k\ell})_{k\ell}$ are unit vectors in k by unitarity.

Left multiplying both sides of (14) by $\rho^{1/2} L_j^*$ and taking the trace we find $B = CU^T = CU$. It follows that the range of the operators B, CU and C coincide and $C^{-1}B = U$ is everywhere defined, unitary and T-symmetric because U is T-symmetric. Moreover, since B is T-symmetric by the cyclic property of the trace, we have also

$$BC^T = CU^T C^T = C(CU)^T = CB^T = CB.$$

Conversely, we show that (i) and (ii) imply the SQDB. To this end notice that, by the spectral theorem we can find a unitary linear transformation $V = (v_{mn})_{m,n\geq 1}$ on k such that V^*CV is diagonal. Therefore, choosing a new GKSL representation of the generator $\mathcal{L}$ by means of the operators $L''_k = \sum_{n\geq 1} v_{nk} L_n$, if necessary, we can suppose that both $(L_\ell \rho^{1/2})_{\ell\geq 1}$ and $(\rho^{1/2} L_k^*)_{k\geq 1}$ are *orthogonal* bases of the same closed linear space. Note that

$$\mathrm{tr}\,(\rho^{1/2}(L'')_k^* \rho^{1/2}(L'')_j^*) = \sum_{m,n\geq 1} \bar{v}_{nk}\bar{v}_{mj}\,\mathrm{tr}\,(\rho^{1/2} L_n^* \rho^{1/2} L_m^*)$$

and the operator B, after this change of GKSL representation, becomes $V^*B(V^*)^T$ which is also T-symmetric.

Writing the expansion of $\rho^{1/2}L_k^*$ with respect to the orthogonal basis $(L_\ell\rho^{1/2})_{\ell\geq 1}$, for all $k\geq 1$ we have

$$\rho^{1/2}L_k^* = \sum_{\ell\geq 1} \frac{\mathrm{tr}\,(\rho^{1/2}L_\ell^*\rho^{1/2}L_k^*)}{\|L_\ell\rho^{1/2}\|_{HS}^2}\, L_\ell\,\rho^{1/2}. \tag{18}$$

In this way we find a matrix Y of complex numbers $y_{k\ell}$ such that $\rho^{1/2}L_k^* = \sum_\ell y_{k\ell}L_\ell\rho^{1/2}$ and the series is Hilbert-Schmidt norm convergent. Clearly, since C is diagonal and B is T-symmetric, $y_{k\ell} = (BC^{-1})_{k\ell} = ((B(C^{-1})^T)_{k\ell} = ((C^{-1}B)^T)_{k\ell}$. It follows from (ii) that Y coincides with the unitary operator $(C^{-1}B)^T$ and (14) holds. Moreover, Y is symmetric because

$$y_{\ell k} = (BC^{-1})_{\ell k} = ((B(C^{-1})^T)_{\ell k} = (C^{-1}B)_{k\ell} = y_{k\ell}.$$

This completes the proof. □

Formula (18) has the following consequence.

Corollary 1. *Suppose that a QMS $\mathcal{T}$ satisfies the SQDB condition. For every special GKSL representation of $\mathcal{L}$ with operators $L_\ell\rho^{1/2}$ that are orthogonal in the Hilbert space of Hilbert-Schmidt operators on h if $tr(\rho^{1/2}L_\ell^*\rho^{1/2}L_k^*)\neq 0$ for a pair of indices $k,\ell\geq 1$, then $tr(\rho L_\ell^* L_\ell) = tr(\rho L_k^* L_k)$.*

Proof. It suffices to note that the matrix $(u_{k\ell})$ with entries

$$u_{k\ell} = \frac{\mathrm{tr}\,(\rho^{1/2}L_\ell^*\rho^{1/2}L_k^*)}{\|L_\ell\rho^{1/2}\|_{HS}^2} = \frac{\mathrm{tr}\,(\rho^{1/2}L_\ell^*\rho^{1/2}L_k^*)}{\mathrm{tr}\,(\rho L_\ell^* L_\ell)}$$

must be T-symmetric. □

Remark 5. The matrix C can be viewed as the covariance matrix of the zero-mean (recall that $\mathrm{tr}\,(\rho L_\ell) = 0$) "random variables" $\{\, L_\ell \mid \ell\geq 1\,\}$ and in a similar way, B can be viewed as a sort of mixed covariance matrix between the previous random variable and the adjoint $\{\, L_\ell^* \mid \ell\geq 1\,\}$. Thus the SQDB condition holds when the random variables L_ℓ right multiplied by $\rho^{1/2}$ and the adjoint variables L_ℓ^* left multiplied by $\rho^{1/2}$ generate the same subspace of Hilbert-Schmidt operators and the mixed covariance matrix B is a left unitary transformation of the covariance matrix C.

If we consider a special GKSL representation of $\mathcal{L}$ with operators $L_\ell\rho^{1/2}$ that are orthogonal, then, by Corollary 1 and the identity $\|L_\ell\rho^{1/2}\|_{HS} = \|L_k\rho^{1/2}\|_{HS}$, the unitary matrix U can be written as $C^{-1/2}BC^{-1/2}$. This, although not positive definite, can be interpreted as a *correlation coefficient* matrix of $\{\, L_\ell \mid \ell\geq 1\,\}$ and $\{\, L_\ell^* \mid \ell\geq 1\,\}$.

The characterisation of generators of symmetric QMSs with respect to the $s=1/2$ scalar product follows along the same lines.

Theorem 7. *A norm-continuous QMS $\mathcal{T}$ is symmetric if and only if there exists a special GKSL representation of the generator $\mathcal{L}$ by means of operators G, L_ℓ such that*

(1) $G\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ *for some* $c \in \mathbb{R}$,
(2) $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}$, *for all* k, *for some unitary* $(u_{k\ell})_{k\ell}$ *on* k *which is also* T*-symmetric.*

Proof. Choose a special GKSL representation of $\mathcal{L}$ by means of operators G, L_k. Theorem 4 allows us to write the symmetric dual $\mathcal{L}'$ in a special GKSL representation by means of operators G', L'_k as in (15).

Suppose first that $\mathcal{T}$ is KMS-symmetric. Comparing the special GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$, by Theorem 2 we find

$$G = G' + ic, \quad L'_k = \sum_j u_{kj}L_j,$$

for some unitary matrix (u_{kj}) and some $c \in \mathbb{R}$. This, together with (15) implies that conditions (1) and (2) hold.

Assume now that conditions (1) and (2) hold. Taking the adjoint of (2) we find immediately $L_k\rho^{1/2} = \sum_k \overline{u}_{k\ell}\rho^{1/2}L_\ell^*$. Then a straightforward computation, by the unitarity of the matrix $(u_{k\ell})$, yields

$$\begin{aligned}\mathcal{L}_*(\rho^{1/2}x\rho^{1/2}) &= G\rho^{1/2}x\rho^{1/2} + \sum_k L_k\rho^{1/2}x\rho^{1/2}L_k^* + \rho^{1/2}x\rho^{1/2}G^* \\ &= \rho^{1/2}G^*x\rho^{1/2} + \sum_{\ell kj} \overline{u}_{k\ell}\, u_{kj}\, \rho^{1/2}L_k^*xL_j\rho^{1/2} + \rho^{1/2}xG\rho^{1/2} \\ &= \rho^{1/2}\mathcal{L}(x)\rho^{1/2}\end{aligned}$$

for all $x \in \mathcal{B}(\mathsf{h})$. Iterating we find $\mathcal{L}_*^n(\rho^{1/2}x\rho^{1/2}) = \rho^{1/2}\mathcal{L}^n(x)\rho^{1/2}$ for all $n \geq 0$, therefore, exponentiating, we find $\mathcal{T}_{*t}(\rho^{1/2}x\rho^{1/2}) = \rho^{1/2}\mathcal{T}_t(x)\rho^{1/2}$ for all $t \geq 0$. This, together with (3), implies that $\mathcal{T}$ is KMS-symmetric. □

Remark 6. Note that condition (2) in Theorem 7 implies that the completely positive part of $\mathcal{L}$ is KMS-symmetric. This makes a parallel with Theorem 4, where condition (2) implies that the completely positive parts of the generators $\mathcal{L}$ and $\mathcal{L}'$ are mutually adjoint.

The above theorem simplifies a previous result by Park ([23], Thm 2.2) where conditions (1) and (2) appear in a much more complicated way.

5. Generators of Standard Detailed Balance (with Time Reversal) QMSs

We shall now study generators of semigroups satisfying the SQDB-θ introduced in Definition 3 involving the time reversal operation. In this section, we always assume that the invariant state ρ and the anti-unitary time reversal θ commute.

The relationship between the QMS satisfying the SQDB-θ, its dual and their generators is clarified by the following

Proposition 5. *A QMS $\mathcal{T}$ satisfies the SQDB-θ if and only if the dual semigroup $\mathcal{T}'$ is given by*

$$\mathcal{T}_t'(x) = \theta \mathcal{T}_t(\theta x \theta)\theta \qquad \textit{for all } x \in \mathcal{B}(\mathsf{h}). \tag{19}$$

In particular, if $\mathcal{T}$ is norm-continuous, then $\mathcal{T}'$ is also norm-continuous. Moreover, in this case $\mathcal{T}'$ is generated by

$$\mathcal{L}'(x) = \theta \mathcal{L}(\theta x \theta)\theta, \qquad x \in \mathcal{B}(\mathsf{h}). \tag{20}$$

Proof. Suppose that $\mathcal{T}$ satisfies the SQDB-θ and put $\sigma(x) = \theta x \theta$. Taking $t = 0$ Eq. (6) reduces to $\mathrm{tr}\,(\rho^{1/2} x \rho^{1/2} y) = \mathrm{tr}\,(\rho^{1/2}\sigma(y^*)\rho^{1/2}\sigma(x^*))$ for all $x, y \in \mathcal{B}(\mathsf{h})$, so that

$$\begin{aligned}
\mathrm{tr}\,(\rho^{1/2} x \rho^{1/2}\mathcal{T}_t(y)) &= \mathrm{tr}\,(\rho^{1/2}\sigma(y^*)\rho^{1/2}\mathcal{T}_t(\sigma(x^*)))\\
&= \mathrm{tr}\,(\rho^{1/2}\sigma(\mathcal{T}_t(\sigma(x^*))^*\rho^{1/2}\sigma(\sigma(y^*)^*))\\
&= \mathrm{tr}\,(\rho^{1/2}\sigma(\mathcal{T}_t(\sigma(x)))\rho^{1/2} y)
\end{aligned}$$

for every $x, y \in \mathcal{B}(\mathsf{h})$ and (19) follows. Therefore, if $\mathcal{T}$ is norm continuous, $\mathcal{T}_t' = (\sigma \circ \mathcal{T}_t \circ \sigma)_t$ is also.

Conversely, if (19) holds, the commutation between ρ and θ implies

$$\begin{aligned}
\mathrm{tr}\,(\rho^{1/2}\mathcal{T}_t'(x)\rho^{1/2} y) &= \mathrm{tr}\left(\rho^{1/2}\theta\mathcal{T}_t(\theta x \theta)\theta\rho^{1/2} y\right)\\
&= \mathrm{tr}\left(\theta\left(\rho^{1/2}\mathcal{T}_t(\theta x\theta)\theta\rho^{1/2} y\theta\right)\theta\right)\\
&= \mathrm{tr}\left(\rho^{1/2}\theta y^*\rho^{1/2}\theta\mathcal{T}_t(\theta x^*\theta)\right)
\end{aligned}$$

and (19) is proved. Now (20) follows from (19) differentiating at $t = 0$. □

We can now describe the relationship between special GKSL representations of $\mathcal{L}$ and $\mathcal{L}'$.

Proposition 6. *If $\mathcal{T}$ satisfies the SQDB-θ then, for every special GKSL representation of $\mathcal{L}$ by means of operators H, L_k, the operators $H' = -\theta H \theta$ and $L_k' = \theta L_k \theta$ yield a special GKSL representation of $\mathcal{L}'$.*

Proof. Consider a special GKSL representation of $\mathcal{L}$ by means of operators H, L_k. Since $\mathcal{L}'(a) = \theta\mathcal{L}(\theta a \theta)\theta$ by Proposition 5, from the antilinearity of θ and $\theta^2 = \mathbb{1}$ we get

$$\begin{aligned}
\theta\mathcal{L}'(a)\,\theta &= i[H, \theta a \theta] - \frac{1}{2}\sum_k \left(L_k^* L_k \theta a \theta - 2 L_k^* \theta a \theta L_k + \theta a \theta L_k^* L_k\right)\\
&= i\theta\,(\theta H \theta a - a\theta H\theta)\,\theta + \sum_k \theta\left((\theta L_k^*\theta) a (\theta L_k \theta)\right)\theta\\
&\quad -\frac{1}{2}\sum_k \theta\left((\theta L_k^*\theta)(\theta L_k\theta) a + a(\theta L_k^*\theta)(\theta L_k \theta)\right)\theta\\
&= \theta\,(-i[\theta H \theta, a]\,)\,\theta - \frac{1}{2}\sum_k \theta\left(L_k'^* L_k' a - 2 L_k'^* a L_k' + a L_k'^* L_k'\right)\theta,
\end{aligned}$$

where $L_k' := \theta L_k \theta$. Therefore, putting $H' = -\theta H \theta$, we find a GKSL representation of $\mathcal{L}'$ which is also special because $\mathrm{tr}\,(\rho L_k') = \mathrm{tr}\,(\theta\rho L_k\theta) = \mathrm{tr}\,(L_k^*\rho) = \overline{\mathrm{tr}\,(\rho L_k)} = 0$.

□

The structure of generators of QMSs satisfying the SQDB-θ is described by the following

Theorem 8. *A QMS $\mathcal{T}$ satisfies the SQDB-θ condition if and only if there exists a special GKSL representation of $\mathcal{L}$, with operators G, L_ℓ, such that:*

1. $\rho^{1/2}\theta G^*\theta = G\rho^{1/2}$,
2. $\rho^{1/2}\theta L_k^*\theta = \sum_j u_{kj} L_j \rho^{1/2}$ *for a self-adjoint unitary* $(u_{kj})_{kj}$ *on* k.

Proof. Suppose that $\mathcal{T}$ satisfies the SQDB-θ condition and consider a special GKSL representation of the generator $\mathcal{L}$ with operators G, L_k. The operators $-\theta H\theta$ and $\theta L_k\theta$ give then a special GKSL representation of $\mathcal{L}'$ by Proposition 6. Moreover, by Theorem 4, we have another special GKSL representation of $\mathcal{L}'$ by means of operators G', L'_k such that $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ for some $c \in \mathbb{R}$, and $L'_k\rho^{1/2} = \rho^{1/2}L_k^*$. Therefore there exists a unitary $(v_{kj})_{kj}$ on k such that $L'_k = \sum_j v_{kj}\theta L_j\theta$, and $\rho^{1/2}L_k^* = \sum_j v_{kj}\theta L_j\theta\rho^{1/2}$. Condition 2 follows then with $u_{kj} = \bar{v}_{kj}$ left and right multiplying by the antiunitary θ.

In order to find condition 1, first notice that by the unitarity of $(v_{kj})_{kj}$,

$$\sum_k L_k'^* L_k' = \sum_k \theta L_k^* L_k \theta. \tag{21}$$

Now, by the uniqueness of G' up to a purely imaginary multiple of the identity in a special GKSL representation, $H' = (G'^* - G')/(2i)$ is equal to $-\theta H\theta + c_1$ for some $c_1 \in \mathbb{R}$. From (21) and $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ we obtain then

$$\begin{aligned}
\rho^{1/2}G^* + ic\rho^{1/2} = G'\rho^{1/2} &= -iH'\rho^{1/2} - \frac{1}{2}\sum_k L_k'^* L_k'\rho^{1/2} \\
&= i\theta H\theta\rho^{1/2} + ic_1\rho^{1/2} - \frac{1}{2}\sum_k \theta L_k^* L_k\theta\rho^{1/2} \\
&= \theta G\theta\rho^{1/2} + ic_1\rho^{1/2}.
\end{aligned}$$

It follows that $\rho^{1/2}\theta G^*\theta = G\rho^{1/2} + ic_2\rho^{1/2}$ for some $c_2 \in \mathbb{R}$. Left multiplying by $\rho^{1/2}$ and tracing we find

$$ic_2 = \operatorname{tr}\left(\theta\rho G^*\theta\right) - \operatorname{tr}(\rho G) = \operatorname{tr}(G\rho) - \operatorname{tr}(\rho G) = 0$$

and condition 1 holds.

Finally we show that the square of the unitary $(u_{kj})_{kj}$ on k is the identity operator. Indeed, taking the adjoint of the identity $\rho^{1/2}\theta L_k^*\theta = \sum_j u_{kj}L_j\rho^{1/2}$, we have

$$\theta L_k\theta\rho^{1/2} = \sum_j \bar{u}_{kj}\rho^{1/2}L_j^*.$$

Left and right multiplying by the antilinear time reversal θ (commuting with ρ) we find

$$L_k\rho^{1/2} = \sum_j \theta\bar{u}_{kj}\rho^{1/2}L_j^*\theta = \sum_j u_{kj}\rho^{1/2}\theta L_j^*\theta.$$

Writing $\rho^{1/2}\theta L_j^*\theta$ as $\sum_m u_{jm}L_m\rho^{1/2}$ by condition 2 we have then

$$L_k\rho^{1/2} = \sum_{j,m} u_{kj}u_{jm}L_m\rho^{1/2} = \sum_m (u^2)_{km}L_m\rho^{1/2}$$

which implies that $u^2 = \mathbb{1}$ by the linear independence of the $L_m \rho^{1/2}$. Therefore, since u is unitary, $u = u^*$.

Conversely, if 1 and 2 hold, we can write $\rho^{1/2}\theta\mathcal{L}(\theta x\theta)\theta\rho^{1/2}$ as

$$\rho^{1/2}\theta G^*\theta x\rho^{1/2} + \sum_k \rho^{1/2}\theta L_k^*\theta x\theta L_k\theta\rho^{1/2} + \rho^{1/2}x\theta G\theta\rho^{1/2}$$
$$= G\rho^{1/2}x\rho^{1/2} + \sum_j L_j\rho^{1/2}x\rho^{1/2}L_j^* + \rho^{1/2}x\rho^{1/2}G^*.$$

This, by Theorem 4, can be written as

$$\rho^{1/2}(G')^*x\rho^{1/2} + \sum_j \rho^{1/2}(L'_j)^*xL'_j\rho^{1/2} + \rho^{1/2}xG'\rho^{1/2} = \rho^{1/2}\mathcal{L}'(x)\rho^{1/2}.$$

It follows that $\theta\mathcal{L}(\theta x\theta)\theta = \mathcal{L}'(x)$ for all $x \in \mathcal{B}(\mathsf{h})$ because ρ is faithful. Moreover, it is easy to check by induction that $\theta\mathcal{L}^n(\theta x\theta)\theta = (\mathcal{L}')^n(x)$ for all $n \geq 0$. Therefore $\theta\mathcal{T}_t(\theta x\theta)\theta = \mathcal{T}'_t(x)$ for all $t \geq 0$ and $\mathcal{T}$ satisfies the SQDB-θ condition by Proposition 5. □

We now provide a geometrical characterisation of the SQDB-θ condition as in Theorem 6. To this end we introduce the trace class operator R on k

$$R_{jk} = \mathrm{tr}\left(\rho^{1/2}L_j^*\rho^{1/2}\theta L_k^*\theta\right). \tag{22}$$

A direct application of Lemma 3 shows that R is trace class. Moreover it is self-adjoint because, by the property $\mathrm{tr}(\theta x\theta) = \mathrm{tr}(x^*)$ of the antilinear time reversal, we have

$$\begin{aligned}\overline{R}_{jk} &= \overline{\mathrm{tr}\left(\rho^{1/2}L_j^*\rho^{1/2}\theta L_k^*\theta\right)}\\ &= \mathrm{tr}\left(\theta(L_k\theta\rho^{1/2}L_j\rho^{1/2}\theta)\theta\right)\\ &= \mathrm{tr}\left(\rho^{1/2}\theta L_j^*\rho^{1/2}\theta L_k^*\right)\\ &= \mathrm{tr}\left((\rho^{1/2}\theta L_j^*\theta)(\rho^{1/2}L_k^*)\right) = R_{kj}.\end{aligned}$$

Theorem 9. *$\mathcal{T}$ satisfies the SQDB-θ if and only if the operators G, L_k of a special GKSL representation of the generator $\mathcal{L}$ fulfill the following conditions:*

1. $\rho^{1/2}\theta G^*\theta = G\rho^{1/2}$,
2. *the closed linear span of* $\{\rho^{1/2}\theta L_\ell^*\theta \mid \ell \geq 1\}$ *and* $\{L_\ell\rho^{1/2} \mid \ell \geq 1\}$ *in the Hilbert space of Hilbert-Schmidt operators on* h *coincide,*
3. *the self-adjoint trace class operators R, C defined by (17) and (22) commute and $C^{-1}R$ is unitary and self-adjoint.*

Proof. It suffices to show that conditions 2 and 3 above are equivalent to condition 2 of Theorem 8.

If $\mathcal{T}$ satisfies the SQBD-θ, then it can be shown as in the proof of Theorem 6 that 2 follows from condition 2 of Theorem 8. Moreover, left multiplying by $\rho^{1/2}L_\ell^*$ the identity $\rho^{1/2}\theta L_k^*\theta = \sum_j u_{kj}L_j\rho^{1/2}$ and tracing, we find

$$\mathrm{tr}\left(\rho^{1/2}L_\ell^*\rho^{1/2}\theta L_k^*\theta\right) = \sum_j u_{kj}\mathrm{tr}\left(\rho L_\ell^*L_j\right)$$

for all k, ℓ, i.e. $R = CU^T$. The operator U^T is also self-adjoint and unitary. Therefore R and C have the same range and, since the domain of C^{-1} coincides with the range of C, the operator $C^{-1}R$ is everywhere defined, unitary and self-adjoint. It follows that the densely defined operator RC^{-1} is a restriction of $(C^{-1}R)^* = C^{-1}R$ and $CR = RC$.

In order to prove, conversely, that 2 and 3 imply condition 2 of Theorem 8, we first notice that, by the spectral theorem there exists a unitary $V = (v_{mn})_{m,n\geq 1}$ on the multiplicity space k such that V^*CV is diagonal. Choosing a new GKSL representation of the generator $\mathcal{L}$ by means of the operators $L''_k = \sum_{n\geq 1} v_{nk}L_n$, if necessary, we can suppose that both $(L_\ell\rho^{1/2})_{\ell\geq 1}$ and $(\rho^{1/2}L_k^*)_{k\geq 1}$ are *orthogonal* bases of the same closed linear space. Note that

$$\mathrm{tr}\left(\rho^{1/2}(L'')_k^*\rho^{1/2}\theta(L'')_j^*\theta\right) = \sum_{m,n\geq 1} \bar{v}_{nk}v_{mj}\,\mathrm{tr}\,(\rho^{1/2}L_n^*\rho^{1/2}\theta L_m^*\theta)$$

and the operator R, in the new GKSL representation, transforms into V^*RV which is also self-adjoint.

Expanding $\rho^{1/2}\theta L_k^*\theta$ with respect to the orthogonal basis $(L_\ell\rho^{1/2})_{\ell\geq 1}$, for all $k\geq 1$, we have

$$\rho^{1/2}\theta L_k^*\theta = \sum_{\ell\geq 1} \frac{\mathrm{tr}\,(\rho^{1/2}L_\ell^*\rho^{1/2}\theta L_k^*\theta)}{\|L_\ell\rho^{1/2}\|_{HS}^2}\, L_\ell\,\rho^{1/2}, \tag{23}$$

i.e. $\rho^{1/2}\theta L_k^*\theta = \sum_\ell y_{k\ell}L_\ell\rho^{1/2}$ with a unitary matrix Y of complex numbers $y_{k\ell}$.

Clearly, we have $y_{k\ell} = (C^{-1}R)_{\ell k}$. It follows then from condition 3 above that Y coincides with the unitary operator $(C^{-1}R)^T$ and condition 2 of Theorem 8 holds. Moreover, Y is self-adjoint because both R and C are. □

As an immediate consequence of the commutation of R and C we have the following parallel of Corollary 1 for the SQDB condition

Corollary 2. *Suppose that a QMS $\mathcal{T}$ satisfies the SQDB-θ condition. For every special GKSL representation of $\mathcal{L}$ with operators $L_\ell\rho^{1/2}$ orthogonal as Hilbert-Schmidt operators on h if $tr\,(\rho^{1/2}L_\ell^*\rho^{1/2}\theta L_k^*\theta)\neq 0$ for a pair of indices $k,\ell\geq 1$, then $tr\,(\rho L_\ell^*L_\ell) = tr\,(\rho L_k^*L_k)$.*

When the time reversal θ is given by the conjugation $\theta u = \bar{u}$ (with respect to some orthonormal basis of h), $\theta x^*\theta$ is equal to the transpose x^T of x and we find the following

Corollary 3. *$\mathcal{T}$ satisfies the SQDB-θ condition if and only if there exists a special GKSL representation of $\mathcal{L}$, with operators G, L_k, such that:*

1. $\rho^{1/2}G^T = G\rho^{1/2}$*;*
2. $\rho^{1/2}L_k^T = \sum_j u_{kj}L_j\rho^{1/2}$ *for some unitary self-adjoint* $(u_{kj})_{kj}$*.*

6. SQDB-θ for QMS on $M_2(\mathbb{C})$

In this section, as an application, we find a standard form of a special GKSL representation of the generator $\mathcal{L}$ of a QMS on $M_2(\mathbb{C})$ satisfying the SQDB-θ.

The faithful invariant state ρ, in a suitable basis of $\mathbb{C}^2$, can be written in the form

$$\rho = \begin{pmatrix} \nu & 0 \\ 0 & 1-\nu \end{pmatrix} = \frac{1}{2}\left(\sigma_0 + (2\nu-1)\sigma_3\right), \qquad 0<\nu<1,$$

where σ_0 is the identity matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The time reversal θ is the usual conjugation in the same basis of $\mathbb{C}^2$.

In order to determine the structure of the operators G and L_k satisfying conditions of Corollary 3 we find first a convenient basis of $M_2(\mathbb{C})$. We choose then a basis of eigenvectors of the linear map $X \to \rho^{1/2} X^T \rho^{-1/2}$ in $M_2(\mathbb{C})$ given by $\sigma_0, \sigma_1^\nu, \sigma_2^\nu, \sigma_3$, where

$$\sigma_1^\nu = \begin{pmatrix} 0 & \sqrt{2\nu} \\ \sqrt{2(1-\nu)} & 0 \end{pmatrix}, \qquad \sigma_2^\nu = \begin{pmatrix} 0 & -i\sqrt{2\nu} \\ i\sqrt{2(1-\nu)} & 0 \end{pmatrix}.$$

Indeed, $\sigma_0, \sigma_1^\nu, \sigma_3$ (resp. σ_2^ν) are eigenvectors of the eigenvalue 1 (resp. -1).

Every special GKSL representation of $\mathcal{L}$ is given by (see [11], Lemma 6.1)

$$L_k = -(2\nu - 1) z_{k3}\sigma_0 + z_{k1}\sigma_1^\nu + z_{k2}\sigma_2^\nu + z_{k3}\sigma_3, \qquad k \in \mathcal{J} \subseteq \{1, 2, 3\}$$

with vectors $z_k := (z_{k1}, z_{k2}, z_{k3})$ ($k \in \mathcal{J}$) linearly independent in $\mathbb{C}^3$.

The SQDB-θ holds if and only if G, L_k satisfy

(i) $G = \rho^{1/2} G^T \rho^{-1/2}$,
(ii) $L_k = \sum_{j\in\mathcal{J}} u_{kj} \rho^{1/2} L_j^T \rho^{-1/2}$ for some unitary self-adjoint $U = (u_{kj})_{k,j\in\mathcal{J}}$.

Now, if $\mathcal{J} \neq \emptyset$, since every unitary self-adjoint matrix is diagonalizable and its spectrum is contained in $\{-1, 1\}$, it follows that $U = W^* D W$ for some unitary matrix $W = (w_{ij})_{i,j\in\mathcal{J}}$ and some diagonal matrix D of the form

$$\mathrm{diag}(\epsilon_1, \ldots, \epsilon_{|\mathcal{J}|}), \qquad \epsilon_i \in \{-1, 1\}, \tag{24}$$

where $|\mathcal{J}|$ denotes the cardinality of $\mathcal{J}$. Therefore, replacing the L_k's by operators $L'_k := \sum_{j\in\mathcal{J}} w_{kj} L_j$ if necessary, we can take U of the form (24).

We now analyze the structure of L_k's corresponding to the different (diagonal) forms of U. By condition (ii) we have either $L_k = \rho^{1/2} L_k^T \rho^{-1/2}$ or $L_k = -\rho^{1/2} L_k^T \rho^{-1/2}$; an easy calculation shows that

$$L_k = \rho^{1/2} L_k^T \rho^{-1/2} \quad \text{if and only if} \quad z_{k2} = 0 \tag{25}$$

and

$$L_k = -\rho^{1/2} L_k^T \rho^{-1/2} \quad \text{if and only if} \quad z_{k1} = z_{k3} = 0. \tag{26}$$

Therefore, the linear independence of $\{z_j : j \in \mathcal{J}\}$ forces U to have at most two eigenvalues equal to 1 and at most one equal to -1 and, with a suitable choice of a phase factor for each L_k, we can write

$$L_k = (1 - 2\nu) r_k \sigma_0 + r_k \sigma_3 + \zeta_k \sigma_1^\nu \text{ for } k = 1, 2 \text{ and } r_k \in \mathbb{R}, \zeta_k \in \mathbb{C} \tag{27}$$

$$L_3 = r_3 \sigma_2^\nu, \quad r_3 \in \mathbb{R}. \tag{28}$$

Clearly L_1 and L_2 are linearly independent if and only if $r_1\zeta_2 \neq r_2\zeta_1$. This, together with non triviality conditions leaves us, up to a change of indices, with the following possibilities:

(a) $|\mathcal{J}| = 1$, $U = 1$ then $\mathcal{J} = \{1\}$ with $r_1\zeta_1 \neq 0$,
(b) $|\mathcal{J}| = 1$, $U = -1$ then $\mathcal{J} = \{3\}$ with $r_3 \neq 0$,
(c) $|\mathcal{J}| = 2$, $U = \mathrm{diag}(1, 1)$ then $\mathcal{J} = \{1, 2\}$ with $r_1\zeta_1 r_2\zeta_2 \neq 0$, $r_1\zeta_2 \neq r_2\zeta_1$,
(d) $|\mathcal{J}| = 2$, $U = \mathrm{diag}(1, -1)$ then $\mathcal{J} = \{1, 3\}$, with $r_3 \neq 0$, $r_1\zeta_1 \neq 0$,
(e) $|\mathcal{J}| = 3$, $U = \mathrm{diag}(1, 1, -1)$ then $\mathcal{J} = \{1, 2, 3\}$ with $r_1\zeta_2 \neq r_2\zeta_1$, $r_3 \neq 0$, $r_1\zeta_1 r_2\zeta_2 \neq 0$.

To conclude, we analyze condition (i). If $G = \left(g_{jk}\right)_{1\leq j,k\leq 2}$ then statement (i) is equivalent to

$$\sqrt{\nu}\, g_{21} = \sqrt{1-\nu}\, g_{12}. \tag{29}$$

Since $G = -iH - 2^{-1}\sum_k L_k^* L_k$ with $H = \sum_{j=1}^3 v_j\sigma_j$, $v_j \in \mathbb{R}$, and $\sum_k L_k^* L_k$ is equal to the sum of a term depending only on σ_0 and σ_3 plus

$$\sum_{k=1,2} 2r_k \begin{pmatrix} 0 & \zeta_k\sqrt{2\nu}(1-\nu) - \bar{\zeta}_k\nu\sqrt{2(1-\nu)} \\ \bar{\zeta}_k\sqrt{2\nu}(1-\nu) - \zeta_k\nu\sqrt{2(1-\nu)} & 0 \end{pmatrix},$$

in the case $\mathcal{J} \neq \emptyset$ the identity (29) holds if and only if

$$\begin{cases} v_1\left(\sqrt{1-\nu} - \sqrt{\nu}\right) = -\sqrt{2\nu(1-\nu)}\left(\sqrt{1-\nu} + \sqrt{\nu}\right)^2 \sum_{k=1}^2 r_k \Im\zeta_k \\ v_2\left(\sqrt{1-\nu} + \sqrt{\nu}\right) = -\sqrt{2\nu(1-\nu)}\left(\sqrt{1-\nu} - \sqrt{\nu}\right)^2 \sum_{k=1}^2 r_k \Re\zeta_k \end{cases}. \tag{30}$$

On the other hand, when $\mathcal{J} = \emptyset$, condition (29) is equivalent to $\sqrt{\nu}(v_1 + iv_2) = \sqrt{1-\nu}(v_1 - iv_2)$, i.e.

$$v_1\left(\sqrt{1-\nu} - \sqrt{\nu}\right) = 0, \qquad v_2 = 0, \tag{31}$$

Therefore we have the following possible standard forms for $\mathcal{L}$.

Theorem 10. *Let L_1, L_2, L_3 be as in (27), (28), $H = \sum_{j=1}^3 v_j\sigma_j$ with v_1, v_2 as in (30) and $v_3 \in \mathbb{R}$. The QMS $\mathcal{T}$ satisfies the SQDB-θ if and only if there exists a special GKSL representation of $\mathcal{L}$ given, up to phase factors multiplying L_1, L_2, L_3, in one of the following ways:*

(o) *H with $v_1 = v_2 = 0$ if $\nu \neq 1/2$, and $v_1 \in \mathbb{R}$, $v_2 = 0$ if $\nu = 1/2$,*
(a) *H, L_1 with $r_1\zeta_1 \neq 0$,*
(b) *H, L_3 with $r_3 \neq 0$,*
(c) *H, L_1, L_2 with $r_1\zeta_1 r_2\zeta_2 \neq 0$ and $r_1\zeta_2 \neq r_2\zeta_1$,*
(d) *H, L_1, L_3 with $r_3 \neq 0$ and $r_1\zeta_1 \neq 0$,*
(e) *H, L_1, L_2, L_3 with $r_1\zeta_2 \neq r_2\zeta_1$, $r_1\zeta_1 r_2\zeta_2 \neq 0$ and $r_3 \neq 0$.*

Roughly speaking, the standard form of $\mathcal{L}$ corresponds, up to degeneracies when some of the parameter vanish or when some linear dependence arises, to the case e).

We know that a QMS satisfying the usual (i.e. with pre-scalar product with $s = 0$) QDB-θ condition must commute with the modular group. Moreover, when this happens, the SQDB-θ and QDB-θ conditions are equivalent (see e.g. [6,11]).

We finally show how the generators of a QMSs on $M_2(\mathbb{C})$ satisfying the usual QDB-θ condition can be recovered by a special choice of the parameters $r_1, r_2, r_3, \zeta_1, \zeta_2$ in Theorem 10 describing the generator of a QMS satisfying the SQDB-θ condition.

To this end, we recall that $\mathcal{T}$ fulfills the QDB-θ when $\mathrm{tr}\,(\rho x \mathcal{T}_t(y)) = \mathrm{tr}\,(\rho\theta y^*\theta\mathcal{T}_t(\theta x^*\theta))$ for all $x, y \in \mathcal{B}(\mathsf{h})$. In [11] we classified generators of QMS on $M_2(\mathbb{C})$ satisfying the QDB condition without time reversal (i.e., formally, replacing θ by the identity operator, that is, of course, not antiunitary). The same type of arguments show that, disregarding trivialisations that may occur when some of the parameters below vanishes, QMSs on $M_2(\mathbb{C})$ satisfying the QDB-θ condition have the following standard form

$$\mathcal{L}(x) = i[H, x] - \frac{|\eta|^2}{2}\left(L^2x - 2LxL + xL^2\right)$$
$$-\frac{|\lambda|^2}{2}\left(\sigma^-\sigma^+x - 2\sigma^-x\sigma^+ + x\sigma^-\sigma^+\right) - \frac{|\mu|^2}{2}\left(\sigma^+\sigma^-x - 2\sigma^+x\sigma^- + x\sigma^+\sigma^-\right), \quad (32)$$

where $H = h_0\sigma_0 + h_3\sigma_3$ ($h_0, h_3 \in \mathbb{R}$), $L = -(2\nu - 1)\sigma_0 + \sigma_3$, $\sigma^\pm = (\sigma_1 \pm i\sigma_2)/2$ and, changing phases if necessary, λ, μ, η can be chosen as *non-negative real* numbers satisfying

$$\lambda^2(1-\nu) = \nu\mu^2. \quad (33)$$

Choosing $r_1 = \eta$, $\zeta_1 = 0$ we find immediately that the operator L in (32) coincides with the operator L_1 in (27). Moreover, choosing $r_2 = 0$ we find $v_2 = 0$ and also $v_1 = 0$ for $\nu \neq 1/2$. A straightforward computation yields

$$\begin{pmatrix} \lambda\,\sigma_+ \\ \mu\,\sigma_- \end{pmatrix} = \begin{pmatrix} \lambda/(2\zeta_2\sqrt{2\nu}) & i\lambda/(2r_3\sqrt{2\nu}) \\ \mu/(2\zeta_2\sqrt{2(1-\nu)}) & -i\mu/(2r_3\sqrt{2(1-\nu)}) \end{pmatrix} \begin{pmatrix} L_2 \\ L_3 \end{pmatrix}$$

and the above 2×2 matrix is unitary if we choose $\zeta_2 = \lambda/(2\sqrt{\nu})$, $r_3 = i\mu/(2\sqrt{1-\nu}) = i\zeta_2$ because of (33) and changing the phase of r_3 in order to find a unitary that is also self-adjoint.

This shows that we can recover the standard form (32) choosing H, L_1, L_2, L_3 as in Theorem 10 e) with $r_1 = \eta$, $\zeta_1 = 0$, $r_2 = 0$, $\zeta_2 = \lambda/(2\sqrt{\nu})$, $r_3 = i\mu/(2\sqrt{1-\nu})$, $v_1 = v_2 = 0$.

Appendix

We denote by $\ell^2(J)$ the Hilbert space of complex-valued, square summable sequences indexed by a finite or countable set J.

Lemma 3. *Let $\mathcal{J}$ be a complex separable Hilbert space and let $(\xi_j)_{j\in J}$, $(\eta_j)_{j\in J}$ be two Hilbertian bases of $\mathcal{J}$ satisfying $\sum_{j\in J}\|\xi_j\|^2 < \infty$, $\sum_{j\in J}\|\eta_j\|^2 < \infty$. The complex matrices $A = (a_{jk})_{j,k\in J}$, $B = (b_{jk})_{j,k\in J}$, $C = (c_{jk})_{j,k\in J}$ given by*

$$a_{jk} = \langle\xi_j, \xi_k\rangle, \quad b_{jk} = \langle\xi_j, \eta_k\rangle, \quad c_{jk} = \langle\eta_j, \eta_k\rangle$$

define trace class operators on $\ell^2(J)$ satisfying $B^*A^{-1}B = C$. Moreover A and C are self-adjoint and positive.

Proof. Note that

$$\sum_{j,k\geq 1}|b_{jk}|^2 \leq \sum_{j,k\geq 1}\|\xi_j\|^2\cdot\|\eta_k\|^2 = \sum_j\|\xi_j\|^2\cdot\sum_k\|\eta_k\|^2 < \infty.$$

Therefore B defines a Hilbert-Schmidt operator on $\ell^2(J)$.

In a similar way A and C define Hilbert-Schmidt operators on $\ell^2(J)$ that are obviously self-adjoint. These are also positive because for any sequence $(z_m)_{m\in J}$ of complex numbers with $z_m \neq 0$ for a finite number of indices m at most we have

$$\sum_{m,n\in J} \bar{z}_m a_{mn} z_n = \sum_{m,n\in J} \bar{z}_m \langle \xi_m, \xi_n \rangle z_n = \left\| \sum_{m\in J} z_m \xi_m \right\|^2 \geq 0.$$

Moreover, they are trace class because

$$\sum_{j\in J} a_{jj} = \sum_{j\in J} \|\xi_j\|^2 < \infty, \qquad \sum_{j\in J} c_{jj} = \sum_{j\in J} \|\eta_j\|^2 < \infty.$$

Finally, we show that B is also trace class. By the spectral theorem, we can find a unitary $V = (v_{kj})_{k,j\in J}$ on $\ell^2(J)$ such that V^*AV is diagonal. The series $\sum_{m\in J} v_{mj}\xi_m$ is norm convergent because

$$\left\| \sum_{m} v_{mj}\xi_m \right\|^2 = \sum_{m,n\in J} \bar{v}_{nj} a_{nm} v_{mj} = (V^*AV)_{jj}.$$

The series $\sum_{m\in J} v_{mj}\xi_m$ is norm convergent as well for a similar reason. Therefore, putting $\xi'_j = \sum_{m\in J} v_{mj}\xi_m$ and $\eta'_j = \sum_{m\in J} v_{mj}\eta_m$ we find immediately $(V^*AV)_{kj} = \langle \xi'_k, \xi'_j \rangle = 0$ for $j \neq k$, $(V^*AV)_{jj} = \left\| \xi'_j \right\|^2$ and

$$(V^*BV)_{kj} = \sum_{m,n} \bar{v}_{mk} v_{nj} \langle \xi_m, \eta_j \rangle = \langle \xi'_k, \eta'_j \rangle,$$
$$(V^*CV)_{kj} = \sum_{m,n} \bar{v}_{mk} v_{nj} \langle \eta_m, \eta_j \rangle = \langle \eta'_k, \eta'_j \rangle.$$

As a consequence, the following identity

$$\begin{aligned}\left(V^*B^*A^{-1}BV\right)_{kj} &= \left((V^*B^*V)(V^*AV)^{-1}(V^*BV)\right)_{kj} \\ &= \sum_{m\in J} (V^*B^*V)_{km} \left((V^*AV)_{mm}\right)^{-1} (V^*BV)_{mj} \\ &= \sum_{m\in J} \left\langle \eta'_k, \frac{\xi'_m}{\|\xi'_m\|} \right\rangle \left\langle \frac{\xi'_m}{\|\xi'_m\|}, \eta'_j \right\rangle \\ &= \langle \eta'_k, \eta'_j \rangle = (V^*CV)_{kj}\end{aligned}$$

holds because $(\xi'_m/\|\xi'_m\|)_{m\in J}$ is an orthonormal basis of $\mathcal{J}$.

This proves that $V^*B^*A^{-1}BV = V^*CV$ i.e. $B^*A^{-1}B = C$. It follows that $|A^{-1/2}B| = C^{1/2}$ is Hilbert-Schmidt as well as $A^{-1/2}B$ and $B = A^{1/2}(A^{-1/2}B)$ is trace class being the product of two Hilbert-Schmidt operators. □

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